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On the trigonometric subtraction and addition formulas

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This paper is dedicated to Professor János Aczél on his 75th birthday

Summary. We solve extensions to groups of the trigonometric addition and subtraction formulas, in which the plus/minus has been replaced by an involutive group automorphism. The groups need not be abelian.

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I. Introduction

The trigonometric addition and subtraction formulas and their relations have been studied from the point of view of functional equations by a number of mathematicians. Let us mention Wilson [13], Vietoris [11] and Vincze [12]. The monographs by Aczél [1, Section 3.2.3], by Aczél and Dhombres [2, Ch. 13] and by Székelyhidi [10, Chap. 12] have references and detailed discussions of the classical results. All the above references deal with functions defined on abelian groups. The functional equations $f(xy) = f(x)g(y) + g(x)f(y)$, $x, y \in G$, and $g(xy) = g(x)g(y) + f(x)f(y)$, $x, y \in G$, in which the group G need not be abelian, were solved by Chung, Kannappan and Ng [3] during their discussion of the cosine-sine functional equation

$$f(xy) = f(x)g(y) + g(x)f(y) + h(x)h(y), \quad x, y \in G. \tag{1}$$

We continue these investigations.

To formulate our results we introduce the following notation and assumptions that will be used throughout the paper: G is a topological group, $C(G)$ the algebra of continuous, complex valued functions on G, and $\sigma : G \to G$ a continuous homomorphism such that $\sigma \circ \sigma = I$ were I denotes the identity map.

We find the solutions $f, g \in C(G)$ of each of the following versions of the addition and subtraction formulas for sine and cosine:

$$f(x\sigma(y)) = f(x)g(y) - g(x)f(y), \quad \forall x, y \in G, \tag{2}$$
$$f(x\sigma(y)) = f(x)g(y) + g(x)f(y), \quad \forall x, y \in G, \tag{3}$$
$$g(x\sigma(y)) = g(x)g(y) + f(x)f(y), \quad \forall x, y \in G, \tag{4}$$

(Theorems II.2, II.3 and II.4 below). The classical addition and subtraction formulas, where the group G is abelian, correspond to $\sigma = I$ and $\sigma = -I$ respectively (replace possibly f by if). Other examples of σ are transposition of $n \times n$ matrices and reflection in a hyperplane. To solve the functional equations (2), (3) and (4) we reduce them to the special case of $\sigma = I$ where the solutions are given in Section III.

The two classical functional equations

$$\begin{aligned} g(x+y) &= g(x)g(y) + f(x)f(y), \quad \forall x, y \in G, \\ g(x-y) &= g(x)g(y) + f(x)f(y), \quad \forall x, y \in G, \end{aligned} \tag{5}$$

are in our set up unified by (4) and solved simultaneously by Theorem II.4. So (4) demonstrates the relation between the two equations in (5), and is interesting for as well $\sigma = I$ as $\sigma = -I$.

Our discussion of the addition and subtraction formulas in this general setting is new, as are the explicit solution formulas for $\sigma \neq \pm I$. We take continuity of the solutions into account in contrast to the papers [3] and [12] that describe the set of all solutions without specifying the continuous ones. We can of course get all solutions, continuous or not, by considering the special instance of the discrete topology on G.

We want to point out the following: What generalizes the subtraction formulas from the abelian to the non-abelian case is that the map $x \to -x$ is a homomorphism the square of which is the identity map. We do not discuss functional equations like $f(xy^{-1}) = \cdots$ in the non-abelian case, because $\sigma(y) = y^{-1}$ is an antihomomorphism, not a homomorphism.

The related functional equation $f(y^{-1}x) = (a(x) \mid a(y))$ was studied by Gajda [5] and earlier by O'Connor [6].

II. The subtraction and addition formulas

Notation II.1. The set of continuous homomorphisms $a : G \to (\mathbb{C}, +)$ will be denoted by $\mathcal{A}(G)$. Those $a \in \mathcal{A}(G)$ for which $a \circ \sigma = a$, resp. $a \circ \sigma = -a$ will be denoted $\mathcal{A}^+(G)$, resp. $A^-(G)$. In the classical case of $\sigma = -I$ the spaces $\mathcal{A}^-(G)$ and $\mathcal{A}^+(G)$ become $\mathcal{A}(G)$ and $\{0\}$ respectively. $\mathbb{C}^*$ is the multiplicative group $(\mathbb{C}\backslash\{0\}, \cdot)$ of non-zero complex numbers.

Theorem II.2 (The sine subtraction formula). *The continuous solutions $f, g \in C(G)$ of*

$$f(x\sigma(y)) = f(x)g(y) - g(x)f(y), \quad \forall x, y \in G, \tag{6}$$

are the following, where $m : G \to \mathbb{C}^*$ denotes a continuous homomorphism and c and c_1 complex constants:

(i) $f = 0, g$ *any function.*
(ii) $f = c_1(m - m \circ \sigma)/2, g = (m + m \circ \sigma)/2 + c(m - m \circ \sigma)/2$, *where* $m \neq m \circ \sigma$.
(iii) $f = ma^-, g = m(1 + ca^-)$, *where* $a^- \in \mathcal{A}^-(G)$, *and* $m = m \circ \sigma$.

Proof. Verifying that the stated pairs of functions are solutions consists in simple computations that we leave out. To see that any solution $f, g \in C(G)$ of (6) is contained in one of the three cases we proceed as follows.

Since the right hand side of the functional equation (6) changes sign when x and y are interchanged we see that $f(x\sigma(y)) = -f(y\sigma(x))$ for all $x, y \in G$. Putting $y = e$ we find that $f \circ \sigma = -f$. Using this and the identity (6) we find that

$$\begin{aligned} &f(x)[g(y) - g(\sigma(y))] - f(y)[g(x) - g(\sigma(x))] \\ &= f(x)g(y) - g(x)f(y) - f(x)g(\sigma(y)) + g(\sigma(x))f(y) \\ &= f(x\sigma(y)) + f(\sigma(x))g(\sigma(y)) - g(\sigma(x))f(\sigma(y)) \\ &= f(x\sigma(y)) + f(\sigma(x)\sigma(\sigma(y))) = f(x\sigma(y)) + f(\sigma(x)y) \\ &= f(x\sigma(y)) - f(x\sigma(y)) = 0, \end{aligned} \tag{7}$$

so

$$f(x)[g(y) - g(\sigma(y))] = f(y)[g(x) - g(\sigma(x))] \text{ for all } x, y \in G. \tag{8}$$

If $f = 0$ we deal with case (i). So from now on we will assume that there exists an $x_0 \in G$ for which $f(x_0) \neq 0$.

Defining g^+ and g^- by $g^\pm := (g \pm g \circ \sigma)/2$ we have $g = g^+ + g^-$, and from (8) that $f(x)g^-(y) = g^-(x)f(y), \forall x, y \in G$. In particular that $g^- = cf$ where $c = g^-(x_0)/f(x_0)$. When we substitute this into (6) we get that

$$f(x\sigma(y)) = f(x)g^+(y) - g^+(x)f(y) \quad \text{for all } x, y \in G, \tag{9}$$

and so - replacing y by $\sigma(y)$ - that

$$f(xy) = f(x)g^+(y) + g^+(x)f(y) \quad \text{for all } x, y \in G. \tag{10}$$

This functional equation is solved by Proposition III.1 below according to which we have the following 4 possibilities, using the notation in Proposition III.1:

(a): $f = 0$ and g^+ anything. This case does not occur here due to our assumption that $f \neq 0$.

(b): $g^+ = m/2$ and $f = c_1 m$ where $c_1 \in \mathbb{C}\backslash\{0\}$. Via (9) we see that $f = 0$, so this case does not occur either.

(c): $g^+ = (m + M)/2$ and $f = c_1(m - M)$ where $c_1 \in \mathbb{C}\backslash\{0\}$. Since $f \circ \sigma = -f$ and $g^+ \circ \sigma = g^+$ we get that $m \circ \sigma - M \circ \sigma = M - m$ and $m \circ \sigma + M \circ \sigma = M + m$ implying that $M = m \circ \sigma$. Thus $g^+ = (m + m \circ \sigma)/2$ and $f = c + i(m - m \circ \sigma)$.

Combining this with the relation $g^- = cf$ derived above, we see that we have case (ii).

(d): $g^+ = m$ and $f = ma$. From $g^+ \circ \sigma = g^+$ we infer that $m \circ \sigma = m$, and from $f \circ \sigma = -f$ that $a \circ \sigma = -a$, i.e., that $a \in \mathcal{A}^-(G)$. Combining this with the relation $g = cf$ derived above, we see that we have case (iii). $\square$

Theorem II.3 (The sine addition formula)**.** *The set of solutions $f, g \in C(G)$ of the functional equation*

$$f(x\sigma(y)) = f(x)g(y) + g(x)f(y), \quad \forall x, y \in G, \tag{11}$$

can listed as follows where $m, M : G \to \mathbb{C}^*$ denote continuous homomorphisms such that $m \circ \sigma = m$ and $M = M \circ \sigma$, and where c denotes non-zero complex constants:

(i) *$f = 0$ and g arbitrary.*
(ii) *$f = m/2$ and $f = cm$.*
(iii) *$g = (m + M)/2$ and $f = c(m - M)$.*
(iv) *$g = m$ and $f = ma^+$, where $a^+ \in \mathcal{A}^+(G)$.*

Proof. It is elementary to check that the cases stated in the Theorem define solutions, so it is left to show that any solution $f, g \in C(G)$ of (11) falls into one of these cases.

If $f = 0$ then we have case (i), so we shall from now on assume that $f \neq 0$.

Since the right hand side of (11) is invariant under interchange of x and y we get that $f(x\sigma(y)) = f(y\sigma(x))$. Taking $y = e$ we get in particular that $f = f \circ \sigma$.

Using (11) we get

$$\begin{aligned} &f(x)g(y) + g(x)f(y) = f(x\sigma(y)) = (f \circ \sigma)(x\sigma(y)) = f((\sigma(x))\sigma(\sigma(y))) \\ &= f(\sigma(x))g(\sigma(y)) + g(\sigma(x))f(\sigma(y)) = f(x)g(\sigma(y)) + g(\sigma(x))f(y), \end{aligned} \tag{12}$$

so $f(x)[g(\sigma(y)) - g(y)] = f(y)[g(x) - g(\sigma(x))]$ for all $x, y \in G$. The left hand side does not change if x is replaced by $\sigma(x)$, but the right hand side does. Hence both sides are 0. Since $f \neq 0$ this means that $g = g \circ \sigma$.

Using the σ-invariance of f and g we find

$$f(xy) = f(x\sigma(\sigma(y)) = f(x)g(\sigma(y)) + g(x)f(\sigma(y)) = f(x)g(y) + g(x)f(y). \tag{13}$$

The solutions of the functional equation $f(xy) = f(x)g(y) + g(x)f(y)$ are written down in Proposition III.1 below. However the solutions here have the extra invariance property that $g = g \circ \sigma$ and $f = f \circ \sigma$. We work our way through the 4 possibilities (a)–(d) presented by Proposition III.1 to see what the invariance property entails. We use the notation of Proposition III.1.

(a): $f = 0$. By assumption $f \neq 0$ so this case is excluded.

(b): $g = m/2$ and $f = cm$. Here we get that $m = m \circ \sigma$, so that we have case (ii).

(c): $g = (m + M)/2$ and $f = c(m - M)$. Here we get from the σ-invariance of g and f that $m + M = m \circ \sigma + M \circ \sigma$ and $m - M = m \circ \sigma - M \circ \sigma$. By addition and subtraction we find that $m = m \circ \sigma$ and $M = M \circ \sigma$. This is case (iii).

(d): $g = m$ and $f = ma$. Here we find that $m = m \circ \sigma$. Combined with $f = f \circ \sigma$ it implies that $a = a \circ \sigma$, i.e. that $a \in \mathcal{A}^+(G)$. This is case (iv). □

Theorem II.4 below seemingly contains more solutions than found in Vietoris [11] and Wilson [13]. Remark II.5 explains why.

Theorem II.4 (The cosh addition and the cosine subtraction formulas)**.** *The continuous solutions $f, g \in C(G)$ of*

$$g(x\sigma(y)) = g(x)g(y) + f(x)f(y), \quad \forall x, y \in G, \tag{14}$$

are the following, where $m, M : G \to \mathbb{C}^*$ denote continuous homomorphisms:

(i) $g = 0, f = 0$.

(ii) $g = (1+c^2)^{-1}m,\ f = c(1+c^2)^{-1}m$, *where* $m = m \circ \sigma$, *and* $c \in \mathbb{C}\backslash\{\pm i\}$.

(iii)
$$g = \frac{m + c^2 M}{1+c^2} \text{ and } f = c\frac{m - M}{1+c^2}, \tag{15}$$
where $m = m \circ \sigma$ *and* $M = M \circ \sigma$ *and* $c \in \mathbb{C}\backslash\{0, i, -i\}$

(iv) $g = m(1+ia), f = ma$, *where* $a \in \mathcal{A}^+(G)$ *and* $m = m \circ \sigma$.

(v) $g = m(1-ia), f = ma$, *where* $a \in \mathcal{A}^+(G)$ *and* $m = m \circ \sigma$.

(vi) $g = (m + m\circ\sigma)/2, f = i(m - m\circ\sigma)/2$, *where* $m \neq m \circ \sigma$.

Proof. Checking that the stated pairs of functions satisfy (14) is done by elementary calculations, that we leave out. It is left to show that each solution $f, g \in C(G)$ of (14) falls into one of the categories (i)–(vi).

We note that $g = g \circ \sigma$ because the right hand side of (14) is invariant under interchange of x and y.

If $f = 0$ then we get from (14) that g is zero or a homomorphism into $\mathbb{C}^*$, so we are in case (i) or (ii) with $c = 0$. We can from now on assume that there exists an $x_0 \in G$ such that $f(x_0) \neq 0$.

Using the identity (14) and that $g = g \circ \sigma$ we find that

$$\begin{aligned} g(x)g(y) + f(x)f(y) &= g(x\sigma(y)) = (g\circ\sigma)(x\sigma(y)) = g(\sigma(x)\sigma(\sigma(y))) \\ &= g(\sigma(x))g(\sigma(y)) + f(\sigma(x))f(\sigma(y)) = g(x)g(y) + f(\sigma(x))f(\sigma(y)). \end{aligned} \tag{16}$$

Comparing the left and right hand sides we see that

$$f(x)f(y) = f(\sigma(x))f(\sigma(y)), \quad \forall x, y \in G. \tag{17}$$

Letting $x = y = x_0$ in (17) we get that $f(x_0) = f(\sigma(x_0))$ or $f(x_0) = -f(\sigma(x_0))$. Again by (17),

$$f(x) = \frac{f(\sigma(x_0))}{f(x_0)} f(\sigma(x)) \text{ for all } x \in G, \tag{18}$$

so that either $f \circ \sigma = f$ or $f \circ \sigma = -f$.

Suppose $f \circ \sigma = f$. Replacement of y by $\sigma(y)$ in (14) yields the addition formula $g(xy) = g(x)g(y) + f(x)f(y)$, the solutions of which are given in Proposition III.2.

When we analyse its solutions to find those for which f and g are σ-invariant, we find that only the ones stated in the cases (i) through (v) of Theorem II.4 survive.

Suppose $f \circ \sigma = -f$. Replacing y by $\sigma(y)$ in (14) we find that $g(xy) = g(x)g(y) - f(x)f(y)$ which is the equation (23) in Proposition III.2, except that f should be replaced by if to get the notation to match. Once again we go through the various possibilities (a) - (e) listed in Proposition III.2. Several possibilities will be ruled out by our assumption $f \neq 0$.

(a): $g = 0$ and $f = 0$ is excluded because $f \neq 0$.

(b): $g = (1 + c^2)^{-1}m, f = ic(1 + c^2)^{-1}m$, where $c \in \mathbb{C}\backslash\{\pm i\}$. Since $g \circ \sigma = g$ we get $m = m \circ \sigma$, and so $f \circ \sigma = f$. But $f \circ \sigma = -f$ so we conclude that $f = 0$. But this case is excluded.

(c):

$$g = \frac{m + c^2 M}{1 + c^2} \text{ and } f = ic\frac{m - M}{1 + c^2}, \text{ where } c \in \mathbb{C}\backslash\{0, i, -i\}. \tag{19}$$

From $g \circ \sigma = g$ and $f \circ \sigma = -f$ we get, respectively, that $m \circ \sigma + c^2 M \circ \sigma = m + c^2 M$ and $m \circ \sigma - M \circ \sigma = -m + M$. Since characters are linearly independent (see, e.g., Lemma 29.41 of [7]) the last identity means that at least one of the following four cases occur:

(1) $m \circ \sigma = M \circ \sigma$ or, equivalently, $m = M$.
(2) $m \circ \sigma = m$.
(3) $M = M \circ \sigma$.
(4) $m \circ \sigma = M$.

(1) implies that $f = ic(1+c^2)^{-1}(m - M) = 0$ which is excluded by assumption.
(2) implies by $m \circ \sigma + c^2 M \circ \sigma = m + c^2 M$ that $M = M \circ \sigma$ and so from $m \circ \sigma - M \circ \sigma = -m + M$ that $m = M$. But then we are back in case (1).
(3) implies via $m \circ \sigma + c^2 M \circ \sigma = m + c^2 M$ that $m = m \circ \sigma$ which is the just treated case (2).
(4) implies that $(1 - c^2)M = (1 - c^2)M \circ \sigma$, so either $M = M \circ \sigma$ or $c^2 = 1$. The first possibility is case (3), so only the possibility $c^2 = 1$ remains. We find that $g = (m + c^2 M)(1 + c^2)^{-1} = (m + m \circ \sigma)/2$ and $f = \pm i(m - m \circ \sigma)/2$. Possibly replacing m by $m \circ \sigma$ to get rid of the minus sign we have case (vi) of Theorem II.4.

(d): $g = m(1 + ia), f = ma$, where $a \in \mathcal{A}(G)$. From $g = g \circ \sigma$ and $f = -f \circ \sigma$ we find that $m + iam = m \circ \sigma + i(a \circ \sigma)(m \circ \sigma)$ and $ma = -(m \circ \sigma)(a \circ \sigma)$, implying that $m(1 + 2ia) = m \circ \sigma$. With $M := (m \circ \sigma)/m$ we get that $1 + 2ia = M$. For any $x \in G$ and $n \in \mathbb{Z}$ we find that $1 + 2ina(x) = M(x)^n$. Since exponentials grow faster than linear functions we see that $|M(x)| = 1$, and then the identity shows that $a(x) = 0$. But then $f = ma = 0$ which is excluded by assumption, so this case does not occur.

(e): $g = m(1 - ia), f = ma$, where $a \in \mathcal{A}(G)$. This can be treated just like case (d), so this case does not occur either. $\square$

Remark II.5. If $G = \{x\sigma(x^{-1}) \in G \mid x \in G\}$ then the solutions (ii) through (v) of Theorem II.4 all reduce to the single case (ii), i.e. to $g = (1+c^2)^{-1}, f = c(1+c^2)^{-1}$, where $c \in \mathbb{C}\backslash\{\pm i\}$. This happens in the classical case where G is an abelian group such that $G = 2G$ and where $\sigma = -I$.

Theorem II.3 becomes uninteresting when $G = \{x\sigma(x^{-1}) \in G \mid x \in G\}$, because f then reduces to a constant in all of its cases (i)–(iv). The above explains why the fine structures (i) through (iv) of Theorem II.3 and (ii) through (v) of Theorem II.4 have not been singled out before.

III. Auxiliary results on addition formulas

In this section we exhibit the solutions of the sine and cosh addition formulas that were needed in Section II. These two trigonometric functional equations (20) and (23) were solved by Chung, Kannappan and Ng [3, Lemma 5 and 4], but their results were formulated differently from what we need and they did not take continuity into account, so we present modified proofs of the solution formulas below. For abelian G the functional equations (20) and (23) were solved in Chapter 12 of [10].

Proposition III.1. *The solutions $f, g \in C(G)$ of*

$$f(xy) = f(x)g(y) + g(x)f(y), \quad x, y \in G, \tag{20}$$

are the following, where $m, M : G \to \mathbb{C}^*$ denote continuous homomorphisms and c a non-zero complex constant:

(a) *g any function and $f = 0$.*
(b) *$g = m/2$ and $f = cm$.*
(c) *$g = (m+M)/2$ and $f = c[m-M]$.*
(d) *$g = m$ and $f = ma$, where $a \in \mathcal{A}(G)$.*

Proof. Checking that the stated pairs of functions satisfy (20) is done by elementary calculations, that we leave out. It is left to show that each solution $f, g \in C(G)$ of (20) falls into one of the categories (a) - (d).

Since $f = 0$ is case (a) we shall in the remainder of the proof assume that $f \neq 0$.

If there exists a constant $\alpha \in \mathbb{C}$ such that $g = \alpha f$, then $\alpha \neq 0$ because $g = 0$ implies $f = 0$. The functional equation (20) says that $m := 2\alpha f$ is a continuous homomorphism of G into $\mathbb{C}^*$. Now $f = m/(2\alpha)$ and $g = \alpha f = m/2$, so we have case (b).

We may thus assume that f and g are linearly independent. According to Lemma V.1 of [9] (or the proof of Lemma 5 of [3]) there exists a constant $\kappa \in \mathbb{C}$ such that

$$g(xy) = g(x)g(y) + \kappa^2 f(x)f(y), \quad x, y \in G. \tag{21}$$

Combining this with (20) we get that

$$(g \pm \kappa f)(xy) = (g \pm \kappa f)(x)(g \pm \kappa f)(y), \tag{22}$$

so $g \pm \kappa f$ are continuous homomorphisms of G into $\mathbb{C}^*$. If $\kappa \neq 0$ we have case (c) with $m := g + \kappa f$ and $M := g - \kappa f$.

If $\kappa = 0$ then $m := g$ is according to (21) a continuous homomorphism of G into $\mathbb{C}^*$. Dividing in (20) by $m(xy) = m(x)m(y)$ we get case (d) with $a := f/m$. □

Proposition III.2. *The continuous solutions $f, g \in C(G)$ of*

$$g(xy) = g(x)g(y) + f(x)f(y), \quad \forall x, y \in G, \tag{23}$$

are the following, where $m, M : G \to \mathbb{C}^*$ denote continuous homomorphisms:

(a) $g = 0, f = 0$.
(b) $g = (1 + c^2)^{-1}m$, $f = c(1 + c^2)^{-1}m$, *where* $c \in \mathbb{C}\backslash\{\pm i\}$.
(c)

$$g = \frac{m + c^2 M}{1 + c^2} \text{ and } f = \frac{m - M}{1 + c^2}, \text{ where } c \in \mathbb{C}\backslash\{0, i, -i\}. \tag{24}$$

(d) $g = m(1 + ia)$, $f = ma$, *where* $a \in \mathcal{A}(G)$.
(e) $g = m(1 - ia)$, $f = ma$, *where* $a \in \mathcal{A}(G)$.

Proof. Checking that the stated pairs of functions satisfy (23) is done by elementary calculations, that we leave out. It is left to show that each solution $f, g \in C(G)$ of (23) falls into one of the categories (a) - (e).

If $g = 0$ then so is f. This is case (a). From now on we assume $g \neq 0$.

If there exists a constant $c \in \mathbb{C}$ such that $f = cg$, then

$$g(xy) = (1 + c^2)g(x)g(y), \quad x, y \in G. \tag{25}$$

Here $1 + c^2 \neq 0$ since otherwise $g = 0$ which by assumption is not the case. Thus $m := (1 + c^2)g$ is a non-zero solution of Cauchy's functional equation $\chi(xy) = \chi(x)\chi(y)$ so m is a continuous homomorphism of G into $\mathbb{C}^*$. This is case (b).

From now on we can assume that f and g are linearly independent. According to Lemma 4 of [3] (or Lemma V.3 of [9]) there exists a constant $\alpha \in \mathbb{C}$ such that

$$f(xy) - f(x)g(y) - g(x)f(y) = 2\alpha f(x)f(y), \quad x, y \in G. \tag{26}$$

Combining the identities (26) and (23) we find for any $z \in \mathbb{C}$ that

$$(g - zf)(xy) = (g - zf)(x)(g - zf)(y) - (z^2 + 2\alpha z - 1)f(x)f(y). \tag{27}$$

Let λ and μ denote the two roots of the polynomial $z^2 + 2\alpha z - 1$. Then $\lambda \neq 0, \mu = -\lambda^{-1}$, and so $m := g - \lambda f$ and $M := g + \lambda^{-1} f$ are by (27) solutions of Cauchy's functional equation. Furthermore m and M are non-zero solutions, because f and g are linearly independent. Thus m and M are continuous homomorphisms of G into $\mathbb{C}^*$.

If $\lambda \neq \mu$ then we can express f and g by m and M. This is case (c). If λ and μ coincide then $\lambda = \mu = \alpha = \pm i$ so $g = m \pm if$. From (26) we get in each of the two cases $\alpha = \pm i$ that f satisfies the sine addition formula

$$f(xy) = f(x)m(y) + m(x)f(y), \quad x, y \in G. \tag{28}$$

Dividing by $m(xy) = m(x)m(y)$ in the identity (28) we find that $a := f/m \in \mathcal{A}(G)$. Thus $f = ma$ and $g = m \pm if = m \pm ima = m(1 \pm ia)$. So we have either case (d) or (e). $\square$

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