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An Interior Regularity Criterion for an Axially Symmetric Suitable Weak Solution to the Navier–Stokes Equations

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Abstract. We show that if $\mathbf{v}$ is an axially symmetric suitable weak solution to the Navier–Stokes equations (in the sense of L. Caffarelli, R. Kohn & L. Nirenberg – see [2]) such that either v_ρ (the radial component of $\mathbf{v}$) or v_θ (the tangential component of $\mathbf{v}$) has a higher regularity than is the regularity following from the definition of a weak solution in a sub-domain D of the time-space cylinder Q_T then all components of $\mathbf{v}$ are regular in D.

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1. Introduction

We suppose that Ω is either $\mathbb{R}^3$ or an axially symmetric (about the z axis) bounded domain in $\mathbb{R}^3$ with its boundary $\partial\Omega$ of the class $C^{2+\mu}$ for some $\mu > 0$. Further, we suppose that T is a positive number. We denote $Q_T = \Omega \times \,]0,T[$.

We will deal with the Navier–Stokes initial-boundary value problem for a viscous incompressible fluid with the homogeneous Dirichlet-type boundary condition, which is defined by the Navier–Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \nabla p + \nu\Delta\mathbf{v} \quad \text{in } Q_T, \tag{1}$$

by the equation of continuity

$$\operatorname{div}\mathbf{v} = 0 \quad \text{in } Q_T, \tag{2}$$

by the boundary condition

$$\mathbf{v} = \mathbf{0} \quad \text{on } \partial\Omega \times \,]0,T[\tag{3}$$

and by the initial condition

$$\mathbf{v}|_{t=0} = \mathbf{v}_0 \tag{4}$$

where $\mathbf{v}$ and p denote the unknown velocity and pressure, $\mathbf{f}$ is an external body force and $\nu > 0$ is the viscosity coefficient. We will further suppose for simplicity that $\mathbf{f} = \mathbf{0}$.

There exist many results on existence, uniqueness and regularity of solutions to the problem (1)–(4). However, the most important question of existence of strong solutions (provided that the input data of the problem are sufficiently smooth) is still open. The existence of a strong solution of the problem (1)–(4) has up to now been proved only locally in time (see K. K. Kiselev & O. A. Ladyzhenskaya [10], V. A. Solonnikov [25]) or under the assumption that $\mathbf{v}_0$ and $\mathbf{f}$ are "small enough" (see O. A. Ladyzhenskaya [14], J. G. Heywood [8]).

The existence of a weak solution of the problem (1)–(4) was proved by J. Leray [17] and E. Hopf [9]). Its uniqueness has up to now been proved only in the class $L^{r,s}(Q_T)^3$ with $2/r + 3/s \le 1$, $r \in [2, +\infty]$, $s \in [3, +\infty]$ (see G. Prodi [22], H. Sohr & W. von Wahl [24], H. Kozono & H. Sohr [11], H. Kozono [12], G. P. Galdi [6]). Moreover, if the weak solution finds itself in $L^{r,s}(Q_T)^3$ with $2/r + 3/s \le 1$, $r \in [2, +\infty]$, $s \in \,]3, +\infty]$ and the input data are "smooth enough" then it is already a strong solution. (See Y. Giga [7], G. P. Galdi [6].) The question whether the weak solution which finds itself in $L^{\infty,3}(Q_T)^3$ is a strong solution is still open.

The situation is much simpler in the case of planar (i.e. two-dimensional) flows where the existence of strong solutions and their uniqueness is known. (See e.g. J. Leray [17] and O. A. Ladyzhenskaya [13].) There arises a natural question whether the same also holds for axially symmetric flows. A positive answer is known if $\Omega = \mathbb{R}^3$, the external force $\mathbf{f}$ as well as the initial velocity $\mathbf{v}_0$ are axially symmetric and the tangential components f_θ and $v_{0\theta}$ are equal to zero (see O. A. Ladyzhenskaya [15], M. R. Uchovskii & B. I. Yudovich [26] and S. Leonardi, J. Málek, J. Nečas & M. Pokorný [16]). The same result can also be proved for general axially symmetric flows in the case that the axis of symmetry is outside Ω with a positive distance from Ω (see the note in O. A. Ladyzhenskaya [13]).

The question whether the components of velocity are coupled in such a way that some information about a higher regularity of one of them already implies the higher regularity of all of them was studied in the papers of J. Neustupa & P. Penel [19] and J. Neustupa, A. Novotný & P. Penel [20]. In [19], the authors proved that if $\mathbf{v}$ is a so called suitable weak solution then the essential boundedness of the cartesian velocity component v_3 on a sub-domain D of Q_T implies the regularity of all components in D. This result was improved in [20] where the assumption about the essential boundedness of v_3 was replaced by a weaker assumption that v_3 belongs to $L^{r,s}(D)$ with $2/r + 3/s \le \frac{1}{2}$, $r \in [4, +\infty]$, $s \in \,]6, +\infty]$.

This paper deals with a similar problem as the mentioned papers [19] and [20], however we study an axially symmetric flow. We use the Navier–Stokes equations written in the cylindrical coordinates ρ, θ, z and we show that a higher regularity of one of the velocity components v_ρ, v_θ implies the regularity of all components. The main theorems, proved in this paper, say:

Theorem 1. *Let* $\mathbf{v}$ *be an axially symmetric suitable weak solution to the problem* (1)–(4) *with* $\mathbf{f} = \mathbf{0}$. *Suppose that there exists a sub-domain* D *of* Q_T *such that the radial component* v_ρ *of* $\mathbf{v}$ *has its negative part* v_ρ^- *in* $L^{r,s}(D)$ *for some* $r \in [2, +\infty]$, $s \in \,]3, +\infty]$, $2/r + 3/s \le 1$. *Then* $\mathbf{v}$ *has no singular points in* D.

(The negative part of v_ρ is defined: $v_\rho^- = \max\{-v_\rho;\, 0\}$. A function which is defined a.e. in Q_T belongs to $L^{r,s}(D)$ if its product with the characteristic function of set D belongs to $L^r(0,T;\, L^s(\Omega))$.)

Theorem 2. *Let* $\mathbf{v}$ *be an axially symmetric suitable weak solution to the problem* (1)–(4) *with* $\mathbf{f} = \mathbf{0}$. *Suppose that there exists a sub-domain* D *of* Q_T *such that the tangential component* v_θ *of* $\mathbf{v}$ *belongs to* $L^{r,s}(D)$ *where*

1. *either* $s \in [6, +\infty]$, $r \in [20/7, +\infty]$ *and* $2/r + 3/s \le 7/10$
2. *or* $s \in \,]24/5,\, 6[$, $r \in \,]10, +\infty]$ *and* $2/r + 3/s \le 1 - 9/(5s)$.

Then $\mathbf{v}$ *has no singular points in* D.

2. Auxiliary results

A point $(x,t) \in Q_T$ is called a *regular point* of the weak solution $\mathbf{v}$ if there exists a neighbourhood U of (x,t) in Q_T such that $\mathbf{v} \in L^\infty(U)^3$. Points of Q_T which are not regular are called *singular*. Let us denote by $S(\mathbf{v})$ the set of all singular points of $\mathbf{v}$. It is obvious that $S(\mathbf{v})$ is closed in Q_T.

The notion of a suitable weak solution was introduced by L. Caffarelli, R. Kohn & L. Nirenberg in [2]. A weak solution $\mathbf{v}$ of the problem (1)–(4) (with $\mathbf{f} = \mathbf{0}$) is called a *suitable weak solution* if an associated pressure p belongs to $L^{5/4}(Q_T)$ and if the pair $(\mathbf{v}; p)$ satisfies the so called *generalized energy inequality*

$$2\nu \int_0^T \int_\Omega |\nabla \mathbf{v}|^2\, \phi\, dx\, dt \;\le\; \int_0^T \int_\Omega \left[|\mathbf{v}|^2 \left(\frac{\partial \phi}{\partial t} + \nu \Delta \phi \right) + (|\mathbf{v}|^2 + 2p)\, \mathbf{v} \cdot \nabla \phi \right] dx\, dt$$

for every infinitely differentiable function ϕ on Q_T with a compact support in Q_T. L. Caffarelli, R. Kohn & L. Nirenberg [2] proved the existence of a suitable weak solution of the problem (1)–(4) (with $\mathbf{f} = \mathbf{0}$) under the assumption that the initial velocity $\mathbf{v}_0$ is in $L^2_\sigma(\Omega)^3$ and if Ω is bounded, then it is also in $W^{2/5,5/4}(\Omega)^3$ (the space of vector functions whose components have fractional derivatives up to the order $\frac{2}{5}$ in $L^{5/4}(\Omega)^3$). Moreover, it is also proved in [2] that if $\mathbf{v}$ is a suitable weak solution of the problem (1)–(4) then its singular set $S(\mathbf{v})$ has the 1-dimensional parabolic measure equal to zero. The parabolic measure dominates the Hausdorff measure and so this result implies that the 1-dimensional Hausdorff measure of $S(\mathbf{v})$ equals zero. A new proof of the same result was later published by F. Lin in [18] and the result was recently improved by L. H. Choe & J. Lewis, who have shown that the Hausdorff dimension of $S(\mathbf{v})$ is strictly less than one. (See [3].)

Suppose that $\mathbf{v}$ is an axially symmetric suitable weak solution of the problem (1)–(4) with $\mathbf{f} = \mathbf{0}$ and p is an associated pressure. It means that neither $\mathbf{v}$ nor p depend on the angular cylindrical coordinate θ. It also implies that domain Ω is a circular domain whose axis is the z-axis. Suppose that D is a sub-domain of Q_T where one of the components v_ρ or v_θ of $\mathbf{v}$ has a higher regularity – see Theorem 1 or Theorem 2. Particularly, D can be equal to the whole time-space cylinder Q_T. Due to the axial symmetry of $\mathbf{v}$, we can assume without loss of generality that D is also axially symmetric about the z-axis.

It follows e.g. from the works of J. Heywood [8] and C. Foias & R. Temam [4] that the weak solution $\mathbf{v}$ can be redefined on a set of measure zero in Q_T so that it can have singular points in D only in a set G of instants of time in the interval $]0, T[$ such that the $\frac{1}{2}$-dimensional Hausdorff measure of G is finite and the complement of G in $]0, T[$ is a set of at most countably many disjoint open intervals $]a_\gamma, b_\gamma[$; $\gamma \in \Gamma$. Functions $\mathbf{v}$ and p are (after a suitable redefinition on a set of measure zero in Q_T) infinitely differentiable and satisfy equations (1) and (2) in a classical sense on each of the time intervals $]a_\gamma, b_\gamma[$. We suppose that the suitable weak solution $\mathbf{v}$ we deal with has already been redefined on a set of measure zero so that it has the properties mentioned above. We will call the time instants b_γ such that $\mathbf{v}$ has a singularity in D at time b_γ *D-epochs of irregularity.*

We will further suppose that t_0 is a D-epoch of irregularity (i.e. $t_0 = b_\gamma$ for some $\gamma \in \Gamma$) and (x_0, t_0) is a singular point of $\mathbf{v}$ in D. We will show that this assumption is in contradiction with the assumptions of Theorem 1 (respectively Theorem 2) in Section 3 (respectively in Section 4).

Due to the axial symmetry of $\mathbf{v}$ about the z-axis, x_0 lays on the z-axis. Otherwise the singular set $S(\mathbf{v})$ would contain a circle consisting of all points (x, t) such that $t = t_0$ and x is any point on the circle which arises if x_0 revolves about the z-axis. However, it would be a contradiction with the results of L. Caffarelli, R. Kohn & L. Nirenberg [2], F. Lin [18] and L. H. Choe & J. Lewis [3] about the 1-dimensional Hausdorff measure of $S(\mathbf{v})$.

Lemma 1. *There exist positive numbers τ, R_1, R_2 such that $R_1 < R_2$ and*

1. *τ is so small that $a_\gamma < b_\gamma - \tau = t_0 - \tau$,*
2. $\overline{B_{R_2}(x_0)} \times [t_0 - \tau, t_0 + \tau] \subset D$,
3. $\left\{ \left(\overline{B_{R_2}(x_0)} - B_{R_1}(x_0) \right) \times [t_0 - \tau, t_0 + \tau] \right\} \cap S(\mathbf{v}) = \emptyset$,
4. *$\mathbf{v}$, $\partial \mathbf{v}/\partial t$ and p are, together with all their space derivatives, continuous on* $\left(\overline{B_{R_2}(x_0)} - B_{R_1}(x_0) \right) \times [t_0 - \tau, t_0 + \tau]$.

$B_{R_1}(x_0)$ and $B_{R_2}(x_0)$ denote open balls in $\mathbb{R}^3$ with the center x_0 and radii R_1, R_2. The proof of Lemma 1 can be found in [19]. It uses essentially the fact that the 1-dimensional Hausdorff measure of $S(\mathbf{v})$ is equal to zero. Since this result is not generally known for any weak solution of the problem (1)–(4) and is known to hold only for a suitable weak solution, we deal with a suitable weak solution in this paper.

Let us further denote for simplicity $B_1 = B_{R_1}(x_0)$ and $B_2 = B_{R_2}(x_0)$. Let us remind that x_0 lays on the z-axis. Put $R_3 = (2R_1 + R_2)/3$, $R_4 = (R_1 + 2R_2)/3$ and $B_3 = B_{R_3}(x_0)$, $B_4 = B_{R_4}(x_0)$. Suppose that η is an infinitely differentiable axially symmetric function on $\mathbb{R}^3$ such that its values are in the interval $[0,1]$, $\eta = 0$ on $\mathbb{R}^3 - B_4$ and $\eta = 1$ on B_3.

We will also denote by $\|\,.\,\|_k$ the norm in $L^k(B_2)$ (or in $L^k(B_2)^n$ for some $n \in \mathbb{N}$) and by $|\!|\!|\,.\,|\!|\!|_{l,k}$ the norm in $L^l(t_0-\tau, t_0;\, L^k(B_2))$ (or in $L^l(t_0-\tau, t_0;\, L^k(B_2)^n)$).

The restrictions of functions defined a.e. in Q_T to subsets of Q_T will be denoted by the same letters. Thus, for example, $\mathbf{v} \in L^\infty(t_0-\tau, t_0;\, W^{1,2}(B_2)^3)$ is the statement about the restriction of $\mathbf{v}$ to $B_2 \times \,]t_0-\tau, t_0[$.

Further, if $\mathbf{w}$ is a vector function then we will denote by w_1, w_2, w_3 its cartesian components and by w_ρ, w_θ and w_z its cylindrical components. The relations between these components are well known:

$$w_\rho = w_1 \cos\theta + w_2 \sin\theta \tag{5}$$

$$w_\theta = -w_1 \sin\theta + w_2 \cos\theta \tag{6}$$

$$w_z = w_3\,. \tag{7}$$

We will also denote by $D\mathbf{w}$ the tensor $\left(\dfrac{\partial w_i}{\partial x_j}\right)_{i,\,j=1,2,3}$.

Lemma 2. *To each $q \in]1, +\infty[$ there exists $c_1(q) > 0$ such that*

$$\|D\mathbf{w}\|_q \le c_1(q)\Big(\|\mathrm{curl}\,\mathbf{w}\|_q + \|\mathrm{div}\,\mathbf{w}\|_q\Big) \tag{8}$$

for every vector function $\mathbf{w} \in W_0^{1,q}(B_2)^3$.

Proof. We can extend $\mathbf{w}$ by zero to the vector function in $\mathbb{R}^3$ and then we can use the Fourier transform and the Marcinkiewicz multiplier theorem. □

Lemma 3. *To each $q \in \,]1, +\infty[$ there exists $c_2(q) > 0$ such that*

$$\|\nabla w_\rho\|_q + \left\|\frac{w_\rho}{\rho}\right\|_q \le c_2(q)\,\|(\mathrm{curl}\,\mathbf{w})_\theta\|_q \tag{9}$$

for every axially symmetric divergence-free vector function $\mathbf{w} \in W_0^{1,q}(B_2)^3$.

Proof. Since w_θ does not appear in estimate (9), we can assume without loss of generality that $w_\theta = 0$. Differentiating equation (6) with respect to ρ and z and using the formula

$$\frac{\partial}{\partial\rho} = \cos\theta\,\frac{\partial}{\partial x_1} + \sin\theta\,\frac{\partial}{\partial x_2}$$

which follows from the relation between the cylindrical coordinates ρ, θ, z and the cartesian coordinates x_1, x_2, x_3, we obtain:

$$\frac{\partial w_\rho}{\partial \rho} = \frac{\partial w_1}{\partial x_1}\cos^2\theta + \left(\frac{\partial w_1}{\partial x_2} + \frac{\partial w_2}{\partial x_1}\right)\cos\theta\,\sin\theta + \frac{\partial w_2}{\partial x_2}\sin^2\theta \tag{10}$$

$$\frac{\partial w_\rho}{\partial z} = \frac{\partial w_1}{\partial x_3}\cos\theta + \frac{\partial w_2}{\partial x_3}\sin\theta\,. \tag{11}$$

The equation of continuity $\operatorname{div}\mathbf{w} = 0$ in the cylindrical coordinates has the form

$$\frac{\partial w_\rho}{\partial \rho} + \frac{w_\rho}{\rho} + \frac{\partial w_z}{\partial z} = 0. \tag{12}$$

Thus, we also have:

$$\begin{aligned}\frac{w_\rho}{\rho} &= -\frac{\partial w_\rho}{\partial \rho} - \frac{\partial w_z}{\partial z} = -\frac{\partial w_\rho}{\partial \rho} + \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} \\ &= \frac{\partial w_1}{\partial x_1}\sin^2\theta - \left(\frac{\partial w_1}{\partial x_2} + \frac{\partial w_2}{\partial x_1}\right)\cos\theta\,\sin\theta + \frac{\partial w_2}{\partial x_2}\cos^2\theta\,. \end{aligned}\tag{13}$$

Estimate (9) now easily follows from (10), (11), (13), Lemma 2 and the fact that all components of $\operatorname{curl}\mathbf{w}$ except for the component $(\operatorname{curl}\mathbf{w})_\theta$ are equal to zero. □

We will now localize equations (1) and (2) to the ball B_2 in the space variables. We put $\mathbf{u} = \eta\mathbf{v} - \mathbf{V}$ where $\mathbf{V}$ is an axially symmetric function whose divergence is the same as the divergence of $\eta\mathbf{v}$. Then we will have $\operatorname{div}\mathbf{u} = 0$. The existence of an appropriate function $\mathbf{V}$ is given by the following lemma:

Lemma 4. *There exists a linear operator R from $L^2(B_2)$ into $W_0^{1,2}(B_2)^3$ with the properties:*

1. $\operatorname{div} Rf = f$ *for all* $f \in L^2(B_2)$ *such that* $\int_{B_2} f\,dx = 0$
2. *To each $m \in \mathbb{N}\cup\{0\}$ there exists $c_3 > 0$ such that* $\|\nabla^{m+1} Rf\|_2 \le c_3\,\|\nabla^m f\|_2$ *for all* $f \in W_0^{m,2}(B_2)$.
3. *If f is axially symmetric then Rf is also axially symmetric.*
4. *If f has a compact support in $B_2 - \overline{B_1}$ then Rf has also a compact support in $B_2 - \overline{B_1}$.*

Items 1 and 2 of Lemma 4 follow from G. P. Galdi [5], Theorem 3.2, Chap. III.3 and from W. Borchers & H. Sohr [1], Theorem 2.4. Item 3 is an easy consequence of the fact that domain B_2 is axially symmetric. Item 4 follows from the way how Rf can be constructed – see W. Borchers & H. Sohr [1], pp. 73–76, for details. (In fact, we use the construction from [1] on $B_2 - \overline{B_1}$ and we extend the obtained function by zero to $\overline{B_1}$.) We put $\mathbf{V}(.\,,t) = R(\mathbf{v}(.\,,t)\cdot\nabla\eta)$. Since

$$\int_{B_2}\mathbf{v}\cdot\nabla\eta\,dx = \int_{B_2}\operatorname{div}(\eta\mathbf{v})\,dx = \int_{\partial B_2}\eta\mathbf{v}\cdot\mathbf{n}\,dS = 0$$

(where $\mathbf{n}$ is an outer normal vector to ∂B_2), we have $\operatorname{div} \mathbf{V} = \mathbf{v} \cdot \nabla\eta$ in $B_2 \times [t_0 - \tau, t_0 + \tau]$. It follows also from item 2 of Lemma 4 and the smoothness of $\mathbf{v}$ on $\operatorname{supp}(\nabla\eta) \times [t_0 - \tau, t_0 + \tau]$ (see item 4 of Lemma 1) that $\mathbf{V}$ and $\partial\mathbf{V}/\partial t$ are, together with all their space derivatives, continuous on $\overline{B_2} \times [t_0 - \tau, t_0 + \tau]$. Moreover, since $\mathbf{v}(.\,, t) \cdot \nabla\eta$ has a compact support in $B_2 - \overline{B_1}$, $\mathbf{V}(.\,, t)$ has a compact support in $B_2 - \overline{B_1}$, too.

It can be verified that $\mathbf{u}$ satisfies in a classical sense the equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \,=\, \mathbf{h} - \nabla(\eta\, p) + \nu\, \Delta\mathbf{u} \tag{14}$$

$$\operatorname{div} \mathbf{u} \,=\, 0 \tag{15}$$

where

$$\begin{aligned}\mathbf{h} \,=\, &-\frac{\partial \mathbf{V}}{\partial t} - (\eta\mathbf{v} \cdot \nabla)\mathbf{V} - (\mathbf{V} \cdot \nabla)(\eta\mathbf{v}) + (\mathbf{V} \cdot \nabla)\mathbf{V} + (\eta\mathbf{v} \cdot \nabla\eta)\mathbf{v} \\ &- \eta\,(1-\eta)\,(\mathbf{v} \cdot \nabla)\mathbf{v} - 2\nu\,(\nabla\eta \cdot \nabla)\mathbf{v} - \nu\,\mathbf{v}\,\Delta\eta + \nu\,\Delta\mathbf{V} + p\,\nabla\eta\end{aligned}$$

on $B_2 \times \,]t_0 - \tau, t_0[$. It also satisfies the boundary condition

$$\mathbf{u} = \mathbf{0} \tag{16}$$

on $\partial B_2 \times \,]t_0 - \tau, t_0[$. Since $\eta\,\mathbf{v}(.\,, t)$ and $\mathbf{V}(.\,, t)$ have a compact support in B_2 for all $t \in \,]t_0 - \tau, t_0[$, $\mathbf{u}$ has all derivatives equal to zero on $\partial B_2 \times \,]t_0 - \tau, t_0[$. Function $\mathbf{h}$ and all its space derivatives are continuous on $\overline{B_2} \times [t_0 - \tau, t_0 + \tau]$.

The cylindrical components of $\mathbf{u}$ will be denoted by u_ρ, u_θ and u_z. It is obvious that $u_\rho^- \in L^{r,s}(B_2)$ for $r \in [2, +\infty]$, $s \in \,]3, +\infty]$, $2/r + 3/s \le 1$ in the situation assumed in Theorem 1 and u_θ satisfies the same assumptions as v_θ in the case of Theorem 2.

3. Proof of Theorem 1

Equation (14) can be written as a system of the following three equations in the cylindrical coordinates ρ, θ, z:

$$\frac{\partial u_\rho}{\partial t} + u_\rho \frac{\partial u_\rho}{\partial \rho} + u_z \frac{\partial u_\rho}{\partial z} - \frac{1}{\rho} u_\theta^2 + \frac{\partial(\eta p)}{\partial \rho} \,=\, h_\rho + \nu \left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\Big(\rho \frac{\partial u_\rho}{\partial \rho}\Big) + \frac{\partial^2 u_\rho}{\partial z^2} - \frac{u_\rho}{\rho^2}\right] \tag{17}$$

$$\frac{\partial u_\theta}{\partial t} + u_\rho \frac{\partial u_\theta}{\partial \rho} + u_z \frac{\partial u_\theta}{\partial z} + \frac{1}{\rho} u_\theta u_\rho \,=\, h_\theta + \nu \left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\Big(\rho \frac{\partial u_\theta}{\partial \rho}\Big) + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{\rho^2}\right] \tag{18}$$

$$\frac{\partial u_z}{\partial t} + u_\rho \frac{\partial u_z}{\partial \rho} + u_z \frac{\partial u_z}{\partial z} + \frac{\partial(\eta p)}{\partial z} \,=\, h_z + \nu \left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\Big(\rho \frac{\partial u_z}{\partial \rho}\Big) + \frac{\partial^2 u_z}{\partial z^2}\right]. \tag{19}$$

We remind that the equation of continuity in the cylindrical coordinates is

$$\frac{\partial u_\rho}{\partial \rho} + \frac{u_\rho}{\rho} + \frac{\partial u_z}{\partial z} = 0. \tag{20}$$

Put $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$ and $\mathbf{g} = \operatorname{curl} \mathbf{h}$. The cylindrical components of $\boldsymbol{\omega}$ (respectively $\mathbf{g}$) will be denoted ω_ρ, ω_θ and ω_z (respectively g_ρ, g_θ and g_z). Thus,

$$\omega_\rho = -\frac{\partial u_\theta}{\partial z}, \quad \omega_\theta = \frac{\partial u_\rho}{\partial z} - \frac{\partial u_z}{\partial \rho}, \quad \omega_z = \frac{1}{\rho}\,\frac{\partial(\rho u_\theta)}{\partial \rho}. \tag{21}$$

Applying operator curl to equation (14), we obtain a vector equation for $\boldsymbol{\omega}$. It is equivalent to the system

$$\begin{aligned} &\frac{\partial \omega_\rho}{\partial t} + u_\rho \frac{\partial \omega_\rho}{\partial \rho} + u_z \frac{\partial \omega_\rho}{\partial z} - \frac{\partial u_\rho}{\partial \rho}\omega_\rho - \frac{\partial u_\rho}{\partial z}\omega_z \\ &= g_\rho + \nu\Big[\frac{1}{\rho}\frac{\partial}{\partial \rho}\Big(\rho\frac{\partial \omega_\rho}{\partial \rho}\Big) + \frac{\partial^2 \omega_\rho}{\partial z^2} - \frac{\omega_\rho}{\rho^2}\Big] \end{aligned} \tag{22}$$

$$\begin{aligned} &\frac{\partial \omega_\theta}{\partial t} + u_\rho \frac{\partial \omega_\theta}{\partial \rho} + u_z \frac{\partial \omega_\theta}{\partial z} - \frac{u_\rho}{\rho}\omega_\theta - 2\frac{u_\theta}{\rho}\omega_\rho \\ &= g_\theta + \nu\Big[\frac{1}{\rho}\frac{\partial}{\partial \rho}\Big(\rho\frac{\partial \omega_\theta}{\partial \rho}\Big) + \frac{\partial^2 \omega_\theta}{\partial z^2} - \frac{\omega_\theta}{\rho^2}\Big] \end{aligned} \tag{23}$$

$$\begin{aligned} &\frac{\partial \omega_z}{\partial t} + u_\rho \frac{\partial \omega_z}{\partial \rho} + u_z \frac{\partial \omega_z}{\partial z} - \frac{\partial u_z}{\partial \rho}\omega_\rho - \frac{\partial u_z}{\partial z}\omega_z \\ &= g_z + \nu\Big[\frac{1}{\rho}\frac{\partial}{\partial \rho}\Big(\rho\frac{\partial \omega_z}{\partial \rho}\Big) + \frac{\partial^2 \omega_z}{\partial z^2}\Big] \end{aligned} \tag{24}$$

Step 1. Assume that q is an even natural number and $t \in \,]t_0 - \tau, t_0[$. Let us multiply equation (18) by u_θ^{q-1} and integrate over B_2. In order to keep a simple notation, we will shortly write

$$\int_{B_2} \ldots \quad \text{instead of} \quad \int_{B_2} \ldots \rho\, d\rho\, d\theta\, dz.$$

If we also apply the integration by parts with respect to ρ and z and use equation (20), we obtain:

$$\begin{aligned} &\frac{d}{dt}\frac{1}{q}\int_{B_2} u_\theta^q + \int_{B_2} u_\rho \frac{1}{q}\frac{\partial u_\theta^q}{\partial \rho} + \int_{B_2} u_z \frac{1}{q}\frac{\partial u_\theta^q}{\partial z} + \int_{B_2} \frac{u_\rho}{\rho}\, u_\theta^q \\ &+ \nu \int_{B_2} (q-1)\Big[\Big(\frac{\partial u_\theta}{\partial \rho}\Big)^2 + \Big(\frac{\partial u_\theta}{\partial z}\Big)^2\Big] u_\theta^{q-2} + \nu\int_{B_2}\frac{u_\theta^q}{\rho^2} = \int_{B_2} h_\theta\, u_\theta^{q-1}, \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\frac{1}{q}\,\|u_\theta\|_q^q + \int_{B_2}\frac{u_\rho}{\rho}\,u_\theta^q + \nu\int_{B_2}\frac{q-1}{(q/2)^2}\Big[\Big(\frac{\partial u_\theta^{q/2}}{\partial \rho}\Big)^2 + \Big(\frac{\partial u_\theta^{q/2}}{\partial z}\Big)^2\Big] + \nu\int_{B_2}\frac{u_\theta^q}{\rho^2} \\ \le \int_{B_2} u_\theta^q + \Big(\frac{q-1}{q}\Big)^{q-1}\frac{1}{q}\int_{B_2} h_\theta^q, \end{aligned}$$

$$\frac{d}{dt}\|u_\theta\|_q^q + \frac{4(q-1)}{q}\nu\int_{B_2}\left[\left(\frac{\partial u_\theta^{q/2}}{\partial\rho}\right)^2 + \left(\frac{\partial u_\theta^{q/2}}{\partial z}\right)^2\right] + \nu q\int_{B_2}\frac{u_\theta^q}{\rho^2}$$
$$\leq q\int_{B_2}\frac{u_\rho^-}{\rho}u_\theta^q + q\|u_\theta\|_q^q + \|h_\theta\|_q^q. \tag{25}$$

The first integral on the right-hand side can be estimated as follows:

$$\begin{aligned}\int_{B_2}\frac{u_\rho^-}{\rho}u_\theta^q &\leq \delta_1\int_{B_2}\frac{u_\theta^q}{\rho^2} + \frac{1}{4\delta_1}\int_{B_2}(u_\rho^-)^2u_\theta^q\\ &\leq \delta_1\int_{B_2}\frac{u_\theta^q}{\rho^2} + \frac{1}{4\delta_1}\left(\int_{B_2}(u_\rho^-)^s\right)^{2/s}\left(\int_{B_2}|u_\theta|^{\frac{sq}{s-2}}\right)^{\frac{s-2}{s}}\\ &\leq \delta_1\int_{B_2}\frac{u_\theta^q}{\rho^2} + \frac{1}{4\delta_1}\|u_\rho^-\|_s^2\|u_\theta\|_q^{q\frac{s-3}{s}}\|u_\theta\|_{3q}^{3q/s}\\ &\leq \delta_1\int_{B_2}\frac{u_\theta^q}{\rho^2} + \delta_2\|u_\theta\|_{3q}^q + c_4\|u_\rho^-\|_s^{\frac{2s}{s-3}}\|u_\theta\|_q^q\\ &\leq \delta_1\int_{B_2}\frac{u_\theta^q}{\rho^2} + c_5\delta_2\int_{B_2}\left[\left(\frac{\partial u_\theta^{q/2}}{\partial\rho}\right)^2 + \left(\frac{\partial u_\theta^{q/2}}{\partial z}\right)^2\right] + c_4\|u_\rho^-\|_s^{\frac{2s}{s-3}}\|u_\theta\|_q^q\end{aligned}$$

where

$$c_4 = \frac{s-3}{s}\left(\frac{3}{s\delta_2}\right)^{\frac{3}{s-3}}\left(\frac{1}{4\delta_1}\right)^{\frac{s}{s-3}}.$$

(We have used the Young inequality

$$ab \leq \delta_2\, a^\alpha + \frac{\alpha-1}{\alpha}\left(\frac{1}{\alpha\delta_2}\right)^{\frac{1}{\alpha-1}} b^{\frac{\alpha}{\alpha-1}}$$

which holds for all $a\geq 0$, $b\geq 0$, $\delta_2>0$ and $\alpha>1$.) Substituting these estimates into (25), we obtain the inequality

$$\frac{d}{dt}\|u_\theta\|_q^q + \left(\frac{4(q-1)}{q}\nu - c_5\delta_2 q\right)\int_{B_2}\left[\left(\frac{\partial u_\theta^{q/2}}{\partial\rho}\right)^2 + \left(\frac{\partial u_\theta^{q/2}}{\partial z}\right)^2\right] + (\nu-\delta_1)q\int_{B_2}\frac{u_\theta^q}{\rho^2}$$
$$\leq c_4 q\|u_\rho^-\|_s^{\frac{2s}{s-3}}\|u_\theta\|_q^q + q\|u_\theta\|_q^q + \|h_\theta\|_q^q.$$

Choosing $\delta_1=\nu$ and $\delta_2 = 4\nu(q-1)/c_5q^2$, we get:

$$c_4 = \frac{s-3}{s}\left(\frac{1}{4\nu}\right)^{\frac{s+3}{s-3}}\left(\frac{3c_5}{s}\right)^{\frac{3}{s-3}}\left(\frac{q^2}{q-1}\right)^{\frac{3}{s-3}}$$

and

$$\frac{d}{dt}\,\|u_\theta\|_q^q \leq q\,c_6(t,q)\,\|u_\theta\|_q^q + \|h_\theta\|_q^q \tag{26}$$

where

$$c_6(t,q) = c_4\,\|u_\rho^-(\,.\,,t)\|_s^{\frac{2s}{s-3}} + 1.$$

Integrating inequality (26) with respect to t from $t_0-\tau$ to t, we obtain:

$$\|u_\theta(.\,,t)\|_q^q \le e^{\int_{t_0-\tau}^{t} qc_6(\sigma,q)\,d\sigma}\|u_\theta(.\,,t_0-\tau)\|_q^q + \int_{t_0-\tau}^{t} e^{\int_\xi^t qc_6(\sigma,q)\,d\sigma}\|h_\theta(.\,,\xi)\|_q^q\,d\xi,$$

$$\|u_\theta(.\,,t)\|_q \le e^{\int_{t_0-\tau}^{t_0} c_6(\sigma,q)\,d\sigma}\Big[\|u_\theta(.\,,t_0-\tau)\|_q + \tau^{1/q}|\!|\!|h_\theta|\!|\!|_{\infty,\infty}\Big].$$

Due to our assumption about u_ρ, $c_6(.\,,q)$ is integrable on the time interval $]\,t_0-\tau,t_0\,[$. Hence

$$|\!|\!|u_\theta|\!|\!|_{\infty,q} \le c_7(q) < +\infty \quad \text{for each } q\in\mathbb{N}. \tag{27}$$

Thus, we can observe that u_θ satisfies assumptions of Theorem 2. However, we cannot simply finish the proof of Theorem 1 on this place and refer to Theorem 2. It would be a logical loop because some estimates we will still derive in this section and use in order to complete the proof of Theorem 1 will also be used in the proof of Theorem 2.

Step 2. Let $0<\epsilon<1$ and let us multiply equation (23) by $\omega_\theta/\rho^{2-\epsilon}$ and integrate over B_2. If we also apply integration by parts and use equation (12), we obtain:

$$\frac{d}{dt}\frac{1}{2}\int_{B_2}\frac{\omega_\theta^2}{\rho^{2-\epsilon}} + \frac{1}{2}\int_{B_2}\frac{u_\rho}{\rho^{2-\epsilon}}\frac{\partial\omega_\theta^2}{\partial\rho} + \frac{1}{2}\int_{B_2}\frac{u_z}{\rho^{2-\epsilon}}\frac{\partial\omega_\theta^2}{\partial z} - \int_{B_2}\frac{u_\rho}{\rho^{3-\epsilon}}\omega_\theta^2 - 2\int_{B_2}\frac{u_\theta}{\rho^{3-\epsilon}}\omega_\rho\omega_\theta$$
$$= \int_{B_2} g_\theta\frac{\omega_\theta}{\rho^{2-\epsilon}} + \nu\int_{B_2}\frac{\omega_\theta}{\rho^{2-\epsilon}}\frac{\partial^2\omega_\theta}{\partial\rho^2} + \nu\int_{B_2}\frac{\omega_\theta}{\rho^{2-\epsilon}}\frac{\partial^2\omega_\theta}{\partial z^2} + \nu\int_{B_2}\frac{\omega_\theta}{\rho^{3-\epsilon}}\frac{\partial\omega_\theta}{\partial\rho} - \nu\int_{B_2}\frac{\omega_\theta^2}{\rho^{4-\epsilon}}$$

$$\frac{d}{dt}\frac{1}{2}\int_{B_2}\frac{\omega_\theta^2}{\rho^{2-\epsilon}} + \nu\int_{B_2}\frac{1}{\rho^{2-\epsilon}}\bigg[\Big(\frac{\partial\omega_\theta}{\partial\rho}\Big)^2 + \Big(\frac{\partial\omega_\theta}{\partial z}\Big)^2\bigg]$$
$$= \epsilon\int_{B_2} u_\rho\frac{\omega_\theta^2}{\rho^{3-\epsilon}} + \int_{B_2}\frac{2u_\theta}{\rho^{3-\epsilon}}\omega_\rho\omega_\theta + \nu\bigg[\frac{(2-\epsilon)^2}{2}-1\bigg]\int_{B_2}\frac{\omega_\theta^2}{\rho^{4-\epsilon}} + \int_{B_2} g_\theta\frac{\omega_\theta}{\rho^{2-\epsilon}}. \tag{28}$$

The axial symmetry of the flow implies that ω_θ, u_ρ and u_ρ behave like $O(\rho)$ as $\rho\to 0+$ and so the presence of parameter ϵ assures the convergence of the above integrals. The terms on the right-hand side of (28) can be rewritten or estimated in this way:

$$\begin{aligned}\int_{B_2}\frac{2u_\theta}{\rho^{3-\epsilon}}\omega_\rho\,\omega_\theta &= -\int_{B_2}\frac{2u_\theta}{\rho^{3-\epsilon}}\,\frac{\partial u_\theta}{\partial z}\,\omega_\theta\\ &= \int_{B_2}\frac{u_\theta^2}{\rho^{3-\epsilon}}\,\frac{\partial\omega_\theta}{\partial z}\\ &\le \frac{\nu}{2}\int_{B_2}\frac{1}{\rho^{2-\epsilon}}\Big(\frac{\partial\omega_\theta}{\partial z}\Big)^2 + \frac{1}{2\nu}\int_{B_2}\frac{u_\theta^4}{\rho^{4-\epsilon}},\end{aligned}$$

$$\int_{B_2}\frac{1}{\rho^{2-\epsilon}}\Big(\frac{\partial\omega_\theta}{\partial\rho}\Big)^2 = \frac{(1-\epsilon)(2-\epsilon)}{2}\int_{B_2}\frac{\omega_\theta^2}{\rho^{4-\epsilon}} + \int_{B_2}\bigg[\frac{\partial}{\partial\rho}\Big(\frac{\omega_\theta}{\rho^{1-\epsilon}}\Big)\bigg]^2\frac{1}{\rho^\epsilon}$$

and

$$\int_{B_2} g_\theta \frac{\omega_\theta}{\rho^{2-\epsilon}} \le \left(\int_{B_2}\left|\frac{\omega_\theta}{\rho^{1-\epsilon}}\right|^6\right)^{1/6}\left(\int_{B_2}\left|\frac{g_\theta}{\rho}\right|^{6/5}\right)^{5/6} \le c_8\left[\int_{B_2}\left|\nabla\left(\frac{\omega_\theta}{\rho^{1-\epsilon}}\right)\right|^2\right]^{1/2}$$
$$\le c_9\left[\int_{B_2}\left|\nabla\left(\frac{\omega_\theta}{\rho^{1-\epsilon}}\right)\right|^2\frac{1}{\rho^\epsilon}\right]^{1/2} \le \frac{\nu}{4}\int_{B_2}\left|\nabla\left(\frac{\omega_\theta}{\rho^{1-\epsilon}}\right)\right|^2\frac{1}{\rho^\epsilon} + c_{10}.$$

Substituting these equalities and estimates to (28), we can obtain:

$$\frac{d}{dt}\frac{1}{2}\int_{B_2}\frac{\omega_\theta^2}{\rho^{2-\epsilon}} + \frac{\nu}{4}\int_{B_2}\left\{\left[\frac{\partial}{\partial\rho}\left(\frac{\omega_\theta}{\rho^{1-\epsilon}}\right)\right]^2 + \left[\frac{\partial}{\partial z}\left(\frac{\omega_\theta}{\rho^{1-\epsilon}}\right)\right]^2\right\}\frac{1}{\rho^\epsilon}$$
$$\le \epsilon\int_{B_2}|u_\rho|\frac{\omega_\theta^2}{\rho^{3-\epsilon}} + r + \frac{1}{2\nu}\int_{B_2}\frac{u_\theta^4}{\rho^{4-\epsilon}} + c_{10}$$
$$\le \epsilon\int_{B_2}|u_\rho|\frac{\omega_\theta^2}{\rho^3}R_2^\epsilon + \frac{1}{2\nu}\int_{B_2}\frac{u_\theta^4}{\rho^4}R_2^\epsilon + c_{10}.$$

(Remind that R_2 is the radius of ball B_2.) Passing now to zero with ϵ, we get

$$\frac{d}{dt}\frac{1}{2}\int_{B_2}\frac{\omega_\theta^2}{\rho^2} + \frac{\nu}{4}\int_{B_2}\left\{\left[\frac{\partial}{\partial\rho}\left(\frac{\omega_\theta}{\rho}\right)\right]^2 + \left[\frac{\partial}{\partial z}\left(\frac{\omega_\theta}{\rho}\right)\right]^2\right\} \le \frac{1}{2\nu}\int_{B_2}\frac{u_\theta^4}{\rho^4} + c_{10}. \quad (29)$$

Step 3. In order to estimate the integral on the right-hand side of (29), we multiply equation (18) by u_θ^3/ρ^2 and integrate over B_2. We obtain:

$$\frac{d}{dt}\frac{1}{4}\int_{B_2}\frac{u_\theta^4}{\rho^2}dx + \int_{B_2}u_\rho\frac{u_\theta^3}{\rho^2}\frac{\partial u_\theta}{\partial\rho} + \int_{B_2}u_z\frac{u_\theta^3}{\rho^2}\frac{\partial u_\theta}{\partial z} + \int_{B_2}\frac{u_\theta^4}{\rho^3}u_\rho$$
$$= \int_{B_2}h_\theta\frac{u_\theta^3}{\rho^2} + \nu\int_{B_2}\frac{u_\theta^3}{\rho^3}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u_\theta}{\partial\rho}\right) + \nu\int_{B_2}\frac{u_\theta^3}{\rho^2}\frac{\partial^2 u_\theta}{\partial z^2}dx - \nu\int_{B_2}\frac{u_\theta^4}{\rho^4},$$

$$\frac{d}{dt}\frac{1}{4}\int_{B_2}\frac{u_\theta^4}{\rho^2}dx - \int_{B_2}\frac{\partial u_\rho}{\partial\rho}\frac{u_\theta^4}{4\rho^2} + \int_{B_2}\frac{u_\rho u_\theta^4}{4\rho^3} - \int_{B_2}\frac{\partial u_z}{\partial z}\frac{u_\theta^4}{4\rho^2} + \int_{B_2}\frac{u_\rho u_\theta^4}{\rho^3}$$
$$= \int_{B_2}h_\theta\frac{u_\theta^3}{\rho^2} - 3\nu\int_{B_2}\frac{u_\theta^2}{\rho^2}\left(\frac{\partial u_\theta}{\partial\rho}\right)^2 + \nu\int_{B_2}\frac{u_\theta^4}{\rho^4} - 3\nu\int_{B_2}\frac{u_\theta^2}{\rho^2}\left(\frac{\partial u_\theta}{\partial z}\right)^2 - \nu\int_{B_2}\frac{u_\theta^4}{\rho^4},$$

$$\frac{d}{dt}\frac{1}{4}\int_{B_2}\frac{u_\theta^4}{\rho^2} + \frac{3}{2}\int_{B_2}\frac{u_\rho u_\theta^4}{\rho^3} + \frac{3\nu}{4}\int_{B_2}\frac{1}{\rho^2}\left[\left(\frac{\partial u_\theta^2}{\partial\rho}\right)^2 + \left(\frac{\partial u_\theta^2}{\partial z}\right)^2\right] = \int_{B_2}h_\theta\frac{u_\theta^3}{\rho^2}.$$

Since

$$\frac{3\nu}{4}\int_{B_2}\frac{1}{\rho^2}\left(\frac{\partial u_\theta^2}{\partial\rho}\right)^2 = \frac{3\nu}{4}\int_{B_2}\left[\frac{\partial}{\partial\rho}\left(\frac{u_\theta^2}{\rho}\right)\right]^2 + \frac{3\nu}{4}\int_{B_2}\frac{u_\theta^4}{\rho^4},$$

we obtain

$$\frac{d}{dt}\frac{1}{4}\int_{B_2}\frac{u_\theta^4}{\rho^2}+\frac{3\nu}{4}\int_{B_2}\left\{\left[\frac{\partial}{\partial\rho}\left(\frac{u_\theta^2}{\rho}\right)\right]^2+\left[\frac{\partial}{\partial z}\left(\frac{u_\theta^2}{\rho}\right)\right]^2\right\}+\frac{3\nu}{4}\int_{B_2}\frac{u_\theta^4}{\rho^4}$$
$$=-\frac{3}{2}\int_{B_2}\frac{u_\rho u_\theta^4}{\rho^3}+\int_{B_2}h_\theta\frac{u_\theta^3}{\rho^2}.$$

Using the estimate

$$\int_{B_2}h_\theta\frac{u_\theta^3}{\rho^2}\leq\frac{\nu}{4}\int_{B_2}\left|\frac{u_\theta}{\rho}\right|^4+c_{11}|||h_\theta|||_{\infty,\infty}^4,$$

we get

$$\frac{d}{dt}\frac{1}{4}\int_{B_2}\frac{u_\theta^4}{\rho^2}+\frac{3\nu}{4}\int_{B_2}\left\{\left[\frac{\partial}{\partial\rho}\left(\frac{u_\theta^2}{\rho}\right)\right]^2+\left[\frac{\partial}{\partial z}\left(\frac{u_\theta^2}{\rho}\right)\right]^2\right\}+\frac{\nu}{2}\int_{B_2}\frac{u_\theta^4}{\rho^4}$$
$$=-\frac{3}{2}\int_{B_2}\frac{u_\rho u_\theta^4}{\rho^3}+c_{11}|||h_\theta|||_{\infty,\infty}^4\leq\frac{3}{2}\int_{B_2}\frac{|u_\rho|u_\theta^4}{\rho^3}+c_{12}. \tag{30}$$

Step 4. Multiplying (30) by $2/\nu^2$ and adding it to (29), we obtain:

$$\frac{d}{dt}\frac{1}{2}\int_{B_2}\frac{\omega_\theta^2}{\rho^2}+\frac{d}{dt}\frac{1}{2\nu^2}\int_{B_2}\frac{u_\theta^4}{\rho^2}+\frac{\nu}{4}\int_{B_2}\left\{\left[\frac{\partial}{\partial\rho}\left(\frac{\omega_\theta}{\rho}\right)\right]^2+\left[\frac{\partial}{\partial z}\left(\frac{\omega_\theta}{\rho}\right)\right]^2\right\}$$
$$+\frac{3}{2\nu}\int_{B_2}\left\{\left[\frac{\partial}{\partial\rho}\left(\frac{u_\theta^2}{\rho}\right)\right]^2+\left[\frac{\partial}{\partial z}\left(\frac{u_\theta^2}{\rho}\right)\right]^2\right\}+\frac{1}{2\nu}\int_{B_2}\frac{u_\theta^4}{\rho^4}\leq\frac{3}{\nu^2}\int_{B_2}\frac{|u_\rho|u_\theta^4}{\rho^3}+c_{13}. \tag{31}$$

The first term on the right-hand side of (31) can be estimated as follows:

$$\frac{3}{\nu^2}\int_{B_2}\frac{|u_\rho|u_\theta^4}{\rho^3}\leq\frac{3}{\nu^2}\left(\int_{B_2}\frac{u_\rho^2}{\rho^2}\right)^{1/4}\left(\int_{B_2}u_\rho^6\right)^{1/12}\left(\int_{B_2}\frac{u_\theta^4}{\rho^4}\right)^{5/8}\left(\int_{B_2}u_\theta^{36}\right)^{1/24}$$
$$\leq\frac{1}{2\nu}\int_{B_2}\frac{u_\theta^4}{\rho^4}+c_{14}\left(\int_{B_2}\frac{u_\rho^2}{\rho^2}\right)^{2/3}\left(\int_{B_2}u_\rho^6\right)^{2/9}\left(\int_{B_2}u_\theta^{36}\right)^{1/9}.$$

Using Lemma 3 and estimate (27), we obtain:

$$\frac{3}{\nu^2}\int_{B_2}\frac{|u_\rho|u_\theta^4}{\rho^3}\leq\frac{1}{2\nu}\int_{B_2}\frac{u_\theta^4}{\rho^4}+c_{15}\left(\int_{B_2}\omega_\theta^2\right)^{2/3}\left(\int_{B_2}u_\rho^6\right)^{2/9}$$
$$\leq\frac{1}{2\nu}\int_{B_2}\frac{u_\theta^4}{\rho^4}+c_{16}\left(\int_{B_2}\frac{\omega_\theta^2}{\rho^2}\right)\left(\int_{B_2}u_\rho^6\right)^{1/3}+c_{17}.$$

Substituting this estimate into (31), we get:

$$\frac{d}{dt}\frac{1}{2}\int_{B_2}\frac{\omega_\theta^2}{\rho^2}+\frac{d}{dt}\frac{1}{2\nu^2}\int_{B_2}\frac{u_\theta^4}{\rho^2}+\frac{\nu}{4}\int_{B_2}\left\{\left[\frac{\partial}{\partial\rho}\left(\frac{\omega_\theta}{\rho}\right)\right]^2+\left[\frac{\partial}{\partial z}\left(\frac{\omega_\theta}{\rho}\right)\right]^2\right\}$$
$$+\frac{3}{2\nu}\int_{B_2}\left\{\left[\frac{\partial}{\partial\rho}\left(\frac{u_\theta^2}{\rho}\right)\right]^2+\left[\frac{\partial}{\partial z}\left(\frac{u_\theta^2}{\rho}\right)\right]^2\right\}\leq c_{18}\left(\int_{B_2}\frac{\omega_\theta^2}{\rho^2}\right)\left(\int_{B_2}u_\rho^6\right)^{1/3}+c_{19}. \tag{32}$$

$\|u_\rho\|_6^2$ is an integrable function of t on the interval $]t_0-\tau, t_0[$ and so estimate (32) implies, except others, the boundedness of $\|\omega_\theta/\rho\|_2$ (and consequently also the boundedness of $\|\omega_\theta\|_2$) on the time interval $]t_0-\tau, t_0[$.

Step 5. Finally, let us multiply equation (22) by ω_ρ, equation (24) by ω_z, sum these products and integrate over B_2. We obtain:

$$\frac{d}{dt}\frac{1}{2}\Big(\|\omega_\rho\|_2^2+\|\omega_z\|_2^2\Big)+\nu\Big(\|\nabla\omega_\rho\|_2^2+\|\nabla\omega_z\|_2^2\Big)+\nu\int_{B_2}\frac{\omega_\rho^2}{\rho^2}$$
$$=\int_{B_2}(g_\rho\omega_\rho+g_z\omega_z)+\int_{B_2}\frac{\partial u_\rho}{\partial\rho}\omega_\rho^2+\int_{B_2}\left(\frac{\partial u_\rho}{\partial z}+\frac{\partial u_z}{\partial\rho}\right)\omega_\rho\omega_z+\int_{B_2}\frac{\partial u_z}{\partial z}\omega_z^2. \quad (33)$$

We can now estimate the terms on the right-hand side:

$$\int_{B_2}(g_\rho\omega_\rho+g_z\omega_z)\le\frac{1}{2}\Big(\|\omega_\rho\|_2^2+\|\omega_z\|_2^2+\|\mathbf{g}\|_2^2\Big),$$

$$\int_{B_2}\frac{\partial u_\rho}{\partial\rho}\omega_\rho^2\le\left\|\frac{\partial u_\rho}{\partial\rho}\right\|_2\|\omega_\rho\|_4^2\le c_2(2)\|\omega_\theta\|_2\|\omega_\rho\|_4^2\le\delta\|\omega_\rho\|_6^2+c_{20}(\delta)\|\omega_\rho\|_2^2$$
$$\le c_{21}\delta\|\nabla\omega\|_2^2+c_{20}(\delta)\|\omega_\rho\|_2^2,$$

$$\begin{aligned}\int_{B_2}\left(\frac{\partial u_\rho}{\partial z}+\frac{\partial u_z}{\partial\rho}\right)\omega_\rho\omega_z&\le\left\|\frac{\partial u_\rho}{\partial z}+\frac{\partial u_z}{\partial\rho}\right\|_2\|\omega_\rho\|_4\|\omega_z\|_4\\&=\left\|-\omega_\theta+2\frac{\partial u_\rho}{\partial z}\right\|_2\|\omega_\rho\|_4\|\omega_z\|_4\\&\le c_{22}\|\omega_\theta\|_2\Big(\|\omega_\rho\|_4^2+\|\omega_z\|_4^2\Big)\\&\le\delta\Big(\|\omega_\rho\|_6^2+\|\omega_z\|_6^2\Big)+c_{23}(\delta)\left\|\frac{\omega_\theta}{\rho}\right\|_2^4\Big(\|\omega_\rho\|_2^2+\|\omega_z\|_2^2\Big)\\&\le c_{21}\delta\Big(\|\nabla\omega_\rho\|_2^2+\|\nabla\omega_z\|_2^2\Big)+c_{24}(\delta)\Big(\|\omega_\rho\|_2^2+\|\omega_z\|_2^2\Big).\end{aligned}$$

(We have used Lemma 3 in order to estimate $\|\partial u_\rho/\partial\rho\|_2$ and $\|\partial u_\rho/\partial z\|_2$.)

$$\int_{B_2}\frac{\partial u_z}{\partial z}\omega_z^2=-\int_{B_2}\left(\frac{\partial u_\rho}{\partial\rho}+\frac{u_\rho}{\rho}\right)\omega_z^2\le\left(\left\|\frac{\partial u_\rho}{\partial\rho}\right\|_2+\left\|\frac{u_\rho}{\rho}\right\|_2\right)\|\omega_z\|_4^2$$
$$\le c_2(2)\|\omega_\theta\|_2\|\omega_z\|_4^2\le c_2(2)\left\|\frac{\omega_\theta}{\rho}\right\|_2\|\omega_z\|_2^{1/2}\|\omega_z\|_6^{3/2}$$
$$\le\delta\|\omega_z\|_6^2+c_{25}(\delta)\|\omega_z\|_2^2\le c_{21}\delta\|\nabla\omega_z\|_2^2+c_{25}(\delta)\|\omega_z\|_2^2.$$

(We have also used Lemma 3 in order to estimate the L^2-norms of $\partial u_\rho/\partial\rho$ and u_ρ/ρ.) Choosing now δ so small that $3c_{21}\delta\le\nu/2$ and substituting all the above estimates to (33), we obtain:

$$\frac{d}{dt}\frac{1}{2}\Big(\|\omega_\rho\|_2^2+\|\omega_z\|_2^2\Big)+\frac{\nu}{2}\Big(\|\nabla\omega_\rho\|_2^2+\|\nabla\omega_z\|_2^2\Big)+\nu\int_{B_2}\frac{\omega_\rho^2}{\rho^2}$$
$$\leq\Big[\|\mathbf{g}\|_2^2+c_{20}(\delta)+c_{24}(\delta)+c_{25}(\delta)\Big]\Big(\|\omega_\rho\|_2^2+\|\omega_z\|_2^2\Big).$$

Integrating this inequality on the time interval $]t_0-\tau,t_0[$, we obtain the boundedness of $\|\omega_\rho\|_2$ and $\|\omega_z\|_2$ on this interval. This fact, together with the boundedness of $\|\omega_\theta\|_2$ and Lemma 2 implies the boundedness of $\|D\mathbf{u}\|_2$ on $]t_0-\tau,t_0[$. Using the technique explained e.g. by V. A. Solonnikov in [25], $\mathbf{u}$ can be extended to a strong solution of the problem (14)–(16) on the interval $]t_0-\tau,t_0+\tau'[$ for some $\tau'>0$. Due to the smoothness of function $\mathbf{h}$, this solution is a classical solution on $]t_0-\tau,t_0+\tau'[$. Hence (x_0,t_0) cannot be a singular point of solution $\mathbf{u}$ of the problem (14)–(16) or solution $\mathbf{v}$ of the problem (1)–(4). □

4. Proof of Theorem 2

Step 1. We can derive estimate (31) in the same way as in the previous section. However, the integral on the right-hand side of (31) must be estimated in a different way. We will estimate this integral later, in Step 3.

Step 2. We multiply equation (23) by $\omega_\theta^{7/5}$ and integrate over B_2. We get:

$$\frac{d}{dt}\frac{5}{12}\int_{B_2}\omega_\theta^{12/5}+\frac{35\nu}{36}\int_{B_2}\Big|\nabla\omega_\theta^{6/5}\Big|^2+\nu\int_{B_2}\frac{\omega_\theta^{12/5}}{\rho^2}$$
$$=\int_{B_2}\frac{u_\rho}{\rho}\omega_\theta^{12/5}+2\int_{B_2}\frac{u_\theta}{\rho}\omega_\theta^{7/5}\omega_\rho+\int_{B_2}g_\theta\omega_\theta^{7/5}. \tag{34}$$

The terms on the right-hand side can be estimated in this way:

$$\int_{B_2}\frac{u_\rho}{\rho}\omega_\theta^{12/5}\leq\|u_\rho\|_\infty\left\|\frac{\omega_\theta}{\rho}\right\|_2\|\omega_\theta\|_{14/5}^{7/5}\leq\|u_\rho\|_\infty\left\|\frac{\omega_\theta}{\rho}\right\|_2\|\omega_\theta\|_2^{11/13}\|\omega_\theta\|_{36/5}^{36/65}. \tag{35}$$

The norm $\|u_\rho\|_\infty$ can be estimated by means of interpolation inequalities (see L. Nirenberg [21]):

$$\|u_\rho\|_\infty\leq c_{26}\|u_\rho\|_{12}^{7/10}\ \|\nabla u_\rho\|_{36/5}^{3/10},$$
$$\|u_\rho\|_\infty\leq c_{27}\|u_\rho\|_6^{7/13}\ \|\nabla u_\rho\|_{36/5}^{6/13}.$$

The norms $\|u_\rho\|_{12}$ and $\|u_\rho\|_6$ can be estimated by $\|\nabla u_\rho\|_{12/5}$ and $\|\nabla u_\rho\|_2$. Thus, if we also use Lemma 3, we obtain:

$$\|u_\rho\|_\infty\leq c_{28}\|\omega_\theta\|_{12/5}^{7/10}\ \|\omega_\theta\|_{36/5}^{3/10}, \tag{36}$$

$$\|u_\rho\|_\infty\leq c_{29}\|\omega_\theta\|_2^{7/13}\ \|\omega_\theta\|_{36/5}^{6/13}. \tag{37}$$

If we raise inequality (36) to the power $\frac{12}{35}$, inequality (37) to the power $\frac{23}{35}$, multiply the two inequalities and substitute the product into (35), we obtain:

$$\begin{aligned}
\int_{B_2} \frac{u_\rho}{\rho}\omega_\theta^{12/5} &\le c_{30}\left\|\frac{\omega_\theta}{\rho}\right\|_2 \|\omega_\theta\|_{12/5}^{\frac{7}{10}\cdot\frac{12}{35}} \|\omega_\theta\|_2^{\frac{11}{13}+\frac{7}{13}\cdot\frac{23}{35}} \|\omega_\theta\|_{36/5}^{\frac{36}{65}+\frac{3}{10}\cdot\frac{12}{35}+\frac{6}{13}\cdot\frac{23}{35}} \\
&= c_{30}\left\|\frac{\omega_\theta}{\rho}\right\|_2 \|\omega_\theta\|_{12/5}^{6/25}\|\omega_\theta\|_2^{6/5}\|\omega_\theta\|_{36/5}^{24/25} \\
&\le \delta\|\omega_\theta\|_{36/5}^{12/5} + c_{31}(\delta)\|\omega_\theta\|_2^2\|\omega_\theta\|_{12/5}^{2/5}\left\|\frac{\omega_\theta}{\rho}\right\|_2^{5/3} \\
&\le \delta c_{32}\|\nabla\omega_\theta^{6/5}\|_2^2 + c_{33}(\delta)\|\omega_\theta\|_2^2\left(\left\|\frac{\omega_\theta}{\rho}\right\|_2^2 + \|\omega_\theta\|_{12/5}^{12/5}\right). \qquad (38)
\end{aligned}$$

$$\begin{aligned}
&2\int_{B_2}\frac{u_\theta}{\rho}\omega_\theta^{7/5}\omega_\rho = -2\int_{B_2}\frac{u_\theta}{\rho}\omega_\theta^{7/5}\frac{\partial u_\theta}{\partial z} = \frac{7}{6}\int_{B_2}\frac{u_\theta^2}{\rho}\frac{\partial\omega_\theta^{6/5}}{\partial z}\omega_\theta^{1/5} \\
&\le \frac{\nu}{4}\int_{B_2}\left|\frac{\partial\omega_\theta^{6/5}}{\partial z}\right|^2 + \frac{49}{36\nu}\int_{B_2}\frac{u_\theta^4}{\rho^2}\omega_\theta^{2/5} \\
&\le \frac{\nu}{4}\int_{B_2}\left|\nabla\omega_\theta^{6/5}\right|^2 + \frac{49}{36\nu}\left(\int_{B_2}\frac{u_\theta^{12}}{\rho^6}\right)^{1/10}\left(\int_{B_2}\frac{u_\theta^4}{\rho^2}\right)^{7/10}\left(\int_{B_2}\omega_\theta^2\right)^{1/5} \\
&\le \frac{\nu}{4}\int_{B_2}\left|\nabla\omega_\theta^{6/5}\right|^2 + \delta\left(\int_{B_2}\frac{u_\theta^{12}}{\rho^6}\right)^{1/3} + c_{34}(\delta)\left(\int_{B_2}\frac{u_\theta^4}{\rho^2}\right)\left(\int_{B_2}\omega_\theta^2\right)^{2/7} \\
&\le \frac{\nu}{4}\int_{B_2}\left|\nabla\omega_\theta^{6/5}\right|^2 + c_{35}\delta\int_{B_2}\left|\nabla\left(\frac{u_\theta^2}{\rho}\right)\right|^2 + c_{34}(\delta)\left(\int_{B_2}\frac{u_\theta^4}{\rho^2}\right)\|\omega_\theta\|_2^{4/7}. \qquad (39)
\end{aligned}$$

$$\int_{B_2} g_\theta\omega_\theta^{7/5} \le \int_{B_2}\omega_\theta^{12/5} + c_{36}. \qquad (40)$$

Substituting estimates (38)–(40) to (34) and adding inequality (31), we obtain:

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\left\|\frac{\omega_\theta}{\rho}\right\|_2^2 + \frac{1}{2\nu^2}\left\|\frac{u_\theta^2}{\rho}\right\|_2^2 + \frac{5}{12}\|\omega_\theta\|_{12/5}^{12/5}\right) + \frac{\nu}{4}\left\|\nabla\left(\frac{\omega_\theta}{\rho}\right)\right\|_2^2 \\
&\quad+\left(\frac{3}{2\nu} - \delta c_{35}\right)\left\|\nabla\left(\frac{u_\theta^2}{\rho}\right)\right\|_2^2 + \frac{1}{2\nu}\left\|\frac{u_\theta}{\rho}\right\|_4^4 \\
&\quad+\left(\frac{26\nu}{36} - \delta c_{32}\right)\left\|\nabla\omega_\theta^{6/5}\right\|_2^2 + \nu\int_{B_2}\frac{|\omega_\theta|^{12/5}}{\rho^2} \le \frac{3}{\nu^2}\int_{B_2}\frac{|u_\rho|u_\theta^4}{\rho^3} \\
&\quad+c_{33}(\delta)\|\omega_\theta\|_2^2\left(\left\|\frac{\omega_\theta}{\rho}\right\|_2^2 + \|\omega_\theta\|_{12/5}^{12/5}\right) + c_{34}(\delta)\left\|\frac{u_\theta^2}{\rho}\right\|_2^2\|\omega_\theta\|_2^{4/7} + \|\omega_\theta\|_{12/5}^{12/5} + c_{37}. \qquad (41)
\end{aligned}$$

Step 3. We will now estimate the first term on the right-hand side of (41). We will distinguish between the two cases: $s \ge 6$ and $s < 6$.

Step 3.1. $s \geq 6$

$$\frac{3}{\nu^2}\int_{B_2}\frac{|u_\rho|u_\theta^4}{\rho^3} \leq \frac{3}{\nu^2}\left(\int_{B_2}\left|\frac{u_\rho}{\rho}\right|^q\right)^{1/q}\left(\int_{B_2}\left|\frac{u_\theta}{\rho}\right|^4\right)^{\alpha/4}\left(\int_{B_2}\left|\frac{u_\theta^2}{\rho}\right|^2\right)^{\beta/4}\left(\int_{B_2}|u_\theta|^s\right)^{\gamma/s} \tag{42}$$

where $\alpha+\beta+\gamma = 4$, $\alpha + \frac{\beta}{2} = 2$, $\frac{1}{q}+\frac{\alpha}{4}+\frac{\beta}{4}+\frac{\gamma}{s} = 1$ and $\frac{12}{5} \leq q \leq \frac{36}{5}$.

Interpolating the L^q-norm between the $L^{12/5}$- and the $L^{36/5}$-norms, we obtain:

$$\left\|\frac{u_\rho}{\rho}\right\|_q \leq \left\|\frac{u_\rho}{\rho}\right\|_{12/5}^{\frac{36-5q}{10q}}\left\|\frac{u_\rho}{\rho}\right\|_{36/5}^{\frac{15q-36}{10q}}.$$

Using this estimate, applying the Young inequality, estimating the $L^{12/5}$-norm of u_ρ/ρ by means of Lemma 3 and using finally the continuous imbedding of $W_0^{1,2}(B_2)$ into $L^6(B_2)$, we can obtain that the right-hand side of (42) is less than or equal to

$$\delta\left\|\frac{u_\rho}{\rho}\right\|_{36/5}^{12/5} + \delta\left\|\frac{u_\theta}{\rho}\right\|_4^4 + c_{38}(\delta)\left\|\frac{u_\rho}{\rho}\right\|_{12/5}^{\frac{4(36-5q)}{5(12+3q-2\alpha q)}}\left\|\frac{u_\theta^2}{\rho}\right\|_2^{\frac{4\beta q}{12+3q-2\alpha q}}\|u_\theta\|_s^{\frac{8\gamma q}{12+3q-2\alpha q}}$$

$$\leq \delta c_{39}\left\|\nabla\omega_\theta^{6/5}\right\|_2^2 + \delta\left\|\frac{u_\theta}{\rho}\right\|_4^4 + c_{40}(\delta)\|u_\theta\|_s^{\frac{8\gamma q}{12+3q-2\alpha q}}\left(\|\omega_\theta\|_{12/5}^{12/5} + \left\|\frac{u_\theta^2}{\rho}\right\|_2^{\frac{6\beta}{7-3\alpha}}\right). \tag{43}$$

The left-hand side of (41) contains the norm $\|u_\theta^2/\rho\|_2$ in power 2 and so we need the power on the right-hand side to be also 2. So we get the condition: $6\beta/(7-3\alpha) = 2$. This condition, together with already mentioned equations for α, β, γ and q, gives: $\alpha = \frac{5}{3}$, $\beta = \frac{2}{3}$, $\gamma = \frac{5}{3}$ and $q = 12s/(5s-20)$. Moreover,

$$\frac{8\gamma q}{12+3q-2\alpha q} = \frac{20s}{7s-30} \quad \text{and} \quad \frac{12}{5} \leq \frac{12s}{5s-20} \leq \frac{36}{5} \iff s \geq 6.$$

It follows from the assumptions of Theorem 2 (item 1) that $\|u_\theta\|_s^{20s/(7s-30)}$ is integrable, as a function of time, on the interval $]t_0-\tau, t_0[$.

We can now substitute the computed values of α, β, γ and q to (43), (42) and to (41). Choosing δ sufficiently small and integrating then (41) with respect to time, we obtain the boundedness of the norms $\|\omega_\theta/\rho\|_2$, $\|u_\theta^2/\rho\|_2$ and $\|\omega_\theta\|_{12/5}$ on the time interval $]t_0-\tau, t_0[$.

Step 3.2. $s < 6$ If we multiply equation (18) by $u_\theta^{q-1}\rho^q$ (where q is an even natural number) and integrate over B_2, we obtain:

$$\frac{d}{dt}\frac{1}{q}\int_{B_2}u_\theta^q\rho^q + \nu(q-1)\int_{B_2}|\nabla(u_\theta\rho)|^2(u_\theta\rho)^{q-2}$$
$$= \int_{B_2}h_\theta u_\theta^{q-1}\rho^q \leq c_{41}\int_{B_2}|u_\theta\rho|^{q-1} \leq c_{42}\int_{B_2}|u_\theta\rho)|^q + \frac{c_{43}}{q}.$$

If we integrate this inequality with respect to t, we can show that the norm $\|u_\theta\rho\|_q$ is bounded on the interval $]t_0-\tau, t_0[$ and moreover, the upper bound does not depend on q. Thus, we can pass to $+\infty$ with q and we obtain the boundedness of $\|u_\theta\rho\|_\infty$ on $]t_0-\tau,t_0[$. The integral on the right-hand side of (41) can now be estimated:

$$\frac{3}{\nu^2}\int_{B_2}\frac{|u_\rho|u_\theta^4}{\rho^3}\le\frac{3}{\nu^2}\left(\int_{B_2}\left|\frac{u_\rho}{\rho}\right|^{\frac{36}{5}}\right)^{\frac{5}{36}}\left(\int_{B_2}\left|\frac{u_\theta}{\rho}\right|^4\right)^{\frac{\alpha}{4}}\left(\int_{B_2}\left|\frac{u_\theta^2}{\rho}\right|^2\right)^{\frac{\beta}{4}}\left(\int_{B_2}|u_\theta|^s\right)^{\frac{\gamma}{s}}\|u_\theta\rho\|_\infty^\xi \tag{44}$$

where $\alpha+\beta+\gamma+\xi=4$, $\alpha+\dfrac{\beta}{2}=2$, $\dfrac{\alpha}{4}+\dfrac{\beta}{4}+\dfrac{\gamma}{s}=\dfrac{31}{36}$ and $\alpha,\beta,\gamma,\xi\ge 0$. Using the Young inequality as in Step 3.1 and the boundedness of $\|u_\theta\rho\|_\infty$ on $]t_0-\tau,t_0[$, we can further estimate the right-hand side of (44) and we get:

$$\frac{3}{\nu^2}\int_{B_2}\frac{|u_\rho|u_\theta^4}{\rho^3}\le\delta\|\omega_\theta\|_{36/5}^{12/5}+\delta\left\|\frac{u_\theta}{\rho}\right\|_4^4+c_{44}(\delta)\|u_\theta\|_s^{\frac{12\gamma}{7-3\alpha}}\left\|\frac{u_\theta^2}{\rho}\right\|_2^{\frac{6\beta}{7-3\alpha}}. \tag{45}$$

If we put again $6\beta/(7-3\alpha)=2$, we obtain:

$$\alpha=\frac{45-5s}{9},\quad \beta=\frac{5s-24}{9},\quad \gamma=\frac{5s}{18},\quad \xi=\frac{30-5s}{18}\quad\text{and}\quad\frac{12\gamma}{7-3\alpha}=\frac{10s}{5s-24}.$$

It can be easily verified that $\alpha,\beta,\gamma,\xi\ge0$ and $7-3\alpha>0$ if and only if $\frac{24}{5}<s\le 6$. Substituting these values of $\alpha,\dots,\xi$ into (45), we get:

$$\frac{3}{\nu^2}\int_{B_2}\frac{|u_\rho|u_\theta^4}{\rho^3}\le\delta\|\omega_\theta\|_{36/5}^{12/5}+\delta\left\|\frac{u_\theta}{\rho}\right\|_4^4+c_{44}(\delta)\|u_\theta\|_s^{\frac{10s}{5s-24}}\left\|\frac{u_\theta^2}{\rho}\right\|_2^2. \tag{46}$$

It follows from the assumptions of Theorem 2 (item 2) that $\|u_\theta\|_s^{10s/(5s-24)}$ is integrable on the interval $]t_0-\tau,t_0[$.

We can now use estimate (46) in (41). Choosing δ sufficiently small and integrating (41) with respect to time, we obtain again the boundedness of the norms $\|\omega_\theta/\rho\|_2$, $\|u_\theta^2/\rho\|_2$ and $\|\omega_\theta\|_{12/5}$ on the time interval $]t_0-\tau,t_0[$.

Step 4. The proof of Theorem 2 can now be completed in the same way as the proof of Theorem 1 – see Step 5 in Section 3. □

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References

[1] W. Borchers and H. Sohr, On the equations rot $v=g$ and div $u=f$ with zero boundary conditions, *Hokkaido Math. J.* **19** (1990), 67–87.

[2] L. Caffarelli, R. Kohn and L. Nirenberg, Partial regularity of suitable weak solutions of the Navier–Stokes equations, *Comm. on Pure and Apl. Math.* **35** (1982), 771–831.

[3] L. H. Choe and J. Lewis, *On the singular set in the Navier–Stokes equations*, Preprint, Korean Advanced Institute for Science and Technology, 1999.

[4] C. Foias and R. Temam, Some analytic and geometric properties of the solutions of the evolution Navier–Stokes equations, *J. Math. Pures et Appl.* **58** (1979), 339–368.

[5] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier–Stokes Equations, Vol. I: Linearized Steady Problems, Vol. II: Nonlinear Steady Problems*, Springer Tracts in Natural Philosophy, Vol. 38, 39 Springer-Verlag, New York–Berlin–Heidelberg, 1994.

[6] G. P. Galdi, An Introduction to the Navier–Stokes Initial-Boundary Value Problem, to appear in: G. P. Galdi, J. Heywood and R. Rannacher (editors), *Fundamental Directions in Mathematical Fluid Mechanics, Advances in Mathematical Fluid Mechanics, Vol. 1*, Birkhäuser-Verlag, Basel, 2000.

[7] Y. Giga, Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier–Stokes equations, *J. of Diff. Equations* **61** (1986), 186–212.

[8] J. G. Heywood, The Navier–Stokes equations: On the existence, uniqueness and decay of solutions, *Indiana Univ. Math. J.* **29** (1980), 639–681.

[9] E. Hopf, Über die Anfangwertaufgabe für die hydrodynamischen Grundgleichungen, *Math. Nachr.* **4** (1950), 213–231.

[10] K. K. Kiselev and O. A. Ladyzhenskaya, On existence and uniqueness of the solutions of the nonstationary problem for a viscous incompressible fluid, *Izv. Akad. Nauk SSSR* **21** (1957), 655–680 (in Russian).

[11] H. Kozono and H. Sohr, Remark on uniqueness of weak solutions to the Navier–Stokes equations, *Analysis* **16** (1996), 255–271.

[12] H. Kozono, Uniqueness and regularity of weak solutions to the Navier–Stokes equations, *Lecture Notes in Num. and Appl. Anal.* **16** (1998), 161–208.

[13] O. A. Ladyzhenskaya, The global solution of the boundary value problem for the Navier–Stokes equations in the case of two space variables, *Comm. on Pure and Appl. Math.* **12** (1959), 427–433.

[14] O. A. Ladyzhenskaya, Uniqueness and smoothness of generalized solutions of the Navier–Stokes equations, *Zap. Nauch. Sem. LOMI* **5** (1967), 169–185 (in Russian).

[15] O. A. Ladyzhenskaya, On the unique global solvability of the Cauchy problem for the Navier–Stokes equations in the presence of the axial symmetry, *Zap. Nauch. Sem. LOMI* **7** (1968), 155–177 (in Russian).

[16] S. Leonardi, J. Málek, J. Nečas and M. Pokorný, On the results of Uchovskii and Yudovich on axially symmetric flows of a viscous fluid in $\mathbb{R}^3$, *Zeitschrift für Angew. Anal.* **18** (1999), 639–649.

[17] J. Leray, Sur le mouvements d'un liquide visqueux emplissant l'espace, *Acta Math.* **63** (1934), 193–248.

[18] F. Lin, A new proof of the Caffarelli–Kohn–Nirenberg theorem, *Comm. on Pure and Appl. Math.* **51** (1998), 241–257.

[19] J. Neustupa and P. Penel, Regularity of a suitable weak solution to the Navier–Stokes equations as a consequence of regularity of one velocity component, in: H. Beirao da Veiga, A. Sequeira, J. Videman (editors), *Nonlinear Applied Analysis*, Plenum Press, New York, 1999, 391–402.

[20] J. Neustupa, A. Novotný and P. Penel, *A remark to the interior regularity of a suitable weak solution to the Navier–Stokes equation*, Preprint, University of Toulon–Var, 1999.

[21] L. Nirenberg, On elliptic partial differential equations. *Ann. Scuola Norm.* **13** (1959), 115–162.

[22] G. Prodi, Un teorema di unicità per el equazioni di Navier–Stokes, *Ann. Mat. Pura Appl.* **48** (1959), 173–182.

[23] J. Serrin, On the interior regularity of weak solutions of the Navier–Stokes equations, *Arch. Rat. Mech. Anal.* **9**, 187–195.

[24] H. Sohr and W. von Wahl, On the singular set and the uniqueness of weak solutions of the Navier–Stokes equations, *Manuscripta Math.* **49** (1984), 27–59.

[25] V. A. Solonnikov, Estimates of solutions of a non-stationary Navier–Stokes system, *Zap. Nauch. Sem. LOMI* **38** (1973), 153–231 (in Russian).

[26] M. R. Uchovskii and B. I. Yudovich, Axially symmetric flows of an ideal and viscous fluid, *J. Appl. Math. Mech.* **32** (1968), 52–61.

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