

The tadpole conjecture in the interior of moduli space

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ABSTRACT: We revisit moduli stabilization on Calabi-Yau manifolds with a discrete symmetry. Invariant fluxes allow for a truncation to a symmetric locus in complex structure moduli space and hence drastically reduce the moduli stabilization problem in its dimensionality. This makes them an ideal testing ground for the tadpole conjecture. For a large class of fourfolds, we show that an invariant flux with non-zero on-shell superpotential on the symmetric locus necessarily stabilizes at least 60% of the complex structure moduli. In case this invariant flux induces a relatively small tadpole, it is thus possible to bypass the bound predicted by the tadpole conjecture at these special loci. As an example, we discuss a Calabi-Yau hypersurface with $h^{3,1} = 3878$ and show that we can stabilize at least 4932 real moduli with a flux that induces M2-charge $N_{\text{flux}} = 3$.

KEYWORDS: Flux Compactifications, Superstring Vacua, M-Theory

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1 Introduction

String Theory famously allows for an enormously large number of possible compactifications, giving rise to a huge Landscape of vacua and effective lower-dimensional models. The Swampland Program [1–5] aims at finding general criteria that distinguish effective field theories (EFTs) that can be found within the String Theory Landscape from those which fundamentally cannot arise from String Theory. More generally, it tries to determine which a priori consistent EFTs allow for a UV completion in quantum gravity.¹

Evidence for these criteria (often dubbed “Swampland Conjectures”) typically stems from two different ways of reasoning: either from the bottom-up by studying general properties of gravitational systems, usually the semi-classical physics of black holes, or by collecting and generalizing patterns found in explicit top-down String Theory examples. The latter approach, however, is often limited by the fact that controlled and calculable examples are mostly only available at weak coupling and/or unbroken supersymmetry. This of course poses the risk of extrapolating from specific properties of weakly-coupled, supersymmetric compactifications to the whole landscape, falling victim to a “lamppost effect”. Despite these dangers, weak coupling limits play a particularly prominent role within the Swampland Program. It is generally believed that quantum gravity theories do not allow for any free parameters. Therefore, coupling constants must be dynamical and realized as the expectation values of scalar fields. This implies that every weak coupling limit corresponds to an asymptotic boundary of the scalar field space. In particular, the points where a coupling constant vanishes and a global symmetry would be restored lie at infinite distance from any point in the interior of the scalar field space [1].

¹One might of course be tempted to speculate whether these two questions are actually the same.

Even though conceptually important, the often high-dimensional moduli spaces of Calabi-Yau compactifications pose a serious challenge for string phenomenology. Large numbers of massless (or light) scalar fields are not observed and must be stabilized if one wants to obtain stable, realistic string theory vacua. For the complex structure moduli of IIB or F-theory compactifications this can be achieved by introducing non-trivial three or four-form fluxes on the Calabi-Yau manifold’s middle cohomology [6, 7]. The large combinatorial number of such fluxes on topologically complicated Calabi-Yau manifolds has led to statistical estimates on the enormous size of the String Theory Landscape [8, 9]. However, the intuition that any choice of fluxes, even if supported on only a small number of cycles, can give rise to a consistent vacuum where a large number of moduli is stabilized has recently been challenged by the “tadpole conjecture” [10]. This conjecture states that the fluxes that are needed to stabilize a large number of moduli carry a charge that is directly proportional to the number of stabilized moduli. As this charge contributes to the tadpole cancellation condition and is therefore bounded from above, this statement — if true — drastically reduces the size of the landscape of viable vacua.

For phenomenological purposes and in order to obtain a stable compactification, it is generally desirable to stabilize all complex structure moduli by fluxes, with no (or at most a compact) residual moduli space of unstabilized moduli. On the other hand, one might ask whether the linear scaling behavior of the tadpole conjecture persists also for partial moduli stabilization. In this work, we therefore suggest to refine the tadpole conjecture into a stronger and a weaker version. The stronger version states that for the stabilization of any number, n_{stab} , of moduli the required flux induces an M2-brane tadpole is proportional to n_{stab} . Instead, the weaker condition only requires that in case *all* complex structure moduli of a given fourfold are to be stabilized, the required flux has to induce a tadpole of the order of $h^{3,1}$. Clearly, the weaker version is implied by the stronger one.

First evidence for the tadpole conjecture was provided by an extensive, computer-aided search on $K3 \times K3$ backgrounds [11] where it was found that full moduli stabilization can only be obtained within the tadpole bound at special, singular gauge-enhancement points deep in the interior of moduli space. Yet soon after, general stabilization schemes were found that seemingly violate the tadpole conjecture in the large complex structure regime of general three and four-folds [12]. However, it was pointed out in [13–15] that in order to retain control over the large complex structure expansion generically fluxes with large charges are needed, compatible with the tadpole conjecture. Notice, that the requirement of control over the large complex structure expansion is the same as staying sufficiently close to the asymptotic boundary of scalar field space. Subsequently, using recent insights on asymptotic Hodge structure, the linear scaling relation between the induced charge of the fluxes and the number of stabilized moduli was confirmed for all strict asymptotic limits in complex structure moduli space [16].

Consequently, and in the spirit of the discussion above, the question naturally arises if the tadpole conjecture remains generally true even in the interior of moduli space or if it is only a consequence of stabilizing moduli in its asymptotic regions. This is the motivating question of this paper. Unfortunately, explicitly stabilizing a large number of moduli in the interior of the scalar field space, far away from any asymptotic regime such as the large

complex structure region is a difficult task. The required period integrals are typically only known for Calabi-Yau manifolds with a small number of moduli or in the large complex structure limit and are generally expensive to compute.

Therefore, we rely on a technique that has already been used frequently in earlier examples of explicit flux compactifications [17–23]. Calabi-Yau manifolds that exhibit a discrete symmetry acting on their moduli spaces allow for a consistent truncation on the invariant locus in moduli space. This requires a choice of fluxes that are invariant under the discrete symmetry group and reduces the problem drastically in its dimensionality. Instead of solving the vacuum (F-term) equations for all moduli explicitly, one only needs to solve them for the relatively small subset of invariant moduli. The vacuum equations of the non-invariant moduli are solved automatically by virtue of the discrete symmetry. Still, a solution to the vacuum equations does not guarantee stabilization of the non-invariant moduli since there may remain directions orthogonal to the invariant locus that are not obstructed by the flux-induced potential. For special points along the invariant locus, namely the Fermat point, the number of directions obstructed by invariant flux have been counted in [23, 24] with the result that the scaling between number of stabilized moduli and the tadpole is indeed as required by the tadpole conjecture.

In this work, we address the tadpole conjecture at generic points along the invariant locus, away from the Fermat point. To that end, we do not directly address the moduli stabilization problem in type IIB/F-theory but focus on Calabi-Yau fourfold compactifications of M-theory with G_4 -flux. We further concentrate entirely on the complex structure sector by restricting to primitive G_4 -fluxes. Thus, if the fourfold is elliptically fibered, we can translate our results directly to 4d F-theory compactification on the same fourfold. To exploit the symmetry arguments outlined above, we focus on a special class of fourfolds obtained as hypersurfaces in weighted projective space. There are 3462 of such Calabi-Yau fourfolds which typically have a very high-dimensional complex structure moduli space $h^{3,1} \sim \mathcal{O}(1000)$ making them ideally suited to discuss stabilization of large number of complex structure moduli. The loci in complex structure moduli space along which there is an enhanced discrete symmetry, on the other hand, have relatively small dimension of $h_{\text{inv}}^{3,1} \sim \mathcal{O}(10)$. Therefore the reduced moduli stabilization problem on the invariant locus by means of invariant fluxes becomes a tractable problem.

The question pertinent to our analysis in this paper is whether a choice of G -invariant flux, that leads to a solution to the F-term equation with $W_0 \equiv W_{D_a W=0} \neq 0$, is capable of stabilizing a large number of non-invariant moduli. To that end, we do not assume the vacuum to be located at any particular point along the invariant locus. In this regard our analysis differs from previous work [23, 24]. Of particular interest for us is the effect of the non-zero W_0 for the stabilization of the non-invariant complex structure moduli. As was noticed already in [25], a non-zero W_0 automatically obstructs half of the real deformations such that the residual moduli space has at most real dimension $h^{3,1}$. We therefore propose that in this case the tadpole conjecture should relate the flux-induced tadpole to the number, n_{stab} , of real moduli that are not automatically stabilized. Still, naively a non-zero W_0 yields an obstruction to all deformation away from the invariant locus. However, additional obstructing terms can originate from the Hessian D^2W which in principle can cancel an obstruction induced by $W_0 \neq 0$ if the rank of D^2W is sufficiently large.

To see whether it is conceivable that a full cancellation takes place we perform a scan over (a certain subset) of the hypersurface Calabi-Yau fourfolds in weighted projective space and numerically compute the rank of D^2W at a generic point along the invariant locus. As a result we provide a lower bound on the number n_{stab} of non-invariant directions that are obstructed by a choice of a self-dual invariant flux leading to $W_0 \neq 0$:

$$n_{\text{stab}} \gtrsim 0.2 (h^{3,1} - h_{\text{inv.}}^{3,1}). \tag{1.1}$$

Thus, for $h^{3,1} \gg h_{\text{inv.}}^{3,1}$ it is possible to stabilize a large number of complex structure fields just by solving the reduced stabilization problem along the invariant locus. We illustrate that this is indeed possible in an explicit example where it is possible to stabilize at least $n_{\text{stab}} = 1052$ additional real moduli with a flux leading to a tadpole of $N_{\text{flux}} = 3$. This shows explicitly that it is indeed possible to avoid the bound imposed by the strong form of the tadpole conjecture at special loci in the interior of the moduli space.

The rest of this paper is organized as follows: in section 2 we review M-theory flux compactifications and the tadpole conjecture. In section 3 we provide some basics of Calabi-Yau hypersurfaces in weighted projective space and their complex structure moduli spaces. In section 4 we challenge the tadpole conjecture by studying the stabilization of complex structure deformations through symmetric fluxes. To illustrate the difference between the asymptotic regions and the interior points in the moduli space, we compare the moduli stabilization at the symmetric locus with the situation in asymptotic regimes in section 5. Finally, we discuss our results in section 6.

2 Flux vacua and the tadpole problem

In this work we are interested in flux compactifications of M-theory/F-theory on Calabi-Yau fourfolds Y_4 . The relevant flux in this case is the M-theory four-form flux, G_4 , which has to be quantized such that [26]

$$G_4 + \frac{1}{2}c_2(Y_4) \in H^4(Y_4, \mathbb{Z}). \tag{2.1}$$

The middle cohomology for fourfolds can in general be split into a vertical part, a horizontal part and a remainder that is neither vertical nor horizontal [27]. The horizontal part can be generated by varying the holomorphic four-form Ω and allows for an orthogonal decomposition as

$$H_{\text{hor.}}^4 = H^{4,0} \oplus H^{3,1} \oplus H_{\text{hor.}}^{2,2} \oplus H^{1,3} \oplus H^{0,4}. \tag{2.2}$$

Of particular interest to us are primitive fluxes satisfying $G_4 \wedge J = 0$ for J the Kähler form on Y_4 .² For a primitive G_4 -flux, the induced classical superpotential is given by [28]

$$W = \int_{Y_4} G_4 \wedge \Omega, \tag{2.3}$$

²In case the second Chern class of Y_4 is not even, the quantization condition (2.1) forces us to add a vertical component to the flux. In the following we ignore any such component and only focus on the primitive component of G_4 which by (2.1) has to be integer quantized.

which through Ω depends on the complex structure deformations Φ^α of Y_4 of which there are $h^{3,1}$. In this work, we exclusively focus on the complex structure sector and ignore any mixing with the Kähler sector. The F-term supersymmetry conditions

$$D_\alpha W = 0, \tag{2.4}$$

for the complex structure moduli are solved by self-dual four-form flux $G_4 = *G_4$.³ In addition, for a consistent compactification the M2-brane tadpole needs to be cancelled. For M-theory on four-folds the tadpole cancellation condition reads

$$\frac{1}{2} \int_{Y_4} G_4 \wedge G_4 + N_{M2} = \frac{1}{24} \chi(Y_4), \tag{2.5}$$

where N_{M2} is the number of space-time filling M2-branes and $\chi(Y_4)$ denotes the Euler-characteristic of Y_4 . Since for a supersymmetric theory $N_{M2} \geq 0$, the possible flux choices are constrained by

$$N_{\text{flux}} \equiv \frac{1}{2} \int_{Y_4} G_4 \wedge G_4 \leq \frac{\chi(Y_4)}{24}. \tag{2.6}$$

Of particular interest for us is the dimension of the locus \mathcal{M}_{vac} in field space along which $D_\alpha W = 0$ does have a solution for a given G_4 flux. More precisely, we consider the number of directions in moduli space that are obstructed at first order, i.e., that correspond to massive directions. In the following we refer to this number as the *co-dimension* of \mathcal{M}_{vac} . In [24] this statement was formulated in a mathematically more precise way and expressed in terms of the Zariski dimension of \mathcal{M}_{vac} .

In order to find this co-dimension let us assume that we found a solution to the equations $D_a W = 0$ at a point Φ_{crit} in moduli space. Then the number of first-order obstructed directions, i.e. the co-dimension of \mathcal{M}_{vac} , is determined by the rank of the matrix

$$M = \begin{pmatrix} D_\alpha D_\beta W & \bar{D}_{\bar{\alpha}} D_\beta W \\ D_\alpha \bar{D}_{\bar{\beta}} \bar{W} & \bar{D}_{\bar{\alpha}} \bar{D}_{\bar{\beta}} \bar{W} \end{pmatrix} \Big|_{\Phi_{\text{crit}}} = \begin{pmatrix} D_\alpha D_\beta W & g_{\bar{\alpha}\beta} W \\ g_{\alpha\bar{\beta}} \bar{W} & \bar{D}_{\bar{\alpha}} \bar{D}_{\bar{\beta}} \bar{W} \end{pmatrix} \Big|_{\Phi_{\text{crit}}}, \tag{2.7}$$

where in the last step we used holomorphicity of W . The discussion of the rank of M depends significantly on whether W_0 , i.e. the value of W at Φ_{crit} , vanishes or not. If $W_0 = 0$, the rank of the matrix M is only determined by the rank of $D_\alpha D_\beta W$ and the residual moduli space is complex. On the other hand, as pointed out in [25], if $W_0 \neq 0$ and the metric $g_{\alpha\bar{\beta}}$ is non-degenerate, the residual moduli space is real and at most of dimension $h^{3,1}$. However, the actual dimension of \mathcal{M}_{vac} again depends on the details of $D_\alpha D_\beta W$.

In general, one would expect that the flux G_4 needs to be sufficiently complicated in order to solve the F-term conditions $D_\alpha W = 0$ while at the same time ensuring that $D_\alpha W$ is non-constant in many directions. However, such a complicated flux would typically lead to a large contribution to the M2-brane tadpole and stabilizing many moduli may thus not be possible within the bound (2.6). In [10] it has been conjectured that the co-dimension of

³If we want to solve *all* F-term equations including the Kähler direction and hence get actual supersymmetric vacua, the condition on the fluxes is more constraining than just self-duality as has been discussed in [29].

\mathcal{M}_{vac} , or equivalently the number, n_{stab} , of stabilized moduli, and the flux-induced tadpole are related linearly. In its original formulation [10] considers n_{stab} to be the number of stabilized *complex* fields. However, given our previous discussion, for $W_0 \neq 0$ it is more natural to count real scalar fields. In particular, half of the real directions in moduli space are automatically stabilized. Thus, in order to formulate the tadpole conjecture in a sensible way, in the case $W_0 \neq 0$ we should only count the real directions that are stabilized *in addition* to those stabilized automatically. In the following, we will therefore work with the following form of the tadpole conjecture:

Tadpole Conjecture (Strong Form).⁴ *The fluxes that stabilize a large number, $n_{\text{stab}} \gg 1$, of complex structure moduli in a smooth Calabi-Yau fourfold compactification of M-theory lead to an M2-brane tadpole that grows at least linearly in n_{stab} , i.e.*

$$N_{\text{flux}} \gtrsim \alpha n_{\text{stab}}, \quad \alpha \sim \mathcal{O}(1), \tag{2.8}$$

where n_{stab} is defined as

$$W_0 = 0 : \quad n_{\text{stab}} \equiv \text{codim}_{\mathbb{C}}(\mathcal{M}_{\text{vac}}), \tag{2.9a}$$

$$W_0 \neq 0 : \quad n_{\text{stab}} \equiv \text{codim}_{\mathbb{R}}(\mathcal{M}_{\text{vac}}) - h^{3,1}. \tag{2.9b}$$

In both cases n_{stab} can take values in $\{0, \dots, h^{3,1}\}$ with full moduli stabilization corresponding to $n_{\text{stab}} = h^{3,1}$. In fact, oftentimes the tadpole conjecture is interpreted as an obstruction to stabilize *all* complex structure moduli of a given Calabi-Yau fourfold. This motivates a formulation of a weaker form of the tadpole conjecture that only applies to the case $n_{\text{stab}} = h^{3,1}$:

Tadpole Conjecture (Weak Form). *For a fixed, smooth Calabi-Yau fourfold with $h^{3,1} \gg 1$, the fluxes that stabilize all complex structure moduli (i.e. $n_{\text{stab}} = h^{3,1}$) induce an M2-brane tadpole that is bounded as*

$$N_{\text{flux}} \gtrsim \beta h^{3,1}, \quad \beta \sim \mathcal{O}(1). \tag{2.10}$$

The tadpole conjecture has been investigated in various settings starting with [11]. Much focus has been on confirming or disproving the stronger version of the tadpole conjecture in asymptotic limits in field space [12–16]. Instead, in the following we want to study the validity of the two forms of the tadpole conjecture away from asymptotic limits in the interior of the moduli space. Progress in this direction has recently been made in the type IIB context for orientifold compactifications with $h^{1,1} = 0$ [24]. Here, we focus on a class of Calabi-Yau fourfolds that generically has $h^{3,1} \gg 1$ but still allows for explicit moduli stabilization at special points in the interior of the moduli space. We introduce some basic properties of these fourfolds in the next section before we study moduli stabilization in section 4.

⁴A related formulation of the tadpole conjecture for partial moduli stabilization (without distinguishing between the cases $W_0 = 0$ and $W_0 \neq 0$) was recently put forward in [24].

3 Hypersurface Calabi-Yau fourfolds

In this and the following section, we focus on a simple class of Calabi-Yau fourfolds that can be obtained as a hypersurface in a weighted projective space

$$\mathbb{P}_{a_1, \dots, a_6}^5 = (\mathbb{C}^6 \setminus \{0\}) / \mathbb{C}^*, \tag{3.1}$$

where the \mathbb{C}^* acts on the coordinates $x_{n=1, \dots, 6}$ of \mathbb{C}^6 as

$$(x_1, \dots, x_6) \sim (\lambda^{a_1} x_1, \dots, \lambda^{a_6} x_6). \tag{3.2}$$

A hypersurface in this weighted projective space can be described as the zero locus of a polynomial P for which the Calabi-Yau condition reads

$$d \equiv \sum_{n=1}^6 a_n = \deg P, \tag{3.3}$$

where $\deg P$ is the degree of P under the re-scaling (3.2). A simple class of Calabi-Yau fourfolds Y_4 can hence be obtained from Fermat-type polynomials of the form

$$P_0 = \sum_{n=1}^6 x_n^{d/a_n}. \tag{3.4}$$

For the Calabi-Yau defined by $P_0 = 0$ to be transverse, i.e. $P_0 = 0$ and $dP_0 = 0$ not to have a common solution, we have to require that each a_n is a divisor of d [30]. Based on these Fermat-type Calabi-Yau four-folds we can further consider the families of CYs obtained by deformations of P_0 of the form

$$P = P_0 + \sum_{\underline{\alpha}} \Phi_{\underline{\alpha}} \prod_{n=1}^6 x_n^{\alpha_n}, \tag{3.5}$$

where we sum over $\underline{\alpha} = (\alpha_1, \dots, \alpha_6)$ subject to the constraint

$$\sum_{n=1}^6 \alpha_n a_n = d.$$

The complex parameters $\Phi_{\underline{\alpha}}$ can be interpreted as complex structure deformations of the Calabi-Yau defined by P_0 . Notice that not all $\underline{\alpha}$ lead to independent deformations of P_0 . To single-out the independent deformations, we need to divide out the ideal of P_0 generated by

$$\langle \partial_{x_n} P_0 \rangle = \langle x_1^{d/a_1-1}, \dots, x_6^{d/a_6-1} \rangle. \tag{3.6}$$

We denote the number of independent polynomial deformations of P_0 by $h_{\text{poly}}^{3,1}$. In general, the \mathbb{C}^* -action of the weighted projective space in (3.1) has fixed points on the hypersurface $\{P = 0\}$ leading to orbifold singularities. These singularities can be resolved by turning on vevs for massless twisted states leading eventually to a smooth CY which has additional

moduli associated to these resolution modes.⁵ The total number of deformations can be obtained from the Landau-Ginzburg formula [33, 34]

$$\mathrm{tr} \, t^{dJ_0} \bar{t}^{d\bar{J}_0} = \sum_{l=0}^{d-1} \prod_{l \frac{a_n}{d} \in \mathbb{Z}} \frac{1 - (t\bar{t})^{d-a_n}}{1 - (t\bar{t})^{a_n}} \prod_{l \frac{a_n}{d} \notin \mathbb{Z}} (t\bar{t})^{d/2-a_n} \left(\frac{t}{\bar{t}}\right)^{d(l \frac{a_n}{d} \bmod \mathbb{Z} - 1/2)}. \quad (3.7)$$

Here t is an auxiliary variable, d the degree of P as in (3.3) and a_n are the weights of the weighted projective space as in (3.2). The degeneracy of states with (J_0, \bar{J}_0) charges $(d-p, q)$ gives the Hodge number $h^{p,q}$ of the resolved CY fourfold. Depending on the details of the resolution, the twisted moduli can contribute to both $h^{3,1}$ or $h^{1,1}$.

In this note, we will merely focus on the polynomial deformations introduced in (3.5). Among the families of CYs given by the hypersurface $\{P = 0\}$, the Fermat-type CYs are special since they are invariant under a discrete symmetry group G . We can write the generators of G as $g^{(k)} = (g_1^{(k)}, \dots, g_6^{(k)})$ which act on the coordinate x_n of $\mathbb{P}_{a_1, \dots, a_6}^5$ as

$$g^{(k)} : x_n \mapsto \exp\left(2\pi i g_n^{(k)} \frac{a_n}{d}\right) x_n. \quad (3.8)$$

For the Fermat CY to be symmetric under the action of $g^{(k)}$ the $g_n^{(k)}$ further need to satisfy

$$\sum_{n=1}^6 g_n^{(k)} a_n = 0. \quad (3.9)$$

Since the action of the generators $g^{(k)}$ has to be understood modulo the projective rescaling $x_n \rightarrow \lambda^{a_n} x_n$ not all $g^{(k)}$ satisfying (3.9) are in fact independent. In section 4.2 we illustrate this for an example with $(a_1, \dots, a_6) = (1, 1, 1, 1, 8, 12)$.

The deformations of the polynomial P_0 as in (3.5) are in general not G -invariant unless

$$\sum_{n=1}^6 \alpha_n g_n^{(k)} \frac{a_n}{d} \in \mathbb{Z}. \quad (3.10)$$

Let us denote by $h_{\mathrm{inv}}^{3,1}$ the number of such G -invariant polynomials. We may then split the complex scalar fields $\Phi_{\underline{\alpha}}$ into two sets

$$\Phi_{\underline{\alpha}} \rightarrow (\psi^a, \phi^i), \quad (3.11)$$

where the ψ^a parametrize the G -invariant deformations whereas ϕ^i are the parameters associated to polynomial deformations of P_0 not invariant under G . We are particularly interested in the locus within the deformation space of P_0 defined by $\phi^i = 0$. The polynomial P in (3.5) is G -invariant along this locus to which we refer to as *symmetric locus*. Recall that in addition to the polynomial deformations of P there can be further complex structure deformations associated to the resolution of orbifold singularities. In general the group G does not have a well-defined action on these complex structure moduli. Since in the following we want to use the symmetry G to constrain the periods the Calabi-Yau fourfold we restrict to the case that the resolution of the hypersurface $P = 0$ does not yield any additional complex structure deformations. We hence restrict to the case $h^{3,1} = h_{\mathrm{poly}}^{3,1}$.

⁵Whereas for threefolds all resolution modes have a geometric interpretation [31] this is not necessarily the case for fourfolds [32].

The symmetric locus in moduli space. Given a CY of the type discussed above with $h^{3,1} = h_{\text{poly}}^{3,1}$ we now specialize to the vicinity of the symmetric locus $(\psi^a, \phi^i) = (\psi^a, 0)$. Under $g \in G$ the non-invariant deformations ϕ^i transform as

$$\phi^i \xrightarrow{g} \beta_i(g) \phi^i, \quad \text{no summation,} \quad (3.12)$$

for some discrete complex phases $\beta_i(g)$. In a similar way to the deformations, we can split the periods of the holomorphic $(4, 0)$ -form Ω into invariant and non-invariant periods as⁶

$$(\Pi^A, \tilde{\Pi}^I) \xrightarrow{g} (\Pi^A, \gamma_I(g) \tilde{\Pi}^I), \quad \text{no summation.} \quad (3.13)$$

Here, the $\gamma_I(g)$ are again complex phases and the indices A and I run over a basis of invariant and non-invariant four-cycles, respectively. Notice that the basis $(\Pi^A, \tilde{\Pi}^I)$ does not necessarily correspond to an integer basis of periods, but for our purposes is just *some* basis of periods with nice transformation properties under G . For the following discussion, it is, however, important to note that there exists an integer basis that is compatible with this split of the periods.

Given the transformation properties of the deformations in (3.12), we can infer the dependence of the periods on the ϕ^i . For instance, around the locus $\phi^i = 0$ we have

$$\Pi^A = \pi^{A,0}(\psi^a) + \pi_{ij}^{A,2}(\psi^a) \phi^i \phi^j + \dots, \quad (3.14)$$

where $\pi_{ij}^{A,2}(\psi^a)$ is only non-zero if $\beta_i(g)\beta_j(g) = 1$ for all $g \in G$. On the other hand, the dependence of $\pi_{ij}^{A,2}$ on the invariant deformations ψ^a cannot be determined purely from symmetry considerations. In principle, it is therefore possible that $\pi_{ij}^{A,2}$ vanishes identically even if $\beta_i(g)\beta_j(g) = 1$ for all $g \in G$. On the other hand, the periods $\tilde{\Pi}^I$ have the expansion

$$\tilde{\Pi}^I = \tilde{\pi}_i^{I,1}(\psi^a) \phi^i + \tilde{\pi}_{ij}^{I,2}(\psi^a) \phi^i \phi^j + \tilde{\pi}_{ijk}^{I,3}(\psi^a) \phi^i \phi^j \phi^k + \tilde{\pi}_{ijkl}^{I,4}(\psi^a) \phi^i \phi^j \phi^k \phi^l + \dots \quad (3.15)$$

At each order in the expansions, the coefficients $\pi_{ij}^{A,2}, \tilde{\pi}_{ij}^{I,2}, \tilde{\pi}_{ijk}^{I,3}$, etc. are not all independent but satisfy certain linear relations. To see this we notice that these terms can be associated to the squares (cubes) of the polynomials in (3.5). But these polynomials are defined modulo the ideal $\langle \partial P \rangle$. At a given point along the symmetric locus $\{\phi^i = 0\}$ the ideal depends on ψ^a dictating the relative ψ^a -dependence of the different expansion coefficients in $\Pi^A, \tilde{\Pi}^I$. Moreover, the expansion coefficients can be arranged into representations of the outer automorphism group of G . Invariance with respect to outer automorphism of G requires all coefficients that lie in the same representation to have the same functional dependence on ψ^a , reducing the number of independent components further.

Notice that along the symmetric locus $\{\phi^i = 0\}$ all non-invariant periods $\{\tilde{\Pi}^I\}$ vanish as there is no constant term in the expansion (3.15) since such a term would spoil the transformation (3.13). However, the vanishing of $\tilde{\Pi}^I$ does not lead to a singularity in field space with additional massless degrees of freedoms as, e.g., there is no supersymmetric

⁶In principle the action of G on the periods could also be accompanied by a non-trivial Kähler transformation as pointed out in [19]. However, in the remainder of this work we only consider Kähler-covariant expressions such that our results are independent of this subtlety.

cycle that vanishes at the supersymmetric point. To see this, note that $dP \neq 0$ for generic ψ^a and hence the Calabi-Yau fourfold defined by $P = 0$ is non-singular along the locus $\{\phi^i = 0\}$, indicating that the volume of all 4-cycles is non-vanishing.⁷ Alternatively, we can consider the field space metric

$$g_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \left(\log \int_{Y_4} \Omega \wedge \bar{\Omega} \right). \quad (3.16)$$

Given the expansion of the periods in terms of the deformations, we infer that to leading order

$$\int_{Y_4} \Omega \wedge \bar{\Omega} = K^{(0)}(\psi^a, \bar{\psi}^b) + K_{i\bar{j}}^{(2)}(\psi^a, \bar{\psi}^b) \phi^i \bar{\phi}^{\bar{j}} + \left(K_{i\bar{j}}^{(2)}(\psi^a, \bar{\psi}^b) \phi^i \phi^{\bar{j}} + \text{c.c.} \right). \quad (3.17)$$

The Kähler potential is invariant under G up to a Kähler transformation. Since $K_{i\bar{j}}^{(2)}$ is a function of the invariant moduli ψ^a only, it is invariant under G . Thus $K_{i\bar{j}}^{(2)} = 0$ unless the product $\phi^i \bar{\phi}^{\bar{j}}$ is invariant under G . Similarly, $K_{i\bar{j}}^{(2)}$ can only be made up of terms stemming from the invariant periods Π^A , i.e.

$$K_{i\bar{j}}^{(2)}(\psi^a, \bar{\psi}^b) = \pi_{i\bar{j}}^{A,2} \eta_{AB} \bar{\pi}^{B,0}, \quad (3.18)$$

where η_{AB} is the intersection form restricted to the invariant sector. From the Kähler potential, we can derive the metric on field space along the symmetric locus

$$g_{i\bar{j}} = \frac{K_{i\bar{j}}^{(2)}(\psi^a, \bar{\psi}^b)}{K^{(0)}(\psi^a, \bar{\psi}^b)^2}, \quad (3.19)$$

which for generic $K_{i\bar{j}}^{(2)}$ and generic values of ψ^a is non-degenerate and in particular does not feature a logarithmic divergence in the ϕ^i as would be expected for a point in field space where a supersymmetric cycle vanishes.

Setting the non-invariant deformations $\phi^i = 0$, we are left with a lower-dimensional deformation space spanned by the ψ^a . In certain cases, this space can be identified with the *full* complex structure moduli space of the mirror X_4 of Y_4 . To see this, let us recall that by the Greene-Plesser construction [35] the mirror of Y_4 can be obtained by quotienting the hypersurface $P = 0$ by the group G , i.e. $X_4 = \{P = 0\}/G$. Of course, in order to be able to quotient by G , the hypersurface $P = 0$ needs to have G as one of its symmetries which is precisely the case along $\{\phi^i = 0\}$. The G -orbifolding renders the deformations ϕ^i massive whereas the deformations ψ^a remain as massless complex structure deformations of X_4 . Still, when orbifolding by G additional singularities are induced. Resolving these orbifold singularities might again introduce additional complex structure deformations. Therefore in general the complex structure moduli space of the mirror X_4 is larger than just the invariant locus spanned by the ψ^a . However, in case there are no further complex structure deformations of X_4 associated to the resolution of the orbifold singularities the symmetric locus $\{\phi^i = 0\}$ of Y_4 is identical to the complex structure moduli space of X_4 . This implies

⁷For primitive four-cycles the volume is measured by $e^{K/2} \int |\Omega|$ which can be non-zero even if all period integrals $\int \Omega$ vanish as it happens e.g. at the Fermat point.

in particular that in this case the invariant periods Π^A can be identified with the periods of the mirror X_4 .

Given an integer basis of periods Π^A of X_4 one might be worried whether these also correspond to an integer basis of periods on the invariant locus on Y_4 or whether an integer basis of periods of Y_4 is related to that of X_4 via a rescaling by the order $|G|$ of G . To see that this is not case, let us take a closer look at the effect of the orbifolding. Therefore consider the holomorphic $(4,0)$ -form Ω which can be chosen to be of the form (cf. [36, 37] for the analogous CY 3-fold case)

$$\Omega = -\frac{f(\psi^a)}{(2\pi i)^4} \frac{x_6 dx_1 dx_2 dx_3 dx_4}{\frac{\partial P}{\partial x_5}}, \tag{3.20}$$

for some suitable function f of the invariant deformations ψ^a . Using this $(4,0)$ -form for both, Y_4 and its mirror X_4 , when integrating Ω over a basis of cycles in X_4 the orientifold identifications will lead to an extra factor of $|G|^{-1}$ [36, 37]. To identify the periods of Y_4 with the periods of X_4 we should therefore rescale $\Omega(X_4) \rightarrow |G|\Omega(Y_4)$. A rescaling of Ω does, however, not change the physics and in particular not the quantization condition. Therefore for each integer basis of periods on X_4 there exists an equivalent integer basis of periods on Y_4 without the need to introduce extra factors of $|G|$.⁸ Notice that since the periods of Y_4 on the locus $\{\phi^i = 0\}$ are identified with the periods of X_4 also $K^{(0)}(\psi^a, \bar{\psi}^b)$ appearing in (3.17) is just the Kähler potential on the mirror X_4 .

To summarize, along the symmetric locus in the complex structure moduli space of Y_4 the moduli field space metric is non-degenerate in the ϕ^i directions indicating no additional degrees of freedom. On the other hand, in the special cases discussed above, the remaining directions along the symmetric locus can be identified with the complex structure moduli space of the mirror X_4 of Y_4 . We will use these properties in the next section to analyze flux vacua along this symmetric locus.

4 Moduli stabilization at the symmetric locus

We now turn to flux compactifications of M-/F-theory on the Calabi-Yau fourfold Y_4 obtained as the hypersurface $P = 0$ in the weighted projective space $\mathbb{P}_{a_1, \dots, a_6}^6$. Therefore let us expand $G_4 = m_A \alpha^A + \tilde{m}_I \tilde{\alpha}^I$ in a basis of four-forms dual to the cycles labelled by (A, I) in the previous section. The superpotential then reads

$$W = m_A \Pi^A + \tilde{m}_I \tilde{\Pi}^I. \tag{4.1}$$

In order to test the validity of the strong form of the tadpole conjecture introduced in section 2, we take an approach to find solutions to $D_\alpha W = 0$ first employed by [17] and

⁸To illustrate this point consider the elliptic curve described by the cubic

$$Y_2 : x_1^3 + x_2^3 + x_3^3 + \psi x_1 x_2 x_3 = 0.$$

Upon orbifolding by the $G = \mathbb{Z}_3$ symmetry one obtains the same topological manifold (an elliptic curve) on which hence another integer basis of cycles with equivalent periods and intersections exists.

subsequently used in e.g. [18–20, 23]. We therefore turn on flux only along G -invariant cycles, i.e. $\tilde{m}_I = 0$, such that the superpotential reads

$$W_{\text{inv}} = m_A \Pi^A, \quad (4.2)$$

where we used the basis of periods introduced in (3.13) and chose an integer basis out of the G -invariant periods. Using the expansion of the Π^A periods in (3.14) one finds

$$\partial_{\phi^i} W|_{\{\phi^i=0\}} = 0, \quad \forall i, \quad (4.3)$$

and using that the Kähler potential is at least quadratic in the ϕ^i one deduces

$$\partial_{\phi^i} K|_{\{\phi^i=0\}} = 0, \quad \forall i. \quad (4.4)$$

Hence, along the symmetric locus, $\{\phi^i = 0\}$, we automatically solve the F-term conditions for a superpotential of the form (4.2).

Suppose we find a set of fluxes $m_A \neq 0$ for which we can also solve the remaining F-term constraints $D_a W = 0$ at a generic point along the locus $\{\phi^i = 0\}$. Let us further assume that $W_0 \neq 0$. Given these assumptions, to determine the rank of M in (2.7) let us focus on the block corresponding to $(\alpha, \beta) = (i, j)$ which we split as

$$M = \underbrace{\begin{pmatrix} D_i D_j W & 0 \\ 0 & \bar{D}_i \bar{D}_j \bar{W} \end{pmatrix}}_{\equiv M_1} + \underbrace{\begin{pmatrix} 0 & g_{ij} W \\ g_{ij} \bar{W} & 0 \end{pmatrix}}_{\equiv M_2}. \quad (4.5)$$

We saw in the previous section that the moduli space metric at a generic point along the locus $\{\phi^i = 0\}$ is non-degenerate. From our discussion in section 2 we recall that in the case $W_0 \neq 0$ indeed half of the real directions in moduli space are obstructed automatically. From the perspective of the tadpole conjecture we are thus not interested in the full rank of M but in n_{stab} as defined in (2.9b). Using that the number of cancellations that can occur between M_1 and M_2 depends on the rank of $D_i D_j W$ we can give the following lower bound on n_{stab} ,

$$n_{\text{stab}} \geq (h^{3,1} - h_{\text{inv}}^{3,1}) - \text{rank}(D_i D_j W). \quad (4.6)$$

This bound does not yet use any assumption on the number of G -invariant moduli that are stabilized directly. Assuming, however, that fluxes can be chosen in such a way that all $h_{\text{inv}}^{3,1}$ invariant moduli obtain a mass, the bound (4.6) becomes stronger,

$$n_{\text{stab}} \geq h^{3,1} - \text{rank}(D_i D_j W). \quad (4.7)$$

In the regime $n_{\text{stab}} \gg 1$ the tadpole conjecture predicts that the G_4 -flux induces a large tadpole bounded as in (2.8). To test this in the present setting, we thus require $h^{3,1} \gg 1$. Luckily, the Calabi-Yau four-folds obtained as hyper-surfaces in weighted-projective space provide a large set of examples with large $h^{3,1}$. However, in order to achieve $n_{\text{stab}} \gg 1$ we still need to provide an upper bound on the rank of the Hessian $D_i D_j W$. Importantly, following [23], we can again exploit the presence of the symmetry G to restrict $D_i D_j W$

considerably since the requirement of G -invariance enforces many elements of the matrix $D_i D_j W$ to vanish identically and hence often prohibits it from having full rank.

To see this a bit more concretely, recall that for our flux choice $\tilde{m}_I = 0$, and with the expansion (3.14) of the periods Π^A , the second covariant derivatives of the superpotential are given by

$$D_i D_j W = \pi_{ij}^{A,2} \left(m_A + \eta_{AB} \frac{\bar{\pi}^{B,0} \pi^{C,0}}{(K^{(0)})^2} m_C \right). \quad (4.8)$$

Here, we used (4.4) and from (3.17) deduced

$$g_{ij}|_{\{\phi^i=0\}} = \frac{K_{ij}^{(2)}}{(K^{(0)})^2}. \quad (4.9)$$

Hence, a given element of the Hessian $D_i D_j W$ can only be non-vanishing if $\pi_{ij}^{A,2} \neq 0$ for at least one A . On the other hand, since all Π^A are G -invariant, a necessary condition for $\pi_{ij}^{A,2}$ to be non-zero is that

$$\beta_i(g) \beta_j(g) = 1, \quad \forall g \in G, \quad (4.10)$$

where the $\beta_i(g)$ characterize the transformation of the ϕ^i under $g \in G$ as in (3.12). In particular, we have

$$\beta_i(g_0) \beta_j(g_0) \neq 1 \text{ for some } g_0 \in G \quad \Rightarrow \quad D_i D_j W \propto \pi_{ij}^{A,2} = 0. \quad (4.11)$$

Moreover, we recall from the previous section that not all $\pi_{ij}^{A,2}$ are independent but related via the Jacobi-ideal dP of P . Therefore, modding out by dP generates linear relationships between the elements of $D_i D_j W$ that potentially reduce its rank further.

For a given Fermat-type Calabi-Yau hypersurface with known symmetry group G and group actions $\beta_i(g)$, the invariant combinations (i, j) for which (4.10) is satisfied can be determined explicitly. This yields all elements of $D_i D_j W$ that are not identically zero by symmetry reasons. Assuming that these elements are all non-vanishing and independent, and taking into account the linear relations imposed by the Jacobi-ideal, one can use this information to compute the maximal possible rank of $D_i D_j W$. We do this systematically using the computer algebra system Mathematica for a large number of Fermat-type Calabi-Yau four-folds in our dataset.⁹ Hereby, we restrict to hypersurfaces with $h^{3,1} = h_{\text{poly}}^{3,1}$ and $h^{3,1} - h_{\text{inv}}^{3,1} < 5000$, resulting in a dataset of 609 different Calabi-Yau four-folds¹⁰ The results are illustrated in figure 1.

Let us now denote the rank of the matrix $D_i D_j W$ by r . We are particularly interested in the scaling behavior of r with respect to the number of non-invariant moduli $h^{3,1} - h_{\text{inv}}^{3,1}$. From figure 1 one sees that r grows approximately linear in $h^{3,1} - h_{\text{inv}}^{3,1}$. More precisely, we find

$$r \sim \gamma \left(h^{3,1} - h_{\text{inv}}^{3,1} \right), \quad \text{with} \quad \gamma \in [0.33, 0.80]. \quad (4.12)$$

⁹Our Mathematica code and data is publicly available at [38].

¹⁰We furthermore considered only the first 3000 Fermat-type four-folds and restricted each computation to maximally 8GB of memory. Therefore, there are a few Calabi-Yau four-folds with $h^{3,1} - h_{\text{inv}}^{3,1} < 5000$ that are not contained in our list.

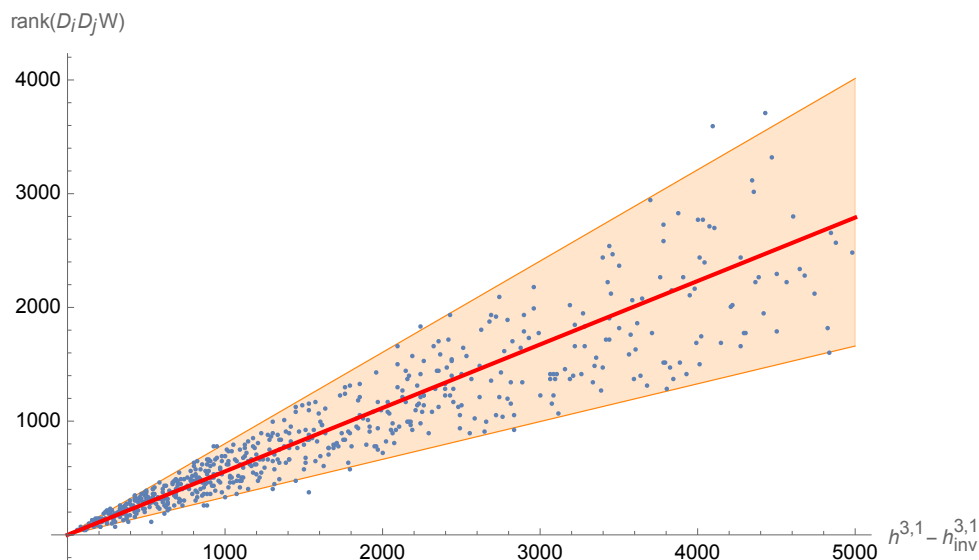


Figure 1. Shown is the maximal rank $r = \text{rk } D_i D_j W$ of the Hessian of W at a generic point along the locus $\{\phi^i = 0\}$ in relation to $h^{3,1} - h_{\text{inv}}^{3,1}$ for the cases with $h^{3,1} = h_{\text{poly}}^{3,1}$. 95% of the data points lie within the shaded region bounded by the two lines with slope $r = 0.80 (h^{3,1} - h_{\text{inv}}^{3,1})$ and $r = 0.33 (h^{3,1} - h_{\text{inv}}^{3,1})$. The red, solid line indicates the mean behavior of $r \sim 0.56 (h^{3,1} - h_{\text{inv}}^{3,1})$.

The latter bound on γ is satisfied for 95% of all Fermat-type four-folds we considered and corresponds to the shaded area in figure 1. We can use this bound and (4.6) to give a lower-bound on the number of stabilized moduli,

$$n_{\text{stab}} \geq (1 - \gamma) (h^{3,1} - h_{\text{inv}}^{3,1}) \gtrsim 0.2 (h^{3,1} - h_{\text{inv}}^{3,1}). \quad (4.13)$$

Accordingly, just by turning on a flux in the invariant directions, we can achieve a large number of stabilized moduli as long as $W_0 \neq 0$. Notice that this bound is saturated if all cancellations between the two matrices in (4.5), that are allowed by G -invariance, are actually realized. It is conceivable that in the generic case the number of such cancellations is considerably smaller or even zero, resulting in a much larger number of stabilized moduli than required by this bound.

Even though here we do not attempt to calculate the exact physical masses of the moduli, let us remark that the mass of the scalar deformations that lie the kernel of $D_i D_j W$ is set by the value of W_0 whereas for the rest the actual mass depends also on $D_i D_j W$ and thus on the form of $\pi_{ij}^{A,2}$.¹¹ On the other hand, the fermion mass matrix is given simply by $D_i D_j W$ such that we expect at least $0.2 (h^{3,1} - h_{\text{inv}}^{3,1})$ massless fermions.

Let us finally contrast the $W_0 \neq 0$ situation with the case in which we find a supersymmetric vacuum $W_0 = 0$ along the symmetric locus $\{\phi^i = 0\}$. As discussed in section 2, in

¹¹This implies that the mass of the scalar deformation modes is not universal, but depends on the exact form of the periods in the vicinity of the symmetric locus. This should be contrasted with the results of [21, 22] where it was found that the non-invariant deformations have a universal mass. To our understanding the difference is that [21, 22] use the LCS expressions also for the non-invariant periods which is not justified as the symmetric locus is far away from the LCS phase.

this case we should indeed count the number of stabilized complex scalar fields, i.e. use the definition of n_{stab} in (2.9a). For $W_0 = 0$, the matrix M_2 does not provide an obstruction for any complex structure deformations. Therefore any first-order obstruction needs to arise from $D_i D_j W$. In this case we simply have

$$n_{\text{stab}} \Big|_{W_0=0} = \text{rk}(D_i D_j W) \lesssim 0.8 (h^{3,1} - h_{\text{inv}}^{3,1}). \tag{4.14}$$

Notice that this bound assumes that all entries of $D_i D_j W$, that can be non-zero by symmetry considerations, are indeed non-vanishing. Thus, whereas for cases with $W_0 \neq 0$ we can give a lower bound on n_{stab} , for $W_0 = 0$ we only have an upper bound on n_{stab} for invariant fluxes. This matches with the results of [23, 24] where it was found that in general supersymmetric vacua at the Fermat point obtained from symmetric fluxes have a large number of flat directions.

4.1 Consequences for the tadpole conjecture

Given the bound on the co-dimension n_{stab} of \mathcal{M}_{vac} we now want to discuss the consequences for the tadpole conjecture of [10] reviewed and refined in section 2. As we discussed in the previous section, the fourfold Y_4 is in general smooth along the symmetric locus unless we tune the ψ^a to special values. Therefore at a generic point along the symmetric locus we meet the requirements of both, the strong and the weak version of the tadpole conjecture (as introduced above). In its stronger form this conjecture predicts a bound as in (2.8) relating n_{stab} and the M2-brane tadpole induced by the choice of flux. To check the validity of this statement in our setup we thus need to give an estimate for the tadpole generated by turning on invariant fluxes m_A . Following the discussion at the end of section 3, in favorable cases this is equivalent to turning on generic fluxes on the mirror X_4 since for each symmetric flux on Y_4 with tadpole N_{flux} there exists a choice of fluxes on X_4 with the same tadpole and equivalent superpotentials. The latter points follows from the fact that the periods of Y_4 and X_4 are the same. Therefore the moduli stabilization problem on the invariant locus of Y_4 is equivalent, in the favorable cases, to the moduli stabilization problem on the full complex structure moduli space of X_4 . Our problem thus reduces to finding self-dual fluxes for X_4 with $W_0 \neq 0$ and relatively small N_{flux} .

Since typically $h_{\text{inv}}^{3,1} \ll h^{3,1}$ this is a much simpler task. In particular if $h^{3,1} \lesssim \mathcal{O}(10)$ this can be done explicitly for instance in the limit $\psi^a \rightarrow \infty$, i.e., the large complex structure regime, using, e.g., the expressions derived in [12]. To find solutions to $D_a W = 0$ for such a small number of moduli in this regime, we expect that it is sufficient to turn on a flux with $N_{\text{flux}} \sim \mathcal{O}(10)$. In particular, the strong version of the tadpole conjecture itself is consistent with such an expectation. Let us therefore *assume* that there exists a bound of the form (2.8) for the $h_{\text{inv}}^{3,1}$ parameters that can marginally be satisfied. The tadpole conjecture would then require

$$\alpha h_{\text{inv}}^{3,1} \stackrel{?}{\geq} (1 - \gamma)(h^{3,1} - h_{\text{inv}}^{3,1}). \tag{4.15}$$

For $\alpha \sim \mathcal{O}(1)$ we notice that this cannot be satisfied for $h^{3,1} \gg h_{\text{inv}}^{3,1}$. Among the Fermat-type hypersurfaces there are indeed examples with $h^{3,1}/h_{\text{inv}}^{3,1} \sim \mathcal{O}(1000)$ implying that (4.15)

does not hold in general. We therefore conclude that on the symmetric locus of Y_4 the strong version of the tadpole conjecture might not hold in general. To see how this works explicitly, we present a simple example in the next section.

As reviewed in section 2, a weaker form of the tadpole conjecture only states that there is an obstruction to stabilize *all* complex structure moduli within the tadpole bound, leading to the bound (2.10). In order to apply our results to this case, we need to ensure that if the rank of the matrix M_1 along the locus $\{\phi^i = 0\}$ is non-zero, there are no cancellations between M_1 and M_2 leading to a reduction of the rank of M altogether. To check for this possibility, let us compare the two matrices at the locus $\{\phi^i = 0\}$:

$$\begin{aligned}
 (M_1)_{i,j} &= \pi_{ij}^{A,2}(\psi^a) \left(m_A + \eta_{AB} \frac{\bar{\pi}^{B,0}(\bar{\psi}^a) \pi^{C,0}(\psi^a)}{(K^{(0)}(\psi^a, \bar{\psi}^b))^2} m_C \right), \\
 (M_2)_{i\bar{j}} &= \frac{K_{i\bar{j}}^{(2)}(\psi^a, \bar{\psi}^b)}{K^{(0)}(\psi^a, \bar{\psi}^b)^2} \left[m_A \pi^{A,0}(\psi^a) \right].
 \end{aligned}
 \tag{4.16}$$

For generic ψ^a and fluxes m_A we do not expect a cancellation between these two matrices. Still, this might be the case for the special values of ψ^a for which $D_a W = 0$. We notice, however, that solving $D_a W = 0$ along the locus $\{\phi^i = 0\}$ is completely independent of the form of the functions $\pi_{ij}^{A,2}$ and $K_{i\bar{j}}^{(2)}$. We therefore do not expect that along the locus $D_a W = 0$ these two functions in general conspire to lead to a decrease of the rank of M .¹² We therefore expect that indeed the rank of M is $h^{3,1}$ if $W_0 \neq 0$. To scrutinize this expectation we would need to have the exact ψ^a -dependence of the relevant functions for general ψ^a at our disposal. To date the exact form of these functions is, however, not known for general CYs. Therefore, at this stage we cannot conclusively show that full moduli stabilization of a large number of moduli at the symmetric point is possible even though our argument indicates that there should be no fundamental obstruction to achieve that. Whereas the strong form of the tadpole conjecture turns out not be realized at the symmetric locus, it is hence still possible that the weaker form holds.

4.2 Example: $\mathbb{P}_{1,1,1,1,8,12}^5$ [24]

Let us consider a Calabi-Yau fourfold Y_4 obtained as the degree-24 hypersurface in $\mathbb{P}_{1,1,1,1,8,12}^5$ determined by the zero locus of the polynomial in Fermat form

$$P_0 = x_1^{24} + x_2^{24} + x_3^{24} + x_4^{24} + x_5^3 + x_6^2 = 0.
 \tag{4.17}$$

Naïvely, the resulting Calabi-Yau hypersurface is invariant under a $\tilde{G} = \mathbb{Z}_{24}^3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$ symmetry. The generators of \tilde{G} can be found by solving (3.9), with a possible choice given by

$$\begin{aligned}
 g^{(1)} &= (-1, 1, 0, 0, 0, 0), & g^{(2)} &= (-1, 0, 1, 0, 0, 0), & g^{(3)} &= (-1, 0, 0, 1, 0, 0), \\
 g^{(4)} &= (-8, 0, 0, 0, 1, 0), & g^{(5)} &= (-12, 0, 0, 0, 0, 1),
 \end{aligned}
 \tag{4.18}$$

¹²In contrast to that, in the next section we discuss how such a cancellation can occur in asymptotic limits.

acting on the coordinates of $\mathbb{P}_{1,1,1,1,8,12}^5$ as in (3.8). However, when taking the rescaling invariance (3.2) of the projective coordinates into account, these five generators are not all independent. For example, the following combinations can be seen to act trivially

$$\left[g^{(1)} g^{(2)} g^{(3)} \right]^4 g^{(4)} = \left[g^{(1)} g^{(2)} g^{(3)} \right]^3 g^{(5)} = 1, \tag{4.19}$$

implying that $g^{(4)}$ and $g^{(5)}$ are not independent but can be expressed in terms of the first three generators. Moreover, squaring the second equation in (4.19) gives $\left[g^{(1)} g^{(2)} g^{(3)} \right]^6 = 1$, reducing one of the \mathbb{Z}_{24} factors to \mathbb{Z}_6 . Therefore, the actual symmetry of the Calabi-Yau defined in (4.17) is

$$G = \mathbb{Z}_{24}^2 \times \mathbb{Z}_6, \tag{4.20}$$

and is spanned by the first three generators $g^{(1)}$, $g^{(2)}$, and $g^{(3)}$ in (4.18).

In total the hypersurface $\{P_0 = 0\}$ allows for 3878 polynomial complex structure deformations. This can be seen as follows: the space of independent deformations is spanned by all degree-24 polynomials in the variables x_1, \dots, x_6 , divided by the Jacobian ideal $\langle \partial P_0 \rangle = \langle x_1^{23}, x_2^{23}, x_3^{23}, x_4^{23}, x_5^2, x_6 \rangle$. A suitable basis is hence given by the monomials

$$x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4}, \quad x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3} x_4^{\beta_4} x_5, \tag{4.21}$$

where $\sum_i \alpha_i = 24$, $\alpha_i < 23$ and $\sum_i \beta_i = 16$. Since the number of degree- p monomials in $n + 1$ variables is $\binom{p+n}{n}$ we find that the total number of deformations is given by

$$\binom{27}{3} - 4 - 4 \times 3 + \binom{19}{3} = 3878, \tag{4.22}$$

where we accounted for the over-counting of monomials that violate the condition $\alpha_i < 23$.

Resolving the orbifold singularities induced by the \mathbb{C}^* -action of the projective space introduces an exceptional divisor E but no further complex structure deformations. We thus have $(h^{1,1}, h^{3,1}) = (2, 3878)$. Out of the complex structure deformations there are two deformations which respect the discrete symmetry. Using the generators $g^{(1)}$, $g^{(2)}$, and $g^{(3)}$ of $G = \mathbb{Z}_{24}^2 \times \mathbb{Z}_6$ in (4.18), one sees that they are of the form (4.21) with all powers of x_1, \dots, x_4 equal. Therefore, a degree-24 hypersurface respecting the discrete symmetry takes the form

$$P_{\text{inv.}}(\psi^0, \psi^1) = x_1^{24} + x_2^{24} + x_3^{24} + x_4^{24} + x_5^3 + x_6^2 - \psi^0 x_1^6 x_2^6 x_3^6 x_4^6 - \psi^1 x_1^4 x_2^4 x_3^4 x_4^4 x_5 = 0, \tag{4.23}$$

where we only kept terms modulo the ideal $\langle \partial P_0 \rangle$. Restricting the 3878-dimensional moduli space to the symmetric locus, where we set all polynomial deformations of P not invariant under G to zero, we are effectively left with a two-dimensional moduli space spanned by (ψ^0, ψ^1) . Notice that the hypersurface $\{P = 0\} \subset \mathbb{P}_{1,1,1,1,8,12}^5$ can be thought of as a smooth elliptic fibration over \mathbb{P}^3 .¹³ Since Y_4 has $h^{1,1} = 2$, its mirror X_4 has $h^{3,1} = 2$ such that we can treat the symmetric locus spanned by (ψ^0, ψ^1) as the full complex structure moduli space of X_4 or, equivalently, as the Kähler moduli space of Y_4 .

¹³Since it is an elliptic fibration, it allows for an uplift to F-theory such that the results we obtain in this appendix can also be lifted to the 4d case and are not a special effect of the underlying 3d M-theory.

We take this latter perspective and analyze moduli stabilization in the large volume/large complex structure regime where the mirror map identifies the complexified Kähler moduli of Y_4 can be identified with the invariant deformations (ψ^0, ψ^1) as

$$b^a + it^a = -\frac{1}{2\pi i} \log(\psi^a) + \dots, \quad \text{for } \psi^a \rightarrow \infty, \quad a = 0, 1. \quad (4.24)$$

The dots indicate terms polynomial in $1/\psi^a$ that are suppressed in the large complex structure limit. This two-dimensional moduli space has already been investigated in some detail in [12, 39] in the context of flux vacua. Let us review some details. The Kähler cone of Y_4 is generated by two divisor classes D_0, D_1 which can be identified as

$$D_0 = E + 4\pi^*H, \quad D_1 = \pi^*H, \quad (4.25)$$

where E is the exceptional divisor, i.e. the zero section of the elliptic fibration $\pi : T^2 \rightarrow \mathbb{P}^3$ and H the hyperplane class of \mathbb{P}^3 . Their intersection numbers can be summarized in the intersection polynomial

$$I(Y_4) = 64D_0^4 + 16D_0^3D_1 + 4D_0^2D_1^2 + D_0D_1^3. \quad (4.26)$$

The dual Mori cone generators $\mathcal{C}^{0,1}$ can accordingly be identified with the generic elliptic fiber and the single Mori cone generator of \mathbb{P}^3 , respectively. In addition, we have two classes of four-cycles in Y_4

$$[H_1] = [D_0 \cdot D_1], \quad [H^1] = \pi^*[\mathcal{C}^1]. \quad (4.27)$$

Together with the point class and the class of the full Calabi-Yau we thus get the K-theory basis

$$(\mathcal{O}_{\text{pt}}, \iota! \mathcal{O}_{\mathcal{C}^0}(K_{\mathcal{C}^0}^{1/2}), \iota! \mathcal{O}_{\mathcal{C}^1}(K_{\mathcal{C}^1}^{1/2}), \mathcal{O}_{H^1}, \mathcal{O}_{H_1}, \mathcal{O}_{D_1}, \mathcal{O}_{D_0}, \mathcal{O}_{X_4}), \quad (4.28)$$

where \mathcal{O}_{Y_4} is the structure sheaf on Y_4 . Via mirror symmetry we can identify this basis with an integer basis of fluxes on X_4 which, following [12], we denote by

$$(e, e_0, e_1, m_1, \hat{m}^1, m^1, m^0, m). \quad (4.29)$$

In order to apply the analysis of [12] in the large structure regime, we also need to specify some additional data of Y_4 such as the Chern classes

$$K_{ij}^{(2)} = \frac{1}{24} \int_{Y_4} c_2(Y_4) \wedge D_i \wedge D_j, \quad K_i^{(3)} = \frac{\zeta(3)}{8\pi^3} \int_{Y_4} c_3(Y_4) \wedge D_i, \quad (4.30)$$

which in the present case are given by

$$\begin{aligned} K_{00}^{(2)} &= \frac{91}{3}, & K_{01}^{(2)} &= K_{10}^{(2)} = \frac{91}{12}, & K_{11}^{(2)} &= 2, \\ K_0^{(3)} &= -3860 \frac{\zeta(3)}{(2\pi)^3}, & K_1^{(3)} &= -960 \frac{\zeta(3)}{(2\pi)^3}. \end{aligned} \quad (4.31)$$

Notice that $c_2(Y_4)$ is even such that we do not need to introduce a vertical flux in order to satisfy the flux-quantization condition (2.1). We want to find solutions to $D_0W = D_1W = 0$ in the large complex structure regime. As discussed in [12] to find such solutions while

keeping the tadpole relatively small, we should set $m^0 = m^1 = m = 0$. In this case the tadpole is simply given by

$$N_{\text{flux}} = \hat{m}^1(m_1 + 2\hat{m}^1). \quad (4.32)$$

The F-term conditions now amount to [12, eqs. (6.44), (6.46)] which can be solved for t^0 and t^1 . For consistency of the LCS approximation we require that the solutions for t^i lie sufficiently deep inside the LCS phase. For instance taking the flux choice

$$(e, e_0, e_1, m_1, \hat{m}^1, m^1, m^0, m) = (-61, 4, 15, 1, 1, 0, 0, 0), \quad (4.33)$$

we obtain a solution to the F-terms at

$$(t^0, t^1) \sim (5.18, 16.77), \quad (4.34)$$

which lies within the region where we can approximate the actual moduli space geometry by its LCS expressions.¹⁴ This flux choice contributes to the tadpole as

$$N_{\text{flux}} = 3, \quad (4.35)$$

which is well below the available tadpole

$$\frac{\chi(Y_4)}{24} = 972. \quad (4.36)$$

On the other hand, the on-shell value of W is given by

$$|W_0| = \frac{1}{2} \left[\hat{m}^1(t^1)^2 + m_1 \left(2t^0 t^1 + 4(t^0)^2 \right) \right] \sim 237.79 \neq 0. \quad (4.37)$$

Following our general discussion in this section, we can thus expect that through this flux n_{stab} is bounded by

$$n_{\text{stab}} \geq h^{3,1} - \text{rk}(D_i D_j W), \quad (4.38)$$

where we used that the chosen flux stabilizes all invariant moduli $\psi^{0,1}$.

To find the elements $D_i D_j W$ that are not identically zero, we have to determine all products of two monomials of the form (4.21) that are invariant with respect to G . After dividing by the Jacobian ideal $\langle \partial P \rangle$, there are only two such independent, rank-48 monomials,

$$x_1^{12} x_2^{12} x_3^{12} x_4^{12}, \quad \text{and} \quad x_1^{10} x_2^{10} x_3^{10} x_4^{10} x_5, \quad (4.39)$$

reflecting the fact that $h_{\text{inv}}^{2,2} = 2$. Importantly, while the number of independent polynomials is independent of the point at which $\langle \partial P \rangle$ is evaluated, it does play a role for the computation of $D_i D_j W$. For example, at the Fermat point, i.e. $\psi^a = 0$, all monomials containing x_5^2 are elements of $\langle \partial P \rangle$ and the corresponding elements of $D_i D_j W$ vanish identically. On the other hand, at a generic point in moduli space, $\psi^a \neq 0$, dividing by $\langle \partial P \rangle$ creates a linear relationship between such monomials and other terms but does not set them to

¹⁴To see this, we note that these values for the t^i correspond to values for $|\psi^i| \gg 10^{10}$. Since there are no singular loci for $\psi^0 > 864$ and $\psi^1 > 256$ (cf. for instance [39]) at which the LCS approximation breaks down, we are well inside the regime of applicability of the expressions of [12].

zero. Therefore, away from the Fermat point $D_i D_j W$ can have generically additional non-vanishing entries.

Using our Mathematica code we can go through all the possible combinations of monomials of the form (4.21). We find that there are 2956 G -invariant combinations at the Fermat point, while at a generic point on the symmetric locus this number is increased to 5986. We furthermore compute the rank of the resulting matrix $(D_i D_j W)$, assuming that all its elements are non-vanishing, and find that it is bounded by 2824. Therefore, according to our formula (4.38) this flux choice leads to

$$n_{\text{stab}} \geq 1052, \tag{4.40}$$

while inducing a tadpole of just $N_{\text{flux}} = 3$. This example thus shows explicitly that at the symmetric locus we can achieve

$$\frac{n_{\text{stab}}}{N_{\text{flux}}} \geq \frac{1052}{3} \gg 1, \tag{4.41}$$

which violates (2.8) indicating that the strong form of the tadpole conjecture is not valid at the symmetric locus.

5 Comparison to other regimes

The validity of the tadpole conjecture has previously been under investigation in other regimes of the complex structure deformation space such as the large complex structure regime [12–15, 40] or even more general asymptotic limits [16]. The hall-mark of these limits is that there is a unipotent monodromy around an asymptotic locus in moduli space that can be exploited to analyze the moduli dependence of the Hodge star operator involved in the self-duality constraint $G_4 = *G_4$. This monodromy can be associated to a continuous shift-symmetry that is approximately realized in the vicinity of the asymptotic locus. The shift symmetry can be made explicit by splitting the complex structure fields into a saxionic and an axionic component which is reflected in the asymptotic form of the periods. By contrast around the symmetric locus $\{\phi^i = 0\}$ of the families of fourfolds considered in this work, we do not have such a shift-symmetry. Instead, the form of the periods in the vicinity of the locus $\{\phi^i = 0\}$ is now constrained by the discrete symmetry G . Hence, the loci considered so far in this work do not fall in the class of asymptotic loci considered in [16].

Given the decomposition of the primitive middle cohomology of Y_4 as

$$H^4 = H^{4,0} \oplus H^{3,1} \oplus H_{\text{prim}}^{2,2} \oplus H^{1,3} \oplus H^{0,4}, \tag{5.1}$$

self-duality of the flux implies that at the critical point $D_a W = 0$ the G_4 -flux cannot have any $(3, 1)$ and $(1, 3)$ components. Since $W = \int \Omega \wedge G_4$, the condition $W_0 \neq 0$ further implies that G_4 has a component in $H^{0,4}$. In [16] it was shown that if one wants to solve the F-term condition in strict asymptotic limits using the leading expression for the periods, it is almost irrelevant whether G_4 has a $(0, 4)$ component or not in order to achieve $n_{\text{stab}} \gg 1$. In these limits most of the directions in complex structure moduli are obstructed by fluxes that are

exclusively in $H^{2,2}$. More precisely, the $(0, 4)$ component of G_4 can only obstruct up to four directions in the complex structure moduli space. This is in stark contrast to our findings for the case of the symmetric locus: we find that a large number of non-invariant complex structure deformations are obstructed merely by the presence of a $(0, 4)$ component of G_4 implying a non-zero W_0 . The comparison to the asymptotic regimes investigated in [16] thus nicely illustrates that the physics in asymptotic regimes can differ significantly from the physics in the deep interior of moduli space.

In order to understand this difference better, let us consider the matrix M in (2.7) in an asymptotic regime with approximate continuous shift symmetry. In order to avoid confusion, let us denote the complex scalar fields in this regime by T^α , $\alpha = 1, \dots, h^{3,1}$, on which the shift symmetry acts as $T^\alpha \rightarrow T^\alpha + c$ for $c \in \mathbb{R}$. In case we have such a shift symmetry, the Kähler potential is a function of the imaginary part of T^α , i.e.

$$K(T^\alpha, \bar{T}^\alpha) = K(T^\alpha - \bar{T}^\alpha). \tag{5.2}$$

Again, M_2 as in (4.5) naively leads to an obstruction for all complex structure deformations. However, $D_\alpha D_\beta W$ contains a term

$$D_\alpha D_\beta W \ni g_{\alpha\beta} W, \tag{5.3}$$

with $g_{\alpha\beta} = \partial_{T^\alpha} \partial_{T^\beta} K$. Due to the shift symmetry we have $g_{\alpha\beta} = -g_{\alpha\bar{\beta}}$ such that

$$\ker \begin{pmatrix} g_{\alpha\beta} W & g_{\bar{\alpha}\beta} W \\ g_{\alpha\bar{\beta}} \bar{W} & g_{\bar{\alpha}\bar{\beta}} \bar{W} \end{pmatrix} = \text{span}_{\alpha=1, \dots, h^{3,1}} \langle v^\alpha \rangle, \tag{5.4}$$

with

$$v^\alpha = (0, \dots, 0, \text{Re } T^\alpha, 0, \dots, 0; 0, \dots, 0, \text{Re } T^\alpha, 0, \dots, 0). \tag{5.5}$$

Hence, this contribution to $D_\alpha D_\beta W$ cancels half of the obstructions induced by M_2 . In particular, at this point there is no obstruction to changing the value of the axionic component of T^α . On the other hand, the saxionic directions of T^α naively are obstructed unless the metric $g_{\alpha\bar{\beta}}$ is degenerate. This reflects the fact that for $W_0 \neq 0$ half of the real moduli are automatically stabilized and the residual moduli space is real and of maximal dimension $h^{3,1}$ [25]. However, the results of [16] imply that at a point away from the singular locus there are further cancellations between M_1 and M_2 reducing the rank of M further unless the flux is chosen suitably to have enough components in the different subspaces of $H^{2,2}$. In particular this implies that $D_\alpha D_\beta W$ also has almost full rank allowing for such a cancellation. This is precisely the difference to the symmetric locus analyzed in the previous sections where we showed that M_1 in general has rank considerably smaller than $h^{3,1}$ therefore allowing us to infer that many additional directions in the field space are automatically obstructed just by the non-vanishing of W_0 .

While we do not attempt to show how this cancellation works in the general asymptotic considered in [16] let us briefly discuss this cancellation in one of the large complex structure scenarios discussed in [12]: in the large complex structure regime, the Kähler potential (5.2) can be conveniently expressed in terms of the intersection numbers $\mathcal{K}_{\alpha\beta\gamma\delta}$ of the mirror as

$$K = -\log \left(\frac{2}{3} \mathcal{K}_{\alpha\beta\gamma\delta} t^\alpha t^\beta t^\gamma t^\delta \right). \tag{5.6}$$

Here we split $T^\alpha = b^\alpha + it^\alpha$. Hence, $g_{\alpha\beta} = -g_{\alpha\bar{\beta}}$ and the rank of the matrix in (5.4) is indeed reduced by a factor of two. In particular, this contribution to M now only gives a mass to the imaginary part of T^α , i.e. the saxionic part. Let us focus on the generic scenario of [12]. In this case the superpotential is given by

$$W = \bar{e} + \bar{e}_\alpha T^\alpha + \frac{1}{2} \zeta_{\mu,\alpha\beta} m^\mu T^\alpha T^\beta, \quad (5.7)$$

where $\zeta_{\mu,\alpha\beta}$ is a tensor that depends on the topological data of the CY fourfold and \bar{e} , \bar{e}_α and m^μ are related to quantized fluxes in a certain basis. As shown in [12] the condition $D_\alpha W = 0$ enforces (without taking into account any corrections to the leading order periods)

$$\bar{e} = -\frac{1}{2} e_\alpha b^\alpha, \quad e_\alpha = -\zeta_{\mu,\alpha\beta} m^\mu b^\beta, \quad (\mathcal{K} \zeta_{\mu,\alpha} - \mathcal{K}_\alpha \zeta_\mu) m^\mu = 0. \quad (5.8)$$

Here we introduced $\mathcal{K}_\alpha = \mathcal{K}_{\alpha\beta\gamma\delta} t^\beta t^\gamma t^\delta$ and $\mathcal{K} = t^\alpha \mathcal{K}_\alpha$, as well as $\zeta_{\mu,\alpha} = \zeta_{\mu,\alpha\beta} t^\beta$ and $\zeta_\mu = \zeta_{\mu,\alpha\beta} t^\alpha t^\beta$. Accordingly, on-shell we find

$$W_0 = -\frac{1}{2} \zeta_{\mu,\alpha\beta} m^\mu t^\alpha t^\beta. \quad (5.9)$$

The matrix M is now given by

$$M = \left(\zeta_{\mu,\alpha\beta} m^\mu - \frac{2\mathcal{K}_\alpha \mathcal{K}_\beta}{\mathcal{K}^2} \zeta_{\mu,\gamma\delta} m^\mu t^\gamma t^\delta \right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\zeta_{\mu,\gamma\delta} m^\mu t^\gamma t^\delta}{2} \begin{pmatrix} g_{\alpha\beta} & g_{\alpha\bar{\beta}} \\ g_{\alpha\bar{\beta}} & g_{\bar{\alpha}\bar{\beta}} \end{pmatrix},$$

with

$$g_{\alpha\beta} = -g_{\alpha\bar{\beta}} = -4 \frac{\mathcal{K}_\alpha \mathcal{K}_\beta}{\mathcal{K}^2} + 3 \frac{\mathcal{K}_{\alpha\beta}}{\mathcal{K}}. \quad (5.10)$$

Since the second term in M does not stabilize the real part of T^α , the obstruction for these axionic directions comes entirely from the first part. And indeed the rank of the first term in M is essentially determined by the rank of $\zeta_{\mu,\alpha\beta} m^\mu$ in accordance with the analysis of [12]. On the other hand, generically we expect all saxions t^α to get a mass from the second term in M unless there is a cancellation. And indeed we find that there is a flat direction since

$$M \begin{pmatrix} t^\alpha \\ -t^\alpha \end{pmatrix} = 0. \quad (5.11)$$

To see this, we calculate explicitly

$$\zeta_{\mu,\beta} m^\mu - 2 \frac{\mathcal{K}_\beta}{\mathcal{K}} \zeta_\mu m^\mu + \frac{\mathcal{K}_\beta}{\mathcal{K}} \zeta_\mu m^\mu = \frac{1}{\mathcal{K}} (\mathcal{K} \zeta_{\mu,\beta} - \mathcal{K}_\beta \zeta_\mu) m^\mu = 0, \quad (5.12)$$

where in the last step we used (5.8). Hence, in this approximation there is indeed a flat direction which gets lifted by including polynomial corrections to the periods. Thus, including corrections we do not expect any cancellations between the two contributions, M_1 and M_2 , to M anymore. As discussed in the previous section also for the case of stabilization on the symmetric locus we do not expect such a cancellation between M_1 and M_2 in general.

6 Discussion

In this note we investigated moduli stabilization and the tadpole conjecture [10] in the interior of the complex structure moduli space of M-theory flux-compactifications on Calabi-Yau fourfolds. We suggested to differentiate between a strong and a weak version of the tadpole conjecture. The strong version states that stabilizing any number $n_{\text{stab}} \leq h^{3,1}$ of directions in moduli space requires G_4 -flux with an M2-brane tadpole that is directly proportional to n_{stab} . On the other hand, the weak version states that this only applies to the case $n_{\text{stab}} = h^{3,1}$ implying that there is only a problem to achieve *full* moduli stabilization for $h^{3,1} \gg 1$. Whereas there is evidence [16] that the strong tadpole conjecture is satisfied in strict asymptotic limits in the vicinity of infinite distance points, it is a priori not clear how these result extend to the interior of the moduli space.

In this work, we mostly focused on the case of $W_0 \neq 0$ vacua. In this case, half of the real deformations are automatically stabilized. Therefore, we proposed that the number n_{stab} in the tadpole conjecture should only count *additionally* stabilized real directions in moduli space. Following this convention, we proceeded to test the strong version of the tadpole conjecture in the interior of the moduli space. To that end, we relied on a strategy first put forward in [17] and exploited the discrete symmetry enhancement of certain Calabi-Yau fourfolds along special loci in the interior of the moduli space. These loci are at finite distance with respect to the moduli space metric and correspond to smooth fourfolds. They are, hence, qualitatively different from the singular loci at which a possible violation of the tadpole conjecture was observed in [11]. In particular, they are not expected to give rise to any additional gauge or other massless degrees of freedom.

On the symmetric locus, the F-term equations can be solved by turning on flux that is invariant under the discrete symmetry group. Phrased differently, it is possible to consistently set a large number of non-invariant flux quanta to zero while still solving the supersymmetry conditions. Still, it is important whether such a flux choice also generates obstructions to many of the deformations orthogonal to the symmetric locus. By scanning over a large class of Calabi-Yau fourfolds we found that an invariant flux with $W_0 \neq 0$ in fact leads to n_{stab} that is (at least) proportional to the number of non-invariant deformations, cf. (4.13). On the other hand, there is no reason for the invariant flux to generate a large M2-brane tadpole, as can be seen, for example, via mirror symmetry. Hence, one does not expect the strong version of the tadpole conjecture to hold at the symmetric loci. We illustrated this in an explicit example where it is possible to find consistent fluxes with $n_{\text{stab}}/N_{\text{flux}} \sim \mathcal{O}(100)$, which violates the strong form of the tadpole conjecture. However, even though we give a strict lower bound on the number of stabilized moduli, we cannot conclusively decide if actually all of them can be stabilized. Still, we provide some arguments for why we believe that also the weak version of the tadpole conjecture might be evaded for $W_0 \neq 0$ vacua along the symmetric locus. We intend to revisit this issue in future work.

We can thus summarize the state of affairs in the following way: whereas in asymptotic regions in the vicinity of points with unipotent monodromy the linear scaling predicted of the tadpole conjecture has been confirmed [16], at other special points in field space, namely the symmetric loci considered in this note, the bound predicted by the tadpole

conjecture can be violated. This is consistent with the analysis of $K3 \times K3$ in [11] where it was shown that the tadpole conjecture bound can be violated at gauge enhancement points in the moduli space of $K3 \times K3$. The symmetric loci considered in this work can be viewed as the analogue of these gauge enhancement point for compactifications on strict $SU(4)$ holonomy manifolds. The difference is that the enhanced symmetry in the latter case is discrete and the corresponding fourfold is smooth such that there are no additional massless degrees of freedom.

The results of this paper show that extrapolating results from the asymptotic regimes to the interior of the moduli space may not be justified in general. Still, one might wonder whether the strong version of the tadpole conjecture holds at generic points in the interior of the moduli space or whether it is only possible to evade it at special loci such as the ones discussed here. In fact, to find consistent vacua along the symmetric locus, it was crucial that we can consistently set to zero a large number of fluxes which allowed us to find vacua with relatively small N_{flux} . It is an open question whether this is still possible when moving away from the symmetric locus or whether in absence of any symmetry generically a large number of integer flux quanta must be non-vanishing.

In a similar spirit, it seems conceivable that in type IIB/M-theory compactifications there is a fundamental obstruction to finding solutions that satisfy the tadpole cancellation condition while at the same time keeping a perturbative expansion under control. This would for instance explain why in the vicinity of asymptotic regimes in moduli space it is very difficult to stabilize a large number of moduli using just the leading order periods, since the asymptotic expansion of the periods can be viewed as a perturbative expansion via mirror symmetry. On the other hand, the complex structure sector is classically exact such that we do not need to rely on such a perturbative expansions.

The situation is different once we also take the Kähler moduli sector into account, which is, of course, inevitable if we aim to find realistic, stable vacua.¹⁵ When including Kähler moduli, we *have* to rely on a perturbative expansion in α' since in a genuine four-dimensional $\mathcal{N} = 1$ theory (or equivalently $\mathcal{N} = 2$ in three dimensions) as of now we do not have a way to do computations that are exact in α' . Let us stress that in this work we exclusively focused on the complex structure sector and secretly assumed that we can treat this sector completely decoupled from the Kähler sector. However, in a genuine $\mathcal{N} = 1$ theory the Kähler and complex structure sectors do not decouple at finite volume (see, e.g., [44] for a recent discussion in F-theory). Therefore, any discussion of complex structure moduli stabilization is strictly speaking only valid in the large volume or decompactification limit, unless amended by an analysis of the Kähler moduli sector. Since such an analysis necessarily relies on a perturbative α' -expansion, we would in general expect a tension between control over this perturbative expansion and the tadpole cancellation condition. It might not come as a surprise that constraints arising from the tadpole cancellation are intimately related to the validity of a perturbative expansion, most prominently the α' -expansion. After all, in M-theory the tadpole cancellation condition is

¹⁵Kähler moduli stabilization can be avoided for special “non-geometric” world-sheet constructions with $h^{1,1} = 0$, as discussed in [24, 41–43].

derived from a higher-derivative term and is therefore part of the α' -expansion. The tension between tadpole cancellation and perturbative control is essentially what was identified as a possible obstruction to the type IIB LVS scenario [45] in [46–48]. Similarly, in the context of the KKLT scenario [49], a tension between tadpole cancellation and the existence of perturbatively controlled supersymmetric AdS flux vacua and potential issues with a controlled antibrane uplift were pointed out in [29, 50–58].

Flux compactifications with $W_0 \neq 0$ play a particularly prominent role in the construction of vacua à la KKLT. There, the non-vanishing, classical contribution W_0 to the superpotential gets balanced against additional, non-perturbative terms in order to achieve also Kähler moduli stabilization. In this context it is crucial that W_0 takes an exponentially small value such that control over the non-perturbative correction terms can be maintained (see [20] for a recent construction of flux vacua with $|W_0| \ll 1$). On the other hand, in our setup we have seen that there is generally a large number of complex structure moduli that obtain classical masses of order $m \sim |W_0|$. To be more specific, this is the case for all non-invariant moduli that lie in the kernel of $D_i D_j W$. However, according to figure 1 the dimension of $\ker(D_i D_j W)$ is directly proportional to the number of moduli, with an order-one proportionality factor. Therefore, there is generally a large number of complex structure moduli with masses comparable to those of the Kähler moduli and the size of the non-perturbative corrections. For the moduli stabilization scenario on the symmetric locus discussed in this work it is hence not clear if it is possible to integrate these moduli out and to discuss Kähler moduli and complex structure moduli stabilization independently, as originally proposed in [49] (for a related discussion of KKLT in the presence of very light complex structure moduli see, e.g., [51, 55, 58]).

In general it would be interesting to see whether the symmetric loci discussed in this work actually do survive as part of the moduli space even away from large volume or whether the mixture between Kähler and complex structure sector becomes important in the vicinity of the symmetric loci.

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