Arch. Math. 76 (2001) 366–376
0003-889X/01/050366-11 $ 3.70/0

Archiv der Mathematik

Positive L^p solutions of Hammerstein integral equations

By

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Abstract. In this paper we use Krasnoselskii's fixed point theorem to establish the existence of at least one positive solution $y \in L^p[0,T]$ of the integral equation $y(t) = h(t) + \int_0^T k(t,s) f(s, y(s))\, ds$, a.e. $t \in [0,T]$ and related equations.

1. Introduction. Much work has been carried out on the existence of positive, continuous solutions of various types of Hammerstein integral equations (see references). One of the most common approaches to a problem of this nature is to use Krasnoselskii's fixed point theorem. In this paper we consider the Hammerstein integral equation

$$y(t) = h(t) + \int_0^T k(t,s) f(s, y(s))\, ds, \text{ a.e. } t \in [0,T], \tag{1.1}$$

and use Krasnoselskii's theorem on this occasion, to establish conditions under which (1.1) will have at least one positive solution $y \in L^p[0,T]$, $1 \leqq p < \infty$. In fact we first consider (1.1) when $h \equiv 0$, that is,

$$y(t) = \int_0^T k(t,s) f(s, y(s))\, ds, \text{ a.e. } t \in [0,T], \tag{1.2}$$

since the analysis is slightly more straight forward in this case. Half the battle in applying Krasnoselskii's fixed point theorem to a problem, is establishing an appropriate cone in which the desired solution will lie and ensuring that the integral operator behaves correctly on this cone. In both of the equations mentioned above we place conditions mainly on the kernel to ensure that the hypotheses of Krasnoselskii's fixed point theorem are satisfied.

We lastly consider the nonlinear integral equation on the half line, namely

$$y(t) = h(t) + \int_0^\infty k(t,s) f(s, y(s))\, ds, \text{ a.e. } t \in [0,\infty), \tag{1.3}$$

which can be dealt with in a similar manner to (1.1) and (1.2).

We complete the introduction by stating the following theorem which will be used throughout the next section.

Theorem 1.1 Krasnoselskii's Fixed Point Theorem [5]**.** *Let E be a Banach space and let $C \subset E$ be a cone in E. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$K : C \cap (\overline{\Omega}_2 \backslash \Omega_1) \to C$$

Mathematics Subject Classification (1991): 45B05, 45G10.

be a completely continuous operator such that either

(i) $||Ku|| \leqq ||u||,\ u \in C \cap \partial\Omega_1$ *and* $||Ku|| \geqq ||u||,\ u \in C \cap \partial\Omega_2$

or

(ii) $||Ku|| \geqq ||u||,\ u \in C \cap \partial\Omega_1$ *and* $||Ku|| \leqq ||u||,\ u \in C \cap \partial\Omega_2$

is true. Then K has a fixed point in $C \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

2. Positive L^p solutions. Consider the nonlinear integral equation

$$y(t) = \int_0^T k(t,s) f(s, y(s))\, ds, \text{ a.e. } t \in [0,T]. \tag{2.1}$$

We would like to know what conditions one requires on k and f in order for this equation to have a positive solution $y \in L^p[0,T]$ where $1 \leqq p < \infty$. Here by a positive function y we mean $y(t) > 0$ for a.e. $t \in [0,T]$.

Notation. We will use $||.||_p$ to denote the norm on $L^p[0,T]$ with

$$||y||_p := \left(\int_0^T |y(t)|^p \, dt \right)^{\frac{1}{p}} \text{ if } 1 \leqq p < \infty,$$

while

$$||y||_p := \text{ess} - \sup_{t \in [0,T]} |y(t)| \text{ if } p = \infty.$$

Ultimately we will apply Krasnoselskii's fixed point theorem to obtain the desired result. This means that we must construct an appropriate cone in $L^p[0,T]$ and ensure that the relevant integral operator satisfies the conditions of this fixed point theorem on the cone.

We start by defining the operators $K_L : L^{p_2}[0,T] \to L^p[0,T]$ and $F : L^p[0,T] \to L^{p_2}[0,T]$ by

$$K_L y(t) := \int_0^T k(t,s) y(s)\, ds, \text{ a.e. } t \in [0,T] \tag{2.2}$$

and

$$Fy(t) := f(t, y(t)), \text{ a.e. } t \in [0,T] \tag{2.3}$$

respectively. Here $1 \leqq p_2 < \infty$ and p_1 is such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$.

Concentrating first on the operator $F : L^p[0,T] \to L^{p_2}[0,T]$ suppose that

$$\begin{cases} f : [0,T] \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function; that is} \\ \text{(i) the map } t \mapsto f(t,y) \text{ is measurable for all } y \in \mathbb{R}, \\ \text{(ii) the map } y \mapsto f(t,y) \text{ is continuous for almost all } t \in [0,T], \end{cases} \tag{2.4}$$

and

$$\begin{aligned} &\text{there exists } a_1 \in L^{p_2}[0,T] \text{ and } a_2 > 0 \\ &\text{such that } |f(t,y)| \leqq a_1(t) + a_2 |y|^{\frac{p}{p_2}}, \text{ a.e. } t \in [0,T] \end{aligned} \tag{2.5}$$

hold. Then we have that

$$f(t, y(t)) \in L^{p_2}[0,T] \text{ for } y \in L^p[0,T]$$

and from [4], we see that

(2.6) $F : L^p[0,T] \to L^{p_2}[0,T]$ is continuous and bounded

is true. From (2.5) we obtain

$$|f(t,y(t))|^{p_2} \leqq 2^{p_2-1}\left(a_1^{p_2}(t) + a_2^{p_2}|y(t)|^p\right), \text{ a.e. } t \in [0,T]$$

and hence

$$\int_0^T |f(t,y(t))|^{p_2}\,dt \leqq 2^{p_2-1}\left(\int_0^T a_1^{p_2}(t)\,dt + a_2^{p_2}\int_0^T |y(t)|^p\,dt\right).$$

Thus defining $\psi \in C[0,\infty)$ by

(2.7) $$\psi(t) := 2^{\frac{p_2-1}{p_2}}\left(||a_1||_{p_2}^{p_2} + a_2^{p_2}t^p\right)^{\frac{1}{p_2}}, \ t \in [0,\infty)$$

we see that

(2.8) there exists $\psi \in C[0,\infty)$ such that $||Fy||_{p_2} \leqq \psi(||y||_p)$ for all $y \in L^p[0,T]$

holds. Finally since we are interested in positive solutions of (2.1) we will assume that

(2.9) $f(t,y) > 0$ for all $y > 0$ and a.e. $t \in [0,T]$

is satisfied. Thus defining the cones

$$C_p := \{y \in L^p[0,T] : y(t) \geqq 0 \text{ a.e. } t \in [0,T]\}$$

and

$$C_{p_2} := \{y \in L^{p_2}[0,T] : y(t) \geqq 0 \text{ a.e. } t \in [0,T]\}$$

we see from (2.6) and (2.9) that

(2.10) $F : C_p \to C_{p_2}$ is a bounded, continuous operator.

We now turn our attention to the operator $K_L : L^{p_2}[0,T] \to L^p[0,T]$. Suppose that

(2.11) $k : [0,T] \times [0,T] \to \mathbb{R}$ is such that $(t,s) \mapsto k(t,s)$ is measurable

and

(2.12) $$\begin{cases} \text{there exists } 0 < M \leqq 1,\ k_1 \in L^p[0,T] \text{ and } k_2 \in L^{p_1}[0,T] \\ \text{such that } 0 < k_1(t),\ k_2(t) \text{ a.e. } t \in [0,T] \text{ and} \\ Mk_1(t)k_2(s) \leqq k(t,s) \leqq k_1(t)k_2(s) \text{ a.e. } t \in [0,T], \text{ a.e. } s \in [0,T]. \end{cases}$$

hold. From (2.12) we have immediately that

$$\left(\int_0^T \left(\int_0^T |k(t,s)|^p\,dt\right)^{\frac{p_1}{p}} ds\right)^{\frac{1}{p_1}} \leqq ||k_1||_p ||k_2||_{p_1} \equiv M_0 < \infty\,.$$

Thus from [9, pp. 47–49] we have that

$$K_L : L^{p_2}[0,T] \to L^p[0,T] \text{ is a completely continuous operator.}$$

In fact due to the positivity of the kernel k we have that

(2.13) $\qquad K_L : C_{p_2} \to C_p$ is a completely continuous operator

is true. However we can be more specific about the range of K_L.

If $y \in C_{p_2}$ then we know from (2.12) that

$$|K_L y(t)|^p \leqq k_1^p(t)\left(\int_0^T k_2(s)y(s)\,ds\right)^p, \text{ a.e. } t \in [0,T],$$

that is,

$$\|K_L y\|_p \leqq \|k_1\|_p \int_0^T k_2(s)y(s)\,ds. \tag{2.14}$$

However we also have from (2.12) that for $y \in C_{p_2}$,

$$K_L y(t) \geqq M\,k_1(t)\int_0^T k_2(s)y(s)\,ds, \text{ a.e. } t \in [0,T].$$

Combining this inequality with (2.14) we have that for $y \in C_{p_2}$,

$$K_L y(t) \geqq M\frac{k_1(t)}{\|k_1\|_p}\|K_L y\|_p, \text{ a.e. } t \in [0,T]. \tag{2.15}$$

Thus defining

$$a(t) := M\frac{k_1(t)}{\|k_1\|_p}, \text{ a.e. } t \in [0,T], \tag{2.16}$$

we see that $a \in L^p[0,T]$, $a(t) > 0$, a.e. $t \in [0,T]$ and $\|a\|_p = M \leqq 1$. Therefore if we define the cone $C_{p,a}$ by

$$C_{p,a} := \{y \in L^p[0,T] : y(t) \geqq a(t)\|y\|_p, \text{ a.e. } t \in [0,T]\},$$

then by (2.15) and (2.13) we have

(2.17) $\qquad K_L : C_{p_2} \to C_{p,a}$ is a completely continuous operator.

Finally since $C_{p,a} \subseteqq C_p$, if we let

$$Ky(t) := K_L F y(t) = \int_0^T k(t,s)f(s,y(s))\,ds, \text{ a.e. } t \in [0,T],$$

then (2.10) and (2.17) combine to give

(2.18) $\qquad K : C_{p,a} \to C_{p,a}$ is a completely continuous operator.

Now that it is clearer what operator and cone we intend to use in Krasnoselskii's fixed point theorem, we state the following existence result for (2.1).

Theorem 2.1. *Assume that* p, p_1 *and* p_2 *are such that* $1 \leqq p_1 \leqq p < \infty$ *and* $1/p_1 + 1/p_2 = 1$. *Suppose that* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *satisfies* (2.4), (2.5), (2.9) *and*

(2.19) $\qquad f(t,y)$ *is nondecreasing in* y *for a.e.* $t \in [0,T]$, *while* $k : [0,T] \times [0,T] \to \mathbb{R}$ *satisfies* (2.11) *and* (2.12). *In addition assume that*

$$\text{(2.20)} \qquad \textit{there exists } \alpha > 0 \textit{ such that } 1 < \frac{\alpha}{||k_1||_p ||k_2||_{p_1} \psi(\alpha)}$$

and

$$\text{(2.21)} \qquad \textit{there exists } \beta > 0,\ \beta \neq \alpha,\ \textit{such that } 1 > \frac{\beta}{M ||k_1||_p \int\limits_0^T k_2(s) f(s, a(s)\beta)\, ds}$$

(Here $\psi \in C[0,\infty)$ *is as described in* (2.7) *and* $a \in L^p[0,T]$ *is as described in* (2.16).*) Then* (2.1) *has at least one positive solution* $y \in L^p[0,T]$ *and either*

(A) $0 < \alpha < ||y||_p < \beta$ *and* $y(t) \geqq a(t)\alpha$ *a.e.* $t \in [0,T]$ *if* $\alpha < \beta$

or

(B) $0 < \beta < ||y||_p < \alpha$ *and* $y(t) \geqq a(t)\beta$ *a.e.* $t \in [0,T]$ *if* $\beta < \alpha$

holds.

Proof. Define

$$\Omega_\alpha := \{ y \in L^p[0,T] : ||y||_p < \alpha \}$$

and

$$\Omega_\beta := \{ y \in L^p[0,T] : ||y||_p < \beta \}.$$

Assume that $\beta < \alpha$. (A similar argument holds if $\alpha < \beta$.) It is clear from (2.18) that

$$K : C_{p,a} \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta) \to C_{p,a} \text{ is a completely continuous operator.}$$

If we show that in addition

$$\text{(2.22)} \qquad ||Ky||_p \leqq ||y||_p \text{ for } y \in C_{p,a} \cap \partial\Omega_\alpha$$

and

$$\text{(2.23)} \qquad ||Ky||_p \geqq ||y||_p \text{ for } y \in C_{p,a} \cap \partial\Omega_\beta$$

are true, then by Krasnoselskii's fixed point theorem the operator K has a fixed point in $C_{p,a} \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta)$. This in turn implies that (2.1) has at least one solution $y \in L^p[0,T]$ such that $\beta \leqq ||y||_p \leqq \alpha$ and $y(t) \geqq a(t)\beta$ for a.e. $t \in [0,T]$.

Suppose then that $y \in C_{p,a} \cap \partial\Omega_\alpha$. In particular $||y||_p = \alpha$. Then (2.12), Hölder's inequality, (2.8) and (2.20) give

$$\begin{aligned} ||Ky||_p &\leqq ||k_1||_p ||k_2||_{p_1} ||Fy||_{p_2} \leqq ||k_1||_p ||k_2||_{p_1} \psi(||y||_p) \\ &= ||k_1||_p ||k_2||_{p_1} \psi(\alpha) < \alpha = ||y||_p \end{aligned}$$

and thus (2.22) is satisfied.

Next suppose that $y \in C_{p,a} \cap \partial\Omega_\beta$. Then $||y||_p = \beta$ and note also that $y(t) \geqq a(t)\beta$ for a.e. $t \in [0,T]$. Conditions (2.12) and (2.19) give

$$\int_0^T |Ky(t)|^p\,dt \geqq M^p ||k_1||_p^p \left(\int_0^T k_2(s) f(s, y(s))\,ds \right)^p$$
$$\geqq M^p ||k_1||_p^p \left(\int_0^T k_2(s) f(s, a(s)\beta)\,ds \right)^p.$$

This inequality along with (2.21) implies that

$$||Ky||_p \geqq M ||k_1||_p \left(\int_0^T k_2(s) f(s, a(s)\beta)\,ds \right) > \beta = ||y||_p$$

– thus (2.23) holds and the theorem is proved. □

Example 2.1. Let p, p_1 and p_2 be as in Theorem 2.1 and suppose that $f(s, y) = y^n$ where $0 \leqq n < 1$. It is immediate that f satisfies (2.4), (2.9) and (2.19). To ensure that (2.5) holds (with $a_1 = 0$ and $a_2 = 1$), we must make the additional assumption that $np_2 \leqq p$. This fact, along with Hölder's inequality gives

$$\int_0^T |f(s, y(s))|^{p_2}\,ds = \int_0^T |y(s)|^{np_2}\,ds \leqq T^m ||y||_p^{np_2} \tag{2.24}$$

where $m := (p - np_2)/p$. In addition from (2.24) we can define $\psi(t) := Ct^n$ where $C := T^{m/p_2}$. It remains to show that there exist α and β that satisfy (2.20) and (2.21) respectively, with the above choice of f and ψ. Since $0 \leqq n < 1$, it is possible to choose $0 < \alpha$ (large enough) and $0 < \beta$ (small enough) to satisfy

$$1 < \frac{\alpha^{1-n}}{C ||k_1||_p ||k_2||_{p_1}}$$

and

$$1 > \frac{\beta^{1-n}}{M ||k_1||_p \int_0^T k_2(s) a^n(s)\,ds}$$

respectively. In addition we see that $0 < \beta < \alpha$ since

$$\beta^{1-n} < MC ||k_1||_p ||k_2||_{p_1} ||a||_p^n = M^{1+n} C ||k_1||_p ||k_2||_{p_1} \leqq C ||k_1||_p ||k_2||_{p_1} < \alpha^{1-n}.$$

For integral equations with similar nonlinearities, we refer the reader to the references. In particular [2, 3, 6–8, 12] discuss integral equations where the nonlinearity f exhibits sublinear growth as above.

Example 2.2. Let p, p_1 and p_2 be as in Theorem 2.1 and suppose that $f(s, y) = y^n$ where now $n > 1$ and we assume again that $np_2 \leqq p$. Arguing as in Example 2.1, we see that f satisfies (2.4), (2.5), (2.9) and (2.19) and ψ can once again be given by $\psi(t) := Ct^n$ where C is as defined in the above example. Now since $n > 1$, one can choose $0 < \alpha$ (small enough) to ensure that

$$1 < \frac{1}{C ||k_1||_p ||k_2||_{p_1} \alpha^{n-1}}.$$

In addition β can then be chosen (large enough) such that $0 < \alpha < \beta$ and

$$1 > \frac{1}{M\|k_1\|_p \left(\int\limits_0^T k_2(s) a^n(s)\, ds \right) \beta^{n-1}}.$$

Integral equations whose nonlinear part exhibits superlinear growth are also discussed in [2, 3, 6–8].

While the results in [2, 3] also rely on Krasnoselskii's fixed point theorem, the advantage of Theorem 2.1 for the two examples discussed here, is that we are only required to know how the nonlinearity f behaves at two points, α and β.

Remark 2.1. In Theorem 2.1 we sought the existence of a positive solution $y \in L^p[0, T]$ of (2.1). It is possible to modify the hypotheses of this theorem to obtain the existence of at least one *nonnegative* solution of (2.1), that is, a solution $y \in L^p[0, T]$ such that $y(t) \geqq 0$, a.e. $t \in [0, T]$. Replacing condition (2.12) with

$$(2.12^\star) \qquad \begin{cases} \text{there exists } 0 < M \leqq 1,\ k_1 \in L^p[0, T] \text{ and } k_2 \in L^{p_1}[0, T] \\ \text{such that } 0 \leqq k_1(t),\ k_2(t) \text{ a.e. } t \in [0, T],\ 0 < \|k_1\|_p,\ \|k_2\|_{p_1} \text{ and} \\ M k_1(t) k_2(s) \leqq k(t, s) \leqq k_1(t) k_2(s) \text{ a.e. } t \in [0, T],\ \text{a.e. } s \in [0, T] \end{cases}$$

and assuming that $\int\limits_0^T k_2(s) f(s, a(s)\beta)\, ds > 0$, for all $\beta > 0$, will ensure the result.

For the remainder of this section we will concentrate on establishing results analogous to Theorem 2.1, for two variations of equation (2.1). Firstly we consider the Hammerstein integral equation

$$(2.25) \qquad y(t) = h(t) + \int\limits_0^T k(t, s) f(s, y(s))\, ds, \text{ a.e. } t \in [0, T].$$

The hypotheses and proof of Theorem 2.1 can be modified easily to ensure that this nonlinear equation has at least one positive solution $y \in L^p[0, T]$. We give some details of the proof here since it is necessary to consider a slightly different cone to the one used in the proof of Theorem 2.1.

Theorem 2.2. *Suppose that* $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ *satisfies* (2.4), (2.5), (2.9) *and* (2.19), *while* $k : [0, T] \times [0, T] \to \mathbb{R}$ *satisfies* (2.11) *and* (2.12). *In addition assume that*

$$(2.26) \qquad \begin{cases} h \in L^p[0, T] \textit{ and } h(t) \geqq a(t)\|h\|_p, \textit{ a.e. } t \in [0, T]; \\ \textit{here } a \in L^p[0, T] \textit{ is as described in } (2.16), \end{cases}$$

$$(2.27) \qquad \textit{there exists } \alpha > 0 \textit{ such that } 1 < \frac{\alpha}{4^{\frac{p-1}{p}} (\|h\|_p + \|k_1\|_p \|k_2\|_{p_1} \psi(\alpha))}$$

and

$$(2.28) \qquad \textit{there exists } \beta > 0,\ \beta \neq \alpha, \textit{ such that } 1 > \frac{\beta}{2^{\frac{1-p}{p}} \left(\|h\|_p + M\|k_1\|_p \int\limits_0^T k_2(s) f(s, \tilde{a}(s)\beta)\, ds \right)}$$

are true. (The function ψ *in* (2.27) *is such that* (2.8) *holds, while* $\tilde{a}$ *in condition* (2.28) *is as* *given in* (2.33).*) Then* (2.25) *has at least one positive solution* $y \in L^p[0,T]$ *and either*

$$\text{(A)}\quad 0 < \alpha < ||y||_p < \beta \text{ and } y(t) \geqq a(t)\alpha \text{ a.e. } t \in [0,T] \text{ if } \alpha < \beta$$

or

$$\text{(B)}\quad 0 < \beta < ||y||_p < \alpha \text{ and } y(t) \geqq a(t)\beta \text{ a.e. } t \in [0,T] \text{ if } \beta < \alpha$$

holds.

Proof. From the proof of Theorem 2.1 we see that (2.10) is true, that is,

$$F : C_p \to C_{p_2} \text{ is a bounded, continuous operator,}$$

where $Fy(t) := f(t, y(t))$, a.e. $t \in [0,T]$. Defining the operator $\tilde{K} : L^{p_2}[0,T] \to L^p[0,T]$ by

$$\tilde{K}y(t) := h(t) + \int_0^T k(t,s)y(s)\,ds, \text{ a.e. } t \in [0,T],$$

we can show (once again using [9, pp. 47–49] in addition to (2.26)) that

$$\tilde{K} : L^{p_2}[0,T] \to L^p[0,T] \text{ is a completely continuous operator.}$$

Indeed, due to the positive nature of the functions involved, it is clear that

$$\tilde{K} : C_{p_2} \to C_p \text{ is a completely continuous operator.} \tag{2.29}$$

However, as in the proof of Theorem 2.1, we want to be more specific about the range of $\tilde{K}$ in this case.

If $y \in C_{p_2}$, then from (2.12), (2.26) and using the facts that

$$(a+b)^p \leqq 2^{p-1}(a^p + b^p) \text{ and } (a+b)^{\frac{1}{p}} \leqq 2^{\frac{p-1}{p}}(a^{\frac{1}{p}} + b^{\frac{1}{p}}) \text{ for } a, b > 0,\ p \geqq 1, \tag{2.30}$$

we see that

$$|\tilde{K}y(t)|^p \leqq 2^{p-1}\left(|h(t)|^p + k_1^p(t)\left(\int_0^T k_2(s)y(s)\,ds\right)^p\right), \text{ a.e. } t \in [0,T],$$

that is,

$$||\tilde{K}y||_p^p \leqq 2^{p-1}\left(||h||_p^p + ||k_1||_p^p\left(\int_0^T k_2(s)y(s)\,ds\right)^p\right),$$

and therefore

$$||\tilde{K}y||_p \leqq 4^{\frac{p-1}{p}}\left(||h||_p + ||k_1||_p \int_0^T k_2(s)y(s)\,ds\right). \tag{2.31}$$

However we also have from (2.12) and (2.26) that for $y \in C_{p_2}$,

$$\begin{aligned}\tilde{K}y(t) &\geqq a(t)||h||_p + Mk_1(t)\int_0^T k_2(s)y(s)\,ds \\ &= M\frac{k_1(t)}{||k_1||_p}\left(||h||_p + ||k_1||_p\int_0^T k_2(s)y(s)\,ds\right), \text{ a.e. } t \in [0,T].\end{aligned}$$

Combining this with (2.31) yields

$$\tilde{K}y(t) \geqq 4^{\frac{1-p}{p}}M\frac{k_1(t)}{||k_1||_p}||\tilde{K}y||_p, \text{ a.e. } t \in [0,T]. \tag{2.32}$$

Thus defining

$$\tilde{a}(t) := 4^{\frac{1-p}{p}} M \frac{k_1(t)}{||k_1||_p}, \text{ a.e. } t \in [0, T], \tag{2.33}$$

we see that $\tilde{a} \in L^p[0, T]$, $\tilde{a}(t) > 0$, a.e. $t \in [0, T]$ and $||\tilde{a}||_p = 4^{\frac{1-p}{p}} M \leqq 1$. Therefore if we define the cone $C_{p,\tilde{a}}$ by

$$C_{p,\tilde{a}} := \{y \in L^p[0, T] : y(t) \geqq \tilde{a}(t)||y||_p, \text{ a.e. } t \in [0, T]\},$$

then by (2.29) and (2.32) we have

$$K_L : C_{p_2} \to C_{p,\tilde{a}} \text{ is a completely continuous operator.}$$

Finally if we let

$$Ky(t) := \tilde{K}Fy(t) = h(t) + \int_0^T k(t, s) f(s, y(s))\, ds, \text{ a.e. } t \in [0, T],$$

we obtain

$$K : C_{p,\tilde{a}} \to C_{p,\tilde{a}} \text{ is a completely continuous operator.}$$

In order to apply Krasnoselskii's fixed point theorem to obtain the desired result it remains to show that

$$||Ky||_p \leqq ||y||_p \text{ for } y \in C_{p,\tilde{a}} \cap \partial\Omega_\alpha \tag{2.34}$$

and

$$||Ky||_p \geqq ||y||_p \text{ for } y \in C_{p,\tilde{a}} \cap \partial\Omega_\beta \tag{2.35}$$

are true. (Here Ω_α and Ω_β are as defined in the proof of Theorem 2.1.) Suppose first that $y \in C_{p,\tilde{a}} \cap \Omega_\alpha$, implying in particular that $||y||_p = \alpha$. This fact, (2.31), Hölder's inequality, (2.8) and (2.27) give

$$\begin{aligned} ||\tilde{K}y||_p &\leqq 4^{\frac{p-1}{p}}(||h||_p + ||k_1||_p ||k_2||_{p_1} \psi(||y||_p) \\ &= 4^{\frac{p-1}{p}}(||h||_p + ||k_1||_p ||k_2||_{p_1} \psi(\alpha)) < \alpha = ||y||_p \end{aligned}$$

– thus (2.34) holds.

For $y \in C_{p,\tilde{a}} \cap \Omega_\beta$ we have that $||y||_p = \beta$ and $y(t) \geqq \tilde{a}(t)\beta$, a.e. $t \in [0, T]$. This information, along with (2.12) and (2.19), gives

$$\begin{aligned} |Ky(t)|^p &\geqq |h(t)|^p + M^p k_1^p(t) \left(\int_0^T k_2(s) f(s, y(s))\, ds \right)^p \\ &\geqq |h(t)|^p + M^p k_1^p(t) \left(\int_0^T k_2(s) f(s, \tilde{a}(s)\beta)\, ds \right)^p. \end{aligned}$$

If we recall that $(a + b)^{\frac{1}{p}} \geqq 2^{\frac{1-p}{p}}(a^{\frac{1}{p}} + b^{\frac{1}{p}})$, for $a, b > 0$ and $p \geqq 1$, we now obtain from (2.28),

$$||Ky||_p \geqq 2^{\frac{1-p}{p}} \left(||h||_p + M||k_1||_p \int_0^T k_2(s) f(s, \tilde{a}(s)\beta)\, ds \right) > \beta = ||y||_p,$$

– and (2.35) is proved. □

Finally we state a result for a second variation of (2.1), namely,

$$y(t) = h(t) + \int_0^\infty k(t,s)f(s,y(s))\,ds, \text{ a.e. } t \in [0,\infty). \tag{2.36}$$

Theorem 2.3. *Suppose that* $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ *satisfies* (2.4), (2.5), (2.9) *and* (2.19), $k : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *satisfies* (2.11) *and* (2.12) *and* $h : [0,\infty) \to \mathbb{R}$ *satisfies* (2.26), *in addition to* (2.27) *and* (2.28) *being true, where in all of the above,* $T = \infty$ *and* $[0,T]$ *is replaced by* $[0,\infty)$. *Then* (2.36) *has at least one positive solution* $y \in L^p[0,\infty)$ *and either*

$$\text{(A)}\quad 0 < \alpha < \|y\|_p < \beta \textit{ and } y(t) \geqq a(t)\alpha \textit{ a.e. } t \in [0,\infty) \textit{ if } \alpha < \beta$$

or

$$\text{(B)}\quad 0 < \beta < \|y\|_p < \alpha \textit{ and } y(t) \geqq a(t)\beta \textit{ a.e. } t \in [0,\infty) \textit{ if } \beta < \alpha$$

holds.

Notation. Here $\|.\|_p$ denotes the norm on $L^p[0,\infty)$ with

$$\|y\|_p := \left(\int_0^\infty |y(t)|^p\,dt \right)^{\frac{1}{p}} \text{ if } 1 \leqq p < \infty,$$

while

$$\|y\|_p := \text{ess} - \sup_{t \in [0,\infty)} |y(t)| \text{ if } t = \infty.$$

Proof. Proving that (2.36) has at least one positive solution $y \in L^p[0,\infty)$ follows almost identically to the proof of Theorem 2.1 and Theorem 2.2. The only real difference arises when we want to show that

$$K_\infty : L^{p_2}[0,\infty) \to L^p[0,\infty) \text{ is a completely continuous operator,} \tag{2.37}$$

where now

$$K_\infty y(t) := h(t) + \int_0^\infty k(t,s)y(s)\,ds, \text{ a.e. } t \in [0,\infty). \tag{2.38}$$

In the proofs of Theorem 2.1 and Theorem 2.2 we refer the reader to [9, pp. 47–49], where the Riesz compactness criteria are used, to show that both $K_L : L^{p_2}[0,T] \to L^p[0,T]$ and $\tilde{K} : L^{p_2}[0,T] \to L^p[0,T]$ are completely continuous. Slightly different compactness criteria are required to show that (2.37) is true, however the reader can find the details of this argument in [9, pp. 66–68]. We omit the rest of the details since they are identical to those already outlined in the above proofs. □

References

[1] P. J. Bushell, On a class of Volterra and Fredholm nonlinear integral equations. Math. Proc. Cambridge Philos. Soc. **79,** 329–335 (1976).

[2] L. H. Erbe and H. Wang, On the existence of positive solutions of ordinary differential equations. Proc. Amer. Math. Soc., **120,** 3, 743–748 (1994).

[3] L. H. Erbe, S. Hu and H. Wang, Multiple positive solutions of some boundary value problems. J. Math. Anal. Appl. **184,** 640–648 (1994).

[4] M. A. KRASNOSELSKII, Topological Methods in the Theory of Nonlinear Integral Equations. Oxford 1964.
[5] M. A. KRASNOSELSKII, Positive Solutions of Operator Equations. Groningen 1964.
[6] M. MEEHAN and D. O'REGAN, Multiple nonnegative solutions of nonlinear integral equations on compact and semi-infinite intervals. Appl. Anal. **74,** 413–427 (2000).
[7] M. MEEHAN and D. O'REGAN, Positive solutions of singular and nonsingular Fredholm integral equations. J. Math. Anal. Appl. **240,** 416–432 (1999).
[8] M. MEEHAN and D. O'REGAN, Positive solutions of singular integral equations. J. Integral Equations Appl. To appear.
[9] D. O'REGAN and M. MEEHAN, Existence Theory for Nonlinear Integral and Integrodifferential Equations. Dordrecht 1998.
[10] C. A. STUART, Integral equations with decreasing nonlinearities and applications. J. Differential Equations **18,** 202–217 (1975).
[11] C. A. STUART, Concave solutions of singular nonlinear differential equations. Math. Z. **136,** 117–135 (1974).
[12] C. A. STUART, Existence theorems for a class of nonlinear integral equations. Math. Z. **137,** 49–66 (1974).
[13] L. VON WOLFERSDORF, Travelling wave solutions of a nonlinear diffusion equation with integral term. Z. Anal. Anwendungen **9,** 4, 303–312 (1990).
[14] P. P. ZABREYKO, A. I. KOSHELEV, M. A. KRASNOSELSKII, S. G. MILKIN, L. S. RAKOV-SHCHIK and V. YA. STETSENKO, Integral Equations – a Reference Text. Leyden 1975.

Eingegangen am 12. 11. 1999

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