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Locally finite varieties

GURAM BEZHANISHVILI

Abstract. In this paper we present a new and useful criterion for a variety to be locally finite. Many examples are given to justify the effectiveness of the criterion.

1. Introduction

The only characterization of locally finite varieties which can be found in a textbook of universal algebra states that a given variety $\mathcal{V}$ is locally finite iff all finitely generated free $\mathcal{V}$-algebras are finite. No other more effective criteria for a variety to be locally finite have been known. A. Malcev was the first who partly filled in this gap. He introduced the notions of *locally finite algebras* and *uniformly locally finite classes of algebras*, and proved that in order for a variety $\mathcal{V}$ of a finite signature to be locally finite it is necessery and sufficient that $\mathcal{V}$ be generated by a uniformly locally finite class (see Malcev [24]). However, several unsolved questions still remained: How to generalise Malcev's theorem to an arbitrary variety (not necesserily of a finite signature)? Is it possible to find out more effective criterion for a variety to be locally finite? This paper is devoted to answer these and other related questions.

The paper is organized as follows. §2 has an auxiliary purpose. In it we introduce the notions of a *regularly locally finite class (in the weak sense)* and a *uniformly locally finite class (in the weak sense)*, which will play a central role in §3. We prove that every regularly locally finite class (in the weak sense) is also uniformly locally finite (in the weak sense), and that the converse holds only if we deal with the classes of algebras of a finite signature. §3 is the core of the paper. In it we prove that the operations **H** and **S** preserve local finiteness and give a necessery and sufficient condition for **P** to preserve local finiteness[1]. We extend Malcev's theorem to any variety (not necesserily of a finite signature), and also present more effective criterion for a variety to be locally finite. Moreover, several useful consequences of these results are established. Finally, in §4 we consider many (known and new) examples of locally finite varieties and show how effectively the criterion works.

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[1] Here and below **H**, **S** and **P** denote the operations of taking homomorphic images, extracting subalgebras and forming direct products.

2. Regular and uniform local finiteness

A universal algebra $\mathcal{A} = (A, \{f_i\}_{i \in I})$ is said to be *locally finite* if every finitely generated subalgebra of $\mathcal{A}$ is finite (cf. Malcev [24]). A class $\mathcal{K}$ of (the same type) universal algebras is called *locally finite* if every algebra from $\mathcal{K}$ is locally finite (cf. Malcev [24]). Below we consider only the classes of universal algebras of the same type.

If a class $\mathcal{K}$ is locally finite, then obviously every finitely generated algebra from $\mathcal{K}$ is finite. Conversely, we have that if every finitely generated algebra from $\mathcal{K}$ is finite and $\mathbf{S}(\mathcal{K}) \subseteq \mathcal{K}$ (in particular, if $\mathcal{K}$ forms a variety), then $\mathcal{K}$ is locally finite. However, in general there are classes whose finitely generated algebras are finite, but they are not locally finite. A witness is a class which contains a single universal algebra $\mathcal{A}$ which is neither locally finite nor finitely generated. Therefore, it is natural to call the classes whose all finitely generated algebras are finite *locally finite in the weak sense*.

DEFINITION 2.1. a) A class $\mathcal{K}$ is said to be *regularly locally finite* if $\mathcal{K}$ is locally finite and for any $n \in \omega$ there exist only finitely many nonisomorphic n-generated subalgebras of algebras from $\mathcal{K}$.

b) A class $\mathcal{K}$ is called *regularly locally finite in the weak sense* if $\mathcal{K}$ is locally finite in the weak sense and for any $n \in \omega$ there exist only finitely many nonisomorphic n-generated algebras from $\mathcal{K}$.

c) (See Malcev [24]) A class $\mathcal{K}$ is said to be *uniformly locally finite* if for any $n \in \omega$ there exists $m(n) \in \omega$ such that the cardinality of every n-generated subalgebra of an algebra from $\mathcal{K}$ is less than or equal to $m(n)$.

d) A class $\mathcal{K}$ is called *uniformly locally finite in the weak sense* if for any $n \in \omega$ there exists $m(n) \in \omega$ such that the cardinality of every n-generated algebra from $\mathcal{K}$ is less than or equal to $m(n)$.

It is obvious that if $\mathcal{K}$ is regularly locally finite, then $\mathcal{K}$ is uniformly locally finite as well. In the same way, if $\mathcal{K}$ is regularly locally finite in the weak sense, then $\mathcal{K}$ is uniformly locally finite in the weak sense.

Conversely, if the signature of algebras from $\mathcal{K}$ is finite, then uniform local finiteness of $\mathcal{K}$ implies regular local finiteness of $\mathcal{K}$. Indeed, first note that if $\mathcal{K}$ is uniformly locally finite, then all n-generated algebras from $\mathbf{S}(\mathcal{K})$ are finite. Further, since a finite set gives rise only to finitely many nonisomorphic algebras of a given finite signature, any set whose cardinality is less than or equal to $m(n) \in \omega$ gives rise only to finitely many nonisomorphic algebras of the signature of $\mathcal{K}$. Now since we have only finitely many nonequivalent sets of the cardinality less than or equal to $m(n)$, there exist only finitely many nonisomorphic algebras of the signature of $\mathcal{K}$ whose cardinalities are less than or equal to $m(n)$. Hence, there are only finitely many nonisomorphic n-generated subalgebras of algebras from $\mathcal{K}$.

Analogously we can prove that if the signature of algebras from $\mathcal{K}$ is finite and $\mathcal{K}$ is uniformly locally finite in the weak sense, then $\mathcal{K}$ is also regularly locally finite in the weak sense.

On the other hand, the example presented in Page 361 of Section 14.1 of Chapter VI of Malcev [24] (in English translation it is Page 286) serves as an example of a uniformly locally finite class of algebras of infinite signature which is not regularly locally finite. It also serves as an example of a class of algebras which is uniformly locally finite in the weak sense, but is not regularly locally finite in the weak sense.

It is evident that regular local finiteness is much stronger condition than local finiteness. However, if all n-generated free algebras $\mathcal{F}(n)$ $(n \in \omega)$ of the variety $\mathbf{Var}(\mathcal{K}) = \mathbf{HSP}(\mathcal{K})$ belong to $\mathbf{S}(\mathcal{K})$ (in particular, if $\mathcal{K}$ forms a variety), then local finiteness of $\mathcal{K}$ is equivalent to regular local finiteness of $\mathcal{K}$. Indeed, if $\mathcal{F}(n) \in \mathcal{K}$ and $\mathcal{K}$ is locally finite, then since all n-generated algebras $\mathcal{A}(n)$ from $\mathbf{Var}(\mathcal{K})$ and, in particular, all n-generated algebras from $\mathbf{S}(\mathcal{K})$ are homomorphic images of $\mathcal{F}(n)$, all of them are finite. Moreover, since every $\mathcal{A}(n)$ is isomorphic to $\mathcal{F}(n)/\theta$ for a suitable congruence θ of $\mathcal{F}(n)$, and $\mathcal{F}(n)$ has only a finite number of different congruences (for $\mathcal{F}(n)$ is finite itself), there is only a finite number of nonisomorphic n-generated subalgebras of algebras from $\mathcal{K}$, and $\mathcal{K}$ is regularly locally finite.

In the same way, if $\mathcal{F}(n) \in \mathcal{K}$ for every $n \in \omega$, then local finiteness of $\mathcal{K}$ in the weak sense is equivalent to regular local finiteness of $\mathcal{K}$ in the weak sense. Needless to say that if $\mathcal{K}$ has a finite signature and $\mathcal{F}(n) \in \mathbf{S}(\mathcal{K})$ ($\mathcal{F}(n) \in \mathcal{K}$) for every $n \in \omega$, then local finiteness of $\mathcal{K}$ (in the weak sense) is equivalent to uniform local finiteness of $\mathcal{K}$ (in the weak sense).

3. The criterion

DEFINITION 3.1. Let $\{\mathcal{A}_i : i \in I\}$ be a family of (the same type) universal algebras, $\mathcal{A} \in \mathbf{S}(\prod_{i \in I} \mathcal{A}_i)$ and $\pi_i : \prod_{i \in I} \mathcal{A}_i \to \mathcal{A}_i$ denote the i-th projection. $\{\mathcal{A}_i : i \in I\}$ is said to be *regularly (uniformly) locally finite with respect to* $\mathcal{A}$ if the family $\{\pi_i(\mathcal{A}) : i \in I\}$ is regularly (uniformly) locally finite.

It is obvious that if the family $\{\mathcal{A}_i : i \in I\}$ is regularly (uniformly) locally finite, then $\{\mathcal{A}_i : i \in I\}$ is regularly (uniformly) locally finite with respect to any $\mathcal{A} \in \mathbf{S}(\prod_{i \in I} \mathcal{A}_i)$, and that if $\mathcal{A}$ is a subdirect product of $\{\mathcal{A}_i : i \in I\}$ (in particular, if $\mathcal{A} = \prod_{i \in I} \mathcal{A}_i$), then $\{\mathcal{A}_i : i \in I\}$ is regularly (uniformly) locally finite iff $\{\mathcal{A}_i : i \in I\}$ is regularly (uniformly) locally finite with respect to $\mathcal{A}$.

It is also worth mentioning that for a finitely generated $\mathcal{A} \in \mathbf{S}(\prod_{i \in I} \mathcal{A}_i)$, we have that $\{\pi_i(\mathcal{A}) : i \in I\}$ is regularly (uniformly) locally finite iff $\{\pi_i(\mathcal{A}) : i \in I\}$ is regularly (uniformly) locally finite in the weak sense. Let us prove this statement for a regularly locally finite class. The case of a uniformly locally finite class is proved analogously. Suppose $\mathcal{A}$ is n-generated and $\{\pi_i(\mathcal{A}) : i \in I\}$ is regularly locally finite in the weak sense. Then all $\pi_i(\mathcal{A})$ $(i \in I)$ are also n-generated. Therefore, all of them are finite and there exists only a finite number of nonisomorphic $\pi_i(\mathcal{A})$ $(i \in I)$. But then the cardinality of every subalgebra of an algebra $\pi_i(\mathcal{A})$ $(i \in I)$ is also finite and there exists only a finite number of nonisomorphic subalgebras of algebras $\pi_i(\mathcal{A})$ $(i \in I)$.

THEOREM 3.2. a) *If $\mathcal{A}_1$ is locally finite and $\mathcal{A}_2 \in \mathbf{S}(\mathcal{A}_1)$, then $\mathcal{A}_2$ is also locally finite.*

b) *If* $\mathcal{A}_1$ *is locally finite and* $\mathcal{A}_2 \in \mathbf{H}(\mathcal{A}_1)$*, then* $\mathcal{A}_2$ *is locally finite as well.*

c) *If* $\mathcal{A} = \prod_{i\in I} \mathcal{A}_i$*, then* $\mathcal{A}$ *is locally finite iff* $\{\mathcal{A}_i : i \in I\}$ *is regularly locally finite with respect to any n-generated subalgebra* $\mathcal{A}(n)$ *of* $\mathcal{A}$*. (In particular, if* $\{\mathcal{A}_i : i \in I\}$ *is regularly locally finite, then* $\mathcal{A}$ *is locally finite.)*

d) *If the signature of algebras* $\mathcal{A}_i$ $(i \in I)$ *is finite, then* $\mathcal{A} = \prod_{i\in I} \mathcal{A}_i$ *is locally finite iff* $\{\mathcal{A}_i : i \in I\}$ *is uniformly locally finite with respect to any n-generated subalgebra* $\mathcal{A}(n)$ *of* $\mathcal{A}$*. (In particular, if* $\{\mathcal{A}_i : i \in I\}$ *is uniformly locally finite, then* $\mathcal{A}$ *is locally finite.)*

Proof. a) Since $\mathcal{A}_2$ is a subalgebra of $\mathcal{A}_1$, then for every n-generated subalgebra $\mathcal{A}(n)$ of $\mathcal{A}_2$, $\mathcal{A}(n)$ is also a subalgebra of $\mathcal{A}_1$. Now since $\mathcal{A}_1$ is locally finite, $\mathcal{A}(n)$ is finite. Hence, $\mathcal{A}_2$ is also locally finite.

b) Suppose $\mathcal{A}_2 \in \mathbf{H}(\mathcal{A}_1)$ and $\mathcal{A}_2[g_1,\ldots,g_n]$ is an n-generated subalgebra of $\mathcal{A}_2$. Denote by h a homomorphism from $\mathcal{A}_1$ onto $\mathcal{A}_2$. Then we have $h^{-1}(g_i) \neq \emptyset$ for every $i \in [1,n]$. Suppose $p_i \in h^{-1}(g_i)$ and show that $\mathcal{A}_2[g_1,\ldots,g_n] \in \mathbf{H}(\mathcal{A}_1[p_1,\ldots,p_n])$. Indeed, for every $a \in \mathcal{A}_1[p_1,\ldots,p_n]$ we have that there exists an n-placed term t such that $a = t(p_1,\ldots,p_n)$. But then $h(a) = h(t(p_1,\ldots,p_n)) = t(h(p_1),\ldots,h(p_n)) = t(g_1,\ldots,g_n) \in \mathcal{A}_2[g_1,\ldots,g_n]$. Hence, the restriction of h to $\mathcal{A}_1[p_1,\ldots,p_n]$ is a homomorphism from $\mathcal{A}_1[p_1,\ldots,p_n]$ into $\mathcal{A}_2[g_1,\ldots,g_n]$. Let us prove that it is an onto homomorphism. For every $b \in \mathcal{A}_2[g_1,\ldots,g_n]$ we have $b = t'(g_1,\ldots,g_n) = t'(h(p_1),\ldots,h(p_n)) = h(t'(p_1,\ldots,p_n))$ and hence $(h|_{\mathcal{A}_1[p_1,\ldots,p_n]})^{-1}(b) \neq \emptyset$. Therefore, $\mathcal{A}_2[g_1,\ldots,g_n] \in \mathbf{H}(\mathcal{A}_1[p_1,\ldots,p_n])$. Now since $\mathcal{A}_1$ is locally finite, $\mathcal{A}_1[p_1,\ldots,p_n]$ is finite and hence $\mathcal{A}_2[g_1,\ldots,g_n]$ is finite too. Therefore, $\mathcal{A}_2$ is locally finite.

c) If $\mathcal{A} = \prod_{i\in I} \mathcal{A}_i$ is locally finite, then for any $n \in \omega$ and for any n-generated subalgebra $\mathcal{A}(n)$ of $\mathcal{A}$, all $\pi_i(\mathcal{A}(n))$ are finite (for all of them are homomorphic images of $\mathcal{A}(n)$). Moreover, since all homomorphic images of $\mathcal{A}(n)$ are determined by the congruences of $\mathcal{A}(n)$, and since the number of congruences of $\mathcal{A}(n)$ is finite (for $\mathcal{A}(n)$ is finite itself), there are only finitely many nonisomorphic $\pi_i(\mathcal{A}(n))$, which means that the family $\{\pi_i(\mathcal{A}(n)) : i \in I\}$ is regularly locally finite (in the weak sense).

Conversely, suppose $\{\mathcal{A}_i : i \in I\}$ is regularly locally finite with respect to any $\mathcal{A}(n)$. We shall prove that all $\mathcal{A}(n)$ are finite for any $n \in \omega$. Indeed, for any $\mathcal{A}(n)$ we have that $\mathcal{A}(n)$ is embedded into $\prod_{i\in I} \pi_i(\mathcal{A}(n))$. For each $i \in I$, let θ_i be the kernel of the homomorphism $\mathcal{A}(n)$ onto $\pi_i(\mathcal{A}(n))$. Let us regard two elements i and j of I as equivalent if $\theta_i = \theta_j$. This equivalence relation partitions I. Let J be a transversal of this partition. So J is a subset of I and it has exactly one element from each equivalence class. Now the family $\{\theta_j : j \in J\}$ seperates the points of $\mathcal{A}(n)$. This means that $\mathcal{A}(n)$ is embeddable into $\prod_{j\in J} \pi_j(\mathcal{A}(n))$ and the θ_j's will be the kernels of the underlying system of homomorphisms. Therefore, $|\mathcal{A}(n)| \leq |\prod_{j\in J} \mathcal{A}(n)/\theta_j|$ and we only need to show that $|\prod_{j\in J} \mathcal{A}(n)/\theta_j|$ is finite. Since all $\mathcal{A}(n)/\theta_j$ are n-generated, all of them are finite. Therefore, it remains to show that J is finite itself. But since the family $\{\pi_i(\mathcal{A}(n)) : i \in I\}$ is regularly locally finite, the number of nonisomorphic $\mathcal{A}(n)/\theta_j$ is finite too. Hence it remains to show that for every $j \in J$, the set $J_j = \{k \in J : \mathcal{A}(n)/\theta_j \simeq \mathcal{A}(n)/\theta_k\}$ is finite as well. For this we need two additional lemmas:

LEMMA 3.3. *For any universal algebra $\mathcal{A}$ and any two congruence relations θ_i and θ_j of $\mathcal{A}$ the following two conditions are equivalent:*

1. *There exists an isomorphism $\psi : \mathcal{A}/\theta_j \to \mathcal{A}/\theta_i$ such that $\psi \circ h_j = h_i$, where $h_j : \mathcal{A}(n) \to \mathcal{A}(n)/\theta_j$ and $h_i : \mathcal{A}(n) \to \mathcal{A}(n)/\theta_i$ are canonical morphisms;*
2. *$\theta_i = \theta_j$.*

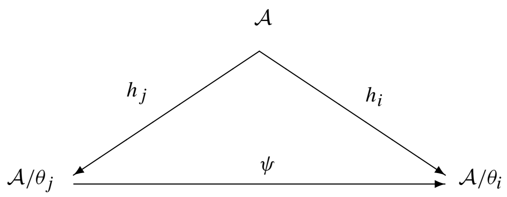

Figure 1

Proof. Suppose $\theta_i = \theta_j$. Then it is obvious that the identical morphism $\psi : \mathcal{A}/\theta_j \to \mathcal{A}/\theta_i$ is an isomorphism and $\psi \circ h_j = h_i$. Conversely, suppose $\theta_i \neq \theta_j$. Then either $\theta_i \not\subseteq \theta_j$ or $\theta_j \not\subseteq \theta_i$. For certainty suppose $\theta_i \not\subseteq \theta_j$. Then there exist $a, b \in A$ such that $(a, b) \in \theta_i$ and $(a, b) \notin \theta_j$. But then $h_i(a) = h_i(b)$ and $h_j(a) \neq h_j(b)$. Now for any isomorphism $\psi : \mathcal{A}/\theta_j \to \mathcal{A}/\theta_i$, if $\psi \circ h_j = h_i$ then $\psi \circ h_j(a) = h_i(a) = h_i(b) = \psi \circ h_j(b)$, which contradicts the fact that ψ is injective. Hence, $\psi \circ h_j \neq h_i$. $\square$

LEMMA 3.4. *Suppose $\mathcal{A}$ is a finitely generated universal algebra and $g_1, \ldots, g_n$ are generators of $\mathcal{A}$. Let also $h_{\mathcal{B}}$ and $h_{\mathcal{C}}$ be homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ and $\mathcal{C}$ respectively, and ψ be such a homomorphism from $\mathcal{C}$ to $\mathcal{B}$ that $\psi \circ h_{\mathcal{C}} \neq h_{\mathcal{B}}$. Then there exists $i \in [1, n]$ such that $\psi \circ h_{\mathcal{C}}(g_i) \neq h_{\mathcal{B}}(g_i)$.*

Proof. Suppose $\psi \circ h_{\mathcal{C}}(g_i) = h_{\mathcal{B}}(g_i)$ for any $i \in [1, n]$. Then for any $a \in A$ we have $h_{\mathcal{B}}(a) = h_{\mathcal{B}}(t(g_1, \ldots, g_n)) = t(h_{\mathcal{B}}(g_1), \ldots, h_{\mathcal{B}}(g_n)) = t(\psi \circ h_{\mathcal{C}}(g_1), \ldots, \psi \circ h_{\mathcal{C}}(g_n)) = \psi \circ h_{\mathcal{C}}(t(g_1, \ldots, g_n)) = \psi \circ h_{\mathcal{C}}(a)$, which contradicts our assumption. Hence, there exists $i \in [1, n]$ such that $\psi \circ h_{\mathcal{C}}(g_i) \neq h_{\mathcal{B}}(g_i)$. $\square$

Now let us return to the proof of Theorem 3.2. Suppose $k \in J_j$ and $k \neq j$. Then Lemma 3.3 implies that $\psi \circ h_k \neq h_j$. From Lemma 3.4 it follows that there exists $m \in [1, n]$ such that $\psi \circ h_k(g_m) \neq h_j(g_m)$, and hence $(\psi \circ h_k(g_1), \ldots, \psi \circ h_k(g_n)) \neq (h_j$

$(g_1), \ldots, h_j(g_n))$. Now since both $(\psi \circ h_k(g_1), \ldots, \psi \circ h_k(g_n))$ and $(h_j(g_1), \ldots, h_j(g_n))$ belong to $(\mathcal{A}(n)/\theta_j)^n$, and since $(\mathcal{A}(n)/\theta_j)^n$ is finite (for $\mathcal{A}(n)/\theta_j$ is finite itself), there exist only finitely many different n-tuples, and hence, there exist only finitely many ks which differ from j. Therefore, J_j is finite for any $j \in J$. As a result we obtain that $\mathcal{A}(n)$ is finite for any $n \in \omega$.

(It is obvious that if $\{\mathcal{A}_i : i \in I\}$ is regularly locally finite, then $\{\pi_i(\mathcal{A}(n)) : i \in I\}$ is regularly locally finite (in the weak sense) for any $\mathcal{A}(n) \in \mathbf{S}(\prod_{i \in I} \mathcal{A}_i)$. But then the above arguments imply that $\mathcal{A} = \prod_{i \in I} \mathcal{A}_i$ is locally finite; though regular local finiteness of $\{\mathcal{A}_i : i \in I\}$ does not follow from local finiteness of $\mathcal{A}$.)

d) follows from c) and the fact that for any family of algebras of a finite signature the notions of regular local finiteness (in the weak sense) and uniform local finiteness (in the weak sense) coincide. □

For a class $\mathcal{K}$ of universal algebras we denote by $\mathbf{P_S}(\mathcal{K})$ the class of all subdirect products of algebras from $\mathcal{K}$. A family $\{\mathcal{A}_i : i \in I\}$ of subdirectly irreducible algebras is said to be a *subdirect decomposition* or a *subdirect representation* of $\mathcal{A}$, if each $\mathcal{A}_i$ is a factor of $\mathcal{A}$ and $\mathcal{A} \in \mathbf{P_S}\{\mathcal{A}_i : i \in I\}$. In other words, it means that there is a system $\{h_i\}_{i \in I}$ of homomorphisms such that

1. h_i maps $\mathcal{A}$ onto $\mathcal{A}_i$ for each $i \in I$;
2. The system $\{h_i\}_{i \in I}$ separates the points of $\mathcal{A}$.

As a direct consequence of Theorem 3.2 we get the following

THEOREM 3.5. *Let $\{\mathcal{A}_i : i \in I\}$ be a subdirect decomposition of $\mathcal{A}$ into subdirectly irreducible factors. Then*

1) *$\mathcal{A}$ is locally finite iff the class $\{\mathcal{A}_i : i \in I\}$ is regularly locally finite with respect to all n-generated subalgebras of $\mathcal{A}$. In particular, if $\{\mathcal{A}_i : i \in I\}$ is regularly locally finite, then $\mathcal{A}$ is locally finite.*
2) *If the signature of $\mathcal{A}$ is finite, then $\mathcal{A}$ is locally finite iff the class $\{\mathcal{A}_i : i \in I\}$ is uniformly locally finite with respect to all n-generated subalgebras of $\mathcal{A}$. In particular, if $\{\mathcal{A}_i : i \in I\}$ is uniformly locally finite, then $\mathcal{A}$ is locally finite.*

Another immediate consequence of Theorem 3.2 is the following

COROLLARY 3.6. 1) *If $\mathcal{K}$ is locally finite, then the classes $\mathbf{S}(\mathcal{K})$ and $\mathbf{H}(\mathcal{K})$ are also locally finite.*
2) *If $\mathcal{K}$ is regularly locally finite (in the weak sense), then $\mathbf{P}(\mathcal{K})$ is locally finite (in the weak sense). Moreover, if $\mathcal{K}$ is regularly locally finite in the weak sense, then $\mathbf{P_S}(\mathcal{K})$ is locally finite in the weak sense.*

3) *If $\mathcal{K}$ is a class of algebras of a finite signature, then uniform local finiteness of $\mathcal{K}$ (in the weak sense) implies local finiteness of* $\mathbf{P}(\mathcal{K})$ *(in the weak sense). Moreover, uniform local finiteness of $\mathcal{K}$ in the weak sense implies local finiteness of* $\mathbf{P_S}(\mathcal{K})$ *in the weak sense.*

Now we are in a position to prove our criterion.

THEOREM 3.7. 1) *For a given variety $\mathcal{V}$ of universal algebras, $\mathcal{V}$ is locally finite iff $\mathcal{V}$ is generated by a regularly locally finite class.*

2) *If $\mathcal{V}$ has a finite signature, then $\mathcal{V}$ is locally finite iff $\mathcal{V}$ is generated by a uniformly locally finite class.*

3) *A variety $\mathcal{V}$ is locally finite iff the class $\mathcal{V}_{SI}$ of all subdirectly irreducible $\mathcal{V}$-algebras is regularly locally finite in the weak sense.*

4) *If $\mathcal{V}$ has a finite signature, then $\mathcal{V}$ is locally finite iff $\mathcal{V}_{SI}$ is uniformly locally finite in the weak sense.*

Proof. 1) As was already mentioned, if a variety $\mathcal{V}$ is locally finite, then it is regularly locally finite. Hence, every subclass of $\mathcal{V}$ is also regularly locally finite. Conversely, if $\mathcal{V}$ is generated by a regularly locally finite class $\mathcal{K}$, then $\mathcal{V} = \mathbf{HSP}(\mathcal{K})$, and using Corollary 3.6 we obtain that $\mathcal{V}$ is locally finite.

2) directly follows from 1).

3) It is obvious that if $\mathcal{V}$ is locally finite, then $\mathcal{V}_{SI}$ is regularly locally finite in the weak sense. Conversely, suppose $\mathcal{V}_{SI}$ is regularly locally finite in the weak sense. From Birkhoff's theorem we have $\mathcal{V} = \mathbf{P_S}(\mathcal{V}_{SI})$. From Corollary 3.6 it follows that $\mathcal{V}$ is locally finite in the weak sense. Now since $\mathbf{S}(\mathcal{V}) \subseteq \mathcal{V}$ and $\mathcal{V}$ is locally finite in the weak sense, we obtain that $\mathcal{V}$ is locally finite.

4) directly follows from 3).

□

A variety $\mathcal{V}$ is said to be *finitely generated* if $\mathcal{V}$ is generated by a finite family of finite algebras, or which is the same, by a single finite algebra. It is obvious that the class $\{\mathcal{A}\}$ is regularly locally finite for every finite algebra $\mathcal{A}$. Consequently, we arrive at the following

COROLLARY 3.8. *If a variety $\mathcal{V}$ is finitely generated, then it is locally finite.*

It should be mentioned that Item 2 of Theorem 3.7 and Corollary 3.8 were first proved in Malcev [24]. However, the proofs offered above are much simpler. Moreover, Items 3 and 4 prove to be the most useful in applications. Indeed, they allow to restrict our attention to the class $\mathcal{V}_{FGSI}$ of finitely generated subdirectly irreducible $\mathcal{V}$-algebras, which one can handle quite easily, while the best you can achieve by Malcev's criterion is to deal with the class $\mathcal{V}_{SI}$, which is still rather huge to be able to work with (especially if the variety under consideration is not residually small). For practical use of our criterion see §4 below.

It is known that a variety generated by a locally finite algebra is not always locally finite. Theorem 3.7 offers us the following criterion to recognize whether a variety generated by a locally finite algebra is locally finite.

THEOREM 3.9. *Let $\mathcal{V}$ be the variety generated by a locally finite algebra $\mathcal{A}$. Then:*

1) *$\mathcal{V}$ is locally finite iff $\{\mathcal{A}\}$ is regularly locally finite iff, for any $n \in \omega$, there exists only a finite number of nonisomorphic algebras in the family of all n-generated subalgebras of $\mathcal{A}$.*
2) *If the signature of $\mathcal{A}$ is finite, then $\mathcal{V}$ is locally finite iff $\{\mathcal{A}\}$ is uniformly locally finite iff, for any $n \in \omega$, there exists $m(n) \in \omega$ such that the cardinality of every n-generated subalgebra of $\mathcal{A}$ is less than or equal to $m(n)$.*

It also follows that the join of two subvarieties $\mathcal{V}_1$ and $\mathcal{V}_2$ of a variety $\mathcal{V}$ is locally finite iff both $\mathcal{V}_1$ and $\mathcal{V}_2$ are locally finite. Indeed, it is obvious that if $\mathcal{V}_1 \vee \mathcal{V}_2$ is locally finite, then so are both $\mathcal{V}_1$ and $\mathcal{V}_2$. Conversely, suppose both $\mathcal{V}_1$ and $\mathcal{V}_2$ are locally finite. Then they are regularly locally finite too. By Corollary 3.6, **HSP** $(\mathcal{V}_1 \cup \mathcal{V}_2)$ is locally finite. Now since $\mathcal{V}_1 \vee \mathcal{V}_2 = \mathbf{HSP}(\mathcal{V}_1 \cup \mathcal{V}_2)$, we obtain that $\mathcal{V}_1 \vee \mathcal{V}_2$ is locally finite. Hence, the class $\Lambda_{LF}(\mathcal{V})$ of locally finite subvarieties of a variety $\mathcal{V}$ always forms a lattice which is a (non-bounded) sublattice of the lattice $\Lambda(\mathcal{V})$ of all subvarieties of $\mathcal{V}$.

It is known from Blok [8] that every proper subvariety $\mathcal{V}_0$ of a congruence-distributive finitely approximated variety $\mathcal{V}$ has a cover in the lattice $\Lambda(\mathcal{V})$. Now we show that if $\mathcal{V}_0$ is locally finite, then its cover is locally finite too. Indeed, since $\mathcal{V}_0 \neq \mathcal{V}$ and $\mathcal{V}$ is finitely approximated, there exists a finite algebra $\mathcal{A}$ in $\mathcal{V} - \mathcal{V}_0$. Consider the variety $\mathbf{Var}(\mathcal{A}) \vee \mathcal{V}_0$. From Jonsson's lemma it follows that $(\mathbf{Var}(\mathcal{A}) \vee \mathcal{V}_0)_{SI} \subseteq (\mathcal{V}_0)_{SI} \cup \mathbf{HS}(\mathcal{A})$, and since $(\mathcal{V}_0)_{SI}$ is regularly locally finite in the weak sense and $\mathbf{HS}(\mathcal{A})$ is finite, $(\mathbf{Var}(\mathcal{A}) \vee \mathcal{V}_0)_{SI}$ is also regularly locally finite in the weak sense. Therefore, $\mathbf{Var}(\mathcal{A}) \vee \mathcal{V}_0$ is locally finite and there exist only finitely many varieties in the interval $[\mathcal{V}_0, \mathbf{Var}(\mathcal{A}) \vee \mathcal{V}_0]$. It should be clear that one of them will cover $\mathcal{V}_0$.

4. Examples

4.1. *Boolean algebras*

Let us denote the variety of all Boolean algebras by **BA**. It is well known that the only subdirectly irreducible Boolean algebra is $\mathbf{2} = \{0, 1\}$ (see e. g. Sikorski [30]). Therefore, $\mathbf{BA} = \mathbf{Var}(\mathbf{2})$ and **BA** is a finitely generated variety. Now using Corollary 3.8 we obtain that **BA** is locally finite.

4.2. *Monadic Boolean algebras*

Let us recall that a pair $(B, \exists)$ is called a *monadic Boolean algebra*, written $(B, \exists) \in$ **MBA**, if $B \in$ **BA** and $\exists$ is a unary operator on B satisfying the following identities (see Halmos [15]):

$\exists 0 = 0$;
$a \leq \exists a$;
$\exists(\exists a \wedge b) = \exists a \wedge \exists b$.

It is known that every monadic Boolean algebra $(B, \exists)$ is represented as the pair (B, B_0) where B_0 is a *relatively complete* subalgebra of B (that is the set $\{b \in B_0 : a \leq b\}$ has a least element for every $a \in B$), and that (B, B_0) is subdirectly irreducible iff $B_0 = \mathbf{2}$. Therefore, for any n-generated subdirectly irreducible monadic Boolean algebra $(B, \exists)$ we have that B is n-generated as a Boolean algebra, and hence, there exists $m(n)$ such that $|B| \leq m(n)$. But then the class $\mathbf{MBA}_{SI}$ is uniformly locally finite in the weak sense, and Theorem 3.7 implies that **MBA** is locally finite.

4.3. *Distributive lattices*

Let us denote the variety of all distributive lattices by **DL**. It is well known that the only subdirectly irreducible distributive lattice is $\mathbf{2} = \{0, 1\}$ (see e.g. Birkhoff [5] or Grätzer [14]). Therefore, $\mathbf{DL} = \mathbf{Var}(\mathbf{2})$ and **DL** is a finitely generated variety. As a direct consequence of Corollary 3.8 we get that **DL** is locally finite.

4.4. *Monadic distributive lattices*

A pair $(D, \exists)$ is called a *monadic distributive lattice*, written $(D, \exists) \in$ **MDL**, if $D \in$ **DL** and $\exists$ is a unary operator on D satisfying the following identities (the definition can be found in Cignoli [9], N. Bezhanishvili [4], and implicitely in Ono [28]):

$\exists 0 = 0$;
$a \leq \exists a$;
$\exists(\exists a \wedge b) = \exists a \wedge \exists b$;
$\exists(a \vee b) = \exists a \vee \exists b$.

It is known from [4] that every monadic distributive lattice $(D, \exists)$ is represented as the pair (D, D_0) where D_0 is a relatively complete sublattice of D satisfying the condition $a \wedge \bigwedge D_b = \bigwedge D_{a \wedge b}$ for every $a \in D_0$ and $b \in D$ (here for any $a \in D$, D_a denotes the set $\{b \in D_0 : a \leq b\}$, and $\bigwedge D_a$ – a least element of D_a). Moreover, we have that if (D, D_0) is subdirectly irreducible, then $D_0 = \mathbf{2}$ (however, unlike **MBA**, the converse does not hold in general case). Therefore, for any n-generated subdirectly irreducible monadic distributive lattice $(D, \exists)$, D is n-generated as a distributive lattice, and hence, there exists $m(n)$ such that $|D| \leq m(n)$. But then the class $\mathbf{MDL}_{SI}$ is uniformly locally finite in the weak sense, and by Theorem 3.7 **MDL** is locally finite. (The construction of finitely generated free **MDL**-algebras can be found in Cignoli [10].)

4.5. *Implicational semilattices*

Let us denote the variety of all implicational semilattices by **IS**. The investigation of implicational semilattices can be found in Diego [11], Köhler [18] and Rasiowa [29]. It is known that an implicational semilattice $(A, \wedge, \to) \in \mathbf{IS}$ is subdirectly irreducible iff the set $A - \{1\}$ contains a greatest element, say a, and that $A - \{a\}$ forms an implicational semilattice of A. Therefore, for any n-generated subdirectly irreducible implicational semilattice A we have that $A - \{a\}$ is an $(n-1)$-generated implicational semilattice. Hence, for any n-generated subdirectly irreducible implicational semilattice A, $|A| \leq |\mathcal{F}_{n-1}| + 1$, where $\mathcal{F}_{n-1}$ denotes the $(n-1)$-generated free implicational semilattice. Now since $|\mathcal{F}_0| = 1$, we can prove by induction that the class of all subdirectly irreducible implicational semilattices is uniformly locally finite in the weak sense, which together with Theorem 3.7 imply that **IS** is locally finite.

4.6. *Heyting algebras*

Let us denote the variety of all Heyting algebras by **HA**. It is known that a Heyting algebra $(H, \wedge, \vee, \to, 0) \in \mathbf{HA}$ is subdirectly irreducible iff the set $H - \{1\}$ contains a greatest element, say a (see e.g. Esakia [12]). Let us consider the following list of formulas

$$P_1 : \ q_1 \vee \neg q_1;$$
$$P_{n+1} : \ q_n \vee (q_n \to P_{n-1}), \quad n \geq 1;$$

and prove that the variety $\mathbf{HA} + (P_n = 1)$ is locally finite for any $n \in \omega$ (this fact was first established by Kuznetsov [19], [20] and Komori [17]). Indeed, $\mathbf{HA} + (P_1 = 1) = \mathbf{BA}$ and therefore is locally finite. Suppose $\mathbf{HA} + (P_n = 1)$ is locally finite and let us prove that so is $\mathbf{HA} + (P_{n+1} = 1)$. For any m-generated subdirectly irreducible algebra $H \in \mathbf{HA} + (P_{n+1} = 1)$, consider the algebra $H/_{[a)}$ (here $[a)$ denotes the filter generated by a). We have that $H/_{[a)}$ is also an m-generated algebra and that $H/_{[a)} \in \mathbf{HA} + (P_n = 1)$ (see Komori [17]). By the induction hypothesis, there exists $k(m)$ such that $|H/_{[a)}| \leq k(m)$. Now since $|H| = |H/_{[a)}| + 1$, then $|H| \leq k(m) + 1$. Therefore, the class of all subdirectly irreducible algebras of $\mathbf{HA} + (P_{n+1} = 1)$ is uniformly locally finite in the weak sense, and from Theorem 3.7 it follows that $\mathbf{HA} + (P_{n+1} = 1)$ is locally finite.

4.7. *Monadic Heyting algebras*

Let us recall that a triple $(H, \forall, \exists)$ is called a *monadic Heyting algebra*, written $(H, \forall, \exists) \in$ **MHA**, if $H \in$ **HA** and $\forall$ and $\exists$ are unary operators on H satisfying the following identities (see Monteiro and Varsavsky [27] and Bezhanishvili [1], [2]):

(1) $\forall a \leq a$ $\quad a \leq \exists a$;
(2) $\forall(a \wedge b) = \forall a \wedge \forall b$ $\quad \exists(a \vee b) = \exists a \vee \exists b$;
(3) $\forall 1 = 1$ $\quad \exists 0 = 0$;
(4) $\exists \forall a = \forall a$ $\quad \forall \exists a = \exists a$;
(5) $\exists(\exists a \wedge b) = \exists a \wedge \exists b$.

In the same way as for monadic Boolean algebras we have that every monadic Heyting algebra $(H, \forall, \exists)$ is represented as the pair (H, H_0) where H_0 is a relatively complete Heyting subalgebra of H (note that this time relatively complete means that the sets $\{b \in H_0 : a \leq b\}$ and $\{b \in H_0 : b \leq a\}$ have a least and a greatest elements, respectively), and that (H, H_0) is subdirectly irreducible iff H_0 is subdirectly irreducible as a Heyting algebra (see [1], [2]).

In the same way as for Heyting algebras we can prove that **MHA** $+ (P_n = 1)$ is locally finite for any $n \in \omega$ (this fact was first established in [1]). For this with every P_n let us associate the formula $P_n[\forall q_i/q_i]$ which is obtained from P_n by substituting every occurence of q_i by $\forall q_i$ $(i \leq n)$. Now for every $n < \omega$ consider the list of varieties **MHA** $+ (P_n = 1) + (P_1[\forall q_i/q_i] = 1), \ldots,$ **MHA** $+ (P_n = 1) + (P_n[\forall q_i/q_i] = 1)$ and prove that every variety from this list is locally finite. Since **MHA** $+ (P_n = 1) + (P_n[\forall q_i/q_i] = 1) =$ **MHA** $+ (P_n = 1)$, it will follow that **MHA** $+ (P_n = 1)$ is locally finite too.

It is obvious that if $(H, H_0) \in$ **MHA** $+ (P_n = 1) + (P_1[\forall q_i/q_i] = 1)$, then $H_0 \in$ **HA** $+ (P_1 = 1) =$ **BA**, and if (H, H_0) is subdirectly irreducible, then $H_0 = \mathbf{2}$. Therefore, if (H, H_0) is n-generated, H is n-generated as a Heyting algebra, and since $H \in$ **HA** $+ (P_n = 1)$, there exists $m(n)$ such that $|H| \leq m(n)$. Hence, the class $(\mathbf{MHA} + (P_n = 1) + (P_1[\forall q_i/q_i] = 1))_{SI}$ is uniformly locally finite in the weak sense and by Theorem 3.7 **MHA** $+ (P_n = 1) + (P_1[\forall q_i/q_i] = 1)$ is locally finite.

Now suppose **MHA** $+ (P_n = 1) + (P_{n-1}[\forall q_i/q_i] = 1)$ is locally finite and let us prove that **MHA** $+ (P_n = 1) + (P_n[\forall q_i/q_i] = 1)$ is locally finite too. For this suppose $(H, H_0) \in$ **MHA** $+ (P_n = 1) + (P_n[\forall q_i/q_i] = 1)$ is an m-generated subdirectly irreducible algebra. Then H_0 is subdirectly irreducible as a Heyting algebra and there exists a greatest element, say a, in $H_0 - \{1\}$. Consider the algebra $(H/_{[a)}, H_0/_{[a) \cap H_0})$. Since $(H/_{[a)}, H_0/_{[a) \cap H_0})$ is a homomorphic image of (H, H_0), $(H/_{[a)}, H_0/_{[a) \cap H_0})$ is also m-generated. Moreover, $(H/_{[a)}, H_0/_{[a) \cap H_0}) \in$ **MHA** $+ (P_n = 1) + (P_{n-1}[\forall q_i/q_i] = 1)$, and from the induction hypothesis it follows that there exists $k(m)$ such that $|H_0/_{[a) \cap H_0}| \leq k(m)$. Since $|H_0| = |H_0/_{[a) \cap H_0}| + 1$, $|H_0| \leq k(m) + 1$. But then we can easily prove that H is no more than $(m + k(m) + 1)$-generated as a Heyting algebra and since $H \in$ **HA** $+ (P_n = 1)$, there exists $r(m)$

such that $|H| \leq r(m)$. Therefore, $(\mathbf{MHA} + (P_n = 1) + (P_n[\forall q_i/q_i] = 1))_{SI}$ is uniformly locally finite in the weak sense and by Theorem 3.7 $\mathbf{MHA} + (P_n = 1) + (P_n[\forall q_i/q_i] = 1)$ is locally finite. As a result we obtain that $\mathbf{MHA} + (P_n = 1)$ is locally finite for every $n < \omega$.

4.8. *Closure algebras and Grzegorczyk algebras*

Let us recall that a couple (B, C) is called a *closure algebra*, written $(B, C) \in \mathbf{CA}$, if $B \in \mathbf{BA}$ and C is a *closure operator* on B satisfying Kuratowski's identities (see e.g. McKinsey and Tarski [25], Blok [6] and Esakia [12]):

$C0 = 0$;
$a \leq Ca$;
$CCa \leq Ca$;
$C(a \vee b) = Ca \vee Cb$.

Let $Ia = -C - a$. I is called an *interior operator* and it satisfies the identities which are dual to the identities defining a closure operator. That is why closure algebras are sometimes called equivalently *interior algebras*.

It is known that the set $H = \{Ia : a \in B\}$ is a sublattice of B which constitutes a Heyting algebra, and that (B, C) is represented as the pair (B, H), where H is a relatively complete sublattice of B. It is also known that (B, C) is subdirectly irreducible iff H is subdirectly irreducible as a Heyting algebra.

Consider the following list of formulas:

$P_1^I : Iq_1 \vee I\neg Iq_1$;
$P_n^I : Iq_n \vee I(Iq_n \to P_{n-1}^I)$.

It is obvious that P_n^I is valid in (B, H) iff P_n is valid in H. We are in a position now to prove that $\mathcal{V} \in \Lambda(\mathbf{CA})$ is locally finite iff there exists a natural number n such that $\mathcal{V} \subseteq \mathbf{CA} + (P_n^I = 1)$ (this theorem was first established by Blok [6] and Maksimova [22]). Indeed, since $\mathbf{CA} + (P_1^I = 1) = \mathbf{MBA}$, $\mathbf{CA} + (P_1^I = 1)$ is locally finite. Suppose $\mathbf{CA} + (P_{n-1}^I = 1)$ is locally finite and let us prove that so is $\mathbf{CA} + (P_n^I = 1)$. For every subdirectly irreducible $(B, H) \in \mathbf{CA} + (P_n^I = 1)$, there exists a greatest element, say a, in H. It is obvious that $(B/_{[a)}, H/_{[a)\cap H})$ is a homomorphic image of (B, H), and that $(B/_{[a)}, H/_{[a)\cap H}) \in \mathbf{CA} + (P_{n-1}^I = 1)$. Moreover, if (B, H) is m-generated, so is $(B/_{[a)}, H/_{[a)\cap H})$. But then there exists $k(m)$ such that $|H/_{[a)\cap H}| \leq k(m)$, and since $|H| = |H/_{[a)\cap H}| + 1$, $|H| \leq k(m) + 1$. Further, B is at most $(m + k(m) + 1)$-generated as a Boolean algebra, and hence, there exists $r(m)$ such that $|B| \leq r(m)$. Therefore, the class $(\mathbf{CA} + (P_n^I = 1))_{SI}$ is uniformly locally finite in the weak sense, and by Theorem 3.7 $\mathbf{CA} + (P_n^I = 1)$ is locally finite.

Thus, for every variety $\mathcal{V} \in \Lambda(\mathbf{CA})$, if there exists a natural number n such that $\mathcal{V} \subseteq \mathbf{CA} + (P_n^I = 1)$, then $\mathcal{V}$ is locally finite. Conversely, if $\mathcal{V} \not\subseteq \mathbf{CA} + (P_n^I = 1)$ for any n, then

CA+**grz**+**lin** $\subseteq \mathcal{V}$ (here **grz** denotes Grzegorczyk's formula $I(I(a \to Ia) \to a) \leq a$ and **lin** denotes Dummet's formula $I(Ia \to b) \vee I(Ib \to a) = 1$), and since **CA** + **grz** + **lin** is not locally finite, neither is $\mathcal{V}$. As a result we obtain that $\mathcal{V}$ is locally finite iff there exists a natural number n such that $\mathcal{V} \subseteq$ **CA** + $(P_n^I = 1)$ iff **CA** + **grz** + **lin** $\not\subseteq \mathcal{V}$.

Denote by **Grz** the subvariety of **CA** whose algebras satisfy Grzegorczyk's formula. It is obvious that **Grz** = **CA**+**grz**. The algebras from **Grz** are called *Grzegorczyk algebras*. As a direct consequence of the previous theorem we obtain that a variety $\mathcal{V} \in \Lambda($**Grz**$)$ is locally finite iff there exists a natural number n such that $\mathcal{V} \subseteq$ **Grz** + $(P_n^I = 1)$ iff **Grz**+**lin** $\not\subseteq \mathcal{V}$. Later we will see that this theorem can be extended to the variety of monadic Grzegorczyk algebras.

4.9. *Derivative algebras and Magari algebras*

Let us recall that a *derivative algebra* is a pair (B, δ), written $(B, \delta) \in$ **DA**, where $B \in$ **BA** and δ is a unary operator on B which satisfies the following identities (see McKinsey and Tarski [25]):

$\delta 0 = 0.$

$\delta(a \vee b) = \delta a \vee \delta b;$

$\delta\delta a \leq \delta a.$

Let $\tau a = -\delta - a$. It is known that τ satisfies the identities which are dual to the identities defining δ, and that derivative algebras can be defined in terms of τ as well.

With every derivative algebra (B, δ) is naturally associated the closure algebra (B, C_δ) by putting $C_\delta a = a \vee \delta a$ for every $a \in B$. (Dually $I_\tau a = a \wedge \tau a$.) Therefore, there exists a natural functor Φ : **DA** $\to$ **CA** which sends every $(B, \delta) \in$ **DA** to $(B, C_\delta) \in$ **CA**. The important property of Φ is the fact that (B, δ) is subdirectly irreducible iff (B, C_δ) is subdirectly irreducible. It is also easy to prove that Φ is not injective, and that for every class $\mathcal{K} \subseteq$ **DA** we have $\Phi\mathbf{H}(\mathcal{K}) = \mathbf{H}\Phi(\mathcal{K})$, $\Phi\mathbf{S}(\mathcal{K}) \subset \mathbf{S}\Phi(\mathcal{K})$ and $\Phi\mathbf{P}(\mathcal{K}) = \mathbf{P}\Phi(\mathcal{K})$. Therefore, Φ-image of a variety is not always a variety. For $\mathcal{V} \in \Lambda($**DA**$)$ denote by $\hat{\Phi}(\mathcal{V})$ the least variety generated by $\Phi(\mathcal{V})$. It is obvious that $\hat{\Phi}(\mathcal{V}) = \mathbf{S}\Phi(\mathcal{V})$, and that $\hat{\Phi}$ is a morphism from $\Lambda($**DA**$)$ to $\Lambda($**CA**$)$. Note that $\hat{\Phi}$ is an order-preserving morphism, but it is not a lattice morphism.

Now we are ready to prove that $\mathcal{V} \in \Lambda($**DA**$)$ is locally finite iff $\hat{\Phi}(\mathcal{V})$ is locally finite (see Maksimova [23]).

Consider the following list of formulas:

P_1^τ : $(q_1 \wedge \tau q_1) \vee [\neg(q_1 \wedge \tau q_1) \wedge \tau\neg(q_1 \wedge \tau q_1)]$;
P_n^τ : $(q_n \wedge \tau q_n) \vee [((q_n \wedge \tau q_n) \to P_n^\tau) \wedge \tau((q_n \wedge \tau q_n) \to P_n^\tau)]$.

We have that P_n^τ is valid in (B, δ) iff P_n^I is valid in (B, C_δ) (it directly follows from the identity $I_\tau a = a \wedge \tau a$). Let us prove that $\mathbf{DA} + (P_n^\tau = 1)$ is locally finite for every $n \in \omega$ (see Blok [7]). Indeed, $\mathbf{DA} + (P_1^\tau = 1) = \mathbf{DA} + \delta(p \wedge \tau p) \leq p$, and hence $(\mathbf{DA}+(P_1^\tau = 1))_{SI} = \mathbf{MBA}_{SI} \cup \{(\mathbf{2}, \delta)\}$, where $\delta 1 = \delta 0 = 0$. But then $(\mathbf{DA}+(P_1^\tau = 1))_{SI}$ is uniformly locally finite in the weak sense and by Theorem 3.7 $\mathbf{DA} + (P_1^\tau = 1)$ is locally finite.

Now suppose $\mathbf{DA}+(P_{n-1}^\tau = 1)$ is locally finite and let us prove that so is $\mathbf{DA}+(P_n^\tau = 1)$. For a given m-generated subdirectly irreducible algebra (B, δ) we have that (B, C_δ) is also subdirectly irreducible. Therefore, there exists an element $a \in B$ such that $a \wedge \tau a$ is a greatest element in the set $H - \{1\}$. But then the algebra $(B/_{[a\wedge\tau a)}, \delta_{[a\wedge\tau a)})$ is a homomorphic image of (B, δ) and it belongs to $\mathbf{DA} + (P_{n-1}^\tau = 1)$. Since $(B/_{[a\wedge\tau a)}, \delta_{[a\wedge\tau a)})$ is also m-generated, by the induction hypothesis there exists $k(m)$ such that $|B/_{[a\wedge\tau a)}| \leq k(m)$. But then $|H| \leq k(m) + 1$. Now since $I\tau a = \tau a$ for every $a \in B$, we have that $\tau a \in H$ for every $a \in B$. Therefore, B is no more than $(m + k(m) + 1)$-generated as a Boolean algebra, and hence there exists $r(m)$ such that $|B| \leq r(m)$. But then $(\mathbf{DA} + (P_n^\tau = 1))_{SI}$ is uniformly locally finite in the weak sense and on the base of Theorem 3.7 we conclude that $\mathbf{DA} + (P_n^\tau = 1)$ is locally finite.

Conversely, suppose $\mathcal{V} \not\subseteq \mathbf{DA} + (P_n^\tau = 1)$ for any $n \in \omega$. Then $\hat{\Phi}(\mathcal{V}) \not\subseteq \mathbf{CA} + (P_n^I = 1)$ for any $n \in \omega$, and hence $\hat{\Phi}(\mathcal{V})$ is not locally finite. Now since $\hat{\Phi}(\mathcal{V}) = \mathbf{S}\Phi(\mathcal{V})$, Corollary 3.6 implies that $\Phi(\mathcal{V})$ is not locally finite as well. Hence, there exists $(B, C_\delta) \in \Phi(\mathcal{V})$ which is not locally finite. But then neither is $(B, \delta) \in \mathcal{V}$, and $\mathcal{V}$ is not locally finite as well.

As a result we obtain that the following conditions are equivalent:

1) $\mathcal{V} \in \Lambda(\mathbf{DA})$ is locally finite.
2) There exists $n \in \omega$ such that $\mathcal{V} \subseteq \mathbf{DA} + (P_n^\tau = 1)$.
3) There exists $n \in \omega$ such that $\hat{\Phi}(\mathcal{V}) \subseteq \mathbf{CA} + (P_n^I = 1)$.
4) $\mathbf{Grz} + \mathbf{lin} \not\subseteq \hat{\Phi}(\mathcal{V})$.
5) $\hat{\Phi}(\mathcal{V})$ is locally finite.

Denote by **Mag** the subvariety of **DA** whose algebras satisfy Löb's formula $\tau(\tau a \to a) \leq \tau a$. The algebras from **Mag** are called *Magari algebras*. Note that $\hat{\Phi}(\mathbf{Mag}) = \mathbf{Grz}$, and hence $\mathcal{V} \in \Lambda(\mathbf{Mag})$ implies $\hat{\Phi}(\mathcal{V}) \in \Lambda(\mathbf{Grz})$ (see Kuznetsov and Muravitskij [21]). As a direct consequence of the previous theorem we obtain that a variety $\mathcal{V} \in \Lambda(\mathbf{Mag})$ is locally finite iff there exists $n \in \omega$ such that $\mathcal{V} \subseteq \mathbf{Mag} + (P_n^\tau = 1)$ iff $\hat{\Phi}(\mathcal{V})$ is locally finite. Later we will see that this theorem can be extended to the variety of monadic Magari algebras.

4.10. *Monadic Grzegorczyk algebras*

A triple $(B, C, \exists)$ is called a *monadic closure algebra*, written $(B, C, \exists) \in \mathbf{MCA}$, if $(B, C) \in \mathbf{CA}$, $(B, \exists) \in \mathbf{MBA}$ and $\exists Ca \leq C\exists a$ (see Monteiro [26], Esakia [13] and

Bezhanishvili [3]). For a given monadic closure algebra $(B, C, \exists)$, the couple (B_0, C_0), where $B_0 = \{\exists a : a \in B\}$ and C_0 is the restriction of C to B_0, forms a subalgebra of (B, C), and we have that every monadic closure algebra $(B, C, \exists)$ can be represented as the pair $((B, C), (B_0, C_0))$, where (B_0, C_0) is a relatively complete subalgebra of (B, C). Moreover, $(B, C, \exists)$ is subdirectly irreducible iff (B_0, C_0) is subdirectly irreducible as a closure algebra (see [3]).

Note that there exists a close correspondence between monadic closure algebras and monadic Heyting algebras. Indeed, with every monadic closure algebra $(B, C, \exists)$ we can associate the monadic Heyting algebra $(H, \forall_H, \exists_H)$, where $H = \{Ia : a \in B\}$, $\forall_H a = I\forall a$ and $\exists_H a = \exists a$ for every $a \in H$. Here $Ia = -C - a$ and $\forall a = -\exists - a$ for every $a \in B$. Now we have that $H_0 = \{Ia : a \in B_0\}$ and that the following conditions are equivalent:

1) $(B, C, \exists)$ is subdirectly irreducible.
2) (B_0, C_0) is subdirectly irreducible as a closure algebra.
3) $(H, \forall_H, \exists_H)$ is subdirectly irreducible as a monadic Heyting algebra.
4) H_0 is subdirectly irreducible as a Heyting algebra (see [3]).

Let us call an algebra $(B, C, \exists) \in$ **MCA** *a monadic Grzegorczyk algebra*, written $(B, C, \exists) \in$ **MGrz**, if $(B, C) \in$ **Grz** (see [13] and [3]). We are in a position now to prove that a variety $\mathcal{V}$ of monadic Grzegorczyk algebras is locally finite iff there exists $n \in \omega$ such that $\mathcal{V} \subseteq \mathbf{MGrz} + (P_n^I = 1)$. Denote by $P_n^I[\forall q_i/q_i]$ the polynomial which is obtained from P_n^I by replacing every occurence of q_i in P_n^I by $\forall q_i$ $(1 \le i \le n)$, and let us prove that the variety $\mathbf{MGrz} + (P_n^I = 1) + (P_k^I[\forall q_i/q_i] = 1)$ is locally finite for every $k \le n$. Suppose $(B, C, \exists)$ is an m-generated subdirectly irreducible algebra from $\mathbf{MGrz} + (P_n^I = 1) + (P_1^I[\forall q_i/q_i] = 1)$. Then (B_0, C_0) is a subdirectly irreducible algebra from $\mathbf{Grz} + (P_1^I = 1)$, and hence $B_0 = \mathbf{2}$ and C_0 is discrete. But then (B, C) is m-generated as a closure algebra, and since $(B, C) \in \mathbf{Grz} + (P_n^I = 1)$ and $\mathbf{Grz} + (P_n^I = 1)$ is locally finite, there exists $k(m)$ such that $|B| \le k(m)$. Therefore, the class $(\mathbf{MGrz} + (P_n^I = 1) + (P_1^I[\forall q_i/q_i] = 1))_{SI}$ is uniformly locally finite in the weak sense, and by Theorem 3.7 $\mathbf{MGrz} + (P_n^I = 1) + (P_1^I[\forall q_i/q_i] = 1)$ is locally finite. Now suppose $\mathbf{MGrz} + (P_n^I = 1) + (P_{n-1}^I[\forall q_i/q_i] = 1)$ is locally finite and let us prove that so is $\mathbf{MGrz} + (P_n^I = 1) + (P_n^I[\forall q_i/q_i] = 1)$. Consider an m-generated subdirectly irreducible algebra $(B, C, \exists) \in \mathbf{MGrz} + (P_n^I = 1) + (P_n^I[\forall q_i/q_i] = 1)$. Then $(H, \forall_H, \exists_H)$ is a subdirectly irreducible monadic Heyting algebra from $\mathbf{MHA} + (P_n = 1)$. Hence, there exists a greatest element, say a, in the set $H_0 - \{1\}$. Consider the filter $[a) \subseteq B$ and the algebra $(B/_{[a)}, C_{[a)}, \exists_{[a)})$. Since $(B/_{[a)}, C_{[a)}, \exists_{[a)})$ is a homomorphic image of $(B, C, \exists)$, it is also m-generated. Moreover, it belongs to $\mathbf{MGrz} + (P_n^I = 1) + (P_{n-1}^I[\forall q_i/q_i] = 1)$, and by the induction hypothesis there exists $k(m)$ such that $|B/_{[a)}| \le k(m)$. But then $|H_0| \le k(m) + 1$ and since $(B_0, C_0) \in \mathbf{Grz} + (P_n^I = 1)$, B_0 is generated by H_0 as a Boolean algebra. Therefore, there exists $r(m)$ such that $|B_0| \le r(m)$. Hence, (B, C) is no more than $(m + r(m))$-generated as a closure algebra, and since $(B, C) \in \mathbf{Grz} + (P_n^I = 1)$ and $\mathbf{Grz} + (P_n^I = 1)$ is locally finite, there exists $l(m)$ such that $|B| \leq l(m)$. Thus, $(\mathbf{MGrz} + (P_n^I = 1) + (P_n^I[\forall q_i/q_i] = 1))_{SI}$ is uniformly locally finite in the weak sense, and by Theorem 3.7 $\mathbf{MGrz} + (P_n^I = 1) + (P_n^I[\forall q_i/q_i] = 1)$ is locally finite. Now since $\mathbf{MGrz} + (P_n^I = 1) + (P_n^I[\forall q_i/q_i] = 1) = \mathbf{MGrz} + (P_n^I = 1)$, we obtain that $\mathbf{MGrz} + (P_n^I = 1)$ is locally finite for every $n \in \omega$.

Conversely, if $\mathcal{V} \not\subseteq \mathbf{MGrz} + (P_n^I = 1)$ for any $n \in \omega$, then obviously the $\exists$-free reduct of $\mathcal{V}$ is not locally finite and, all the more, neither is $\mathcal{V}$. As a result we arrive at the following theorem:

The following two conditions are equivalent:

1) $\mathcal{V} \in \Lambda(\mathbf{MGrz})$ is locally finite.
2) There exists $n \in \omega$ such that $\mathcal{V} \subseteq \mathbf{MGrz} + (P_n^I = 1)$.

REMARK 4.1. Unfortunately, it is impossible to extend this theorem to the lattice $\Lambda(\mathbf{MCA})$, for, as was noticed by A. Tarski (see e.g. Henkin, Monk, Tarski [16]), already $\mathbf{MCA} + P_1^I$ is not locally finite.

4.11. *Monadic Magari algebras*

A triple $(B, \delta, \exists)$ is called a *monadic derivative algebra*, written $(B, \delta, \exists) \in \mathbf{MDA}$, if $(B, \delta) \in \mathbf{DA}$, $(B, \exists) \in \mathbf{MBA}$ and $\exists\delta a \leq \delta\exists a$ (see Esakia [13] and Bezhanishvili [3]). In the same way as for $\mathbf{MCA}$ we have that for a given monadic derivative algebra $(B, \delta, \exists)$, the couple (B_0, δ_0), where $B_0 = \{\exists a : a \in B\}$ and δ_0 is the restriction of δ to B_0, forms a subalgebra of (B, δ), and that every monadic derivative algebra $(B, \delta, \exists)$ is represented as the pair $((B, \delta), (B_0, \delta_0))$, where (B_0, δ_0) is a relatively complete subalgebra of (B, δ). Moreover, $(B, \delta, \exists)$ is subdirectly irreducible iff (B_0, δ_0) is subdirectly irreducible as a derivative algebra (see [3]).

As in the case of $\mathbf{DA}$ and $\mathbf{CA}$, there exists a close correspondence between $\mathbf{MDA}$ and $\mathbf{MCA}$. For every $(B, \delta, \exists) \in \mathbf{MDA}$ we have that $(B, C_\delta, \exists) \in \mathbf{MCA}$ and that $(B, \delta, \exists)$ is subdirectly irreducible iff $(B, C_\delta, \exists)$ is subdirectly irreducible (see [3]). Denote $(B, C_\delta, \exists)$ by $\Phi(B, \delta, \exists)$ and let $\hat{\Phi}(\mathcal{V})$ denote the variety of monadic closure algebras generated by the class $\Phi(\mathcal{V})$.

Call an algebra $(B, \delta, \exists) \in \mathbf{MDA}$ *a monadic Magari algebra* and write $(B, \delta, \exists) \in \mathbf{MMag}$, if $(B, \delta) \in \mathbf{Mag}$ (see [13] and [3]). As in the case of $\mathbf{Mag}$, we have that if $(B, \delta, \exists) \in \mathbf{MMag}$, then $(B, C_\delta, \exists) \in \mathbf{MGrz}$, and that $\hat{\Phi}(\mathcal{V}) \in \Lambda(\mathbf{MGrz})$ for $\mathcal{V} \in \Lambda(\mathbf{MMag})$. However, unlike $\mathbf{Mag}$ and $\mathbf{Grz}$, here we have that $\hat{\Phi}(\mathbf{MMag})$ is properly contained in $\mathbf{MGrz}$ (see [3]).

We are in a position now to prove that a variety $\mathcal{V}$ of monadic Magari algebras is locally finite iff $\hat{\Phi}(\mathcal{V})$ is locally finite. Consider the varieties $\mathbf{MMag} + (P_n^\tau = 1)$ for $n \in \omega$. If

$(B, \delta, \exists) \in (\mathbf{MMag} + (P_1^\tau = 1))_{SI}$, then $(B_0, \delta_0) \in (\mathbf{Mag} + (P_1^\tau = 1))_{SI}$, and hence $B_0 = \mathbf{2}$. But then $(B, \delta, \exists)$ is m-generated iff (B, δ) is m-generated as a Magari algebra, and since $\mathbf{Mag} + (P_1^\tau = 1)$ is locally finite, there exists $k(m)$ such that $|B| \leq k(m)$. Therefore, $(\mathbf{MMag} + (P_1^\tau = 1))_{SI}$ is uniformly locally finite in the weak sense, and by Theorem 3.7 $\mathbf{MMag} + (P_1^\tau = 1)$ is locally finite. Suppose $\mathbf{MMag} + (P_{n-1}^\tau = 1)$ is locally finite and let us prove that so is $\mathbf{MMag} + (P_n^\tau = 1)$. For an m-generated subdirectly irreducible algebra $(B, \delta, \exists) \in \mathbf{MMag} + (P_n^\tau = 1)$ we have that $(B, C_\delta, \exists)$ is also subdirectly irreducible. But then there exists $a \in B$ such that $\forall a \wedge \tau\forall a$ is a greatest element in the set $H_0 - \{1\}$. Therefore, $(B_{[\forall a \wedge \tau\forall a)}, \delta_{[\forall a \wedge \tau\forall a)}, \exists_{[\forall a \wedge \tau\forall a)})$ is a homomorphic image of $(B, \delta, \exists)$ and $(B_{[\forall a \wedge \tau\forall a)}, \delta_{[\forall a \wedge \tau\forall a)}, \exists_{[\forall a \wedge \tau\forall a)}) \in \mathbf{MMag} + (P_{n-1}^\tau = 1)$. Hence, there exists $k(m)$ such that $|B_{[\forall a \wedge \tau\forall a)}| \leq k(m)$. But then $|H_0| \leq k(m) + 1$ and since B_0 is generated as a Boolean algebra by H_0, there exists $r(m)$ such that $|B_0| \leq r(m)$. It directly implies that $(B, \delta, \exists)$ is no more than $(m + r(m))$-generated as a Magari algebra, and since $\mathbf{Mag} + (P_n^\tau = 1)$ is locally finite, there exists $l(m)$ such that $|B| \leq l(m)$. But then $(\mathbf{MMag} + (P_n^\tau = 1))_{SI}$ is uniformly locally finite in the weak sense and by Theorem 3.7 $\mathbf{MMag} + (P_n^\tau = 1)$ is locally finite.

Conversely, if $\mathcal{V} \not\subseteq \mathbf{MMag} + (P_n^\tau = 1)$ for any $n \in \omega$, then $\hat{\Phi}(\mathcal{V}) \not\subseteq \mathbf{MGrz} + (P_n^I = 1)$. Therefore, $\Phi(\mathcal{V})$ is not locally finite, and neither is $\mathcal{V}$. As a result we arrive at the following theorem:

For a given variety $\mathcal{V} \in \Lambda(\mathbf{MMag})$ the following conditions are equivalent:

1) $\mathcal{V}$ is locally finite.
2) There exists $n \in \omega$ such that $\mathcal{V} \subseteq \mathbf{MMag} + (P_n^\tau = 1)$.
3) There exists $n \in \omega$ such that $\hat{\Phi}(\mathcal{V}) \subseteq \mathbf{MGrz} + (P_n^I = 1)$.
4) $\hat{\Phi}(\mathcal{V})$ is locally finite.

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G. Bezhanishvili
Department of Mathematical Sciences
New Mexico State University
Las Cruces, NM 88003-0001
USA
e-mail: gbezhani@nmsu.edu