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Regular projective polyhedra with planar faces I

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Summary. This is the first of two papers in which we classify the regular projective polyhedra in $\mathbb{P}^3$ with planar faces. Here, we develop the basic notions; we introduce a new diophantine trigonometric equation, which plays a key role in the classification theorem, relating the combinatorial and geometric parameters of such polyhedra, and conclude with the case in which the polyhedron is an embedded surface.

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1. Introduction

Intuitively, a projective polyhedron is the drawing of a combinatorial polyhedron in the projective 3-dimensional space $\mathbb{P}^3$; and two are considered to be the same if they differ by a projective isometry. If the combinatorial polyhedron is regular (as symmetric as possible) and all of its symmetries are realized by geometric isometries of the ambient space, then the projective polyhedron is called regular. In this paper and its sequel [2] we center our attention on the case where each face lies on a projective plane, leaving the case of non planar faces for further studies [3].

It is surprising that in the literature of geometric regular polyhedra there is no systematic study of them in terms of projective geometry; in spite of the fact that Coxeter in [6] took the projective geometry approach for the study of elliptic honeycombs, but that was it. This is the program we start, or retake from Coxeter, with this paper. One may think of it as mainly motivated by the question: what and who are the analogues in projective space of the classic regular euclidean polyhedra?

For each regular projective polyhedron, its natural double cover in the 3-sphere can be seen as a regular euclidean polyhedron in $\mathbb{R}^4$. Conversely, a bounded regular euclidean polyhedron in $\mathbb{R}^4$ projects to a projective one. So, in principle, the study of these objects is equivalent. However, we found it easier to visualize and to work with them in $\mathbb{P}^3$; moreover, we think it is the natural approach to the classification problem. This belief is validated by the fact that using this approach one of the authors classified (in the sense of Grünbaum [11]) all regular projective polyhedra in $\mathbb{P}^3$ (see [3]), giving as a corollary the classification of regular polyhedra in $\mathbb{R}^4$.

From the euclidean point of view, a lot of work about these objects and closely related ones has been done, mainly by Coxeter [5, 7, ...], Grünbaum [11], Dress [9, 10], McMullen [13], McMullen–Schulte [14] and Schulte–Wills [16, 17]. This brief list would have to grow considerably (see, e.g. [12]) if we were not focused on polyhedra which are geometrically regular. In particular, the first interesting examples are Coxeter's regular skew polyhedra in $\mathbb{R}^4$ [5], whose "halves" or projective versions we describe and characterize. Therefore, let us emphasize what, besides the approach, is new in this paper.

After establishing general terminology and some basic facts about regular projective polyhedra in general, we prove that those with planar faces are determined by three parameters, which are rational if the polyhedron is finite. Then, we obtain a diophantine trigonometric equation [4], called the *waist equation*, on these three parameters and another, the waist, which is a necessary condition for their existence. The integer solutions of this equation give rise to the projected (and hence to) Coxeter's regular skew polyhedra, yielding a new proof of their uniqueness. The program is similar to Coxeter's in [5], but with a different equation. Some rational solutions are presented and correspond to tori with self intersections (polyhedra of the Kepler–Poinsot type); combinatorially, they are described in [8], but their geometric realization appears explicitly here for the first time. In [2] the waist equation is exploited further, it turns out that not all rational solutions of the waist equation come from finite regular projective polyhedra, but they play a key role for their classification. Also, "taking the opposite" is a new, intrinsically projective, construction introduced in this paper. Given a bounded polyhedron in $\mathbb{R}^4$, projecting it, taking the opposite and lifting it back gives another polyhedron in $\mathbb{R}^4$. In [2], this construction is studied in more generality and detail. Here, we use it for a brief description of Grünbaum's polyhedra in $\mathbb{R}^3$ [11], but, for the sake of brevity, we don't digress on its obvious and implicit applications to other polyhedra to obtain new ones. Finally, we should remark that our description of the projected Coxeter's $\{4, 6\}$ is interesting, projectivelly natural and, to the best of our knowledge, new. Combinatorially, it generalizes naturally to regular polyhedra whose group of automorphisms is the symmetric group S_n, see [1].

2. Definitions

Although some definitions and general arguments may be carried out in the greater generality of incidence polytopes, we shall restrict to polyhedra. Also, we don't digress into the general notion of projective polyhedra but go directly to the regular ones. However, it should be remarked that our implicit definition of projective polyhedra is analogous to that of Grünbaum in [12] for euclidean spaces.

2.1. Regular projective polyhedra

A *combinatorial polyhedron* $\mathcal{P}$ consists of a graph, called its 1-skeleton and denoted $\mathrm{Sk}^1(\mathcal{P})$, together with a collection of cycles, called the *faces* of $\mathcal{P}$, satisfying some additional properties stated below. First, the 1-skeleton is connected and has no loops; but double edges should be allowed in view of our projective geometry interest, because there, two points may be joined by two different segments. For each vertex v define a graph called the *vertex figure* of v, whose vertices are the edges incident to v with two of them adjacent if there is a face containing them. The second condition is that all the vertex figures are cycles. Observe that this implies that each edge is in two faces, and that these conditions are the combinatorial translation of asking that when the faces are viewed as 2-cells attached to the 1-skeleton one obtains a connected surface.

A *flag* is any incident triplet (vertex, edge, face). A combinatorial polyhedron $\mathcal{P}$ is said to be *regular* if its group of automorphisms, $\mathrm{Aut}(\mathcal{P})$, acts transitively on the set of all flags. In particular, this implies that all faces of a regular polyhedron are cycles of the same length, p say, and that all the vertices have the same degree, q say. The ordered pair $\{p, q\}$ is called the Schläfli symbol of $\mathcal{P}$.

The n-dimensional projective space $\mathbb{P}^n$, also known as elliptic space, is the n-dimensional sphere S^n with antipodes identified. Its metric, as well as other geometric notions, come from this identification. Thus, its group of isometries is $\mathrm{Iso}(\mathbb{P}^n) = \mathrm{PO}(n+1) = \mathrm{O}(n+1)/\{I, -I\}$, where I is the identity matrix. For any pair of points in $\mathbb{P}^n$ there are exactly two line segments joining them. Such segments, which together form a line, will be called *opposite*. Given a projective subspace Π of $\mathbb{P}^n$, its *polar space* is the set of points of maximal distance $\pi/2$ from Π; it is also a projective subspace of dimension $n-1$.

A *projective graph* is a finite set of points in $\mathbb{P}^n$, called vertices, together with a set of line segments, called edges, joining some pairs of these vertices. Clearly, there is an underlying combinatorial graph with no loops and, at most, double edges. Given a projective graph G, the *opposite* graph G^{op} has the same vertices as G, and for each edge we choose the opposite segment between the corresponding vertices. Thus, G^{op} is combinatorially isomorphic to G, but in general it is a different projective graph. A *linear map* $g : G \to \mathbb{P}^n$ of a graph G to $\mathbb{P}^n$ is a surjective graph homomorphism (in the sense that vertices go to vertices and edges to edges) to a projective graph. Clearly, any linear map has an *opposite* linear map onto the opposite graph.

A *regular projective polyhedron* $\mathcal{P}$ consists of a regular combinatorial polyhedron $\mathcal{P}_c$ together with a linear map

$$g : \mathrm{Sk}^1(\mathcal{P}_c) \to \mathbb{P}^n$$

for which there exists an injective homomorphism

$$\gamma : \mathrm{Aut}(\mathcal{P}_c) \to \mathrm{Iso}(\mathbb{P}^n),$$

such that for every $\rho \in \mathrm{Aut}(\mathcal{P}_c)$ we have

$$g \circ \rho = \gamma(\rho) \circ g.$$

Two regular projective polyhedra are called *equivalent* (and will be regarded as being the same) if their combinatorial polyhedra are isomorphic and if there exists an isometry of $\mathbb{P}^n$ making the linear maps commute with the isomorphism. A regular projective polyhedron $\mathcal{P}$ in $\mathbb{P}^n$ is *degenerate* if its associated projective graph $g(\mathrm{Sk}^1(\mathcal{P}_c))$ lies in a non trivial projective subspace, that is, if it can be considered as a projective polyhedron in $\mathbb{P}^k$ with $k < n$.

There is an important class of regular projective polyhedra which we call *Platonic* because they resemble their euclidean analogues. Consider a projective graph in $\mathbb{P}^3$ with a distinguished collection of cycles, each of which lies in a projective plane and bounds there a topological disk. If the union of these closed planar disks is a surface, we have a combinatorial polyhedron with a geometric realization. It is a *Platonic projective polyhedron* if it is combinatorially regular and every automorphism may be realized by an isometry of the ambient space. The obvious examples are the platonic solids thinking of projective space as euclidean space plus a plane at infinity. Of course, they will be projectively different according to their "size" or *radius*, which is the distance of a vertex to the center of symmetry, which can take values from 0 to $\pi/2$.

2.2. Relation to euclidean polyhedra

We will see how regular projective polyhedra are in natural correspondence to bounded regular euclidean polyhedra one dimension higher.

Given a regular projective polyhedron $\mathcal{P}$ in $\mathbb{P}^n$ as above, we can *lift* it to a regular euclidean polyhedron $\tilde{\mathcal{P}}$ in $\mathbb{R}^{n+1}$ as follows. Let $\pi : S^n \to \mathbb{P}^n$ be the natural double cover. The inverse image under π of the projective graph $G = g(\mathrm{Sk}^1(\mathcal{P}_c))$ is a "geodesic" graph $\tilde{G}$ in S^n, and we have a double cover $\pi : \tilde{G} \to G$. Let H be the pullback of π by g, that is, the vertices (and edges) of H are pairs (α, β) with $\alpha \in \mathrm{Sk}^1(\mathcal{P}_c)$ and $\beta \in \tilde{G}$ such that $g(\alpha) = \pi(\beta)$. H is a double cover of $\mathrm{Sk}^1(\mathcal{P}_c)$ on which we have a natural collection of cycles as follows. For every face of $\mathcal{P}_c$, consider its cycle in $\mathrm{Sk}^1(\mathcal{P}_c)$. It lifts to either two isomorphic cycles in H or to a single one of twice its length. Declare both of them, or it, as the case may be, as distinguished cycles of H. It is easy to see that these cycles satisfy the properties of being faces of a polyhedron $\tilde{\mathcal{P}_c}$, except, possibly, that it may not be connected (corresponding to whether $\tilde{H}$ and $\tilde{G}$ are not connected); in this case let $\tilde{\mathcal{P}_c}$ be one component. Clearly, $\tilde{\mathcal{P}_c}$ comes equipped with a map of its vertices to $\mathbb{R}^{n+1}$. And moreover, by the construction we have an injective homomorphism $\tilde{\gamma} : \mathrm{Aut}(\tilde{\mathcal{P}_c}) \to O(n+1)$ (which covers $\gamma : \mathrm{Aut}(\mathcal{P}_c) \to \mathrm{Iso}(\mathbb{P}^n)$). These are the ingredients of a regular euclidean polyhedron (in McMullen's terminology, [13], a faithful realization of a regular incidence-polyhedron; see also [9]).

Conversely, consider a bounded regular euclidean polyhedron $\tilde{\mathcal{P}}$ in $\mathbb{R}^{n+1}$; it consists of a regular combinatorial polyhedron $\tilde{\mathcal{P}_c}$, an injective homomorphism

$\tilde{\gamma} : \mathrm{Aut}(\tilde{\mathcal{P}}_c) \to O(n+1)$ and a compatible non-trivial map of its vertices to $\mathbb{R}^{n+1}$ (see [13] and [9]). Since $\tilde{\mathcal{P}}$ is bounded, then up to a scalar factor we may assume its vertices lie on the unit sphere S^n. Now, the 1-skeleton may be mapped uniquely to a rectilinear graph, then projected out to S^n from the origin and down to $\mathbb{P}^n$ to obtain a linear map to $\mathbb{P}^n$. Two cases must be considered. First, if the antipodal map, $-\mathrm{I}$, is not a symmetry of $\tilde{\mathcal{P}}$, then a regular projective polyhedron $\mathcal{P}$ is obtained by the simple composition with the projection (for example the tetrahedron). And second, if the antipodal map $-\mathrm{I}$ is a symmetry of $\tilde{\mathcal{P}}$, then define the combinatorial polyhedron $\mathcal{P}_c$ to be the quotient $\tilde{\mathcal{P}}_c/\{\mathrm{I}, -\mathrm{I}\}$ observing that its 1-skeleton and automorphism group map naturally to $\mathbb{P}^n$.

3. Planar polyhedra

To fix ideas and to illustrate the projective approach, in this section we briefly describe the projective polyhedra in $\mathbb{P}^2$, which, according to the previous section, correspond to the finite regular polyhedra in $\mathbb{R}^3$. Starting from the Platonic solids and projecting them down to $\mathbb{P}^2$, a quick glance at their projective graphs and their opposites suggests the construction of four more, corresponding to the Kepler–Poinsot polyhedra. Their opposite polyhedra yield 9 more which lift to Grünbaum's polyhedra with skew faces [11].

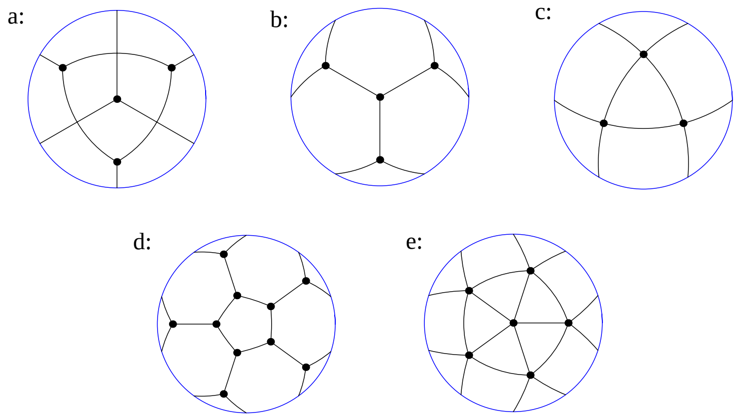

Figure 1. The 1-skeletons of the projected Platonic Solids.

Consider the five platonic solids in $\mathbb{R}^3$ and project them, as in 1.1, to $\mathbb{P}^2$. Let us denote them $[\![3,3]\!]$, $[\![4,3]\!]$, $[\![3,4]\!]$, $[\![5,3]\!]$ and $[\![3,5]\!]$. Their 1-skeletons are drawn respectively in Figure 1 by stereographic projection to $\mathbb{R}^2$ and thus the boundary has to be antipodally identified.

Observe that on the graph (e) we may take the pentagons that surround each vertex as faces for a new polyhedron, which we denote $[\![5, 5/2]\!]$. (The precise meaning of the notation we are using will be given in Lemma 1 of Section 4.1). Now, observe that (a) and (b) are opposite graphs and that (c) is opposite to itself. The opposite graphs of (d) and (e) are respectively (f) and (g) of Figure 2.

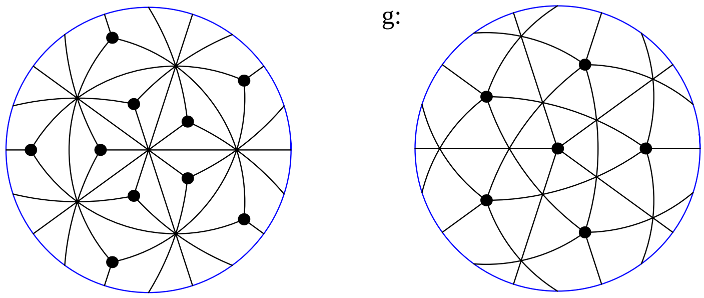

Figure 2. The opposite graphs of $\mathrm{Sk}^1[\![5, 3]\!]$ and $\mathrm{Sk}^1[\![3, 5]\!]$.

Consider the five outermost vertices on the graph (f). They form a regular polygon of type $[\![5/2]\!]$ (a "pentagram"), with all such pentagrams as faces we obtain the polyhedron $[\![5/2, 3]\!]$. Finally, on the graph (g) we may take the pentagrams around each vertex to obtain $[\![5/2, 5]\!]$, or the big triangles (obtained by fixing one edge and one of its sides then at its ends skip the next edge on the same side and take the following one) to get the polyhedron $[\![3, 5/2]\!]$. The last four we have encountered lift to the stellated polyhedra of Kepler–Poinsot.

Given any regular projective polyhedron $\mathcal{P}$, we may obtain another one $\mathcal{P}^{op}$, called its *opposite*, by keeping the same combinatorial and group information but taking the opposite linear map of the 1-skeleton. For example, $[\![3, 3]\!]^{op}$ has the graph (b) as 1-skeleton, but its faces are the cycles of length three; they don't bound disks (and so will be called essential), but, as Grünbaum has pointed out [11], they deserve to be considered as polyhedra, and indeed satisfy the definition. Observe that the lift to $\mathbb{R}^3$ of $[\![3, 3]\!]^{op}$ has the 1-skeleton of the cube but its faces are the "equatorial hexagons" or Petrie-polygons; it is the Petrie polyhedron of the cube. Finally, the opposites of the 9 projective polyhedra we have described above complete the list of regular projective polyhedra in $\mathbb{P}^2$. Their lifts are the 18 bounded regular euclidean polyhedra in $\mathbb{R}^3$, [11].

4. Regular polygons in $\mathbb{P}^3$

A *polygon* $\mathcal{L}$ in $\mathbb{P}^n$ is a projective graph which is combinatorially a cycle and it is *regular* if there exists a compatible inclusion of its combinatorial automorphisms as isometries of the ambient space. It is *degenerate* if it lies in a non-trivial projective subspace. The simplest regular polygons lie in the projective line $\mathbb{P}^1$. They are classified by rational numbers p/q (where in such expressions we always assume p and q are relatively prime) with $p/q > 1$ as follows. Let $[\![p/q]\!]^1$ consist of p successive segments in $\mathbb{P}^1$ of length $(q/p)\pi$. Recall that the length of $\mathbb{P}^1$ is π so that $[\![p/q]\!]^1$ is combinatorially a cycle of length p which winds q times around the projective line. Observe that the opposite of $[\![p/q]\!]^1$ is $[\![p/(p-q)]\!]^1$.

Let $\mathcal{L}$ be a regular polygon in $\mathbb{P}^n$. If we fix a flag, that is a vertex v and an incident edge e, we obtain canonical generators of the dihedral group $\mathrm{Aut}(\mathcal{L})$. These are ρ_0 and ρ_1, where ρ_1 fixes the vertex v and transposes its two edges, and ρ_0 fixes e as a segment but transposes its two vertices. Without confusion we may consider ρ_0 and ρ_1 as isometries of $\mathbb{P}^n$, and they satisfy the relations $\rho_0^2 = \rho_1^2 = (\rho_0\rho_1)^p = id$, where p is the length of the cycle and id is the identity of $\mathbb{P}^n$.

4.1. Planar polygons

In the projective plane $\mathbb{P}^2$, the canonical generators ρ_0 and ρ_1 of a non degenerate regular polygon $\mathcal{L}$, are reflections along lines ℓ_0 and ℓ_1. These lines meet at a point, called the center of symmetry, at a rational angle of the form $(q/p)\pi$ with $q/p < 1/2$. The distinguished vertex v of $\mathcal{L}$ lies in ℓ_1 at a distance r from the center (with $0 < r < \pi/2$); r is called the *radius*. Finally, the distinguished edge of $\mathcal{L}$ (going from v to $\rho_0(v)$) may cross ℓ_0 orthogonally or it may be the opposite segment that passes through the polar point of ℓ_0. Let us denote the first case by $[\![p/q\,;r]\!]$. It is a projective version of the classic euclidean $\{p/q\}$ and will be called an *inessential regular polygon of type* $[\![p/q]\!]$ (see Figure 3 for a polygon of type $[\![7/3]\!]$). The other case is simply its opposite $[\![p/q\,;r]\!]^{op}$, and will be called *essential* because every line intersects it.

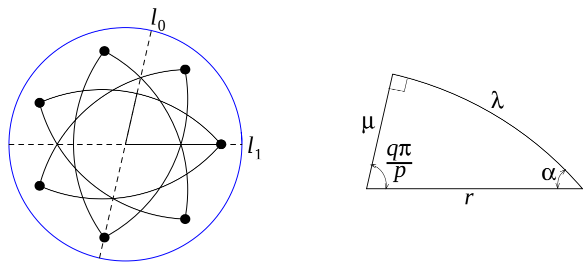

Figure 3. An inessential regular polygon and its basic triangle.

There are other real invariants of $[\![p/q\,;r]\!]$, namely, the length 2λ of the edge, the internal angle 2α between consecutive sides and the distance μ of the center to the edge (see Figure 3). By the spherical law of cosines they are related by the following equations:

$$\begin{aligned} \cos(q\pi/p) &= \sin(\alpha)\cos(\lambda) \\ \cos(r) &= \cos(\mu)\cos(\lambda). \end{aligned} \tag{1}$$

Observe that $[\![p/q\,;r]\!]$ projects from its center to a regular polygon $[\![p/2q]\!]^1$ in its polar line. This projection is a combinatorial isomorphism or a double cover according to whether p is odd or even.

4.2. Polygons in projective space

In the projective space $\mathbb{P}^3$ there are two types of non-degenerate regular polygons. They are best characterized by the "dimensions" of their group generators. Let $\rho \in \mathrm{Iso}(\mathbb{P}^3)$ be an involution, that is $\rho^2 = id$. Define $\dim(\rho)$ to be -1 if ρ has no fixed point, and otherwise the maximum dimension of a pointwise fixed projective subspace. If $\dim(\rho) = 2$, it is a reflection along a plane and at the same time an inversion on its polar point. If $\dim(\rho) = 1$, it is a π-rotation along a line and also along its polar line. And if $\dim(\rho) = -1$, it is a $\pi/2$ translation along a pair of polar lines; however, this case does not arise in our present context because our involutions have fixed points.

Let $\mathcal{L}$ be a regular polygon in $\mathbb{P}^3$ with distinguished flag v, e, and canonical generators ρ_0 and ρ_1. Suppose $\dim(\rho_0) = 2$ and let Π_0 be the reflection plane. Consider the plane Π generated by the segments e and $\rho_1(e)$ (which meet at v). Since e is orthogonal to Π_0, so is Π, and therefore ρ_0 and ρ_1 fix Π (as a set, not pointwise). We may conclude that $\mathcal{L}$ lies in Π and so it is degenerate. This leaves us with only two possibilities when $\mathcal{L}$ is non-degenerate: either $\dim(\rho_0, \rho_1) = (1, 2)$ and we call it *skew*, or $\dim(\rho_0, \rho_1) = (1, 1)$ and we call it a *helicoid*.

4.2.1. Skew polygons Consider a skew regular polygon $\mathcal{L}$ as above. Let ℓ_0 be the pointwise fixed line of ρ_0 that meets e at its midpoint, and let ℓ_0' be its polar line. Let Π_1 be the reflection plane of ρ_1 and let $c = \ell_0 \cap \Pi_1$, $c' = \ell_0' \cap \Pi_1$. Let Π be the polar plane of c' and observe that ρ_0 and ρ_1 fix it. Thus, the projection of $\mathcal{L}$ from c' to the plane Π yields a planar regular polygon, called the *symmetry polygon* of $\mathcal{L}$ on its *symmetry plane* Π; it is inessential because ℓ_0 intersects its basic edge. The type $[\![p/q]\!]$ of this planar polygon will be called the *type* of the skew polygon $\mathcal{L}$.

Conversely, a skew regular polygon of type $[\![p/q]\!]$ is obtained from a fixed planar polygon $[\![p/q\,;r]\!]$, by moving the vertices alternatively over and under the plane a fixed distance along orthogonal lines, taking the corresponding edges that intersect the plane. If p is even this process results in a skew polygon of p sides called *antiprismatic*. But if p is odd, after one turn we are on the other side of the plane and have to turn once more yielding a polygon of $2p$ sides called skew *prismatic*. These correspond, respectively, to the cases in which the projection to the symmetry polygon is a single or a double cover.

Note that we have changed the classic usage of notation. The euclidean analogue of our "skew of type $[\![p/q]\!]$", is the classic prismatic skew polyhedron of type $\{2p/q\}$ (p odd), whose notation is based on the fact that it is combinatorially

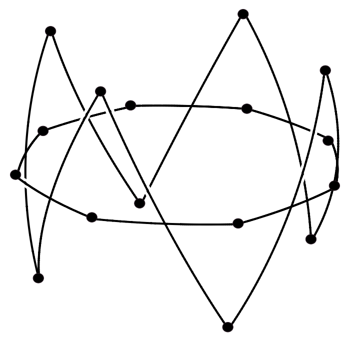

Figure 4. A skew regular polygon of type $[\![8]\!]$ and its symmetry polygon.

of length $2p$. However, we have dared to change established notation because it seems to fit better in the general theory, see for example the uniqueness principle (Lemma 1, below) where no assumption has to be made on the parity. The rule of translation is simple: skew of type $[\![p/q]\!]$ is prismatic of length $2p$ if p is odd, and antiprismatic of length p if p is even.

Observe finally, that the opposite of a skew regular polygon is again skew, and that for a fixed type there is a two parameter family of geometrically different regular polygons.

4.2.2. Helicoids For the sake of completeness let us finally describe the helicoids, although we will not consider them in this paper. Suppose that $\mathcal{L}$ is a non-degenerate regular polygon with $\dim(\rho_0, \rho_1) = (1, 1)$. Let ℓ_0 and ℓ_0' be the polar lines about which ρ_0 is a π-rotation, and likewise define ℓ_1 and ℓ_1' for ρ_1. If ℓ_0 and ℓ_1 meet, $\mathcal{L}$ is easily seen to be degenerate, so that if $\mathcal{L}$ is a helicoid no pair of these four lines intersect. Then there is a pair of polar lines ℓ and ℓ' which intersect the previous four orthogonally. The projection of $\mathcal{L}$ from ℓ to ℓ' and from ℓ' to ℓ give regular polygons in $\mathbb{P}^1$ over which $\mathcal{L}$ may wind around several times. However, they are of the form $[\![p/q]\!]^1$ and $[\![p'/q']\!]^1$. These linear polygons define a *type* of the helicoid, and a real number giving the distance from a vertex to ℓ say, specifies it geometrically.

5. Polyhedra in $\mathbb{P}^3$

5.1. Generalities, definitions and notation

Let $\mathcal{P}$ be a regular polyhedron in $\mathbb{P}^3$ with distinguished flag (v, e, f) (where, recall, v is a vertex incident to the edge e, contained in the face f), and let ρ_0, ρ_1, ρ_2 be the canonical generators of its automorphism group with respect to this flag (that is, ρ_0 fixes e and f but moves v; ρ_1 fixes v and f but moves e, and ρ_2 fixes v and e but moves f). We will think of ρ_i as an isometry of $\mathbb{P}^3$. The face f is a regular polygon in $\mathbb{P}^3$ with canonical generators ρ_0 and ρ_1. The type of this polygon is the coarsest classification of such polyhedra. Later in this work we will mainly analyse the case of polyhedra with planar faces.

There is another regular polygon associated to $\mathcal{P}$, called its *vertex figure*, $VF(\mathcal{P})$, defined combinatorially in Section 1. Its vertices are the barycenters, or midpoints, of the edges in $\mathrm{Sk}^1(\mathcal{P})$ incident to v, and two of them are combinatorially adjacent if their edges lie in a common face. To choose the appropriate segment, observe that two adjacent vertices in $VF(\mathcal{P})$ are the endpoints of a path made of two half-edges meeting at v. Choose the segment that makes this triangle homotopically trivial, that is, that defines an ordinary triangle in the plane of the three points.

We claim that $VF(\mathcal{P})$ *is not a helicoid*. Observe that ρ_1 and ρ_2 are the canonical generators of $VF(\mathcal{P})$. Since ρ_1 and ρ_2 fix the vertex v, all the automorphisms do. It is not hard to see that in a helicoid the isometry that corresponds to a generating rotation has no fixed points. Observe also that v is the center of symmetry of the vertex figure and that if it is planar it must be inessential.

We have therefore described $VF(\mathcal{P})$ as a regular polygon which is planar-inessential or skew of type $[\![q_1/q_2]\!] =: [\![q]\!]$, for some rational $q > 2$.

5.2. Planar faced regular polyhedra

Now, suppose f is planar. We may also assume it is inessential, for otherwise we may change $\mathcal{P}$ for its opposite. Then, the face f is $[\![p_1/p_2\,; r]\!] =: [\![p\,; r]\!]$ for some rational $p > 2$ and $0 < r < \pi/2$.

Lemma 1 (Uniqueness principle). *There is at most one regular polyhedron with planar faces $[\![p\,; r]\!]$ and vertex figure of type $[\![q]\!]$. If it exists it is denoted by $[\![p, q\,; r]\!]$.*

Proof. The main idea is that there is at most one way to fit copies of the prescribed face $[\![p\,; r]\!]$ around a vertex v in such a way that the vertex figure turns out to be of type $[\![q]\!]$. The details follow.

Let 2λ be the length of the edge in $[\![p\,; r]\!]$, and let ν be the length of the segment that joins inessentially two barycenters of incident edges (observe that ν must be the length of the edge in the vertex figure). Consider the polygon $[\![q\,; \lambda]\!]$ centered at v on a plane Π', and let $2\lambda'$ be the length of its side.

There are three cases to analyse. First, if $\nu = 2\lambda'$, the vertex figure is planar and lies in the same plane of the face, so that the polyhedron becomes planar. Given p and q it is easy to see that there is at most one way to choose r for this to happen, namely when the internal angle of $[\![p\,; r]\!]$ is equal to $2\pi/q$. Thus we may simplify the notation for the polyhedron and call it $[\![p, q]\!]$ as we did in Section 2.

Second, if $\nu < 2\lambda'$, the vertex figure must be planar and it is obtained by moving simultaneously all the vertices of $[\![q\,;\lambda]\!]$ to the same (local) side of Π' along orthogonal planes passing through them and v and keeping them at distance s from v. In such a process the vertices define planar polygons of type $[\![q]\!]$ and the edge shrinks until we reach ν, that polygon is the vertex figure VF.

And third, if $\nu > 2\lambda'$, the vertex figure must be skew. Analogously, move the vertices alternately to the two sides of Π' forming skew polygons whose vertices are at distance s from v. In this process the length of the side grows monotonically up to the upper bound 2λ when the vertices cluster at the orthogonal line to Π' at v. If $\nu < 2\lambda$, there is a unique skew polygon, VF say, of side ν with vertices at distance λ from its center of symmetry.

Suppose that the given data (p, q and r) produces, as above, a vertex figure. Let f be a fixed polygon $[\![p\,;r]\!]$ on a plane Π with distinguished flag $v < e$, and let ℓ_0, ℓ_1 be its canonical symmetry lines (see Figure 3). Corresponding to the additional data $[\![q]\!]$, construct the vertex figure VF with center v and with distinguished flag corresponding to e and f. Let ρ_2 be its second canonical generator (the reflection along the plane orthogonal to Π' and passing through e). Let ρ_0 be the reflection on the plane orthogonal to Π at ℓ_0. Finally, if VF is planar let ρ_1 be the reflection on the plane orthogonal to Π at ℓ_1, and if VF is skew let ρ_1 be the π-rotation along ℓ_1. By definition, ρ_0 and ρ_1 serve as canonical generators for f, while ρ_1 and ρ_2 are the canonical generators for VF.

Consider the subgroup of $\mathrm{Iso}(\mathbb{P}^3)$ generated by ρ_0, ρ_1 and ρ_2. The Wythoff's construction (cf. [7], [14]) on this group yields a combinatorial polyhedron, and its action on the vertex v and the segment e gives a linear map of its one skeleton to projective space. This is $[\![p, q\,;r]\!]$. □

A natural question that arises is to classify the $[\![p, q; r]\!]$ which are finite polyhedra. Observe also that from the proof we obtain the following.

Corollary 1. *Let $\mathcal{P}$ be a regular non-degenerate polyhedron in $\mathbb{P}^3$ with planar face, and let ρ_0, ρ_1 and ρ_2 be canonical generators of its group. Then*
- *$VF(\mathcal{P})$ is planar if and only if* $\dim(\rho_0, \rho_1, \rho_2) = (2, 2, 2)$.
- *$VF(\mathcal{P})$ is skew if and only if* $\dim(\rho_0, \rho_1, \rho_2) = (2, 1, 2)$.

5.3. Planar-planar polyhedra

Let $\mathcal{P}$ be a non-degenerate regular polyhedron with planar-inessential face and planar vertex figure. By the previous corollary the three canonical generators are reflections in planes, and these planes meet at a point c which is fixed by the automorphism group. Let Π be the polar plane of c. The projection of $\mathcal{P}$ from c to Π is a double (or single) cover of a planar polyhedron with inessential faces. And then it is easy to see that $\mathcal{P}$ is a projective embedding of one of the classic nine (Platonic and Kepler–Poinsot) euclidean polyhedra, which depends on a radius parameter (the distance from a vertex to c).

6. Planar-skew polyhedra

From now on we shall assume that $\mathcal{P}$ is a regular polyhedron in $\mathbb{P}^3$ with planar-inessential face and skew-antiprismatic vertex figure. Then $\mathcal{P} = [\![p, q\,; r]\!]$ for some $0 < r < \pi/2$ and $p = p_1/p_2$, $q = q_1/q_2$ rational numbers greater than 2. Combinatorially, $\mathcal{P}$ has Schläfli symbol $\{p_1, q_1\}$. Observe that q_1 is even because the vertex figure is skew antiprismatic. Likewise, p_1 is even. To see this, consider a face f and observe that because the vertex figure is skew the faces adjacent to f lie alternatively on one side and the other of the plane of f. If we travel once around the polygon f we must return to the starting adjacent face and thus p_1 must be even.

The combinatorial dual $\mathcal{P}^*$ of $\mathcal{P}$ can be geometrically realized as $[\![q, p\,; r]\!]$. Indeed, consider the vertex figure $VF(\mathcal{P})$ around a vertex v. The centers of the faces incident to v lie on the symmetry plane of $VF(\mathcal{P})$, and naturally form a polygon of type $[\![q]\!]$. The radius of this polygon is the distance of the center of a face to v, which is precisely r. This defines the vertices, edges and faces of $\mathcal{P}^*$. Finally, observe that its vertex figure is skew of type $[\![p]\!]$.

It is interesting to note that $(\mathcal{P}^*)^* = \mathcal{P}$ geometrically and not only combinatorially, as in the classic case in $\mathbb{R}^3$, or in the planar-planar case.

In the rest of this section we give examples of planar-skew polyhedra $[\![p, q\,; r]\!]$ with integer p and q. In the last section, we will prove they are the only ones.

6.1. Tori and euclidean planes, $[\![4, 4; r]\!]$

Consider a square $[\![4; r]\!]$. Its internal angle is greater than $\pi/2$, so that to match four of them regularly around a vertex they produce a skew vertex figure of type $[\![4]\!]$. By Lemma 1, we obtain a polyhedron $[\![4, 4; r]\!]$ whose universal cover is the plane tiling $\{4, 4\}$.

Lemma 2. *The polyhedron* $\mathcal{P} = [\![4, 4; r]\!]$ *is finite if and only if* r/π *is rational.*

Proof. Consider a vertex v of $\mathcal{P}$, and the four faces around it. Note that the diagonal segment of a face is collinear with the diagonal of the opposite face at v; in fact, they lie on the symmetry plane of the vertex figure. If we follow this line from v we will find vertices of $\mathcal{P}$ after each segment of length $2r$. If r/π is irrational they will be dense and hence infinite.

On the other hand, suppose that $r = (s/k)\pi$. Then these diagonals with vertices of $\mathcal{P}$ form a linear polygon $[\![k/2s]\!]^1$, which has combinatorial length k if k is odd, and $k/2$ if it is even. In the first case, it is the regular polyhedral torus $\{4, 4\}_{k,0}$ and in the second the "slanted" regular polyhedral torus $\{4, 4\}_{k/2,k/2}$ (see, e.g. [8]). □

Observe that the first argument of the preceding proof holds for any planar-skew polyhedron. Let us state this in general:

Corollary 2. *If the planar-skew polyhedron* $[\![p, q; r]\!]$ *is finite then* r/π *is rational.* □

Observe also that the only case when the polyhedron $[\![4, 4\,; r]\!]$ is an embedded torus is when $r = \pi/2k$, with integer $k > 1$. Otherwise the surface which it defines has selfintersections. See Figure 5.

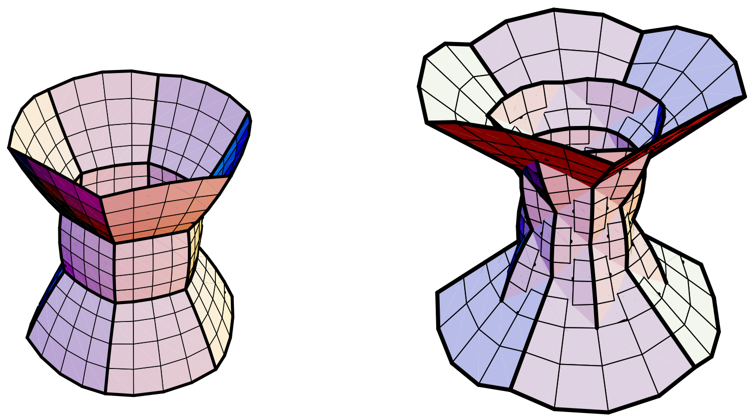

Figure 5. Stereographic projections of $[\![4, 4\,; \pi/6]\!]$ and $[\![4, 4\,; \pi/5]\!]$. In the first one, the two boundaries are to be antipodally identified. In the second, also the outermost faces are antipodally identified.

The natural geometric way to look at these polyhedra is with their vertices on the quadric surface Q in $\mathbb{P}^3$ defined by the equation $x_1^2 + x_2^2 = x_3^2 + x_4^2$, where $[x_1 : x_2 : x_3 : x_4]$ are homogeneous coordinates. In this quadric, the lines of opposite rulings meet at an angle of $\pi/2$, and generate the tangent plane to Q at their intersection point. Thus, the square $[\![4\,; r]\!]$ centered at this point and with diagonals on the rules has vertices on Q. From this square proceed to build $[\![4, 4\,; r]\!]$. The diagonal lines at any given vertex become precisely the two rules of Q. Finally, observe that $[\![4, 4\,; r]\!]$ is isometric (and hence equivalent) to its dual.

6.2. The pentecaidecahedron and the decahedron

There is a dual pair $[\![4, 6\,; \pi/4]\!]$ and $[\![6, 4\,; \pi/4]\!]$ with 15 and 10 faces respectively, hence their names. We proceed to describe the pentecaidecahedron, $[\![4, 6\,; \pi/4]\!]$. Its dual is then constructed in the standard way.

Consider K_5, the complete graph on 5 vertices. Its edge graph has 10 vertices (one for each edge of K_5) and two are adjacent if the corresponding edges are incident, hence it is a regular graph of degree 6. For every cycle of length 4 in K_5, attach the corresponding quadrilateral face to the edge graph. It is easy to see that this is an abstract regular polyhedron, $\mathcal{P}$ say, with automorphism group S_5 (the symmetric group on 5 letters), and Schläfli symbol $\{4, 6\}$. Now, to describe its natural embedding in $\mathbb{P}^3$, consider the five basic vectors of $\mathbb{R}^5$ as the vertices of K_5. To each edge of K_5, there corresponds a line, which translated to the origin defines a point in $\mathbb{P}^4$. These ten lines lie in a hyperplane. Thus, the corresponding points lie in a 3-dimensional flat which may be considered as $\mathbb{P}^3$. The edges of $\mathcal{P}$ are taken to be the segments of length $\pi/3$, and then the radius of each quadrilateral turns out to be $\pi/4$. To see this, consider the same construction with K_3 and K_4.

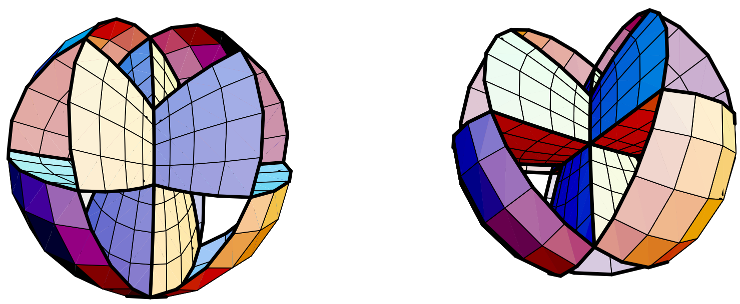

Figure 6. Two views of projections of $[\![4, 6\,; \pi/4]\!]$ without one face, to see the interior. Combinatorially, the outermost faces are to be identified with their antipodes.

Topologically, $[\![4, 6\,; \pi/4]\!]$ is a non-orientable surface of genus 7. Its euclidean double cover is the skew $\{4, 6\}$ in $\mathbb{R}^4$ discovered by Coxeter in [5]. See Figure 6, which is the stereographic projection of $\mathbb{P}^3$ (with $[\![4, 6\,; \pi/4]\!]$ drawn in it) to $\mathbb{R}^3$; therefore the unit sphere has to be antipodally identified. Distances are distorted but, at least, not the angles. The symmetries of this drawing, S_4, are only a subgroup of the original, S_5, and thus it fails to be geometrically regular. If

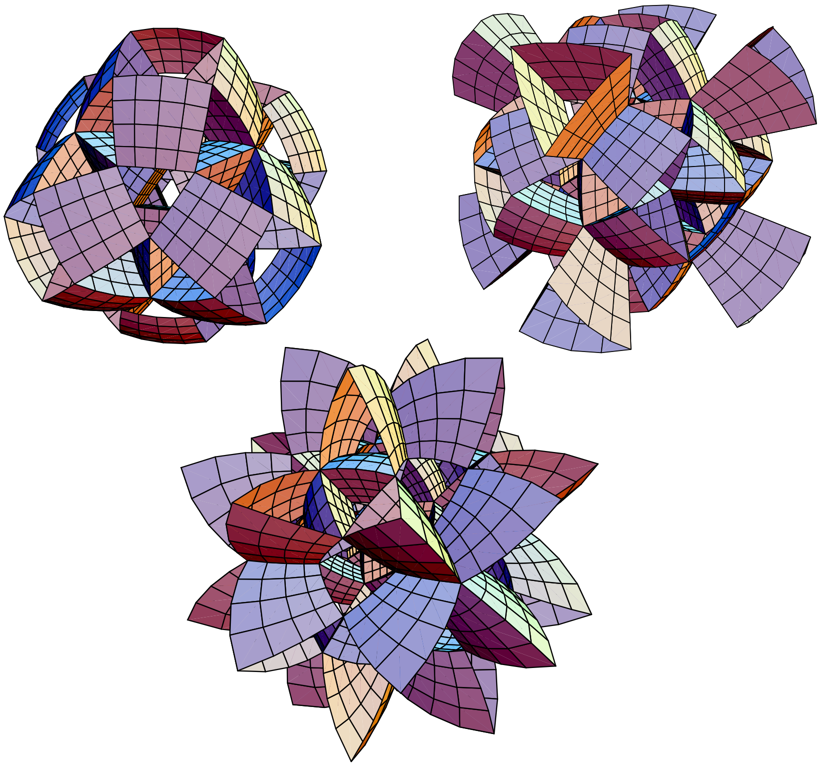

Figure 7. Three stages of $[\![4,8\,;\pi/8]\!]$ projected to R^3. In the last one, the outermost faces are cut by the sphere at infinity (not drawn); thus the outermost vertices have to be identified with their antipodes, which appear with only six faces.

one draws the faces planarly, it becomes a part (the inner half) of Schulte–Wills embedding of $\{4,6\}$ in $\mathbb{R}^3$, [16]. The analogue applies to Figure 7.

6.3. The pachyhedron and the hemipachyhedron

They are the dual pair $[\![4,8\,;\pi/8]\!]$, $[\![8,4\,;\pi/8]\!]$, with 144 and 72 faces respectively. ("pachy" means "thick" and, in a secondary sense, "gross" which is also used for "144"). We follow the description in [15] of $[\![4,8\,;\pi/8]\!]$; see also [16] and [17].

Consider the 12-cell (see [6]), which is a self-dual polytope consisting of 12 solid octahedra, giving a 3-dimensional tiling or honeycomb of $\mathbb{P}^3$. Shrink each octahedron uniformly and insert triangular prisms with quadrilateral faces between formerly adjacent pairs of octahedra. (These prisms will be called "waists" below.) The quadrilaterals are the faces of $[\![4, 8\,; \pi/8]\!]$.

7. The waist equation

Let $\mathcal{P} = [\![p, q\,; r]\!]$ be a finite regular projective polyhedron with planar-inessential face and skew-antiprismatic vertex figure. For the rest of this section let us call it a *skew projective polyhedron.*

A *combinatorial belt* is a simple cyclic sequence of faces $f_1, f_2, \ldots, f_{c_1}$, such that each face is adjacent to its neighbours through opposite edges (recall that the faces have an even number of sides). The *combinatorial waist* of the polyhedron $\mathcal{P}$ is the length, c_1, of a belt. By regularity, every edge defines a belt and all belts have the same length.

Now we give it a geometric meaning. Consider the plane Π orthogonal to the common edge of f_1 and f_2 at its midpoint. It is orthogonal to the planes of f_1 and f_2 and then, it also intersects the common edge of f_2 and f_3 orthogonally at its midpoint, and so on. The corresponding *geometric belt* is then the planar polygon in Π with vertices at the midpoints of the edges and the segments that contain the centers of the faces. It is therefore of type $[\![c]\!]$ or $[\![c]\!]^{op}$ for some rational $c = c_1/c_2 > 2$. This c is called the *waist* of $\mathcal{P}$.

The geometric belt is inessential if and only if $p_1 \equiv 0 \bmod (4)$. To see this, consider a planar polygon and an edge on it. This segment defines locally two sides. Observe that if the polygon is inessential the two adjacent edges lie on the same side. Conversely, if it is essential then they lie on opposite sides. Recall that the faces adjacent to a given one, say f, in the polyhedron $\mathcal{P}$ lie on alternating sides of the plane of f as we travel around the polygon f. When we get to the opposite edge we started with, we are on the same side if and only if $p_1 \equiv 0 \bmod (4)$. This proves the claim on the geometric belt.

Theorem 1 (The waist equation). *Let $\mathcal{P} = [\![p, q\,; r]\!]$ be a skew projective polyhedron with waist c. If $p_1 \equiv 0 \bmod (4)$, where $p = p_1/p_2$, then*

$$\cos(\pi/p)\cos(\pi/c) = \sin(\pi/q)\cos(r).$$

Proof. Let α be half the internal angle of the face $[\![p\,; r]\!]$, and let β be half the dihedral angle among faces. We claim that

$$\sin(\pi/q) = \sin(\alpha)\sin(\beta). \tag{2}$$

To see this, let b_0, b_1 and b_2 be respectively the barycenters of a distinguished flag. In the tangent space of the vertex b_0, consider the infinitesimal euclidean tetrahedron defined by the plane of the distinguished face, the bisecting plane of the dihedral angle at the edge, the symmetry plane of the vertex figure and an orthogonal plane to the edge (infinitesimally close to b_0), (Figure 8). Then from the various right triangles formed, equation (2) follows.

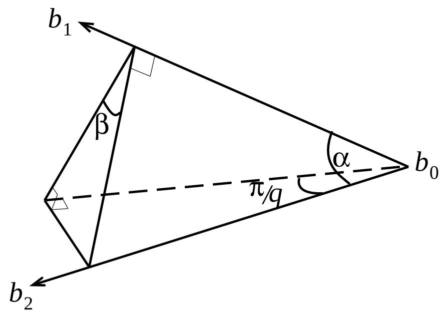

Figure 8. Tetrahedron in the tangent space to b_0.

On the plane of the face, we have a right triangle b_0, b_1, b_2 as in Figure 3, where $\lambda = d(b_0, b_1)$ and $\mu = d(b_1, b_2)$. Equations (1) now become

$$\begin{aligned} \cos(\pi/p) &= \sin(\alpha)\cos(\lambda) \\ \cos(r) &= \cos(\mu)\cos(\lambda) \end{aligned} \tag{3}$$

Finally, consider the geometric belt of $\mathcal{P}$. Since $p_1 \equiv 0 \bmod(4)$, this polygon is of type $[\![c]\!]$ with side 2μ. Thus the corresponding first equation in (1) yields

$$\cos(\pi/c) = \sin(\beta)\cos(\mu). \tag{4}$$

The theorem follows by expressing $\cos(\pi/p)\cos(\pi/c)$ in terms of equations (3) and (4) and then using (2) and (3). □

Remark. If $p_1 \equiv 2 \bmod(4)$, similar reasoning leads to the equation $\sin(\pi/2c) = \sin(\pi/q)\sin(r)$. However, we will not use it here.

Proposition 1. *Let $\mathcal{P} = [\![p, q\,; r]\!]$ be a skew projective polyhedron such that $q_1 \equiv 2 \bmod(4)$, where $q = q_1/q_2$. Then $p = 4$ and $r = \pi/4$.*

Proof. Observe that the hypothesis implies that opposite faces at a vertex are coplanar (because opposite vertices of a skew antiprismatic polygon with $q_1 \equiv 2 \bmod(4)$ sides are collinear with the center of symmetry). Let Π be the plane of a face f. Π contains all opposing faces at the vertices of f and their corresponding ones too, and so on. Draw a point at the barycenter of each face so obtained, and its p_1 radii line segments. Each radius matches with an opposing one to form an edge of a projective graph in Π which is regular (in the strongest geometric sense), and thus is the 1-skeleton of a planar polyhedron. Since p_1 is the number of edges at a vertex of this polyhedron, and it is even, then by Section 2 this polyhedron must be $[\![3,4]\!]$. This proves that $p = 4$. Moreover, $2r$ is the size of the edge of $[\![3,4]\!]$. Thus $r = \pi/4$. (See Figure 6 for different views of the plane Π and the 3 faces on it.) □

Theorem 2. *Let $\mathcal{P} = [\![p, q\,; r]\!]$ be a skew projective polyhedron with integer p, q and waist c. Then $\mathcal{P}$ is one of the following: $[\![4, 4\,; \pi/k]\!]$ with integer $k \geq 3$, $[\![4, 6\,; \pi/4]\!]$, $[\![6, 4\,; \pi/4]\!]$, $[\![4, 8\,; \pi/8]\!]$ or $[\![8, 4\,; \pi/8]\!]$.*

Proof. If p or q is congruent to 2 mod(4), we may assume by duality that it is q, and then, by Proposition 2, that $p = 4$. Otherwise, p and q are congruent to 0 mod(4). In either case, the waist equation (Proposition 1) holds. From it, since $c \geq 3$ and $\cos r < 1$, we obtain that $\cos \pi/p < 2 \sin \pi/q$. This inequality implies that if $p = 4$ then $q < 9$; that if $p = 8$ then $q < 7$, and that there are no solutions for $p \geq 12$. It is then easy to see that the only integer solutions of the waist equation with our congruence requirements are the following. If $p = q = 4$ then $r = \pi/c$ for any $c \geq 3$. If $p = 4$ and $q = 6$ then $c = 3$ and $r = \pi/4$. If $p = 4$ and $q = 8$ then $c = 3$ and $r = \pi/8$. If $p = 8$ then $q = 4$, $c = 4$ and $r = \pi/8$. The existence of skew projective polyhedra with such invariants was proved in Section 5. □

Observe that the only non-embedded polyhedra of the above list are those in the family $[\![4, 4\,; \pi/k]\!]$ with k odd. However, they lift to the embedded tori $\{4, 4\,|\,k\}$ in $\mathbb{R}^4$ (see [5] and [16]), with the projection being a combinatorial isomorphism. For k even, $[\![4, 4\,; \pi/k]\!]$ is embedded in $\mathbb{P}^3$ and lifts to its double cover $\{4, 4\,|\,k\}$ in $\mathbb{R}^4$. This is so because $-\mathrm{I}$ is a symmetry of $\{4, 4\,|\,k\}$ precisely when k is even. Thus, the lifting to $\mathbb{R}^4$ of the list in the preceding theorem is the same as Coxeter's [5].

Finally, observe that if $[\![p, q\,; r]\!]$ is an embedded surface, then its parameters, including the waist, must be integers.

Corollary 3. *The regular projective polyhedra in $\mathbb{P}^3$ with planar face and skew vertex figure that define embedded surfaces are: $[\![4, 4\,; \pi/2k]\!]$ with integer $k \geq 2$, $[\![4, 6\,; \pi/4]\!]$, $[\![6, 4\,; \pi/4]\!]$, $[\![4, 8\,; \pi/8]\!]$ and $[\![8, 4\,; \pi/8]\!]$.*

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