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Homogeneity Criterion for the Navier–Stokes Equations in the Whole Spaces

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Abstract. This paper is concerned with the Navier–Stokes flows in the homogeneous spaces of degree -1, the critical homogeneous spaces in the study of the existence of regular solutions for the Navier–Stokes equations by means of linearization. In order to narrow the gap for the existence of small regular solutions in $\dot{B}^{-1}_{\infty,\infty}(R^n)^n$, the biggest critical homogeneous space among those embedded in the space of tempered distributions, we study small solutions in the homogeneous Besov space $\dot{B}^{-1+n/p}_{p,\infty}(R^n)^n$ and a homogeneous space defined by $\hat{M}_n(R^n)^n$, which contains the Morrey-type space of measures $\tilde{M}_n(R^n)^n$ appeared in Giga and Miyakawa [20]. The earlier investigations on the existence of small regular solutions in homogeneous Morrey spaces, Morrey-type spaces of finite measures, and homogeneous Besov spaces are strengthened. These results also imply the existence of small forward self-similar solutions to the Navier–Stokes equations. Finally, we show alternatively the uniqueness of solutions to the Navier–Stokes equations in the critical homogeneous space $C([0,\infty);L_n(R^n)^n)$ by applying Giga–Sohr's $L_p(L_q)$ estimates on the Stokes problem.

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1. Introduction

Consider the incompressible viscous fluid motion governed by the Navier–Stokes equations in R^n, $n \geq 2$:

$$\begin{aligned} \partial u/\partial t - \Delta u + \nabla\cdot(u\otimes u) + \nabla\pi &= 0, \\ \nabla\cdot u &= 0, \\ u(0) &= a \end{aligned} \tag{1}$$

with unknown velocity $u = (u_1(t,x),\ldots,u_n(t,x))$ and pressure $\pi = \pi(t,x)$. Here $\nabla =$ the gradient $(\partial_{x_1},\ldots,\partial_{x_n})$ with $\partial_{x_i} = \partial/\partial x_i$ and $\Delta =$ the Laplacian $\nabla\cdot\nabla$.

In order to understand the regular solutions of the Navier–Stokes equations from the point of view of homogeneity, we will consider problem (1) in homogeneous spaces. In this paper, a normed space $X^\alpha(R^n)$ of functions $u(x)$ defined on R^n is said to be homogeneous of degree $\alpha \in R$ if

$$\|u(\lambda\cdot)\|_{X^\alpha} = \lambda^\alpha \|u(\cdot)\|_{X^\alpha} \quad \text{for } \lambda > 0, \; u \in X^\alpha(R^n),$$

and, a normed space $Y^\alpha(R_+ \times R^n)$ of functions $u(t,x)$ defined on the parabolic domain $R_+ \times R^n$ is said to be homogeneous of degree α if

$$\|u(\lambda^2\cdot, \lambda\cdot)\|_{Y^\alpha} = \lambda^\alpha \|u(\cdot,\cdot)\|_{Y^\alpha} \quad \text{for } \lambda > 0, \; u \in Y^\alpha(R_+ \times R^n).$$

Moreover, for a point $(t,x) \in R_+ \times R^n$ and a parabolic ball

$$Q_r(t,x) = \{(s,y) \in R \times R^n | \; t - r^2 < s \le t, \; |x-y| < r\}$$

a local space $Y^\alpha(Q_r(t,x))$ centered at (t,x) is said to be homogeneous of degree α if

$$\limsup_{r\to 0} \|u(\lambda^2\cdot, \lambda\cdot)\|_{Y^\alpha(Q_r(t,x))} = \lambda^\alpha \limsup_{r\to 0} \|u(\cdot,\cdot)\|_{Y^\alpha(Q_r(\lambda^2 t, \lambda x))}.$$

The typical homogeneous spaces are the Lebesgue spaces $L_q(0,\infty; L_p(R^n))$ and $L_p(R^n)$, of which the degrees are $-(n/p + 2/q)$ and $-n/p$, respectively.

In this paper we are particularly interested in the homogeneous spaces of degree -1, due to the scaling invariant property of the Navier–Stokes system. Indeed, a function $u(t,x)$ solves the Navier–Stokes equations in Eq. (1) for $t > 0$ if and only if $u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$ does so too for each given constant λ. Thus, if $u \in Y^{-1}(R_+ \times R^n)$, a homogeneous space of degree -1, then

$$\|u\|_{Y^{-1}} = \|u_\lambda\|_{Y^{-1}} \quad \text{for all } \lambda > 0.$$

Additionally, since linearization is still one of the key techniques in the regularity theory for the problem (1), it is convenient to establish a priori estimates in homogeneous spaces for the Stokes equations in R^n, $n \ge 2$:

$$\begin{aligned} \partial u/\partial t - \Delta u + \nabla \pi &= \nabla \cdot f \qquad (f = (f_{ij}(t,x))_{n\times n}), \\ \nabla \cdot u &= 0, \\ u(0) &= a. \end{aligned} \tag{2}$$

Thus for a homogeneous space of degree α, $X^\alpha(R^n)$, it is natural to look for the homogeneous spaces $Y^\alpha(R_+ \times R^n)$ and $Y^{\alpha-1}(R_+ \times R^n)$ such that the following a priori estimate on the Stokes flows holds true

$$\|u\|_{Y^\alpha} \le C\|a\|_{X^\alpha} + C\|f\|_{Y^{\alpha-1}}.$$

If one shows

$$\|u \otimes u\|_{Y^{\alpha-1}} \le C\|u\|_{Y^\alpha}\|u\|_{Y^{-1}},$$

then the Navier–Stokes flows are subject to the following sharp estimate in the homogeneous spaces of degree -1:

$$\|u\|_{Y^{-1}} \le C\|a\|_{X^{-1}} + C\|u\|^2_{Y^{-1}}. \tag{3}$$

Taking the contraction mapping principle into account, we obtain the global existence and uniquness of solutions when the initial velocity is sufficiently small in X^{-1}.

The motivation of this paper is the understanding of the question of knowing whether the Navier–Stokes equations are well-posed in Homogeneous Besov space $\dot{B}^{-1}_{\infty,\infty}(R^n)^n$. Eq. (3) will be obtained for $X^{-1}(R^n)$ chosen respectively as the homogeneous Besov spaces $\dot{B}^{-1+n/p}_{p,\infty}(R^n)^n$ and a homogeneous space defined as $\hat{M}_n(R^n)^n$ which seems "close" to $\dot{B}^{-1}_{\infty,\infty}(R^n)^n$. In fact, with a slight modification, the local regular solutions can be obtained when the initial data are in homogeneous spaces of degree α with $-1 < \alpha \leq 0$. However, generally speaking, it seems that neither the global existence of regular solutions for initial data in a single homogeneous space of degree $\alpha \neq -1$ nor the local existence of regular solutions for the initial data in a single homogeneous space of degree $\alpha < -1$ is established. This is, in fact, due to the fact that Eq. (3) with X^{-1} and Y^{-1} replaced respectively by X^{α} and Y^{α} ($\alpha \neq 1$) is not true. This also suggests that homogeneous spaces of degree -1 are critical spaces in studying regular solutions of Navier–Stokes equations.

The mathematical researches on the incompressible Navier–Stokes flows start from the fundamental paper [27] in 1934, where Leray obtained the existence of weak solutions of problem (1) satisfying the energy inequality

$$\|u\|^2_{L_\infty(0,\infty;L_2)} + \|\nabla u\|^2_{L_2(0,\infty;L_2)} \leq \|a\|^2_{L_2}. \tag{4}$$

However, the homogeneous space involved in this estimate is of degree $-n/2 \leq -1$, where the equality holds only when $n = 2$. Thus this energy inequality may not be enough to ensure the regularity of the weak solutions when $n \geq 3$. Inspired by the investigation of Prodi [31], Serrin [33, 34] provided an interior regularity criterion showing the regularity of the weak solution u to problem (1) when u is in the homogeneous space of degree $-n/p - 2/q > -1$:

$$u \in L_q(0,\infty; L_p(R^n)^n),\ \frac{n}{p} + \frac{2}{q} < 1,\ n < p \leq \infty.$$

This interior regularity result was extended to the critical case $-(n/p+2/q) = -1$ with $2 < q$ by Struwe [39] and Takahashi [40, 41], and was extended as a global regularity result by Fabes *et al.* [12] in the critical case

$$u \in L_q(0,\infty; L_p(R^n)^n),\ \frac{n}{p} + \frac{2}{q} = 1,\ n < p < \infty.$$

The first existence result of regular solutions with small initial data in a homogeneous space of degree -1 is due to Fujita and Kato [15] on an initial boundary value problem for Navier–Stokes equations. Weissler [47] gives a detailed $L_p(R^n_+)^n$ theory in R^n_+ (half-space) for local solutions. Inspired by Weissler [47], Kato [24] and Giga–Miyakawa [19] obtained independently the existence of small L_n-solutions to Eq. (1) such that

$$\|u(t)\|_{L_n} + t^{1/2}\|u(t)\|_{L_\infty} \leq \text{ const. } (a \in L_n(R^n)^n).$$

Although Giga and Miyakawa [19] dealt with the case of bounded domains, the theory of [19] applies to whole-space problem in the space $L_n(R^n)^n$ and in the homogeneous space of Bessel potentials $\dot{H}_p^{-1+n/p}(R^n)^n$ with $n < p < \infty$.

This result has now been extensively studied and there is a large literature on the existence of global small regular solutions in some homogeneous spaces of degree -1. In particular, Giga–Miyakawa [20] and Giga–Miyakawa–Osada [21] dealt with the case that the initial vorticity $\nabla \times a$ is in the Morrey-type space of measures $\tilde{M}_{n/2}(R^n)^n$. Kato [25] and Taylor [42] deal with the small initial velocity in the Morrey space $M_n(R^n)^n$ and the Morrey–Campanato space $M_{n,q}(R^n)^n$. Generalizing a result by Cannone [7], Planchon [30] obtains the existence of regular solutions in the homogeneous Besov spaces $\dot{B}_{p,\infty}^{-1+n/p}(R^n)^n \cap \dot{B}_{q,\infty}^{-1+n/q}(R^n)^n$ with $(n \le p < q < \infty)$. This result is now improved by our Theorem 2.1, stated in Section 2, which concerns the existence of regular solutions in $\dot{B}_{p,\infty}^{-1+n/p}(R^n)^n$ with $n \le p < \infty$. It should be pointed out that our approach to this theorem is different from those used in ([30]). The existence results from [20, 25, 42] mentioned above are now covered by the second main result, Theorem 2.2, stated in Section 2, which gives the existence of a small regular solution when the initial velocity is in the space $\hat{M}_n(R^n)^n$ (see Definition 2.2 in Section 2), a space containing $\tilde{M}_n(R^n)^n$ and $M_n(R^n)^n$. One can also refer to Barraza [3, 4] for the existence of small regular solutions in the Lorentz space $L_{n,\infty}(R^n)^n$, which is also contained in $M_n(R^n)^n$.

For the local existence of regular solutions with initial velocities in supercritical homogeneous spaces, we refer to Giga [17] in $L_p(R^n)^n$ $(n \le p < \infty)$, Giga, Inui and Matsui [18] in $L_\infty(R^n)^n$, Federbush [13] in $M_{p,2}(R^n)^n$ $(n < p < \infty)$, and Cannone [7] and Cannone and Meyer [8] for a general algorithm with respect to some supercritical homogeneous spaces.

As for the uniqueness of solutions in the critical homogeneous space $C([0,\infty); L_3(R^3)^3)$, one may refer to Furioli, Lemarié-Rieusset and Terraneo [16] and Lions and Masmoudi [29].

Recently, Amann [1] provided a systematic study on the well-posedness of Navier–Stokes equations over a domain $\Omega \subset R^n$ with $n \ge 3$ in mainly answering the question that whether Cannone's result could be proven for other domains as well; for example, if Ω is a bounded or an exterior domain (see [1, page 16]). In particular, he obtained that the Navier–Stokes equations has a unique regular solution on an interval $[0,T)$ if the initial divergence free velocity is in $H_q^{-1+n/q}(\Omega)^n$ and is small in the little Nikol'skii space $n_r^{-1+n/r}(\Omega)^n$ ($n/3 < q \le r < \infty$ and $n < r$), which denotes the closure of $H_r^{-1+n/r}(\Omega)^n$ in $B_{r,\infty}^{-1+n/r}(\Omega)^n$. Here T is presupposed to be any positive number when Ω is an exterior domain and $T = \infty$ when Ω is a bounded domain. However, the results of [1] do not seem to be comparable to Theorem 2.2 on the well-posedness in the space $\hat{M}_n(R^n)^n$.

On the other hand, the problem on the existence of regular solutions in the critical homogeneous spaces is closely related to the theory of partial regularity started by Scheffer in [32], and well developed in [6], where Caffarelli, Kohn and Nirenberg proved that a suitable weak solution u is regular at the point (t,x) if the following local homogeneous norm of degree -1 at (t, x) is sufficiently small:

$$\limsup_{r\to 0}\left(\frac{1}{r}\int_{t-r^2}^{t+r^2}\int_{|y-x|<r}|\nabla u|^2 dyds\right)^{1/2} < \epsilon. \tag{5}$$

Recently, the proof of the main result of [6] has been simplified by Lin [28]. If u is a suitable weak solution, Tian and Xin [43] obtain the local L_∞ estimate for ∇u under the assumption of Eq. (5) or any one among the following:

$$\limsup_{r\to 0}\left(\frac{1}{r^3}\int_{t-r^2}^{t}\int_{|y-x|<r}|u|^2 dyds\right)^{1/2} < \epsilon \text{ and } \limsup_{r\to 0}\left(\frac{1}{r}\int_{|y-x|<r}|u|^2 dy\right)^{1/2} < \infty,$$

$$\limsup_{r\to 0}\left(\frac{1}{r}\int_{t-r^2}^{t}\int_{|y-x|<r}|\nabla\times u|^2 dyds\right)^{1/2} < \epsilon,$$

$$\limsup_{r\to 0}\left(\frac{1}{r^2}\int_{t-r^2}^{t}\int_{|y-x|<r}|u|^3 dyds\right)^{1/3} < \epsilon.$$

It should be noted that these local Morrey-type norms at the point (t, x) are also homogeneous of degree -1. Since the linearization for the Navier–Stokes equations around the point (t, x) is used in [6, 28, 43], similar estimates to Eq. (3) are , in fact, implicitly obtained for the local homogeneous norms of degree -1 at (t, x) in arriving the regularity at the point (t, x).

This paper is organized as follows: In Section 2, we state the main results, Theorems 2.1 and 2.2, which yield respectively the global existence of small regular solutions to the problem (1) in $\dot{B}_{p,\infty}^{-1+n/p}(R^n)^n$ and $\hat{M}_n(R^n)^n$ respectively. Section 3 gives the connections of our results and others by comparing $\dot{B}_{p,\infty}^{-1+n/p}(R^n)^n$ and $\hat{M}_n(R^n)^n$ with other function spaces. Section 4 collects some basic lemmas. Section 5 contains a priori estimates for the Stokes problem in homogeneous Besov spaces, and a variant of the $L_p(L_q)$ estimate for the Stokes problem due to Giga and Sohr [22]. The main results Theorems 2.1 and 2.2 are proved respectively in Sections 6 and 7. Section 8 describes the existence of regular small forward self-similar solutions. The results contained in this section cover the earlier investigations on the existence of forward self-similar solutions deduced in [3, 20, 9]. Finally, in Section 9, we give a remark on the result due to Furioli, Lemarié-Rieusset and Terraneo [16] and Lions and Masmoudi [29] on the uniqueness of solutions of Eq. (1) in $C([0,\infty); L_3(R^3)^3)$, a homogeneous space of degree -1. We shall show that this uniqueness result is an easy consequence of the Giga–Sohr's $L_p(L_q)$ estimates on the Stokes flows. For the connection with this problem, we refer to the limit case of Serrin's regularity criterion obtained by Sohr and von Wahl [36] and von Wahl [46] on the uniqueness of solutions both in the critical homogeneous space $C([0,\infty); L_3(R^3)^3)$ and in the subcritical homogeneous spaces $L_\infty(0,\infty; L_2(R^n)^n) \cap L_2(0,\infty; \dot{H}_2^1(R^n)^n)$ involved in the energy inequality (4).

2. Statement of main results

In order to state our main results, we introduce the following notations:

$\mathcal{S}(R^n)$ = the Schwartz space of scalar rapidly decreasing functions in $C^\infty(R^n)$,
$\mathcal{S}'(R^n)$ = the space of all tempered distributions on R^n, i.e. the dual of $\mathcal{S}(R^n)$,
$h_t(x)$ = the heat kernel $(4\pi t)^{-n/2}e^{-|x|^2/(4t)}$,
$e^{t\Delta}$ = the heat semigroup such that $e^{t\Delta}u$ = the convolution $h_t * u$ on R^n,
$\mathcal{P}(R^n)$ = the set of all scalar polynomials defined on R^n,
$\mathcal{F}$ = the Fourier transformation on R^n,
$(-\Delta)^\alpha$ = $\mathcal{F}^{-1}|\xi|^{2\alpha}\mathcal{F}$, the fractional power of the Laplacian,

P is the Leray projection operator defined by $(Pu)_i = \sum_{j=1}^{n}(\delta_{ij} - \partial_{x_i}\partial_{x_j}\Delta^{-1})u_j$,

$\|\cdot\|_{L_p}$ = the norm of the Lebesgue spaces $L_p(R^n)$,
C = a generic constant independent of the quantities u, v, w, f, a, ϵ, T, τ and $t > 0$.

By using the homogeneous counterpart of [45, Theorem 2.12.2] and the lifting property [45, Theorem 5.2.3/1], one may define the homogeneous Besov spaces in the following form.

Definition 2.1. For $1 \le p, q \le \infty$ and $-\infty < \alpha < \infty$, the homogeneous Besov spaces are defined to be

$$\dot{B}^\alpha_{p,q}(R^n) = \left\{ u \in \mathcal{S}'(R^n)/\mathcal{P}(R^n) \;\middle|\; \|u\|_{\dot{B}^\alpha_{p,q}} < \infty \right\}$$

where

$$\|u\|_{\dot{B}^\alpha_{p,q}} = \begin{cases} \left(\int_0^\infty t^{(1-\alpha/2)q} \|\Delta e^{t\Delta}u\|^q_{L_p} \dfrac{dt}{t} \right)^{1/q} & \text{for } 0 < \alpha < 2 \ (q \neq \infty), \\ \sup\limits_{t>0} t^{1-\alpha/2} \|\Delta e^{t\Delta}u\|_{L_p} & \text{for } 0 < \alpha < 2 \ (q = \infty), \\ \|(-\Delta)^{(\alpha-1)/2}u\|_{\dot{B}^1_{p,q}} & \text{for others.} \end{cases}$$

This definition ensures the following homogeneous property

$$\|u(\lambda\cdot)\|_{\dot{B}^\alpha_{p,q}} = \lambda^{\alpha - n/p}\|u(\cdot)\|_{\dot{B}^\alpha_{p,q}}.$$

In particular, we have the homogeneous space of degree -1:

$$\dot{B}^{-1}_{\infty,\infty}(R^n)^n = \left\{ u \in \mathcal{S}'(R^n)^n \;\middle|\; \|u\|_{\dot{B}^{-1}_{\infty,\infty}} = \sup_{t>0} t^{1/2}\|e^{t\Delta}u\|_{L_\infty} < \infty \right\}.$$

It should be mentioned that not only is $\dot{B}^{-1}_{\infty,\infty}(R^n)$ the biggest critical homogeneous space, but, as pointed out for instance in Auscher and Tchamitchian [2] or, in a more general framework, in Frazier, Jawerth and Weiss [14] that any critical homogeneous space continuously embedded in $\mathcal{S}'(R^n)$ is also continuously embedded into $\dot{B}^{-1}_{\infty,\infty}(R^n)$.

It is unknown whether there exists a global regular solution if the initial velocity is in $\dot{B}^{-1}_{\infty,\infty}(R^n)$ and small. To narrow the gap in the understanding of this problem, we introduce the following homogeneous spaces of degree $-n/p$.

Definition 2.2. For $0 < p < \infty$,

$$\hat{M}_p(R^n)^n = \left\{ u \in \mathcal{S}'(R^n)^n \;\middle|\; |u| \in \mathcal{S}'(R^n),\ \|u\|_{\hat{M}_p} = \sup_{t>0} t^{n/(2p)} \|e^{t\Delta}|u|\,\|_{L_\infty} < \infty \right\}.$$

Among them the homogeneous space of degree -1 is $\hat{M}_n(R^n)$, which is continuously imbedded in the space $\dot{B}^{-1}_{\infty,\infty}(R^n)$.

With the use of the Leray projection operator P, we write Eq. (1) in the integral form

$$u(t) = e^{t\Delta}Pa - \int_0^t e^{(t-s)\Delta} P\nabla \cdot (u(s) \otimes u(s))ds. \tag{6}$$

We are now in the position to state the main results of this paper.

Theorem 2.1. *Let $n \geq 2$, $n \leq p < \infty$, $2 - n/p < \alpha < 2$, $a \in \dot{B}^{-1+n/p}_{p,\infty}(R^n)^n$, $\nabla \cdot a = 0$, and $\|a\|_{\dot{B}^{-1+n/p}_{p,\infty}} \leq \epsilon$ for some small constant $\epsilon = \epsilon(n, p, \alpha) > 0$. Then Eq. (6) admits a unique regular solution satisfying*

$$\|u(t)\|_{\dot{B}^{-1+n/p}_{p,\infty}} + t^{1/2}\|u(t)\|_{L_\infty} + t^{\alpha/2}\|u(t)\|_{\dot{B}^{\alpha-1+n/p}_{p,\infty}} \leq C\|a\|_{\dot{B}^{-1+n/p}_{p,\infty}}.$$

Theorem 2.2. *Let $n \geq 2$, $1 < \alpha < 2$, $a \in \hat{M}_n(R^n)^n$, $\nabla \cdot a = 0$ and $\|a\|_{\hat{M}_n} < \epsilon$ for some small constant $\epsilon = \epsilon(\alpha, n)$. Then Eq. (6) admits a unique regular solution u satisfying*

$$\|u(t)\|_{\hat{M}_n} + t^{1/2}\|u(t)\|_{L_\infty} + t^{\alpha/2}\|u(t)\|_{\dot{B}^{\alpha-1}_{\infty,\infty}} \leq C\|a\|_{\hat{M}_n}.$$

Remark 2.1. The proof of Theorem 2.2 is based on the following estimate

$$\|e^{t\Delta}P\nabla u\|_{\hat{M}_n} \leq Ct^{-1/2}\|u\|_{\hat{M}_n} \quad (u \in \hat{M}_n(R^n)^n \cap L_\infty(R^n)^n),$$

which follows from the estimate

$$\|e^{\tau\Delta}|e^{t\Delta}P\nabla u|\,\|_{L_\infty} \leq Ct^{-1/2}\|e^{\tau\Delta}|u|\,\|_{L_\infty}. \tag{7}$$

This becomes, when $\tau = 0$,

$$\|e^{t\Delta}P\nabla u\|_{L_\infty} \leq Ct^{-1/2}\|u\|_{L_\infty}. \tag{8}$$

It should be mentioned that $L_\infty(R^n)^n$ is a "poor" space, in which the projection operator P is not bounded, and thus Eqs. (7, 8) are not simple. In fact, Eqs. (7, 8) are based on the following bound shown in Carpio [10]:

$$\|P\nabla h_t\|_{L_1} \le Ct^{-1/2}. \tag{9}$$

We shall provide alternatively an approach to this bound by developing the technique of Chen [11] in obtaining the estimate

$$\|P\nabla u\|_{L_1} \le C\left(\|\nabla u\|_{L_1} + \|u\|_{\dot{B}^{\epsilon}_{1,\infty}} + \|u\|_{\dot{B}^{1-\epsilon}_{1,\infty}}\right) \quad u \in \mathcal{S}(R^n)^n$$

for examining the strong Navier–Stokes solutions in another "poor" space $L_1(R^n)^n$. Moreover, Eq. (8) has been proved in Giga, Inui and Matsui [18] based on Eq. (9) in the study of the local existence of solutions of Eq. (1) with the initial velocity in $L_\infty(R^n)^n$. Shimizu [35] gives the estimates similar to Eq. (8) for the half-space Stokes flows in $L_\infty(R^n_+)$ and the Hardy space $H_1(R^n_+)$. One can also refer to Cannone [7] and Cannone–Meyer [8] for the estimates of $e^{t\Delta}P\nabla u$ in $L_p(R^3)^3$ with $3 < p \le \infty$ and in the Hölder space $C^\alpha(R^3)^3$ with $\alpha > 0$.

Theorems 2.1 and 2.2 give the existence of small regular solutions of the Navier–Stokes equations when the initial velocity is in $\dot{B}^{-1+n/p}_{p,\infty}(R^n)^n$ and $\hat{M}_n(R^n)^n$ respectively. As is mentioned in Section 1, the existence with respect to some other homogeneous spaces of degree -1 has been studied by many authors, whereas our results provide additionally the sharp estimates on the homogeneous spaces of degree -1. In next section, we shall describe more precisely on the relations between Theorems 2.1 and 2.2 and the results in other homogeneous spaces of degree -1.

3. Other function spaces

Let us now introduce some other homogeneous spaces, in which $\mathcal{S}(R^n)$ is not dense, and give the connections of the above theorems and some other known results concerning those spaces.

- The Lorentz space $(1 \le p < \infty)$

$$L_{p,\infty}(R^n) = \left\{ u \in L_{1,loc}(R^n) \;\middle|\; \|u\|_{L_{p,\infty}} = \sup_E |E|^{-1+1/p} \int_E |u(y)|dy < \infty \right\}$$

  where $|E|$ denotes the Lebesgue measure of E, and the supremum is taken over all Lebesgue measurable sets of R^n.
- The weak Lebesgue space $(1 \le \infty)$

$$L^*_p(R^n) = \left\{ u \;\middle|\; u \text{ is a measurable function, and } \sup_{r>0} r|\{x \in R^n \,|\, |u(x)| > r\}|^{1/p} < \infty \right\}$$

  which equals the Lorentz space $L_{p,\infty}(R^n)$ (see [5]).
- The Morrey space $(1 \le p < \infty)$

$$M_p(R^n) = \left\{ u \in L_{1,loc}(R^n) \;\middle|\; \|u\|_{M_p} = \sup_{x\in R^n,\, r>0} r^{n/p-n} \int_{|x-y|<r} |u(y)|dy < \infty \right\}.$$

- The Morrey-type space of measures ($1 \le p < \infty$)

$$\tilde{M}_p(R^n) = \left\{ u \in \mathcal{S}'(R^n) \;\middle|\; u \text{ is a locally finite Radon measure },\ \|u\|_{\tilde{M}_p} < \infty \right\}$$

with

$$\|u\|_{\tilde{M}_p} = \sup_{x\in R^n,\, r>0} r^{n/p-n} |u|[\{y \in R^n \mid |x-y| < r\}],$$

where $|u|$ denotes the total variation of u.

- The Morrey–Campanato space ($1 \le q < p < \infty$)

$$M_{p,q}(R^n) = \left\{ u \in L_{q,loc}(R^n) \;\middle|\; \|u\|_{M_{p,q}} < \infty \right\}$$

with

$$\|u\|_{M_{p,q}} = \sup_{x\in R^n,\, r>0} r^{n/p-n/q} \left(\int_{|x-y|<r} |u(y)|^q dy \right)^{1/q}.$$

- The homogeneous Triebel–Lizorkin spaces ($1 \le p < \infty$, $1 \le q \le \infty$, $1 < s < \infty$, $-\infty < \alpha < \infty$) (see [45, Section 5])

$$\dot{F}^{\alpha}_{p,q}(R^n) = \left\{ u \in \mathcal{S}'(R^n)/\mathcal{P}(R^n) \;\middle|\; \left\| \left(\sum_{j=-\infty}^{\infty} 2^{\alpha j q} |\mathcal{F}^{-1}\phi_j \mathcal{F} u|^q \right)^{1/q} \right\|_{L_p} < \infty \right\},$$

$$\dot{F}^{\alpha}_{\infty,s}(R^n) = \text{ the dual space of } \dot{F}^{-\alpha}_{1,s/(s-1)}(R^n),$$

where the system $\{\phi_j\}_{j=-\infty}^{\infty} \subset \mathcal{S}(R^n)$ satisfies the following properties:

$$\mathrm{supp}\phi_j \subset \{x \in R^n |\ 2^{j-1} \le |x| \le 2^{j+1}\}, \quad \sum_{j=-\infty}^{\infty} \phi_j = 1 \text{ in } R^n \setminus \{0\},$$

and

$$2^{jk} |\nabla^k \phi_j(x)| \le C_k \text{ for all integers } j \text{ and } k \text{ and } x \in R^n.$$

Let us show that the following homogeneous spaces of degree -1

$$L_n(R^n),\ L_{n,\infty}(R^n),\ M_{n,q}(R^n),\ M_n(R^n),\ \tilde{M}_n(R^n),$$
$$\dot{F}^{-1+n/p}_{p,\infty}(R^n),\ \dot{B}^{-1+n/p}_{p,\infty}(R^n),\ \hat{M}_n(R^n)$$

are continuously imbedded in $\dot{B}^{-1}_{\infty,\infty}(R^n)$.

If we denote by "$\subset$" the continuous imbedding, it then follows from the definition that

$$\hat{M}_n(R^n) \subset \dot{B}^{-1}_{\infty,\infty}(R^n).$$

Additionally, we have the following inclusion relations.

$$L_n(R^n) \subset L_{n,\infty}(R^n) \subset M_{n,q}(R^n) \subset M_{n,1}(R^n) = M_n(R^n) \subset \tilde{M}_n(R^n) \ \ (1 \le q < n), \tag{10}$$

$$L_{n,\infty}(R^n) \subset \dot{B}^{-1+n/p}_{p,\infty}(R^n) \ (n < p < \infty), \tag{11}$$

$$L_n(R^n) \subset \dot{F}^{-1+n/p}_{p,\infty}(R^n) \subset \dot{B}^{-1+n/p}_{p,\infty}(R^n) \ (n \le p < \infty), \tag{12}$$

$$\tilde{M}_n(R^n) \subset \hat{M}_n(R^n). \tag{13}$$

Equation (10) except the second inclusion follows from the definitions, while Eq. (11) is shown in [30], and Eq. (12) follows from [45, Eq. (2.3.2/9) and Subsection 5.2.5], since $L_n(R^n) = \dot{F}^0_{n,2}(R^n)$ (see [45, Remark 2.3.5]). The relation $L_{n,\infty}(R^n) \subset M_{n,q}(R^n)$ is shown in the following.

$$\begin{aligned}
\|u\|_{M_{n,q}} &\le \sup_E |E|^{1/n-1/q} \left(\int_E |u(y)|^q dy \right)^{1/q} \quad (u \in L_{n,\infty}(R^n)\,) \\
&= \left(\sup_E |E|^{q/n-1} \int_E |u(y)|^q dy \right)^{1/q} \\
&\cong \left(\sup_{r>0} r|\{x \in R^n \mid |u(x)|^q > r\}|^{q/n} \right)^{1/q} \\
&= \sup_{r>0} r|\{x \in R^n \mid |u(x)| > r\}|^{1/n} \\
&\cong \|u\|_{L_{n,\infty}}.
\end{aligned}$$

Finally, Eq. (13) is given from the following.

Proposition 3.1. *For* $1 \le p < \infty$,

$$\tilde{M}_p(R^n) = \{u \in \hat{M}_p(R^n) \mid u \textit{ is a locally finite Radon measure}\}.$$

Proof. Assuming $u \in \tilde{M}_p(R^n)$, we have, by Fubini's theorem, that

$$\begin{aligned}
e^{t\Delta}|u| &= \int_{R^n} h_t(x-y)|u|(dy) \\
&= (4\pi t)^{-n/2} \int_{R^n} \int_0^{e^{-|x-y|^2/(4t)}} ds|u|(dy) \\
&= (4\pi t)^{-n/2} \int_0^1 |u|[\{y \in R^n| \ |x-y| < 2t^{1/2}|\ln s|^{1/2}\}]ds \\
&\le Ct^{-n/(2p)} \|u\|_{\tilde{M}_p} \int_0^1 |\ln s|^{(n-n/p)/2} ds \\
&\le Ct^{-n/(2p)} \|u\|_{\tilde{M}_p}.
\end{aligned}$$

This gives

$$\|u\|_{\hat{M}_p} \le C\|u\|_{\tilde{M}_p}.$$

On the other hand, if $u \in \hat{M}_p(R^n)$ is a locally finite Radon measure, then

$$\begin{aligned} |u|[\{y \in R^n \mid |x-y| < t^{1/2}\}] &= \int_{|x-y|<t^{1/2}} |u|(dy) \\ &\le e^{1/4}(4\pi t)^{n/2} \int_{|x-y|<t^{1/2}} h_t(x-y)|u|(dy) \\ &\le Ct^{n/2}e^{t\Delta}|u| \\ &\le Ct^{(n-n/p)/2}\|u\|_{\hat{M}_p}, \end{aligned}$$

which gives the desired estimate. The proof is complete.

Note that Planchon [30] has obtained the existence of small regular solutions when the initial data

$$a \in \dot{B}^{-1+n/p}_{p,\infty}(R^3)^3 \cap \dot{B}^{-1+n/q}_{q,\infty}(R^3)^3,\ \|a\|_{\dot{B}^{-1+3/q}_{q,\infty}} < \epsilon\ (3 \le p < q < \infty). \tag{14}$$

As is pointed in [30] that this result covers those concerning the homogeneous Besov spaces in Cannone [7]. It is clear that Eq. (14) implies the initial condition imposed in Theorem 2.1. Furthermore, our approach is different from that used in [30]. As far as Theorem 2.2 is concerned, it should be noted that the global existence of small regular solutions has been obtained by Kato [25] and Taylor [42] when the intial velocity $a \in M_n(R^n)^n$, and by Giga–Miyakawa–Osada [21] and Giga–Miyakawa [20] for the initial voticity $\nabla \times a \in \tilde{M}_{n/2}(R^n)^n$. [21] deals with large solutions in dimension two. It follows from Eqs. (10,13) that these results in [20, 25, 42] have now been strengthened by Theorem 2.2, since a variant of the Sobolev imbedding theorem (see [20]) gives

$$\|a\|_{\tilde{M}_n} = \|\Delta^{-1}\nabla \times (\nabla \times a)\|_{\tilde{M}_n} \le C\|\nabla \times a\|_{\tilde{M}_{n/2}},$$

that is,

$$\{a \in \mathcal{S}'(R^n)^n \mid \nabla \cdot a = 0,\ \nabla \times a \in \tilde{M}_{n/2}(R^n)^n\} \subset \{a \in \tilde{M}_n(R^n)^n \mid \nabla \cdot a = 0\}.$$

It should be mentioned that [20, 21] provide the global existence of regular solutions with Radon measures as initial vorticity, whereas Theorem 2.2 shows the global existence of small regular solutions with Radon measures as initial velocity. We remark that it follows from Eq. (12) and the proof of Theorem 2.1 in Section 6 that instead of the homogeneous Besov space $\dot{B}^{-1+n/p}_{p,\infty}(R^n)^n$, the result parallel to Theorem 2.1 holds true in the homogeneous Triebel–Lizorkin spaces $\dot{F}^{-1+n/p}_{p,\infty}(R^n)^n$ with $n \le p < \infty$. One can also refer to Barraza [3, 4] for global small solutions in $L_{n,\infty}(R^n)^n$.

We note also that the following divergence free vector fields from [20]

$$\left(0, -\frac{x_3}{|x|^2}, \frac{x_2}{|x|^2}\right),\ \left(\frac{x_3}{|x|^2}, 0, -\frac{x_1}{|x|^2}\right) \text{ and } \left(\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}, 0\right)$$

are in the space

$$L_3^*(R^3)^3 = L_{3,\infty}(R^3)^3 \subset B_{p,\infty}^{-1+3/p}(R^3)^3 \cap \hat{M}_3(R^3)^3 \ \ (3 < p < \infty).$$

As is well known, a solution u is regular as long as $u \in L_\infty(0,\infty; L_\infty(R^n)^n)$, a homogeneous space of degree 0. Compared with the homogeneous spaces of degree -1, the homogeneous spaces of degree α with $-1 < \alpha \le 0$ is more "close" to $L_\infty(0,\infty; L_\infty(R^n)^n)$. One thus may apply the similar approach to show the local existence result on the homogeneous Besov spaces of degree α with $-1 < \alpha \le 0$, although from the criterion of linearization, it seems impossible to give a global existence result when the initial data are only in a homogeneous space of degree α with $-1 < \alpha \le 0$, no matter how small the initial data are.

Remark 3.1. After the previous version of this paper was submitted for publication, we learned from a referee that Auscher and Tchamitchian [2] has presented a general algorithm, developed from [7, 8], on the well-posedness of the Cauchy problem in homogeneous spaces of degree -1. The theory of [2] seems applicable to the existence of regular solutions when the initial data are small in the space $\dot{B}_{p,\infty}^{-1+n/p}(R^n)^n$ $(n \le p < \infty)$ or $\tilde{M}_n(R^n)^n$. What is more, we learned from the referee that Koch and Tataru [26] have obtained the global existence of small regular solutions with the initial velocities in $BMO^{-1}(R^n)^n$, which equals $\dot{F}_{\infty,2}^{-1}(R^n)^n$ and so (see [45])

$$BMO^{-1}(R^n) \subset \dot{B}_{\infty,\infty}^{-1}(R^n).$$

Moreover (see [26])

$$BMO^{-1}(R^n)^n \supset \dot{B}_{p,\infty}^{-1+n/p}(R^n)^n \text{ for } n \le p < \infty$$

and

$$BMO^{-1}(R^n)^n \supset M_{n,q}(R^n)^n \text{ for } 1 < q \le n.$$

Thus Theorem 2.1 is strengthened by [26]. However, the spaces $\hat{M}_n(R^n)^n$ and $BMO^{-1}(R^n)^n$ are different and no inclusion relation between them.

4. Preliminary lemmas

For the reader's convenience, we state the following known results with respect to homogeneous Besov spaces.

Lemma 4.1 (see [45, Theorem 5.2.3/1 (i)])**.** *Let* $-\infty < \alpha, \beta < \infty$ *and* $1 \le p, q \le \infty$*. Then the operator* $(-\Delta)^{\alpha/2} = \mathcal{F}^{-1}|\xi|^\alpha \mathcal{F}$ *maps* $\dot{B}_{p,q}^{\beta}(R^n)$ *isomorphically onto* $\dot{B}_{p,q}^{\beta+\alpha}(R^n)$*.*

Lemma 4.2 (see [45, Theorems 2.4.2, 2.4.7 and Subsection 5.2.5])**.** *Let* $0 < \theta < 1$*,* $1 \le p, q \le \infty$*,* $-\infty < \alpha < \beta < \infty$*, and*

$$\gamma = (1-\theta)\alpha + \theta\beta.$$

Then

$$\left(\dot{B}^{\alpha}_{p,\infty}(R^n),\, \dot{B}^{\beta}_{p,\infty}(R^n)\right)_{\theta,q} = \dot{B}^{\gamma}_{p,q}(R^n)$$

where and in what follows $(\cdot,\,\cdot)_{\theta,q}$ *denotes the real interpolation functor* (*see* [5, 45]).

It follows from the definition of the Besov spaces, the Sobolev imbedding theorem in Lebesgue spaces and the elementary estimate

$$\|\Delta e^{t\Delta}u\|_{L_\infty} \le Ct^{-n/(2p)}\|\Delta e^{t\Delta/2}u\|_{L_p}$$

that the following variant of Sobolev's imbedding theorem holds true.

Lemma 4.3 (see [45, Theorem 2.7.1 and Subsection 5.2.5])**.** *If* $1 \le p \le q \le \infty$, $1 \le s \le \infty$, $-\infty < \alpha \le \beta < \infty$, *and* $\beta - n/p = \alpha - n/q$, *then*

$$\|\cdot\|_{\dot{B}^{\alpha}_{q,s}} \le C\|\cdot\|_{\dot{B}^{\beta}_{p,s}}. \tag{15}$$

Following result is concerned with a characterization of the norms for the homogeneous Besov spaces.

Lemma 4.4 (see [45, Theorem 5.2.3])**.** *For* $0 < \alpha < 1$ *and* $1 \le p \le \infty$, *then*

$$\|u\|_{\dot{B}^{\alpha}_{p,\infty}} \cong \sup_{y\ne 0}\frac{\|u(\cdot+y)-u(\cdot)\|_{L_p}}{|y|^{\alpha}} \quad \text{in } \dot{B}^{\alpha}_{p,\infty}(R^n).$$

Next, we state a variant of Mikhlin theorem on Fourier multipliers.

Lemma 4.5 (see [45, Theorem 5.2.2])**.** *Let* $-\infty < \alpha < \infty$, *and let* $\phi(x)$ *be a complex-valued infinitely differentiable function on* $R^n \setminus \{0\}$ *so that*

$$\sup_{j\le k}\sup_{x\in R^n}|x|^j|\nabla^j\phi(x)| < \infty$$

for a sufficiently large positive integer k. *Then*

$$\|\mathcal{F}^{-1}\phi\mathcal{F}u\|_{\dot{B}^{\alpha}_{p,q}} \le C\|u\|_{\dot{B}^{\alpha}_{p,q}} \quad (u\in \dot{B}^{\alpha}_{p,q}(R^n), \quad 1\le p,\, q\le\infty).$$

Finally, we give a variant of the well-known L_p-L_q estimates for the heat semigroup.

Lemma 4.6. *Let* $1 \le p \le q \le \infty$, $1 \le r,\, s \le \infty$, $-\infty < \alpha \le \beta < \infty$, *and* $r = s$ *if* $n/p - n/q + \beta - \alpha = 0$. *Then*

$$\|e^{t\Delta}a\|_{\dot{B}^{\beta}_{q,s}} \le Ct^{-(n/p-n/q+\beta-\alpha)/2}\|a\|_{\dot{B}^{\alpha}_{p,r}} \quad (a\in \dot{B}^{\alpha}_{p,r}(R^n)). \tag{16}$$

Proof. Let $\gamma = \beta - \alpha + n/p - n/q$. It follows from Lemma 4.3 that

$$\begin{aligned}\|e^{t\Delta}a\|_{\dot{B}^{\beta}_{q,s}} &= \|e^{t\Delta}a\|_{\dot{B}^{\alpha-n/p+n/q+\gamma}_{q,s}} \\ &\le C\|e^{-t\Delta}a\|_{\dot{B}^{\alpha+\gamma}_{p,s}}.\end{aligned}$$

When $\gamma = 0$, this is obviously bounded by $C\|a\|_{\dot{B}^{\alpha}_{p,r}}$. When $\gamma \neq 0$, we see that, by Lemma 4.2,

$$(\dot{B}^{\alpha}_{p,r}(R^n), \dot{B}^{\alpha+\gamma+1}_{p,r}(R^n))_{\gamma/(\gamma+1),s} = \dot{B}^{\alpha+\gamma}_{p,s}(R^n).$$

Thus the interpolation property (see [45, Proposition 2.4.1]) yields

$$\begin{aligned}\|e^{t\Delta}a\|_{\dot{B}^{\alpha+\gamma}_{q,s}} &\le C\|e^{t\Delta}a\|^{1-\gamma/(\gamma+1)}_{\dot{B}^{\alpha}_{p,r}}\|e^{t\Delta}a\|^{\gamma/(\gamma+1)}_{\dot{B}^{\alpha+\gamma+1}_{p,r}} \\ &\le Ct^{-\gamma/2}\|a\|_{\dot{B}^{\alpha}_{p,r}},\end{aligned}$$

where one has used the following elementary estimate

$$\|(-\Delta)^{(\gamma+1)/2}e^{t\Delta}a\|_{L_p} \le Ct^{-(\gamma+1)/2}\|a\|_{L_p}.$$

The proof is complete.

5. Estimates for the Stokes problem

In order to prove the main results, it is convenient to give a priori estimate in homogeneous Besov spaces for the Stokes equations.

Lemma 5.1. *Let* $0 < \alpha < 2$, $2 \le n \le p \le \infty$, $1 \le q \le \infty$, *and*

$$a \in \mathcal{S}'(R^n)^n, \quad f(t) \in \dot{B}^{\alpha-2+n/p}_{p,\infty}(R^n)^{n\times n}.$$

Then the solution of Eq. (2) *in the following integral formulation*

$$u(t) = e^{t\Delta}Pa + \int_0^t e^{(t-s)\Delta}P\nabla\cdot f(s)ds$$

satisfies the estimates

$$\|u(t)\|_{\dot{B}^{-1+n/p}_{p,q}} \le C\|a\|_{\dot{B}^{-1+n/p}_{p,q}} + C\sup_{0<s<t} s^{\alpha/2}\|f(s)\|_{\dot{B}^{\alpha-2+n/p}_{p,\infty}},$$

$$t^{\alpha/2}\|u(t)\|_{\dot{B}^{\alpha-1+n/p}_{p,\infty}} \le C\|a\|_{\dot{B}^{-1+n/p}_{p,\infty}} + C\sup_{0<s<t} s^{\alpha/2}\|f(s)\|_{\dot{B}^{\alpha-2+n/p}_{p,\infty}},$$

provided that the right-hand sides of the above inequalities are finite respectively.

Proof. Note that Lemma 4.5 gives

$$\|Pv\|_{\dot{B}^{\alpha}_{p,q}} + \|\nabla(-\Delta)^{-1/2}v\|_{\dot{B}^{\alpha}_{p,q}} \le C\|v\|_{\dot{B}^{\alpha}_{p,q}} \tag{17}$$

and there holds the following elementary estimate

$$\|\nabla^k e^{t\Delta}v\|_{L_p} \le Ct^{-k/2}\|v\|_{L_p} \quad (k \ge 0). \tag{18}$$

Thus, by Lemma 4.1, we have

$$
\begin{aligned}
&\|u(t)-e^{-tA}Pa\|_{\dot B^{\alpha-1+n/p}_{p,\infty}}\\
&= \sup_{\tau>0}\tau^{1-(\alpha-1+n/p)/2}\|\Delta e^{\tau\Delta}\int_0^t e^{(t-s)\Delta}P\nabla\cdot f(s)ds\|_{L_p}\\
&\le C\sup_{\tau>0}\tau^{1-(\alpha-1+n/p)/2}\int_0^t\|\Delta^2 e^{(t-s+\tau)\Delta}\Delta^{-1}P\nabla\cdot f(s)\|_{L_p}ds\\
&\le C\sup_{\tau>0}\tau^{1-(\alpha-1+n/p)/2}\int_0^t(t+\tau-s)^{-1}\|\Delta e^{(t+\tau-s)\Delta/2}\Delta^{-1}P\nabla\cdot f(s)\|_{L_p}ds\\
&\le C\sup_{\tau>0}\tau^{1-(\alpha-1+n/p)/2}\int_0^t(t+\tau-s)^{-2+(\alpha-1+n/p)/2}\|\Delta^{-1}P\nabla\cdot f(s)\|_{\dot B^{\alpha-1+n/p}_{p,\infty}}ds\\
&\le C\sup_{\tau>0}\tau^{1-(\alpha-1+n/p)/2}\int_0^t(t+\tau-s)^{-2+(\alpha-1+n/p)/2}\\
&\quad\times\|P(-\Delta)^{-1/2}\nabla\cdot f(s)\|_{\dot B^{\alpha-2+n/p}_{p,\infty}}ds\\
&\le C\sup_{\tau>0}\tau^{1-(\alpha-1+n/p)/2}\int_0^t(t+\tau-s)^{-2+(\alpha-1+n/p)/2}\|f(s)\|_{\dot B^{\alpha-2+n/p}_{p,\infty}}ds\\
&\le C\sup_{\tau>0}\tau^{1-(\alpha-1+n/p)/2}\left(\int_{t/2}^t+\int_0^{t/2}\right)(t+\tau-s)^{-2+(\alpha-1+n/p)/2}s^{-\alpha/2}ds\\
&\quad\times\sup_{0<s<t}s^{\alpha/2}\|f(s)\|_{\dot B^{\alpha-2+n/p}_{p,\infty}}\\
&\le C\sup_{\tau>0}\tau^{1-(\alpha-1+n/p)/2}t^{-\alpha/2}\int_0^t(t+\tau-s)^{-2+(\alpha-1+n/p)/2}ds\\
&\quad\times\sup_{0<s<t}s^{\alpha/2}\|f(s)\|_{\dot B^{\alpha-2+n/p}_{p,\infty}}\\
&\quad+C\sup_{\tau>0}\tau^{1-(\alpha-1+n/p)/2}(t+\tau)^{-2+(\alpha-1+n/p)/2}\int_0^t s^{-\alpha/2}ds\\
&\quad\times\sup_{0<s<t}s^{\alpha/2}\|f(s)\|_{\dot B^{\alpha-2+n/p}_{p,\infty}}\\
&\le Ct^{-\alpha/2}\sup_{\tau>0}\int_0^{t/\tau}(1+s)^{-2+(\alpha-1+n/p)/2}ds\sup_{0<s<t}s^{\alpha/2}\|f(s)\|_{\dot B^{\alpha-2+n/p}_{p,\infty}}\\
&\quad+Ct^{-\alpha/2}\sup_{0<s<t}s^{\alpha/2}\|f(s)\|_{\dot B^{\alpha-2+n/p}_{p,\infty}}\\
&\le Ct^{-\alpha/2}\sup_{0<s<t}s^{\alpha/2}\|f(s)\|_{\dot B^{\alpha-2+n/p}_{p,\infty}}.
\end{aligned}
$$

By Eqs. (17, 18) and Lemmas 4.1, 4.6,

$$\begin{aligned}\|u(t) - e^{-tA}Pa\|_{\dot{B}^{-1+n/p}_{p,q}} &\le C\int_0^t \|e^{(t-s)\Delta}\nabla\cdot f(s)\|_{\dot{B}^{-1+n/p}_{p,q}}ds \\ &\le C\int_0^t \|e^{(t-s)\Delta} f(s)\|_{\dot{B}^{n/p}_{p,q}}ds \\ &\le C\int_0^t (t-s)^{-1+\alpha/2}\|f(s)\|_{\dot{B}^{\alpha-2+n/p}_{p,\infty}}ds \\ &\le C\int_0^t (t-s)^{-1+\alpha/2}s^{-\alpha/2}ds \sup_{0<s<t} s^{\alpha/2}\|f(s)\|_{\dot{B}^{\alpha-2+n/p}_{p,\infty}} \\ &\le C\sup_{0<s<t} s^{\alpha/2}\|f(s)\|_{\dot{B}^{\alpha-2+n/p}_{r,\infty}}.\end{aligned}$$

Collecting terms, we arrive at

$$\begin{aligned}t^{\alpha/2}\|u(t)\|_{\dot{B}^{\alpha-1+n/p}_{p,\infty}} &\le t^{\alpha/2}\|e^{t\Delta}Pa\|_{\dot{B}^{\alpha-1+n/p}_{p,\infty}} + C\sup_{0<s<t} s^{\alpha/2}\|f(s)\|_{\dot{B}^{\alpha-2+n/p}_{p,\infty}}, \\ \|u(t)\|_{\dot{B}^{-1+n/p}_{p,q}} &\le \|e^{t\Delta}Pa\|_{\dot{B}^{-1+n/p}_{p,q}} + C\sup_{0<s<t} s^{\alpha/2}\|f(s)\|_{\dot{B}^{\alpha-2+n/p}_{p,\infty}}.\end{aligned}$$

Consequently, by Lemma 4.6 and Eq. (17), we obtain the desired estimates and complete the proof.

It should be noted that the estimates in Lemma 5.1 are convenient to be applied to the existence of regular solutions of Eq. (1) in the class where $t^{1/2}\|u(t)\|_{L_\infty}$ is bounded, but not suitable in the study of uniqueness for the solutions in the class $C([0,T]; L_n(R^n)^n)$. To provide an alternative approach to this problem, we will give an $L_p(L_q)$ estimate for the Stokes problem, a variant of the estimate from Giga and Sohr [22], which goes back to the L_p estimate from Solonnikov [37, 38].

Let us recall the estimate of [22].

Lemma 5.2 ([22, Theorem 2.7]). *Let $1 < p, q < \infty$, $T > 0$, and $f \in L_q(0,T; L_p(R^n)^n)$. Then there exists a unique solution u of the Stokes problem*

$$\begin{aligned}\partial_t u - \Delta u &= Pf \quad \text{in } L_p(R^n)^n \text{ for a.e. } t\in(0,T), \\ u(0) &= 0\end{aligned} \tag{19}$$

satisfying the property

$$\int_0^T \|\Delta u(s)\|^q_{L_p}ds \le C\int_0^T \|f(s)\|^q_{L_p}ds.$$

Applying the operator $(-\Delta)^{-1/2}$ to Eq. (19), and using Calderón–Zygmund's estimate and Sobolev's imbedding estimate, we obtain immediately the following lemma, which will be used in Section 9.

Lemma 5.3. *Let $1 < p, q < \infty$, $T > 0$. The for every $f \in L_q(0,T; L_p(R^n)^{n\times n})$ there exist a unique solution $v = (-\Delta)^{-1/2}u$ of the problem*

$$\begin{aligned}\partial_t v - \Delta v &= P(-\Delta)^{-1/2}\nabla\cdot f \quad \text{in } L_p(R^n)^n \text{ for a.e. } t \in (0,T),\\ v(0) &= 0\end{aligned}$$

satisfying the properties

$$\int_0^T \|\nabla u(s)\|_{L_p}^q ds \le C\int_0^T \|f(s)\|_{L_p}^q ds$$

and

$$\int_0^T \|u(s)\|_{L_{np/(n-p)}}^q ds \le C\int_0^T \|f(s)\|_{L_p}^q ds \quad (1 < p < n).$$

6. Proof of Theorem 2.1

Set

$$U = \Big\{u \in L_\infty(0,\infty; \dot B_{p,\infty}^{-1+n/p}(R^n)^n) \mid \nabla\cdot u = 0,\ \|u\|_U < \infty\Big\}$$

with

$$\|u\|_U = \sup_{t>0}\Big(\|u(t)\|_{\dot B_{p,\infty}^{-1+n/p}} + t^{\alpha/2}\|u(t)\|_{\dot B_{p,\infty}^{\alpha-1+n/p}}\Big)$$

and

$$Mu(t) = e^{t\Delta}a - \int_0^t e^{(t-s)\Delta}P\nabla\cdot(u(s)\otimes u(s))ds.$$

It will be shown that M is a contraction operator mapping a ball of U into itself.

Note that, by Lemmas 4.2 and 4.3 and [45, Proposition 2.5.7],

$$\dot B_{p,\infty}^{\alpha-2}(R^n)^n = (\dot B_{p,\infty}^{-1}(R^n)^n,\, \dot B_{p,\infty}^{\alpha-1}(R^n)^n)_{(\alpha-1)/\alpha,\infty},$$

$$\dot B_{\infty,\infty}^{0}(R^n)^n \supset L_\infty(R^n)^n \supset \dot B_{\infty,1}^{0}(R^n)^n = (\dot B_{\infty,\infty}^{-1}(R^n)^n,\, \dot B_{\infty,\infty}^{\alpha-1}(R^n)^n)_{1/\alpha,1},$$

which contains the space

$$(\dot B_{p,\infty}^{-1+n/p}(R^n)^n,\, \dot B_{p,\infty}^{\alpha-1+n/p}(R^n)^n)_{1/\alpha,1}.$$

Thus

$$\begin{aligned}&\|u(t,t^{1/2}\cdot)\|_{L_\infty} + \|u(t,t^{1/2}\cdot)\|_{\dot B_{p,\infty}^{\alpha-2+n/p}}\\ &\le C(\|u(t,t^{1/2}\cdot)\|_{\dot B_{p,\infty}^{-1+n/p}} + \|u(t,t^{1/2}\cdot)\|_{\dot B_{p,\infty}^{\alpha-1+n/p}}),\end{aligned}$$

which becomes, after a variable transformation,

$$\begin{aligned}&t^{1/2}\|u(t)\|_{L_\infty} + t^{(\alpha-1)/2}\|u(t)\|_{\dot B_{p,\infty}^{\alpha-2+n/p}}\\ &\le C(\|u(t)\|_{\dot B_{p,\infty}^{-1+n/p}} + t^{\alpha/2}\|u(t)\|_{\dot B_{p,\infty}^{\alpha-1+n/p}}).\end{aligned} \tag{20}$$

Thus with the use of Lemmas 4.4 and 5.1, we have, for $u \in U$,

$$\begin{aligned} &\|Mu(t)\|_{\dot{B}^{-1+n/p}_{p,\infty}} + t^{\alpha/2}\|Mu(t)\|_{\dot{B}^{\alpha-1+n/p}_{p,\infty}} \\ &\le C\|a\|_{\dot{B}^{-1+n/p}_{p,\infty}} + C \sup_{0<s<t} s^{\alpha/2}\|u(s)\otimes u(s)\|_{\dot{B}^{\alpha-2+n/p}_{p,\infty}} \\ &\le C\|a\|_{\dot{B}^{-1+n/p}_{p,\infty}} + C \sup_{0<s<t} s^{\alpha/2}\|u(s)\|_{L_\infty}\|u(s)\|_{\dot{B}^{\alpha-2+n/p}_{p,\infty}} \\ &\le C\|a\|_{\dot{B}^{-1+n/p}_{p,\infty}} + C\|u\|_U^2, \end{aligned}$$

since $0 < \alpha - 2 + n/p < 1$ and $n \le p < \infty$.

Likewise, we have

$$\|Mu - Mv\|_U \le C(\|u\|_U + \|v\|_U)\|u - v\|_U \ \text{ for } \ u, v \in U.$$

Since $PMu(t) = Mu(t)$, we have $\nabla \cdot Mu(t) = 0$.

Define a complete metric space by

$$U_\epsilon = \{u \in U | \, \|u\|_U \le \epsilon\}.$$

Then the previous analysis shows that

$$\|Mu\|_U \le C\|a\|_{\dot{B}^{-1+n/p}_{p,\infty}} + C\epsilon\|u\|_U \ \text{ for } \ u \in U_\epsilon,$$

$$\|Mu - Mv\|_U \le C\epsilon\|u - v\|_U \ \text{ for } \ u, v \in U_\epsilon.$$

As a consequence of the contraction mapping principle, Eq. (6) admits a unique solution $u \in U_\epsilon$, provided that $C\|a\|_{\dot{B}^{-1+n/p}_{p,\infty}} \le \epsilon/2$ and $\epsilon > 0$ is sufficiently small. Obviously $u(t)$ is regular for all $t > 0$.

The proof of Theorem 2.1 is complete.

7. Proof of Theorem 2.2

Let us begin with the derivation of a crucial estimate by developing a technique from [11]. However, the estimate in $L_\infty(R^n)^n$ rather than in the space $\hat{M}_n(R^n)^n$ has been verified in [18].

Lemma 7.1. *For u, $v \in L_\infty(R^n)^n \cap \hat{M}_n(R^n)^n$, there holds*

$$\|e^{t\Delta}P\nabla \cdot (u \otimes v)\|_{\hat{M}_n} \le Ct^{-1/2}\|u\|_{L_\infty}\|v\|_{\hat{M}_n}.$$

Proof. First, we prove the known estimate, shown in [10],

$$\|\partial_{x_i}\partial_{x_j}\Delta^{-1}\partial_{x_k}h_t\|_{L_1} \le Ct^{-1/2} \ \ (i,\, j,\, k = 1, \dots, n)$$

with h_t the heat kernel defined in Section 2.

This can be done alternatively by following the proof of [11, Lemma 2.7]. Set $\nu_i(x) = x_i/|x|$, $\omega_n = |\{x \in R^n | \, |x| \le 1\}|$ and

$$\Gamma(x-y) = \begin{cases} \dfrac{1}{n(2-n)\omega_n}|x-y|^{2-n}, & n > 2, \\ \\ \dfrac{1}{2\pi}\log|x-y|, & n = 2, \end{cases}$$

which is the fundamental solution of the Laplacian equation in the following form:

$$-\Delta^{-1}u(x) = \int_{R^n} \Gamma(x-y)u(y)dy.$$

Standard potential estimates, see [23, Lemma 4.1], imply that

$$\begin{aligned} \partial_{x_i}\partial_{x_j}\Delta^{-1}\partial_{x_k}h_t(x) &= \partial_{x_i}\partial_{x_j}\int_{R^n}\Gamma(x-y)\partial_{y_k}h_t(y)dy \\ &= \partial_{x_i}\int_{R^n}\partial_{x_j}\Gamma(x-y)\partial_{y_k}h_t(y)dy. \end{aligned}$$

Moreover, taking a cut off function $\eta \in C^1(R^n)$ so that $0 \le \eta \le 1$, $0 \le \eta' \le 2$, and

$$\eta(s) = \begin{cases} 0, & \text{for } s \le 1, \\ 1, & \text{for } s \ge 2, \end{cases}$$

and setting, for $\epsilon > 0$,

$$v_\epsilon = \int_{R^n} \eta\left(|x-y|/\epsilon\right)\partial_{x_j}\Gamma(x-y)\partial_{y_k}h_t(y)dy,$$

we have, for $\lambda > 2\epsilon$,

$$\begin{aligned} \partial_{x_i}v_\epsilon &= \int_{R^n}\partial_{x_i}\left(\eta\left(|x-y|/\epsilon\right)\partial_{x_j}\Gamma(x-y)\right)\partial_{y_k}h_t(y)dy \\ &= \int_{|x-y|<\lambda}\partial_{x_i}\left(\eta\left(|x-y|/\epsilon\right)\partial_{x_j}\Gamma(x-y)\right)\partial_{y_k}h_t(y)dy \\ &\quad + \int_{|x-y|>\lambda}\partial_{x_i}\partial_{x_j}\Gamma(x-y)\partial_{y_k}h_t(y)dy \\ &= \int_{|x-y|<\lambda}\partial_{x_i}\left(\eta\left(|x-y|/\epsilon\right)\partial_{x_j}\Gamma(x-y)\right)(\partial_{y_k}h_t(y) - \partial_{x_k}h_t(x))dy \\ &\quad -\partial_{x_k}h_t(x)\int_{|x-y|<\lambda}\partial_{y_i}\left(\eta\left(|x-y|/\epsilon\right)\partial_{x_j}\Gamma(x-y)\right)dy \\ &\quad + \int_{|x-y|>\lambda}\partial_{x_k}\partial_{x_i}\partial_{x_j}\Gamma(x-y)\,h_t(y)dy \\ &\quad - \int_{|x-y|=\lambda}\partial_{x_i}\partial_{x_j}\Gamma(x-y)\,\nu_k(x-y)h_t(y)dy \end{aligned}$$

$$\begin{aligned}
&= \int_{|x-y|<\lambda} \partial_{x_i} \left(\eta\left(|x-y|/\epsilon\right) \partial_{x_j} \Gamma(x-y)\right) \left(\partial_{y_k} h_t(y) - \partial_{x_k} h_t(x)\right) dy \\
&\quad -\partial_{x_k} h_t(x) \int_{|x-y|=\lambda} \partial_{x_j} \Gamma(x-y) \nu_i(x-y) dy \\
&\quad + \int_{|x-y|>\lambda} \partial_{x_k} \partial_{x_i} \partial_{x_j} \Gamma(x-y)(h_t(y) - h_t(x)) dy \\
&\quad - \int_{|x-y|=\lambda} \partial_{x_i} \partial_{x_j} \Gamma(x-y)\, \nu_k(x-y)(h_t(y) - h_t(x)) dy.
\end{aligned}$$

Passing to the limit as $\epsilon \to 0$ yields

$$\begin{aligned}
\partial_{x_i} \partial_{x_j} \Delta^{-1} \partial_{x_k} h_t(x) \;=\; & \int_{|x-y|>\lambda} \partial_{x_k} \partial_{x_i} \partial_{x_j} \Gamma(x-y)\, (h_t(y) - h_t(x)) dy \\
& + \int_{|x-y|<\lambda} \partial_{x_i} \partial_{x_j} \Gamma(x-y)\, (\partial_{x_k} h_t(y) - \partial_{x_k} h_t(x)) dy \\
& - \int_{|x-y|=\lambda} \partial_{x_i} \partial_{x_j} \Gamma(x-y)\, \nu_k(x-y)(h_t(y) - h_t(x)) dy \\
& - \partial_{x_k} h_t(x) \int_{|x-y|=\lambda} \partial_{x_j} \Gamma(x-y)\, \nu_i(x-y) dy.
\end{aligned}$$

This together with the simple estimate

$$|x|^2 |\nabla^3 \Gamma(x)| + |x|\, |\nabla^2 \Gamma(x)| + |\nabla \Gamma(x)| \le C |x|^{-n+1}$$

implies, for $\lambda = t^{1/2}$,

$$\begin{aligned}
\|\partial_{x_i} \partial_{x_j} \Delta^{-1} \partial_{x_k} h_t\|_{L_1} \;\le\; & C \int_{|y|>t^{1/2}} |y|^{-n-1} \|h_t(\cdot - y) - h_t(\cdot)\|_{L_1} dy \\
& + C \int_{|y|<t^{1/2}} |y|^{-n} \|\nabla h_t(\cdot - y) - \nabla h_t(\cdot)\|_{L_1} dy \\
& + C \int_{|y|=t^{1/2}} |y|^{-n} \|h_t(\cdot - y) - h_t(\cdot)\|_{L_1} dy + C \|\nabla h_t\|_{L_1} \\
\le\; & C \int_{|y|>t^{1/2}} |y|^{-n-1} \|h_1(\cdot - t^{-1/2} y) - h_1(\cdot)\|_{L_1} dy \\
& + C t^{-1/2} \int_{|y|<t^{1/2}} |y|^{-n} \|\nabla h_1(\cdot - t^{-1/2} y) - \nabla h_1(\cdot)\|_{L_1} dy \\
& + C t^{-1/2} \|h_1\|_{L_1} + C t^{-1/2} \|\nabla h_1\|_{L_1}
\end{aligned}$$

$$
\begin{aligned}
&\le Ct^{-1/4}\int_{|y|>t^{1/2}}|y|^{-n-1/2}dy\|h_1\|_{\dot B^{1/2}_{1,\infty}}+Ct^{-1/2}\|h_1\|_{L_1}\\
&\quad +Ct^{-3/4}\int_{|y|<t^{1/2}}|y|^{-n+1/2}dy\|h_1\|_{\dot B^{3/2}_{1,\infty}}+Ct^{-1/2}\|\nabla h_1\|_{L_1}\\
&\le Ct^{-1/2}.
\end{aligned}
$$

Next, observing that the operation with respect to the convolution is commutative, on setting

$$G_{ijk,t}(x)=(\delta_{ij}-\partial_{x_i}\partial_{x_j}\Delta^{-1})\partial_{x_k}h_t(x),$$

one can estimate as, for $s>0$,

$$
\begin{aligned}
&\|e^{-s\Delta}|e^{t\Delta}P\nabla\cdot(u\otimes v)|\,\|_{L_\infty}\\
&\le \sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\|h_s*|G_{ijk,t}*(u_kv_j)|\,\|_{L_\infty}\\
&\le \sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\|h_s*|G_{ijk,t}|*|u_kv_j|\,\|_{L_\infty}\\
&= \sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\|\,|G_{ijk,t}|*h_s*|u_kv_j|\,\|_{L_\infty}\\
&\le \sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\|G_{ijk,t}\|_{L_1}\|h_s*|u_kv_j|\,\|_{L_\infty}\\
&\le \left(\|\nabla h_t\|_{L_1}+\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\|\partial_{x_i}\partial_{x_j}\Delta^{-1}\partial_{x_k}h_t\|_{L_1}\right)\|e^{s\Delta}|u_kv_j|\,\|_{L_\infty}\\
&\le Ct^{-1/2}\|u\|_{L_\infty}\|e^{s\Delta}|v|\,\|_{L_\infty}.
\end{aligned}
$$

This gives

$$\sup_{s>0}s^{1/2}\|e^{s\Delta}|e^{t\Delta}P\nabla\cdot(u\otimes v)|\,\|_{L_\infty}\le Ct^{-1/2}\|u\|_{L_\infty}\sup_{s>0}s^{1/2}\|e^{s\Delta}|v|\,\|_{L_\infty},$$

and hence completes the proof.

Proof of Theorem 2.2. Similar to the proof of Theorem 2.1, we set

$$W=\left\{u:[0,\infty)\mapsto\hat M_n(R^n)^n|\ \nabla\cdot u=0,\ \|u\|_W<\infty\right\}$$

with

$$\|u\|_W=\sup_{t>0}\left(\|u(t)\|_{\hat M_n}+t^{\alpha/2}\|u(t)\|_{\dot B^{\alpha-1}_{\infty,\infty}}\right).$$

Recall that

$$Mu(t) = e^{t\Delta}a - \int_0^t e^{(t-s)\Delta} P\nabla \cdot (u(s) \otimes u(s))ds.$$

The goal is to show that M is a contraction operator mapping a ball of W into itself.

Similar argument as for Eq. (20) gives

$$t^{(\alpha-1)/2}\|v(t)\|_{\dot{B}^{\alpha-2}_{\infty,\infty}} + t^{1/2}\|v(t)\|_{L_\infty} \le C\|v(t)\|_{\dot{B}^{-1}_{\infty,\infty}} + Ct^{\alpha/2}\|v(t)\|_{\dot{B}^{\alpha-1}_{\infty,\infty}}.$$

With this observation in mind, we obtain from Lemma 5.1 that

$$\begin{aligned}
&t^{\alpha/2}\|Mu(t)\|_{\dot{B}^{\alpha-1}_{\infty,\infty}}\\
&\le C\|a\|_{\dot{B}^{-1}_{\infty,\infty}} + C\sup_{0<s<t} s^{\alpha/2}\|u(s)\otimes u(s)\|_{\dot{B}^{\alpha-2}_{\infty,\infty}}\\
&\le C\|a\|_{\dot{B}^{-1}_{\infty,\infty}} + C\sup_{0<s<t} s^{1/2}\|u(s)\otimes u(s)\|_{\dot{B}^{-1}_{\infty,\infty}} + C\sup_{0<s<t} s^{(\alpha+1)/2}\|u(s)\otimes u(s)\|_{\dot{B}^{\alpha-1}_{\infty,\infty}}\\
&\le C\|a\|_{\hat{M}_n} + C\sup_{0<s<t} s^{1/2}\|u(s)\otimes u(s)\|_{\hat{M}_n} + C\sup_{0<s<t} s^{(\alpha+1)/2}\|u(s)\otimes u(s)\|_{\dot{B}^{\alpha-1}_{\infty,\infty}}\\
&\le C\|a\|_{\hat{M}_n} + C\sup_{0<s<t} s^{1/2}\|u(s)\|_{L_\infty}\|u(s)\|_{\hat{M}_n} + C\sup_{0<s<t} s^{(\alpha+1)/2}\|u(s)\|_{L_\infty}\|u(s)\|_{\dot{B}^{\alpha-1}_{\infty,\infty}}\\
&\le C\|a\|_{\hat{M}_n} + C\|u\|_W^2.
\end{aligned}$$

Moreover, Lemma 7.1 leads to

$$\begin{aligned}
\|Mu(t)\|_{\hat{M}_n} &\le \sup_{s>0} s^{1/2}\|e^{t\Delta}e^{s\Delta}|a|\,\|_{L_\infty} + \int_0^t \|e^{(t-s)\Delta}P\nabla\cdot(u(s)\otimes u(s))\|_{\hat{M}_n} ds\\
&\le C\|a\|_{\hat{M}_n} + C\int_0^t (t-s)^{-1/2}\|u(s)\|_{\hat{M}_n}\|u(s)\|_{L_\infty} ds\\
&\le C\|a\|_{\hat{M}_n} + C\|u\|_W^2.
\end{aligned}$$

Thus

$$\|Mu\|_W \le C\|a\|_{\hat{M}_n} + C\|u\|_W^2 \;\; (u \in W)$$

and, similarly,

$$\|Mu - Mv\|_W \le C(\|u\|_W + \|v\|_W)\|u - v\|_W \;\; (u, v \in W).$$

Obviously, $PMu(t) = Mu(t)$. By the contraction mapping principle, Eq. (6) admits a unique regular solution in the ball $W_\epsilon = \{u \in W \mid \|u\|_W \le \epsilon\}$, provided that $C\|a\|_{\hat{M}_n} \le \epsilon/2$ for some small constant $\epsilon > 0$.

The proof of Theorem 2.2 is complete.

From the proof of Theorems 2.1 and 2.2, we obtain the following existence result of the Navier–Stokes flows with the initial velocities in an abstract critical homogeneous space.

Corollary 7.1. *Assume $1 < \alpha < 2$ and $n \le p \le \infty$ such that $\alpha + n/p > 2$ whenever $p \ne \infty$. Let X and Y be the critical homogeneous spaces such that*

$$X \subset \dot{B}^{-1+n/p}_{p,\infty}(R^n)^n,$$
$$Y = \{u \in L_\infty(0,\infty;X) \mid \|u\|_Y < \infty\}$$

with

$$\|u\|_Y = \sup_{t>0}\|u(t)\|_X + \sup_{t>0} t^{1/2}\|u(t)\|_{L_\infty} + \sup_{t>0} t^{\alpha/2}\|u(t)\|_{\dot{B}^{\alpha-1+n/p}_{p,\infty}}$$

such that

$$\|e^{t\Delta}a\|_Y \le C\|a\|_X \quad (a \in X), \tag{21}$$

and, for u, $v \in Y$,

$$\begin{aligned}&\left\|\int_0^t e^{(t-s)\Delta}P\nabla\cdot(u\otimes v)ds\right\|_X \\ &\le C(\|u\|_Y + \|v\|_Y)\sup_{0<s<t} s^{1/2}(\|u(s)\|_{L_\infty} + \|v(s)\|_{L_\infty}).\end{aligned} \tag{22}$$

Then for every $a \in X$ with $\nabla\cdot a = 0$, Eq. (6) admits a unique regular solution u satisfying the persistence property

$$\|u(t)\|_X + t^{\alpha/2}\|u(t)\|_{\dot{B}^{\alpha-1+n/p}_{p,\infty}} \le C\|a\|_X$$

provided $\|a\|_X \le \epsilon$ for some small ϵ.

In particular, we may specify the existence results on the the critical homogeneous spaces given by Eqs. (10)–(13) in the following.

Corollary 7.2. *Let X be any one of the critical homogeneous spaces*

$$L_n(R^n)^n,\ L_{n,\infty}(R^n)^n,\ \dot{F}^{-1+n/p}_{p,\infty}(R^n)^n,\ M_n(R^n)^n,\ \tilde{M}_n(R^n)^n\ \ (2 \le n \le p < \infty).$$

Then for every $a \in X$ with $\nabla\cdot a = 0$ and $\|a\|_X \le \epsilon$ for some small ϵ, Eq. (6) admits a unique regular solution u such that

$$\sup_{t>0}\left(\|u(t)\|_X + t^{1/2}\|u(t)\|_{L_\infty} + t^{\alpha/2}\|u(t)\|_{\dot{B}^{\alpha-1+n/q}_{q,\infty}}\right) \le C\|a\|_X$$

where $q = n$ when $X = L_n(R^n)^n$ or $L_{n,\infty}(R^n)^n$, $q = p$ when $X = \dot{F}^{-1+n/p}_{p,\infty}(R^n)^n$, and $q = \infty$ for $X = M_n(R^n)^n$ or $\tilde{M}_n(R^n)^n$.

Proof. By Corollary 7.1, it remains to verify the validity of Eqs. (21, 22). Note that the theory of homogeneous Triebel–Lizorkin spaces is parallel to those of homogeneous Besov spaces (see [45]). Thus by the proof of Lemma 4.6 and Eq. (12), one has

$$\|e^{t\Delta}a\|_X + t^{1/2}\|e^{t\Delta}a\|_{L_\infty} + t^{\alpha/2}\|e^{t\Delta}a\|_{\dot{B}^{\alpha-1+n/p}_{p,\infty}} \le C\|a\|_X$$

with $X = \dot{F}^{-1+n/p}_{p,\infty}(R^n)^n$. For the remaining choice of X, this inequality is well known. Thus Eq. (21) is valid.

To verify Eq. (22), we apply Eqs. (10–12, 20), Lemmas 4.4, 5.1 and the observation (see [45, Eqs. (2.3.2/9), (2.3.5/6) and Subsection 5.2.5])

$$L_n(R^n)^n = \dot{F}^0_{n,2}(R^n)^n \supset \dot{B}^0_{n,2}(R^n)^n$$

to obtain, for $X = L_n(R^n)^n$ or $L_{n,\infty}(R^n)^n$,

$$\begin{aligned}
&\left\|\int_0^t e^{(t-s)\Delta} P\nabla\cdot(u(s)\otimes v(s))ds\right\|_X \\
&\le \left\|\int_0^t e^{(t-s)\Delta} P\nabla\cdot(u(s)\otimes v(s))ds\right\|_{L_n} \\
&\le C\|\int_0^t e^{(t-s)\Delta} P\nabla\cdot(u(s)\otimes v(s))ds\|_{\dot{B}^0_{n,2}} \\
&\le C\sup_{t>0} t^{\alpha/2}\|u(t)\otimes v(t)\|_{\dot{B}^{\alpha-1}_{n,\infty}} \\
&\le C\sup_{t>0} t^{\alpha/2}(\|u(t)\|_{L_\infty}+\|v(t)\|_{L_\infty})(\|u(t)\|_{\dot{B}^{\alpha-1}_{n,\infty}}+\|v(t)\|_{\dot{B}^{\alpha-1}_{n,\infty}}) \\
&\le C\sup_{t>0} t^{1/2}(\|u(t)\|_{L_\infty}+\|v(t)\|_{L_\infty})(\|u\|_Y+\|v\|_Y),
\end{aligned}$$

for $X = \dot{F}^{-1+n/p}_{p,\infty}(R^n)^n$,

$$\begin{aligned}
&\left\|\int_0^t e^{(t-s)\Delta} P\nabla\cdot(u(s)\otimes v(s))ds\right\|_X \\
&\le C\left\|\int_0^t e^{(t-s)\Delta} P\nabla\cdot(u(s)\otimes v(s))ds\right\|_{\dot{B}^{-1+n/p}_{p,p}} \\
&\le C\sup_{t>0} t^{\alpha/2}\|u(t)\otimes v(t)\|_{\dot{B}^{\alpha-2+n/p}_{p,\infty}} \\
&\le C\sup_{t>0} t^{\alpha/2}(\|u(t)\|_{L_\infty}+\|v(t)\|_{L_\infty})(\|u(t)\|_{\dot{B}^{\alpha-2+n/p}_{p,\infty}}+\|v(t)\|_{\dot{B}^{\alpha-2+n/p}_{p,\infty}}) \\
&\le C\sup_{t>0} t^{1/2}(\|u(t)\|_{L_\infty}+\|v(t)\|_{L_\infty})(\|u\|_Y+\|v\|_Y),
\end{aligned}$$

and for $X = M_n(R^n)^n$ or $\hat{M}_n(R^n)^n$, by the proofs of Proposition 3.1 and Lemma 7.1,

$$\begin{aligned}
\left\|\int_0^t e^{(t-s)\Delta} P\nabla\cdot(u(s)\otimes v(s))ds\right\|_X &\le \int_0^t \|e^{(t-s)\Delta} P\nabla\cdot(u(s)\otimes v(s))\|_X ds \\
&\le C\int_0^t (t-s)^{-1/2}\|u(s)\otimes v(s)\|_X ds \\
&\le C\sup_{s>0} s^{1/2}\|u(s)\|_{L_\infty}\|v(s)\|_X \\
&\le C\sup_{s>0} s^{1/2}\|u(s)\|_{L_\infty}\|v\|_Y.
\end{aligned}$$

We thus obtain Eq. (22).

The proof is complete.

8. Self-similar solutions

As an easy application of Theorems 2.1 and 2.2, we may give the existence of small forward self-similar solutions of Eq. (1).

Definition 8.1. u is said to be a forward self-similar solution of Eq. (6) if u solves Eq. (6) in the sense of distribution and has the scaling invariance:

$$\lambda u(\lambda^2 t, \lambda x) = u(t, x) \quad (x \in R^n,\ t > 0,\ \lambda > 0).$$

This implies $u(t,x) = \sqrt{s/t}\, u(s, \sqrt{s/t}\, x)$ and $v(x) = u(s,x)$ with $s > 0$ a constant satisfies, for some pressure π, the steady-state equations

$$\begin{aligned} &-\Delta v - \frac{1}{2s}(v + (x \cdot \nabla)v) + \nabla \cdot (v \otimes v) + \nabla \pi = 0, \\ &\nabla \cdot v = 0. \end{aligned}$$

Let us formulate a simple result which is implicitly given in [20].

Lemma 8.1. *Let X be a metric space of functions defined on R^n, and Y be a ball centered at the origin of a critical homogeneous space of functions defined on $R_+ \times R^n$. Suppose that for every initial velocity $a \in X$ Eq. (6) admits a unique and global solution $u \in Y$. Then for every initial velocity $a \in X$ satisfying the scaling invariance*

$$\lambda a(\lambda x) = a(x) \text{ for all } x \in R^n,\ \lambda > 0, \tag{23}$$

Eq. (6) admits a unique forward self-similar solution $u \in Y$.

Proof. Let $u \in Y$ be the solution of Eq. (6) with the initial velocity $a \in X$ satisfying Eq. (23). By the scaling invariance of Eq. (1), we see that $u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$ solves Eq. (6) as well. By the assumption on the space Y, we have $\|u_\lambda\|_Y = \|u\|_Y$, and so $u_\lambda \in Y$ for all $\lambda > 0$. Thus Eq. (23) and the uniqueness assumption imply $u_\lambda = u$ for all $\lambda > 0$.

The proof is complete.

As immediate consequences of this lemma and Theorems 2.1, 2.2, we have the following results on the existence of self-similar solutions.

Corollary 8.1. *Let $2 \le n \le p < \infty$, $\max\{1, 2 - n/p\} < \alpha < 2$, and $\epsilon > 0$ be the small constant specified by Theorem 2.1. Assume that the initial velocity a is in* *the metric space*

$$X = \left\{ a \in \dot{B}^{-1+n/p}_{p,\infty}(R^n)^n \middle| \ \nabla \cdot a = 0, \ \|a\|_{\dot{B}^{-1+n/p}_{p,\infty}} \le \epsilon, \ \ a \text{ satisfies Eq. } (23) \right\}.$$

Then Eq. (6) *admits a unique regular forward self-similar solution u in the metric space*

$$Y = \left\{ u \in L_\infty(0,\infty; \dot{B}^{-1+n/p}_{p,\infty}(R^n)^n) \ \middle| \ \sup_{t>0} \|u(t)\|_{\dot{B}^{-1+n/p}_{p,\infty}} + \sup_{t>0} t^{\alpha/2} \|u(t)\|_{\dot{B}^{\alpha-1+n/p}_{p,\infty}} \le C\epsilon \right\}.$$

Corollary 8.2. *For $1 < \alpha < 2 \le n$ and $\epsilon > 0$ the small constant specified by Theorem 2.2, assume the initial velocity*

$$a \in X = \left\{ a \in \hat{M}_n(R^n)^n \middle| \ \nabla \cdot a = 0, \ \|a\|_{\hat{M}_n} \le \epsilon \right\}$$

satisfying Eq. (23)*. Then Eq.* (6) *admits a unique regular forward self-similar solution $u \in Y$ defined as*

$$\left\{ u \in L_\infty(0,\infty; \hat{M}_n(R^n)^n) \ \middle| \ \|u(t)\|_{\hat{M}_n} + t^{\alpha/2} \|u(t)\|_{\dot{B}^{\alpha-1}_{\infty,\infty}} \le C\epsilon \right\}.$$

The existence of a small regular forward self-similar solution initiated from a divergence vector field a satisfying Eq. (23) has been obtained respectively by Giga and Miyakawa [20] when $\nabla \times a \in \tilde{M}_{3/2}(R^3)^3$, by Cannone and Planchon [9] when $a \in \dot{B}^0_{3,\infty}(R^3)^3$, and by Barraza [4] when $a \in L_{3,\infty}(R^3)^3$. With the use of the observations in Section 3, Corollaries 8.1 and 8.2 strengthen those existence results.

Finally, let us mention that the backward self-similar solutions of Eq. (1) in the form

$$u(t,x) = \sqrt{\frac{s}{T-t}}\, v\left(\sqrt{\frac{s}{T-t}}\, x\right), \quad 0 < t < T, \ s > 0$$

with v subject to the steady-state equations, for some pressure π,

$$\begin{aligned} &-\Delta v + \frac{1}{2s}(v + (x \cdot \nabla) v) + \nabla \cdot (v \otimes v) + \nabla \pi = 0, \\ &\nabla \cdot v = 0. \end{aligned}$$

The problem on the existence of backward self-similar solutions arises from seeking a Navier–Stokes flow with a smooth initial velocity developing singularity in a finite time.

We remark that Tian and Xin [44] obtained that

$$u(t,x) = \sqrt{\frac{s}{T-t}}\, u_0\left(\sqrt{\frac{s}{T-t}}\, x\right), \tag{24}$$

with

$$u_0(x) = \frac{1}{|x|}\left(\frac{2|x|^2 - |x|x_1 + 2x_1^2}{(2|x|-x_1)^2}, \frac{x_1(2x_1-|x|)}{(2|x|-x_1)^2}, \frac{x_3(2x_1-|x|)}{(2|x|-x_1)^2}\right),$$

is an exact solution of Eq. (1). This solution is independent of t and is singular at the point $(0, t)$ for all t. Moreover $u \in C([0,\infty); L_{3,\infty}(R^3)^3)$, a homogeneous space of degree -1.

9. A remark on the uniqueness of solutions

In this section we will provide a simple proof for a uniqueness result from Furioli, Lemarié-Rieusset and Terraneo [16] and Lions and Masmoudi [29].

Theorem 9.1 ([16, 29])**.** *For $n \geq 3$, assume that u, $v \in C([0,\infty); L_n(R^n)^n)$ solve Eq.* (6) *such that*

$$u(t) - v(t) = -\int_0^t e^{(t-s)\Delta} P\nabla\cdot(u(s)\otimes u(s))ds + \int_0^t e^{(t-s)\Delta} P\nabla\cdot(v(s)\otimes v(s))ds.$$

Then $u = v$.

Proof. As in [29], we use the decomposition $u = u_1 + u_2$ and $v = v_1 + v_2$ such that, for a constant $T > 0$,

$$\|u_1\|_{C([0,T];L_n(R^n)^n)} + \|v_1\|_{C([0,T];L_n(R^n)^n)} < \epsilon,$$

and

$$\|u_2\|_{L_\infty((0,T)\times R^n)} + \|v_2\|_{L_\infty((0,T)\times R^n)} < C_\epsilon.$$

Let $w = u - v$,

$$w_1(t) = -\int_0^t e^{(t-s)\Delta} P\nabla\cdot(w(s)\otimes u_1(s) + v_1(s)\otimes w(s))ds,$$

and

$$w_2(t) = -\int_0^t e^{(t-s)\Delta} P\nabla\cdot(w(s)\otimes u_2(s) + v_2(s)\otimes w(s))ds.$$

This gives, by elementary calculations,

$$\begin{aligned}
\|w_2(t)\|_{L_n} &= C\int_0^t (t-s)^{-1/2}\|w(s)\otimes u_2(s) + v_2(s)\otimes w(s)\|_{L_n} ds \\
&\leq C\int_0^t (t-s)^{-1/2}\|w(s)\|_{L_n}(\|u_2(s)\|_{L_\infty} + \|v_2(s)\|_{L_\infty})ds \\
&\leq C_\epsilon \int_0^t (t-s)^{-1/2}\|w(s)\|_{L_n} ds \\
&\leq C_\epsilon t^{1/4}\left(\int_0^t \|w(s)\|_{L_n}^4 ds\right)^{1/4},
\end{aligned}$$

and hence,

$$\int_0^T \|w_2(s)\|_{L_n}^4 ds \le C_\epsilon T \int_0^T \int_0^s \|w(\tau)\|_{L_n}^4 d\tau ds. \tag{25}$$

On the other hand, by Lemma 5.3,

$$\begin{aligned}\int_0^T \|w_1(s)\|_{L_n}^4 ds &\le C \int_0^T \|w(s) \otimes u_1(s) + v_1(s) \otimes w(s)\|_{L_{n/2}}^4 ds \\ &\le C \left(\|u_1\|_{C([0,T];L_n(R^n)^n)} + \|v_1\|_{C([0,T];L_n(R^n)^n)}\right) \int_0^T \|w(s)\|_{L_n}^4 ds \\ &\le C\epsilon \int_0^T \|w(s)\|_{L_n}^4 ds.\end{aligned}$$

This together with Eq. (25) implies,

$$\int_0^T \|w(s)\|_{L_n}^4 ds \le C_\epsilon T \int_0^T \int_0^s \|w(\tau)\|_{L_n}^4 d\tau ds,$$

provided that ϵ is sufficiently small. By Gronwall's inequality we have

$$\int_0^T \|w(s)\|_{L_n}^4 ds = 0$$

and so $u = v$. The proof is complete.

As is known, one may consider the uniqueness of solutions in a larger critical homogeneous space, for example, $C([0,\infty); L_{3,\infty}(R^3)^3)$, which, however, contains the singular solution u defined by Eq. (24). Thus this space is too large to consider regular solutions.

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Note added to the proof. When this paper was in print, the authors were notified by a referee that the Cauchy problem with initial data in the homogeneous Bessel potentials $\dot{H}_p^{-1+n/p}(R^n)^n$ is treated by Kato and Ponce [48], and the Cauchy problem with the initial data in the spaces strictly larger than the Morrey-type spaces of measures and generalizing the homogeneous Besov spaces is treated alternatively by Kozono and Yamazaki [49]. Moreover, for completeness, it is better to provide an example function, in $\hat{M}_p(R^n)$ but not in $\tilde{M}_p(R^n)$, in showing the former strictly larger than the latter.

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