J. math. fluid mech. 3 (2001) 302–315
1422-6928/01/030302-14 $ 1.50+0.20/0

Journal of Mathematical Fluid Mechanics

Global Existence of Two-Dimensional Navier–Stokes Flow with Nondecaying Initial Velocity

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Dedicated to Professor Takaaki Nishida and Professor Masayasu Mimura on their 60th birthdays

Abstract. A global-in-time unique smooth solution is constructed for the Cauchy problem of the Navier–Stokes equations in the plane when initial velocity field is merely bounded not necessary square-integrable. The proof is based on a uniform bound for the vorticity which is only valid for planar flows. The uniform bound for the vorticity yields a coarse globally-in-time a priori estimate for the maximum norm of the velocity which is enough to extend a local solution. A global existence of solution for a q-th integrable initial velocity field is also established when $q > 2$.

Mathematics Subject Classification (2000). 35Q30, 76D05, 35K55.

Keywords. Nondecaying initial velocity, two-dimentional Navier–Stokes flow, global existence, logarithmic Gronwall inequality, vorticity equation.

1. Introduction

We consider the nonstationary Navier–Stokes equations in the plane:

$$\text{(NS)} \qquad \begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla p = 0 \text{ in } (0,T) \times \mathbb{R}^2, \\ \operatorname{div} u = 0 \text{ in } (0,T) \times \mathbb{R}^2, \\ u|_{t=0} = u_0 \quad (\text{with } \operatorname{div} u_0 = 0) \text{ in } \mathbb{R}^2, \end{cases}$$

where $u = u(t,x) = (u_1(t,x), u_2(t,x))$ and $p = p(t,x)$ stand for the unknown velocity field of the fluid and its pressure field, while $u_0 = u_0(x) = (u_0^1(x), u_0^2(x))$ is a given initial velocity vector field; $x = (x_1, x_2)$ denotes a point of the plane $\mathbb{R}^2$ and $t (\geq 0)$ denotes the time.

*Partly supported by the Grant-in-Aid for Scientific Research No. 10304010, the Japan Society for the Promotion of Science

†Partly supported by the Grant-in-Aid for Scientific Research No. 11640211, the Japan Society for the Promotion of Science.

Our goal is to prove the unique existence of global-in-time smooth solution of (NS) when initial data u_0 is merely bounded uniformly continuous, i.e., $u_0 \in BUC = BUC(\mathbb{R}^2)$ or more generally $u_0 \in L^\infty(\mathbb{R}^2)$. (We do not distinguish spaces of vector-valued and scalar functions.) For a Banach space X and an interval $I \subset \mathbb{R}$ let $C(I;X)$ denote the space of all continuous functions on I with values in X. We are now in position to state our main result.

Theorem 1. *Assume that $u_0 \in BUC$ satisfies $\operatorname{div} u_0 = 0$ in $\mathbb{R}^2$ (in the sense of distribution). Then there exists a unique $u \in C([0,\infty); BUC)$ $(\cap C^\infty((0,\infty)\times\mathbb{R}^2))$ satisfying* (NS) *with $p = \sum_{i,j=1}^2 R_i R_j u_i u_j$, where $R_j = (-\Delta)^{-1/2}\partial/\partial x_j$ is the Riesz operator.*

Remark. If $u_0 \in L^\infty(\mathbb{R}^2)$, it is known in [GIM] that there is a unique local-in-time solution u of (NS) in a suitable sense and $u(t) \in BUC(\mathbb{R}^2)$ for $t > 0$. Thus by Theorem 1 it is extended to a global solution. We do not impose any smallness assumptions on u_0 in Theorem 1.

There is a large literature on local existence of smooth solutions of (NS) even in a various domain of $\mathbb{R}^n (n \geq 2)$. It is also well-known that the solution can be extended globally in time provided that the initial velocity is *small* in various scaling invariant spaces. However, most of results assume a decay at space infinity for initial velocity. A recent paper of Amann [A] includes a nice survey of local solvability for initial data which decays at space infinity. The reader is referred to [A] for the state of arts. For nondecaying initial data there are only a few articles. Cannon and Knightly [CK] constructed a local solution which is continuous up to $t = 0$ for bounded continuous initial data u_0 for $\mathbb{R}^n$. The method is based on the analysis developed by [K1]. Later it is extended in [K2] for bounded initial data. Local solvability for $u_0 \in L^\infty$ is also mentioned in [C]. The method is based on Littlewood–Payley decomposition developed in [CM] and [C]. More recently, K. Inui and the first two authors [GIM] constructed a local solution for $u_0 \in BUC(\mathbb{R}^n)$ or $L^\infty(\mathbb{R}^n)$. Their key estimate can be written as

$$\sup_{t>0} t^{1/2} \cdot ||\nabla E_t||_{L^1(\mathbb{R}^n)} < \infty$$

which yields the crucial estimate

$$\sup_{0<t<1} t^{-1/2} \int_0^t ||\nabla E_s||_{L^1(\mathbb{R}^n)}\, ds < \infty$$

of [CK], where E_t denotes the fundamental solution of the Stokes system $u_t - \Delta u + \nabla p = 0$, $\operatorname{div} u = 0$ in $\mathbb{R}^n$. In [GIM] and [CK] the time T_0 where solution exists in $(0, T_0)$ is estimated by

$$T_0 \geq C/||u_0||_\infty^2 \tag{1.1}$$

with C depending only on n. There are several novelty of [GIM] compared with [CK] or [C]. It proves that $u_0 \in BUC(\mathbb{R}^n)$ implies $u(t,\cdot) \to u_0$ in BUC as $t \to 0$ for the solution u. It also clarifies relation of solutions for the integral equation and the original equation (NS) and discusses the uniqueness of solutions. For nondecaying initial data relation $p = \sum_{i,j} R_i R_j u_i u_j$ is not automatically derived from the Poisson equation $-\Delta p = \operatorname{div}((u, \nabla)u)$, so it is included in our main statement so that the solution is unique. It is curious what condition of p guarantees $p = \sum R_i R_j u_i u_j$. For this direction the reader is referred to a recent paper by J. Kato [Ka]; see also [GIKM]. It seems that there are no results on solvability for exterior domains for nondecaying initial data u_0 although there is a large literature when u_0 is asymptotically constant at the space infinity and $n \geq 3$; see e.g. [BM, KO, S].

It is well-known since Leray's pioneering work [L1] that there exists a global smooth solution if the initial data u_0 is in $L^2(\mathbb{R}^2)$, in other words, the initial kinematic energy is finite. For a such initial data the global existence of solution is proved by a priori estimate called an energy equality:

$$||u||^2_{L^2}(t) + 2\int_0^t ||\nabla u||^2_{L^2}(s)ds = ||u_0||^2_{L^2}, \quad t > 0.$$

This is formally obtained by multiplying u with the first equation of (NS) and integrating by parts. Such an estimate is not expected for $u_0 \in L^\infty(\mathbb{R}^2)$ so we develop a different a priori estimate for the L^∞ norm $||u||_\infty(t)$ of the solution u.

Let us briefly explain main ideas in proving Theorem 1.

(i) The local solution in [GIM] fulfills the integral equation

$$u(t) = e^{t\Delta}u_0 - \int_0^t \operatorname{div}\ (e^{(t-s)\Delta}\mathbb{P}(u(s) \otimes u(s)))ds, \tag{1.2}$$

where $e^{t\Delta}$ denotes the heat semigroup and $\mathbb{P}$ denotes the Helmholtz projection and its ij $(1 \leq j,\ j \leq 2)$ component is of form $\delta_{ij} + R_j R_j$; δ_{ij} denotes Kronecker's delta and $\otimes$ denotes tensor product; $\operatorname{div} F$ for tensor $F = (F_{ij})_{i,j=1,2}$ is defined by a vector $(\sum_{j=1}^2 \partial F_{ij}/\partial x_j)_{i=1,2}$.

(ii) As proved in [GIM] there is a regularizing effect so that $\nabla u(t) \in BUC$ for $t \in (0, T_0)$. Thus we may assume that $\nabla u_0 \in BUC$ to prove the global existence.

(iii) In the plane the vorticity $\omega = \operatorname{rot} u = \partial u_2/\partial x_1 - \partial u_1/\partial x_2$ is scalar and fulfills

$$\omega_t - \Delta\omega + (u, \nabla)\omega = 0. \tag{1.3}$$

(This equation is obtained by applying rot to the first equation of (NS).) The maximum principle yields that $||\omega||_\infty(t) \leq ||\omega_0||_\infty$, where $\omega_0 = \operatorname{rot} u_0$.

(iv) The crucial step is to establish

$$||\operatorname{div} e^{t\Delta}\mathbb{P}(f \otimes f)||_\infty \leq C\Big(1 + \log R + \frac{1}{\sqrt{t}}\Big)||f||_\infty ||\operatorname{rot} f||_\infty + \frac{C}{R}||f||^2_\infty \tag{1.4}$$

for all $R > 1$ and all $t > 0$, where C is a constant independent of t, f, R. Using

(1.4) to estimate the integral of (1.2) with $R = 1 + ||u||_\infty(s)$, we obtain

$$\begin{aligned} ||u||_\infty(t) \le ||u_0||_\infty + C||\omega_0||_\infty \int_0^t (t-s)^{-1/2} \cdot ||u||_\infty(s)\, ds + \\ + C(1 + ||\omega_0||_\infty) \int_0^t \{1 + \log(1 + ||u||_\infty(s))\} \cdot ||u||_\infty(s)\, ds \end{aligned} \tag{1.5}$$

since $||\omega||_\infty(t) \le ||\omega_0||_\infty$.

(v) In the similar way to prove the Gronwall inequality it turns out that (1.5) implies

$$||u||_\infty(t) \le K \exp(Ke^{Kt}) \tag{1.6}$$

with $K > 0$ depending only on $||u_0||_\infty$ and $||\omega_0||_\infty$. Although this estimate looks weak, because of (1.1) this yields the global solvability of (NS). To prove the inequality (1.4) we study the derivatives of the Newton potentials carefully but the proof itself is not so complicated.

Since we do use the vorticity equation (1.3) it is not expected to generalize this method to the Dirichlet problem on an exterior domain or the half space instead of the whole space $\mathbb{R}^2$. If the boundary exists, the property $||\omega||_\infty(t) \le ||\omega_0||_\infty$ is not expected since the vorticity is created near the boundary. The vorticity equation for $\mathbb{R}^3$ is

$$\omega_t - \Delta\omega + (u, \nabla)\omega - (\omega, \nabla)u = 0 \tag{1.7}$$

instead of (1.3). It is not expected to have $||\omega||_\infty(t) \le ||\omega_0||_\infty$ because of the vorticity stretching term $(\omega, \nabla)u$ in (1.7). Thus the present method is not expected to apply the three-dimensional setting. (In $\mathbb{R}^3$ the global existence of smooth solution for non small initial data is a famous open problem since Leray's pioneering work [L2] even for smooth $u_0 \in L^2(\mathbb{R}^3)$.)

For $u_0 \in L^p(\mathbb{R}^2)$, $p > 2$ we also prove that the global existence of smooth solutions. The proof is easier than that of BUC since $\mathbb{P}$ is bounded in $L^q(\mathbb{R}^2)$ for $1 < q < \infty$ by the Calderón–Zygmund inequality. Since the local solution constructed by Amann [A] belongs to $L^q(\mathbb{R}^2)(q > 2)$ for $t > 0$, so his solution can be extended globally in time by our L^p results. For $L^2(\mathbb{R}^2)$ initial data the global existence result goes back to Leray [L1]. For the initial data in the Lorentz space $L^{2,\infty}(\mathbb{R}^2)$ there is a unique global existence result [KY]; their proof is based on a kind of energy estimate.

The logarithmic type Gronwall inequality goes back to a work of Wolibner [W] which starts mathematical analysis on the Euler equation in $\mathbb{R}^2$. It is also found in a paper of Brezis and Gallouet [BG] for proving the global existence of solutions for some semilinear Schrödinger equation. However, in these papers the singular term $(t-s)^{-1/2}$ in (1.5) does not exist so the derivation of (1.6) is much easy. A similar argument to derive (1.5) from (1.4) by setting $R = 1 + ||u||_\infty$ is found in [BG].

This paper is organized as follows. In §2 we recall various properties of the Riesz operators to establish (1.4). In §3 we prove (1.6) by establishing the Gronwall type inequalities. We also prove Theorem 1. In §4 we prove the global existence for L^q initial data ($2 < q < \infty$).

After this work was completed, we were learned of a recent work of Koch and Tataru [KT] who study local well-posedness in the space $\mathrm{BMO}^{-1}(\mathbf{R}^n)$ of derivatives of BMO and related localized space BMO_T^{-1}. Local existence for L^∞ data is proved as a special element of BMO_T^{-1} for $T \in (0,\infty)$. However, for non-small initial data they do not discuss the global solvability for $\mathbb{R}^2$. The authors are grateful to Professor Herbert Koch for informing that [KT] is applicable for L^∞ initial data and for pointing out [W].

After this work was completed, we were learned of a recent work of J. C. Mattingly and Ya G. Sinai [MS] who among other results give a different elementary proof of global existence of smooth solutions for two dimensional Navier–Stokes equations with periodic boundary condition. They use the vorticity equation (1.3) instead of the energy estimate. However, their assumption does not include our setting. The authors are also grateful to Professor Alice Chang for letting them know a recent article [MS]. They are also grateful to Professor Tohru Ozawa for letting them know [BG]. They are also grateful to Professor Hideo Kozono for letting us know the state of arts on the exterior problems.

Before closing this introduction we prepare several notations. Let $e^{t\Delta}$ denote the heat semigroup defined by

$$e^{t\Delta} f = G_t * f,\ G_t(x) = (4\pi t)^{-1} \exp(-|x|^2/4t) \text{ for } t > 0 \text{ and } x \in \mathbb{R}^2,$$

where $*$ denotes convolution of functions defined in $\mathbb{R}^2$. Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^2)$ denote the space of rapidly decreasing functions in the sense of L. Schwartz. The divergence is denoted also by $\nabla\cdot$, for example, $\nabla \cdot F$ for a tensor $F = (F_{ij})_{i,j=1,2}$ is $(\sum_j \partial_j F_{ij})_{i=1,2}$ where $\partial_j = \partial/\partial x_j$.

2. Estimate of the quadratic term

Let R_i be the Riesz operator whose symbol is $\sqrt{-1}\xi_i/|\xi|$ ($i = 1,2$) i.e., $R_i = (-\Delta)^{-1/2}\partial_i$. Let K denote the fundamental solution of the minus Laplacian $-\Delta$ in $\mathbb{R}^2$, i.e., $K(x) = (-1/2\pi)\log|x|$.

We first summarize some properties of the Riesz operator and the fundamental solution of the Laplacian, which may be well-known.

Lemma 1. *The following identities hold for all $\varphi \in \mathcal{S}(\mathbb{R}^2)$ and $i,j,k = 1,2$.*

(1) $R_iR_j\varphi = \mathrm{p.v.}(\partial_i\partial_j K) * \varphi - \dfrac{\delta_{ij}}{2}\varphi.$

(2) $R_iR_j\varphi = \partial_i K * \partial_j\varphi = \partial_j K * \partial_i\varphi.$

(3) $R_iR_j\partial_k\varphi = R_kR_j\partial_i\varphi = R_iR_k\partial_j\varphi.$

(4) $\left(R_1^2 + R_2^2\right)\varphi = -\varphi.$

For an integer $m \geq 1$ there exists a numerical constant $C_m > 0$ such that

(5) $|D^m K(x)| \leq \dfrac{C_m}{|x|^m},$

where $D^m f(x)$ denotes one of m-th derivatives of $f(x)$.

Definition. Let $\chi_{IN}(x)$, $\chi^R_{MID}(x)$ and $\chi^R_{OUT}(x)$ be characteristic functions of $\{0 \leq |x| \leq 1\}$, $\{1 \leq |x| \leq R\}$ and $\{R \leq |x|\}$ respectively for $R > 1$. For the fundamental solution $K(x) = (-1/2\pi)\log|x|$ and $i = 1, 2$ we put

$$J^i_{IN} = \chi_{IN} \cdot \partial_i K, \quad J^i_{MID} = \chi^R_{MID} \cdot \partial_i K, \quad J^i_{OUT} = \chi^R_{OUT} \cdot \partial_i K.$$

By definition $\partial_i K = J^i_{IN} + J^i_{MID} + J^i_{OUT}$. We also define a (vector valued) layer potential $\mathbb{L}^r_i(\varphi) = (L^r_{ij}(\varphi))_j$ by

$$L^r_{ij}(\varphi)(x) = \int_{|y|=r} \partial_i K(y)\varphi(x-y)\frac{y_j}{|y|}dS_y.$$

We shall estimate L^1-norms of these functions.

Lemma 2. *There exists a numerical constant $C > 0$ such that following estimates are valid.*

(0) $||\mathbb{L}^r_i(\varphi)||_1 \leq C||\varphi||_1,$

(1) $||J^i_{IN} * (\nabla\varphi)||_1 \leq C||\nabla\varphi||_1,$

(2) $||J^i_{MID} * (\nabla\varphi)||_1 \leq C(1 + \log R)||\varphi||_1,$

(3) $||\nabla\left\{(\chi^i_{OUT}\partial^2_{ij}K) * \varphi\right\}||_1 \leq \dfrac{C}{R}||\varphi||_1$

for all $\varphi \in \mathcal{S}$, all $R > 1$, all $r > 0$ and $i, j = 1, 2$, where $\partial^2_{ij} = \partial_i\partial_j$.

Proof. The estimates (0) and (1) are obtained by the Young inequality since $\partial_i K$ is integrable on any disk by Lemma 1–(5).

(2) Integrating by parts we have

$$\begin{aligned} J^i_{MID} * (\partial_j\varphi)(x) &= \int_{1\leq|y|\leq R} (\partial_i K)(y) \cdot (-\partial/\partial y_j)\{\varphi(x-y)\}dy \\ &= (\chi^R_{MID} \cdot \partial^2_{ij}K) * \varphi - L^R_{ij}(\varphi) + L^1_{ij}(\varphi). \end{aligned}$$

Here we note that $||(\chi^R_{MID} \cdot \partial^2_{ij}K) * \varphi||_1 \leq ||\chi^R_{MID} \cdot \partial^2_{ij}K||_1 \cdot ||\varphi||_1$. Hence (0) and Lemma 1–(5) imply

$$||J^i_{MID} * (\nabla\varphi)||_1 \leq C(1 + \log R)||\varphi||_1.$$

This shows (2).

(3) Since $\partial_l\left\{(\chi^i_{OUT}\partial^2_{ij}K) * \varphi\right\} = (\chi^i_{OUT}\partial^2_{ij}K) * (\partial_l\varphi)$ holds, we have

$$(\chi^i_{OUT}\partial^2_{ij}K) * (\partial_l\varphi) = (\chi^i_{OUT}\partial^3_{ijl}K) * \varphi + \int_{|y|=R} \partial^2_{ij}K(y)\varphi(x-y)\frac{y_l}{|y|}dS_y.$$

By the Young inequality and Lemma 1–(5) L^1 norm of the righthand side is estimated by $(C/R)||\varphi||_1$ with a positive numerical constant C. The proof is now complete. $\square$

Lemma 3. *There exists a numerical constant C such that*

$$||\nabla \cdot e^{t\Delta}\mathbb{P}(f \otimes f)||_\infty \leq C\left(1 + \log R + \frac{1}{\sqrt{t}}\right)||f||_\infty ||\mathrm{rot}\, f||_\infty + \frac{C}{R}||f||_\infty^2$$

for all $R > 1$, all $t > 0$ and all $f \in C^1(\mathbb{R}^2)$ with $\mathrm{div}\, f = 0$ *in* $\mathbb{R}^2$.

Proof. By the duality L^∞-norm for $F \in \mathcal{S}'$ is characterized by

$$||F||_\infty = \sup\{\, |< F, \varphi >|\,; \quad \varphi \in \mathcal{S} \text{ with } ||\varphi||_1 = 1\},$$

where $< \cdot, \cdot >$ denotes the pairing of (vector-valued) distribution and test functions. Thus to estimate $||F||_\infty$ for $F \equiv \nabla \cdot e^{t\Delta}\mathbb{P}(f \otimes f) \in \mathcal{S}'$ we shall estimate $< F, \varphi >$ assuming that $\varphi \in \mathcal{S}$ satisfies $||\varphi||_1 = 1$.

If a vector field f satisfies $\mathrm{div} f = 0$, then the identity $\nabla \cdot f \otimes f = (f, \nabla)f$ holds. Furthermore we have

$$(f, \nabla)f = \mathrm{rot}\, f \times f + \frac{1}{2}\nabla|f|^2,$$

when $\times$ denotes the exterior product in $\mathbb{R}^3$ and $f = (f^1, f^2)$ is regarded as a vector valued function $(f^1, f^2, 0)$. This identity holds for vector fields in $\mathbb{R}^3$.

Since $\langle \nabla|f|^2, \mathbb{P}e^{t\Delta}\varphi \rangle = 0$,

$$< F, \varphi >= (\mathrm{rot}\, f \times f, \mathbb{P}e^{t\Delta}\varphi),$$

where $(\cdot, \cdot)$ is $L^2(\mathbb{R}^2)$ inner product. Here, we put $\phi = e^{t\Delta}\varphi \in \mathcal{S}$ for a fixed t, which satisfies $||\phi||_1 \leq 1$ and $||\nabla\phi||_1 \leq C/\sqrt{t}$ with a numerical constant $C > 0$.

Since $\mathbb{P} = [\delta_{ij} + R_iR_j]$ i.e. the ij component equals $\delta_{ij} + R_iR_j$, by Lemma 1–(2) we get

$$\begin{aligned} \mathbb{P}\phi &= E\phi + [\partial_i K * \partial_j]_{ij}\phi \\ &= \left(E + [J^i_{IN} * \partial_j] + [J^i_{MID} * \partial_j] + [J^i_{OUT} * \partial_j]\right)\phi \end{aligned}$$

with 2×2 identity matrix E. We now estimate each term. By Lemma 2 the following estimates hold:

$$\begin{aligned} &||E\phi||_1 \leq ||\phi||_1 \leq 1, \\ &||J^i_{IN} * \partial_j\phi||_1 \leq C||\nabla\phi||_1 \leq C'/\sqrt{t}, \\ &||J^i_{MID} * \partial_j\phi||_1 \leq C(1 + \log R)||\phi||_1 \leq C(1 + \log R). \end{aligned}$$

Combining these estimates yields

$$\begin{aligned} &|(\mathrm{rot}\, f \times f, (E + [J^i_{IN} * \partial_j] + [J^i_{MID} * \partial_j])\phi)| \\ &\qquad\qquad \leq C(1 + \log R + 1/\sqrt{t})||\mathrm{rot}\, f||_\infty ||f||_\infty \end{aligned}$$

for $t > 0$ and $R > 1$. Using the identity $\mathrm{rot}\, f \times f = \nabla \cdot (f \otimes f - (1/2)|f|^2 E)$, we have

$$(\mathrm{rot}\, f \times f,\ J^i_{OUT} * \partial_j \phi) = -\Big(f \otimes f - \frac{1}{2}|f^2|E,\ \nabla(J^i_{OUT} * \partial_j \phi)\Big).$$

Here by the integration by parts we obtain $J^i_{OUT} * \partial_j \phi = (\chi^R_{OUT} \partial^2_{ij} K) * \phi + L_{ij}(\phi)$. From this identity we now observe that

$$\nabla(J^i_{OUT} * \partial_j \phi) = \nabla(\chi^R_{OUT} \partial^2_{ij} K) * \phi + \nabla L_{ij}(\phi) \equiv \nabla F_{ij} + \nabla L_{ij}(\phi).$$

Thus

$$\begin{aligned}(\mathrm{rot}\, f \times f,\ J^i_{OUT} * \partial_j \phi) &= -(f \otimes f - (|f^2|/2)E,\ \nabla F_{ij}) \\ &\qquad -(f \otimes f - (|f^2|/2)E,\ \nabla L_{ij}(\phi)) \\ &= -(f \otimes f - (|f^2|/2)E,\ \nabla F_{ij}) \\ &\qquad +(\mathrm{rot}\, f \times f,\ L_{ij}(\phi)).\end{aligned}$$

Lemma 2–(0) and (3) imply $||\nabla F_{ij}||_1$ and $||L_{ij}(\phi)||_1$ are estimated by C/R and C respectively. This proves our lemma. □

Since Lemmas 1 and 2 extend to $\mathbb{R}^n (n \geq 3)$, Lemma 3 also extends for $f \in C^1(\mathbf{R}^n)(n \geq 3)$ by interpreting rot f in a suitable way.

3. Logarithmic Gronwall inequality and a priori bounds

In this section we derive a uniformly (in time) bound for a mild solution of (NS). For this purpose we establish the following logarithmic Gronwall inequality.

Lemma 4. *Let a nonnegative function $a(t,s)$ be continuous in $\{(t,s)|\ 0 \leq s < t \leq T\}$ and satisfy $a(t,\cdot) \in L^1(0,t)$ for all $t \in (0,T]$ with some $T > 0$. Assume that there exists a positive constant ε_0 and $A \in (0,1)$ such that*

$$\sup_{0 \leq t \leq T} \int_{t-\varepsilon_0}^{t} a(t,s)\, ds \leq 1 - A. \tag{3.1}$$

If positive constants α, β and a non negative function $f \in C([0,T])$ satisfy

$$f(t) \leq \alpha + \int_0^t a(t,s) f(s)\, ds + \beta \int_0^t \{1 + \log(1 + f(s))\} \cdot f(s)\, ds, \tag{3.2}$$

for all $t \in [0, T]$. *Then we have*

$$f(t) \leq \begin{cases} \dfrac{\alpha}{A} \cdot e^{\gamma t/A} & (\beta = 0, a(t,s) \not\equiv 0), \\ -1 + \dfrac{[(1+\alpha)e]^{\exp(\beta t)}}{e} & (\beta \geq 0, a(t,s) \equiv 0), \\ -1 + \dfrac{[(1+\alpha/A)e]^{\exp(\beta+\gamma)t/A}}{e} & (\beta > 0, a(t,s) \not\equiv 0) \end{cases}$$

for all $t \in [0, T]$. *Here a positive constant* γ *is defined by*

$$\gamma = \sup_{0 \leq t \leq T} \{ \sup_{0 \leq s \leq t - \varepsilon_0} a(t,s) \}.$$

Remark. In the case of $a(t,s) = B(t-s)^{-\delta}$ with $0 < \delta < 1$ and $B > 0$, it is easy to show that for any $A \in (0,1)$ there exists ε_0 of the form

$$\varepsilon_0 = \left(\frac{(1-\delta)(1-A)}{B} \right)^{1/(1-\delta)} \quad \text{so that } \gamma = B \left(\frac{(1-\delta)(1-A)}{B} \right)^{-\delta/(1-\delta)}.$$

Proof. (i) The case $a(t,s) \equiv 0$:

Let $F(t)$ be the right hand side of (3.2) with $a(t,s) \equiv 0$. Computing $F'(t)$ and using (3.2), we have

$$\int_0^t \frac{F'(s)}{\{1 + \log(1 + F(s))\}(1 + F(s))} \, ds \leq \beta t$$

We change the variable of integration by $y = 1 + \log(1 + F(s))$ and integrate

$$f(t) \leq F(t) \leq -1 + \frac{[(1+\alpha)e]^{\exp(\beta t)}}{e} \tag{3.3}$$

for all $t \in [0, T]$.

(ii) The case $a(t,s) \not\equiv 0$:

The inequality (3.2) implies that $g(t) = \sup_{0 \leq \theta \leq t} f(\theta)$ satisfies

$$\begin{aligned} g(t) &\leq \alpha + \int_{t-\varepsilon_0}^t a(t,s) \, ds \cdot g(t) + \int_0^{t-\varepsilon_0} a(t,s) g(s) \, ds + \\ &\qquad + \beta \int_0^t \{1 + \log(1 + g(s))\} \cdot g(s) \, ds \\ &\leq \alpha + (1-A) \cdot g(t) + \int_0^{t-\varepsilon_0} \gamma g(s) \, ds + \\ &\qquad + \beta \int_0^t \{1 + \log(1 + g(s))\} \cdot g(s) \, ds \\ &\leq \alpha + (1-A) \cdot g(t) + \int_0^t \{(\beta + \gamma) + \beta \log(1 + g(s))\} \cdot g(s) \, ds. \end{aligned}$$

That is,

$$g(t) \leq \frac{\alpha}{A} + \frac{1}{A} \cdot \int_0^t \{(\beta+\gamma) + \beta \log(1+g(s))\} \cdot g(s)\, ds.$$

Applying (3.3) to this inequality with $\beta \neq 0$, we get our assertion because of $f(t) \leq g(t)$. In the case of $\beta = 0$, it is easy to show our assertion by the standard way. This proves our lemma. □

Now we state a priori estimate of solution for (NS). According to remarks (Step (v), (vi)) in Introduction, the following estimate is enough to guarantee that the unique local-in-times solution obtained in [GIM] can be extended to a global solution. Thus we obtain Theorem 1.

Theorem 2. *Let $u(t)$ be a mild solution of* (NS)*, that is, $u(t)$ is a solution of the integral equation* (1.2)*. Assume that u and ∇u belong to $C([t_0, t_0+T]; BUC)$ for $T > 0$ and $t_0 \geq 0$. Then there exists a positive constant K which depends only on $||u(t_0)||_\infty$ and $||\mathrm{rot}\, u(t_0)||_\infty$, such that*

$$||u||_\infty(t) \leq K \exp(Ke^{Kt})$$

for all $t \in [t_0, t_0+T]$.

Proof. We may of course assume that $t_0 = 0$ and $u(t_0) = u_0$.

By (1.2) we have

$$||u||_\infty(t) \leq ||e^{t\Delta}u_0||_\infty + \int_0^t ||\nabla \cdot (e^{(t-s)\Delta}\mathbb{P}(u(s) \otimes u(s)))||_\infty ds.$$

for $t \in [0, T]$. Applying Lemma 3 with $f = ||u||_\infty(s)$ and $R = 1 + ||u||_\infty(s)$ to this inequality, we have

$$\begin{aligned} ||u||_\infty(t) \leq ||u_0||_\infty + \int_0^t CM \cdot (t-s)^{-1/2} ||u||_\infty(s)\, ds + \\ + C(1+M) \cdot \int_0^t \{1 + \log(1 + ||u||_\infty(s))\} \cdot ||u||_\infty(s)\, ds, \end{aligned}$$

where $M = \sup_{0\leq t\leq T} ||\mathrm{rot}\, u(t)||_\infty$. On the other hand, $\omega(t) = \mathrm{rot}\, u(t)$ is a classical solution of the vorticity equation with the bounded coefficient $u \in C([0,T] \times \mathbb{R}^2)$.

$$\begin{cases} \omega_t - \Delta\omega + (u, \nabla)\omega = 0 \quad \text{in } (0,T] \times \mathbb{R}^2, \\ \omega(0) = \mathrm{rot}\, u_0 \quad \text{on } \mathbb{R}^2 \end{cases}$$

Since the maximum principle [PW] yields $\sup_{0\leq t\leq T} ||\omega||_\infty(t) \leq ||\mathrm{rot}\, u_0||_\infty$, the constant M is estimated by $||\mathrm{rot}\, u_0||_\infty$. (See also [KF].) Thus, Lemma 4 and Remark after it imply our assertion. □

The solution obtained in Theorem 1 is of course equal to a local-in-time solution obtained in [GIM] for $0 \le t \le T_0$ for some $T_0 > 0$. Furthermore by the arguments in [GIM] $||\mathrm{rot}\, u(t_0)||_\infty$ with $0 \le t_0 \le T_0$ is estimated by a quantity depending only on $||u_0||_\infty$ from above. Thus we have the following estimate.

Corollary 1. *Let $u(t)$ be the unique solution of* (NS) *obtained in Theorem 1. Then there exists a positive constant K_0 depending only on $||u_0||_\infty$ such that*

$$||u||_\infty(t) \le K_0 \exp(K_0 e^{K_0 t})$$

for all $t \in [0, \infty)$.

4. L^q global estimate

In this section we prove the following the global existence of (NS) for L^q initial data with $2 \le q < \infty$.

Theorem 3. *Assume that $u_0 \in L^q_\sigma(\mathbb{R}^2)$ for $2 \le q < \infty$, where $L^q_\sigma(\mathbb{R}^2)$ is the solenoidal closed subspace of $L^q(\mathbb{R}^2)$. Then there exists a unique solution $u(t)$ of* (NS)*, which belongs to $C([0,\infty); L^q_\sigma(\mathbb{R}^2))$ as well as $C^\infty((0,\infty) \times \mathbb{R}^2)$.*

Remark. Since $u(t)$ decays at space infinity in L^q-sense, the relation

$$p = \sum_{i,j=1,2} R_i R_j u_i u_j$$

is automatically fulfilled if $(u, \nabla p)$ is a solution of (NS).

For initial data $u_0 \in L^q_\sigma(\mathbb{R}^2)$ the first author [G] obtained a unique local-in-time mild solution $u(t)$ of (NS), which belongs to $C([0, T_0); L^q_\sigma(\mathbb{R}^2))$ and $C^\infty((0, T_0) \times \mathbb{R}^2)$ with $2 \le q < \infty$. Here T_0 satisfies $T_0 \ge C||u_0||_q^{2/(2/q-1)}$ with a constant $C > 0$ independent of u_0, where we denote the $L^q(\mathbb{R}^2)$ norm by $||\cdot||_q$. This solution, of course, is a classical solution of (NS) in $(0,\infty) \times \mathbb{R}^2$. Thus as in the proof of Theorem 1, to prove Theorem 3 it is enough to show the following a priori estimate.

Theorem 4. *Let $u(t)$ be a mild solution of* (NS)*. Assume that u and ∇u belong to $C([t_0, t_0 + T]; L^q_\sigma)$ for $T > 0$, $t_0 \ge 0$ and $2 \le q < \infty$. Then there exists a positive constant C which depends only on q, such that*

$$||u||_q(t) \le 2||u(t_0)||_q \exp(C||\mathrm{rot}\, u(t_0)||_q t)$$

for all $t \in [t_0, t_0 + T]$.

Proof. Without loss of generality, we may assume that $t_0 = 0$ and $u(t_0) = u_0$.

By (1.2) we have

$$||u||_q(t) \le ||e^{t\Delta}u_0||_q + \int_0^t ||\nabla \cdot (e^{(t-s)\Delta}\mathbb{P}(u(s)\otimes u(s)))||_q ds.$$

for $t \in [0,T]$. It is well-known that the operator $\mathbb{P}$ is also a projection operator of $L^r_\sigma(\mathbb{R}^2)$ in $L^r(\mathbb{R}^2)$ for any $1 < r < \infty$. Thus, applying Young's inequality and $\operatorname{div} u = 0$ yields

$$\begin{aligned} ||\nabla \cdot (e^{(t-s)\Delta}\mathbb{P}(u(s)\otimes u(s)))||_q &= ||\mathbb{P}e^{(t-s)\Delta}(u(s)\cdot\nabla)u(s)||_q \\ &\le C||e^{(t-s)\Delta}(u(s)\cdot\nabla)u(s)||_q \\ &\le \frac{C}{(4\pi(t-s))^{1/q}}\cdot||(u(s)\cdot\nabla)u(s)||_{q/2} \\ &\le \frac{C}{(4\pi(t-s))^{1/q}}\cdot||u(s)||_q\cdot||\nabla u(s)||_q \end{aligned}$$

with some positive constant depending only on q. We now employ the Calderón–Zygmund inequality $||\nabla f||_r \le C||\operatorname{rot} f||_r$ with a constant $C > 0$ depending only on $1 < r < \infty$ to get

$$||u||_q(t) \le ||u_0||_q + \int_0^t \frac{C}{(t-s)^{1/q}}\cdot||u(s)||_q\cdot||\operatorname{rot} u||_q ds.$$

On the other hand, it is easy to see that $||\operatorname{rot} u||_q(t) \le ||\operatorname{rot} u_0||_q$ using the vorticity equation (see, eg. [GMO]). Hence we have

$$||u||_q(t) \le ||u_0||_q + C||\operatorname{rot} u_0||_q \cdot \int_0^t \frac{1}{(t-s)^{1/q}}\cdot||u(s)||_q\, ds.$$

By the Gronwall inequality (Lemma 4) we have the desired a priori estimate. $\square$

Finally we give some remarks on the n dimensional Navier–Stokes equations with $n \ge 3$;

$$(\mathrm{NS}_n)\qquad \begin{cases} u_t - \Delta u + (u,\nabla)u + \nabla p = 0 \text{ in } (0,T)\times\mathbb{R}^n, \\ \operatorname{div} u = 0 \text{ in } (0,T)\times\mathbb{R}^n, \\ u|_{t=0} = u_0 \text{ in } \mathbb{R}^n. \end{cases}$$

In this case, a unique local-in-time solution which belongs to $C([0,T_0];L^q(\mathbb{R}^n))$ with $n < q \le \infty$, has been obtained (see, eg. [GIM] and [G]). Note that $\nabla u \in C((0,T_0];L^q(\mathbb{R}^n))$ and $T_0 \ge C||u_0||_q^{2/(-1+n/q)}$ with a positive constant depending only on n and q. We state a criterion on a global solvability for the integral equation (1.2), whose solution is called a mild solution.

Theorem 5. *Assume that $n \geq 3$ and $n < q \leq \infty$. Let $u(t)$ be a mild solution $u \in C([0,T); L^q(\mathbb{R}^n))$ of* (NS). *If there exists a positive constant M such that*

$$\sup_{0 \leq t < T} ||\mathrm{rot}\, u||_q(t) \leq M,$$

then $u(t)$ can be extended beyond T as a smooth solution with $u \in C([0,T]; L^q(\mathbb{R}^n))$ and $\nabla u \in C([0,T]; L^q(\mathbb{R}^n))$

Here in the case of $n \geq 4$ we define a vorticity (matrix) $\mathrm{rot}\, u$ of a vector field $u = (u_i)_{i\downarrow}$ by

$$\mathrm{rot}\, u \equiv \nabla u - {}^t\nabla u,$$

where $\nabla u = (\partial_j u_i)_{i\downarrow,\, j\rightarrow}$ is a Jacobi matrix and ${}^t\nabla u$ is its transposed matrix. Then it is easy to see

$$(u, \nabla)u = (\nabla u)u = (\mathrm{rot}\, u)u + ({}^t\nabla u)u = (\mathrm{rot}\, u)u + \frac{1}{2}\nabla|u|^2.$$

The proof of this theorem is along in the line for the proofs of Theorems 1 and 3. So we safely omit the proof.

The result in Theorem 5 is by no means optimal but such a criterion on extendability is not completely included in the classical regularity criteria in the literature (e.g. [G]).

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(accepted: November 26, 2000)