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Sublinear functionals and conical measures

By

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Abstract. The paper is devoted to the concept of conical measures which is central for the Choquet theory of integral representation in its final version. The conical measures need not be continuous under monotone pointwise convergence of sequences on the *lattice subspace* of functions which form their domain. We prove that they indeed become continuous (even in the nonsequential sense) when one restricts that domain to an obvious *subcone*. This result is in accord with the recent representation theory in measure and integration developed by the author. We also prove that one can pass from the subcone in question to a certain natural extended cone.

The main point in the present paper comes from the theory of integral representation due to Choquet 1956–68 in its final version. This version takes place on certain convex cones in certain locally convex topological vector spaces. The fundamental concepts are the *caps* of these cones and the *conical measures* on the spaces. The aim is to represent certain conical measures in terms of appropriate Radon measures. We refer to the expositions in Choquet [3] and Becker [1] and to the survey articles [2] [4]. Our point of interest is the concept of conical measures. In the review [13] on the book [1] the present author commented on that concept and on possible variants, and in the meantime obtained some relevant results which this paper wants to develop. They can be expected to form a sensible combination with our recent work in measure and integration [10] [11] [12].

Let E be a real vector space. We form the usual classes $E^\star$ of all linear and $E^\#$ of all sublinear functionals $E \to \mathbb{R}$. The Choquet theory under consideration assumes a linear subspace $F \subset E^\star$ which separates E. The weak topology $\sigma(E, F)$ on E is Hausdorff with topological dual F. One defines $s(E) = s(E, F) \subset \mathbb{R}^E$ to consist of the pointwise maxima $\vartheta = f_1 \vee \ldots \vee f_r$ of finite families $f_1, \ldots, f_r \in F$, and $h(E) = h(E, F) := s(E, F) - s(E, F) \subset \mathbb{R}^E$. One verifies that $h(E, F)$ is a lattice subspace, while $s(E, F)$ is a maximum-stable convex cone. The *conical measures* on E (with respect to F) are defined to be the positive (:= isotone) linear functionals $\psi : h(E, F) \to \mathbb{R}$. They are of course in one-to-one correspondence with their restrictions to $s(E, F)$, that is with the isotone positive-linear functionals $\varphi : s(E, F) \to \mathbb{R}$.

The conical measures do not well comply with the traditional representation theorems of Daniell-Stone type. One reason is that the lattice subspace $h(E, F)$ does not fulfil the *Stone condition* (because its members are positive-homogeneous functions on E). It can be said that

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Choquet invented the *caps* in order to overcome this drawback. The other reason is that the conical measures need not be *continuous* under monotone pointwise convergence even in the σ (:= sequential) sense, except when E is complete in $\sigma(E, F)$, that is when the canonical map $E \to F^\star$ is onto; see [3] 38.13–14. This unpleasant fact obstructs the extension of conical measures, a procedure in which the Stone condition does not occur.

It is this situation to which our former comments applied. First of all one has

$$F \subset s(E, F) \subset E^{\#}_{\sigma(E,F)} \subset E^{\#} \subset \mathbb{R}^E,$$

where $E^{\#}_{\sigma(E,F)}$ consists of the members of $E^{\#}$ which are continuous in $\sigma(E, F)$. One concrete proposal was to consider the idea to base the concept of conical measures upon $E^{\#}_{\sigma(E,F)}$ in place of $s(E, F)$, a perhaps more flexible function class. The immediate response of Gustave Choquet was that with this one might come to the same conical measures as before. It will be proved below that this suspicion is in fact true: Each isotone positive-linear functional $\varphi : s(E, F) \to \mathbb{R}$ is the restriction of a unique $\phi : E^{\#}_{\sigma(E,F)} \to \mathbb{R}$ of the same sort. It remains to be seen what the benefits of $E^{\#}_{\sigma(E,F)}$ are compared to $s(E, F)$.

The other concrete proposal arose from the experience that in [10] [11] [12] the fundamental representation procedure in measure and integration has been restructured and in particular extended, as to the domains of the basic functionals, from Stonean lattice *subspaces* of functions to Stonean lattice *cones*. Thus it was a natural proposal to remain, as to the domains of conical measures, with *cones* like $s(E, F)$ and $E^{\#}_{\sigma(E,F)}$ and to refrain from the transition to differences. We know from [10] 3.11 and 14.20 that such transitions can result in the destruction of continuity. Therefore the present paper will concentrate on the further question whether and in what sense *continuity* can be asserted for the functionals on those smaller domains. Our result will be the desirable one in the sense of [10] [11] [12]: Each isotone positive-linear functional $\phi : E^{\#}_{\sigma(E,F)} \to \mathbb{R}$, and hence each such $\varphi : s(E, F) \to \mathbb{R}$, is *downward continuous* under pointwise convergence even in the τ (:= nonsequential) sense. Thus the picture is quite different from the former one. But of course the task of actual representation for these functionals in the spirit of [10] [11] [12] remains to be done.

The proofs of the two results in question rest upon certain fundamental properties of the classes $E^\star$ and $E^{\#}$ around the Hahn-Banach theorem. The basic facts are the *maximum* and *sum theorems* after the author's paper [8] of 1972, which was an attempt at systematization. A similar attempt has been made in 1969 in the Lectures on Analysis of Choquet [3] chapter 8. It includes the sum theorem and its connection with the Strassen disintegration theorem, but there is no explicit maximum theorem. The maximum theorem remained almost unnoticed to this day despite its remarkable power, for which we insert one more example with an extension of the decomposition theorem [3] 36.4. In [8] the maximum and sum theorems were simple consequences of a Hahn-Banach separation theorem which is equivalent to the Mazur-Orlicz theorem [15] 2.41. Later on the separation theorem found a wide extension in Fuchssteiner-König [5], and on this basis the maximum and sum theorems in [9] [14], but these extensions will not be needed in the sequel. For the entire context we refer to the historical comments in Fuchssteiner-Lusky [6] pp. 72–86 and [14] pp. 109–115.

Then we also need the two basic tool theorems about weak topologies on vector spaces: On the one hand we adopt from [8] [9] the assertion called the *convex closure theorem*, which is the common substance of the strong separation and bipolar theorems. On the other hand we present the *Alaoglu-Bourbaki theorem* in an unconventional version which might well be its final one. The author produced this version in his courses since 1989 but did not see it in the literature earlier than in 1997 in Pallu de la Barrière [16] amidst of preliminaries p. 17 (an old special case, which is much closer to the usual Alaoglu-Bourbaki theorem, is in [7] excerc. 3 p. 90).

All this will be presented in Section 1, with at least short sketches of proofs even of the old results, in order to make the text self-contained (from the basic Hahn-Banach theorem) and to underline its ease. The author likes the fact that the machinery of topological vector spaces does not occur at all. After this then Section 2 will obtain our main results.

The author wants to express his warmest thanks to Gustave Choquet and Richard Becker for most valuable exchange of ideas and for the benefits which he drew from their works.

1. Some basic properties of sublinear functionals. Let as before E be a real vector space with $E^\star$ and $E^\#$. For $\vartheta \in E^\#$ we define $\Lambda(\vartheta) \subset E^\star$ to consist of all $f \in E^\star$ such that $f \leqq \vartheta$. The basic Hahn-Banach theorem asserts that $\Lambda(\vartheta)$ is nonvoid. It is obvious that $\Lambda(\vartheta)$ is convex.

Theorem 1.1. *Assume that $\vartheta \in E^\#$, and that $A \subset E$ is nonvoid convex.*

1) (Variant of the Mazur-Orlicz Theorem) *If $F : A \to \mathbb{R}$ is concave with $F \leqq \vartheta|A$ then there exists $f \in \Lambda(\vartheta)$ such that $F \leqq f|A$.*
2) *There exists $f \in \Lambda(\vartheta)$ such that* $\inf(f|A) = \inf(\vartheta|A)$.

S k e t c h o f p r o o f. 1) Define $\theta(x) = \inf\{\vartheta(x + tu) - tF(u) : u \in A \text{ and } t \geqq 0\}$ for $x \in E$. One verifies that $\theta \in E^\#$, and for $f \in E^\star$ the equivalence $f \leqq \theta \Leftrightarrow f \leqq \vartheta$ and $F \leqq f|A$.

2) We can assume that $c := \inf(\vartheta|A) > -\infty$. The assertion follows from 1) applied to $F := c$. □

For the sequel we define $\mathbb{P}^r$ to consist of all $t = (t_1, \dots, t_r) \in \mathbb{R}^r$ with $t_1, \dots, t_r \geqq 0$ and $t_1 + \cdots + t_r = 1$.

Theorem 1.2. *Assume that $\vartheta_1, \dots, \vartheta_r \in E^\#$ and $f \in E^\star$, and that $A \subset E$ is nonvoid convex.*

1) (Maximum Theorem) *If $f \leqq \vartheta_1 \vee \ldots \vee \vartheta_r$ on A then there exists $t \in \mathbb{P}^r$ such that $f \leqq t_1\vartheta_1 + \cdots + t_r\vartheta_r$ on A.*
2) (Sum Theorem) *If $f \leqq \vartheta_1 + \cdots + \vartheta_r$ on A then there exist $f_l \in \Lambda(\vartheta_l)$ $(l = 1, \dots, r)$ such that $f \leqq f_1 + \cdots + f_r$ on A.*
3) (Combination) *If $f \leqq \vartheta_1 \vee \ldots \vee \vartheta_r$ on A then there exist $f_l \in \Lambda(\vartheta_l)$ $(l = 1, \dots, r)$ and $t \in \mathbb{P}^r$ such that $f \leqq t_1 f_1 + \cdots + t_r f_r$ on A.*

S k e t c h o f p r o o f. 1) Define $H \in (\mathbb{R}^r)^\#$ to be $H(u) = u_1 \vee \ldots \vee u_r$ $\forall u = (u_1, \dots, u_r) \in \mathbb{R}^r$. Then $\Lambda(H) \subset (\mathbb{R}^r)^\star$ consists of the $h \in (\mathbb{R}^r)^\star$ of the form $h(u) = t_1u_1 + \cdots + t_ru_r$ $\forall u \in \mathbb{R}^r$ with $t \in \mathbb{P}^r$. The assertion follows from 1.1.2) applied to $H \in (\mathbb{R}^r)^\#$ and to the convex subset $\{u \in \mathbb{R}^r : u_l \geqq \vartheta_l(x) - f(x)\ (l = 1, \dots, r) \text{ for some } x \in A\} \subset \mathbb{R}^r$.

2) The case $r = 1$ is obvious. For the induction step $1 \leqq r \Rightarrow r + 1$ assume that $\vartheta_0, \vartheta_1, \dots, \vartheta_r \in E^\#$ fulfil $f \leqq \vartheta_0 + \vartheta_1 + \cdots + \vartheta_r$ on A. The assertion follows from 1.1.1) applied to ϑ_0 and $F := \big(f - (\vartheta_1 + \cdots + \vartheta_r)\big)|A$ and from the induction hypothesis.

3) Combine 1) and 2). □

For later reference we collect the required properties of the formation $\Lambda(\cdot)$. They are all clear after 1.1 and 1.2.

Properties 1.3. 1) $\Lambda(\vartheta_1 + \cdots + \vartheta_r) = \Lambda(\vartheta_1) + \cdots + \Lambda(\vartheta_r)\ \forall \vartheta_1, \ldots, \vartheta_r \in E^{\#}$.
2) $\Lambda(t\vartheta) = t\Lambda(\vartheta)\ \forall \vartheta \in E^{\#}$ and $t \geqq 0$.
3) $\Lambda(\vartheta_1 \vee \ldots \vee \vartheta_r) = \operatorname{conv}(\Lambda(\vartheta_1), \ldots, \Lambda(\vartheta_r))\ \forall \vartheta_1, \ldots, \vartheta_r \in E^{\#}$.
4) For $\alpha, \beta \in E^{\#}$ we have $\alpha \leqq \beta \Leftrightarrow \Lambda(\alpha) \subset \Lambda(\beta)$.
5) Let $M \subset E^{\#}$ be nonvoid and downward directed (in the pointwise order). Then the pointwise infimum $\alpha := \inf\limits_{\vartheta \in M} \vartheta$ is in $E^{\#}$, and we have $\Lambda(\alpha) = \bigcap\limits_{\vartheta \in M} \Lambda(\vartheta)$.

We turn to the relevant weak topologies. The weak⋆ topology $\sigma(E^\star, E)$ on $E^\star$ is defined to be the restriction to $E^\star \subset \mathbb{R}^E$ of the product topology $\sigma(\mathbb{R}^E, E)$ on $\mathbb{R}^E$. For each $\vartheta \in E^{\#}$ the subset $\Lambda(\vartheta) \subset E^\star$ is compact in $\sigma(E^\star, E)$, that is in $\sigma(\mathbb{R}^E, E)$. This follows from $\Lambda(\vartheta) = E^\star \cap \prod\limits_{x \in E} [-\vartheta(-x), \vartheta(x)]$, since $E^\star$ is closed and the subsequent product set is compact in $\sigma(\mathbb{R}^E, E)$.

In contrast to the unique topology $\sigma(E^\star, E)$ on $E^\star$ we have to consider a whole collection of topologies on E. For each linear subspace $F \subset E^\star$ let $\sigma(E, F)$ be the weakest topology on E in which all members $f \in F$ are continuous functions $E \to \mathbb{R}$. Thus $\sigma(E, F)$ is Hausdorff iff F separates E.

Convex Closure Theorem 1.4. *Let $F \subset E^\star$ be a linear subspace. For nonvoid $A \subset E$ then*

$$\overline{\operatorname{conv} A}^{\sigma(E,F)} = \{u \in E : f(u) \leqq \sup_{x \in A} f(x) \ \textit{for all}\ f \in F\}.$$

Sketch of proof. To be shown is $\supset$. We fix $a \notin \overline{\operatorname{conv}}^{\sigma(E,F)}$. There are $f_1, \ldots, f_r \in F$ and $\varepsilon > 0$ such that $\max\{|f_l(x-a)| : l = 1, \ldots, r\} \geqq \varepsilon\ \forall x \in \operatorname{conv} A$. Thus the sublinear functional $\vartheta := f_1 \vee (-f_1) \vee \ldots \vee f_r \vee (-f_r)$ fulfils

$$\inf\{\vartheta(z) : z \in a - \operatorname{conv} A\} \geqq \varepsilon.$$

From 1.1.2) we obtain an $f \in \Lambda(\vartheta)$ such that

$$\inf\{f(z) : z \in a - \operatorname{conv} A\} \geqq \varepsilon.$$

This means that $f(a) \geqq \sup\{f(x) : x \in \operatorname{conv} A\} + \varepsilon > \sup\{f(x) : x \in A\}$. Now 1.2.1) says that $f \in F$. It follows that a is not in the second member. □

Consequence 1.5. *For nonvoid $A \subset E^\star$ we have*

$$\overline{\operatorname{conv} A}^{\sigma(E^\star,E)} = \{h \in E^\star : h(x) \leqq \sup_{f \in A} f(x) \ \textit{for all}\ x \in E\}.$$

Sketch of proof. Follows from 1.4 via the canonical map $E \to (E^\star)^\star$. □

We come to the Alaoglu-Bourbaki theorem. We define $\mathfrak{K}(E)$ to consist of all nonvoid convex and weak⋆ compact subsets of $E^\star$.

Theorem 1.6. *The map $\vartheta \mapsto \Lambda(\vartheta)$ is a bijection $\Lambda : E^{\#} \to \mathfrak{K}(E)$. The inverse map is $K \mapsto \vartheta : \vartheta(x) = \max\{f(x) : f \in K\}$ for $x \in E$.*

P r o o f. We know that $\vartheta \mapsto \Lambda(\vartheta)$ is a map $\Lambda : E^{\#} \to \mathfrak{K}(E)$. We define $I : \mathfrak{K}(E) \to E^{\#}$ as follows. For each $x \in E$ the evaluation $E^{\star} \to \mathbb{R} : f \mapsto f(x)$ is continuous in $\sigma(E^{\star}, E)$, and hence is bounded and attains its supremum on each $K \in \mathfrak{K}(E)$. For $K \in \mathfrak{K}(E)$ thus $\vartheta(x) := \max\{f(x) : f \in K\}\ \forall x \in E$ defines a functional $\vartheta \in E^{\#}$. We put $I(K) := \vartheta$. We have to prove that $I \circ \Lambda$ is the identity on $E^{\#}$ and that $\Lambda \circ I$ is the identity on $\mathfrak{K}(E)$. i) $I \circ \Lambda =$ identity on $E^{\#}$: Let $\vartheta \in E^{\#}$ and $K := \Lambda(\vartheta)$. For each $a \in E$ we obtain from 1.1.2) applied to $\{a\} \subset E$ an $f \in K$ such that $f(a) = \vartheta(a)$. Thus $\vartheta(a) = \max\{h(a) : h \in K\}$. This means that $\vartheta = I(K)$. ii) $\Lambda \circ I =$ identity on $\mathfrak{K}(E)$: Let $K \in \mathfrak{K}(E)$ and $\vartheta := I(K)$. From 1.5 we have

$$K = \overline{\mathrm{conv} K}^{\sigma(E^{\star},E)} = \{h \in E^{\star} : h(x) \leqq \sup_{f \in K} f(x) = \vartheta(x)\ \forall x \in E\} = \Lambda(\vartheta). \quad \square$$

We conclude the section with an application of 1.2 which extends and specifies the decomposition theorem [3] 36.4 (except of course its extra assertion which requires some sort of lattice assumption). We note that an assertion of similar kind occured in [8] p. 507 and [9] 6.2.

Theorem 1.7. *Let $A \subset E^{\star}$ be nonvoid, and assume that $\vartheta(x) := \sup\{|f(x)| : f \in A\} < \infty$ $\forall x \in E$ and hence $\vartheta \in E^{\#}$. Then*

$$\Lambda(\vartheta) = \mathrm{conv}\big(\overline{\mathrm{conv} A}^{\sigma(E^{\star},E)}, -\overline{\mathrm{conv} A}^{\sigma(E^{\star},E)}\big).$$

P r o o f. Define $\alpha(x) := \sup\{f(x) : f \in A\}$ and $\beta(x) := \sup\{-f(x) : f \in A\}$ for $x \in E$, so that $\alpha, \beta \in E^{\#}$ with $\vartheta = \alpha \vee \beta$. We have $\beta(x) = \alpha(-x)\ \forall x \in E$ and hence $\Lambda(\beta) = -\Lambda(\alpha)$. Moreover $\Lambda(\alpha) = \overline{\mathrm{conv} A}^{\sigma(E^{\star},E)}$ from 1.5. The assertion follows from 1.3.3). $\square$

2. The main results. Let as before E be a real vector space with $E^{\star}$ and $E^{\#}$. We fix a linear subspace $F \subset E^{\star}$ which separates E and consider

$$F \subset s(E, F) \subset E^{\#}_{\sigma(E,F)} \subset E^{\#} \subset \mathbb{R}^{E},$$

as defined in the introduction.

Proposition 2.1. *For $\vartheta \in E^{\#}$ the following are equivalent.*
1) *ϑ is upper semicontinuous in $\sigma(E, F)$.*
2) *ϑ is continuous in $\sigma(E, F)$.*
3) *There exists a nonvoid finite $M \subset F$ such that $\vartheta \leqq \max_{f \in M} f$.*
4) *There exists a nonvoid finite $M \subset F$ such that $|\vartheta(u) - \vartheta(v)| \leqq \max\{|f(u - v)| : u, v \in M\}$ for all $u, v \in E$.*
5) *$\Lambda(\vartheta)$ is contained in some finite-dimensional linear subspace of F.*

The proof will be after the scheme 1)⇒3)⇒4)⇒2)⇒1) and 3)⇔5). We start with a useful remark.

R e m a r k 2.2. Let $T \subset E^{\star}$ be a nontrivial finite-dimensional linear subspace and $f_1, \ldots, f_r \in T$ be a basis of T. Define

$$\|f\| := \sum_{l=1}^{r} |t_l| \quad \text{for} \quad f = \sum_{l=1}^{r} t_l f_l \in T,$$

so that $\|\cdot\|$ is a norm on T which of course produces $\sigma(E^{\star}, E)|T$. Also define $\omega := f_1 \vee (-f_1) \vee \ldots \vee f_r \vee (-f_r)$, so that $\omega \in E^{\#}$ with $\Lambda(\omega) \subset T$. Then

$$\Lambda(\omega) = \{f \in T : \|f\| \leqq 1\},$$

and hence is a 0 neighbourhood in $\sigma(E^{\star}, E)|T$.

Proof of 2.2. $f \in \Lambda(\omega) = \operatorname{conv}(\pm f_1, \dots, \pm f_r, 0)$ means that

$$f = \sum_{l=1}^{r} (s_l - t_l) f_l + c0 \quad \text{with } s_1, \dots, s_r, t_1, \dots, t_r \geqq 0$$
$$\text{and } \sum_{l=1}^{r} (s_l + t_l) + c = 1,$$

which is equivalent to

$$f = \sum_{l=1}^{r} c_l f_l \quad \text{with } \sum_{l=1}^{r} |c_l| \leqq 1, \text{ that is to } \|f\| \leqq 1. \qquad \square$$

Proof of 2.1. 1)⇒3). Under 1) $[\vartheta < 1]$ is a 0 neighbourhood in $\sigma(E, F)$. Thus there exists a nonvoid finite $M \subset F$ such that $\max\{|f(x)| < 1 : f \in M\} \Rightarrow \vartheta(x) < 1$ and hence

$$\vartheta(x) \leqq \max\{|f(x)| : f \in M\} = \max\{f(x) \vee (-f(x)) : f \in M\} \quad \text{for all } x \in E.$$

3)⇒4) For $u, v \in E$ we have

$$\begin{aligned} |\vartheta(u) - \vartheta(v)| &= (\vartheta(u) - \vartheta(v)) \vee (\vartheta(v) - \vartheta(u)) \leqq \vartheta(u - v) \vee \vartheta(v - u) \\ &\leqq \max\{f(u - v) \vee f(v - u) : f \in M\} \\ &= \max\{|f(u - v)| : f \in M\}. \end{aligned}$$

4)⇒2) and 2)⇒1) are obvious. 3)⇒5) For $M = \{f_1, \dots, f_r\}$ we see from 1.3.3) that

$$\Lambda(\vartheta) \subset \Lambda(f_1 \vee \dots \vee f_r) = \operatorname{conv}(f_1, \dots, f_r) \subset \operatorname{Lin}(f_1, \dots, f_r).$$

5)⇒3) Assume that $\Lambda(\vartheta)$ is contained in the nontrivial finite-dimensional linear subspace $T \subset F$, and let $f_1, \dots, f_r \in T$ be a basis of T as in 2.2. Then $\|f\| \leqq c < \infty \; \forall f \in \Lambda(\vartheta)$ since $\Lambda(\vartheta)$ is compact in $\sigma(E^\star, E)|T$. Therefore

$$f(x) = \sum_{l=1}^{r} t_l f_l(x) \leqq c \max\{|f_l(x)| : l = 1, \dots, r\} \quad \text{for } f \in \Lambda(\vartheta) \text{ and } x \in E,$$

and hence $\vartheta \leqq c f_1 \vee (-f_1) \vee \dots \vee f_r \vee (-f_r)$. $\square$

Proposition 2.3 (which will not be needed in the sequel). *For $\vartheta \in E^\#$ the following are equivalent.*

1) *ϑ is continuous in the Mackey topology $\tau(E, F)$.*

2) $\Lambda(\vartheta) \subset F$.

Sketch of proof. One of the usual definitions of the Mackey topology $\tau(E, F)$ is that a basis of 0 neighbourhoods consists of the so-called polars $[\max_{f \in K} f \leqq 1]$ of the $K \in \mathfrak{K}(E)$ with $K \subset F$, which in view of 1.6 are the $[\vartheta \leqq 1]$ for the $\vartheta \in E^\#$ with $\Lambda(\vartheta) \subset F$. $\square$

We come to the first main result.

Theorem 2.4. *Assume that $H \subset E^\#$ is a convex cone with $s(E, F) \subset H \subset E^\#_{\sigma(E,F)}$. Then each isotone positive-linear functional $\varphi : H \to \mathbb{R}$ is downward τ continuous under pointwise convergence, that is: If $M \subset H$ is nonvoid and downward directed $\downarrow \alpha \in H$ then $\inf_{\vartheta \in M} \varphi(\vartheta) = \varphi(\alpha)$.*

Proof. In view of 2.1 we can assume that that there is a nontrivial finite-dimensional linear subspace $T \subset F$ such that $\Lambda(\vartheta) \subset T$ for all $\vartheta \in M$, and hence that $\Lambda(\alpha) \subset T$ as well. From 1.3.5) we know that $\Lambda(\alpha) = \bigcap_{\vartheta \in M} \Lambda(\vartheta)$. Let $f_1, \ldots, f_r \in T$ be a basis of T and $\omega \in E^\#$ as in 2.2. For $\varepsilon > 0$ then $\Lambda(\varepsilon\omega) = \varepsilon\Lambda(\omega) \subset T$ is a 0 neighbourhood in $\sigma(E^\star, E)|T$. Thus compactness implies the existence of $\vartheta \in M$ such that $\Lambda(\vartheta) \subset \Lambda(\alpha) + \varepsilon\Lambda(\omega) = \Lambda(\alpha + \varepsilon\omega)$ and hence $\vartheta \leqq \alpha + \varepsilon\omega$. In view of $\omega \in s(E, F) \subset H$ we have $\varphi(\vartheta) \leqq \varphi(\alpha + \varepsilon\omega) = \varphi(\alpha) + \varepsilon\varphi(\omega)$. Thus $\inf_{\vartheta \in M} \varphi(\vartheta) \leqq \varphi(\alpha) + \varepsilon\varphi(\omega)$ for each $\varepsilon > 0$ and hence $\leqq \varphi(\alpha)$. □

Our next aim is the approximation Theorem 2.6 below.

Lemma 2.5. *Let T be a nontrivial finite-dimensional real vector space and $\|\cdot\|$ be a norm on T. For nonvoid $K \subset T$ and $\delta > 0$ define $K(\delta) := \{f \in T : \mathrm{dist}(f, K) \leqq \delta\}$, so that $K(\delta)$ is* i) *compact when K is compact and* ii) *convex when K is convex. If K is compact convex and $\delta > 0$ then there exists a nonvoid finite $M \subset K(\delta)$ such that $K \subset \mathrm{conv}M \subset K(\delta)$.*

Proof. Let $M \subset K(\delta)$ be nonvoid finite such that $K(\delta) \subset M(\delta)$. Then fix $f \in K$. For each $v \in T$ with $\|v\| \leqq \delta$ we have $f + v \in K(\delta) \subset M(\delta)$, so that there exists $u \in M$ with $\|f + v - u\| \leqq \delta$. For $\varphi \in T^\star$ therefore

$$\varphi(f) + \varphi(v) \leqq \varphi(u) + |\varphi(f + v - u)| \leqq \sup_{u \in M} \varphi(u) + \delta\|\varphi\|.$$

From $\sup\{\varphi(v) : v \in T \text{ with } \|v\| \leqq \delta\} = \delta\|\varphi\|$ we obtain $\varphi(f) \leqq \sup_{u \in M} \varphi(u)$ for all $\varphi \in T^\star$. After 1.4 this says that $f \in \overline{\mathrm{conv}M}^{\sigma(T,T^\star)} = \mathrm{conv}M$. □

Theorem 2.6. *Assume that $\vartheta \in E^\#_{\sigma(E,F)}$, and let $T \subset F$ be a nontrivial finite-dimensional linear subspace with $\Lambda(\vartheta) \subset T$. Then there exists a sequence $(\vartheta_n)_n$ in $s(E, F)$ such that $\vartheta_n \downarrow \vartheta$ pointwise on E, and that the $\Lambda(\vartheta_n)$ are $\subset T$ and neighbourhoods of $\Lambda(\vartheta)$ in $\sigma(E^\star, E)|T$.*

Proof. Let $\|\cdot\|$ be a norm on T as in 2.2, and put $K := \Lambda(\vartheta) \subset T$. For $n \in \mathbb{N}$ we obtain from 2.5 a nonvoid finite $M(n) \subset K(\frac{1}{2^n})(\frac{1}{2^n}) = K(\frac{1}{2^{n-1}})$ such that

$$K \subset \ldots \subset K\left(\frac{1}{2^n}\right) \subset \mathrm{conv}M(n) \subset K\left(\frac{1}{2^{n-1}}\right) \subset \ldots \subset K(1) \subset T.$$

Thus $\mathrm{conv}M(n) \subset T$ is a neighbourhood of K in $\sigma(E^\star, E)|T$. After 1.6 there exists $\vartheta_n \in E^\#$ with $\Lambda(\vartheta_n) = \mathrm{conv}M(n) \subset T$. From 1.3.3) it follows that $\Lambda(\vartheta_n) = \mathrm{conv}M(n) = \Lambda(\max_{f \in M(n)} f)$ and hence $\vartheta_n = \max_{f \in M(n)} f \in s(E, F)$. At last from $\Lambda(\vartheta_n) = \mathrm{conv}M(n) \downarrow K = \Lambda(\vartheta)$ and 1.3.4)5) we see that $\vartheta_n \downarrow \vartheta$. □

Proposition 2.7. *Assume that $M \subset s(E, F)$ is nonvoid and downward directed $\downarrow \alpha \in E^\#_{\sigma(E,F)}$. For each isotone positive-linear $\varphi : s(E, F) \to \mathbb{R}$ then*

$$\inf_{\vartheta \in M} \varphi(\vartheta) = \inf\{\varphi(\eta) : \eta \in s(E, F) \text{ with } \eta \geqq \alpha\}.$$

Proof. To be shown is $\leqq$. We can assume that there is a nontrivial finite-dimensional linear subspace $T \subset F$ such that $\Lambda(\vartheta) \subset T$ for all $\vartheta \in M$, and hence that $\Lambda(\alpha) \subset T$ as well. From 2.6 we obtain a sequence $(\vartheta_n)_n$ in $s(E, F)$ such that $\vartheta_n \downarrow \alpha$ pointwise on E, and that the $\Lambda(\vartheta_n) \subset T$ are neighbourhoods of $\Lambda(\alpha)$ in $\sigma(E^\star, E)|T$. For fixed $n \in \mathbb{N}$ then $\Lambda(\alpha) = \bigcap_{\vartheta \in M} \Lambda(\vartheta)$

from 1.3.5) and compactness furnish some $\vartheta \in M$ such that $\Lambda(\vartheta) \subset \Lambda(\vartheta_n)$ and hence $\vartheta \leqq \vartheta_n$, so that $\varphi(\vartheta) \leqq \varphi(\vartheta_n)$. It follows that $\inf_{\vartheta \in M} \varphi(\vartheta) \leqq \varphi(\vartheta_n)$ for all $n \in \mathbb{N}$. Now for fixed $\eta \in s(E, F)$ with $\eta \geqq \alpha$ we have $\vartheta_n \vee \eta \in s(E, F)$ and $\vartheta_n \vee \eta \downarrow \alpha \vee \eta = \eta$, and hence $\varphi(\vartheta_n \vee \eta) \downarrow \varphi(\eta)$ from 2.4 applied to $H = s(E, F)$. It follows that

$$\inf_{\vartheta \in M} \varphi(\vartheta) \leqq \lim_{n\to\infty} \varphi(\vartheta_n) \leqq \lim_{n\to\infty} \varphi(\vartheta_n \vee \eta) = \varphi(\eta). \qquad \square$$

We come to the second main result.

Theorem 2.8. *Each isotone positive-linear functional $\varphi : s(E, F) \to \mathbb{R}$ has a unique isotone positive-linear extension $\phi : E^{\#}_{\sigma(E,F)} \to \mathbb{R}$. It is defined to be*

$$\phi(\alpha) = \inf\{\varphi(\eta) : \eta \in s(E, F) \text{ with } \eta \geqq \alpha\} \quad \text{for } \alpha \in E^{\#}_{\sigma(E,F)}.$$

Proof. i) We define $\phi : E^{\#}_{\sigma(E,F)} \to [-\infty, \infty[$ as above, where 2.6 or 2.1.2)⇒3) has been used. ϕ is isotone and an extension of φ. Also $\phi(\alpha) > -\infty$, because the Hahn-Banach theorem furnishes an $f \in \Lambda(\alpha)$ and hence $\phi(\alpha) \geqq \varphi(f) > -\infty$. After 2.6 there are sequences $(\alpha_n)_n$ in $s(E, F)$ such that $\alpha_n \downarrow \alpha$ pointwise on E, and from 2.7 we have $\varphi(\alpha_n) \downarrow \phi(\alpha)$ each time. It follows that ϕ is additive and hence positive-linear. Thus ϕ is as required.

ii) Assume now that $\theta : E^{\#}_{\sigma(E,F)} \to \mathbb{R}$ is any isotone positive-linear extension of φ. We know from 2.4 that both θ and ϕ are downward σ continuous. Thus the approximation Theorem 2.6 implies that $\theta = \phi$. $\square$

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