Aequationes Math. 59 (2000) 134–149
0001-9054/00/010134-16 $ 1.50+0.20/0

Aequationes Mathematicae

Multiplicative properties of real functions with applications to classical functions

Carlos E. Finol and Marek Wójtowicz

Summary. From the characterisation of geometrically convex and geometrically concave functions defined on $(0, A]$ or $[A, \infty)$ with $A > 0$, by means of their multiplicative conditions, we obtain unified proofs of some known and new inequalities. Functions of class C^2 and strictly increasing on (a, b) fulfil some kind of supermultiplicativity and superadditivity. We have obtained a new constant determining the intervals of sub- and supermultiplicativity for the log function.

Mathematics Subject Classification (1991). 26A09, 26A48, 26A51, 26D05, 26D15, 33B15.

Keywords. Functions: geometrically convex, geometrically concave, submultiplicative, supermultiplicative, monotonic; classical functions; Riemann zeta function; Euler gamma function, log function.

1. Introduction and preliminaries

Let J be an interval in $\mathbb{R}^+ := (0, \infty)$ such that $J \cdot J \subseteq J$. A function $f : J \to \mathbb{R}^+$ is said to be submultiplicative on J if the inequality $f(xy) \le f(x)f(y)$ holds for all $x, y \in J$. If the above inequality holds in reverse, then f is said to be supermultiplicative. There are a few papers of very particular (although important) or general nature concerning multiplicative properties of real functions. The first paper in that subject comes from R. Cooper [2] and dates 1927. Characterizations and applications of submultiplicative Orlicz functions are given in the monograph of Krasnosielskii and Rutickii [13]. Gustavsson, Maligranda and Peetre studied submultiplicative properties of the log function, giving also a general criteria for submultiplicativity on $[1, \infty)$ ([9], Lemma 1).

Submultiplicative functions appear naturally in diverse subjects such as interpolation theory [1], [14] and play a significant role in the theory of operators in Orlicz spaces [13]. Some special results are included in [3], [4], [9], [12], [17]. With the aid of the submultiplicative function $f(x) = \log(x+e^2)$ Zafran [26] constructed a Banach algebra B such that $A(T) \subset B \subset C(T)$, where $A(T)$ is the algebra of absolutely convergent Fourier series, $C(T)$ the algebra of continuous functions on T and T is the unit circle group, all the inclusions being proper, thereby solving the "dichotomy problem". More recently, Gowers [5] uses the submultiplicative function $f(x) = \log_2(x+1)$, $x \geq 1$, to construct an infinite dimensional Banach space not isomorphic to any of its proper subspaces; this gives a negative answer to the celebrated "hyperplane problem" of Banach.

The purpose of this paper is to indicate a large class of real functions which are sub- or supermultiplicative on proper intervals. We study the classes of functions which fulfil the functional inequalities

$$f(x^\alpha y^\beta) \leq f(x)^\alpha f(y)^\beta,$$

for all $\alpha, \beta \geq 0, \alpha + \beta = 1$, $x, y \in I_A$, and $I_A = (0, A]$ or $I_A = [A, \infty)$ with $A > 0$. Following J. Matkowski ([18], p. 108), functions fulfiling the above inequality are called *geometrically convex*, and *geometrically concave* if the reversed inequality holds. It is easy to see that $f : I_A \to \mathbb{R}^+$ is geometrically convex if and only if *the function* $\widehat{f}(t) := \log(f(e^t))$ *is convex on the semiaxis* $\log I_A$, where $\log I_A$ denotes the respective semiaxes $(-\infty, \log A]$ or $[\log A, \infty)$, and $\log x$ denotes the natural logarithm of x. Gronau and Matkowski [7], [8] studied geometrically convex properties of the gamma function Γ in the context of some functional equations, and they obtained generalizations of Bohr–Mollerup type theorems. In 1990 Lucht found a new constant γ determining the intervals on which Γ is geometrically concave and geometrically convex ([16], Satz 1); this problem was treated also in Gronau [6], pp. 68–70.

Geometrically convex [geometrically concave] functions turn out to fulfil a kind of super[sub-]multiplicative inequalities. We give ample characterizations to these functions as well as sufficient conditions for super- or submultiplicativity (Section 2). The characterizations of geometrically convex and concave functions are given in Theorem 1 (the general case) and in Theorem 2 (twice differentiable functions).

Although the idea of using the transformation $f \mapsto \widehat{f}$ is not new (cf. [10] or [7]), it yields new and nontrivial results for classical functions. For example, Theorem 2 leads to the amazing consequences for functions of class C^2: strictly increasing functions fulfil some kind of supermultiplicativity (Theorem 3) and superadditivity (Theorem 4). Applying Theorem 1 to the log function we obtain a new constant determining the intervals on which this function is super- and submultiplicative (Theorem 10); for the function Γ, we complete the above mentioned result of Lucht concerning the constant γ (Theorem 8). Other consequences of Theorems 1 and 2 are presented in Section 3, where we give unified proofs of some classical and new inequalities, in Section 4, dealing with inequalities for power series, and in Section 6, devoted to inequalities for trigonometric and hyperbolic functions.

The main tools of this paper are the classical Petrovič inequality ([15], p. 197):

(P) *if* F *is a convex function on the interval* $J = [0, u_0]$, *and* $x, y, x+y \in J$, *then* $F(x+y) \geq F(x) + F(y) - F(0)$, *with the reversed inequality for concave functions* (*these inequalities are strict for strictly convex or concave functions*),

and a converse of (P) which follows easily from ([15]; Theorem 2, p. 153, and

Theorem 3, p. 154):

Proposition 1. *For a continuous function* $F : [a, \infty) \to \mathbb{R}$ *the following two conditions are equivalent:*

(i) F *is convex on* $[a, \infty)$*;*

(ii) *for every* $t, s \geq 0$ *and* $x \geq a$ *we have*

$$F(t + s + x) + F(x) \geq F(t + x) + F(s + x)$$

(*with the reversed inequality in* (*ii*) *for* F *concave in* (*i*)).

We wish to point out that in many considerations regarding sub- and supermultiplitive functions, the right approach is to consider the domains $(0, 1]$ and $[1, \infty)$ separately. To begin with, some properties do not hold or become trivial when we consider them defined at large; for instance (see [15]; Theorem 2, p. 409, the additive and more general case; rediscovered in [17], Theorem 1.6, for the multiplicative case):

Proposition 2. *If a submultiplicative function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *is differentiable at* $x = 1$ *and* $f(1) = 1$*, then* $f(x) = x^p$ *for all* $x \in \mathbb{R}^+$ *and some* $p \in \mathbb{R}$.

Therefore, *a non-power differentiable function* f *with* $f(1) = 1$ *can never be sub- or supermultiplicative on* $\mathbb{R}^+$. Nevertheless, there are many differentiable functions f, fulfilling the condition $f(1) = 1$, that are submultiplicative on $(0, 1]$ and supermultiplicative on $[1, \infty)$, and so on. The results presented in the next sections yield many such examples; they complete essentially the examples given e.g. in [2], [13], [17].

The above proposition can be viewed as a limitary connection between submultiplicative and power functions: it is known ([19], Lemmas 1 and 2) that if $f : \mathbb{R}^+ \to \mathbb{R}^+$ is submultiplicative and *locally upper bounded* (e.g. *measurable*), then there exist constants $A, B > 0$ and $\alpha, \beta \in \mathbb{R}$ with $Ax^\alpha \leq f(x) \leq Bx^\beta$ for all $x \in \mathbb{R}^+$ (cf. [11]; pp. 241, 244, 250). Similar upper bounds for functions submultiplicative on $(0, 1]$ or $[1, \infty)$ only have been obtained by Cooper ([2], proof of Theorem VI).

2. Main criteria

The theorems of this section yield conditions for a given function to be super- or submultiplicative on a suitable interval. We have separated the differentiable case (Theorem 2) from the general case (Theorem 1), because the characterization of twice differentiable sub[super-]multiplicative functions, via condition (v) in Theorem 2, is more appropriate for classical functions (see next sections).

Recall that I_A denotes the interval $(0, A]$ or $[A, \infty)$, where $A > 0$. Throughout the paper the symbol I will denote the interval I_1, which is understood as $(0, 1]$ when the given theorem describes the case $I_A = (0, A]$, and $I_1 = [1, \infty)$ for $I_A = [A, \infty)$. For a given function $f : (r, s) \to \mathbb{R}$ and fixed real numbers x, a, t, with $x, a > 0$, we define the following three auxiliary functions: $f^{\{x\}}$, $f_{\{a\}}$, and f_t, by the formulas $f^{\{x\}}(a) = f(ax)/f(a)$, $f_{\{a\}}(x) = f(ax)/f(a)$, and $f_t(x) = f(t + x)$, with the domains $(r/x, s/x) \cap (r, s)$ (where $x > 0$), $(r/a, s/a)$ (where $a \in (r, s)$), and $(r - t, s - t)$ (where $t \in \mathbb{R}$), respectively.

The classical theorem of Mulholland ([15]; Lemma 1, p. 198) asserts that if $\varphi : [0, \infty) \to [0, \infty)$ is continuous, strictly increasing, convex, and $\varphi(0) = 0$ (i.e., φ is an an Orlicz function [13]) then φ is geometrically convex iff φ is of the form $\varphi(x) = x \exp(h(\log x))$, for $x > 0$, where $h : \mathbb{R} \to \mathbb{R}$ is continuous, convex and increasing. Theorems 1 and 2 presented below complete this result by giving full characterization of geometrically convex functions, in terms of their both multiplicative and monotonic properties, defined on the intervals I_A.

Theorem 1. *Let $f : I_A \to \mathbb{R}^+$. Consider the following four conditions:*
(i) *The function f is geometrically convex on I_A.*
(ii) *For every $a \in I_A$ the function $f_{\{a\}}$ is supermultiplicative on I.*
(iii) *For every $a \in I_A$ and every sequence $x_1, \ldots, x_n \in (0, a]$ we have*

$$f(a)^{n-1} \cdot f((x_1 \cdot \ldots \cdot x_n)/a^{n-1}) \geq f(x_1) \cdot \ldots \cdot f(x_n).$$

(iv) *For every $x \in I$ the function $f^{\{x\}}$ is*
(a) *nonincreasing on $I_A = (0, A]$,*
(b) *nondecreasing on $I_A = [A, \infty)$.*

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). *The first implication is strict, and if f is continuous, then* (ii) $\Rightarrow$ (i)*, and in this case all four conditions are equivalent.*

These statements remain valid on replacing everywhere "convex" by "concave", "super" by "sub", and "nonincreasing" by "nondecreasing" (*and vice versa*).

Proof. Let us consider the case $I_A = (0, A]$ and $\widehat{f}$ convex, since the other cases one can prove in a similar fashion.

(i) implies (ii): by the Petrovič inequality applied to the function $[0, \infty) \ni t \mapsto \widehat{f}(-t + \log a)$, where $a \in (0, A]$.

(ii) and (iii) are equivalent. Obvious.

(ii) and (iv) are equivalent. Fix $x \in I$; then for $a_1, a_2 \in I_A$ with $a_1 \leq a_2$ we have $a_1 = za_2$ for some $z \in I$, whence, by (ii), $f(a_2) \cdot f(xza_2) \geq f(xa_2) \cdot f(za_2)$, i.e. $f^{\{x\}}(a_1) \geq f^{\{x\}}(a_2)$. Thus (ii) implies(iv), and that (iv) implies (ii) is proved similarly.

We have that (ii) does not imply (i), in general, because if F is an additive and discontinuous function on R, then the function $f(x) = \exp(F(\log x))$, $x \in \mathbb{R}^+$, fulfils the condition (ii) with $\widehat{f} = F$ nonconvex.

If f is continuous, then, by Proposition 1 applied to the function $[-\log A, \infty) \ni t \mapsto \widehat{f}(-t)$, we have that (ii) implies (i). $\square$

Remarks. 1. If f is differentiable on appropriate intervals, then the condition (iv)(a) in the above theorem holds if and only if the function $\phi(x) := xf'(x)/f(x)$ is nondecreasing on $(0, A]$; similarly, (iv)(b) holds iff ϕ is nonincreasing on $[A, \infty)$ ([13], Lemma I.5.2)

2. If $\widehat{f}$ is strictly convex, then by the strict Petrovič inequality all inequalities in Theorem 1 become strict.

In concrete situations we use the following corollary of Theorem 1. The word "separately" in condition (i) below, as well as in Theorems 7, 9, 10, 12, and 13, is justified by Proposition 2, and it means that the given inequality does not hold on the whole semiaxis $(0, \infty)$.

Corollary 1. *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonpower function, and let $A > 0$.*

(i) *If f is geometrically convex on $\mathbb{R}^+$, then for every $a > 0$ the function $f_{\{a\}}$ is supermultiplicative on $(0, 1]$ and $[1, \infty)$ separately.*

(ii) *If $\widehat{f}$ has exactly one inflection point $\log A$ with $\widehat{f}$ strictly convex on $\log(0, A]$ and strictly concave on $\log[A, \infty)$, then $f_{\{a\}}$ is*

  (a) *supermultiplicative on $(0, 1]$ if and only if $a \in (0, A]$;*

  (b) *submultiplicative on $[1, \infty)$ if and only if $a \in [A, \infty)$*

(hence, $f_{\{A\}}$ is supermultiplicative on $(0, 1]$ and submultiplicative on $[1, \infty)$).

These statements remain valid on replacing everywhere "convex" by "concave" and "super" by "sub".

Proof. Since for every $a > 0$ the function $f_{\{a\}}$ is geometrically convex whenever f is such a function, part (a) and parts "if" in (b) are immediate consequences of Theorem 1. To prove parts "only if", assume that $\widehat{f}$ is strictly convex on $\log(0, A]$ and strictly concave on $\log[A, \infty)$. It follows that for $a > A$ the function $\widehat{f_{\{a\}}}$ is strictly concave on $\log(A/a, 1]$ with $\widehat{(f_{\{a\}})}(0) = 0$ whence, by the strict Petrovič inequality, $f_{\{a\}}(xy) < f_{\{a\}}(x)f_{\{a\}}(y)$ for all $x, y \in (\sqrt{A/a}, 1)$ (the reversed inequality holds for $a \in (0, A)$ and $x, y \in (1, \sqrt{A/a})$). Therefore $f_{\{a\}}$ cannot be supermultiplicative on $(0, 1]$ for all $a > A$; similarly, $f_{\{a\}}$ is non-submultiplicative on $[1, \infty)$ for all $a \in (0, A)$.

The geometrically concave case is proved in a similar way. $\square$

Now we shall consider briefly the case of twice differentiable functions. On the class of twice differentiable functions define the operator D by the formula

$$D(F) = F'' \cdot F - (F')^2.$$

It is well known that such a function F, defined on an interval $J = (a, b)$, is log-convex on J iff $D(F)(x) \geq 0$ for all $x \in J$ (the reversed inequality holds for log-concave functions).

Theorem 2. *Let $A > 0$, and let $f : I_A \to \mathbb{R}^+$ be twice differentiable. Then all four conditions in Theorem 1 are equivalent to the following one:*

(v) *the expression* $\delta(f(x)) := f(x) \cdot f'(x) + x \cdot D(f)(x)$

is nonnegative [*nonpositive, for the geometrically concave case*] *on* I_A.

Proof. The condition (iv) in Theorem 1 holds true iff $\phi(x) = xf'(x)/f(x)$ is nondecreasing on I_A (see Remark 1) iff $\delta(f(x)) \geq 0$ on I_A. On the other hand, defining the functions F^+ and F^- on $\log[A, \infty)$ and $\log[1/A, \infty)$, respectively, by the rules $F^{\pm}(t) = f(e^{\pm t})$, we see that $D(F^{\pm})(t) = x\delta(f(x))$, where $x = e^{\pm t} \in I_A$. Hence f is geometrically convex (condition (i) in Theorem 1) iff $\delta(f(x)) \geq 0$ for all $x \in I_A$ (condition (v) above). □

Remark 3. From the above theorem we obtain immediately that *every twice differentiable, log-convex and increasing* [*log-concave and decreasing, resp.*] *function f is geometrically convex* [*geometrically concave, resp.*], but, by the use of Young's inequality, one can easily prove that this implication is true without assuming differentiability of f (cf. [7]; see also [6]; Lemma 1, p. 67). However, the class of geometrically convex functions, defined on a given interval, is essentially larger than the class of log-convex and increasing functions (defined on the same interval). An example is furnished by the Euler gamma function Γ for which $\widehat{\Gamma}$ is convex on $\log(\gamma, \infty)$ for some $\gamma \in (0, 1)$ (see Lemma 2 below) and $\log \Gamma$ is convex on $(0, \infty)$ ([24], Theorem 7.71 and Exercise 8(g), p. 472), but Γ is strictly decreasing on $(0, 1]$ ([24], Theorem 7.71 and Exercise 8(c), p. 472) hence on $(\gamma, 1)$.

As an immediate consequence of Theorem 2 we shall single out a large class of functions, including many classical functions, which locally fulfil some kind of sub- or supermultiplicativity (Theorem 3 below). The term "locally" means that if $f : (a, b) \to \mathbb{R}^+$, then for every $t \in (a, b)$ the shifted function f_t fulfils on a right neighbourhood of the origin (depending on t) the inequality described in Theorem 1 (iii). A more precise description of this property is given in the following definition.

Definition 1. Let (a, b) be an interval with $-\infty \leq a < b \leq \infty$. A function $f : (a, b) \to \mathbb{R}^+$ is said to be *locally supermultiplicative* provided that for every $t \in (a, b)$ there exists $A_t \in (0, b - t)$ such that the function $(f_t)_{\{A_t\}}$ is supermultiplicative on $(0, 1]$; equivalently (see Theorem 1), $f(A_t + t) \cdot f((xy + t)/A_t) \geq f(x + t) \cdot f(y + t)$ for all $x, y \in [0, A_t]$ for local supermultiplicativity, with this inequality reversed for the corresponding notion of *local submultiplicativity.*

The next theorem can be now presented in a concise form.

Theorem 3. *Let (a,b) be an interval with $-\infty \le a < b \le \infty$. Every strictly increasing [strictly decreasing] function $f:(a,b)\to\mathbb{R}^+$ of class C^2 is locally super[sub-]multiplicative.*

Proof. Let f be positive and strictly increasing on (a,b). For every $t\in(a,b)$ we have $\delta(f_t(0)) = f(t)\cdot f'(t) > 0$. Since the function $x\mapsto \delta(f_t(x))$ is continuous on $(0,b-t)$, for every $t\in(a,b)$ there exists A_t with $\delta(f_t(x))>0$ for all $x\in[0,A_t]$. By Theorem 2, the function f is locally supermultiplicative. The "bracketed case" is proved similarly. □

The additive version of Theorem 3 one obtains by considering functions of the form $f=\breve{F}$, where $\breve{F}(x) := \exp F(\log x)$. To shorten the text, we introduce the notion of somewhat super[sub-]additive functions; this notion allows us to construct many nontrivial examples of super[sub-]additive functions on $[0,\infty)$ (Theorem 4, Example 1), completing the examples given by Rosenbaum in [23].

Definition 2. Let (a,b) be an interval with $-\infty\le a<b\le\infty$. A function $f:(a,b)\to\mathbb{R}$ is said to be *somewhat super[sub-]additive* provided that the function $\breve{f}$ is locally super[sub-]multiplicative on (e^a,e^b), i.e. for every $t\in(a,b)$ there exists $A_t\in(0,b-t)$ such that the function $x\to\Psi^t(x):=\psi(x,t,A_t)$ is super[sub-]additive on $[0,\infty)$, where

$$\psi(x,y,z) = f(y+\log(1+z\cdot e^{-y-x})) - f(y+\log(1+z\cdot e^{-y})),$$

and $x\in[0,\infty)$, $y\in(a,b)$, $z\in(0,b-y)$.

Since f and $\breve{f}$ are increasing [decreasing] simultanously, and since for $F=\breve{f}$ we have $\widehat{F}=f$, from Theorem 3 we get immediately

Theorem 4. *Let $-\infty\le a<b\le\infty$, and let a function $f:(a,b)\to\mathbb{R}$ be strictly increasing [decreasing] and of class C^2. Then f is somewhat super[sub-]additive.*

Remark 4. The reader may observe that although $\psi(0,t,A_t)=0$, the direct verification, by means of the Petrovič inequality, if the function Ψ^t from Definition 2 have the additive properties stated in Theorem 4 provided that f is strictly monotonic, becomes very labour-consuming and probably noneffective in the case $f^{'}\cdot f^{''}<0$. Indeed, for f twice diferentiable we have

$$(\Psi^t)''(x) = Y^2\cdot[f^{''}(X)+f^{'}(X)\cdot e^x/C],$$

where $Y=C\cdot e^{-x}/(1+C\cdot e^{-x})$, $X=t+\log(1+C\cdot e^{-x})$, and $C=A_t\cdot e^{-t}$, and so Ψ^t can be both convex and concave on an interval whenever $f^{'}$ and $f^{''}$ have different signs on that interval.

Example 1. For $f(x) = \sin x$ we have that f is strictly increasing on the interval $(-\pi/2, \pi/2)$ and strictly decreasing on $(\pi/2, 3\pi/2)$. From Theorem 4 it follows that for every $t \in (-\pi/2, \pi/2)$ there exists $A_t \in (0, \pi/2 - t)$ such that the function $x \to \Psi^t(x) = \psi(x, t, A_t) =$

$$2 \cdot \sin\left(\log\sqrt{\frac{1 + A_t e^{-t} e^{-x}}{1 + A_t e^{-t}}}\right) \cdot \cos\left(t + \log\sqrt{(1 + A_t e^{-t} e^{-x})(1 + A_t e^{-t})}\right)$$

is superadditive on $[0, \infty)$, and for every $t \in (\pi/2, 3\pi/2)$ there exists $B_t \in (0, 3\pi/2 - t)$ with $\Phi^t(x) := \psi(x, t, B_t)$ subadditive on $[0, \infty)$. We also have that f is strictly concave on $(0, \pi)$ and strictly convex both on $(-\pi/2, 0)$ and $(\pi, 3\pi/2)$, which illustrates the problem described in Remark 4.

3. Applications to multiplicative inequalities

In this section we apply the results presented above to give unified proofs of some known and new inequalities. It appears that these inequalities are nothing more than the results of multiplicative properties of concrete functions.

Theorem 5. *Let I denote the interval $(0, 1]$ or $[1, \infty)$.*

(i) *For $x_1, \ldots, x_n \in I$ we have*

$$\prod_{k=1}^{n} x_k + (n-1) \geq \sum_{k=1}^{n} x_k, \tag{1}$$

$$2^{n-1} \cdot \left(1 + \prod_{k=1}^{n} x_k\right) \geq \prod_{k=1}^{n} (1 + x_k). \tag{2}$$

(ii) *For every $a \in (0, 1)$ and $x_1, \ldots, x_n \in (0, a)$ we have*

$$\left(\frac{1}{a} - 1\right)^{n-1} \cdot \left(a^n - \prod_{k=1}^{n} x_k\right) \leq \prod_{k=1}^{n} (1 - x_k); \tag{3}$$

equivalently, for every $\epsilon > 0$ and $x_1, \cdots, x_n \in (0, 1]$ we have

$$\left(\frac{\epsilon}{1+\epsilon}\right)^{n-1} \cdot \left(1 - \frac{1}{1+\epsilon} \prod_{k=1}^{n} x_k\right) \leq \prod_{k=1}^{n} \left(1 - \frac{x_k}{1+\epsilon}\right). \tag{4}$$

Putting $\epsilon = 1$ in (4) we obtain

Corollary 2. *For every $x_1, \ldots, x_n \in (0, 1]$ we have*

$$2 - \prod_{k=1}^{n} x_k \leq \prod_{k=1}^{n} (2 - x_k). \tag{5}$$

The inequality (1) above is another form of the classical Weierstrass product inequalities, and the inequality (2) is an improvement of $(\prod_{k=1}^{n}(1-x_k))^{-1} \geq \prod_{k=1}^{n}(1+x_k)$ for values of $x_k \in (0,1)$ near to one (see [20], p. 210); the inequalities (3), and (4) seem to be new. Another proof of (5) is given in Example 3.

Proof of Theorem 5. (i) For the function $f(x) = e^{x-1}$ we have $f(1) = 1$ and that $\widehat{f}(t) = e^t - 1$ is convex on $\mathbb{R}$, whence, by Theorem 1, f is supermultiplicative both on $(0,1]$ and $[1,\infty)$, separately; it proves (1). To prove (2) take the function $g(x) = (1+x)/2$ with $x \in (0,\infty)$, for which we have $g(1) = 1$ and $\delta(g(x)) = 1/4$ for all x's; by Theorem 2, g is supermultiplicative both on $(0,1]$ and $[1,\infty)$, separately, which is an equivalent form of (2).

(ii) For the function $h(x) = 1-x$, where $x \in (0,1)$, we have $\delta(h(x)) = -1$ for all x's, whence, by Theorem 2 and Theorem 1 (parts (iii) and (ii)), we get the desired inequalities (3) and (4). □

Remark 5. The convexity of $\widehat{f}$ for $f(x) = e^{x-1}$, just used in the proof of the above theorem, allows also to obtain at once the classical Young's inequality.

Another product inequalities yields the following

Theorem 6. *Let I denote the interval $(0,1]$ or $[1,\infty)$, let $B \in (0,\infty)$, and put $\xi_n = x_1 \cdot \ldots \cdot x_n$, where $x_1, \ldots, x_n \in I$. We have*

(i) *If $B > 1$, then*

$$B^{\xi_n} + (n-1)B \geq \sum_{k=1}^{n} B^{x_k}. \tag{6}$$

(ii) *If $B \in [1/e, 1)$, then (6) holds on $I = (0,1]$.*

(iii) *If $B \in (0, 1/e)$, then the inequality revrsed to (6) holds for $I = [1,\infty)$.*

Proof. It follows from Theorem 1 applied to the function $f(x) = 2^{B^x - B}$. □

4. The case of power series

The results presented here yield simple ("at a glance" in Theorem 7) and useful criteria for super[sub-]multiplicativity of some analytic functions.

Theorem 7. *Let $(a_k)_{k=-\infty}^{\infty}$ be a sequence of nonnegative real numbers with at least two positive elements and $\sum_{k=-\infty}^{\infty} a_k = 1$. Let D_f denote the domain of the function f defined by the formula*

$$f(x) = \sum_{k=-\infty}^{\infty} a_k x^k.$$

Then f is supermultiplicative
(i) *on* $(0,1]$, *if* $D_f \supset (0,1]$;
(ii) *on* $[1,\infty)$, *if* $D_f \supset [1,\infty)$;
(iii) *on* $(0,1]$ *and* $[1,\infty)$ *separately, if* $D_f \supset \mathbb{R}^+$.

Proof. We have that for all $k \in \{j : a_j > 0\}$, the functions $g^k(t) := a_k e^{kt}$ are log-convex on $\mathbb{R}$. By the result assserting that the sum of a finite number of log-convex functions is log-convex also ([22], Theorem 13), we obtain, passing to infinite series, that $\widehat{f}$ is convex. Now we apply Theorem 1, with $A = a = 1$, to the cases $D_f \supset (0,1]$ and $D_f \supset [1,\infty)$, and since $\mathrm{card}\{j : a_j > 0\} \geq 2$, from Proposition 2 it follows that the function f cannot be supermultiplicative on $\mathbb{R}^+$ in the case $D_f \supset \mathbb{R}^+$. □

From this theorem it follows that if f is a non-power function analytic on $\mathbb{R}\setminus\{0\}$ with $f(1) = 1$ and, for all $k \in \mathbb{N}$ we have $f^{(k)}(0) \geq 0$, then f is supermultiplicative both on $(0,1]$ and $[1,\infty)$, separately. Two simple examples of such functions are presented below.

Example 2. For the functions $f(x) = (x^2+1)/x$ and $g(x) = (e^x + e^{1/x})/2e$, analytic on $\mathbb{R}\setminus\{0\}$, we have that $f(1) = 1 = g(1)$, and $g(x) = \sum_{k=-\infty}^{\infty} a_k x^k$ with $a_k = (|k|!)/2e$ for $k \neq 0$ and $a_0 = 1/e$. Hence, by Theorem 7, these functions are supermultiplicative both on $(0,1]$ and $[1,\infty)$, separately only.

We shall now consider polynomials whose coefficients take both signs, positive and negative. The lemma below shows the verification if a power series with arbitrary signs of its coefficients is super- or submultiplicative becomes a very complicated problem, in general. On the other hand, Theorem 2 easily applies for trigonometric functions because these functions fulfil useful differential identities (see Section 6).

Lemma 1. *Let $W_n(x) = \sum_{k=0}^{n} a_n x^k$, where a_k are arbitrary reals. Then*

$$\delta(W_n(x)) = \sum_{k=1}^{n} a_k \cdot x^{k-1} \cdot \sum_{j=0}^{k-1} (k-j)^2 \cdot a_j \cdot x^j. \tag{7}$$

In particular, $\delta(ax^2 + bx + c) = abx^2 + 4acx + bc$.

Proof. For $n = 1$, the equality (7) is obvious. Moreover, it is not hard to check that the following identities hold true:

$$\begin{aligned}\delta(W_{n+1}(x)) &= \delta(W_n(x)) + a_{n+1}\cdot(a_1\cdot n^2\cdot x^{n+1} + a_0\cdot(n+1)^2\cdot x^n)\\ &\quad+\sum_{m=2}^{n}(n+1-m)^2\cdot a_{n+1}\cdot x^{n+m}\\ &= \delta(W_n(x)) + a_{n+1}\cdot x^n\cdot\sum_{m=0}^{n}(n+1-m)^2\cdot a_m\cdot x^m;\end{aligned}$$

now the mathematical induction argument proves (7). □

From Lemma 1 and Theorem 2 we obtain immediately that a polynomial f with positive coefficients and $f(1) = 1$ is supermultiplicative both on $(0, 1]$ and $[1, \infty)$ — a particular case of Theorem 7.

The next example shows how Theorem 2 and Lemma 1 work for concrete polynomials.

Example 3. For $f(x) = 2 - x^2$ we have that $f(x) \geq 0$ on $[-\sqrt{2}, \sqrt{2}]$, $f(1) = 1$, and (by Lemma 1) $\delta(f(x)) = -8x \leq 0$ for $x \in [0, 1]$. Hence, by Theorem 2, the function f is submultiplicative on $(0, 1]$. This is another form of inequality (5).

5. The case of gamma, log, and zeta functions

In this section we examine the multiplicative properties of the classical Euler gamma Γ, log, and the Riemann zeta ζ functions.

The first two theorems deal with the function Γ. The below auxiliary result, interesting in its own right, is due to Lucht ([16], Satz 1; cf. [6], Lemma 1, p. 68).

Lemma 2. *There exists exactly one number $\gamma \in (0, 1)$ such that $\log\gamma$ is an inflection point of the function $\widehat{\Gamma}$, with $\widehat{\Gamma}$ strictly concave on $(-\infty, \log\gamma]$ and strictly convex on $[\log\gamma, \infty)$.*

The constant γ equals $0{,}2160987\ldots$ (and $\log\gamma = -1{,}5320198\ldots$) and it is the positive solution of the equation

$$\sum_{n=1}^{\infty}\frac{2nx + x^2}{n(n+x)^2} = C$$

where C is the Euler's constant $0{,}577215\ldots$.

The above lemma and Corollary 1(ii) yield immediately:

Theorem 8. *Let γ be the number described in Lemma 2. Then the function $\Gamma_{\{a\}}$ is*

(i) *submultiplicative on $(0,1]$ if and only if $a \in (0,\gamma]$;*
(ii) *supermultiplicative on $[1,\infty)$ if and only if $a \in [\gamma,\infty)$.*

Hence, $\Gamma_{\{\gamma\}}$ is both submultiplicative on $(0,1]$ and supermultiplicative on $[1,\infty)$, and Γ is supermultiplicative on $[1,\infty)$.

Remark 6. Since for the shifted function $\Gamma_1(x) = \Gamma(x+1)$ we have $\Gamma_1(x) = x\Gamma(x)$, from the identity $(\widehat{\Gamma_1})'' = (\widehat{\Gamma})''$ and Lemma 2 it follows that the constant γ has the same role for the multiplicative properties of both functions, Γ and Γ_1.

Now let $x_{\min}$ denote the point where the function Γ reaches its only minimum on $(0,\infty)$, i.e.

$$x_{\min} = 1{,}4616\ldots$$

(see e.g. [21], p. 303), and let Γ_t be the shifted function, $t > 0$, as defined in Section 2.

Theorem 9.

(i) *For every $t \geq x_{\min}$ and $a > 0$ we have that the functions*

$$x \mapsto (\Gamma_t)_{\{a\}}(x) = \Gamma(ax+t)/\Gamma(a+t)$$

are supermultiplicative both on $(0,1]$ and $[1,\infty)$, separately.

(ii) *The function Γ is locally sub[super-]multiplicative and somewhat sub-[super-]-additive on the interval $(0,x_{\min}]$ [respectively, on $[x_{\min},\infty)$].*

Proof. (i) Since Γ is log-convex on $(0,\infty)$ ([24], Theorem 7.71 and Exercise 8(g), p. 472) and increasing on $[x_{\min},\infty)$, the function Γ_t shares these properties on $[0,\infty)$. The result now follows from Remark 2 and Theorem 1.

(ii) It follows from Theorems 3 and 4. □

The function $x \mapsto \log(a+x)$ was studied by Gustavsson, Maligranda and Peetre [9], and the authors proved it is submultiplicative on $[1,\infty)$ if and only if $a \geq a_0 = 1{,}755069\ldots$. The next theorem completes the above result to those values of a and intervals for which the functions $L_a(x) := \log(a+x)/\log(a+1)$ are super- or submultiplicative. For this purpose let ξ denote the positive solution of the equation $\log(1+a) = a$, i.e. $\xi = 1{,}2399778876\ldots$.

Theorem 10. *With the above notations, the function L_a is:*

(i) *submultiplicative on $(0,1]$ and $[1,\infty)$ separately, for $a = 1$;*
(ii) *submultiplicative on $[1,\infty)$ if and only if $a \in (0,\xi]$;*
(iii) *supermultiplicative on $(0,1]$ if and only if $a \in [\xi,\infty)$.*

In particular, the function L_ξ is supermultiplicative on $(0,1]$ and submultiplicative on $[1,\infty)$.

Proof. For the function $f(x) = \log x$ we have that the shifted function $f_a(x) = f(a+x)$, with $a > 0$, is positive if and only if $x \in (1-a, \infty)$. Moreover,

$$(a+x)^2 \cdot \delta(f_a(x)) = \phi_a(x) := a\log(a+x) - x. \tag{8}$$

Part (i). If $a = 1$, then $\delta(f_a(x)) \leq 0$ for all $x \geq 0$. By Theorems 1, 2 and Corollary 1, for every $\beta > 0$ the function $x \mapsto \log(1+\beta x)/\log(1+\beta)$ is submultiplicative on $(0,1]$ and $[1,\infty)$ separately. Taking $\beta = 1$ we get (i).

Parts (ii) *and* (iii). Note first that if $a < 1$ [respectively, $a > 1$] then the domain of $\widehat{f_a}$ equals $(\log(1-a), \infty)$ [respectively, $\mathbb{R}$]. Since the function ϕ_a is decreasing and, for every $a > 0$ we have $\lim_{x\to\infty}\phi_a(x) = -\infty$ and $\phi_a(0) = a\log a$, it follows from Theorem 2 that

- $(*)$ $\widehat{f_a}$ is strictly concave on $(\log(1-a), \infty)$ iff $a < 1$, and
- $(**)$ $\widehat{f_a}$ has single inflection point $\log x_a$, where x_a is the solution of the equation $\phi_a(x) = 0$, iff $a \geq 1$ (with f_a geometrically convex on $(-\infty, x_a]$).

By $(*)$ and Theorem 1, for every $y > 1-a$ the function $x \mapsto \log(a+xy)/\log(a+y)$ is submultiplicative on $[1,\infty)$ whenever $a \in (0,1]$; in particular, for $y = 1$, the function L_a is submultiplicative on $[1,\infty)$ if $a \in (0,1]$.

Now consider the case $a > 1$. By $(**)$ and Corollary 1(ii), the functions $x \mapsto \log(a+rx)/\log(a+r)$ are (still) submultiplicative on $[1,\infty)$ for $r \geq x_a$ and supermultiplicative on $(0,1]$ for $r \in (0, x_a]$. Putting $r = 1$ we get that L_a is submultiplicative on $[1,\infty)$ (and supermultiplicative on $(0,1]$, respectively) for those $a > 1$ that fulfil the condition $x_a \leq 1$ (and $x_a \geq 1$, respectively). By the definition of x_a (see $(**)$), the function $a \mapsto y = x_a$ is well defined and it is given in the form $M(a,y) := \phi_a(y) = 0$, where ϕ_a is defined in (8). Since now $a > 1$, we have the deriviative $dy/da = y^{-1}[(a+y)\log(a+y) + a]$ is positive; it follows that $x_a \leq 1$ if and only if $a \in (1, \xi]$ (and $x_a \geq 1$ if and only if $a \in [\xi, \infty)$, respectively). We thus have proved parts "if" both for (ii) and (iii). Parts "only if" hold true by virtue of Corollary 1(ii). □

The last theorem of this section concerns the classical Riemann zeta function $\zeta(x) = \sum_{n=1}^{\infty} n^{-x}$, where $x > 1$.

Theorem 11. *For the Riemann function ζ we have:*

- (i) *ζ is locally submultiplicative and somewhat subadditive on $(1,\infty)$;*
- (ii) *for every $a > 1$ the function $\zeta_{\{a\}}$ is supermultiplicative on $(1,\infty)$;*
- (iii) *for exery $x > 1$ the function $\zeta^{\{x\}}$ is increasing on $(1,\infty)$; in particular, $\zeta(2a)/\zeta(a) \geq \pi^2/15$ for all $a \geq 2$.*

Proof. Part (i) follows easily from Theorems 3 and 4. According to Euler's formula ([25], p. 1), we have $1/\zeta(x) = \prod_{n=1}^{\infty} g_n(x)$, where $g_n(x) = 1 - p_n^{-x}$ and the p_n's denote all prime numbers, $n = 1, 2, \ldots$. Since

$$\delta(g_n(x)) = (1 - p_n^{-x} - x \cdot \log p_n) \cdot p_n^{-x} \log p_n$$

is negative for all n's and $x > 1$, from Theorem 2 it follows that the function $\widehat{\zeta}(t) = -\sum_{n=1}^{\infty} \widehat{g_n}(t)$ is convex on $(0, \infty)$. Now Theorems 1 and 2 prove parts (ii) and (iii).

Taking $a = 2$ and noting that $\zeta(4) = \pi^4/90$ and $\zeta(2) = \pi^2/6$, by Theorem 1(iv) we obtain the particular part of (iii). □

6. Other classical functions

We shall now apply the results of Section 2 to show that trigonometric, cyclometric, and hyperbolic functions satisfy multiplicative inequalities on $(0, 1]$ or $[1, \infty)$.

Theorem 12.

(i) *The functions* $\sin x/\sin 1$, $\sin(\pi x/2)$, $\cot x/\cot 1$, *and* $\cot(\pi x/4)$ *are submultiplicative on* $(0, 1]$.

(ii) *The functions* $\tan x/\tan 1$, *and* $\tan(\pi x/4)$ *are supermultiplicative on* $(0, 1]$.

(iii) *The function* $(2/\pi) \cdot \arcsin x$ *is supermultiplicative, and for every* $a \in (0, 1)$ *the function* $\arccos(ax)/\arccos a$ *is submultiplicative on* $(0, 1]$.

(iv) *The functions* $(4/\pi) \cdot \arctan x$ *and* $(4/\pi) \cdot \operatorname{arccot} x$ *are submultiplicative both on* $(0, 1]$ *and* $[1, \infty)$*, separately.*

Proof. Parts (i) and (ii) follow directly from Theorems 1 and 2. Indeed, for $x \in (0, 1]$ we have: $\delta(\sin x) = \sin x \cdot \cos x - x \le 0$, $\delta(\cos x) = -\sin x \cdot \cos x - x \le 0$, and $\delta(\tan x) = \tan x + \tan^3 x + \tan^4 x - x \ge 0$. Another proof follows from Theorem 7, since on the interval $(-\pi/2, \pi/2)$ the functions $\frac{1}{\cos x}$, $\frac{1}{\sin x} - \frac{1}{x}$, and $\tan x$ have Maclaurin expansions with positive coefficients.

Parts (iii) and (iv) can be obtained from parts (i) and (ii). Alternatively, by means of Theorems 1 and 2 again, for $x \in (0, 1]$ we have $(1 - x^2)^{3/2} \cdot \delta(\arcsin x) = \arcsin x - x\sqrt{1 - x^2} \ge 0$, and $(1 - x^2)^{3/2} \cdot \delta(\arccos x) = -(1 - x^2) \cdot \arccos x - x \cdot (x \cdot \arccos x + \sqrt{1 - x^2}) \le 0$. Moreover, for $x \in (0, \infty)$ we have $(1 + x^2)^2 \cdot \delta(\arctan x) = \arctan x - x - x^2 \cdot \arctan x \le 0$, and $(1 + x^2)^2 \cdot \delta(\operatorname{arccot} x) = x^2 \cdot \operatorname{arccot} x - x - \operatorname{arccot} x := \Psi(x) \le 0$ (because Ψ is decreasing on $(0, \infty)$ and takes value $-\pi/2$ for $x = 0$). □

Similar inequalities can be obtained for hyperbolic functions, e.g. applying Theorem 7 to the functions $\sinh x$ and $\cosh x$ one gets

Theorem 13. *The functions* $(2e/(e^2 - 1)) \cdot \sinh x$ *and* $(2e/(e^2 + 1)) \cdot \sinh x$ *are supermultiplicative both on* $(0, 1]$ *and* $[1, \infty)$*, separately.*

7. Final remarks

We wish to remark that, even though the above inequalities hold for *real* functions, some may be applied to holomorphic functions (i.e. their real and imaginary parts, or their modules). For functions the domains of which contain neither $(0, A]$ nor $[A, \infty)$ for every $A > 0$, one can use proper shifts to obtain similar decompositions. This can be done for functions such as $\zeta(z)$, $\Gamma(z)$, and $\log(z)$, etc.

References

[1] E. A. Bishop, *Holomorphic completion, analytic continuation, and the interpolation of seminorms*, Ann. Math. *78* (1963), 468–500.

[2] R. Cooper, *The converses of the Cauchy–Hölder inequality and the solutions of the inequality $g(x+y) \le g(x) + g(y)$*, Proc. London Math. Soc. *26* (1927), 285–291.

[3] C. E. Finol, *On dilation functions and some applications*, Publicationes Math. *33* (1986), 307–322.

[4] C. E. Finol and L. Maligranda, *On a decomposition of some functions*, Commentationes Math. (Prace Mat.) *30* (1991), 285–291.

[5] W. T. Gowers, *A solution to Banach's hyperplane problem*, Bull. London Math. Soc. *26* (1994), 523–530.

[6] D. Gronau, *Selected topics on functional equations*, Functional Analysis, IV (Dubrownik, 1993), 63–84, Various Publ. Ser. *43*, Aarhus Univ., Aarhus, 1994.

[7] D. Gronau and J. Matkowski, *Geometrical convexity and generalization of the Bohr–Mollerup theorem on the gamma function*, Math. Pannon. *4* (1993), 153–160.

[8] D. Gronau and J. Matkowski, *Geometrically convex solutions of certain difference equations and generalized Bohr–Mollerup type theorems*, Results Math. *26* (1994), 290–297.

[9] J. Gustavsson, L. Maligranda and J. Peetre, *A submultiplicative function*, Indag. Math. *51* (1989), 435–442.

[10] G. A. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1934.

[11] E. Hille and R. S. Philips, *Functional Analysis and Semigroups*, American Mathematical Society, Providence, 1957.

[12] H. Hudzik and L. Maligranda, *Some remarks on submultiplicative Orlicz functions*, Indag. Math. (N.S.) *3* (1992), 313–321.

[13] M. A. Krasnosielskii and Ya. B. Ruticki, *Convex Functions and Orlicz Spaces*, Noordhoff Ltd., Groningen, 1961.

[14] S. G. Krein, Ju. I. Petunin and E. M. Semenov, *Interpolation of Linear Operators*, American Mathematical Society, Providence, 1982.

[15] M. Kuczma, *Introduction to the Theory of Functional Equations and Inequalities, Cauchy's Equation and Jensen's Inequality*, Polish Scientific Publishers, Warszawa–Kraków–Katowice, 1985.

[16] L. G. Lucht, *Mittelwertungleichungen für Lösungen gewisser Differenzengleichungen*, Aequationes Math. *39* (1990), 204–209.

[17] L. Maligranda, *Indices and Interpolation*, Dissertationes Math. (Rozprawy Mat.) *234* (1985), 1–52.

[18] J. Matkowski, *L^p-like paranorms*, Selected Topics in Functional Equations and Iteration Theory, Proceedings of the Austrian-Polish Seminar, Graz 1991, Grazer Math. Ber. *316* (1992), 103–138.

[19] J. Matkowski, *Subadditive functions and relaxation of the homogenity condition of seminorms*, Proc. Amer. Math. Soc. *117* (1993), 991–1000.

[20] D. S. Mitrinovič, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[21] J. Pierpont, *Functions of a Complex Variable*, Dover Publ., New York, 1914.
[22] A. W. Roberts and D. E. Varberg, *Convex Functions*, Academic Press, New York, 1973.
[23] R. A. Rosenbaum, *Subadditive functions*, Duke Math. J. *17* (1950), 227–242.
[24] K. R. Stromberg, *An Introduction to Classical Real Analysis*, Brooks/Coole Publishing Company, Pacific Grove, 1981.
[25] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Oxford University Press, Oxford, 1967.
[26] M. Zafran, *The dichotomy problem for homogeneous Banach algebras*, Ann. Math. *108* (1978), 97–105.

C. E. Finol
Departamento de Matemática
Facultad de Ciencias
Universidad Central de Venezuela
P. O. Box 40645
Caracas 1040-A
Venezuela
e-mail: cfinol@euler.ciens.ucv.ve

M. Wójtowicz
Institute of Mathematics
Pedagogical University
Pl. Słowiański 9
PL–65-069 Zielona Góra
Poland
e-mail: mwojt@lord.wsp.zgora.pl

Manuscript received: May 28, 1998 and, in final form, March 2, 1999.