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On a filtered multiplicative basis of group algebras

By

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Abstract. Let K be a field of characteristic p and G a nonabelian metacyclic finite p-group. We give an explicit list of all metacyclic p-groups G, such that the group algebra KG over a field of characteristic p has a filtered multiplicative K-basis. We also present an example of a non-metacyclic 2-group G, such that the group algebra KG over any field of characteristic 2 has a filtered multiplicative K-basis.

1. Introduction. Let A be a finite-dimensional algebra over a field K and let B be a K-basis of A. Suppose that B has the following properties:

1. if $b_1, b_2 \in B$ then either $b_1 b_2 = 0$ or $b_1 b_2 \in B$;
2. $B \cap \mathrm{rad}\,(A)$ is a K-basis for rad (A), where rad (A) denotes the Jacobson radical of A.

Then B is called a *filtered multiplicative K-basis* of A.

The filtered multiplicative K-basis arises in the theory of representations of algebras and was introduced first by H. Kupisch [5]. In [1] R. Bautista, P. Gabriel, A. Roiter and L. Salmeron proved that if there are only finitely many isomorphism classes of indecomposable A-modules over an algebraically closed field K, then A has a filtered multiplicative K-basis. Note that by Higman's theorem the group algebra KG over a field of characteristic p has only finitely many isomorphism classes of indecomposable KG-modules if and only if all the Sylow p-subgroups of G are cyclic.

Here we study the following question from [1]: *When does exist a filtered multiplicative K-basis in the group algebra KG*?

Let G be a finite abelian p-group. Then $G = \langle a_1 \rangle \times \langle a_2 \rangle \times \ldots \times \langle a_s \rangle$ is the direct product of cyclic groups $\langle a_i \rangle$ of order q_i, the set

$$B = \{(a_1 - 1)^{n_1}(a_2 - 1)^{n_2} \cdots (a_s - 1)^{n_s} \quad | \quad 0 \leqq n_i < q_i\}$$

is a filtered multiplicative K-basis of the group algebra KG over the field K of characteristic p.

Moreover, if KG_1 and KG_2 have filtered multiplicative K-bases, which we call B_1 and B_2 respectively, then $B_1 \times B_2$ is a filtered multiplicative K-basis of the group algebra $K[G_1 \times G_2]$.

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P. Landrock and G. O. Michler [6] proved that the group algebra of the smallest Janko group over a field of characteristic 2 does not have a filtered multiplicative K-basis.

L. Paris [7] gave examples of group algebras KG, which have no filtered multiplicative K-bases. He also showed that if K is a field of characteristic 2 and either a) G is a quaternion group of order 8 and also K contains a primitive cube root of the unity or b) G is a dihedral 2-group, then KG has a filtered multiplicative K-basis. We shall show that for the class of all metacyclic p-groups the groups mentioned in the items a) and b) are exactly those for which a multiplicative K-basis exists.

We also present an example of a non-metacyclic 2-group G, such that the group algebra KG over any field of characteristic 2 has a filtered multiplicative K-basis.

2. Preliminary remarks and notations. Let B be a filtered multiplicative K-basis in a finite-dimensional K-algebra A. In the proof of the main result we use the following simple properties of B:

(I) $B \cap \operatorname{rad}(A)^n$ is a K-basis of $\operatorname{rad}(A)^n$ for all $n \geqq 1$.

Indeed, by the definition of a basis, $B \cap \operatorname{rad}(A)$ is a K-basis of $\operatorname{rad}(A)$ and the subset $B \cap \operatorname{rad}(A)^n$ is linearly independent over K. Since the set of products $b_1 b_2 \cdots b_n$ with $b_i \in B$ is a generator system for $\operatorname{rad}(A)^n$ and each such product is either 0 or belongs to $B \cap \operatorname{rad}(A)^n$, we conclude that $B \cap \operatorname{rad}(A)^n$ is a K-basis of $\operatorname{rad}(A)^n$.

(II) if $u, v \in B \setminus \operatorname{rad}(A)^k$ and $u \equiv v \,(\operatorname{mod} \operatorname{rad}(A)^k)$ then $u = v$.

Indeed, if $u - v = \sum_{w \in B \cap \operatorname{rad}(A)^k} \lambda_w w$ with $\lambda_w \in K$, then by the linearly independency of the basis elements we conclude that $\lambda_w = 0$ and therefore $u = v$.

Recall that the Frattini subalgebra $\Phi(A)$ of A is defined as the intersection of all maximal subalgebras of A if those exist and as A otherwise. G.L. Carns and C.-Y. Chao [2] showed that if A is a nilpotent algebra over a field K, then $\Phi(A) = A^2$. It follows that

(III) If B is a filtered multiplicative K-basis of A and if $B \setminus \{1\} \subseteqq \operatorname{rad}(A)$, then all elements of $B \setminus \operatorname{rad}(A)^2$ are generators of A over K.

Now let G be a group. For $a, b \in G$ we define ${}^b a = bab^{-1}$ and $[a, b] = aba^{-1}b^{-1}$. The ideal

$$I_K(G) = \Big\{ \sum_{g \in G} \alpha_g g \in KG \mid \sum_{g \in G} \alpha_g = 0 \Big\}$$

is called the augmentation ideal of KG. Then the following subgroup

$$\mathfrak{D}_n(G) = \{ g \in G \mid g - 1 \in I_K^n(G) \}.$$

is called the n-th dimensional subgroup of KG.

3. Results. By Theorem 3.11.2 of [3] every metacyclic p-group has the following presentation

$$G = \langle a, b \mid a^{p^n} = 1, \quad b^{p^m} = a^{p^t}, \quad {}^b a = a^r \rangle,$$

where $t \geqq 0$, $r^{p^m} \equiv 1 \,(\operatorname{mod} p^n)$ and $p^t(r-1) \equiv 0 \,(\operatorname{mod} p^n)$. Therefore, every element of G can be written as $g = a^i b^j$, where $0 \leqq i < p^n$ and $0 \leqq j < p^{m-t}$. Using the identity

(1) $$(xy - 1) = (x-1)(y-1) + (x-1) + (y-1),$$

we obtain that every element of the augmentation ideal $I_K(G)$ is a sum of elements of the form $(a-1)^k(b-1)^l$, where $0 \leqq k < p^n$, $0 \leqq l < p^{m-t}$ and $k+l \geqq 1$.

Theorem. *Let G be a finite metacyclic p-group and K be a field of characteristic p. Then the group algebra KG possesses a filtered multiplicative K-basis if and only if* $p=2$ *and exactly one of the following conditions holds:*

1. *G is a dihedral group;*
2. *K contains a primitive cube root of the unity and G is a quaternion group of order* 8.

Proof. Clearly, $I_K(G)$ is a radical of KG. Suppose that $\{1,B\}$ is a filtered multiplicative K-basis of KG. Then B is a filtered multiplicative K-basis of $I_K(G)$. Obviously, $(a-1)^i(b-1)^j \in I_K^2(G)$ if $i+j \geqq 2$ and $a-1, b-1$ are generators of $I_K(G)$ over K. By Jennings theory [4], $(a-1)+I_K^2(G)$ and $(b-1)+I_K^2(G)$ form a K-basis of $I_K(G)/I_K^2(G)$. Therefore, by property (III), the subset $B \setminus B^2$ consists of two elements, which we denote u and v. Thus $K[u,v]=I_K(G)$ and

$$\begin{cases} u \equiv \alpha_1(a-1)+\alpha_2(b-1) \pmod{I_K^2(G)}; \\ v \equiv \beta_1(a-1)+\beta_2(b-1) \pmod{I_K^2(G)}, \end{cases} \tag{2}$$

where $\alpha_i, \beta_i \in K$ and $\Delta = \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$.

Clearly, $c=[b,a] \in \mathfrak{D}_2(G)$ and $c-1 \in I_K^2(G)$. By a simple calculation we get

$$uv \equiv \alpha_1\beta_1(a-1)^2+\alpha_2\beta_2(b-1)^2+(\alpha_1\beta_2+\alpha_2\beta_1)(a-1)(b-1) + \alpha_2\beta_1(c-1) \pmod{I_K^3(G)}, \tag{3}$$

$$vu \equiv \alpha_1\beta_1(a-1)^2+\alpha_2\beta_2(b-1)^2+(\alpha_1\beta_2+\alpha_2\beta_1)(a-1)(b-1) + \alpha_1\beta_2(c-1) \pmod{I_K^3(G)}, \tag{4}$$

$$u^2 \equiv \alpha_1^2(a-1)^2+\alpha_2^2(b-1)^2+2\alpha_1\alpha_2(a-1)(b-1)+\alpha_1\alpha_2(c-1) \pmod{I_K^3(G)}, \tag{5}$$

$$v^2 \equiv \beta_1^2(a-1)^2+\beta_2^2(b-1)^2+2\beta_1\beta_2(a-1)(b-1)+\beta_1\beta_2(c-1) \pmod{I_K^3(G)}. \tag{6}$$

We consider the case when $c-1 \in I_K^3(G)$. Then by (3) and (4) we have $uv \equiv vu \pmod{I_G^3(K)}$. Moreover, $uv, vu \notin I_K^3(G)$. Indeed, if uv or $vu \in I_K^3(G)$ then by (3) or (4) we obtain $\alpha_1\beta_1 = \alpha_2\beta_2 = \alpha_1\beta_2+\alpha_2\beta_1 = 0$ and $\Delta = 0$, which is impossible. Therefore, $uv, vu \notin I_K^3(G)$ and $uv \equiv vu \pmod{I_K^3(G)}$ and by property (II) of the filtered multiplicative K-basis we conclude that $uv = vu$ and $I_K(G)$ is a commutative algebra, which is contradiction.

In the rest of the proof we assume that $c-1 \notin I_K^3(G)$. It is well-known that for all nonabelian p-groups the factor group G/G' is not cyclic (see [3], Theorem 3.7.1). Thus $r-1$ is divisible by p and $r-1 = ps$ for some s. Then $c-1 = (a^s-1)^p \in I_K^3(G)$ for $p>2$ and also for $p=2$ if s is even. We have established that s is odd and G is a 2-group with the following defining presentation: either

$$G = \langle a, b \mid a^{2^n}=1,\ \ b^{2^m}=1,\ \ {}^b a = a^r \rangle, \tag{7}$$

where $r^{2^m} \equiv 1 \pmod{2^n}$ and $r \not\equiv 1 \pmod 4$, or

$$G = \langle a, b \mid a^{2^n}=1,\ \ b^{2^m}=a^{2^{n-1}},\ \ {}^b a = a^r \rangle, \tag{8}$$

where $r^{2^m} \equiv 1 \pmod{2^n}$, $2^{n-1}(r-1) \equiv 0 \pmod{2^n}$ and 4 does not divide $r-1$.

Suppose that G has the defining presentation (7) and $b^2 = 1$. Since $r - 1 = 2s$ and $(s, 2) = 1$, from $r^2 \equiv 1 \pmod{2^n}$ it follows that $s = -1$ or $s = -1 + 2^{n-2}$ for $n \geqq 3$. Then by (1) we have $c + 1 = a^{2s} + 1 \equiv (1 + a)^2 \pmod{I_K^3(G)}$ and it follows from (3)–(6) that

$$
(9) \qquad \begin{cases} uv \equiv \beta_1(\alpha_1 + \alpha_2)(1+a)^2 + \varDelta(1+a)(1+b) & \pmod{I_K^3(G)}; \\ vu \equiv \alpha_1(\beta_1 + \beta_2)(1+a)^2 + \varDelta(1+a)(1+b) & \pmod{I_K^3(G)}; \\ u^2 \equiv \alpha_1(\alpha_1 + \alpha_2)(1+a)^2 & \pmod{I_K^3(G)}; \\ v^2 \equiv \beta_1(\beta_1 + \beta_2)(1+a)^2 & \pmod{I_K^3(G)}. \end{cases}
$$

Clearly $uv, vu \notin I_K^3(G)$ and by $\varDelta \neq 0$ we have that $uv \not\equiv vu \pmod{I_K^3(G)}$. Since the K-dimension of $I_K^2(G)/I_K^3(G)$ equals 2, the elements $uv + I_K^3(G)$ and $vu + I_K^3(G)$ form a K-basis of $I_K^2(G)/I_K^3(G)$ and $u^2, v^2 \in I_K^3(G)$. We conclude that $\alpha_1(\alpha_1 + \alpha_2) = 0$ and $\beta_1(\beta_1 + \beta_2) = 0$, whence it follows that $u = \alpha(a + b)$ and $v = \beta(1 + b)$. Clearly we can set $\alpha = \beta = 1$.

Let $G = \langle a, b \mid a^{2^n} = 1, \ \ b^2 = 1, \ \ {}^b a = a^{-1} \rangle$ with $n \geqq 2$ be a dihedral group of order 2^{n+1}. We shall prove by induction in i that u^i can be written as

$$
(10) \qquad (1+a)^{2i-1} + (1+a)^{2i-2}(1+b) + \beta_1(1+a)^{2i} + \beta_2(1+a)^{2i-1}(1+b) \pmod{I_K^{2i+1}(G)},
$$

where $\beta_1 = \beta_2 = 1$ if i is even and $\beta_1 = \beta_2 = 0$ otherwise.

Base of induction: It is easy to see that this is true for $i = 1, 2$, and the induction step follows by,

$$
\begin{aligned} (1+b)(1+a) \equiv{} & (1+a)(1+b) + (1+a)^2(1+b) + (1+a)^2 \\ & + (1+a)^3(1+b) + (1+a)^3 + (1+a)^4 \pmod{I_K^5(G)} \end{aligned}
$$

and

$$
\begin{aligned} u^i u \equiv{} & (\beta_1 + \beta_2 + 1)[(1+a)^{2i+1} + (1+a)^{2i}(1+b)] \\ & + (1 + \beta_2)[(1+a)^{2i+2} + (1+a)^{2i+1}(1+b)] \equiv u^{i+1} \pmod{I_K^{2i+3}(G)}. \end{aligned}
$$

Hence (10) holds.

Using (10), we obtain that

$$
\begin{aligned} u^i &\equiv (1+a)^{2i-1} + (1+a)^{2i-2}(1+b) \pmod{I_K^{2i}(G)}, \\ vu^i &\equiv (1+a)^{2i} + (1+a)^{2i-1}(1+b) \pmod{I_K^{2i+1}(G)}, \\ u^j v &\equiv (1+a)^{2j-1}(1+b) \pmod{I_K^{2j+1}(G)}, \\ vu^j v &\equiv (1+a)^{2j}(1+b) \pmod{I_K^{2j+2}(G)}, \end{aligned}
$$

where $i = 1, \ldots, 2^{n-1}$ and $j = 1, \ldots, 2^{n-1} - 1$.

Clearly, the factor algebra $I_K^t(G)/I_K^{t+1}(G)$ has the following basis: $(a+1)^t + I_K^{t+1}(G)$ and $(a+1)^{t-1}(b-1) + I_K^{t+1}(G)$.

First, let $t = 2k + 1$, where $k = 1, \ldots, 2^{n-2} - 1$. Then we have

$$
\begin{aligned} u^{k+1} &\equiv (1+a)^{2k+1} + (1+a)^{2k}(1+b) \pmod{I_K^{t+1}(G)}, \\ vu^k v &\equiv (1+a)^{2k}(1+b) \pmod{I_K^{t+1}(G)} \end{aligned}
$$

and it follows that u^{k+1} and $vu^k v$ are linearly independent by modulo $I_K^{t+1}(G)$.

Now, let $t = 2k$, where $k = 1, \dots, 2^{n-2} - 1$. Then we have

$$vu^k \equiv (1+a)^{2k} + (1+a)^{2k-1}(1+b) \pmod{I_K^{t+1}(G)},$$
$$u^k v \equiv (1+a)^{2k-1}(1+b) \pmod{I_K^{t+1}(G)}$$

and, as before, vu^k and $u^k v$ are linearly independent by modulo $I_K^{t+1}(G)$.

Therefore the matrix of decomposition is unitriangle and

$$\{1, v, u^i, vu^i, u^j v, vu^j v \quad | \, i = 1, \dots, 2^{n-1} \text{ and } j = 1, \dots, 2^{n-1} - 1\}$$

form a filtered multiplicative K-basis of KG.

Now let $G = \langle a, b \mid a^{2^n} = b^2 = 1, {}^b a = a^{-1+2^{n-1}} \rangle$ with $n \geqq 3$ be a semidihedral group and set $u = a + b$, $v = 1 + b$. An easy calculation gives $1 + a^{-1} = \sum_{i=1}^{2^n - 1} (1+a)^i$ and

$$u^2 = \sum_{i=2}^{2^{n-1}-1} (1+a)^i (1+b) + \sum_{j=3}^{2^{n-1}-1} (1+a)^j,$$
$$uvu = \sum_{i=2}^{2^{n-1}} (1+a)^i (1+b) + \sum_{j=3}^{2^{n-1}} (1+a)^j.$$

Therefore $u^2 \equiv uvu \pmod{I_K^{2^{n-1}}(G)}$, but $u^2, uvu \notin I_K^4(G)$ and

$$u^2 - uvu = (1+a)^{2^{n-1}}(1+b) + (1+a)^{2^{n-1}} \neq 0,$$

which contradicts property (II).

Suppose that G has the defining presentation (7) with $m > 1$ or (8) with $m > 1$. By (1) we have

$$(1+b)(1+a) \equiv (1+a)(1+b) + (1+a^s)^2 \equiv (1+a)(1+b) + (1+a)^2 \pmod{I_K^3(G)}$$

and it follows from (3)–(6) that

$$(11) \quad \begin{cases} uv \equiv \beta_1(\alpha_1 + \alpha_2)(1+a)^2 + \Delta(1+a)(1+b) + \alpha_2\beta_2(1+b)^2 & \pmod{I_K^3(G)}; \\ vu \equiv \alpha_1(\beta_1 + \beta_2)(1+a)^2 + \Delta(1+a)(1+b) + \alpha_2\beta_2(1+b)^2 & \pmod{I_K^3(G)}; \\ u^2 \equiv \alpha_1(\alpha_1 + \alpha_2)(1+a)^2 + \alpha_2^2(1+b)^2 & \pmod{I_K^3(G)}; \\ v^2 \equiv \beta_1(\beta_1 + \beta_2)(1+a)^2 + \beta_2^2(1+b)^2 & \pmod{I_K^3(G)}. \end{cases}$$

It is easy to see that $uv, vu \notin I_K^3(G)$. Using the fact that $\Delta \neq 0$, we establish that $uv \not\equiv vu \pmod{I_K^3(G)}$. Therefore $uv + I_K^3(G)$ and $vu + I_K^3(G)$ are K-linearly independent. It is easily verified that $u^2 + I_K^3(G)$ and $v^2 + I_K^3(G)$ are nonzero elements of $I_K^2(G)/I_K^3(G)$ and $uv \not\equiv v^2$, $vu \not\equiv v^2$, $uv \not\equiv u^2$, $vu \not\equiv u^2$. Since the K-dimension of $I_K^2(G)/I_K^3(G)$ equals 3, we have $u^2 \equiv v^2 \pmod{I_K^3(G)}$ and by property (II) of the filtered multiplicative K-basis, $u^2 = v^2$. From $u^2 \equiv v^2 \pmod{I_K^3(G)}$ we obtain $\alpha_2^2 = \beta_2^2$ and $\alpha_1(\alpha_1 + \alpha_2) = \beta_1(\beta_1 + \beta_2)$. By $\Delta \neq 0$ we have $\alpha_2 = \beta_2 \neq 0$, whence the equation $\alpha_1^2 + \beta_2\alpha_1 + \beta_1(\beta_1 + \beta_2) = 0$ has a solution $\alpha_1 = \beta_1 + \beta_2$ whence $\Delta = \beta_2^2 \neq 0$. Thus we observe that $u = (1+\lambda)a + b + \lambda$ and $v = \lambda a + b + \lambda + 1$, where $\lambda = \dfrac{\beta_1}{\beta_2}$. Then, keeping the equality $u^2 = v^2$, we conclude that $1 + a^2 + ab + ba = 0$, which is impossible.

Suppose that G has the defining presentation (8) with $m = 1$. As we obtained before, either $r = -1$ or $r = -1 + 2^{n-1}$. By (1) we have

$$(1+b)(1+a) \equiv (1+a)(1+b) + (1+a)^2 \pmod{I_K^3(G)}$$

and we can write the elements u, v in the form (2). It follows from (3)–(6) that (11) hold by modulo $I_K^3(G)$.

We shall consider two cases depending on the values of r and m in (8).

Case 1. Let G be a quaternion group of order 8. Then by (11) we have

$$\begin{cases} uv \equiv (\alpha_1\beta_1 + \alpha_2\beta_1 + \alpha_2\beta_2)(1+a)^2 + \Delta(1+a)(1+b) \,(\operatorname{mod} I_K^3(G)); \\ vu \equiv (\alpha_1\beta_1 + \alpha_1\beta_2 + \alpha_2\beta_2)(1+a)^2 + \Delta(1+a)(1+b) \,(\operatorname{mod} I_K^3(G)); \\ u^2 \equiv (\alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2)(1+a)^2 \,(\operatorname{mod} I_K^3(G)); \\ v^2 \equiv (\beta_1^2 + \beta_1\beta_2 + \beta_2^2)(1+a)^2 \,(\operatorname{mod} I_K^3(G)). \end{cases}$$

Since the K-dimension of $I_K^j(G)/I_K^{j+1}(G)$ $(j = 1, \dots, 4)$ equals 2 and $uv \not\equiv vu \,(\operatorname{mod} I_K^3(G))$, we have $\alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2$ and $\beta_1^2 + \beta_1\beta_2 + \beta_2^2 = 0$. Using the fact that $\Delta \neq 0$, we establish $\frac{\alpha_1}{\alpha_2} = \omega$, $\frac{\beta_2}{\beta_1} = \omega^2$. Thus we observe that $u = \omega(1+a) + (1+b)$ and $v = (1+a) + \omega^2(1+b)$, where ω is a primitive cube root of the unity.

A simple calculation by modulos $I_K^4(G)$, $I_K^5(G)$ shows that

$$\{1, u, v, uv, vu, uvu, vuv, uvuv\}$$

is a filtered multiplicative K-basis for KG.

Case 2. Let G has a presentation

$$(12) \qquad \langle a, b \mid a^{2^n} = 1, \quad b^2 = a^{2^{n-1}}, \quad {}^b a = a^r \rangle$$

with $n > 2$. Then by (11) we have

$$\begin{cases} uv \equiv (\alpha_1 + \alpha_2)\beta_1(1+a)^2 + \Delta(1+a)(1+b) \,(\operatorname{mod} I_K^3(G)); \\ vu \equiv \alpha_1(\beta_1 + \beta_2)(1+a)^2 + \Delta(1+a)(1+b) \,(\operatorname{mod} I_K^3(G)); \\ u^2 \equiv \alpha_1(\alpha_1 + \alpha_2)(1+a)^2 \,(\operatorname{mod} I_K^3(G)); \\ v^2 \equiv \beta_1(\beta_1 + \beta_2)(1+a)^2 \,(\operatorname{mod} I_K^3(G)). \end{cases}$$

Since the K-dimension of $I_K^2(G)/I_K^3(G)$ equals 2 and $\Delta \neq 0$, we have either $\alpha_1 = \alpha_2 \neq 0$ and $\beta_1 = 0$ or $\alpha_1 = 0$ and $\beta_1 = \beta_2 \neq 0$. The second case is similar to first. Therefore, we can put $u = (1+a) + (1+b)$, $v = 1 + b$.

Case 2.1. Let $r = -1$ in (12). Then G is a generalized quaternion group. An easy calculation gives

$$(1+b)(1+a) = \sum_{j=1}^{2^n-1} (1+a)^j(1+b) + \sum_{j=1}^{2^n-1} (1+a)^{j+1}$$

and

$$u^2 = \sum_{j=1}^{2^n-1} (1+a)^{j+1}(1+b) + \sum_{j=1}^{2^n-1} (1+a)^{j+2} + (1+a)^{2^{n-1}},$$

$$uvu = \sum_{j=1}^{2^n-1} (1+a)^{j+1}(1+b) + \sum_{j=1}^{2^n-1} (1+a)^{j+2} + (1+a)^{2^{n-1}}(1+b).$$

Therefore, $u^2 \equiv uvu \,(\mathrm{mod}\, I_K^4(G))$, but $u^2, uvu \notin I_K^4(G)$ and

$$u^2 - uvu = (1+a)^{2^{n-1}} + (1+a)^{2^{n-1}}(1+b) \neq 0,$$

which contradicts property (II).

Case 2.2. Let $G = \langle a, b \mid a^{2^n} = 1,\ \ b^2 = a^{2^{n-1}},\ \ {}^b a = a^{-1+2^{n-1}} \rangle$. It is easy to see that $(ab)^2 = a^{2^{n-1}} b^2 = 1$ and

$$G \cong \langle a, ab \mid a^{2^n} = 1,\ \ (ab)^2 = 1,\ \ {}^{ab}a = a^{-1+2^{n-1}} \rangle,$$

which is a semidihedral group and, as we saw before, KG has no filtered multiplicative K-basis.

Thus our theorem is proved.

4. Example. Now we give an example of a nonmetacyclic 2-group with a filtered multiplicative basis.

Let $G = \langle a, b \mid a^4 = b^4 = 1,\ {}^b a = b^2 a^3,\ {}^a b = a^2 b^3,\ [a^2, b] = [b^2, a] = 1 \rangle$, a group of order 16, and let K be a field of characteristic 2. Then elements

$$\{1, u, v, uv, vu, v^2, uvu, uv^2, vuv, v^3, uvuv, uv^3, vuv^2, uvuv^2, vuv^3, uvuv^3 \mid$$
$$u = a + b, v = \mu_1 a + \mu_2 b + (\mu_1 + \mu_2) \text{ and } \mu_1, \mu_2 \in K, \text{ and } \mu_1 \neq \mu_2\}$$

form a filtered multiplicative K-basis for KG.

Indeed, by (1) we have

$$(1+b)(1+a) \equiv (1+a)(1+b) + (1+a)^2 + (1+b)^2 \,(\mathrm{mod}\, I_K^3(G))$$

and u, v be can writen in the form (2).

By a simple calculation modulo $I_K^3(G)$ we have

$$(13)\qquad \begin{cases} uv \equiv (\alpha_1 + \alpha_2)\beta_1(1+a)^2 + \Delta(1+a)(1+b) + \alpha_2(\beta_1+\beta_2)(1+b)^2; \\ vu \equiv \alpha_1(\beta_1+\beta_2)(1+a)^2 + \Delta(1+a)(1+b) + \beta_2(\alpha_1+\alpha_2)(1+b)^2; \\ u^2 \equiv (\alpha_1+\alpha_2)\alpha_1(1+a)^2 + \alpha_2(\alpha_1+\alpha_2)(1+b)^2; \\ v^2 \equiv (\beta_1+\beta_2)[\beta_1(1+a)^2 + \beta_2(1+b)^2]. \end{cases}$$

It is easy to see that K-dimension of $I_K^2(G)/I_K^3(G)$ equals 3 and $uv \not\equiv vu \,(\mathrm{mod}\, I_K^3(G))$, $uv \not\equiv u^2 \,(\mathrm{mod}\, I_K^3(G))$, $uv \not\equiv v^2 \,(\mathrm{mod}\, I_K^3(G))$, $vu \not\equiv u^2 \,(\mathrm{mod}\, I_K^3(G))$, $vu \not\equiv v^2 \,(\mathrm{mod}\, I_K^3(G))$.

We have the following two cases.

First let $u^2 \equiv v^2 \not\equiv 0 \,(\mathrm{mod}\, I_K^3(G))$. Then by (13) we have $\alpha_1^2 + \alpha_1\alpha_2 = \beta_1^2 + \beta_1\beta_2$ and $\alpha_2^2 + \alpha_1\alpha_2 = \beta_2^2 + \beta_1\beta_2$. It follows that $(\alpha_1+\alpha_2)^2 = (\beta_1+\beta_2)^2$ and $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$. Then by $u^2 \equiv v^2 \not\equiv 0 \,(\mathrm{mod}\, I_K^3(G))$ we have $\Delta = 0$, which is impossible.

Now let $u^2 \equiv 0 \,(\mathrm{mod}\, I_K^3(G))$ or $v^2 \equiv 0 \,(\mathrm{mod}\, I_K^3(G))$. It is easy to see that the second case is symmetric to the first one, so we consider only the first case. Then $\alpha_1 = \alpha_2 \neq 0$ and by (13) we have

$$\begin{cases} uv \equiv \lambda[(1+a)(1+b) + (1+b)^2] \,(\mathrm{mod}\, I_K^3(G)); \\ vu \equiv \lambda[(1+a)^2 + (1+a)(1+b)] \,(\mathrm{mod}\, I_K^3(G)); \\ v^2 \equiv \lambda[\beta_1(1+a)^2 + \beta_2(1+b)^2] \,(\mathrm{mod}\, I_K^3(G)), \end{cases}$$

where $\lambda = \beta_1 + \beta_2 \neq 0$. By a simple calculation modulo $I_K^4(G)$ we obtain

$$\begin{cases} uvu \equiv \lambda[(1+a)^3 + (1+a)^2(1+b) + (1+a)(1+b)^2 + (1+b)^3]; \\ uv^2 \equiv \lambda[\beta_1(1+a)^3 + \beta_1(1+a)^2(1+b) + \beta_2(1+a)(1+b)^2 + \beta_2(1+b)^3]; \\ vuv \equiv \lambda^2[(1+a)^2(1+b) + (1+a)(1+b)^2]; \\ v^3 \equiv \lambda[\beta_1^2(1+a)^3 + \beta_1\beta_2(1+a)^2(1+b) + \beta_1\beta_2(1+a)(1+b)^2 + \beta_2^2(1+b)^3] \end{cases}$$

and modulo $I_K^5(G)$

$$\begin{cases} uvuv \equiv \lambda^2[(1+a)^3(1+b) + (1+a)^2(1+b)^2 + (1+a)(1+b)^3]; \\ uv^3 \equiv \lambda^2[\beta_1(1+a)^3(1+b) + \beta_1(1+a)^2(1+b)^2 + \beta_2(1+a)(1+b)^3]; \\ vuv^2 \equiv \lambda^2[\beta_1(1+a)^3(1+b) + \beta_2(1+a)^2(1+b)^2 + \beta_2(1+a)(1+b)^3]. \end{cases}$$

Similarly

$$\begin{cases} uvuv^2 \equiv \lambda^2[(1+a)^3(1+b)^2 + (1+a)^2(1+b)^3] \pmod{I_K^6(G)}; \\ vuv^3 \equiv \lambda^2[\beta_1(1+a)^3(1+b)^2 + \beta_2(1+a)^2(1+b)^3] \pmod{I_K^6(G)}; \end{cases}$$

and $uvuv^3 \equiv \lambda^3(1+a)^3(1+b)^3 \pmod{I_K^7(G)}$.

Since the number of elements modulo $I_K^j(G)$, $(j = 2, \ldots, 6)$ equals the numbers of the K-dimension of $I_K^j(G)/I_K^{j+1}(G)$, we conclude that the elements $\{1, u, v, uv, vu, v^2, uvu, uv^2, vuv, v^3, uvuv, uv^3, vuv^2, uvuv^2, vuv^3, uvuv^3\}$ form a filtered multiplicative K-basis for KG.

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