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Regular projective polyhedra with planar faces II

Javier Bracho

Summary. The regular projective polyhedra with planar faces and skew vertex figures are classified. There is an infinite family of tori, plus 84 special polyhedra. Correspondingly, but not explicitly, the classification of an important family of regular euclidean polyhedra in 4 dimensional space may be derived. New euclidean regular polytopes in all dimensions are defined.

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1. Introduction

A regular polyhedron $\mathcal{P}$ is a flag-transitive combinatorial surface, where a flag is an incident triplet *vertex* $<$ *edge* $<$ *face*. An euclidean realization of $\mathcal{P}$ is a rectilinear drawing of its underlying graph $\mathcal{P}^{(1)}$ (vertices and edges) in $\mathbb{R}^n$, so that all of its combinatorial automorphisms are realized by isometries, see [10, 8, 14]. Classic examples in $\mathbb{R}^3$ are the Platonic solids, the Kepler–Poinsot polyhedra and their Petrie-duals; and in $\mathbb{R}^4$ the Coxeter skew polyhedra, [3]. In $\mathbb{R}^3$, they have been classified, see [10, 8, 9, 13]. In this paper, we work towards the classification of finite regular polyhedra in $\mathbb{R}^4$ by completing a special, but crucial, case, namely those for which the affine subspace generated by (the vertices of) each face is of dimension 2 or of dimension 3 and contains the center of symmetry.

Given a finite regular polyhedron in $\mathbb{R}^4$, we may assume that its vertices lie in the unit sphere S^3 and then project it to projective space $\mathbb{P}^3$ to obtain a regular projective polyhedron (see [4] for the first examples of this idea). Conversely, given any projective polyhedron we can lift it to an euclidean one, see [1]. Thus their study is equivalent. Here, we adopt the projective point of view. It has two advantages. First, combinatorially speaking the objects are more simple, usually “half” of their euclidean lifting. Thus the relations among them become clearer. One example of this is the *opposite operation* (3.2) which leaves the combinatorics unchanged but not the geometry. The Petrie-duals of the classic regular polyhedra in $\mathbb{R}^3$ may be reinterpreted in this way (see [1]), and new euclidean polytopes arise in all dimensions (see 3.2). And second, working in dimension 3 (although with projective geometry) instead of dimension 4 (euclidean) makes some geometric arguments and constructions easier. One example of this is the definition of the regular projective polyhedra we shall classify: those for which each face lies on a plane —compare with the previous paragraph. For the sake of conciseness, we shall make no further reference to the lifted euclidean polyhedra.

In [1], we defined and started to study projective polyhedra with planar faces and skew-antiprismatic vertex-figure, and completed the “Platonic” case, when the polyhedron is an embedded surface. Now we analyse and classify in general the projective polyhedra with planar face and skew vertex figure. The results in [1] are only essential for Section 3. Elsewhere, the present paper is mostly self contained, because the main definitions (although equivalent to those of [1]) are technically easier to work with when the role of the C-group is stressed. This is what we presently do.

2. Preliminaries

2.1. Regular projective polytopes

Let $\mathcal{P}$ be a partially ordered set with rank function; minimum (which we may assume has rank -1), and maximum of rank $d+1$. A *flag* is a maximal chain; and two flags are *adjacent* if they only differ by one element. $\mathcal{P}$ is *flag connected* if the graph whose vertices are its flags with edges among adjacent pairs is connected. A *proper section* of $\mathcal{P}$ is the strict subposet of elements that lie between two given comparable ones whose rank differs by at least 2. If $d = 0$, $\mathcal{P}$ is a *polytope* if it has exactly two elements of rank 0. Recursively for $d > 0$, $\mathcal{P}$ is a *polytope* (called traditionally an *incidence polytope*, [7, 16, 12]) if it is flag connected and every proper section is a polytope. Its *rank* is $d+1$.

Polytopes of rank $d+1$ can be thought of as geometric or topological objects of dimension d. For example, polytopes of rank 2, called *polygons*, correspond to combinatorial cycles, which are topological circles. The vertices are the elements of rank 0 and the edges are those of rank 1. Polytopes of rank 3 correspond to combinatorial surfaces (sometimes called maps), where the faces are the elements of rank 2. They are our main object of study and, following classic terminology, we call them *polyhedra*. In general, an element of rank i is called an i-face, and those of rank d the *facets*.

Let $\mathcal{P}$ be a polytope of rank $d+1$. It is called *regular* if its automorphism group Γ acts transitively on flags (which is always true for $d = 0, 1$). Suppose from now on that $\mathcal{P}$ is regular. Then, if we fix a basic flag $f_0 < f_1 < \cdots < f_d$ (the minimum and maximum may be neglected), we obtain nontrivial automorphisms $\sigma_0, \sigma_1, \ldots, \sigma_d \in \Gamma$ defined by

$$\sigma_i(f_j) = f_j \iff i \neq j.$$

These automorphisms generate Γ, so that $\Gamma = \langle \sigma_0, \sigma_1, \ldots, \sigma_d \rangle$, and it is easy to see that they satisfy the identities

$$\begin{aligned} \sigma_i^2 &= 1, \\ \sigma_i\sigma_j &= \sigma_j\sigma_i \quad \text{for} \quad |i-j| \geq 2. \end{aligned} \tag{1}$$

Furthermore, they satisfy the *intersection property*:

$$\langle \sigma_i | i \in I \rangle \cap \langle \sigma_i | i \in J \rangle = \langle \sigma_i | i \in I \cap J \rangle \quad \text{for} \quad I, J \subset \{0, \ldots, d\}. \tag{2}$$

Such a group together with its ordered generators is called a C-*group*, the "C" is in honor of Coxeter, see [12]. It defines the regular polytope $\mathcal{P}$ up to isomorphism. Moreover, all C-groups come from regular polytopes, [16]. Its *Schläfli symbol* or *type* is the sequence of integers $\{p_1, p_2, \ldots, p_d\}$ where p_i is the order of $\sigma_{i-1}\sigma_i$.

Now let us define a *faithful projective realization* of a regular polytope $\mathcal{P}$ as a monomorphism of its C-group into the isometries of $\mathbb{P}^n$, the real projective space of dimension n,

$$\Gamma = \langle \sigma_0, \sigma_1, \ldots, \sigma_d \rangle \hookrightarrow \mathrm{Iso}(\mathbb{P}^n) \tag{3}$$

together with a *basic pair* of points $b_0, b_1 \in \mathbb{P}^n$ satisfying

$$\begin{aligned} \sigma_i(b_0) = b_0 &\iff i \neq 0, \\ \sigma_i(b_1) = b_1 &\iff i \neq 1, \\ b_1 \in b_0 * \sigma_0(b_0)&, \end{aligned} \tag{4}$$

where $x * y$ is the line through x and y, and we identify elements of Γ with their corresponding isometry. By a *regular projective polytope*, which will also be denoted by $\mathcal{P}$, we mean a faithful projective realization of a regular polytope. It will usually be given by (3) and a basic pair satisfying (4) where it is understood that Γ is indeed a C-group.

Let $\mathcal{P}$ be a regular projective polytope as above. The point b_0 is the basic geometric vertex (corresponding to the vertex in the basic combinatorial flag), and b_1 indicates which of the two segments from b_0 to $\sigma_0(b_0)$, that together form the projective line $b_0 * \sigma_0(b_0)$, is to be considered as the basic edge. Observe that in the definition of regular euclidean polytopes, see [14] and [8], only the basic vertex, analogue of b_0, appears because segments are unique, but in the projective case specifying b_1 is crucial. So let the *basic edge* e be the line segment from b_0 to $\sigma_0(b_0)$ containing b_1. It satisfies $\sigma_i(e) = e$ for $i \neq 1$, although σ_0 transposes its endpoints. The *geometric 1-skeleton* of $\mathcal{P}$ is then defined by the action of Γ on e, and we can write $\mathrm{Sk}^1(\mathcal{P}) = \Gamma(e)$. Observe that in general, the combinatorial 1-skeleton $\mathcal{P}^{(1)}$ need not be isomorphic as a graph to $\mathrm{Sk}^1(\mathcal{P})$, see e.g. 3.0.2, but it covers it regularly. Thus, our definition is equivalent to that of [1], but the present one is technically easier to work with. It should also be remarked that our definition is philosophically congruent to Grünbaum's [11], in the sense that the geometric 1-skeleton is "labeled" by the combinatorial one, so that repetitions are allowed. However, we are presently focused on geometrically regular polyhedra and thus the group information must be stressed.

$\mathcal{P}$ is *non-degenerate* if $\Gamma(b_0)$ generates $\mathbb{P}^n$, that is, if the geometric vertices of $\mathcal{P}$ are not contained in a flat (a projective subspace of lower dimension).

Two regular projective polytopes are *equivalent* (the "same" for classification purposes) if they differ by an isometry, that is, if their C-groups are conjugate by an isometry which sends one basic pair to the other.

An important invariant of the projective regular polytope $\mathcal{P}$ is its dimension vector. An involution σ of $\mathbb{P}^n$ with at least one fixed point, such as our generators σ_i, has a *polar pair* as fixed point set, that is, two projective subspaces Σ and $\Sigma^\perp$ with $\dim(\Sigma) + \dim(\Sigma^\perp) = n - 1$ at distance $\pi/2$ from each other. The lines through Σ and $\Sigma^\perp$ are perpendicular to both, they cover all of $\mathbb{P}^n$, and σ acts reflecting these lines in their intersection with Σ and $\Sigma^\perp$. For example, if Σ is a hyperplane, then σ is a reflection in it and at the same time an involution in its polar point $\Sigma^\perp$. We call Σ and $\Sigma^\perp$ *the mirrors* of σ.

In the case of a regular projective polytope $\mathcal{P}$, we may *distinguish* one of the mirrors of each generator as follows. For $i \geq 1$, let Σ_i be the mirror of σ_i that contains b_0; and let Σ_0 be the mirror of σ_0 such that $b_1 \in \Sigma_0$. The *dimension vector* of $\mathcal{P}$ is then

$$\dim(\mathcal{P}) = (\dim(\Sigma_0), \dim(\Sigma_1), \ldots, \dim(\Sigma_d)).$$

Observe that the use of $\dim(\mathcal{P})$ in [1] is slightly different.

Using the distinguished mirrors of $\mathcal{P}$, it is easy to see how to lift a projective polytope to an euclidean one $\widetilde{\mathcal{P}}$. A projective isometry σ has two liftings $\widetilde{\sigma}$ and $-\widetilde{\sigma}$. If it has a distinguished mirror Σ, this lifts to a subspace $\widetilde{\Sigma}$. Then, we may choose among $\widetilde{\sigma}$ and $-\widetilde{\sigma}$ the one that has $\widetilde{\Sigma}$ as eigenspace of eigenvalue 1. This tells us how to lift the generators of the C-group. Any of the two liftings of the basic point b_0 completes the information for $\widetilde{\mathcal{P}}$.

Observe that giving the distinguished mirror Σ_0 is equivalent to giving the basic point b_1, because, by the conditions (4), b_1 is the orthogonal projection of b_0 to the distinguished mirror of σ_0, that is, $b_1 = (b_0 * \Sigma_0^\perp) \cap \Sigma_0$. Here for any two subsets $A, B \subset \mathbb{P}^n$, $A * B$ denotes their projective *span*, that is, the minimal *flat* (i.e. projective subspace) that contains both. For example, if x and y are points, $x * y$ is the line through them as before.

Finally, let us establish more geometric notation. The distance in $\mathbb{P}^n$ is denoted by $\mathrm{d}(\ ,\ )$. The angle between two meeting lines, a plane and a line or two planes will be denoted $\angle(\ ,\ )$; all lie between 0 and $\pi/2$.

2.2. Planar-skew polyhedra

For the scope of this work, a *regular projective polyhedron* $\mathcal{P}$ is a faithful realization in $\mathbb{P}^3$ of a regular polytope of rank 3. It is given by a C-group $\Gamma = \langle \sigma_0, \sigma_1, \sigma_2 \rangle \hookrightarrow \mathrm{Iso}(\mathbb{P}^3)$ together with a basic pair of points b_0, b_1 satisfying (4). Let Σ_i be the distinguished mirror of σ_i, that is, $b_j \in \Sigma_i$ for $i \neq j$, and then, $b_1 = (b_0 * \Sigma_0^\perp) \cap \Sigma_0$.

$\mathcal{P}$ has two natural polygonal invariants: its *face* $\mathcal{F}(\mathcal{P})$, which is the projective polygon $\langle \sigma_0, \sigma_1 \rangle$ with basic pair b_0, b_1; and its *vertex figure* (or *link*), $\mathcal{L}(\mathcal{P})$ whose C-group is $\langle \sigma_1, \sigma_2 \rangle$ with basic pair $b_1, (b_1 * \Sigma_1^{\perp}) \cap \Sigma_1$ (the orthogonal projection of b_1 on Σ_1).

A regular polygon in $\mathbb{P}^3$ may be of three types: planar (degenerate), skew or helicoidal, [1]. Thus, a rough classification of regular projective polyhedra is by the types of their face and vertex figure. We say that $\mathcal{P}$ is a *planar-skew polyhedron* if $\mathcal{F}(\mathcal{P})$ is planar and $\mathcal{L}(\mathcal{P})$ is skew.

Lemma 1. *Let $\mathcal{P}$ be a planar-skew polyhedron. Then its dimension vector is* $(0, 1, 2)$ *or* $(2, 1, 2)$.

Proof. Since a skew polygon has dimension vector $(1, 2)$ (see [1]), we must only rule out the case where Σ_0 is a line. Suppose that Σ_0 is a line. If it did not meet Σ_1, then $\mathcal{F}(\mathcal{P})$ would be a helicoid. Thus, it intersects Σ_1 and together they generate a plane $\Pi = \Sigma_0 * \Sigma_1$. Since σ_2 commutes with σ_0, we have that Σ_2 is orthogonal to Σ_0 or contains Σ_0. In either case, σ_2 fixes the plane Π (as a set, not pointwise). Therefore $\mathcal{P}$ is degenerate: all its vertices are in Π. But then its vertex figure is not skew. Therefore $\dim(\Sigma_0)$ is 0 or 2. $\square$

Let $\mathcal{P}$ be a planar-skew polyhedron with $\dim(\mathcal{P}) = (2, 1, 2)$, the other case will be taken care of in 3.2. If $\mathcal{P}$ has finite face, which we will assume from now on, then Σ_0 and Σ_1 must meet at an angle which is rational over π because the order of $\sigma_0\sigma_1$ is finite. Let $p \in \mathbb{Q}$ be such that $\angle(\Sigma_0, \Sigma_1) = \pi/p$. Analogously, if $\mathcal{P}$ has finite vertex figure, there exists $q \in \mathbb{Q}$ such that $\angle(\Sigma_2, \Sigma_1) = \pi/q$. Let $b_2 := \Sigma_0 \cap \Sigma_1$ (observe that $b_0 = \Sigma_2 \cap \Sigma_1$), and define $r := \mathrm{d}(b_0, b_2)$; it is the *radius* of the face. The three invariants p, q, r determine $\mathcal{P}$ up to equivalence, see [1], thus it will be most useful to give it a specific notation. We will write

$$\mathcal{P} = [\![p, q\,; r]\!]. \tag{5}$$

The main problem we will solve is to determine for which values of p, q, r is $[\![p, q\,; r]\!]$ a finite polyhedron. For the moment let us make some general remarks about their values and their geometric and combinatorial meaning.

Observe first that $p > 2$. Indeed, if $p = 2$ then Σ_0 and Σ_1 are orthogonal, which implies that $b_1 \in \Sigma_1$ and thus that $\sigma_1(b_1) = b_1$ which contradicts (4). Also $q > 2$, otherwise σ_0, σ_1 and σ_2 fix Σ_2, thus $\Gamma(b_0) \subset \Sigma_2$ and $\mathcal{P}$ is degenerate so that its vertex figure could not be skew.

The face $\mathcal{F}(\mathcal{P})$ is a regular polygon that lies in the plane $\Pi = \Sigma_0^{\perp} * \Sigma_1$. Let the irreducible expression of p be p_1/p_2. Suppose p_1 is even. Then $\mathcal{F}(\mathcal{P})$ is a regular polygon of p_1 sides that winds p_2 times arround its center of symmetry b_2 with radius r; it was denoted $[\![p\,; r]\!]$ in [1]. If p_1 is odd, then $(\sigma_0\sigma_1)^{p_1}$ is the reflection on the plane Π and not the identity of $\mathbb{P}^3$. So that the face $\mathcal{F}(\mathcal{P})$ winds 2 times arround the polygon $[\![p\,; r]\!]$; combinatorially it is of length $2p_1$, although geometrically it "looks" to be of length p_1. Let us denote it by $2[\![p\,;r]\!]$. Observe, in any case, that as the radius r approaches $\pi/2$, the face becomes a linear polygon. More precisely, if $r = \pi/2$ then $\mathcal{F}(\mathcal{P})$ is contained in the line $\Pi \cap \Sigma_2$, and so $\mathcal{P}$ is degenerate. Therefore, we also have that $r < \pi/2$.

Let q_1/q_2 be the irreducible expression of q. If q_1 is even then the vertex figure $\mathcal{L}(\mathcal{P})$ is antiprismatic of length q_1, if q_1 is odd it is prismatic of length $2q_1$. Therefore the Schläfli symbol of $\mathcal{P} = [\![p, q\,; r]\!]$ is $\{\epsilon(p_1), \epsilon(q_1)\}$, where $\epsilon(n) = n$ if n is even and $\epsilon(n) = 2n$ if n is odd.

It will be convenient to say that a rational number (such as p or q) is *even* (or *odd*) if its irreducible numerator is.

3. Mixing and geometric operations

The basic ideas about mixing operations were introduced in [12]. Here, we need to know the following. A regular polytope $\mathcal{P}$ is determined by its C-group $\Gamma = \langle \sigma_0, \dots, \sigma_d \rangle$. Then one can obtain a new group Ψ by taking as its generators certain suitably chosen products $\rho_0, \dots, \rho_m$, of (or words on) the generators σ_i of Γ. Hence Ψ is a subgroup of Γ. This process is called a *mixing operation* on (the generators of) Γ and is denoted

$$\mu : (\sigma_0, \dots, \sigma_d) \mapsto (\rho_0, \dots, \rho_m)\ . \tag{6}$$

The interesting case is when Ψ is again a C-group (satisfying (1) and (2)). Then a polytope $\mathcal{Q}$ may be obtained by Wythoff's construction on $\Psi = \langle \rho_0, \dots, \rho_m \rangle$. We say that $\mathcal{Q}$ is obtained from $\mathcal{P}$ by the mixing operation μ and denote it $\mathcal{Q} = \mu(\mathcal{P})$ or $\mathcal{Q} = \mathcal{P}^\mu$. Well known examples are the dual of any polytope (take $\rho_i := \sigma_{d-i}$ in (6)), or the *Petrie dual* of a polyhedron defined by the operation

$$\pi : (\sigma_0, \sigma_1, \sigma_2) \mapsto (\sigma_0\sigma_2, \sigma_1, \sigma_2) =: (\rho_0, \rho_1, \rho_2).$$

Suppose furthermore, that $\mathcal{P}$ is geometrically realized, in $\mathbb{P}^n$ say, so that we have (3) and (4). Then $\mathcal{Q}$ comes with a natural inclusion ($\Psi \hookrightarrow \mathrm{Iso}(\mathbb{P}^n)$) as in (3). Sometimes we can also produce naturally a basic pair (of points in $\mathbb{P}^n$) $a_0 = \mu_0(\mathcal{P})$ and $a_1 = \mu_1(\mathcal{P})$ satisfying (4) for $\mathcal{Q}$. Then we say that the projective polytope $\mathcal{Q}$ is obtained by the *geometric operation* μ (which now includes the geometric data) and denote it $\mathcal{Q} = \mu(\mathcal{P})$ or $\mathcal{Q} = \mathcal{P}^\mu$. Examples are the face and vertex figure of polyhedra, which can easily be generalized to the facet and vertex figure of any polytope. We also have the Petrie dual with $a_0 = \pi_0(\mathcal{P}) := b_0$ and $a_1 = \pi_1(\mathcal{P}) := b_1$, because the combinatorial $\pi(\mathcal{P})$ has the same graph as $\mathcal{P}$ and so the geometric $\pi(\mathcal{P})$ should have the same 1-skeleton as $\mathcal{P}$.

3.1. The dual

The combinatorial dual of $\mathcal{P}$, denoted by $\mathcal{P}^*$, is obtained by reversing the order of the generators (and hence inverting the partial order). Geometrically, when the C-group of the facet has a unique fixed point, it can be used as basic vertex, and then project it to the distinguished mirror to obtain a basic pair of points. In our case of interest, when $\mathcal{P} = [\![p, q\,; r]\!]$, this is so, namely, $\mathcal{P}^*$ has basic pair $b_2 := \Sigma_1 \cap \Sigma_0$ and $b_1^* := (b_2 * \Sigma_2^\perp) \cap \Sigma_2$. From the definitions it is clear that

$$[\![p, q\,; r]\!]^* = [\![q, p\,; r]\!].$$

3.2. The opposite polytope

The "opposite operation" was defined in [1], it can be considered as a geometric operation. Given a regular projective polytope $\mathcal{P}$ in $\mathbb{P}^n$, its *opposite*, denoted $\mathcal{P}^{op}$, consists of the same group and it has the same basic vertex b_0, but we take as basic edge the opposite e^{op} of e, going from b_0 to $\sigma_0(b_0)$ the other way arround in $\mathbb{P}^n$. That is, if $b_1 = (b_0 * \Sigma_0^\perp) \cap \Sigma_0$, then we take $b_1^{op} = (b_0 * \Sigma_0) \cap \Sigma_0^\perp$. The distinguished mirror Σ_0 of $\mathcal{P}$ is changed to its polar $\Sigma_0^\perp$. Clearly $(\mathcal{P}^{op})^{op} = \mathcal{P}$.

Although $\mathcal{P}$ and $\mathcal{P}^{op}$ are combinatorially identical, they are geometrically different and when lifting them to euclidean space $\mathbb{R}^{n+1}$ they give rise to different polytopes, sometimes even combinatorially. In fact, projecting an euclidean polytope, taking the opposite and lifting back, has the effect of changing the first generator σ_0 for its antipode $-\sigma_0$. This operation gives rise to new euclidean polytopes in all dimensions greater than 3. The "opposites" of regular simplices are always combinatorially different, with twice as many vertices, and Schläfli symbol $\{6, 3, \ldots, 3\}$. The "opposites" of cubes alternate; in even dimensions they are combinatorially the same but (for dimension greater than 2) with another realization, and in odd dimensions they are combinatorially different, with half the vertices; their Schläfli symbol remains $\{4, 3, \ldots, 3\}$. The "opposites" of orthohedra are combinatorially different, with Schläfli symbol $\{6, 3, \ldots, 3, 4\}$.

Observe that a standard projective planar polygon $[\![p\,; r]\!]$ has dimension vector $(1, 1)$, its basic vertex is chosen in the 1-mirror, and it avoids the polar line to its symmetry center; that is why it was called *inessential* in [1]. Its opposite $[\![p\,; r]\!]^{op}$ has dimension vector $(0, 1)$, meets every projective line, and so is called *essential*. The lift to $\mathbb{R}^3$ of the former is a planar polygon (choose one of the two copies). And the lift of an essential polygon is skew with the origin as center of symmetry. When p is even, it is antiprismatic and has the same length (again, choose one component). When p is odd, it is prismatic of twice the length.

Our polyhedra $[\![p, q\,; r]\!]$ have planar inessential faces. They are in 1-1 correspondence with their opposites which have planar essential face and dimension vector $(0, 1, 2)$. Thus, for classification purposes it will be enough to focus on the former. We can now define the dual of $[\![p, q\,; r]\!]^{op}$ as $[\![q, p\,; r]\!]^{op}$.

3.3. The polar polyhedron

This operation is defined on planar-skew polyhedra in $\mathbb{P}^3$. Suppose that $\mathcal{P}$ is such a polyhedron, with b_0, b_1 its basic pair of points and Σ_i its distinguished mirrors. The *polyhedron* of $\mathcal{P}$, denoted by $\mathcal{P}^\perp$, consists of the same C-group as $\mathcal{P}$ (therefore they are combinatorially identical), but with basic pair $a_0 = \Sigma_1^\perp \cap \Sigma_2$ and $a_1 = (a_0 * \Sigma_0^\perp) \cap \Sigma_0$. It is then obtained by changing the distinguished mirror Σ_1 for its polar line $\Sigma_1^\perp$.

Clearly, $\mathcal{P}^\perp$ is again a planar-skew polyhedron with the same dimension vector, and $(\mathcal{P}^\perp)^\perp = \mathcal{P}$. Moreover, it is easy to see that the dual and the opposite commute with the polar.

Lemma 2. *If $\mathcal{P} = [\![p, q\,; r]\!]$, then $\mathcal{P}^\perp = [\![\bar{p}, \bar{q}\,; \hat{r}]\!]$, where $1/\bar{p} = 1/2 - 1/p$, $1/\bar{q} = 1/2 - 1/q$ and*

$$\cos(\hat{r}) = \cos(r)\frac{\sin(\pi/p)\,\sin(\pi/q)}{\cos(\pi/p)\,\cos(\pi/q)}.$$

Proof. The first part follows because polar lines meet a plane at complementary angles. The second is obtained by analizing the tetrahedron with vertices b_0, b_1, b_2 and b_1^*, where the three parameters have geometric meaning, observing that $\hat{r}$ is the dihedral angle at the edge b_0b_2 of length r. The spherical-trigonometric arguments that lead to the last equation are quite laborious and add little to the present paper, so we omit them. □

3.4. The *j*-hole polyhedron

Given a regular polyhedron $\mathcal{P}$ with C-group $\Gamma = \langle\sigma_0, \sigma_1, \sigma_2\rangle$. Its *j-hole polyhedron*, denoted by $h_j(\mathcal{P})$, is defined by the mixing operation

$$h_j : (\sigma_0, \sigma_1, \sigma_2) \mapsto (\sigma_0, (\sigma_1\sigma_2)^{j-1}\sigma_1, \sigma_2) =: (\rho_0, \rho_1, \rho_2) \qquad (7)$$

keeping the basic pair fixed. For the detailed geometric insight see [12]. Observe, however, that the cyclic order of edges arround a vertex in $\mathcal{P}$, becomes "jumping to the j^{th}" in $h_j(\mathcal{P})$. As examples, the reader may care to check that $h_2([\![3, 5]\!]) = [\![5, 5/2]\!]$; see also other non-trivial applications of h_2 to the polyhedra in Section 3 of [1].

Lemma 3. *Let $\mathcal{P}$ be a planar-skew projective polyhedron. Then, $h_2(\mathcal{P})$ is combinatorially isomorphic to an euclidean polyhedron with planar face.*

Proof. Let $\Gamma = \langle\sigma_0, \sigma_1, \sigma_2\rangle$ be the C-group of $\mathcal{P}$. Since $\rho_1 = \sigma_1\sigma_2\sigma_1$ is the conjugate of a plane reflection, it is also a plane reflection. Thus $\dim(h_2(\mathcal{P})) = (2, 2, 2)$. The three reflection planes meet at a point, v say, so that the C-group Ψ of $h_2(\mathcal{P})$ fixes the point v. Therefore, Ψ can be identified with a C-group acting on the tangent space of $\mathbb{P}^3$ at v, which is euclidean of dimension 3. The direction from v to b_0 clearly acts as geometric vertex (it is fixed by ρ_1 and $\rho_2 = \sigma_2$) to obtain a faithful euclidean realization of $h_2(\mathcal{P})$. It has planar face because its generators are reflections. □

Corollary 1. *Let $\mathcal{P}$ be a planar-skew projective polyhedron with Schläfli symbol $\{p, q\}$. Then $p \leq 10$ and $q \leq 10$.*

Proof. We know that q is even, so $h_2(\mathcal{P})$ has Schläfli symbol $\{s, q/2\}$ for some s. By the preceding lemma it is isomorphic to a finite euclidean polyhedron with planar face, and by their classification ([10, 8, 9, 13]) we have that $q/2 \leq 5$. To see that $p \leq 10$, consider the dual. □

4. Antiprismatic vertex figure

The purpose of this section is to classify the finite regular projective polyhedra with simple planar inessential face and skew antiprismatic vertex figure.

Theorem 1. *The finite polyhedra $\mathcal{P} = [\![p, q\,; r]\!]$ with p and q even are:*

$$[\![4, 4\,; \pi/t]\!] \quad \text{with} \quad t \in \mathbb{Q},\ t > 2\,;$$
$$[\![4, 6\,; \pi/4]\!],\ [\![6, 4\,; \pi/4]\!]\,;$$
$$[\![4, 8\,; \pi/8]\!]\,,\ [\![4, 8/3\,; 3\pi/8]\!],\ [\![8, 8/3\,; \pi/4]\!],$$
$$[\![8, 4\,; \pi/8]\!],\ [\![8/3, 4\,; 3\pi/8]\!],\ [\![8/3, 8\,; \pi/4]\!]\,;$$
$$[\![4, 10\,; \pi/4]\!],\ [\![4, 10/3\,; \pi/4]\!],\ [\![10, 4\,; \pi/4]\!],\ [\![10/3, 4\,; \pi/4]\!].$$

Observe that the grouping by semicolons is associated with the automorphism groups of the polyhedra. The infinite family of tori, $[\![4, 4\,; \pi/t]\!]$, $2 < t \in \mathbb{Q}$, whose combinatorics and groups are defined in [6], was described in [1]. So were the *pentacaidecahedron* $[\![4, 6\,; \pi/4]\!]$ (15 faces) and the *decahedron* $[\![6, 4\,; \pi/4]\!]$ (10 faces), whose automorphism group is the symmetric group on five letters. The next six polyhedra have the automorphism group of the 12-cell, and the last four those of the 60-cell (see [4]).

Of the "Platonic" projective polyhedra, described in [1], we have yet to mention the dual pair *pachyhedron* $[\![4, 8\,; \pi/8]\!]$ (144 faces) and *hemipachyhedron* $[\![8, 4\,; \pi/8]\!]$ (72 faces). The classic way to define them would be as quotients, or "halves", of Coxeter's skew polyhedra in the 4 dimensional euclidean space, [3]. See also [17] for planar face embeddings of the latter into $\mathbb{R}^3$; however, they are not geometrically regular.

4.0.1. Associated to the 12-cell The other four polyhedra with the group of the 12-cell can be obtained from the last two as follows. A dual pair as their polars

$$[\![4, 8/3\,; 3\pi/8]\!] = [\![4, 8\,; \pi/8]\!]^{\perp}$$
$$[\![8/3, 4\,; 3\pi/8]\!] = [\![8, 4\,; \pi/8]\!]^{\perp}$$

are inmersions of the same combinatrorial surface but with faces of three times the radius. They could be called "great pachyhedron" and "great emipachyhedron".

Consider the 3-hole polyhedron of the pachyhedron, which is not hard to describe geometrically as

$$[\![8, 8/3\,; \pi/4]\!] = h_3([\![4, 8\,; \pi/8]\!]).$$

Its dual $[\![8/3, 8\,; \pi/4]\!]$ completes our list. To prove that they are combinatorially isomorphic, observe that they are a polar pair ($[\![8, 8/3\,; \pi/4]\!]^{\perp} = [\![8/3, 8\,; \pi/4]\!]$ by Lemma 2).

4.0.2. Associated to the 60-cell With the automorphism group of the 60-cell, there are two dual pairs. We shall define their representatives made of squares, $\mathcal{P}_{4,10} := [\![4, 10\,; \pi/4]\!]$ and $\mathcal{P}_{4,10/3} := [\![4, 10/3\,; \pi/4]\!]$, which are combinatorially isomorphic. Formally, they will be defined in the next section. For the moment, let us describe them geometrically. Consider the 60-cell, a projective polytope of rank 4 whose facets are 60 dodecahedra, called a honeycomb in [4]. Let an edge grow on both ends. The vertices enter two facets. Stop when they are at the barycenters. It is then a segment of length $\pi/3$. Do this for all edges of the 60-cell to obtain the geometric 1-skeleton of $\mathcal{P}_{4,10}$. Observe that it has vertices at the barycenters of the facets with 20 edges incident to them corresponding to, and passing through, the vertices of its dodecahedron. These edges naturally group together to form 900 regular squares of radius $\pi/4$. They are the faces (combinatorially and geometrically) of $\mathcal{P}_{4,10}$. At each edge there are 6 squares; in fact, if we intersect them with a dodecahedron the edge becomes a vertex and the squares geodesics to the 6 vertices reachable by three steps in Petrie polygons. To make a polyhedron the squares must be formally matched in pairs at each edge. By the way in which the squares fit arround the edge, there are only three matching rules that can be performed regularly on all edges. They give rise to $\mathcal{P}_{4,10}$, $\mathcal{P}_{4,10/3}$ and to 60 copies of $[\![4, 6\,; \pi/4]\!]$. Thus, each geometric edge of $\mathcal{P}_{4,10}$ and $\mathcal{P}_{4,10/3}$ stands for 3 combinatorial edges; the surface passes three times through it.

4.1. Solutions of the waist equation

We have proved the existence part of Theorem 1. Now we prove uniqueness.

Suppose that $\mathcal{P}$ is a finite regular projective polyhedron with simple inessential planar face and skew antiprismatic vertex figure. Then $\mathcal{P} = [\![p, q\,; r]\!]$, and we must prove that the only possible values which p, q and r can take are those of the theorem.

It is not hard to see that the radius r has to be rational over π (Corollary 2 of [1]), so that $t := \pi/r \in \mathbb{Q}$, and $t > 2$. By hypothesis, we know that p_1 and q_1 are even (recall $p = p_1/p_2$ and $q = q_1/q_2$), and from Corollary 1 we have that $q_1 \leq 10$ and $p_1 \leq 10$. Therefore, p and q must be in the set $\{4, 6, 8, 8/3, 10, 10/3\}$. Many of the possible assignments are eliminated by Proposition 1 of [1], which implies that if $q = 6, 10$ or $10/3$ then $p = t = 4$, and its dual (interchange p and q).

Our main tool will be the waist equation (Theorem 1 of [1]), which reads as follows. If $p_1 \equiv 0 \bmod(4)$ then the geometric waist $c \in \mathbb{Q}$ of $\mathcal{P}$, satisfies

$$\cos(\pi/p)\cos(\pi/c) = \sin(\pi/q)\cos(\pi/t). \tag{8}$$

Our task now is to find the rational solutions (p, q, t, c) of (8) when the pair (p, q) takes values in the set

$$\begin{gathered}(4,4),\ (4,6),\ (4,8),\ (4,8/3),\ (4,10),\ (4,\ 10/3),\\ (8,4),\ (8,8),\ (8,8/3),\ (8/3,4),\ (8/3,8),\ (8/3,8/3).\end{gathered} \tag{9}$$

In [15], based on work of Conway and Jones [2], Myerson describes the complete list of rational solutions of the trigonometric equation

$$\cos(x_1\pi)\cos(x_2\pi) = \cos(x_3\pi)\cos(x_4\pi). \tag{10}$$

This list reads as follows:

a) 24 special solutions.

b) A family of solutions coming from the trigonometric identity $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$. Namely, for each $\alpha \in \mathbb{Q}$, take $x_1 = 1/3$, $x_2 = 1/2 - 2\alpha$, $x_3 = \alpha$ and $x_4 = 1/2 - \alpha$.

c) Degenerate solutions $\{x_1, x_2\} = \{x_3, x_4\}$.

For each solution of (10) we may have different assignments of our variables (p, q, t, c) giving us solutions of (8). A quick glance at the 24 special cases, eliminates them because of our restrictions (9) on p and q.

From family (**b**), we get only 4 acceptable solutions for (p, q, t, c):

b.1: $(4, 8, 8, 3)$, **b.2:** $(4, 8/3, 8/3, 3)$,
b.3: $(8, 4, 3, 8/3)$, **b.4:** $(8/3, 4, 3, 8)$.

For the degenerate solutions (**c**), we have two possible assignments. One is taking $1/q = 1/2 - 1/p$ and $t = c$. The other, $t = p$ and $1/c = 1/2 - 1/q$. From the first asingment we get three infinite families, with $2 < \alpha \in \mathbb{Q}$:

c.1.1: $(4, 4, \alpha, \alpha)$, **c.1.2:** $(8, 8/3, \alpha, \alpha)$, **c.1.3:** $(8/3, 8, \alpha, \alpha)$.

Finally, from the second assignment, and eliminating the three particular cases that already appear in the above families, we get:

c.2.1:	$(4, 6, 4, 3)$,	**c.2.2:**	$(4, 8, 4, 8/3)$,
c.2.3:	$(4, 8/3, 4, 8)$,	**c.2.4:**	$(4, 10, 4, 5/2)$,
c.2.5:	$(4, 10/3, 4, 5)$,	**c.2.6:**	$(8, 4, 8, 4)$,
c.2.7:	$(8, 8, 8, 8/3)$,	**c.2.8:**	$(8/3, 4, 8/3, 4)$,
c.2.9:	$(8/3, 8/3, 8/3, 8)$.		

This is the complete list of solutions to the waist equation (8) subject to the restrictions (9) on (p, q). But we still have to eliminate some of the above possibilities.

Observe that if (p, q, t, c) is a solution with $p_1, q_1 \equiv 0 \bmod(4)$ coming from the finite polyhedron $\mathcal{P} = [\![p, q\,; \pi/t]\!]$, then there must exist a solution (q, p, t, c^*) for some c^*, corresponding to its dual $\mathcal{P}^* = [\![q, p\,; \pi/t]\!]$. This criterion eliminates **b.3**, **b.4**, **c.2.2** and **c.2.3**. But not the families **c.1.2** and **c.1.3** which are dual, or the cases **c.2.7** and **c.2.9** which are self-dual. However, of all of these cases, only (8, 8/3, 4, 4) and (8/3, 8, 4, 4) satisfy the following.

Lemma 4. *If $\mathcal{P} = [\![p, q\,; r]\!]$ is finite with $p_1 = 8$ (where $p = p_1/p_2$), then its waist is $c = 4$.*

Proof. Consider the dual $\mathcal{P}^* = [\![q, p\,; r]\!]$. Its 2-hole polyhedron $h_2(\mathcal{P}^*)$ is euclidean with planar face by Lemma 3. Since $p_1 = 8$, it has Schläfli symbol $\{x, 4\}$. Thus it is the octahedron $\{3, 4\}$. Now observe that the 2-hole of the octahedron has length 4 and that it corresponds to the combinatorial waist of $\mathcal{P}$. Therefore the geometric waist c must also be 4. □

Observe now that the list of feasible solutions that remains, corresponds precisely to the polyhedra of Theorem 1 to which the waist equation applies, the remaining three are duals. This concludes the proof of Theorem 1.

Before we proceed, observe that the solutions to the waist equation give us new information about the polyhedra, namely, the value of the waist and when applicable, of the dual waist. The solutions which were discarded at the end, correspond to regular infinite polyhedra with finite face, vertex figure and waist, and which are thus not simply connected.

5. Skew-prismatic vertex figure

In Sections 6.1 and 6.2 of [12] a family of regular polyhedra in the euclidean 4-space $\mathbb{R}^4$ is obtained by the "Petrie–Coxeter" mixing operation performed on the classical regular polytopes of rank 4 in $\mathbb{R}^4$. The polyhedron $\mathcal{P}$ obtained by this operation from the polytope $\mathcal{Q}$, say, is geometrically realized by taking vertices on the barycenters of the edges of $\mathcal{Q}$ and edges along its 2-dimensional faces. The faces of $\mathcal{P}$ then correspond to, are in fact the dual of, the Petrie polygons of the facets of $\mathcal{Q}$. Thus they are planar, and the vertex figure is skew prismatic (see [12] for a detailed geometric description). Of course, these polyhedra may be projected to $\mathbb{P}^3$. The main point of this section is to prove that if we also consider the opposites of the classical polytopes, then all the "odd" projective polyhedra arise in this way.

Let $\mathcal{Q}$ be a projective polytope of rank 4 in $\mathbb{P}^3$ given by the C-group $\Psi = \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle \subset \mathrm{Iso}(\mathbb{P}^3)$ and the basic pair $a_0, a_1 \in \mathbb{P}^3$. The *Petrie–Coxeter* polyhedron of $\mathcal{Q}$, denoted $\mathcal{P} = \mu(\mathcal{Q})$ is defined by the geometric operation:

$$\begin{aligned} \mu : (\rho_0, \rho_1, \rho_2, \rho_3) \mapsto (\rho_1, \rho_0\rho_2, \rho_3) &=: (\sigma_0, \sigma_1, \sigma_2), \\ b_0 = \mu_0(\mathcal{Q}) &:= a_1, \\ b_1 = \mu_1(\mathcal{Q}) &:= (a_1 * \Pi_1^{\perp}) \cap \Pi_1 \end{aligned} \tag{11}$$

where Π_i is the distinguished mirror of ρ_i in $\mathcal{Q}$. It is proved in [12] that $\mathcal{P}$ is combinatorially a polyhedron, and it is easy to see that it further satisfies (4). Thus it is a projective polyhedron.

Lemma 5. *Let $\mathcal{Q}$ be a projective polytope of rank 4 in $\mathbb{P}^3$. If* $\dim(\mathcal{Q}) = (2, 2, 2, 2)$ *or* $(0, 2, 2, 2)$*, then* $\dim(\mu(\mathcal{Q})) = (2, 1, 2)$ *and*

$$\mu(\mathcal{Q}^{op}) = \mu(\mathcal{Q})^{\perp}.$$

Proof. Let $\mathcal{P} = \mu(\mathcal{Q})$ be the Petrie–Coxeter polyhedron of $\mathcal{Q}$ defined by, and with notation as in (11) above. Since ρ_0 and ρ_2 commute, then $\sigma_1 = \rho_0\rho_2$ is a π-rotation along the line $\Sigma_1 := \Pi_0 \cap \Pi_2$ when $\dim(\Pi_0) = 2$ or the line $\Sigma_1 := \Pi_0 * \Pi_2^{\perp}$ when $\dim(\Pi_0) = 0$. Σ_1 is then the distinguished mirror of σ_1. The other distinguished mirrors of $\mathcal{P}$ are $\Sigma_0 := \Pi_1$ and $\Sigma_2 := \Pi_3$. Therefore $\dim(\mathcal{P}) = (2, 1, 2)$.

Now, observe that interchanging the roles of Π_0 and $\Pi_0^{\perp}$ produces the interchange of Σ_1 and $\Sigma_1^{\perp}$, so that the Petrie–Coxeter of the opposite is the polar of the Petrie–Coxeter, that is, $\mu(\mathcal{Q}^{op}) = \mathcal{P}^{\perp}$. □

The classic 4-dimensional euclidean polytopes whose groups are generated by reflections are classified, see [5]; they are denoted by symbols of the form $\{x, y, z\}$. If we denote the corresponding projection by $[\![x, y, z]\!]$, then we obtain the projective polyhedra of Table 1 as their Petrie–Coxeter and their polars. Those in the second column are the projections of the ones described by McMullen and Schulte in [12]. The radius r in each row is the angular distance of the barycenter of the facet to the barycenter of its edge in the corresponding polytope; and $\hat{r}$, if applicable, is then obtained by Lemma 2. Only a few easily expressed radii are explicitly given, and it is understood that r varies from row to row.

Theorem 2. *Let $\mathcal{P} = [\![p, q\,; r]\!]$ be finite and such that $q_1 \equiv 1 \bmod(2)$ or $q_1 \equiv 2 \bmod(4)$, where q_1/q_2 is the irreducible expression of $q \in \mathbb{Q}$. Then there exists a rank 4 regular projective polytope $\mathcal{Q}$ whose C-group generators are reflections and such that $\mathcal{P}$ is its Petrie–Coxeter polyhedron, that is, such that $\mu(\mathcal{Q}) = \mathcal{P}$.*

Proof. Let $\hat{q} = \hat{q}_1/\hat{q}_2$ (its irreducible expression) be complementary to q, that is, $1/q + 1/\hat{q} = 1/2$. Then it is easy to see that $q_1 \equiv 1 \bmod(2)$ if and only if $\hat{q}_1 \equiv 2 \bmod(4)$. Therefore, changing $\mathcal{P}$ for its polar $\mathcal{P}^{\perp}$ if necessary, we may assume that $q_1 \equiv 1 \bmod(2)$.

$\mathcal{Q}$	$\mu(\mathcal{Q})$	$\mu(\mathcal{Q}^{op})$
$[\![3,3,3]\!]$	$[\![4,3\,;\cos^{-1}(1/\sqrt{6})]\!]$	$[\![4,6\,;\pi/4]\!]$
$[\![3,3,4]\!]$	$[\![4,4\,;\pi/4]\!]$	$[\![4,4\,;\pi/4]\!]$
$[\![3,4,3]\!]$	$[\![6,3\,;r]\!]$	$[\![3,6\,;r]\!]$
$[\![4,3,3]\!]$	$[\![6,3\,;r]\!]$	$[\![3,6\,;r]\!]$
$[\![3,3,5]\!]$	$[\![4,5\,;r]\!]$	$[\![4,10/3\,;\pi/4]\!]$
$[\![3,3,5/2]\!]$	$[\![4,5/2\,;r]\!]$	$[\![4,10\,;\pi/4]\!]$
$[\![5,3,3]\!]$	$[\![10,3\,;r]\!]$	$[\![5/2,6\,;\hat{r}]\!]$
$[\![5/2,5,5/2]\!]$	$[\![6,5/2\,;r]\!]$	$[\![3,10\,;\hat{r}]\!]$
$[\![5/2,3,3]\!]$	$[\![10/3,3\,;r]\!]$	$[\![5,6\,;\hat{r}]\!]$
$[\![5,5/2,5]\!]$	$[\![6,5\,;r]\!]$	$[\![3,10/3\,;\hat{r}]\!]$
$[\![5,5/2,3]\!]$	$[\![6,3\,;r]\!]$	$[\![3,6\,;r]\!]$
$[\![5/2,5,3]\!]$	$[\![6,3\,;r]\!]$	$[\![3,6\,;r]\!]$
$[\![3,5,5/2]\!]$	$[\![10,5/2\,;r]\!]$	$[\![5/2,10\,;r]\!]$
$[\![5,3,5/2]\!]$	$[\![10,5/2\,;r]\!]$	$[\![5/2,10\,;r]\!]$
$[\![5/2,3,5]\!]$	$[\![10/3,5\,;r]\!]$	$[\![5,10/3\,;r]\!]$
$[\![3,5/2,5]\!]$	$[\![10/3,5\,;r]\!]$	$[\![5,10/3\,;r]\!]$

Table 1.

To ease notation, let us use q to denote also its numerator q_1, and likewise for p; this should cause no problem, for, the context, either geometrical or combinatorial, tells us if it is the rational or the integer we refer to. Recall that $\mathcal{P}$ is combinatorially a regular polyhedron with Schläfli symbol $\{p, 2q\}$. Let $\sigma_0, \sigma_1, \sigma_2$ be its canonical generators. Then, the mixing operation (defined in [12] pg. 18)

$$\bar{\mu} : (\sigma_0, \sigma_1, \sigma_2) \mapsto ((\sigma_1\sigma_2)^q, \sigma_0, (\sigma_1\sigma_2)^q\sigma_1, \sigma_2) =: (\rho_0, \rho_1, \rho_2, \rho_3) \tag{12}$$

yields a group $\Psi = \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$ that satisfies (1) because of the identity $(\sigma_1\sigma_2)^{2q} = 1$. Clearly, $\bar{\mu}$ is a right inverse to μ of (11), that is $\mu\bar{\mu} = \textit{identity}$. In its full combinatorial generality, it is an open question if Ψ is always a C-group, that is, if it satisfies the intersection property (2). At least, there is no proof of this in [12], and the author doubts it in general. However, in our case we will prove it with geometric arguments.

Let b_0, b_1 be the basic pair of $\mathcal{P}$, and Σ_i the distinguished mirror of σ_i. Consider the prism in $\mathbb{P}^3$ whose vertices are those of the vertex figure $\mathcal{L}(\mathcal{P})$, that is, $\langle\sigma_1,\sigma_2\rangle(b_1)$. It is a prism whose base is a planar (inessential) regular q-gon, with q odd. Observe that $\rho_0 = (\sigma_1\sigma_2)^q$ is the reflection along the symmetry plane of $\mathcal{L}(\mathcal{P})$ (let us call it Π_0) that interchanges the two q-gonal faces; that $\rho_2 = (\sigma_1\sigma_2)^q\sigma_1$ is the reflection along the plane Π_2 orthogonal to Π_0 and to a lateral face. So that $\Pi_0 \cap \Pi_2 = \Sigma_1$ (see Figure 1.a where $q = 3$). Finally, let Π_1 and Π_3 be the reflection planes of $\rho_1 = \sigma_0$ and $\rho_3 = \sigma_2$ respectively, i.e., $\Pi_1 := \Sigma_0$ and $\Pi_3 := \Sigma_2$; and let $a_i := \bigcap_{j\neq i} \Pi_j$ (Figure 1.b).

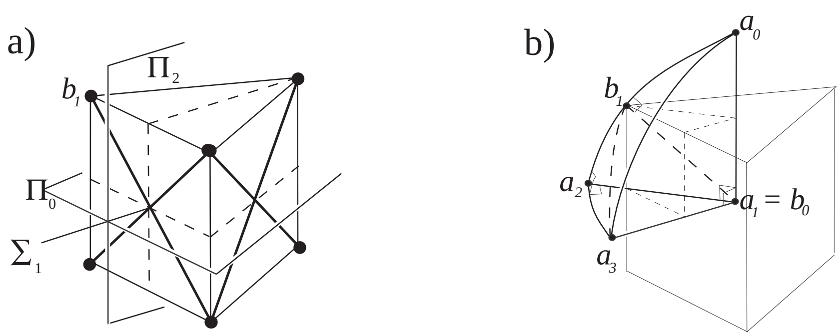

Figure 1. a) The prismatic vertex figure. b) The basic tetrahedron

Suppose for the moment, that $\Psi = \langle\rho_0,\rho_1,\rho_2,\rho_3\rangle$ is the C-group of a combinatorial polytope. It can then be realized in $\mathbb{P}^3$ by taking as basic pair $a_0 =: \bar{\mu}_0(\mathcal{P})$ and $a_1 =: \bar{\mu}_1(\mathcal{P})$, so that $\bar{\mu}$ becomes a geometric operation. If we take $\mathcal{Q} = \bar{\mu}(\mathcal{P})$, it is clear that $\dim(\mathcal{Q}) = (2,2,2,2)$ and that $\mathcal{P} = \mu(\mathcal{Q})$, which completes the proof of the theorem. It remains to prove that Ψ satisfies the intersection property (2).

Let Δ be the tetrahedron with vertices a_i (and faces on Π_i) containing the basic triangle $b_0b_1b_2$ of $\mathcal{P}$ (Figure 1.b), where $b_2 = a_3 = \Sigma_0 \cap \Sigma_1$. We claim that if the diameter of Δ is less than $\pi/2$ then Ψ satisfies the intersection property. To see this, and to fix ideas, let us prove that $\langle\rho_1,\rho_2,\rho_3\rangle \cap \langle\rho_0,\rho_1,\rho_2\rangle = \langle\rho_1,\rho_2\rangle$. All the other cases, for different subsets I and J of indices, are analogous.

Consider $\omega \in \langle\hat{\rho}_0\rangle \cap \langle\hat{\rho}_3\rangle$ (where $\hat{\rho}_i$ means "omit ρ_i"). We must prove that $\omega \in \langle\rho_1,\rho_2\rangle$. Since $\omega \in \langle\hat{\rho}_i\rangle$ implies that $\omega(a_i) = a_i$, then if the length of the segment e from a_0 to a_3 contained in Δ is not $\pi/2$, this segment is also fixed pointwise by ω. Consider a small sphere S^2 centered at a_0. The group $\langle\rho_1,\rho_2,\rho_3\rangle$ acts on this sphere, and can be considered as a group generated by three reflections on S^2, namely by the reflections on the great circles $S^2\cap\Pi_1, S^2\cap\Pi_2, S^2\cap\Pi_3$. These finite groups are well understood and classified, they are either dihedral or the groups of regular euclidean polyhedra. Now, observe that $S^2\cap e \in S^2\cap\Pi_1\cap\Pi_2$, so that ω fixes pointwise the antipodal pair $S^2\cap\Pi_1\cap\Pi_2$. It is true for these finite groups that then ω can be expressed by only those two reflections, that is, $\omega \in \langle \rho_1, \rho_2 \rangle$. This means geometrically that the vertices and edges of the classic euclidean polyhedra are not "double", which is no longer the case in higher dimensions.

To prove that $d(a_i, a_j) < \pi/2$, for all i, j we will use the following simple fact. Consider a projective triangle with angles α, β, γ and respectively opposite sides of lengths a, b, c; if $\alpha = \pi/2$ and $\gamma, b < \pi/2$ then $a, c, \beta < \pi/2$. From $\mathcal{P}$, we know that $d(b_0, b_1) < \pi/2$ and $d(a_1, a_3) < \pi/2$ (in fact, since $a_3 = b_2$ and $a_1 = b_0$ then $d(a_1, a_3) = r$, the radius of $\mathcal{P}$). Since $\angle b_1 a_1 a_0 < \pi/2$ (because $\angle b_1 a_1 a_0 + \angle b_1 a_1 a_2 = \pi/2$) then $d(a_1, a_0) < \pi/2$ and $\angle a_1 a_0 a_2 < \pi/2$. Then, $d(a_0, a_2) < \pi/2$, $d(a_1, a_2) < \pi/2$ (and $\angle a_1 a_2 a_0 < \pi/2$). Since $\angle a_2 a_1 a_3 = \pi/q < \pi/2$ and $\angle a_1 a_2 a_3 = \pi/2$ then $d(a_3, a_2) < \pi/2$. Finally, $d(a_0, a_3) < \pi/2$ because the segment $a_0 a_3$ lies in a right triangle with right legs less than $\pi/2$. □

6. Conclusion

Theorems 1 and 2 yield the classification of finite planar-skew polyhedra in $\mathbb{P}^3$. Suppose $\mathcal{P} = [\![p, q\,; r]\!]$ is finite. If p and q are even (their denominators, we mean) then $\mathcal{P}$ is listed in Theorem 1. If not, then Theorem 2 applies either to $\mathcal{P}$ or to its dual $\mathcal{P}^*$, and then, by the classification of groups generated by reflections in $\mathbb{R}^4$, it, or its dual, appears in Table 1. Observe that three polyhedra, besides a torus, appear in both theorems, and correspondingly, three duals have not been mentioned, namely, $[\![3, 4\,; \cos^{-1}(1/\sqrt{6})]\!]$, $[\![5, 4\,; r]\!]$ and $[\![5/2, 4\,; r]\!]$ for suitable r's. The other duality relations in Table 1 are in the same row when $\hat{r}$ does not appear because p and q are complementary $(1/p + 1/q = 1/2)$, or in contiguous rows, where we further know that r in one is equal to $\hat{r}$ in the other.

Thus, we may conclude that, besides the infinite family of tori parametrized by $t \in \mathbb{Q}, t > 2$, there are 42 finite polyhedra $[\![p, q\,; r]\!]$, which together with their opposites are all the finite regular projective polyhedra with planar face and skew vertex figure.

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J. Bracho
Instituto de Matemáticas, UNAM
Circuito Exterior, C.U.
México D.F., 04510
México
e-mail: jbracho@math.unam.mx

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