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Uniformly differentiable bump functions

By

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Abstract. We present a construction of uniformly smooth norms from uniformly smooth bumb functions without making use of the Implicit Function Theorem.

Introduction. A bump function on a Banach space is a real valued function with bounded nonempty support. It is clear that if a space admits a norm with certain smoothness, then the space admits a Lipschitzian bump function with the same smoothness (cf. [1, I.2.1]). The converse is not true in general, in fact there is a Banach space which admits a Lipschitzian C^1-smooth bump function and admits no Gâteaux smooth norm (cf. [1, Chapter VII]). However, if X is separable, the existence of Fréchet differentiable bump function on X implies that X admits a Fréchet differentiable norm (cf. [1, II.5.3]). Furthermore, it is shown in [3] that if a space X admits a bump function that is uniformly Fréchet differentiable, then X admits a norm with the same kind of smoothness. The proofs of the above theorems made use of the Implicit Function Theorem for Fréchet differentiability. Unfortunately, the Implicit Function Theorem does not seem to work in the uniformly Gâteaux case. In this article, we make use of the duality between uniform smoothness and uniform convexity to construct uniformly Gâteaux differentiable norms from uniformly Gâteaux differentiable bump functions. Our construction also works for the uniformly Fréchet case, and all the intermediate cases.

Let $(X, \|\cdot\|)$ be a Banach space. The unit sphere $S_X\,(\|\cdot\|)$ of X is the set $\{x \in X : \| x \| = 1\}$. Let f be a real valued function on X. We say that f is Gâteaux differentiable at $x \in X$ if there is $l_x \in X^*$ such that

$$(*) \qquad \lim_{t \to 0} \frac{f(x+th) - f(x)}{t} = l_x(h)$$

for every $h \in S_X(\|\cdot\|)$. If moreover, this limit is uniform for $h \in S_X(\|\cdot\|)$, we say that f is Fréchet differentiable at x. We call l_x the Gâteaux or Fréchet derivative of f at x. The function f is said to be Gâteaux (Fréchet) differentiable if it is Gâteaux (Fréchet) differentiable at every $x \in X$. If for every $h \in S_X\,(\|\cdot\|)$, the limit in $(*)$ is uniform in $x \in X$, we call the function f uniformly Gâteaux differentiable (UG). Suppose that f is a norm equivalent to $\|\cdot\|$, we say that the norm f is Gâteaux differentiable if it is Gâteaux differentiable at every point on its unit sphere. If for every $h \in S_X\,(\|\cdot\|)$, the limit in $(*)$ is uniform in $x \in S_X$, we say that f is a UG norm.

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We recall that a dual norm $\|\cdot\|$ is said to be *weak*-uniformly rotund* (W*UR) if $x_n - y_n \to 0$ in the w^*-topology whenever $x_n, y_n \in S_{X^*}(\|\cdot\|)$ and $\left\|\frac{x_n + y_n}{2}\right\| \to 1$. It is a well known result that a dual norm is W*UR if and only if its predual norm is unformly Gâteaux differentiable (cf., for instance, [1, II.6.7]). Given a convex function f defined on X, f^* denotes the Fenchel dual (or conjugate) of f, i.e. $f^*(x^*) = \sup\{(x^*, x) - f(x) : x \in X\}$, for $x^* \in X^*$. We refer the readers to [1] and [6] for all other unexplained results and terminologies used in this note.

The main result of this article is the following theorem.

Theorem 1. *Suppose a Banach space X admits a UG bump function b, then it admits a UG norm.*

We need the following lemma in the proof of this theorem.

Lemma 2. *Suppose X admits a convex function f such that $\frac{1}{2}\|x\|^2 \leqq f(x) \leqq M\|x\|^2 + c$ for some constants $M > 0$ and $c > 0$ and such that f is UG on bounded sets, i.e., given $x \in S_X$, $\varepsilon > 0$ and $m > 0$, there exists a $\delta > 0$ such that*

$$f(y + tx) - f(y - tx) - 2f(y) < \varepsilon|t|,$$

whenever $|t| < \delta$ and $y \in mB_X$, then X admits a UG norm.

Proof. We begin the proof by constructing a dual W*UR norm on X^*. Without loss of generality, we may assume that the function f is symmetric and $f > 0$. Consider the dual function $g = f^*$. Clearly

$$\frac{1}{4M}\|x^*\|^2 - c \leqq g(x^*) \leqq \frac{1}{2}\|x^*\|^2. \tag{1}$$

Let $\{x_n^*\}$ and $\{y_n^*\}$ be two sequences in X^* such that $g(x_n^*) = g(y_n^*) = 0$ for all n, and $g\left(\frac{x_n^* + y_n^*}{2}\right) \longrightarrow 0$ as $n \to \infty$.

We shall see that $x_n^* - y_n^* \to 0$ in the w^*-topology. To this end, let $x_n \in X$ be such that $\left(x_n, \frac{x_n^* + y_n^*}{2}\right) - f(x_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore we have

$$0 = \frac{g(x_n^*) + g(y_n^*)}{2} \geqq \frac{1}{2}[(x_n, x_n^*) - f(x_n)] + \frac{1}{2}[(x_n, y_n^*) - f(x_n)] \longrightarrow 0.$$

Hence,

$$\begin{aligned} &(x_n, y_n^*) - f(x_n) \longrightarrow 0 = g(y_n^*), \quad \text{and} \\ &(x_n, x_n^*) - f(x_n) \longrightarrow 0 = g(x_n^*). \end{aligned} \tag{2}$$

Claim. $\{x_n\}$ is bounded.

Proof of claim. Since $g(x_n^*) = 0$, we have $\|x_n^*\|^2 \leqq 4Mc$ by (1). Therefore $\|x_n^*\| \leqq 2\sqrt{Mc} = K$. From (2)

$$(x_n, x_n^*) - f(x_n) > -\frac{3}{4}, \text{ for large } n.$$

Therefore $f(x_n) \leqq (x_n, x_n^*) + \frac{3}{4} \leqq K\|x_n\| + \frac{3}{4}$.

But $f(x_n) \geqq \frac{1}{2}\|x_n\|^2$. Therefore $\frac{1}{2}\|x_n\|^2 - K\|x_n\| + \frac{3}{4} \leqq 0$. Hence $\|x_n\| \leqq \lambda$ for some constant λ for large n (end of proof of claim).

To see $x_n^* - y_n^* \longrightarrow 0$ in the w^*-topology, let $x \in S_X$, $\varepsilon > 0$ be given. Since f is uniformly Gâteaux differentiable on λB_X, there is a $\delta > 0$ such that

$$f(y + tx) + f(y - tx) - 2f(y) \leqq \varepsilon|t|,$$

for all $|t| < \delta$ and $y \in \lambda B_X$. Therefore we have

$$f(x_n + tx) + f(x_n - tx) - 2f(x_n) \leqq \varepsilon|t|.$$

Hence,

$$(x_n + tx, x_n^*) - g(x_n^*) + (x_n - tx, y_n^*) - g(y_n^*) - 2f(x_n) \leqq \varepsilon|t|.$$

Therefore

$$[(x_n, x_n^*) - g(x_n^*) - f(x_n)] + [(x_n, y_n^*) - g(y_n^*) - f(x_n)] + t(x, x_n^* - y_n^*) \leqq \varepsilon|t|.$$

According to (2) there exists $n_0 \in \mathbb{N}$ such that for all $n \geqq n_0$,

$$|(x_n, x_n^*) - g(x_n^*) - f(x_n)| + |(x_n, y_n^*) - g(y_n^*) - f(x_n)| \leqq \varepsilon\delta.$$

Consequently, for all $n \geqq n_0$, $t(x, x_n^* - y_n^*) \leqq 2\varepsilon\delta$ for all $|t| \leqq \delta$. Which implies $(x, x_n^* - y_n^*) \leqq 2\varepsilon$. Similarly, we can show that $(x, y_n^* - x_n^*) \leqq 2\varepsilon$. Therefore $x_n^* - y_n^*$ converges to zero in the w^* topology.

Now let $B = \{x^* : g(x^*) \leqq 0\}$ and $|||\cdot|||$ be Minkowski's functional of B. The set B is w^*-compact and contains nonempty interior. Thus the norm $|||\cdot|||$ is an equivalent dual norm on X^*.

It remains to verify that the norm $|||\cdot|||$ is W*UR. Let u_n^* and v_n^* be points on $S_{X^*}(|||\cdot|||)$ such that $|||w_n^*||| \to 1$, where $w_n^* = \dfrac{u_n^* + v_n^*}{2}$. It is clear that $g(u_n^*) = g(v_n^*) = 0$ for all $n \in \mathbb{N}$. Moreover, $w_n^* - \dfrac{w_n^*}{|||w_n^*|||} \to 0$, since $|||w_n^*||| \to 1$. By the uniform continuity of g, we have $g(w_n^*) \to 0$. By the property exhibited by g as seen above, we have that $u_n^* - v_n^*$ converges to zero in the w^*-topology. Therefore $|||\cdot|||$ is a W*UR norm and it's predual norm $|||\cdot|||_*$ is UG. □

Proof of Theorem 1. We follow the arguments as in [5]. Let b be a bounded uniformly Gâteaux differentiable bump function. According to [5, 2.1(a)], we may assume that b satisfies the following properties:

(1) $b(x) = 1$ if $\|x\| \leqq 1/3$, and

(2) $b(x) = 0$ if $\|x\| \geqq 1/2$.

Let $\varphi(x) = \sum_{n=1}^{\infty} 3^n\left[1 - b\left(\frac{x}{3^n}\right)\right]$. It is shown in [5, 2.2] that

$$\frac{1}{3}(\|x\| - 3) \leqq \varphi(x) \leqq 9(\|x\| + 1) \quad \text{for all} \quad x \in X,$$

and φ is uniformly Gâteaux differentiable.

The convex function ψ of φ defined by

$$\psi(x) = \inf\left\{\sum_{i=1}^{n} \alpha_i\varphi(x_i) : x = \sum_{i=1}^{n} \alpha_i x_i,\ \alpha_i \geqq 0,\ \sum_{i=1}^{n} \alpha_i = 1, n \in \mathbb{N}\right\}$$

satisfies

$$\frac{1}{3}(\|x\| - 3) \leqq \psi(x) \leqq \varphi(x) \leqq 9(\|x\| + 1) \quad \text{for all } \in X,$$

Furthermore, ψ is uniformly Gâteaux differentiable (cf, [5] or [1, V.3.2]). An appropriate multiple and translate of the function ψ^2 will satisfy the hypothesis of the convex function in Lemma 2. □

Remarks.

1. If X admits a UG norm, then by [4, 2.4], every convex function bounded on bounded sets can be approximated uniformly on bounded sets by UG convex functions. Therefore Theorem 1 generalises [5, Theorem 2.5].

2. Our construction presented above is still valid for the UF case and all the intermediate cases. Therefore, by the same construction, we obtain:

Theorem 3 [3, Theorem 3.2]. *Suppose a Banach space X admits a UF bump function b, then it admits a UF norm.*

We would now show an approximation theorem for UG and UF convex functions. We shall make use of the following lemma.

Lemma 4 [5]. *Let f and g be continuous convex functions defined on X. Suppose g is UG (UF) so is $f \square g$, where $f \square g(x) := \inf\{f(y) + g(x-y) : y \in X\}$ is called the infimal convolution of f and g.*

Proposition 5. *Let ϕ be a convex function defined on X that is bounded on bounded sets. Suppose ϕ is UG (UF), then every convex function $f \leqq \phi$ can be approximated uniformly on bounded sets by UG (UF) convex functions.*

Proof. Without loss of generality, we assume $\phi^* \geqq 0$. By our assumption, $\phi^* \leqq f^*$, therefore we have $\text{dom} f^* \subset \text{dom}\phi^*$. Let $g_n = f \square \left(\left(\frac{\phi^*}{n} \right)^*_{\restriction X} \right)$. By Lemma 5, g_n is uniformly smooth for each $n \in \mathbb{N}$.

We have to show that g_n converges to f uniformly on bounded sets, we first note that $g_n^* = f^* + \frac{\phi^*}{n} \geqq f^*$, therefore $g_n \leqq f$. Given $\varepsilon > 0$, $\lambda \in \mathbb{R}$, let $F = \sup_{x \in \lambda B_X} |f(x)|$ and $x_0 \in \lambda B_X$. Let $x^* \in \partial f(x_0)$, then $\|x^*\| \leqq M$, the Lipschitz constant of f on λB_X. By a property of subdifferentials, we have

$$f(x_0) = (x^*, x_0) - f^*(x^*). \tag{3}$$

Therefore

$$f(x_0) = (x^*, x_0) - g_n^*(x^*) + \frac{\phi^*(x^*)}{n} \leqq g_n(x_0) + \frac{\phi^*(x^*)}{n}.$$

Furthermore, (3) yields

$$f^*(x^*) = (x^*, x_0) - f(x_0) \leqq \|x^*\| \|x_0\| + |f(x_0)| \leqq K\lambda + F =: Z.$$

Consequently, $\phi^*(x^*) \leqq Z$ (as $f \leqq \phi$). Therefore for all $n \geqq \frac{Z}{\varepsilon}$, we have $g_n(x_0) \leqq f(x_0) \leqq g_n(x_0) + 2\varepsilon$ for all $x_0 \in \lambda B_X$. □

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