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Finite-to-finite universal quasivarieties are Q-universal

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Dedicated to the memory of Viktor Aleksandrovich Gorbunov

Abstract. Let **K** be a quasivariety of algebraic systems of finite type. **K** is said to be *universal* if the category **G** of all directed graphs is isomorphic to a full subcategory of **K**. If an embedding of **G** may be effected by a functor $\mathbf{F} : \mathbf{G} \longrightarrow \mathbf{K}$ which assigns a finite algebraic system to each finite graph, then **K** is said to be *finite-to-finite* universal. **K** is said to be *Q-universal* if, for any quasivariety **M** of finite type, $L(\mathbf{M})$ is a homomorphic image of a sublattice of $L(\mathbf{K})$, where $L(\mathbf{M})$ and $L(\mathbf{K})$ are the lattices of quasivarieties contained in **M** and **K**, respectively.

We establish a connection between these two, apparently unrelated, notions by showing that if **K** is finite-to-finite universal, then **K** is Q-universal. Using this connection a number of quasivarieties are shown to be Q-universal.

1. Introduction

An *algebraic system of finite type* is a nonvoid set which admits a finite family of operations and relations. It is an *algebra* if it admits no relations and a *relational system* if it admits no operations.

For a class **K** of algebraic systems of similar type, let $\mathbf{I}(\mathbf{K})$, $\mathbf{H}(\mathbf{K})$, $\mathbf{S}(\mathbf{K})$, $\mathbf{P}(\mathbf{K})$, and $\mathbf{P_u}(\mathbf{K})$ respectively denote the classes of all isomorphic algebraic systems, homomorphic images, subsystems, products, and ultraproducts of algebraic systems in **K**. A class **K** is a *quasivariety* provided $\mathbf{K} = \mathbf{ISPP_u}(\mathbf{K})$ (equivalently, **K** is a universal Horn class that contains a trivial algebraic system) and is a *variety* provided $\mathbf{K} = \mathbf{HSP}(\mathbf{K})$ (thus, every variety is a quasivariety). For further information on quasivarieties see Section 2 and, more generally, Gorbunov [11].

A quasivariety of algebraic systems **K** is *universal* if every category of algebras of finite type (or equivalently, as shown by Pultr [23], Hedrlín and Pultr [16] and Vopěnka, Hedrlín, and Pultr [28], the category **G** of all directed graphs) is isomorphic to a full subcategory of **K**. If an embedding of **G** may be effected by a functor $\mathbf{F} : \mathbf{G} \longrightarrow \mathbf{K}$ which assigns a finite algebraic system to each finite graph, then **K** is said to be *finite-to-finite* universal. Of

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particular interest for a universal quasivariety **K** is the fact that, for every monoid M, there exists a proper class of non-isomorphic algebraic systems belonging to **K** each of which has an endomorphism monoid isomorphic to M. If **K** is finite-to-finite universal, then, in addition, for a finite monoid M, there exists infinitely many non-isomorphic finite algebraic systems in **K** with the preceding property. The literature on universal quasivarieties is extensive and many familiar (quasi)varieties of algebraic systems are known to be finite-to-finite universal. For a detailed background see Pultr and Trnková [24]. More recent results include, for example, a complete characterization of all varieties of $(0, 1)$-lattices that are finite-to-finite universal as given by Goralčík, Koubek, and Sichler in [9]. Their result represents the conclusion of a long sequence of papers beginning with Grätzer and Sichler [14] (see [9] and the references therein).

For a quasivariety **K**, let $L(\mathbf{K})$ denote the lattice (ordered by inclusion) of all quasivarieties contained in **K**. A quasivariety **K** of algebraic systems of finite type is *Q-universal* providing that, for any quasivariety **M** of finite type, $L(\mathbf{M})$ is a homomorphic image of a sublattice of $L(\mathbf{K})$. Sapir introduced the notion in [25] where he showed that the variety of commutative 3-nilpotent semigroups is Q-universal. In [1], it was shown that the existence of a family of finite members in a quasivariety **K** satisfying a set of conditions denoted (P1)–(P4) (see Section 2) is sufficient to guarantee that the ideal lattice of a free lattice with ω free generators is embeddable in $L(\mathbf{K})$ which, in turn, is sufficient to ensure that **K** is Q-universal. An alternative set of conditions guaranteeing Q-universality of a quasivariety is given in Gorbunov [10]. As a consequence, a number of quasivarieties of familiar algebras were thereby seen to be Q-universal (for details, see [1] and [10]).

The principal aim of this paper is to establish the following.

THEOREM 1.1. *If* **K** *is a finite-to-finite universal quasivariety of algebraic systems of finite type, then* **K** *is Q-universal.*

By Theorem 1.1, the following theorem of Sizyĭ is immediate.

COROLLARY 1.2. (Sizyĭ [26]) *The quasivariety of directed graphs* **G** *is Q-universal.*

We will consider the quasivariety of undirected graphs elsewhere and, in particular, there answer a problem of Kravchenko [20] by showing that the quasivariety of undirected graphs is also Q-universal.

At the risk of gross over simplification, for a Q-universal quasivariety **K**, amongst the most interesting properties are that the free lattice on ω free generators is embeddable into $L(\mathbf{K})$ (and, hence, $L(\mathbf{K})$ fails to satisfy any non-trivial lattice identity) and that $|L(\mathbf{K})| = 2^{\omega}$. We remark that, for any finite-to-finite universal quasivariety **K**, it is not difficult to see that $|L(\mathbf{K})| = 2^{\omega}$. Thus, of principal interest for finite-to-finite universal quasivarieties **K** of finite type is that a free lattice on ω free generators is embeddable into $L(\mathbf{K})$ which, therefore, fails to satisfy any non-trivial lattice identity. In fact, what will follow from the proof of Theorem 1.1 is that the ideal lattice of a free lattice on ω free generators is embeddable into $L(\mathbf{K})$ whenever $\mathbf{K}$ is finite-to-finite universal and of finite type.

As stated above, many quasivarieties of algebraic systems are known to be finite-to-finite universal. For example, from Hedrlín and Pultr [16] and Theorem 1.1, we obtain the following.

COROLLARY 1.3. (Gorbunov [10]) *For $n \geq 2$, the quasivariety $\mathbf{A_n}$ of all algebras with n unary operations is Q-universal.*

Even though, as observed in [16], $\mathbf{A_1}$ is not finite-to-finite universal (in fact, not universal), Kartashov [17] showed that the free lattice on ω free generators is embeddable in $L(\mathbf{A_1})$. Later this was strengthened by Gorbunov [10] who showed that $\mathbf{A_1}$ is also Q-universal.

Applying a result of [5] and Theorem 1.1, we also obtain the following.

COROLLARY 1.4. *The quasivariety $\mathbf{P_n}$ of all posets with n distinguished constants is Q-universal iff $n \geq 2$.*

We remark that in [5] it was also shown that the quasivarieties $\mathbf{BD_n}$ of bounded distributive lattices with n distinguished constants and $\mathbf{D_n}$ of distributive lattices with n constants are finite-to-finite universal iff $n \geq 2$ and $n \geq 3$, respectively. Corresponding results for Q-universality have already been established in [2].

From a result of Goralčík, Koubek, and Sichler [9] and Theorem 1.1, we conclude the following.

COROLLARY 1.5. *The following hold*

(i) *if a variety $\mathbf{V}$ of $(0, 1)$-lattices contains a finite non-distributive simple $(0, 1)$-lattice, then $\mathbf{V}$ is Q-universal*;

(ii) *a variety $\mathbf{V}$ of modular $(0, 1)$-lattices is Q-universal iff M_3 belongs to $\mathbf{V}$.*

By inspecting the construction of the functor given in [9] for a variety $\mathbf{V}$ of bounded $(0, 1)$-lattices to be finite-to-finite universal one can obtain a stronger corollary than Corollary 1.5(i). Namely, if a *quasivariety* $\mathbf{V}$ of $(0, 1)$-lattices contains a finite non-distributive simple $(0, 1)$-lattice, then $\mathbf{V}$ is Q-universal.

Compare 1.5 (ii) with the analogous result established in [8] (cf. [1]): a variety $\mathbf{V}$ of modular lattices is Q-universal iff $\mathrm{M}_{3,3}$ belongs to $\mathbf{V}$. Note that, for a variety $\mathbf{V}$ of modular $(0, 1)$-lattices or modular lattices, if M_3 or $\mathrm{M}_{3,3} \notin \mathbf{V}$, respectively, then the lattice of quasivarieties contained in $\mathbf{V}$ forms a countable chain.

For other examples of universal quasivarieties to which Theorem 1.1 can be applied we refer the reader to Demlová and Koubek [6], Koubek [18], Hedrlín and Pultr [15] and [16], as well as Pultr and Trnková [24].

As illustrated by $\mathbf{A_1}$, the converse implication to Theorem 1.1 does not hold. This is not an isolated example. Another is given in Tropin [27] (cf. [1] and [7]) where it is shown that the variety of pseudocomplemented distributive lattices is Q-universal but, as observed in [4], it is not finite-to-finite universal. Here too it is the case that the variety of pseudocomplemented distributive lattices is not actually universal albeit, in a sense that can be made precise, it is almost universal (see [4]).

2. Idea of the proof of Theorem 1.1

For a class $\mathbf{K}$ of algebraic systems of similar type, let $\mathbf{Q}(\mathbf{K})$ denote the quasivariety generated by $\mathbf{K}$ (the smallest quasivariety to contain $\mathbf{K}$): in general, $\mathbf{Q}(\mathbf{K}) = \mathbf{ISPP_u}(\mathbf{K})$. If $\mathbf{K}$ has only finitely many members, say $\mathbf{K} = \{A_0, \ldots, A_{n-1}\}$, then we may write $\mathbf{Q}(A_0, \ldots, A_{n-1})$ rather than $\mathbf{Q}(\{A_0, \ldots, A_{n-1}\})$.

For a set I, we shall denote by $P_{fin}(I)$ the set of all finite subsets of I.

Assume I is an infinite set of cardinality ω and consider an infinite family $(A_W : W \in P_{fin}(I))$ of finite algebraic systems of similar type that satisfy the following conditions, where X, Y, and $Z \in P_{fin}(I)$:

(P1) $A_\emptyset$ is a trivial algebraic system;
(P2) if $X = Y \cup Z$, then $A_X \in \mathbf{Q}(A_Y, A_Z)$;
(P3) if $X \neq \emptyset$ and $A_X \in \mathbf{Q}(A_Y)$, then $X = Y$;
(P4) if A_X is a subsystem of $B \times C$ for finite B and $C \in \mathbf{Q}(\{A_W : W \in P_{fin}(I)\})$, then there exist Y and Z with $A_Y \in \mathbf{Q}(B)$, $A_Z \in \mathbf{Q}(C)$, and $X = Y \cup Z$.

Note that a *trivial algebraic system* A is one whose domain consists of exactly one element, say a, such that, for each function f, $f(a, \ldots, a) = a$ and, for each relation R, $(a, \ldots, a) \in R$.

The method used in this paper for proving that a quasivariety is Q-universal is based on the following proposition which was proved in [1] for algebras. However, a straightforward inspection of the proof given in [1] shows that the proposition is also true for algebraic systems. An alternative method is given in Gorbunov [10].

PROPOSITION 2.1. *If* $\mathbf{K}$ *is a quasivariety of algebraic systems of finite type that contains an infinite family of finite members satisfying* (P1) – (P4)*, then the ideal lattice of a free lattice with* ω *free generators is embeddable in* $L(\mathbf{K})$*. In particular,* $L(\mathbf{K})$ *is* Q*-universal and, therefore, fails every non-trivial lattice identity and is of cardinality* 2^ω.

Let $\mathbf{R}$ denote the quasivariety of commutative rings with a unit. In particular, each member $(R; +, \cdot, -, 0, 1)$ of $\mathbf{R}$ has type $\langle 2, 2, 1, 0, 0\rangle$. Let $\mathbf{K_f}$ denote the full subcategory of $\mathbf{K}$ consisting of all finite systems of $\mathbf{K}$.

A map between two directed graphs is said to be a *strong morphism* in $\mathbf{G}$ if it is onto both on vertices and on edges.

Applying the functors used by Hedrlín and Pultr [16] and [23] to show that every category of algebras of finite type is isomorphic to a full subcategory of **G** we will establish the following in Section 3.

PROPOSITION 2.2. *There exists a functor* **F** *that fully embeds the category* $\mathbf{R_f}$ *into the category* $\mathbf{G_f}$ *such that, whenever* $\varphi : A \longrightarrow B$ *is an onto morphism in* $\mathbf{R_f}$*, then* $\mathbf{F}(\varphi) : \mathbf{F}(A) \longrightarrow \mathbf{F}(B)$ *is a strong morphism in* $\mathbf{G_f}$.

Next, using Proposition 2.1 we will prove the following in Section 4.

PROPOSITION 2.3. *Let* **K** *be a quasivariety of algebraic systems of finite type. If the category* $\mathbf{R_f}$ *is isomorphic to a full subcategory of* $\mathbf{K_f}$ *by a functor* $\mathbf{F} : \mathbf{R_f} \longrightarrow \mathbf{K_f}$ *for which every homomorphism* $\mathbf{F}(\varphi) : \mathbf{F}(A) \longrightarrow \mathbf{F}(B)$ *is onto whenever* $\varphi : A \longrightarrow B$ *is onto, then* **K** *is* Q*-universal.*

Let $\mathcal{G}$ denote the full subcategory of **G** that is determined by the following two objects: the singleton void graph $V = (\{0\}, \emptyset)$ and the two-element graph $E = (\{0, 1\}, \{(0, 1)\})$; morphisms in **G** are mappings that preserve edges. Let $\mathcal{E}(\mathbf{G})$ denote the class of morphisms in **G** that are strong (see above) and $\mathcal{M}(\mathbf{G})$ the class of morphisms in **G** that are injective.

To complete the proof of Theorem 1.1, we will need a proposition which is a particular case of Theorem 1.2 established in Koubek and Sichler [19]. In order to state the proposition, some categorical notations are required.

For a category **K**, let $\mathbf{F} : \mathbf{J} \longrightarrow \mathbf{K}$ be a *diagram* in **K**, that is a functor from an index category **J** to **K**. A *cone* of **F** is an object $\mathcal{B}$ of **K** together with a family $(\psi_i : \mathbf{F}(i) \longrightarrow \mathcal{B} : i \in obj(\mathbf{J}))$ of **K**-morphisms such that, for every arrow $u : i \longrightarrow j$ in **J**, $\psi_i = \psi_j \circ \mathbf{F}(u)$. A *colimit* of a diagram $\mathbf{F} : \mathbf{J} \longrightarrow \mathbf{K}$ is a cone $(\psi_i : \mathbf{F}(i) \longrightarrow \mathcal{B} : i \in obj(\mathbf{J}))$ of **F** such that, for every other cone $(\varphi_i : \mathbf{F}(i) \longrightarrow \mathcal{C} : i \in obj(\mathbf{J}))$ of **F**, there exists a unique **K**-morphism $\sigma : \mathcal{B} \longrightarrow \mathcal{C}$ such that, for every $i \in obj(\mathbf{J})$, $\varphi_i = \sigma \circ \psi_i$. Then **K** is *cocomplete* providing colimits of all diagrams in **K** exist.

A *factorization system* $(\mathcal{E}, \mathcal{M})$ for **K** consists of some category $\mathcal{E}$ of **K**-epimorphisms and some category $\mathcal{M}$ of **K**-monomorphisms such that, for every **K**-morphism f, there exists a decomposition $f = m \circ e$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, and the *diagonalization property* holds (that is, for $h \circ e = m \circ k$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, $g \circ e = k$ and $m \circ g = h$ for some **K**-morphism g).

A family $(\varphi_i : \mathcal{A}_i \longrightarrow \mathcal{B} : i \in I)$ of **K**-morphisms is called a *sink* in **K**. A sink $(\varphi_i : \mathcal{A}_i \longrightarrow \mathcal{B} : i \in I)$ in **K** is said to be an $\mathcal{E}(\mathbf{K})$*-sink* if, for every sink $(\psi_i : \mathcal{A}_i \longrightarrow \mathcal{C} : i \in I)$ in **K** and every **K**-morphism $\sigma : \mathcal{C} \longrightarrow \mathcal{B}$, $\varphi_i = \sigma \circ \psi_i$ for all $i \in I$ implies that σ is in $\mathcal{E}(\mathbf{K})$.

Finally, $(\mathcal{E}(\mathbf{K}), \mathcal{M}(\mathbf{K}))$ *factorizes sinks* in **K** if, for every sink in **K** $(\varphi_i : \mathcal{A}_i \longrightarrow \mathcal{B} : i \in I)$ there exist an $\mathcal{E}(\mathbf{K})$-sink $(\psi_i : \mathcal{A}_i \longrightarrow \mathcal{C} : i \in I)$ and $\sigma : \mathcal{C} \longrightarrow \mathcal{B}$ in $\mathcal{M}(\mathbf{K})$ such that $\varphi_i = \sigma \circ \psi_i$ for all $i \in I$.

The following proposition is a particular case of Theorem 1.2 from Koubek and Sichler [19]; that $\mathbf{\Phi}$ is finite-to-finite follows from the proof of Theorem 1.2 given in [19].

PROPOSITION 2.4. *Suppose that* $\mathbf{K}$ *is a cocomplete category of algebraic systems with homomorphisms as morphisms and with a factorization system* $(\mathcal{E}, \mathcal{M})$. *If* $\mathbf{F} : \mathbf{G} \longrightarrow \mathbf{K}$ *is a full embedding that is finite-to-finite, then there is a functor* $\mathbf{\Phi} : \mathbf{G} \longrightarrow \mathbf{K}$ *having the following properties*:

(i) $\mathbf{\Phi}$ *is finite-to-finite, full, and faithfull*;
(ii) *if* $(\mathcal{E}, \mathcal{M})$ *factorizes sinks in* $\mathbf{K}$ *and if* $\mathbf{F}\restriction\mathcal{G}$ *maps every* $\mathcal{E}(\mathbf{G})$*-sink in* $\mathcal{G}$ *to an* $\mathcal{E}(\mathbf{K})$*-sink, then* $\mathbf{\Phi}(\mathcal{E}(\mathbf{G})) \subseteq \mathcal{E}$.

We will show in Section 5 that, for a finite-to-finite universal quasivariety $\mathbf{K}$ of algebraic systems, all of the assumptions of Proposition 2.4 concerning $\mathbf{K}$ are satisfied by a choice of $\mathcal{E}$ and $\mathcal{M}$ whereby every member of $\mathcal{E}$ is also an onto map. Theorem 1.1 will then follow from Propositions 2.4, 2.2, and 2.3. Indeed, let $\mathbf{K}$ be a finite-to-finite universal quasivariety of algebraic systems of finite type. By Proposition 2.4, $\mathbf{\Phi} : \mathbf{G} \longrightarrow \mathbf{K}$ is a finite-to-finite full embedding such that, for all G and H in $\mathbf{G_f}$, $\mathbf{\Phi}(f) : \mathbf{\Phi}(G) \longrightarrow \mathbf{\Phi}(H)$ is onto whenever $f : G \longrightarrow H$ is strong in $\mathbf{G}$. This, by Proposition 2.2, yields a finite-to-finite and full embedding $\mathbf{\Phi} \circ \mathbf{F} : \mathbf{R_f} \longrightarrow \mathbf{K_f}$ that satisfies the assumptions of Proposition 2.3. Thus, by Proposition 2.3, $\mathbf{K}$ is Q-universal.

3. Proof (Proposition 2.2)

For algebraic systems A and B of similar type, a mapping $\varphi : A \longrightarrow B$ is a *homomorphism* if $\varphi(c) = c$ for each constant, $\varphi(f(a_0, \ldots, a_{n-1})) = f(\varphi(a_0), \ldots, \varphi(a_{n-1}))$ for each function f of non-zero arity, and $\varphi(R) \subseteq R$ for each relation R. A homomorphism $\varphi : A \longrightarrow B$ is *strong* provided that φ is onto and every relation in B is an image of φ of the corresponding relation in A, that is, if S is an n-ary relation in B and the corresponding relation in A is R, then $S = \{(\varphi(a_0), \ldots, \varphi(a_{n-1})) : (a_0, \ldots, a_{n-1}) \in R\}$. (In particular, for an algebra, every onto homomorphism is strong.)

A *directed graph* is a pair $(X; R)$ where $R \subseteq X \times X$. For directed graphs $(X; R)$ and $(Y; S)$ regarded as algebraic systems, a map $\varphi : X \longrightarrow Y$ is *compatible* providing it is a homomorphism (namely, for $a, b \in X$, $(\varphi(a), \varphi(b)) \in S$ whenever $(a, b) \in R$). Thus $\mathbf{G}$ denotes category of all directed graphs together with all compatible maps. Clearly, the category $\mathbf{G_f}$ (all finite directed graphs) is a full subcategory of $\mathbf{G}$. Recall (see above), a compatible map $\varphi : X \longrightarrow Y$ is *strong* if it is onto and $S = \{(\varphi(a), \varphi(b)) : (a, b) \in R\}$.

For a given fixed finite type Δ, let $\mathbf{A}(\Delta)$ denote the category of all algebras of type Δ where morphisms are the usual homomorphisms. In Pultr [23] and Hedrlín and Pultr [16], for any given fixed finite type Δ, a sequence of functors are given to establish that $\mathbf{A}(\Delta)$ is isomorphic to a full subcategory of $\mathbf{G}$. The objective of this section is to consider the functors given by them as applied to $\mathbf{A}(\Delta)$, where throughout Δ has the same fixed finite type $\langle 2, 1, 1, 0, 0\rangle$ as $\mathbf{R}$, and thereby establish 2.2. We remark that in [23], Pultr operates in a set theory that does not allow inaccessible cardinals. Even though this set theoretic assumption was later removed in Vopěnka, Hedrlín and Pultr [28], it in no way affects our considerations since we are concerned only with $\mathbf{R_f}$.

Following [16] (page 394), for $\Delta = \{k_i : 1 \le i \le n\}$, let $\Delta^* = \{k_i + 1 : 1 \le i \le n\}$. If $\mathbf{R}(\Delta^*)$ denotes the category of all relational systems of type Δ^* where morphisms are all maps which preserve the corresponding relations of Δ^*, then there exists a full and faithful functor $\mathbf{F_1} : \mathbf{A}(\Delta) \longrightarrow \mathbf{R}(\Delta^*)$ given as follows. For an algebra $(A; f_1, \dots, f_n)$ in $\mathbf{A}(\Delta)$ and for each $1 \le i \le n$, define a relation R_i by $(a_1, \dots, a_{k_i}, a_{k_i+1}) \in R_i$ in $\mathbf{F_1}(A)$ iff $f_i(a_1, \dots, a_{k_i}) = a_{k_i+1}$ in A. Then $\mathbf{F_1}(A) = (A; R_1, \dots, R_n)$. For a homomorphism $\varphi : A \longrightarrow B$ in $\mathbf{A}(\Delta)$, $\mathbf{F_1}(\varphi) = \varphi$.

Clearly, for a finite algebra A in $\mathbf{A}(\Delta)$, $\mathbf{F_1}(A)$ is a finite relational system. Clearly too, if $\varphi : (A; f_1, \dots, f_n) \longrightarrow (B; f_1, \dots, f_n)$ is an onto homomorphism in $\mathbf{A}(\Delta)$, then $\mathbf{F_1}(\varphi) : \mathbf{F_1}(A) \longrightarrow \mathbf{F_1}(B)$ is a strong homomorphism in $\mathbf{R}(\Delta^*)$.

To summarize, the functor $\mathbf{F_1} : \mathbf{A}(\Delta) \longrightarrow \mathbf{R}(\Delta^*)$ is a full and faithful functor which assigns a finite relational system of $\mathbf{R}(\Delta^*)$ to every finite algebra in $\mathbf{A}(\Delta)$ and which assigns to every onto homomorphism in $\mathbf{A}(\Delta)$ a strong homomorphism in $\mathbf{R}(\Delta^*)$.

Following [16] (page 394, the proof of Theorem 1), for a fixed given set A which will be specified below, let $\mathbf{A}(A)$ denote the category of all unary algebras $(X; \{f_a : a \in A\})$ where the morphisms are the usual homomorphisms. Then there is a full and faithful functor $\mathbf{F_2} : \mathbf{R}(\Delta^*) \longrightarrow \mathbf{A}(A)$ as now prescribed.

For $\Delta^* = \{k_i + 1 : 1 \le i \le n\}$, set

$$A = \{(i, j) : 1 \le i \le n \text{ and } 1 \le j \le k_i + 1\} \cup \{1, 2, 3\}.$$

Then $\mathbf{F_2}$ is given as follows. For $(X; R_1, \dots, R_n)$ an object of $\mathbf{R}(\Delta^*)$, set $\mathbf{F_2}(X) = (X \cup \bigcup(\{i\} \times R_i : 1 \le i \le n) \cup \{u, v\}; \{f_{ij} : 1 \le i \le n \text{ and } 1 \le j \le k_i + 1\} \cup \{f_1, f_2, f_3\})$ where u and v are disinct elements not belonging to $X \cup \bigcup(\{i\} \times R_i : 1 \le i \le n)$ and

$$\begin{aligned} f_{ij}(i, (x_l : 1 \le l \le k_i + 1)) &= x_j \quad \text{for all } (x_l : 1 \le l \le k_i + 1) \in R_i, \\ f_{ij}(\xi) &= u \quad \text{otherwise}, \\ f_1(\xi) &= u \quad \text{for all } \xi, \end{aligned}$$

$$\begin{aligned} f_2(\xi) &= \begin{cases} v & \text{if } \xi \ne v, \\ u & \text{if } \xi = v, \end{cases} \\ f_3(\xi) &= \begin{cases} u & \text{if } \xi \ne u, \\ v & \text{if } \xi = u. \end{cases} \end{aligned}$$

Given a morphism $\varphi : X \longrightarrow Y$ in $\mathbf{R}(\Delta^*)$, let

$$\begin{aligned} \mathbf{F_2}(\varphi)(u) &= u, \\ \mathbf{F_2}(\varphi)(v) &= v, \\ \mathbf{F_2}(\varphi)(x) &= \varphi(x) \quad \text{for all } x \in X, \text{ and} \\ \mathbf{F_2}(\varphi)((i, (x_l : 1 \leq l \leq k_i + 1))) &= (i, (\varphi(x_l) : 1 \leq l \leq k_i + 1)) \\ &\quad \text{for } (x_l : 1 \leq l \leq k_i + 1) \in R_i. \end{aligned}$$

Clearly, for a finite relational system X of $\mathbf{R}(\Delta^*)$, $\mathbf{F_2}(X)$ is a finite algebra $\mathbf{F_2}(X)$ since Δ^* is finite. Suppose that $\varphi : X \longrightarrow Y$ is a strong homomorphism in $\mathbf{R}(\Delta^*)$. We claim that $\mathbf{F_2}(\varphi) : \mathbf{F_2}(X) \longrightarrow \mathbf{F_2}(Y)$ is onto in $\mathbf{A}(A)$. Since φ is onto, it is clear that $Y \cup \{u, v\} \subseteq \mathbf{F_2}(\varphi)(X \cup \{u, v\})$. Thus, to justify the claim it is only necessary to consider elements of the form $(i, (y_l : 1 \leq l \leq k_i + 1))$ for some $1 \leq i \leq n$ and $(y_l : 1 \leq l \leq k_i + 1) \in R_i$ in Y. As φ is strong, $(y_l : 1 \leq l \leq k_i + 1) = (\varphi(x_l) : 1 \leq l \leq k_i + 1)$ for some $(x_l : 1 \leq l \leq k_i + 1) \in R_i$. But then $\mathbf{F_2}(\varphi)(i, (x_l : 1 \leq l \leq k_i + 1)) = (i, (\varphi(x_l) : 1 \leq l \leq k_i + 1)) = (i, (y_l : 1 \leq l \leq k_i + 1))$, showing $\mathbf{F_2}(\varphi)$ is onto as claimed.

To summarize, the functor $\mathbf{F_2} : \mathbf{R}(\Delta^*) \longrightarrow \mathbf{A}(A)$ is a full and faithful functor which assigns a finite algebra of $\mathbf{A}(A)$ to every finite relational system in $\mathbf{R}(\Delta^*)$ and which assigns to every strong homomorphism in $\mathbf{R}(\Delta^*)$ an onto homomorphism in $\mathbf{A}(A)$.

Following [16] (page 394), for the set A given in the definition of $\mathbf{F_2}$, let $\mathbf{R}(A)$ denote the category whose objects are relational systems $(X; \{R_a : a \in A\})$ where, for $a \in A$, $R_a \subseteq X \times X$ and morphisms are the usual homomorphisms (namely, for $(X; \{R_a : a \in A\})$ and $(Y; \{R_a : a \in A\})$, $\varphi : X \longrightarrow Y$ is a homomorphism providing, for every $a \in A$, $(\varphi(x), \varphi(y)) \in R_a$ in Y whenever $(x, y) \in R_a$ in X). Then there is a full and faithful functor $\mathbf{F_3} : \mathbf{A}(A) \longrightarrow \mathbf{R}(A)$ defined as follows. For a unary algebra $(X; \{f_a : a \in A\})$ in $\mathbf{A}(A)$, $\mathbf{F_3}(X) = (X; \{R_a : a \in A\})$ where, for all $a \in A$ and $x, y \in X$, $(x, y) \in R_a$ iff $f_a(x) = y$. For a homomorphism $\varphi : X \longrightarrow Y$ in $\mathbf{A}(A)$, $\mathbf{F_3}(\varphi) = \varphi$.

Clearly, for a finite algebra X in $\mathbf{A}(A)$, $\mathbf{F_3}(X)$ is a finite relational system in $\mathbf{R}(A)$. Equally obvious is the fact that $\mathbf{F_3}(\varphi)$ is an onto homomorphism in $\mathbf{R}(A)$ whenever $\varphi : X \longrightarrow Y$ is an onto homomorphism in $\mathbf{A}(A)$. Furthermore, if, $(x, y) \in R_a$ in $\mathbf{F_3}(Y)$ for some $a \in A$, then $y = f_a(x)$ in Y. Since $\varphi : X \longrightarrow Y$ is an onto homomorphism in $\mathbf{A}(A)$, $x = \varphi(x')$ and $y = \varphi(y')$ for some x' and y' in X with $y' = f_a(x')$. Since $(x', f_a(y')) = (x', y') \in R_a$ in $\mathbf{F_3}(X)$ and $(x, y) = (\varphi(x'), \varphi(y'))$, $\mathbf{F_3}(\varphi)$ is a strong homomorphism in $\mathbf{R}(A)$.

To summarize, the functor $\mathbf{F_3} : \mathbf{A}(A) \longrightarrow \mathbf{R}(A)$ is a full and faithful functor which assigns a finite relational system in $\mathbf{R}(A)$ to each finite algebra in $\mathbf{A}(A)$ and which assigns to every onto homomorphism in $\mathbf{A}(A)$ a strong homomorphism in $\mathbf{R}(A)$.

Following [23] (page 232, the proof of Theorem 2.5), let $\mathbf{G_a}$ denote the category of all directed graphs $(X; R)$ such that $(x, x) \notin R$ for any $x \in X$ and, for each $y \in X$, $(x, y) \in R$ for some $x \in X$. Obviously, $\mathbf{G_a}$ together with all compatible mappings is a full subcategory of $\mathbf{G}$. Then there is a full and faithful functor $\mathbf{F_4} : \mathbf{R}(A) \longrightarrow \mathbf{G_a}$. That functor, as given in [23], will now be defined.

Since A is finite, it is possible to choose a finite *rigid* (in the sense that the only compatible map from the graph to itself is the identity map) directed graph $(B; E)$ belonging to $\mathbf{G_a}$ such that $|B| \geq |A| + 1$. Let $(B; E)$ denote some such fixed choice for which, in addition, the length of any cycle in $(B; E)$ is divisible by either 2 or 3. Let p_i, for $1 \leq i \leq 4$, be mutually different primes each of which is distinct from 2 and 3. Let $U = \bigcup(U_i : 1 \leq i \leq 4)$ where, for $1 \leq i \leq 4$, $U_i = \{u_{i,j} : 1 \leq j \leq p_i\}$ are sets of formally distinct elements. Since $|A| + 1 \leq |B|$, we may choose a one-to-one mapping $\alpha : A \longrightarrow B$ and $b_\alpha \in B \backslash \alpha(A)$. For $i = 1, 2$, set $B_i = \{(b, i) : b \in B\}$ and $\alpha_i : A \longrightarrow B_i$ be given by $\alpha_i(a) = (\alpha(a), i)$.

For $(X; \{R_a : a \in A\})$ an object in $\mathbf{R}(A)$, let $X_a = \{(x, y, a) : (x, y) \in R_a\}$, $X_A = \bigcup(X_a : a \in A)$, and set

$$\mathbf{F_4}(X) = (X \cup X_A \cup U \cup B_1 \cup B_2; R)$$

where $R \subseteq (X \cup X_A \cup U \cup B_1 \cup B_2) \times (X \cup X_A \cup U \cup B_1 \cup B_2)$ will now be specified. R contains precisely the following elements:

$(u_{i,j}, u_{i,j+1})$ for $1 \leq i \leq 4$ and $i \leq j \leq p_i - 1$;

$(u_{i,p_i}, u_{i,1})$ for $1 \leq i \leq 4$;

$(u_{i,1}, (b, 1))$ for $i = 1, 2$ and $(b, 1) \in B_1$;

$(u_{i,1}, (b, 2))$ for $i = 3, 4$ and $(b, 2) \in B_2$;

$((b, i), (b', i))$ for $i = 1, 2$ and $(b, b') \in E$;

$((b_\alpha, i), x)$ for $i = 1, 2$ and $x \in X$;

$(x, (x, y, a)), ((x, y, a), y)$, and $(\alpha_i(a), (x, y, a))$
for $i = 1, 2$, $(x, y) \in R_a$, and $a \in A$.

Given a homomorphism $\varphi : (X; \{R_a : a \in A\}) \longrightarrow (Y; \{S_a : a \in A\})$ in $\mathbf{R}(A)$, let $\mathbf{F_4}(\varphi) : \mathbf{F_4}(X) \longrightarrow \mathbf{F_4}(Y)$ in $\mathbf{G_a}$ be given by

$$\begin{aligned} \mathbf{F_4}(\varphi)(x) &= \varphi(x) && \text{for } x \in X, \\ \mathbf{F_4}(\varphi)((x, y, a)) &= (\varphi(x), \varphi(y), a) && \text{for } x, y \in X \text{ and } a \in A, \\ \mathbf{F_4}(\varphi)(x) &= x && \text{for } x \in U \cup B_1 \cup B_2. \end{aligned}$$

If X is a finite relational system in $\mathbf{R}(A)$, then, since both A and B are finite, X_A, B_1, and B_2 are finite. By the choice of U, $X \cup X_A \cup U \cup B_1 \cup B_2$ is finite and, in particular, $\mathbf{F_4}(X)$ is a finite directed graph in $\mathbf{G_a}$.

Let $\varphi : (X; \{R_a : a \in A\}) \longrightarrow (Y; \{S_a : a \in A\})$ be an onto and strong homomorphism in $\mathbf{R}(A)$, then it is to be shown that the compatible mapping $\mathbf{F_4}(\varphi) : \mathbf{F_4}(X) \longrightarrow \mathbf{F_4}(Y)$ is strong in $\mathbf{G_a}$.

Since $\mathbf{F_4}(\varphi)$ is the identity on $U \cup B_1 \cup B_2$ and, for $x \in X$, $\mathbf{F_4}(\varphi)(x) = \varphi(x)$, to see that $\mathbf{F_4}(\varphi)$ is onto, it is only necessary to consider elements of Y_A. For $(x, y, a) \in Y_A$,

$(x, y) \in S_a$. Since φ is strong, there exist $x', y' \in X$ such that $(x', y') \in R_a$ and $(x, y) = (\varphi(x'), \varphi(y'))$. Thus,

$$\begin{aligned}\mathbf{F_4}(\varphi)((x', y', a)) &= (\varphi(x'), \varphi(y'), a)\\ &= (x, y, a),\end{aligned}$$

as required.

Similar to the above, to see that $\mathbf{F_4}(\varphi)$ is strong, it is only necessary to consider edges in $\mathbf{F_4}(Y)$ that contain an element of Y_A. For $(x, (x, y, a)) \in S$ in $\mathbf{F_4}(Y)$, $(x, y) \in S_a$. Since $\varphi : X \longrightarrow Y$ is strong, $(x, y) = (\varphi(x'), \varphi(y'))$ for some $(x', y') \in R_a$. Thus, $(x', (x', y', a)) \in R$ while $\mathbf{F_4}(\varphi)(x') = \varphi(x') = x$ and $\mathbf{F_4}(\varphi)((x', y', a)) = (\varphi(x'), \varphi(y'), a) = (x, y, a)$, as required. Likewise, for $((x, y, a), y) \in S$ in $\mathbf{F_4}(Y)$, $(x, y) \in S_a$ and $(x, y) = (\varphi(x'), \varphi(y'))$ for some $(x', y') \in R_a$. In particular, as to be shown, $\mathbf{F_4}(\varphi)((x', y', a)) = (\varphi(x'), \varphi(y'), a) = (x, y, a)$ and $\mathbf{F_4}(\varphi)(y') = \varphi(y') = y$. Finally, for $(\alpha_i(a), (x, y, a)) \in S$ with $i = 1, 2$, $(x, y) \in S_a$ and, once more, there are $x', y' \in X$ with $(x, y) = (\varphi(x'), \varphi(y'))$ and $(x', y') \in R_a$. Then $\mathbf{F_4}(\varphi)(\alpha_i(a)) = \alpha_i(a)$ and $\mathbf{F_4}(\varphi)((x', y', a)) = (x, y, a)$, as desired.

To summarize, the functor $\mathbf{F_4} : \mathbf{R}(A) \longrightarrow \mathbf{G_a}$ is a full and faithful functor which assigns a finite directed graph in $\mathbf{G_a}$ to each finite relational system in $\mathbf{R}(A)$ and which assigns a strong compatible map in $\mathbf{G_a}$ to every strong homomorphism in $\mathbf{R}(A)$.

Thus, as shown above, the functor $\mathbf{F} : \mathbf{A}(\Delta) \longrightarrow \mathbf{G_a}$ given by

$$\mathbf{F} = \mathbf{F_4} \circ \mathbf{F_3} \circ \mathbf{F_2} \circ \mathbf{F_1}$$

is a full and faithful functor such that a finite algebra in $\mathbf{A}(\Delta)$ is assigned to a finite directed graph in $\mathbf{G_a}$ and an onto homomorphism in $\mathbf{A}(\Delta)$ is assigned to a strong compatible map in $\mathbf{G_a}$. Since $\mathbf{R}$ is a full subcategory of $\mathbf{A}(\Delta)$ and $\mathbf{G_a}$ is a full subcategory of $\mathbf{G}$, the functor $\mathbf{F} : \mathbf{R_f} \longrightarrow \mathbf{G_f}$ establishes Proposition 2.2, thereby completing this Section.

4. Proof (Proposition 2.3)

The aim of this section is to establish Proposition 2.3.

A homomorphism φ is an *embedding* of A into B if it is one-to-one and, for every n-ary relation R in A and its corresponding relation S in B, for all $a_0, \ldots, a_{n-1} \in A$, $(a_0, \ldots, a_{n-1}) \in R$ iff $(\varphi(a_0), \ldots, \varphi(a_{n-1})) \in S$. A is *embeddable into* B if there is an embedding φ of A into B. An *isomorphism* is an onto embedding. If $A \subseteq B$ and the identity mapping is an embedding of A into B, then A is a *subsystem* of B. Moreover, if $\varphi : A \longrightarrow B$ is a homomorphism, then the subsystem of B determined by $\varphi(A)$ is called a *homomorphic image of* A *under* φ and is often denoted by $Im(\varphi)$.

Let B and $(B_i : i \in I)$ be algebraic systems of similar type and $(f_i : i \in I)$ be a family of homomorphisms such that, for $i \in I$, $f_i : B \longrightarrow B_i$. We shall denote by

$\langle f_i : i \in I\rangle$ the homomorphism from B into $\prod(B_i : i \in I)$ defined as follows: for $x \in B$, $\pi_i(\langle f_i : i \in I\rangle)(x) = f_i(x)$ for all $i \in I$, where π_i denotes the projection map from $\prod(B_i : i \in I)$ onto B_i. In particular, for $i \in I$, $\pi_i \circ \langle f_i : i \in I\rangle = f_i$, that is the diagram of Figure 1 commutes.

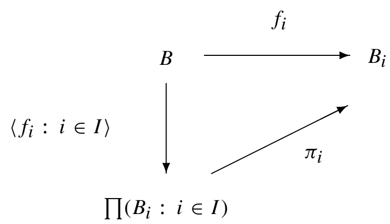

Figure 1

Throughout this section, let $\mathbf{K}$ be a quasivariety of algebraic systems of finite type and $\mathbf{F} : \mathbf{R_f} \longrightarrow \mathbf{K_f}$ be a full and faithful functor for which $\mathbf{F}(\varphi) : \mathbf{F}(A) \longrightarrow \mathbf{F}(B)$ is an onto homomorphism whenever $\varphi : A \longrightarrow B$ is onto.

Using the functor $\mathbf{F}$ we will define a family $(A_W \in \mathbf{K_f} : W \in P_{fin}(I))$ that satisfies (P1)–(P4) of Section 2 for a suitable countably infinite set I. With this in mind, we pause to consider $\mathbf{R_f}$.

For a prime p, let $(Z_p; +, \cdot, -, 0, 1)$ denote the ring of integers modulo p. Since Z_p is simple and the unit is a constant, the only endomorphism of Z_p is the identity map. In particular, for a prime q, if $\varphi : Z_p \longrightarrow Z_q$ is a homomorphism, then $p = q$ and φ is the identity.

Since $\mathbf{R}$ has the Fraser-Horn property (namely, for all $R_0, R_1 \in \mathbf{R}$, whenever Θ is a congruence relation on $R_0 \times R_1$, there exist congruence relations Θ_0 and Θ_1 on R_0 and R_1, respectively, such that Θ is of the form $\Theta_0 \times \Theta_1$), the following is immediate.

LEMMA 4.1. *Let $K \cup \{p\}$ be a finite set of prime numbers. If $\varphi : \prod(Z_i : i \in K) \longrightarrow Z_p$ is a homomorphism, then $p \in K$ and $\varphi = \pi_p$, where π_p is the p-th projection.*

LEMMA 4.2. *Let K and J be finite sets of prime numbers. If $\varphi : \prod(Z_i : i \in K) \longrightarrow \prod(Z_j^{n_j} : j \in J)$ is a homomorphism, where $n_j \geq 1$ for all $j \in J$, then* (i) $J \subseteq K$; (ii) *if φ is one-to-one, then $K \subseteq J$.*

Let I be some fixed infinite set of prime numbers. We now define a suitable family $(A_W \in \mathbf{K_f} : W \in P_{fin}(I))$. Recall that $P_{fin}(I)$ denotes all finite subsets of the set I. Obviously, as required by condition (P1), $A_\emptyset$ is defined to be a trivial algebraic system from $\mathbf{K}$. For $\emptyset \neq W \in P_{fin}(I)$, A_W is defined as the homomorphic image of $\mathbf{F}(\prod(Z_i : i \in W))$ in $\prod(\mathbf{F}(Z_i) : i \in W)$ under the homomorphism $\langle \mathbf{F}(\pi_i) : i \in W\rangle$.

For $\emptyset \neq W \in P_{fin}(I)$, some words of clarification on the definition of A_W are in order. Since the i-th projection $\pi_i : \prod(Z_i : i \in W) \longrightarrow Z_i$ is a ring homomorphism for each $i \in W$, the homomorphism $\mathbf{F}(\pi_i)$ assigned to π_i sends the algebraic system $\mathbf{F}(\prod(Z_i : i \in W))$ into the algebraic system $\mathbf{F}(Z_i)$. In other words, for each $i \in W$, we have a homomorphism

$$\mathbf{F}(\pi_i) : \mathbf{F}\left(\prod(Z_i : i \in W)\right) \longrightarrow \mathbf{F}(Z_i).$$

As such there is a well-defined homomorphism

$$\langle \mathbf{F}(\pi_i) : i \in W \rangle : \mathbf{F}\left(\prod(Z_i : i \in W)\right) \longrightarrow \prod(\mathbf{F}(Z_i) : i \in W).$$

By definition, A_W is the homomorphic image of $\mathbf{F}(\prod(Z_i : i \in W))$ in $\prod(\mathbf{F}(Z_i) : i \in W)$ under this homomorphism. In particular, A_W is a substructure of $\prod(\mathbf{F}(Z_i) : i \in W)$, which we denote by $A_W \leq \prod(\mathbf{F}(Z_i) : i \in W)$, whose operations and relations are defined as follows: if f is an n-ary operation for some $n \geq 1$, then, for $a_0, a_1, \ldots, a_{n-1} \in A_W \leq \prod(\mathbf{F}(Z_i) : i \in W)$,

$$f(a_0, a_1, \ldots, a_{n-1})(i) = f(a_0(i), a_1(i), \ldots, a_{n-1}(i)),$$

where $i \in W$; if R is an n-ary relation for some $n \geq 1$, then for $a_0, a_1, \ldots, a_{n-1} \in A_W \leq \prod(\mathbf{F}(Z_i) : i \in W)$,

$$(a_0, a_1, \ldots, a_{n-1}) \in R \text{ in } A_W \text{ iff } (a_0(i), a_1(i), \ldots, a_{n-1}(i)) \in R \text{ in } \mathbf{F}(Z_i)$$

for each $i \in W$.

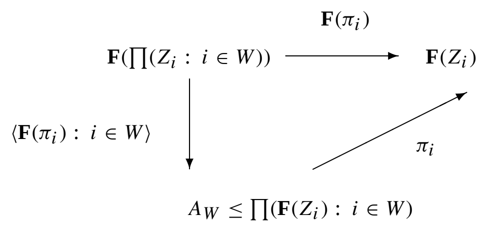

Figure 2

Observe that, for $i \in W$, $\pi_i \circ \langle \mathbf{F}(\pi_i) : i \in W \rangle = \mathbf{F}(\pi_i)$. In particular, the diagram of Figure 2 commutes.

Condition (P2) follows from the next lemma.

LEMMA 4.3. *Let X, Y, and $Z \in P_{fin}(I)$. If $X = Y \cup Z$, then A_X is embeddable in $A_Y \times A_Z$.*

Proof. Since $A_\emptyset$ is a trivial algebraic system, we may assume that $Y \neq \emptyset$ and $Z \neq \emptyset$. Let $\pi_Y : \prod(Z_i : i \in X) \longrightarrow \prod(Z_i : i \in Y)$ and $\pi_Z : \prod(Z_i : i \in X) \longrightarrow \prod(Z_i : i \in Z)$ denote suitable projections of the ring $\prod(Z_i : i \in X)$. Let $\widetilde{\pi}_Y : \prod(\mathbf{F}(Z_i) : i \in X) \longrightarrow \prod(\mathbf{F}(Z_i) : i \in Y)$ and $\widetilde{\pi}_Z : \prod(\mathbf{F}(Z_i) : i \in X) \longrightarrow \prod(\mathbf{F}(Z_i) : i \in Z)$ denote suitable projections of the algebraic system $\prod(\mathbf{F}(Z_i) : i \in X)$. See Figure 3.

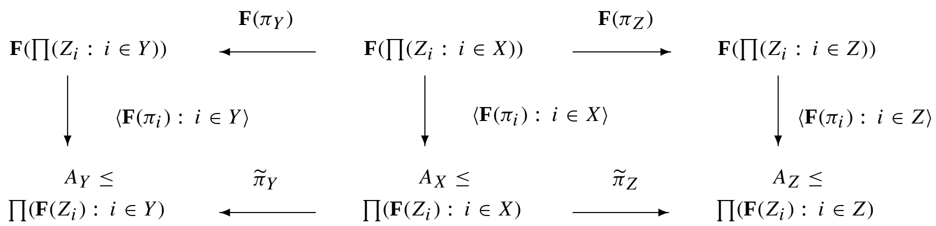

Figure 3

Since $X = Y \cup Z$, for $a, b \in \prod(\mathbf{F}(Z_i) : i \in X)$, $a = b$ iff $\widetilde{\pi}_Y(a) = \widetilde{\pi}_Y(b)$ and $\widetilde{\pi}_Z(a) = \widetilde{\pi}_Z(b)$. Thus, in order to show that A_X is embeddable in $A_Y \times A_Z$, it is sufficient to show the following:

(i) $\widetilde{\pi}_Y(A_X) \subseteq A_Y$ and $\widetilde{\pi}_Z(A_X) \subseteq A_Z$;
(ii) if R is an n-ary relation from the type of $\mathbf{K}$ and $a_0, a_1, \dots, a_{n-1}$ are elements of A_X, then $(a_0, a_1, \dots, a_{n-1}) \in R$ in A_X whenever $((\widetilde{\pi}_Y(a_0), \widetilde{\pi}_Z(a_0)), (\widetilde{\pi}_Y(a_1), \widetilde{\pi}_Z(a_1)), \dots, (\widetilde{\pi}_Y(a_{n-1}), \widetilde{\pi}_Z(a_{n-1}))) \in R$ in $A_Y \times A_Z$.

We begin by showing $\widetilde{\pi}_Y(A_X) \subseteq A_Y$ and merely observe that a similar argument shows that $\widetilde{\pi}_Z(A_X) \subseteq A_Z$.

Let $a \in A_X$. We need to find $b \in \mathbf{F}(\prod(Z_i : i \in Y))$ such that $\langle \mathbf{F}(\pi_i) : i \in Y\rangle(b) = \widetilde{\pi}_Y(a)$. Since $\langle \mathbf{F}(\pi_i) : i \in Y\rangle(b) \in A_Y$, this would give $\widetilde{\pi}_Y(a) \in A_Y$ and, so, $\widetilde{\pi}_Y(A_X) \subseteq A_Y$ as required.

Since $a \in A_X$, $a = \langle \mathbf{F}(\pi_i) : i \in X\rangle(x)$ for some $x \in \mathbf{F}(\prod(Z_i : i \in X))$. We set $b = \mathbf{F}(\pi_Y)(x)$. Thus, it remains to show that

$$\langle \mathbf{F}(\pi_i) : i \in Y\rangle(\mathbf{F}(\pi_Y)(x)) = \widetilde{\pi}_Y(a).$$

Since both sides of this equation belong to $\prod(\mathbf{F}(Z_i) : i \in Y)$, we need to show that, for each projection $\pi_j : \prod(\mathbf{F}(Z_i) : i \in Y) \longrightarrow \mathbf{F}(Z_j)$, where $j \in Y$,

$$\pi_j(\langle \mathbf{F}(\pi_i) : i \in Y\rangle(\mathbf{F}(\pi_Y)(x))) = \pi_j(\widetilde{\pi}_Y(a)).$$

We have

$$\begin{aligned}\pi_j(\langle \mathbf{F}(\pi_i) : i \in Y\rangle(\mathbf{F}(\pi_Y)(x))) &= (\pi_j \circ \langle \mathbf{F}(\pi_i) : i \in Y\rangle)(\mathbf{F}(\pi_Y)(x)) \\ &= \mathbf{F}(\pi_j)(\mathbf{F}(\pi_Y)(x)) \\ &= (\mathbf{F}(\pi_j) \circ \mathbf{F}(\pi_Y))(x) \\ &= \mathbf{F}(\pi_j \circ \pi_Y)(x) \quad \text{(since } \mathbf{F} \text{ is a functor)} \\ &= \mathbf{F}(\pi_j)(x) \\ &= \pi_j(a) \quad \text{(because } a = \langle \mathbf{F}(\pi_i) : i \in X\rangle(x)) \\ &= (\pi_j \circ \widetilde{\pi}_Y)(a) \\ &= \pi_j(\widetilde{\pi}_Y(a)).\end{aligned}$$

Thus, $\widetilde{\pi}_Y(A_X) \subseteq A_Y$.

Assume $((\widetilde{\pi}_Y(a_0), \widetilde{\pi}_Z(a_0)), (\widetilde{\pi}_Y(a_1), \widetilde{\pi}_Z(a_1)), \ldots, (\widetilde{\pi}_Y(a_{n-1}), \widetilde{\pi}_Z(a_{n-1}))) \in R$ in $A_Y \times A_Z$, where $a_0, \ldots, a_{n-1} \in A$. Then $(\widetilde{\pi}_Y(a_0), \widetilde{\pi}_Y(a_1), \ldots, \widetilde{\pi}_Y(a_{n-1})) \in R$ in A_Y and $(\widetilde{\pi}_Z(a_0), \widetilde{\pi}_Z(a_1), \ldots, \widetilde{\pi}_Z(a_{n-1})) \in R$ in A_Z. Since $A_Y \leq \prod(\mathbf{F}(Z_i) : i \in Y)$ and $A_Z \leq \prod(\mathbf{F}(Z_i) : i \in Z)$, this implies that, for each $i \in Y \cup Z$, $(\pi_i(a_0), \pi_i(a_1), \ldots, \pi_i(a_{n-1})) \in R$ in $\mathbf{F}(Z_i)$. However, $X = Y \cup Z$ and A_X is a subsystem of $\prod(\mathbf{F}(Z_i) : i \in X)$. It follows that $(a_0, a_1, \ldots, a_{n-1}) \in R$ in A_X because $a_0, \ldots, a_{n-1} \in A_X$. □

For a category **C** and a family of **C** morphisms $f_i : B \longrightarrow B_i$ for $i \in U$, the family is a *mono source in* **C** *for the object* B if, for all morphisms $f : A \longrightarrow B$ and $g : A \longrightarrow B$, the following implication holds: if $f_i \circ f = f_i \circ g$ for all $i \in U$, then $f = g$.

LEMMA 4.4. *If, for* $i \in U$, $f_i : B \longrightarrow B_i$ *is a mono source in the category* $\mathbf{R_f}$, *then* $\langle f_i : i \in U\rangle$ *is a one-to-one homomorphism from* B *into* $\prod(B_i : i \in U)$.

Proof. Since f_i is a homomorphism for each $i \in U$, $\langle f_i : i \in U\rangle : B \longrightarrow \prod(B_i : i \in U)$ is a homomorphism. We will show that $\langle f_i : i \in U\rangle$ is one-to-one. To this effect it is enough to show that, for distinct $a, b \in B$, $f_j(a) \neq f_j(b)$ for some $j \in U$. Suppose, to the contrary, that for some $a, b \in B$ with $a \neq b$ we have $f_i(a) = f_i(b)$ for all $i \in U$. Consider the free ring S in the variety generated by B that is freely generated by a single element x. Since B is finite, so too is the ring S. In particular, S belongs to $\mathbf{R_f}$.

Let $f : S \longrightarrow B$ and $g : S \longrightarrow B$ be ring homomorphisms satisfying $f(x) = a$ and $g(x) = b$. Since S is finite, $f : S \longrightarrow B$ and $g : S \longrightarrow B$ are morphisms in $\mathbf{R_f}$. Since $f_i(a) = f_i(b)$ for all $i \in U$, $f_i \circ f = f_i \circ g$ for all $i \in U$. However, $f_i : B \longrightarrow B_i$ for $i \in U$ is a mono source for B in $\mathbf{R_f}$. It follows that $f = g$, which is a contradiction. Thus, as required, $\langle f_i : i \in U\rangle$ is one-to-one. □

Condition (P3) will be derived in the next lemma.

LEMMA 4.5. *For* X *and* $Y \in P_{fin}(I)$, *if* $X \neq \emptyset$ *and* $A_X \in \mathbf{Q}(A_Y)$, *then* $X = Y$.

Proof. Since $A_X \in \mathbf{Q}(A_Y)$, there exists a finite set U and an embedding h such that $h : A_X \longrightarrow A_Y^U$. For each pair $(s, k) \in Y \times U$, we have a homomorphism

$$\pi_s \circ \pi_k \circ h : A_X \longrightarrow \mathbf{F}(Z_s)$$

where π_k is the k-th projection of A_Y^U onto A_Y and π_s is the s-th projection of $\prod(\mathbf{F}(Z_i) : i \in Y)$ onto $\mathbf{F}(Z_s)$ (recall that A_Y is a substructure of $\prod(\mathbf{F}(Z_i) : i \in Y)$). Since $\langle \mathbf{F}(\pi_i) : i \in X\rangle : \mathbf{F}(\prod(Z_i : i \in X)) \longrightarrow A_X$, we have, for $(s, k) \in Y \times U$, a homomorphism

$$\pi_s \circ \pi_k \circ h \circ \langle \mathbf{F}(\pi_i) : i \in X\rangle : \mathbf{F}\left(\prod(Z_i : i \in X)\right) \longrightarrow \mathbf{F}(Z_s).$$

Consequently, since $\mathbf{F}$ is a full embedding of $\mathbf{R_f}$ into $\mathbf{K_f}$, it follows that there exists a ring homomorphism $\psi_{(s,k)} : \prod(Z_i : i \in X) \longrightarrow Z_s$ such that

$$\mathbf{F}(\psi_{(s,k)}) = \pi_s \circ \pi_k \circ h \circ \langle \mathbf{F}(\pi_i) : i \in X\rangle.$$

We show that the family $\psi_{(s,k)} : \prod(Z_i : i \in X) \longrightarrow Z_s$, where $(s, k) \in Y \times U$, is a mono source for $\prod(Z_i : i \in X)$ in $\mathbf{R_f}$. Let $f : A \longrightarrow \prod(Z_i : i \in X)$ and $g : A \longrightarrow \prod(Z_i : i \in X)$ be morphisms in $\mathbf{R_f}$ such that $\psi_{(s,k)} \circ f = \psi_{(s,k)} \circ g$ for all $(s, k) \in Y \times U$. Since $\mathbf{F}$ is a functor, the last equation implies

$$\mathbf{F}(\psi_{(s,k)}) \circ \mathbf{F}(f) = \mathbf{F}(\psi_{(s,k)}) \circ \mathbf{F}(g)$$

for all $(s, k) \in Y \times U$. Hence,

$$\pi_s \circ \pi_k \circ h \circ \langle \mathbf{F}(\pi_i) : i \in X\rangle \circ \mathbf{F}(f) = \pi_s \circ \pi_k \circ h \circ \langle \mathbf{F}(\pi_i) : i \in X\rangle \circ \mathbf{F}(g)$$

for all $(s, k) \in Y \times U$. Since $h : A_X \longrightarrow A_Y^U$ is an embedding, $\bigwedge(\mathit{Ker}(\pi_s \circ \pi_k \circ h) : (s, k) \in Y \times U) = \omega_{A_X}$. Thus,

$$\langle \mathbf{F}(\pi_i) : i \in X\rangle \circ \mathbf{F}(f) = \langle \mathbf{F}(\pi_i) : i \in X\rangle \circ \mathbf{F}(g).$$

In turn, this implies $\mathbf{F}(\pi_i) \circ \mathbf{F}(f) = \mathbf{F}(\pi_i) \circ \mathbf{F}(g)$ for all $i \in X$ and, in particular, $\mathbf{F}(\pi_i \circ f) = \mathbf{F}(\pi_i \circ g)$ for all $i \in X$. Since $\mathbf{F}$ is a faithful functor, we conclude that $\pi_i \circ f = \pi_i \circ g$ for all $i \in X$. Now observe that the family of homomorphisms $\pi_i : \prod(Z_i : i \in X) \longrightarrow Z_i$ for $i \in X$ is a mono source for $\prod(Z_i : i \in X)$ in $\mathbf{R_f}$. Thus, $f = g$ and, so, the family $\psi_{(s,k)} : \prod(Z_i : i \in X) \longrightarrow Z_s$ for $(s, k) \in Y \times U$ is a mono source for $\prod(Z_i : i \in X)$ in $\mathbf{R_f}$. By 4.4, the homomorphism

$$\langle \psi_{(s,k)} : (s, k) \in Y \times U\rangle : \prod(Z_i : i \in X) \longrightarrow \prod(Z_s^U : s \in Y)$$

is one-to-one. By 4.2, $X = Y$ as required. □

If $f : A \longrightarrow B$ is a morphism in $\mathbf{K_f}$, then we denote by $\mathit{Im}(f)$ the substructure of B that is the homomorphic image of A under f.

LEMMA 4.6. *If* $f : A_X \longrightarrow \mathbf{F}(Z_{i_1})^{m_1} \times \cdots \times \mathbf{F}(Z_{i_k})^{m_k}$ *is a morphism in* $\mathbf{K_f}$, *then* $\{i_1, \ldots, i_k\} \subseteq X$ *and the image of* A_X *under* f (*that is, Im*(f)) *is isomorphic to* A_Y, *where* $Y = \{i_1, \ldots, i_k\}$.

Proof. We first show that $\{i_1, \ldots, i_k\} \subseteq X$. Consider the homomorphisms $\langle \mathbf{F}(\pi_i) : i \in X\rangle : \mathbf{F}(\prod(Z_i : i \in X)) \longrightarrow A_X$, $f : A_X \longrightarrow \mathbf{F}(Z_{i_1})^{m_1} \times \cdots \times \mathbf{F}(Z_{i_k})^{m_k}$, and $\pi_{pq} : \mathbf{F}(Z_{i_1})^{m_1} \times \cdots \times \mathbf{F}(Z_{i_k})^{m_k} \longrightarrow \mathbf{F}(Z_{i_p})$ where $1 \leq p \leq k$, $1 \leq q \leq m_p$, and π_{pq} denotes the composite of the projection from $\mathbf{F}(Z_{i_1})^{m_1} \times \cdots \times \mathbf{F}(Z_{i_k})^{m_k}$ onto $\mathbf{F}(Z_{i_p})^{m_p}$ and the projection from $\mathbf{F}(Z_{i_p})^{m_p}$ onto its q-th component. In particular, for each $1 \leq p \leq k$ and $1 \leq q \leq m_p$, there is a homomorphism $\pi_{pq} \circ f \circ \langle \mathbf{F}(\pi_i) : i \in X\rangle : \mathbf{F}(\prod(Z_i : i \in X)) \longrightarrow \mathbf{F}(Z_{i_p})$. Since the functor $\mathbf{F}$ is full, we have a ring homomorphism $\varphi : \prod(Z_i : i \in X) \longrightarrow Z_{i_p}$ such that $\mathbf{F}(\varphi) = \pi_{pq} \circ f \circ \langle \mathbf{F}(\pi_i) : i \in X\rangle$. By 4.1, $i_p \in X$ which establishes that $Y = \{i_1, \ldots, i_k\} \subseteq X$.

In fact, by 4.1, we know that $\varphi = \pi_{i_p}$ and, so,

$$\mathbf{F}(\pi_{i_p}) = \pi_{pq} \circ f \circ \langle \mathbf{F}(\pi_i) : i \in X\rangle$$

for $1 \leq p \leq k$ and $1 \leq q \leq m_p$.

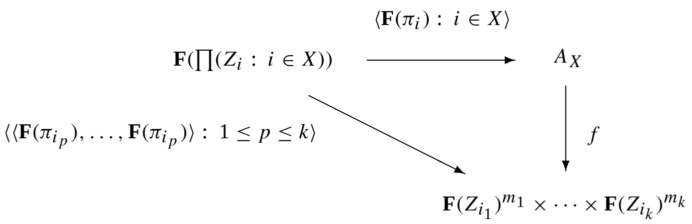

Figure 4

We claim that the diagram of Figure 4 commutes where, for each $1 \leq p \leq k$, $\langle \mathbf{F}(\pi_{i_p}), \ldots, \mathbf{F}(\pi_{i_p})\rangle$ is understood to be an m_p-tuple.

To see this it is enough to show that, for every $1 \leq s \leq k$ and $1 \leq t \leq m_p$,

$$\pi_{st} \circ \langle\langle \mathbf{F}(\pi_{i_p}), \ldots, \mathbf{F}(\pi_{i_p})\rangle : 1 \leq p \leq k\rangle = \pi_{st} \circ f \circ \langle \mathbf{F}(\pi_i) : i \in X\rangle.$$

However, $\pi_{st} \circ \langle\langle \mathbf{F}(\pi_{i_p}), \ldots, \mathbf{F}(\pi_{i_p})\rangle : 1 \leq p \leq k\rangle = \mathbf{F}(\pi_{i_s})$. Since, by the above, $\mathbf{F}(\pi_{i_s}) = \pi_{st} \circ f \circ \langle \mathbf{F}(\pi_i) : i \in X\rangle$, the diagram commutes, as required.

For each $1 \leq p \leq k$, let $d_p : Z_{i_p} \longrightarrow Z_{i_p}^{m_p}$ denote the diagonal embedding where, for all $x \in Z_{i_p}$ and $1 \leq q \leq m_p$, $\pi_q(d_p(x)) = x$.

We now define a homomorphism $e : \mathbf{F}(Z_{i_1}) \times \cdots \times \mathbf{F}(Z_{i_k}) \longrightarrow \mathbf{F}(Z_{i_1})^{m_1} \times \cdots \times \mathbf{F}(Z_{i_k})^{m_k}$ by

$$e = \langle\langle \mathbf{F}(\pi_p^q) : 1 \le q \le m_p\rangle \circ \mathbf{F}(d_p) \circ \widetilde{\pi}_p : 1 \le p \le k\rangle$$

where $\widetilde{\pi}_p$ is the projection of $\mathbf{F}(Z_{i_1}) \times \cdots \times \mathbf{F}(Z_{i_k})$ onto $\mathbf{F}(Z_{i_p})$ and π_p^q is the projection of $Z_{i_p}^{m_p}$ onto the q-th component of $Z_{i_p}^{m_p}$.

The proof of 4.6 will be completed by 4.7, 4.8, and 4.9.

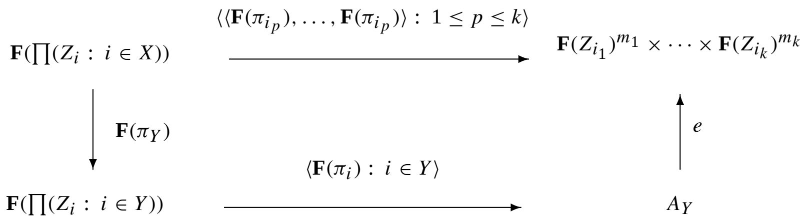

Figure 5

LEMMA 4.7. *The diagram of Figure* 5 *commutes where* $\langle \mathbf{F}(\pi_{i_p}), \ldots, \mathbf{F}(\pi_{i_p})\rangle$ *is regarded as an* m_p*-tuple for* $1 \le p \le k$, *and* π_Y *is the projection of* $\prod(Z_i : i \in X)$ *onto* $\prod(Z_i : i \in Y)$.

Proof. It is sufficient to show that, for each $1 \le s \le k$ and $1 \le t \le m_s$,

$$\pi_{st} \circ e \circ \langle \mathbf{F}(\pi_i) : i \in Y\rangle \circ \mathbf{F}(\pi_Y) = \pi_{st} \circ \langle\langle \mathbf{F}(\pi_{i_p}), \ldots, \mathbf{F}(\pi_{i_p})\rangle : 1 \le p \le k\rangle.$$

Since $\pi_{st} \circ \langle\langle \mathbf{F}(\pi_{i_p}), \ldots, \mathbf{F}(\pi_{i_p})\rangle : 1 \le p \le k\rangle = \mathbf{F}(\pi_{i_s})$, it is sufficient to show that $\pi_{st} \circ e \circ \langle \mathbf{F}(\pi_i) : i \in Y\rangle \circ \mathbf{F}(\pi_Y) = \mathbf{F}(\pi_{i_s})$. Before we do so, recall that π_{pq} is the composite of two projections: one acting from $\mathbf{F}(Z_{i_1})^{m_1} \times \cdots \times \mathbf{F}(Z_{i_k})^{m_k}$ onto $\mathbf{F}(Z_{i_p})^{m_p}$ and another from $\mathbf{F}(Z_{i_p})^{m_p}$ onto the q-th component of $\mathbf{F}(Z_{i_p})^{m_p}$. Denoting these projections by ρ_p and η_q^p, respectively, we have that $\pi_{st} = \eta_t^s \circ \rho_s$. We now show that $\pi_{st} \circ e \circ \langle \mathbf{F}(\pi_i) : i \in Y\rangle \circ \mathbf{F}(\pi_Y) = \mathbf{F}(\pi_{i_s})$.

$$\begin{aligned}
&\pi_{st} \circ e \circ \langle \mathbf{F}(\pi_i) : i \in Y\rangle \circ \mathbf{F}(\pi_Y)\\
&= (\pi_{st} \circ e) \circ \langle \mathbf{F}(\pi_i) : i \in Y\rangle \circ \mathbf{F}(\pi_Y)\\
&= (\pi_{st} \circ \langle\langle \mathbf{F}(\pi_p^q) : 1 \le q \le m_p\rangle \circ \mathbf{F}(d_p) \circ \widetilde{\pi}_p : 1 \le p \le k\rangle) \circ \langle \mathbf{F}(\pi_i) : i \in Y\rangle \circ \mathbf{F}(\pi_Y)\\
&\quad \text{(by the definition of } e\text{)}
\end{aligned}$$

$= (\eta_t^s \circ \rho_s \circ \langle\langle \mathbf{F}(\pi_p^q) : 1 \le q \le m_p\rangle \circ \mathbf{F}(d_p) \circ \widetilde{\pi}_p : 1 \le p \le k\rangle) \circ \langle \mathbf{F}(\pi_i) : i \in Y\rangle$
$\circ \mathbf{F}(\pi_Y)$ (since $\pi_{st} = \eta_t^s \circ \rho_s$)
$= (\eta_t^s \circ (\langle \mathbf{F}(\pi_s^q) : 1 \le q \le m_s\rangle \circ \mathbf{F}(d_s) \circ \widetilde{\pi}_s)) \circ \langle \mathbf{F}(\pi_i) : i \in Y\rangle \circ \mathbf{F}(\pi_Y)$
$= ((\eta_t^s \circ \langle \mathbf{F}(\pi_s^q) : 1 \le q \le m_s\rangle) \circ \mathbf{F}(d_s) \circ \widetilde{\pi}_s) \circ \langle \mathbf{F}(\pi_i) : i \in Y\rangle \circ \mathbf{F}(\pi_Y)$
$= \mathbf{F}(\pi_s^t) \circ \mathbf{F}(d_s) \circ \widetilde{\pi}_s \circ \langle \mathbf{F}(\pi_i) : i \in Y\rangle \circ \mathbf{F}(\pi_Y)$
$= \mathbf{F}(\pi_s^t \circ d_s) \circ \widetilde{\pi}_s \circ \langle \mathbf{F}(\pi_i) : i \in Y\rangle \circ \mathbf{F}(\pi_Y)$
$= \widetilde{\pi}_s \circ \langle \mathbf{F}(\pi_i) : i \in Y\rangle \circ \mathbf{F}(\pi_Y)$
(as $\pi_s^t \circ d_s$ is the identity map $id_{Z_{i_s}}$ on Z_{i_s} and $\mathbf{F}(id_{Z_{i_s}})$ is the identity map on $\mathbf{F}(Z_{i_s})$)
$= \mathbf{F}(\pi_{i_s}) \circ \mathbf{F}(\pi_Y)$ (since $\widetilde{\pi}_s$ is the projection of $\mathbf{F}(Z_{i_1}) \times \cdots \times \mathbf{F}(Z_{i_k})$ onto $\mathbf{F}(Z_{i_s})$)
$= \mathbf{F}(\pi_{i_s} \circ \pi_Y)$
$= \mathbf{F}(\pi_{i_s})$ (since $i_s \in Y$),

as required. □

LEMMA 4.8. *The subsystems of* $\mathbf{F}(Z_{i_1})^{m_1} \times \cdots \times \mathbf{F}(Z_{i_k})^{m_k}$ *which are images of* A_X *under* f *and of* A_Y *under* e*, respectively, coincide.*

Proof. It is sufficient to show that the domain of the image of A_X under f and the domain of the image of A_Y under e coincide.

Let x be in the image of A_X under f. Since the homomorphism $\langle \mathbf{F}(\pi_i) : i \in X\rangle : \mathbf{F}(\prod(Z_i : i \in X)) \longrightarrow A_X$ is onto (by the definition of A_X) and the diagram of Figure 4 commutes, $\langle\langle \mathbf{F}(\pi_{i_p}), \ldots, \mathbf{F}(\pi_{i_p})\rangle : 1 \le p \le k\rangle(y) = x$ for some element $y \in \mathbf{F}(\prod(Z_i : i \in X))$. Since the diagram of Figure 5 commutes, $e(z) = x$, where $z = \langle \mathbf{F}(\pi_i) : i \in Y\rangle \circ \mathbf{F}(\pi_Y)(y)$. Thus, x is in the image of A_Y under e.

Suppose now that x is in the image of A_Y under e. Since $\pi_Y : \prod(Z_i : i \in X) \longrightarrow \prod(Z_i : i \in Y)$ is onto, it follows, from the hypothesis of 1.1 (namely, $\mathbf{F}(\varphi) : \mathbf{F}(A) \longrightarrow \mathbf{F}(B)$ is onto whenever $\varphi : A \longrightarrow B$ is onto), that $\mathbf{F}(\pi_Y) : \mathbf{F}(\prod(Z_i : i \in X)) \longrightarrow \mathbf{F}(\prod(Z_i : i \in Y))$ is also onto. Since $\langle \mathbf{F}(\pi_i) : i \in Y\rangle : \mathbf{F}(\prod(Z_i : i \in Y)) \longrightarrow A_Y$ is onto, it follows from 4.7 that there exists an element y in $\mathbf{F}(\prod(Z_i : i \in X))$ for which $\langle\langle \mathbf{F}(\pi_{i_p}), \ldots, \mathbf{F}(\pi_{i_p})\rangle : 1 \le p \le k\rangle(y) = x$. As the diagram of Figure 4 commutes, $f(z) = x$ where z is the element of A_X given by $z = \langle \mathbf{F}(\pi_i) : i \in X\rangle(y)$. In particular, x is in the domain of the image of A_X under f. □

LEMMA 4.9. *The image of* A_Y *under* e *is isomorphic to* A_Y.

Proof. Since e is a homomorphism, we need to show the following:

(i) e is one-to-one;
(ii) if R is an n-ary relation in the type of $\mathbf{K}$ and $a_0, a_1, \ldots, a_{n-1}$ are elements of A_Y, then $(a_0, a_1, \ldots, a_{n-1}) \in R$ in A_Y whenever $(e(a_0), e(a_1), \ldots, e(a_{n-1})) \in R$ in $\mathbf{F}(Z_{i_1})^{m_1} \times \cdots \times \mathbf{F}(Z_{i_k})^{m_k}$.

Let x and $y \in A_Y$ and assume $e(x) = e(y)$. Since A_Y is a subsystem of $\prod(\mathbf{F}(Z_{i_p}) : 1 \le p \le k)$ (recall that $Y = \{i_1, \dots, i_k\}$), we need to show that $x(p) = y(p)$ for every $1 \le p \le k$, where $x(p)$ and $y(p)$ denote the p-th components of x and y, respectively. Since $e(x)$ and $e(y) \in \mathbf{F}(Z_{i_1})^{m_1} \times \dots \times \mathbf{F}(Z_{i_k})^{m_k}$, it follows from the assumption $e(x) = e(y)$ that $\pi_{st} \circ e(x) = \pi_{st} \circ e(y)$ for all $1 \le s \le k$ and $1 \le t \le m_s$. But, for each $1 \le s \le k$ and $1 \le t \le m_s$, we have

$$\begin{aligned}
\pi_{st} \circ e(x) &= \pi_{st} \circ \langle\langle \mathbf{F}(\pi_p^q) : 1 \le q \le m_p\rangle \circ \mathbf{F}(d_p) \circ \widetilde{\pi}_p : 1 \le p \le k\rangle(x) \\
&= \mathbf{F}(\pi_s^t) \circ \mathbf{F}(d_s) \circ \widetilde{\pi}_s(x) \\
&= \mathbf{F}(\pi_s^t \circ d_s) \circ \widetilde{\pi}_s(x) \\
&= \mathbf{F}(id_{Z_{i_s}}) \circ \widetilde{\pi}_s(x) \\
&= id_{\mathbf{F}(Z_{i_s})} \circ \widetilde{\pi}_s(x) \\
&= \widetilde{\pi}_s(x) \\
&= x(s).
\end{aligned}$$

Similarly one shows that $\pi_{st} \circ e(y) = y(s)$. Thus, since $\pi_{st} \circ e(x) = \pi_{st} \circ e(y)$ for all $1 \le s \le k$ and $1 \le t \le m_s$, $x = y$ and, as required, e is one-to-one.

Assume R is an n-ary relation in the type of $\mathbf{K}$, $a_0, a_1, \dots, a_{n-1}$ are elements of A_Y, and $(e(a_0), e(a_1), \dots, e(a_{n-1})) \in R$ in $\mathbf{F}(Z_{i_1})^{m_1} \times \dots \times \mathbf{F}(Z_{i_k})^{m_k}$. Since A_Y is a subsystem of $\mathbf{F}(Z_{i_1}) \times \dots \times \mathbf{F}(Z_{i_k})$, in order to show that $(a_0, a_1, \dots, a_{n-1}) \in R$ in A_Y it suffices to show that $(\widetilde{\pi}_p(a_0), \widetilde{\pi}_p(a_1), \dots, \widetilde{\pi}_p(a_{n-1})) \in R$ in $\mathbf{F}(Z_{i_p})$ for every $1 \le p \le k$ where, we recall, $\widetilde{\pi}_p$ is the p-th projection of $\prod(\mathbf{F}(Z_{i_p}) : 1 \le p \le k)$ onto $\mathbf{F}(Z_{i_p})$. Since $(e(a_0), e(a_1), \dots, e(a_{n-1})) \in R$ in $\mathbf{F}(Z_{i_1})^{m_1} \times \dots \times \mathbf{F}(Z_{i_k})^{m_k}$, we infer from the definition of e that, for every $1 \le p \le k$ and $1 \le q \le m_p$,

$$\begin{aligned}
&(\mathbf{F}(\pi_p^q) \circ \mathbf{F}(d_p) \circ \widetilde{\pi}_p(a_0), \mathbf{F}(\pi_p^q) \circ \mathbf{F}(d_p) \circ \widetilde{\pi}_p \\
&\quad (a_1), \dots, \mathbf{F}(\pi_p^q) \circ \mathbf{F}(d_p) \circ \widetilde{\pi}_p(a_{n-1})) \in R
\end{aligned}$$

in the q-th component of $\mathbf{F}(Z_{i_p})^{m_p}$, that is, in $\mathbf{F}(Z_{i_p})$. However, $\mathbf{F}(\pi_p^q) \circ \mathbf{F}(d_p) = \mathbf{F}(\pi_p^q \circ d_p) = \mathbf{F}(id_{Z_{i_p}}) = id_{\mathbf{F}(Z_{i_p})}$. In particular, for every $1 \le p \le k$, $(\widetilde{\pi}_p(a_0), \widetilde{\pi}_p(a_1), \dots, \widetilde{\pi}_p (a_{n-1})) \in R$ in $\mathbf{F}(Z_{i_p})$, as required. □

By 4.8 and 4.9, the image of A_X under f is isomorphic to A_Y and, so, the proof of 4.6 is complete. □

The following will complete the proof of 1.1 by establishing (P4).

LEMMA 4.10. *If $\emptyset \ne X \in P_{fin}(I)$ and A_X is a subsystem of $B \times C$ for finite B and $C \in \mathbf{Q}(\{\mathbf{F}(Z_i) : i \in I\})$, then $Im(\pi_B) \cong A_Y$ and $Im(\pi_C) \cong A_Z$ for some Y and $Z \subseteq X$ with $Y \cup Z = X$, where π_B and π_C are the projections of A_X into B and C, respectively.*

Proof. Since B and C are finite, there exist $i_1, \dots, i_k, j_1, \dots, j_l$ in I and numbers $m_1, \dots, m_k, n_1, \dots, n_l \ge 1$ such that B is embeddable into $\mathbf{F}(Z_{i_1})^{m_1} \times \dots \mathbf{F}(Z_{i_k})^{m_k}$ and C is embeddable into $\mathbf{F}(Z_{j_1})^{n_1} \times \dots \mathbf{F}(Z_{j_l})^{n_l}$, say

$$g : B \hookrightarrow \mathbf{F}(Z_{i_1})^{m_1} \times \dots \times \mathbf{F}(Z_{i_k})^{m_k} \text{ and } h : C \hookrightarrow \mathbf{F}(Z_{j_1})^{n_1} \times \dots \times \mathbf{F}(Z_{j_l})^{n_l}.$$

Set

$$Y = \{i_1, \ldots, i_k\} \text{ and } Z = \{j_1, \ldots, j_l\}.$$

The proof of 4.10 will follow from 4.6 together with 4.11.

LEMMA 4.11. $Y \cup Z = X$.

Proof. Since

$$g \circ \pi_B :\longrightarrow \mathbf{F}(Z_{i_1})^{m_1} \times \cdots \times \mathbf{F}(Z_{i_k})^{m_k} \text{ and } h \circ \pi_C \longrightarrow \mathbf{F}(Z_{j_1})^{n_1} \times \cdots \times \mathbf{F}(Z_{j_l})^{n_l}$$

are morphisms in $\mathbf{K_f}$, by 4.6, $Y \subseteq X$ and $Z \subseteq X$.

For each $1 \leq p \leq k$ and $1 \leq q \leq m_p$, π_{pq} denotes the composition of two projections: one acting from $\mathbf{F}(Z_{i_1})^{m_1} \times \cdots \times \mathbf{F}(Z_{i_k})^{m_k}$ onto $\mathbf{F}(Z_{i_p})^{m_p}$ and another from $\mathbf{F}(Z_{i_p})^{m_p}$ onto the q-th component of $\mathbf{F}(Z_{i_p})^{m_p}$. Thus,

$$\pi_{pq} : \mathbf{F}(Z_{i_1})^{m_1} \times \cdots \times \mathbf{F}(Z_{i_k})^{m_k} \longrightarrow \mathbf{F}(Z_{i_p}).$$

Likewise, for each $1 \leq r \leq l$ and $1 \leq s \leq n_r$, π_{rs} denotes the composition of two projections: one acting from $\mathbf{F}(Z_{j_1})^{n_1} \times \cdots \times \mathbf{F}(Z_{j_l})^{n_l}$ onto $\mathbf{F}(Z_{j_r})^{n_r}$ and another from $\mathbf{F}(Z_{j_r})^{n_r}$ onto the s-th component of $\mathbf{F}(Z_{j_r})^{n_r}$. Thus,

$$\pi_{rs} : \mathbf{F}(Z_{j_1})^{n_1} \times \cdots \times \mathbf{F}(Z_{j_l})^{n_l} \longrightarrow \mathbf{F}(Z_{j_r}).$$

Since $\langle \mathbf{F}(\pi_i) : i \in X \rangle : \mathbf{F}(\prod(Z_i : i \in X)) \longrightarrow A_X$, $g \circ \pi_B : A_X \longrightarrow \mathbf{F}(Z_{i_1})^{m_1} \times \cdots \times \mathbf{F}(Z_{i_k})^{m_k}$, and $h \circ \pi_C : A_X \longrightarrow \mathbf{F}(Z_{j_1})^{n_1} \times \cdots \times \mathbf{F}(Z_{j_l})^{n_l}$, we have a family of homomorphisms

$$\pi_{pq} \circ g \circ \pi_B \circ \langle \mathbf{F}(\pi_i) : i \in X \rangle : \mathbf{F}\left(\prod(Z_i : i \in X)\right) \longrightarrow \mathbf{F}(Z_{i_p})$$

and

$$\pi_{rs} \circ h \circ \pi_C \circ \langle \mathbf{F}(\pi_i) : i \in X \rangle : \mathbf{F}\left(\prod(Z_i : i \in X)\right) \longrightarrow \mathbf{F}(Z_{j_r})$$

where $1 \leq p \leq k$, $1 \leq q \leq m_p$, $1 \leq r \leq l$, and $1 \leq s \leq n_r$.

Since $\mathbf{F}$ is a full embedding, for each p, q, r, and s as above, there exist ring homomorphisms φ_{pq} and ψ_{rs} in $\mathbf{R_f}$ such that $\varphi_{pq} : \prod(Z_i : i \in X) \longrightarrow Z_{i_p}$, $\mathbf{F}(\varphi_{pq}) = \pi_{pq} \circ g \circ \pi_B \circ \langle \mathbf{F}(\pi_i) : i \in X \rangle$, $\psi_{rs} : \prod(Z_i : i \in X) \longrightarrow Z_{j_r}$, and $\mathbf{F}(\psi_{rs}) = \pi_{rs} \circ h \circ \pi_C \circ \langle \mathbf{F}(\pi_i) : i \in X \rangle$. We show that the family consisting of all φ_{pq}'s and ψ_{rs}'s is a mono source for $\prod(Z_i : i \in X)$ in $\mathbf{R_f}$.

Let ε_0 and $\varepsilon_1 : A \longrightarrow \prod(Z_i : i \in X)$ be ring homomorphisms in $\mathbf{R_f}$ such that $\varphi_{pq} \circ \varepsilon_0 = \varphi_{pq} \circ \varepsilon_1$ and $\psi_{rs} \circ \varepsilon_0 = \psi_{rs} \circ \varepsilon_1$ for all p, q, r, and s as above. It is to be shown that $\varepsilon_0 = \varepsilon_1$. Since **F** is a functor, it follows that $\mathbf{F}(\varphi_{pq}) \circ \mathbf{F}(\varepsilon_0) = \mathbf{F}(\varphi_{pq}) \circ \mathbf{F}(\varepsilon_1)$ and $\mathbf{F}(\psi_{rs}) \circ \mathbf{F}(\varepsilon_0) = \mathbf{F}(\psi_{rs}) \circ \mathbf{F}(\varepsilon_1)$ for all p, q, r, and s as above. Consequently,

$$\pi_{pq} \circ g \circ \pi_B \circ \langle \mathbf{F}(\pi_i) : i \in X \rangle \circ \mathbf{F}(\varepsilon_0) = \pi_{pq} \circ g \circ \pi_B \circ \langle \mathbf{F}(\pi_i) : i \in X \rangle \circ \mathbf{F}(\varepsilon_1)$$

and

$$\pi_{rs} \circ h \circ \pi_C \circ \langle \mathbf{F}(\pi_i) : i \in X \rangle \circ \mathbf{F}(\varepsilon_0) = \pi_{rs} \circ h \circ \pi_C \circ \langle \mathbf{F}(\pi_i) : i \in X \rangle \circ \mathbf{F}(\varepsilon_1)$$

for all p, q, r, and s. However, A_X is a subsystem of $B \times C$ and g and h are embeddings. Thus,

$$\bigwedge(Ker(\pi_{pq} \circ g \circ \pi_B) : 1 \leq p \leq k \text{ and } 1 \leq q \leq m_p) \wedge \bigwedge(Ker(\pi_{rs} \circ h \circ \pi_C) : 1 \leq r \leq l \text{ and } 1 \leq s \leq n_r) = \omega_{A_X}.$$

Consequently,

$$\langle \mathbf{F}(\pi_i) : i \in X \rangle \circ \mathbf{F}(\varepsilon_0) = \langle \mathbf{F}(\pi_i) : i \in X \rangle \circ \mathbf{F}(\varepsilon_1).$$

In particular, for each $i \in X$,

$$\mathbf{F}(\pi_i) \circ \mathbf{F}(\varepsilon_0) = \mathbf{F}(\pi_i) \circ \mathbf{F}(\varepsilon_1).$$

Since **F** is a faithful embedding, we obtain $\pi_i \circ \varepsilon_0 = \pi_i \circ \varepsilon_1$ for all $i \in X$. However, the family $\pi_i : \prod(Z_i : i \in X) \longrightarrow Z_i$ for $i \in X$ is a mono source for $\prod(Z_i : i \in X)$ in $\mathbf{R_f}$ and, so, $\varepsilon_0 = \varepsilon_1$. In particular, we conclude that the family of all φ_{pq}'s and ψ_{rs}'s is a mono source for $\prod(Z_i : i \in X)$ in $\mathbf{R_f}$. By 4.4, there is a one-to-one ring homomorphism from $\prod(Z_i : i \in X)$ into $Z_{i_1}^{m_1} \times \cdots \times Z_{i_k}^{m_k} \times Z_{j_1}^{n_1} \times \cdots \times Z_{j_l}^{n_l}$ which, by 4.2(ii), implies that $X \subseteq \{i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_l\}$. In other words, $X \subseteq Y \cup Z$ as required. □

To summarize, $Y = \{i_1, \ldots, i_k\}$, $Z = \{j_1, \ldots, j_l\}$, $X = Y \cup Z$,

$$\pi_B : A_X \longrightarrow B \text{ and } g : B \hookrightarrow \mathbf{F}(Z_{i_1})^{m_1} \times \cdots \times \mathbf{F}(Z_{i_k})^{m_k},$$

and

$$\pi_C : A_X \longrightarrow C \text{ and } h : C \hookrightarrow \mathbf{F}(Z_{j_1})^{n_1} \times \cdots \times \mathbf{F}(Z_{j_l})^{n_l}.$$

where both g and h are embeddings. By 4.6, $Im(\pi_B) \cong A_Y$ and $Im(\pi_C) \cong A_Z$, thereby completing the proof of 4.10. □

Concluding this section we want to mention that $\mathbf{R_f}$ can be replaced in 2.2 and in 2.3 by many other classes of finite algebras and still lead to the same conclusions. The only one requirement is that each must have the properties of the rings Z_p which were used in the proof of 2.3.

5. Proof (Proposition 2.4)

For a finite-to-finite universal quasivariety of algebras, the assumptions of Proposition 2.4 are readily verified (indeed, well known). For a quasivariety of algebraic systems some subtleties remain which may be overcome by the use of the concept of a congruence as given by Gorbunov and Tumanov in [13].

In this section, a similarity type is regarded as a triple $\Omega = (F, R, a)$, where F and R are the sets of functional and relational symbols of Ω, respectively, and $a : F \cup R \longrightarrow \omega$ is the arity function of Ω, where it is understood that $a(r) > 0$ for $r \in R$. The symbol $\approx$ is meant to be the identity symbol and it is assumed that $\approx \notin R$. Of course, the arity of $\approx$ is 2. We denote by R^+ the set $R \cup \{\approx\}$.

An algebraic system of type Ω is any triple of the form $\mathcal{A} = (A; \{f_{\mathcal{A}} : f \in F\}, \{r_{\mathcal{A}} : r \in R\})$ such that A is a non-empty set, $f_{\mathcal{A}} : A^{a(f)} \longrightarrow A$ is a function, and $r_{\mathcal{A}} \subseteq A^{a(r)}$ is a relation for all $f \in F$ and $r \in R$. We denote the algebra part $(A; \{f_{\mathcal{A}} : f \in F\})$ of $\mathcal{A}$ by $alg(\mathcal{A})$ and the relational part of $(A; \{r_{\mathcal{A}} : r \in R\})$ by $rel(\mathcal{A})$, respectively.

Given a non-empty set A. A function H that assigns to every element r of R^+ a subset of $A^{a(r)}$ is called an *R^+-indexed family on A*. Congruences on an algebraic system $\mathcal{A}$ are special types of R^+-indexed families on A.

Given an algebraic system $\mathcal{A}$ of type Ω. An R^+-indexed family Θ on A is said to be a *congruence* on $\mathcal{A}$ (see [13] or [11]) if Θ satisfies the following conditions:

(i) $\Theta(\approx)$ is a congruence on $alg(\mathcal{A})$;
(ii) for each $r \in R$, $r_{\mathcal{A}} \subseteq \Theta(r)$;
(iii) for each $r \in R$, if $x_i \equiv y_i(\Theta(\approx))$ for all $1 \leq i \leq a(r)$ and $(x_1, \ldots, x_{a(r)}) \in \Theta(r)$, then $(y_1, \ldots, y_{a(r)}) \in \Theta(r)$.

The *quotient system* $\mathcal{A}/\Theta$ of $\mathcal{A}$ by Θ is defined as follows:

(i) $alg(\mathcal{A}/\Theta)$ is $alg(\mathcal{A})/\Theta(\approx)$;
(ii) for $r \in R$, $([x_1]\Theta, \ldots, [x_{a(r)}]\Theta) \in r_{\mathcal{A}/\Theta}$ iff $(x_1, \ldots, x_{a(r)}) \in \Theta(r)$.

Notice that if $\Theta(\approx) = \{(x, x) : x \in A\}$, then $\mathcal{A}/\Theta$ is $(A; \{f_{\mathcal{A}} : f \in F\}, \{\Theta(r) : r \in R\})$.

From now on $\mathbf{K}$ is assumed to be a quasivariety of algebraic systems of finite type Ω and is regarded as a category with homomorphisms as morphisms.

A congruence Θ on an algebraic system $\mathcal{A}$ of type Ω not necessarily belonging to $\mathbf{K}$ is said to be a **K**-*congruence* if $\mathcal{A}/\Theta \in \mathbf{K}$. Since $\mathbf{K}$ is closed under subdirect products and contains a trivial system, for every R^+-indexed family H on A there exists a smallest $\mathbf{K}$-congruence Θ such that $H \subseteq \Theta$, that is $H(r) \subseteq \Theta(r)$ for all $r \in R^+$. The smallest congruence containing H is denoted by $\Theta_{\mathbf{K}}(H)$. The following lemma characterizes $\Theta_{\mathbf{K}}(H)$, where $0_{\mathcal{A}}$ means an R^+-indexed family on $\mathcal{A}$ such that $0_{\mathcal{A}}(\approx) = \{(a, a) : a \in A\}$ and $0_{\mathcal{A}}(r) = r_{\mathcal{A}}$ for $r \in R$ and where in a quasi-identity r_i and r are permitted to represent $\approx$ as well.

LEMMA 5.1. (see [11]) *Let $\mathcal{A}$ be an algebraic system of type Ω, and let H be an R^+-indexed family on $\mathcal{A}$. Then, for $r \in R^+$ and $(c_1, \dots, c_{a(r)})$ in A, $(c_1, \dots, c_{a(r)}) \in \Theta_{\mathbf{K}}(H)(r)$ iff there exist a quasi-identity $\forall \overline{x}[\bigwedge_{i<k} r_i(t_{i_1}(\overline{x}), \dots, t_{i_{a(r_i)}}(\overline{x})) \Longrightarrow r(t_1(\overline{x}), \dots, t_{a(r)}(\overline{x}))]$ valid in $\mathbf{K}$ and a sequence $\overline{b}$ of elements in A such that $(t_{i_1}(\overline{b}), \dots, t_{i_{a(r_i)}}(\overline{b})) \in (H \cup 0_{\mathcal{A}})(r_i)$ for $i < k$ and $(t_1(\overline{b}), \dots, t_{a(r)}(\overline{b})) = (c_1, \dots, c_{a(r)})$.*

The lemma differs slightly from its counterpart in [11]: the difference is more convenient in our context.

We first show that $\mathbf{K}$ is *cocomplete*, that is colimits of all diagrams in $\mathbf{K}$ exist.

Let $\mathbf{F} : \mathbf{J} \longrightarrow \mathbf{K}$ be a *diagram* in $\mathbf{K}$, that is a functor from an index category $\mathbf{J}$ to $\mathbf{K}$. A *cone* of $\mathbf{F}$ is an object $\mathcal{B}$ of $\mathbf{K}$ together with a family $(\psi_i : \mathbf{F}(i) \longrightarrow \mathcal{B} : i \in obj(\mathbf{J}))$ of $\mathbf{K}$-morphisms such that, for every arrow $u : i \longrightarrow j$ in $\mathbf{J}$, $\psi_i = \psi_j \circ \mathbf{F}(u)$. A *colimit* of a diagram $\mathbf{F} : \mathbf{J} \longrightarrow \mathbf{K}$ is a cone $(\psi_i : \mathbf{F}(i) \longrightarrow \mathcal{B} : i \in obj(\mathbf{J}))$ of $\mathbf{F}$ such that, for every other cone $(\varphi_i : \mathbf{F}(i) \longrightarrow \mathcal{C} : i \in obj(\mathbf{J}))$ of $\mathbf{F}$, there exists a unique $\mathbf{K}$-morphism $\sigma : \mathcal{B} \longrightarrow \mathcal{C}$ such that, for every $i \in obj(\mathbf{J})$, $\varphi_i = \sigma \circ \psi_i$.

Let $\mathbf{F} : \mathbf{J} \longrightarrow \mathbf{K}$ be a diagram in $\mathbf{K}$. For an object i in $\mathbf{J}$, let A_i denote the universe of $\mathbf{F}(i)$. We assume that $A_i \cap A_j = \emptyset$ whenever $i \neq j$; this assumption is made only in order to simplify the construction given below.

Let $\mathcal{F}$ be a free algebraic system in $\mathbf{K}$ with $\bigcup(A_i : i \in obj(\mathbf{J}))$ as the set of free generators. For $i, j \in obj(\mathbf{J})$ and an arrow $u : i \longrightarrow j$ in $\mathbf{J}$, we set

$$\begin{aligned} D_i(\approx) &= \{(c, f_{\mathcal{F}}(c_1, \dots, c_{a(f)})) : c, c_1, \dots, c_{a(f)} \in A_i, c = f_{\mathbf{F(i)}}(c_1, \dots, c_{a(f)}), \\ &\qquad \text{and } f \in F\} \\ E_u &= \{(c, \mathbf{F}(u)(c)) : c \in A_i\}. \end{aligned}$$

Next, we set

$$\begin{aligned} H(\approx) &= \bigcup(D_i(\approx) : i \in obj(\mathbf{J})) \cup \bigcup(E_u : u \text{ is an arrow in } \mathbf{J}) \text{ and} \\ H(r) &= \bigcup(r_{\mathbf{F}(i)} : i \in obj(\mathbf{J})) \text{ for } r \in R. \end{aligned}$$

Notice that H is an R^+-indexed family on F.
For $i \in obj(\mathbf{J})$, we define $\varphi_i : \mathbf{F}(i) \longrightarrow \mathcal{F}/\Theta_{\mathbf{K}}(H)$ by

$$\varphi_i(x) = [x]\Theta_{\mathbf{K}}(H) \text{ for } x \in A_i.$$

A routine verification shows that $\varphi_i : \mathbf{F}(i) \longrightarrow \mathcal{F}/\Theta_{\mathbf{K}}(H)$ is a $\mathbf{K}$-morphism.

PROPOSITION 5.2. *$\mathcal{F}/\Theta_{\mathbf{K}}(H)$ together with $(\varphi_i : \mathbf{F}(i) \longrightarrow \mathcal{F}/\Theta_{\mathbf{K}}(H) : i \in obj(\mathbf{J}))$ is a colimit of $F : \mathbf{J} \longrightarrow \mathbf{K}$.*

Proof. That $(\varphi_i : \mathbf{F}(i) \longrightarrow \mathcal{F}/\Theta_{\mathbf{K}}(H) : i \in obj(\mathbf{J}))$ is a cone of $\mathbf{F}$ is obvious.

Let $(\psi_i : \mathbf{F}(i) \longrightarrow \mathcal{A} : i \in obj(\mathbf{J}))$ be a cone of $\mathbf{F}$. Let $\gamma : \mathcal{F} \longrightarrow \mathcal{A}$ be a $\mathbf{K}$-morphism satisfying $\gamma(a) = \psi_i(a)$ for each $a \in A_i$ and each $i \in obj(\mathbf{J})$. It is obvious that γ exists and is uniquely determined by the family $(\psi_i : i \in obj(\mathbf{J}))$. Define an R^+-indexed family E on $\mathcal{F}$ as follows:

$$E(\approx) = \{(x, y) : \gamma(x) = \gamma(y)\} \text{ and}$$
$$E(r) = \{(x_1, \ldots, x_{a(r)}) : (\gamma(x_1), \ldots, \gamma(x_{a(r)})) \in r_{\mathcal{A}}\} \text{ for } r \in R.$$

We claim

(i) $\Theta_{\mathbf{K}}(H)(\approx) \subseteq E(\approx)$
(ii) $\Theta_{\mathbf{K}}(H)(r) \subseteq E(r)$ for $r \in R$.

From the definitions of $H(\approx)$ and $H(r)$, the definition of γ, and from the assumption that $(\psi_i : \mathbf{F}(i) \longrightarrow \mathcal{A} : i \in obj(\mathbf{J}))$ is a cone of $\mathbf{F}$ it follows that $H(\approx) \subseteq E(\approx)$ and $H(r) \subseteq E(r)$ for $r \in R$. We conclude that (i) and (ii) hold.

Define $\sigma : \mathcal{F}/\Theta_{\mathbf{K}}(H) \longrightarrow \mathcal{A}$ by

$$\sigma([x]\Theta_{\mathbf{K}}(H)) = \gamma(x).$$

By Claim (i), σ is well-defined. Obviously, σ preserves operations. By Claim (ii), it also preserves relations. Thus σ is a homomorphism.

For $i \in obj(\mathbf{J})$ and $a \in A_i$, we have $\sigma \circ \varphi_i(a) = \sigma(\varphi_i(a)) = \sigma([a]\theta_{\mathbf{K}}(H)) = \gamma(a) = \psi_i(a)$. The uniqueness of σ is easy to establish. Thus $\mathcal{F}/\Theta_{\mathbf{K}}(H)$ together with $(\varphi_i : \mathbf{F}(i) \longrightarrow \mathcal{F}/\Theta_{\mathbf{K}}(H) : i \in obj(\mathbf{J}))$ is a colimit of $\mathbf{F}$. □

Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a $\mathbf{K}$-morphism. Let $\mathcal{C}$ be the subsystem of $\mathcal{B}$ determined by $\varphi(A)$, that is

$$\mathcal{C} = (\varphi(A), \{f_{\mathcal{B}} : f \in F\}, \{r_{\mathcal{B}} \cap \varphi(A)^{a(r)} : r \in R\}.$$

Define an R^+-indexed family $Im\,\varphi$ on $\varphi(A)$ as follows:

$$(Im\,\varphi)(\approx) = \{(c, c) : c \in C\} \text{ and}$$
$$(Im\,\varphi)(r) = \varphi(r_{\mathcal{A}}) \text{ for } r \in R.$$

We have

LEMMA 5.3. $\Theta_{\mathbf{K}}(Im\,\varphi)(\approx) = (Im\,\varphi)(\approx)$.

Proof. Let $(a_1, a_2) \in \Theta_{\mathbf{K}}(Im\,\varphi)(\approx)$. Then, by 5.1,

$$(t_{i_1}(\overline{b}), \ldots, t_{i_{a(r_i)}}(\overline{b})) \in (Im\,\varphi)(r_i) \text{ for } i < k, \text{ where } 0_{\varphi(\mathcal{A})} = Im\,\varphi, \text{ and}$$
$$(t_1(\overline{b}), t_2(\overline{b})) = (a_1, a_2)$$

for some quasi-identity $\forall \overline{x}[\bigwedge_{i<k} r_i(t_{i_1}(\overline{x}), \dots, t_{i_{a(r_i)}}(\overline{x})) \Longrightarrow t_1(\overline{x}) \approx t_2(\overline{x})]$ valid in **K** and a sequence $\overline{b}$ of elements in C. Since $(Im\,\varphi)(\approx) = \{(c, c) : c \in C\}$ and $(Im\,\varphi)(r_i) \subseteq r_{\mathcal{C}}$, we have

$$(t_{i_1}(\overline{b}), \dots, t_{i_{a(r_i)}}(\overline{b})) \in r_{\mathcal{C}} \text{ for } i < k.$$

As the above quasi-identity is valid in $\mathcal{C}$ because $\mathcal{C} \in \mathbf{K}$, we obtain $t_1(\overline{b}) = t_2(\overline{b})$ and, consequently, $a_1 = a_2$. Thus $\Theta_{\mathbf{K}}(Im\,\varphi)(\approx) \subseteq (Im\,\varphi)(\approx)$. The converse inclusion is obvious. □

It follows from 5.3 that $\mathcal{C}/\Theta_{\mathbf{K}}(Im\,\varphi)$ is of the form

$$(\varphi(A), \{f_{\mathcal{B}} : f \in F\}, \{\Theta_{\mathbf{K}}(Im\,\varphi)(r) : r \in R\})$$

and, as $\Theta_{\mathbf{K}}(Im\,\varphi)$ is a **K**-congruence on $\mathcal{C}$, the system belongs to **K**. In what follows it will be denoted by $M(Im\,\varphi, \mathcal{B})$.

Let $\mathcal{E}(\mathbf{K})$ and $\mathcal{M}(\mathbf{K})$ denote the classes of all **K**-homomorphisms $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ such that $\mathcal{B}$ coincides with $M(Im\,\varphi, \mathcal{B})$ and $\psi : \mathcal{C} \longrightarrow \mathcal{D}$ such that ψ is injective, respectively. Notice that every member of $\mathcal{E}(\mathbf{K})$ is a surjective map.

A *factorization system* $(\mathcal{E}, \mathcal{M})$ for **K** consists of some category $\mathcal{E}$ of **K**-epimorphisms and some category $\mathcal{M}$ of **K**-monomorphisms such that, for every **K**-morphism f, there exists a decomposition $f = m \circ e$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, and the *diagonalization property* holds (that is, for $h \circ e = m \circ k$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, $g \circ e = k$ and $m \circ g = h$ for some **K**-morphism g).

PROPOSITION 5.4. *$(\mathcal{E}(\mathbf{K}), \mathcal{M}(\mathbf{K}))$ is a factorization system for* **K**.

Proof. Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a **K**-homomorphism. Define $\rho : \mathcal{A} \longrightarrow M(Im\,\varphi, \mathcal{B})$ and $\psi : M(Im\,\varphi, \mathcal{B}) \longrightarrow \mathcal{B}$ by $\rho(x) = \varphi(x)$ for $x \in A$ and $\psi(x) = x$ for $x \in \varphi(A)$. Since $M(Im\,\varphi, \mathcal{B}) \in \mathbf{K}$, ρ and ψ are **K**-morphisms. Obviously, $\varphi = \psi \circ \rho$, proving that $(\mathcal{E}(\mathbf{K}), \mathcal{M}(\mathbf{K}))$ has the factorization property.

To prove that $(\mathcal{E}(\mathbf{K}), \mathcal{M}(\mathbf{K}))$ has the diagonalization property, let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$, $\sigma : \mathcal{A} \longrightarrow \mathcal{C}$, $\rho : \mathcal{B} \longrightarrow \mathcal{D}$, $\psi : \mathcal{C} \longrightarrow \mathcal{D}$ be **K**-morphisms such that φ is in $\mathcal{E}(\mathbf{K})$, ψ is in $\mathcal{M}(\mathbf{K})$, and $\rho \circ \varphi = \psi \circ \sigma$. Since ψ is injective, $Ker\,\varphi \le Ker\,\sigma$. So, the map $\varepsilon : \mathcal{B} \longrightarrow \mathcal{C}$ is well-defined, where $\varepsilon(x) = \sigma(y)$, $y \in A$ and $\varphi(y) = x$. The facts that ε preserves operations, $\varepsilon \circ \varphi = \sigma$, and $\psi \circ \varepsilon = \rho$ are obvious; they follow from the definition of ε. We show that ε preserves relations which will complete the proof.

Let $r \in R$ and $(c_1, \dots, c_{a(r)}) \in r_{\mathcal{B}}$. As φ is in $\mathcal{E}(\mathbf{K})$, $\mathcal{B}$ coincides with $M(Im\,\varphi, \mathcal{B})$. This, by 5.1, implies

$$(t_{i_1}(\overline{b}), \dots, t_{i_{a(r_i)}}(\overline{b})) \in (Im\,\varphi)(r_i) \text{ for } i < k, \text{ where } 0_{\varphi(\mathcal{A})} = Im\,\varphi, \text{ and}$$

$$(t_1(\overline{b}), \dots, t_{a(r_i)}(\overline{b})) = (c_1, \dots, c_{a(r)})$$

for some quasi-identity

$$Q: \quad \forall \overline{x}[\bigwedge_{i<k} r_i(t_{i_1}(\overline{x}), \ldots, t_{i_{a(r_i)}}(\overline{x})) \Longrightarrow r(t_1(\overline{x}), \ldots, t_{a(r)}(\overline{x}))]$$

valid in $\mathbf{K}$ and a sequence $\overline{b}$ of elements in B. Let $\overline{d}$ be a sequence of elements of A such that $\varphi(\overline{d}) = \overline{b}$. Since φ preserves terms, we have

$$(\varphi(t_{i_1}(\overline{d})), \ldots, \varphi(t_{i_{a(r_i)}}(\overline{d}))) \in (Im\,\varphi)(r_i) \text{ for } i < k \text{ and}$$

$$(\varphi(t_1(\overline{d})), \ldots, \varphi(t_{a(r)}(\overline{d}))) = (\varphi(z_1), \ldots, \varphi(z_{a(r)})),$$

where $z_1, \ldots, z_{a(r)}$ are elements of A such that $\varphi(z_i) = c_i$ for $i = 1, \ldots, a(r)$. Recall that $(Im\,\varphi)(\approx) = \{(b, b) : b \in B\}$ and $(Im\,\varphi)(r) = \varphi(r_{\mathcal{A}})$ for $r \in R$. Thus, for each $i < k$, there is $(y_{i_1}, \ldots, y_{i_{a(r_i)}})$ in $r_{i\mathcal{A}}$ such that $t_{i_j}(\overline{d}) \equiv y_{i_j}(Ker\,\varphi)$ for $1 \le j \le a(r_i)$. But $Ker\,\varphi \le Ker\,\sigma$. So $t_{i_j}(\overline{d}) \equiv y_{i_j}(Ker\,\sigma)$ for $1 \le j \le a(r_i)$ and $t_j(\overline{d}) \equiv z_j(Ker\,\sigma)$ for $1 \le j \le a(r)$. As σ is a $\mathbf{K}$-morphism and $(y_{i_1}, \ldots, y_{i_{a(r_i)}}) \in r_{i\mathcal{A}}$, we obtain $(t_{i_1}(\sigma(\overline{d})), \ldots, t_{i_{a(r)}}(\sigma(\overline{d}))) \in r_{i\mathcal{C}}$. But the quasi-identity Q is valid in $\mathcal{C}$. So $(t_1(\sigma(\overline{d})), \ldots, t_{a(r)}(\sigma(\overline{d}))) \in r_{\mathcal{C}}$. Thus $(\sigma(z_1), \ldots, \sigma(z_{a(r)})) \in r_{\mathcal{C}}$, that is $(\varepsilon(c_1), \ldots, \varepsilon(c_{a(r)})) \in r_{\mathcal{C}}$, proving that ε preserves relations. □

A family $(\varphi_i : \mathcal{A}_i \longrightarrow \mathcal{B} : i \in I)$ of $\mathbf{K}$-morphisms is called a *sink* in $\mathbf{K}$. A sink $(\varphi_i : \mathcal{A}_i \longrightarrow \mathcal{B} : i \in I)$ in $\mathbf{K}$ is said to be an $\mathcal{E}(\mathbf{K})$-*sink* if, for every sink $(\psi_i : \mathcal{A}_i \longrightarrow \mathcal{C} : i \in I)$ in $\mathbf{K}$ and every $\mathbf{K}$-morphism $\sigma : \mathcal{C} \longrightarrow \mathcal{B}$, $\varphi_i = \sigma \circ \psi_i$ for all $i \in I$ implies that σ is in $\mathcal{E}(\mathbf{K})$.

Let $(\varphi_i : \mathcal{A}_i \longrightarrow \mathcal{B} : i \in I)$ be a sink in $\mathbf{K}$. Let $\mathcal{C}$ be the subalgebra of $alg(\mathcal{B})$ generated by $\bigcup(\varphi_i(A_i) : i \in I)$. Let $Im \bigcup \varphi_i$ denote the following R^+-indexed family on C:

$$\left(Im \bigcup \varphi_i\right)(\approx) = \{(c, c) : c \in C\} \text{ and}$$
$$\left(Im \bigcup \varphi_i\right)(r) = \bigcup(\varphi_i(r_{\mathcal{A}_i}) : i \in I) \text{ for } r \in R.$$

One may verify as in the proof of 5.3 that $\Theta_{\mathbf{K}}(Im \bigcup \varphi_i)(\approx) = (Im \bigcup \varphi_i)(\approx)$. This gives that the quotient system $\mathcal{C}/\Theta_{\mathbf{K}}(Im \bigcup \varphi_i)$ coincides with

$$\left(C, \{f_{\mathcal{B}} : f \in F\}, \left\{\Theta_{\mathbf{K}}\left(Im \bigcup \varphi_i\right)(r) : r \in R\right\}\right).$$

In what follows the system will be denoted by $M(Im \bigcup \varphi_i, \mathcal{B})$.

LEMMA 5.5. *A sink in* $\mathbf{K}$ $(\varphi_i : \mathcal{A}_i \longrightarrow \mathcal{B} : i \in I)$ *is an* $\mathcal{E}(\mathbf{K})$-*sink iff* $\mathcal{B}$ *coincides with* $M(Im \bigcup \varphi_i, \mathcal{B})$.

Proof. Consider the cone $(\psi_i : \mathcal{A}_i \longrightarrow M(Im \bigcup \varphi_i, \mathcal{B}) : i \in I)$, where $\psi_i(x) = \varphi_i(x)$ for $x \in A_i$, and the **K**-morphism $\sigma : M(Im \bigcup \varphi_i, \mathcal{B}) \longrightarrow \mathcal{B}$, where $\sigma(x) = x$ for x in $M(Im \bigcup \varphi_i, \mathcal{B})$. If $(\varphi_i : \mathcal{A}_i \longrightarrow \mathcal{B} : i \in I)$ is an $\mathcal{E}(\mathbf{K})$-sink, then σ belongs to $\mathcal{E}(\mathbf{K})$. In particular, σ is the identity map on $\mathcal{B}$ and $r_{\mathcal{B}} = \Theta_{\mathbf{K}}(Im\,\sigma)(r)$ for $r \in R$. But $\Theta_{\mathbf{K}}(Im\,\sigma) = \Theta_{\mathbf{K}}(\Theta_{\mathbf{K}}(Im \bigcup \varphi_i))$ and, as $\Theta_{\mathbf{K}}(\Theta_{\mathbf{K}}(Im \bigcup \varphi_i)) = \Theta_{\mathbf{K}}(Im \bigcup \varphi_i)$, $\Theta_{\mathbf{K}}(Im\,\sigma) = \Theta_{\mathbf{K}}(Im \bigcup \varphi_i)$. Thus $\mathcal{B}$ coincides with $M(Im \bigcup \varphi_i, \mathcal{B})$.

Assume now that $\mathcal{B}$ coincides with $M(Im \bigcup \varphi_i, \mathcal{B})$. Let $(\psi_i : \mathcal{A}_i \longrightarrow \mathcal{C} : i \in I)$ be a sink in **K** and $\sigma : \mathcal{C} \longrightarrow \mathcal{B}$ be a **K**-morphism such that $\psi_i \circ \sigma = \varphi_i$ for all $i \in I$. Notice that $\sigma(C)$ coincides with B. Define

$$(Im\,\sigma)(\approx) = \{(b, b) : b \in B\} \text{ and}$$
$$(Im\,\sigma)(r) = \sigma(r_{\mathcal{C}}) \text{ for } r \in R.$$

Notice that $(Im\,\sigma)(\approx) = \Theta_{\mathbf{K}}(Im \bigcup \varphi_i)(\approx)$ and, for $r \in R$,

$$\left(Im \bigcup \varphi_i\right)(r) \subseteq (Im\,\sigma)(r) \subseteq \Theta_{\mathbf{K}}\left(Im \bigcup \varphi_i\right)(r).$$

This gives that $\Theta_{\mathbf{K}}(Im\,\sigma) = \Theta_{\mathbf{K}}(Im \bigcup \varphi_i)$ which, in turn, implies that $M(Im\,\sigma, \mathcal{B})$ coincides with $M(Im \bigcup \varphi_i, \mathcal{B})$. Thus $M(Im\,\sigma, \mathcal{B})$ coincides with $\mathcal{B}$, showing that σ is in $\mathcal{E}(\mathbf{K})$. □

We say that $(\mathcal{E}(\mathbf{K}), \mathcal{M}(\mathbf{K}))$ *factorizes sinks* in **K** if, for every sink in **K** $(\varphi_i : \mathcal{A}_i \longrightarrow \mathcal{B} : i \in I)$ there exist an $\mathcal{E}(\mathbf{K})$-sink $(\psi_i : \mathcal{A}_i \longrightarrow \mathcal{C} : i \in I)$ and $\sigma : \mathcal{C} \longrightarrow \mathcal{B}$ in $\mathcal{M}(\mathbf{K})$ such that $\varphi_i = \sigma \circ \psi_i$ for all $i \in I$.

PROPOSITION 5.6.

(i) $(\mathcal{E}(\mathbf{K}), \mathcal{M}(\mathbf{K}))$ *factorizes sinks in* **K**;
(ii) *If* $\mathbf{F} : \mathbf{G} \longrightarrow \mathbf{K}$ *is a functor, then* $\mathbf{F}\restriction\mathcal{G}$ *maps every* $\mathcal{E}(\mathbf{G})$*-sink in* $\mathcal{G}$ *to an* $\mathcal{E}(\mathbf{K})$*-sink.*

Proof. (i) As $\mathcal{C}$, take $M(Im \bigcup \varphi_i, \mathcal{B})$, as ψ_i, take $\psi_i(x) = \varphi_i(x)$ for all $x \in A_i$, and, as σ, take $\sigma(x) = x$ for all x in $M(Im \bigcup \varphi_i, \mathcal{B})$. Obviously, $\sigma : M(Im \bigcup \varphi_i, \mathcal{B}) \longrightarrow \mathcal{B}$ is in $\mathcal{M}(\mathbf{K})$ and, by 5.5, $(\psi_i : \mathcal{A}_i \longrightarrow M(Im \bigcup \varphi_i, \mathcal{B}) : i \in I)$ is an $\mathcal{E}(\mathbf{K})$-sink.

(ii) Notice that among $\mathcal{G}$-morphisms of an $\mathcal{E}(\mathbf{G})$-sink in $\mathcal{G}$ at least one must be an isomorphism. Thus the image of the sink by **F** must be an $\mathcal{E}(\mathbf{K})$-sink. □

It follows from Propositions 5.2, 5.4, and 5.6 that if **K** a finite-to-finite universal quasivariety of algebraic systems, then **K** together with $\mathcal{E}(\mathbf{K})$ and $\mathcal{M}(\mathbf{K})$ satisfies the assumptions of Proposition 2.4. Thus, for every finite-to-finite universal quasivariety of algebraic systems $\mathbf{K}$ there exists a finite-to-finite and full embedding $\mathbf{\Phi} : \mathbf{G} \longrightarrow \mathbf{K}$ such that $\mathbf{\Phi}(f) : \mathbf{\Phi}(G) \longrightarrow \mathbf{\Phi}(H)$ is onto whenever $f : G \longrightarrow H$ is a strong morphism in $\mathbf{G}$.

6. Posets with constants (Corollary 1.4)

In [5], it was proved that $\mathbf{P_n}$ is finite-to-finite universal iff $n \geq 2$. Clearly, $\mathbf{P_0}$ has precisely two subquasivarieties (itself and the trivial quasivariety). Thus, by Theorem 1.1, the proof of Corollary 1.4 will be completed if we show that $L(\mathbf{P_1})$ is a finite lattice.

For a finite non-trivial algebraic system A, let $A_0, \ldots, A_{n-1}$ denote the set of all proper subsystems of A. Then A is *critical* providing $A \notin \mathbf{Q}(A_0, \ldots, A_{n-1}) = \mathbf{ISP}(A_0, \ldots, A_{n-1})$. In particular, $(P; \leq, p)$ in $\mathbf{P_1}$ is not critical providing, for every $x, y \in P$ with $y \not\leq x$, there exist a proper subsystem Q of P and an order-preserving map $\varphi : P \longrightarrow Q$ such that $\varphi(p) = p$ and $\varphi(y) \not\leq \varphi(x)$. A quasivariety of relational systems and, hence, any subquasivariety of $\mathbf{P_1}$, is generated by its critical relational systems. Thus, classification of the critical relational systems of $\mathbf{P_1}$ will lead automatically to a determination of $L(\mathbf{P_1})$.

Let $(P_1; \leq, p)$ denote the trivial algebraic system in $\mathbf{P_1}$. With the preceding remarks in mind, let $(P; \leq, p)$ be a critical relational system in $\mathbf{P_1}$. We will show that, for some $2 \leq i \leq 7$, $(P; \leq, p)$ is isomorphic to $(P_i; \leq, p)$ where $P_2 = P_3 = P_4 = \{p, a\}$ with $a \leq p$ in P_2 and $p \leq a$ in P_4, and $P_5 = P_6 = P_7 = \{p, a, b\}$ where $a \leq b$ in P_5, $a \leq b, p$ in P_6, and $b, p \leq a$ in P_7.

Suppose that both P_2 and P_4 are subsystems of $(P; \leq, p)$. For any $x, y \in P$ with $y \not\leq x$, there exists an order-preserving map φ onto any 2-element chain in P with $\varphi(y) \not\leq \varphi(x)$. In particular, depending on the value of $\varphi(p)$, φ may be considered as a morphism from P to P_2 or P_4 and, so, P is not critical.

Assume then that either P_2 or P_4 is not a subsystem of P. If P_6 is a subsystem (and, so, P_4 is not a subsystem), then, for $y \not\leq x$, if $y \leq p$ let $\varphi([y)) = p$ and $\varphi(P \setminus [y)) = a$, and if $y \not\leq p$ let $\varphi([y)) = b$, $\varphi(([y)]\setminus[y)) = a$ and $\varphi(P \setminus ([y) \cup ([y)])) = p$. Either way, we conclude that $P = P_6$. A similar argument shows that $P = P_7$ in the event that P_7 is a subsystem of P.

Assume then that neither P_6, nor P_7, nor one of P_2 or P_4 is a subsystem of P. Suppose, P_2 is a subsystem. If $P = (p]$, then, for $y \not\leq x$, let $\varphi(P) \longrightarrow P_2$ be given by $\varphi((x]) = a$ and $\varphi(P \setminus (x]) = p$. It follows that $P = P_2$. If $P \neq (p]$, then P_3 is a subsystem of P. For $y \not\leq x$, φ as just defined will serve unless $x = p$. In which case, let $\varphi : P \longrightarrow P_3$ be given by $\varphi((p]) = p$ and $\varphi(P \setminus (p]) = a$. Since P is critical, this situation can not arise. Thus, we conclude that $P = P_2$ if P_2 is a subsystem. A similar argument holds should P_4 be a subsystem.

Should neither P_2 nor P_4 be subsystems of P, then, for non-trivial P, either P contains a 2-element chain and we conclude that $P = P_5$ or else $P = P_3$.

In summary, any critical system in $\mathbf{P_1}$ is of the form P_i for some $2 \leq i \leq 7$.

Since the only proper subsystems of P_2, P_3, and P_4 are trivial, P_i is critical for $2 \le i \le 4$. For any homomorphism φ from P_5 to a proper subsystem of itself, $\varphi(a) = \varphi(b)$ and, so, it too is critical. Finally, $\varphi(b)$ is comparable with p for any homomorphism from P_6 or P_7 to a proper subsystem of itself and, so, P_6 and P_7 are also critical.

By the above, $(P_i; \le, p)$ for $2 \le i \le 7$ is a complete description of the critical relational systems of $\mathbf{P_1}$.

Clearly, $\mathbf{Q}(P_1)$ is covered by $\mathbf{Q}(P_i)$ for $2 \le i \le 4$ and $\mathbf{Q}(P_3)$ is covered by $\mathbf{Q}(P_5)$. However, since $|\varphi(P_i)| = 1$ for any homomorphism $\varphi : P_i \longrightarrow P_5$ with $i = 2$ or 4, $\mathbf{Q}(P_2)$ and $\mathbf{Q}(P_4) \not\subseteq \mathbf{Q}(P_5)$. Likewise, $|\varphi(P_2)| = 1$ if $\varphi : P_2 \longrightarrow P_7$ and $|\varphi(P_4)| = 1$ if $\varphi : P_4 \longrightarrow P_6$. Thus, $\mathbf{Q}(P_2) \not\subseteq \mathbf{Q}(P_7)$ and $\mathbf{Q}(P_4) \not\subseteq \mathbf{Q}(P_6)$. Since $P_5 \in \mathbf{Q}(P_6)$ and $\mathbf{Q}(P_7)$, $\mathbf{Q}(P_2) \vee \mathbf{Q}(P_3) = \mathbf{Q}(P_2) \vee \mathbf{Q}(P_5)$, $\mathbf{Q}(P_4) \vee \mathbf{Q}(P_3) = \mathbf{Q}(P_4) \vee \mathbf{Q}(P_5)$, $P_6 \in \mathbf{Q}(P_2) \vee \mathbf{Q}(P_4)$, and $P_7 \in \mathbf{Q}(P_2) \vee \mathbf{Q}(P_4)$, we conclude the following.

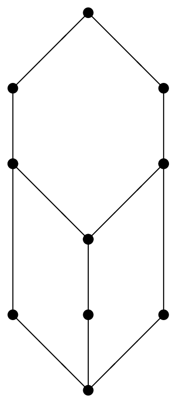

Figure 6

PROPOSITION 6.1. *$L(\mathbf{P_1})$ is diagrammed by Figure* 6.

Acknowledgments

Theorem 1.1 was originally proved (see [3]) under the assumption that $\mathbf{K}$ is finite-to-finite universal due to a functor $\mathbf{F}$ such that, for all finite directed graphs G and H, $\mathbf{F}(f) : \mathbf{F}(G) \longrightarrow \mathbf{F}(H)$ is onto whenever $f : G \longrightarrow H$ is a strong morphism in $\mathbf{G}$. The authors wish to thank Václav Koubek and Jiří Sichler who, after reading an earlier version, generously suggested that Theorem 1.2 of their paper [19] could be used as a means of eliminating the extra assumption imposed on the functor.

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