

Dynamical Abelianization and anomalies in chiral gauge theories

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ABSTRACT: We explore the idea that in some class of strongly-coupled chiral $SU(N)$ gauge theories the infrared dynamics might be characterized by a bifermion condensate in the adjoint representation of the color gauge group. As an illustration, in this work we revisit an $SU(N)$ chiral gauge theory with Weyl fermions in a symmetric (ψ) and anti-symmetric (χ) tensor representations, together with eight fermions in the anti-fundamental representations (η), which we called $\psi\chi\eta$ model in the previous investigations. We study the infrared dynamics of this system more carefully, by assuming dynamical Abelianization, a phenomenon familiar from $\mathcal{N} = 2$ supersymmetric gauge theories, and by analyzing the way various continuous and discrete symmetries are realized at low energies. We submit then these ideas to a more stringent test, by taking into account some higher-form symmetries and the consequent mixed anomalies. A detailed analysis of the mixed anomalies involving certain 0-form $U(1)$ symmetries and the color-flavor locked 1-form \mathbb{Z}_N symmetry in the $\psi\chi\eta$ system shows that the proposed infrared dynamics is consistent with it.

KEYWORDS: Nonperturbative Effects, Anomalies in Field and String Theories, Effective Field Theories

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1 Introduction

Dynamics of strongly-coupled chiral gauge theories is still largely unknown, after many years of studies [1]-[14] and in spite of their potential role in constructing the theory of the fundamental interactions, beyond the standard $SU(3)_{\text{QCD}} \times (SU(2)_L \times U(1)_Y)_{\text{GWS}}$ model of the strong and electroweak interactions.

In the last few years some renewed efforts to understand better this class of gauge theories have been made [15]-[18], mainly by using the idea of anomaly-matching consistency requirement, both based on the conventional 't Hooft anomalies [19], and on the recently found generalized (e.g., 1-form) symmetries and mixed anomalies [20]-[36]. Also, the importance of the strong anomaly and its implications has been pointed out in the context of a large class of (generalized Bars-Yankielowicz and Georgi-Glashow) models, recently [37].

In the present work, we start to explore earnestly the idea that in some chiral gauge theories bifermion condensates in the adjoint representation of the (strong) gauge group form, and play a central role in determining the infrared physics. A possible consequence of such a condensate is dynamical Abelianization, a phenomenon familiar from the exact Seiberg-Witten solution of $\mathcal{N} = 2$ supersymmetric theories [38]-[40], where elementary adjoint scalar fields are present in the theory whose vacuum expectation values (VEV) play the crucial role in the dynamics of the theories. In theories of our interest (a class of non-supersymmetric chiral gauge theories), such an adjoint scalar emerges as a composite field, but nothing forbids it to acquire dynamically nonvanishing vacuum expectation value (VEV), breaking either part of color gauge symmetry, part of the flavor symmetry, or both.

An interesting possibility is that it leads to dynamical Abelianization, i.e., the gauge group is broken as

$$SU(N) \rightarrow U(1)^{N-1} , \tag{1.1}$$

leaving a weakly-coupled, IR free, Abelian theory with a number of massless fermions.

Such a scenario emerged in our previous studies [15, 16] as a way of finding a possible solution to the anomaly matching equations in the $\psi\chi\eta$ and in some other models. Let us note that in many chiral gauge theories as those studied in [15–17] even the conventional ’t Hooft anomaly matching requirement represents a highly nontrivial constraint on the possible infrared dynamics, in general not easy to satisfy.

In this work we revisit the physics of the $\psi\chi\eta$ model more carefully, assuming dynamical Abelianization and reviewing the massless degrees of freedom, consistent with the conventional ’t Hooft anomaly argument. The structure of the low-energy effective action is studied, by taking into account all the anomalous and nonanomalous symmetries as well as the effects of the strong anomalies. We then examine the generalized symmetries and the consequent, mixed anomalies involving some 0-form U(1) symmetries and 1-form color-flavor locked center \mathbb{Z}_N symmetry. Nontrivial anomalies found indicate that there is an obstruction in gauging simultaneously one of the 0-form U(1) symmetries together with the center \mathbb{Z}_N symmetry (generalized ’t Hooft anomalies), implying that some of the symmetries involved must be broken. The pattern of the symmetry breaking predicted by the assumption of dynamical Abelianization is found to fit nicely with these expectations.

2 $\psi\chi\eta$ model and its symmetries

The $\psi\chi\eta$ model was studied earlier in [5, 6, 12] and more recently in [15, 16]. It is an $SU(N)$ gauge theory with left-handed fermion matter fields

$$\psi^{\{ij\}} , \quad \chi_{[ij]} , \quad \eta_i^A , \quad A = 1, 2, \dots, 8 , \tag{2.1}$$

a symmetric tensor, an anti-antisymmetric tensor and eight anti-fundamental multiplets of $SU(N)$, or

$$\square\square \oplus \overline{\square} \oplus 8 \times \overline{\square} . \tag{2.2}$$

The model has a global $SU(8)$ symmetry. It is asymptotically free, the first coefficient of the beta function being,

$$b_0 = \frac{1}{3} [11N - (N + 2) - (N - 2) - 8] = \frac{9N - 8}{3} . \tag{2.3}$$

Such a β function suggests that it is a very strongly coupled theory in the infrared: it is unlikely that it flows into an infrared-fixed CFT. But then some very nontrivial dynamical phenomenon must take place towards the infrared: confinement, tumbling (dynamical gauge symmetry breaking), or something else. The option that the system confines, with no global symmetry breaking and with some massless “baryons” saturating the ’t Hooft anomalies, does not appear to be plausible [5, 6, 12], as it would require an order $\propto N$ of

the underlying fermions to form gauge-invariant baryons. The wish to understand what happens in the (after all, simple) systems such as the $\psi\chi\eta$ model, was the driving motivation for the renewed studies on this model [15, 16]. Several possible dynamical scenarios have been found which are all compatible with 't Hooft's anomaly matching conditions, but the results of the analysis remained not quite conclusive.

The system has three U(1) symmetries, $U(1)_\psi$, $U(1)_\chi$, $U(1)_\eta$, of which two combinations are anomaly-free. For convenience we will take them below as

$$\tilde{U}(1) : \quad \psi \rightarrow e^{2i\alpha}\psi, \quad \chi \rightarrow e^{-2i\alpha}\chi, \quad \eta \rightarrow e^{-i\alpha}\eta, \quad (2.4)$$

and

$$U(1)_{\psi\chi} : \quad \psi \rightarrow e^{i\frac{N-2}{N^*}\beta}\psi, \quad \chi \rightarrow e^{-i\frac{N+2}{N^*}\beta}\chi, \quad \eta \rightarrow \eta, \quad (2.5)$$

where

$$N^* = GCD(N+2, N-2) \quad \text{and} \quad \alpha, \beta \in (0, 2\pi). \quad (2.6)$$

Any combination of the three classical U(1) symmetries which cannot be expressed as a linear combination of the above two, suffers from the strong anomaly. As is well known, the consideration of such an anomalous symmetry also provides us with an important information about the infrared physics. The famous $U_A(1)$ problem and its solution [44]–[50] are an example of this. For related considerations in the context of chiral gauge theories, see [4, 37]. For the present model, we will take

$$U(1)_{\text{an}} : \quad \psi \rightarrow e^{i\delta}\psi, \quad \chi \rightarrow e^{-i\delta}\chi, \quad \eta \rightarrow \eta. \quad (2.7)$$

A nonvanishing instanton amplitude

$$\langle \underbrace{\psi\psi\dots\psi}_{N+2} \underbrace{\chi\chi\dots\chi}_{N-2} \underbrace{\eta\dots\eta}_8 \rangle \neq 0 \quad (2.8)$$

involving $N+2$ ψ 's, $N-2$ χ 's and 8 η 's, is indeed not invariant under $U(1)_{\text{an}}$ while it is invariant under (2.4) and (2.5).

There are also anomaly-free discrete subgroups $(\mathbb{Z}_{N+2})_\psi \times (\mathbb{Z}_{N-2})_\chi \times (\mathbb{Z}_8)_\eta$ of $U(1)_\psi \times U(1)_\chi \times U(1)_\eta$. Under these \mathbb{Z} 's the fields transform as

$$\begin{aligned} \psi &\rightarrow e^{2\pi i \frac{k}{N+2}} \psi, & k &= 1, 2, \dots, N+2, \\ \chi &\rightarrow e^{-2\pi i \frac{\ell}{N-2}} \chi, & \ell &= 1, 2, \dots, N-2, \\ \eta &\rightarrow e^{-2\pi i \frac{m}{8}} \eta, & m &= 1, 2, \dots, 8, \end{aligned} \quad (2.9)$$

which are not broken by the instantons. However, they are not independent. It turns out, in fact, that $(\mathbb{Z}_{N+2})_\psi \times (\mathbb{Z}_{N-2})_\chi \times (\mathbb{Z}_8)_\eta$ is entirely contained inside $SU(8) \times \tilde{U}(1) \times U(1)_{\psi\chi}$, as is easy to check. The global symmetry group is connected.

Furthermore, $\tilde{U}(1) \times U(1)_{\psi\chi}$ and $(\mathbb{Z}_8)_\eta \subset SU(8)$ has an intersection:¹

$$(\tilde{U}(1) \times U(1)_{\psi\chi}) \cap (\mathbb{Z}_8)_\eta = \mathbb{Z}_{8/N^*}. \quad (2.10)$$

¹This can be understood in a simple way. For $e^{i\alpha} \in \tilde{U}(1)$ and $e^{i\beta} \in U(1)_{\psi\chi}$, the composition of the two transformations acts only on η if and only if $2\alpha + \frac{(N-2)}{N^*}\beta = 2\pi\mathbb{Z}$ and $-2\alpha - \frac{(N+2)}{N^*}\beta = 2\pi\mathbb{Z}$. Combining the two equations one obtains $\frac{8}{N^*}\alpha = 2\pi\mathbb{Z}$ (here we use that $\frac{(N+2)}{N^*}A - \frac{(N-2)}{N^*}B = 1$ has integer solutions for A and B , as $(N-2)/N^*$ and $(N+2)/N^*$ are co-primes). Thus $\eta \rightarrow e^{2\pi i \frac{N^*}{8}k} \eta$, which, for $k = 1, \dots, 8/N^*$ forms the \mathbb{Z}_{8/N^*} subgroup of $(\mathbb{Z}_8)_\eta \subset SU(8)$.

This leads to the symmetry of the $\psi\chi\eta$ model:

$$G = \frac{\text{SU}(N) \times \text{U}(1)_{\psi\chi} \times \tilde{\text{U}}(1) \times \text{SU}(8)}{\mathbb{Z}_N \times \mathbb{Z}_{8/N^*}} . \quad (2.11)$$

The division by \mathbb{Z}_N is due to the fact that the color \mathbb{Z}_N center is shared by a subgroup of the flavor $\text{U}(1)$ groups. To see this, it is sufficient to choose the angles $\alpha = \frac{2\pi k}{N}$, $k \in \mathbb{Z}_N$, in (2.4); it indeed reduces to the center $\mathbb{Z}_N \subset \text{SU}(N)$ transformations of the matter fermions,

$$\psi \rightarrow e^{2\frac{2\pi i}{N}} \psi, \quad \chi \rightarrow e^{-2\frac{2\pi i}{N}} \chi, \quad \eta \rightarrow e^{-\frac{2\pi i}{N}} \eta . \quad (2.12)$$

3 Dynamical Abelianization

The aim of this work is to study the consistency of the assumption that bifermion condensates in the adjoint representation

$$\langle \psi^{ik} \chi_{kj} \rangle = \Lambda^3 \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_N \end{pmatrix}^i_j, \quad \langle \psi^{ij} \eta_j^A \rangle = 0, \quad (3.1)$$

$$c_n \in \mathbb{C}, \quad \sum_n c_n = 0, \quad c_m - c_n \neq 0, \quad m \neq n, \quad (3.2)$$

(with no other particular relations among c_j 's) form in the infrared, inducing dynamical Abelianization of the system.

The condition of dynamical Abelianization must be made more precise. We require that the condensate (3.1), (3.2) induce the symmetry breaking

$$\text{SU}(N) \rightarrow \text{U}(1)^{N-1} . \quad (3.3)$$

As the effective composite scalar fields $\phi \sim \psi\chi$ are in the adjoint representation, it is convenient to describe them as a linear combination,

$$\phi \sim \psi\chi = \phi^A T^A = \phi^{(\alpha)} E_\alpha + \phi^{(-\alpha)} E_{-\alpha} + \phi^{(i)} H^i, \quad (3.4)$$

where ϕ^A are complex fields and T^A are the Hermitian generators of $\text{SU}(N)$ in the fundamental representation ($A = 1, 2, \dots, N^2 - 1$). In (3.4) we have introduced the $\text{SU}(N)$ generators in the Cartan-Weyl basis. $E_{\pm\alpha}$ are the raising and lowering operators associated with positive root vectors, α ; H^i ($i = 1, 2, \dots, N - 1$) are the generators in the Cartan subalgebra.

A field in the adjoint representation transforms under $\text{SU}(N)$ as

$$\phi \rightarrow U \phi U^\dagger, \quad U = e^{i\beta^A T^A}, \quad (3.5)$$

i.e., as

$$\phi \rightarrow \phi + i\beta^A [T^A, \phi] + \dots . \quad (3.6)$$

We recall also that the diagonal generators $T^A = H^i$ are those in the Cartan subalgebra, whereas the nondiagonal ones correspond to the pairs,

$$T^A = \frac{1}{\sqrt{2\alpha^2}}(E_\alpha + E_{-\alpha}), \quad \frac{-i}{\sqrt{2\alpha^2}}(E_\alpha - E_{-\alpha}). \quad (3.7)$$

H^i 's commute with each other, and the rest of the $SU(N)$ algebra is of the form:

$$\begin{aligned} [H^i, E_\alpha] &= \alpha_i E_\alpha, & [E_\alpha, E_{-\alpha}] &= \alpha \cdot H = \sum_i \alpha_i H^i, \\ [E_\alpha, E_\beta] &= \begin{cases} N_{\alpha+\beta} E_{\alpha+\beta}, & \text{if } \alpha + \beta \text{ is a root vector,} \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (3.8)$$

The condition of dynamical Abelianization, (3.3), is clearly that the fields that condense are in the Cartan subalgebra,

$$\phi \sim \psi_\chi = \phi^{(i)} H^i, \quad \langle \phi^{(i)} \rangle \neq 0, \quad \forall i, \quad (3.9)$$

whereas

$$\langle \phi^{(\alpha)} \rangle = \langle \phi^{(-\alpha)} \rangle = 0, \quad \forall \alpha. \quad (3.10)$$

See below, section 3.2, for more about the associated (would-be) NG bosons.

The gauge and flavor symmetries are reduced as:

$$SU(N)_c \times SU(8)_f \times \tilde{U}(1) \times U(1)_{\psi\chi} \longrightarrow \prod_{\ell=1}^{N-1} U(1)_\ell \times SU(8)_f \times \tilde{U}(1), \quad (3.11)$$

where $\tilde{U}(1)$ is given in (2.4), with charges

$$\psi : 2, \quad \chi : -2, \quad \eta : -1. \quad (3.12)$$

The unbroken gauge group $\prod_{\ell=1}^{N-1} U(1)_\ell$ is generated by the Cartan subalgebra,

$$\begin{aligned} t^1 &= \frac{1}{2} \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}, & t^2 &= \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -2 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}, \\ \dots, & & t^{N-1} &= \frac{1}{\sqrt{2N(N-1)}} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -(N-1) \end{pmatrix}. \end{aligned} \quad (3.13)$$

By taking into account also the full global structure of the groups, the symmetry breaking pattern due to the (3.1) condensate is actually

$$\mathrm{SU}(N) \times \frac{\mathrm{SU}(8)_f \times \tilde{U}(1) \times \mathrm{U}(1)_{\psi\chi}}{\mathbb{Z}_N \times \mathbb{Z}_{8/N^*}} \longrightarrow \frac{\prod_{\ell=1}^{N-1} \mathrm{U}(1)_\ell \times \mathrm{SU}(8)_f \times \tilde{U}(1)}{\prod_{\ell=1}^{N-1} \mathbb{Z}_\ell \times \mathbb{Z}_N \times \mathbb{Z}_2}, \quad (3.14)$$

where

$$\mathbb{Z}_N = \mathrm{U}(1)_{N-1} \cap \tilde{U}(1) = \mathrm{SU}(N) \cap \tilde{U}(1). \quad (3.15)$$

$\mathrm{U}(1)_{N-1}$ is generated by t^{N-1} in (3.13).²

The condensate (3.1) leaves unbroken a discrete subgroup $\mathbb{Z}_{4/N^*} \subset \mathrm{U}(1)_{\psi\chi}$:

$$\mathbb{Z}_{4/N^*} : \quad \psi \rightarrow e^{i\frac{N-2}{N^*}\alpha}\psi; \quad \chi \rightarrow e^{-i\frac{N+2}{N^*}\alpha}\chi, \quad (3.16)$$

so that

$$\psi\chi \rightarrow e^{-i\frac{4}{N^*}\alpha}\psi\chi, \quad (3.17)$$

with

$$\alpha = 2\pi k \frac{N^*}{4}, \quad k = 1, 2, \dots, \frac{4}{N^*}. \quad (3.18)$$

Note that $4/N^*$ is always an integer, as $N^* = \mathrm{GCD}(N+2, N-2)$ can be only one of 1, 2, 4. We however note that \mathbb{Z}_{4/N^*} is a subgroup of $\mathrm{SU}(8) \times \tilde{U}(1)$.³ Thus the global unbroken symmetry group in (3.14) is still connected.

Another discrete symmetry which remains unbroken by the condensate is \mathbb{Z}_{N^*} ,

$$\mathbb{Z}_{N^*} : \quad \psi \rightarrow e^{2\pi i \frac{p}{N^*}} \psi, \quad \chi \rightarrow e^{-2\pi i \frac{p}{N^*}} \chi, \quad (3.19)$$

($p = 1, \dots, N^*$). This is a subgroup⁴ of the nonanomalous, discrete $(\mathbb{Z}_{N+2})_\psi \times (\mathbb{Z}_{N-2})_\chi$ symmetries, both of which are broken by the condensate. \mathbb{Z}_{N^*} is also a subgroup of nonanomalous, unbroken continuous $\mathrm{SU}(8) \times \tilde{U}(1)$.⁵

The pattern of the gauge symmetry breaking is somewhat reminiscent of what happens in the $\mathcal{N} = 2$ supersymmetric gauge theories. Indeed the massive spectrum will contain 't Hooft-Polyakov magnetic monopoles, as well as the massive $\mathrm{SU}(N)/\mathrm{U}(1)^{N-1}$ gauge bosons. Note however that these monopoles are not in a semiclassical regime. The coupling constant at the scale of symmetry breaking is not small but of order one, $g^2 \sim 1$. Thus the monopole size and its Compton length are comparable; it is a soliton in a highly quantum regime. Our system is analogous to the $\mathcal{N} = 2$ supersymmetric gauge theories in the so-called quark vacua, where the bare quark mass is cancelled by the adjoint field VEV. In the absence of the moduli space of vacua here it is reasonable to assume that our low-energy system describes the photons of the electric $\prod_{\ell=1}^{N-1} \mathrm{U}(1)_\ell$ theory. Our system is analogous to the

²By choosing $\alpha_{\ell-1} = \alpha_\ell = \frac{2\pi\kappa}{\ell}$, it is easily seen that $\mathbb{Z}_\ell = \mathrm{U}(1)_\ell \cap \mathrm{U}(1)_{\ell-1}$. Also $\mathbb{Z}_2 = \tilde{U}(1) \cap \mathrm{SU}(8)$.

³To see this, note first that $\mathrm{SU}(8) \times \tilde{U}(1)$ contains a discrete subgroup \mathbb{Z}_4 acting on ψ and χ by phases $\pm 2\pi k/4$, $k = 1, \dots, 4$. Depending on N , \mathbb{Z}_{4/N^*} of (3.17) can be seen to be $\mathbf{1}$, \mathbb{Z}_2 or \mathbb{Z}_4 , always in $\mathbb{Z}_4 \subset \mathrm{SU}(8) \times \tilde{U}(1)$.

⁴Just take $k = p \frac{N+2}{N^*}$ and $\ell = p \frac{N-2}{N^*}$ in (2.9).

⁵This can be seen by taking $e^{i\frac{2\pi p}{2N^*}} \in \tilde{U}(1)$ and $e^{2\pi i \frac{p}{2N^*}} \in \mathbb{Z}_8 \subset \mathrm{SU}(8)$.

	fields	SU(8)	$\tilde{U}(1)$
UV	ψ	$\frac{N(N+1)}{2} \cdot (\cdot)$	$\frac{N(N+1)}{2} \cdot (2)$
	χ	$\frac{N(N-1)}{2} \cdot (\cdot)$	$\frac{N(N-1)}{2} \cdot (-2)$
	η^A	$N \cdot \square$	$8N \cdot (-1)$
IR	ψ^{ii}	$N \cdot (\cdot)$	$N \cdot (2)$
	η^A	$N \cdot \square$	$8N \cdot (-1)$

Table 1. Full dynamical Abelianization in the $\psi\chi\eta$ model.

$\mathcal{N} = 2$ Seiberg-Witten theories outside the so-called marginal stability curves,⁶ though perhaps not far from one.

3.1 't Hooft anomaly matching

The fields η_i^A which do not participate in the condensate remain massless and weakly coupled to the gauge bosons from the Cartan subalgebra which we will refer to as the photons, in the infrared. Also, some of the fermions ψ^{ij} do not participate in the condensates. Due to the fact that $\psi^{\{ij\}}$ are symmetric whereas $\chi_{[ij]}$ are antisymmetric, actually only the non-diagonal elements of $\psi^{\{ij\}}$ condense and get mass. The diagonal fields $\psi^{\{ii\}}$, $i = 1, 2, \dots, N$, together with all of η_i^a remain massless. Also there is one physical NG boson (see section 3.2 below). All of the anomaly triangles, $[\text{SU}(8)]^3$, $\tilde{U}(1) - [\text{SU}(8)]^2$, $[\tilde{U}(1)]^3$, $\tilde{U}(1) - [\text{gravity}]^2$, $\mathbb{Z}_{N^*} - [\text{SU}(8)]^2$, $\mathbb{Z}_{N^*} - [\text{gravity}]^2$ are easily seen to match, on inspection of table 1. Perhaps the only nontrivial ones are the ones that do not involve SU(8). For $\tilde{U}(1) - [\text{gravity}]^2$ we have

$$\underbrace{2 \cdot \frac{N(N+1)}{2} - 2 \cdot \frac{N(N-1)}{2} - 8 \cdot N}_{\text{UV}} = \underbrace{2 \cdot N - 8N}_{\text{IR}} = -6N, \quad (3.20)$$

for $[\tilde{U}(1)]^3$ we have

$$\underbrace{8 \cdot \frac{N(N+1)}{2} - 8 \cdot \frac{N(N-1)}{2} - 8 \cdot N}_{\text{UV}} = \underbrace{8 \cdot N - 8N}_{\text{IR}} = 0, \quad (3.21)$$

for $\mathbb{Z}_{N^*} - [\text{gravity}]^2$ we have

$$\underbrace{1 \cdot \frac{N(N+1)}{2} - 1 \cdot \frac{N(N-1)}{2}}_{\text{UV}} = \underbrace{1 \cdot N}_{\text{IR}} = N. \quad (3.22)$$

The massless fermions in the infrared are shown again in table 2, where their quantum numbers with respect to the weak $\prod_{\ell=1}^{N-1} \text{U}(1)_\ell \subset \text{SU}(N)$ are also shown.

⁶Due to the phenomenon of isomonodromy the spectrum of the stable particles of the system changes when crossing some subspace of the vacuum moduli space. The phenomenon has been studied in detail for SU(2) Seiberg-Witten theory [41]-[43].

	fields	U(1) ₁	U(1) ₂	⋯	U(1) _{N-1}	SU(8)	$\tilde{U}(1)$	U(1) _{an}
ψ	ψ^{11}	1	$\frac{1}{\sqrt{3}}$	⋯	$\frac{2}{\sqrt{2N(N-1)}}$	(·)	2	1
	ψ^{22}	-1	$\frac{1}{\sqrt{3}}$	⋯	$\frac{2}{\sqrt{2N(N-1)}}$	(·)	2	1
	ψ^{33}	0	$-\frac{2}{\sqrt{3}}$	⋯	$\frac{2}{\sqrt{2N(N-1)}}$	(·)	2	1
	⋮				⋮	⋮	⋮	⋮
	ψ^{NN}	0	0	⋯	$-\frac{2(N-1)}{\sqrt{2N(N-1)}}$	(·)	2	1
η	η_1^a	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	⋯	$-\frac{1}{\sqrt{2N(N-1)}}$	□	-1	0
	η_2^a	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	⋯	$-\frac{1}{\sqrt{2N(N-1)}}$	□	-1	0
	η_3^a	0	$\frac{1}{\sqrt{3}}$	⋯	$-\frac{1}{\sqrt{2N(N-1)}}$	□	-1	0
	⋮				⋮	⋮	⋮	⋮
	η_N^a	0	0	⋯	$\frac{N-1}{\sqrt{2N(N-1)}}$	□	-1	0
π	$\tilde{\phi} \sim (\psi\chi)_1^1$	0	0	⋯	0	(·)	0	0

Table 2. Massless fermions in the infrared in the $\psi\chi\eta$ model and their charges with respect to the unbroken symmetry groups. The massless NG boson carries no charges with respect to the unbroken symmetries, see section 3.2. The two (nonanomalous and anomalous) U(1) symmetries $\tilde{U}(1)$ and U(1)_{an} which are not affected by the $\psi\chi$ condensate are defined in (2.7). $u(1)^{N-1} \subset su(N)$ are taken in Cartan subalgebra, satisfying the orthogonality relations, $\text{Tr}(T^a T^b) \propto \delta^{ab}$.

3.2 Nambu-Goldstone (NG) bosons

Discussion of the infrared physics implied by the dynamical Abelianization requires also understanding of the massless bosonic degrees of freedom, besides the fermions in table 2. As will be seen below, there is one physical massless U(1) NG boson in this system.

Before discussing the U(1) NG boson, however, let us briefly comment on the SU(N) gauge symmetry breaking, discussed in section 3. The diagonal generators in Cartan subalgebra H^i correspond to the unbroken U(1)^{N-1} Abelian group, whereas the broken generators of SU(N)/U(1)^{N-1} are the nondiagonal E_α and $E_{-\alpha}$'s. Recalling the commutation relations (3.8), it is easy to see the NG boson (interpolating) fields $\tilde{\phi}$ which get transformed into the ones which condense, see (A.1)–(A.4), are $\phi^{(\alpha)}$, $\phi^{(-\alpha)}$ of (3.4). They are just the nondiagonal $N(N-1)/2 \cdot 2 = N^2 - N$ components of $(\psi\chi)_j^i$ composite scalars. The would-be NG bosons π^α associated with the currents $J_\mu^\alpha, J_\mu^{-\alpha}$,

$$\langle 0 | J_\mu^\alpha | \pi^\beta \rangle = F_\pi \delta^{\alpha\beta}, \quad \langle \pi^\beta | \phi^{-\alpha}(0) | 0 \rangle \propto \delta^{\alpha\beta}, \quad (3.23)$$

are eaten by the Englert-Brout-Higgs mechanism making the SU(N)/U(1)^{N-1} gauge bosons massive.

Let us now focus our attention on the U(1) NG boson. As noted already, there are three U(1) symmetries in the model, the two nonanomalous ones $\tilde{U}(1)$ and U(1) _{$\psi\chi$} in (2.4), (2.5)

and an anomalous $U(1)_{\text{an}}$ in (2.7). The associated currents are

$$J_{\psi\chi}^\mu = i \left\{ \frac{N-2}{N^*} \bar{\psi} \bar{\sigma}^\mu \psi - \frac{N+2}{N^*} \bar{\chi} \bar{\sigma}^\mu \chi \right\}, \quad \partial_\mu J_{\psi\chi}^\mu = 0, \quad (3.24)$$

$$\tilde{J}^\mu = i \left\{ 2 \bar{\psi} \bar{\sigma}^\mu \psi - 2 \bar{\chi} \bar{\sigma}^\mu \chi - \bar{\eta}^a \bar{\sigma}^\mu \eta^a \right\}, \quad \partial_\mu \tilde{J}^\mu = 0, \quad (3.25)$$

$$J_{\text{an}}^\mu = i \bar{\psi} \bar{\sigma}^\mu \psi - i \bar{\chi} \bar{\sigma}^\mu \chi, \quad \partial_\mu J_{\text{an}}^\mu = \frac{2g^2}{32\pi^2} G_{\mu\nu} \tilde{G}^{\mu\nu}, \quad (3.26)$$

and the associated charges are⁷

$$\begin{aligned} Q_{\psi\chi} &= \int d^3x \left(\frac{N-2}{N^*} \bar{\psi} \psi - \frac{N+2}{N^*} \bar{\chi} \chi \right), \\ \tilde{Q} &= \int d^3x \left(2 \bar{\psi} \psi - 2 \bar{\chi} \chi - \bar{\eta}^a \eta^a \right), \\ Q_{\text{an}} &= \int d^3x (\bar{\psi} \psi - \bar{\chi} \chi). \end{aligned} \quad (3.27)$$

It follows from the standard quantization rule that $((\psi\chi)_m^n \equiv \psi^{nk} \chi_{km})$

$$\begin{aligned} [Q_{\psi\chi}, (\psi\chi)_m^n] &= \frac{4}{N^*} (\psi\chi)_m^n, \\ [\tilde{Q}, (\psi\chi)_m^n] &= (2-2) (\psi\chi)_m^n = 0, \\ [Q_{\text{an}}, (\psi\chi)_m^n] &= 0. \end{aligned} \quad (3.28)$$

It is seen from these that the condensates (3.1), (3.2) (i.e., the diagonal $\langle (\psi\chi)_n^n \rangle$) break spontaneously only the nonanomalous $U(1)_{\psi\chi}$ symmetry. Both $\tilde{U}(1)$ and $U(1)_{\text{an}}$ remain unbroken. There is only one massless (physical) NG boson in the system.

Also, it is seen easily that among the diagonal $(\psi\chi)_n^n$'s there is only one *independent* component, which can be taken e.g.,

$$U(x) = (\psi\chi)_1^1(x), \quad (3.29)$$

that is transformed into a field which acquires a nonvanishing VEV.⁸ A simple chiral Ward-Takahashi identity (see appendix A) then shows that $J_{\psi\chi}^\mu$ generates from the vacuum the massless boson, which can be described by the interpolating field, $U(x) = (\psi\chi)_1^1(x)$.

It is instructive to compare the situation here with the fate of the $U(1)$ symmetries in QCD. In QCD, the bifermion condensate is of the form, $\bar{\psi}_R \psi_L$. It is invariant under $U(1)_V$ and noninvariant under $U(1)_A$. By appropriately choosing the phases of ψ_L and ψ_R the condensate $\langle \bar{\psi}_R \psi_L \rangle$ can be chosen to be real; the (would-be) NG boson of the broken $U(1)_A$ symmetry then corresponds to the imaginary part of $\bar{\psi}_R \psi_L$. Due to the effects of the strong anomaly, this would-be NG boson gets mass.

Here the condensate $\langle \psi\chi \rangle$ is invariant under the $\tilde{U}(1)$ as well as under the anomalous $U(1)_{\text{an}}$. Only the nonanomalous $U(1)_{\psi\chi}$ is broken by the condensate: the quantum fluctuations of $(\psi\chi)_1^1(x)$ contain one physical, massless NG boson, π .

⁷Note that in the two-component spinor notation of Wess and Bagger, $\bar{\sigma}^0 = -i$, and $\bar{\psi} \equiv \psi^\dagger$.

⁸This can be seen by considering the linear combinations such as $c_2(\psi\chi)_1^1 - c_1(\psi\chi)_2^2$, $c_3(\psi\chi)_2^2 - c_2(\psi\chi)_3^3$, etc., whose VEV's all vanish.

3.3 The low-energy effective action

The massless degrees of freedom in the infrared are thus the gauge bosons A_μ^k (the photons) of the $U(1)^{N-1}$ gauge group, the fermions listed in table 2 and the “pion”, π . The effective low-energy Lagrangian has the form,

$$\mathcal{L}^{(eff)} = \mathcal{L}(\psi, \eta, A_\mu^{(i)}) + \mathcal{L}(\pi) - \mathcal{V}(\pi, \psi, \eta) + \dots, \quad (3.30)$$

where ψ, η are the fermions in table 2. $\mathcal{L}(\psi, \eta, A_\mu)$ is the Lagrangian of the $U(1)^{N-1}$ theory with “electrons” ψ, η , minimally coupled to the $U(1)^{N-1}$ gauge fields. $\mathcal{L}(\pi)$ is the Lagrangian containing only the pion. Our task now is to learn about $\mathcal{L}^{(eff)}$ of (3.30) as much as we can from symmetries, either broken, unbroken, anomalous or non anomalous.

In particular, upon condensation the composite scalar field $\psi\chi$ can be written as

$$U(x) = (\psi\chi)_1^1(x) = \text{const. } \Lambda^3 e^{i\pi(x)/F} \quad (3.31)$$

where F is the analogue of the pion decay constant. $\mathcal{L}(\pi)$ contains the kinetic term

$$\mathcal{L}(\pi) = \partial^\mu U(x)^\dagger \partial_\mu U(x) + \dots, \quad (3.32)$$

with possible higher order terms.

Let us recapitulate the symmetries and their low-energy realizations

$$SU(N) \times \frac{SU(8)_f \times \tilde{U}(1) \times U(1)_{\psi\chi}}{\mathbb{Z}_N \times \mathbb{Z}_{8/N^*}} \longrightarrow \frac{\prod_{\ell=1}^{N-1} U(1)_\ell \times SU(8)_f \times \tilde{U}(1)}{\prod_{\ell=1}^{N-1} \mathbb{Z}_\ell \times \mathbb{Z}_N \times \mathbb{Z}_2}. \quad (3.33)$$

A possible local interaction Lagrangian, consistent with the symmetries of the system, (3.33), is

$$\mathcal{V}(\pi, \psi, \eta)(x) \sim U(x)^{N-2} \underbrace{\psi \cdots \psi}_4 \underbrace{\eta \eta \cdots \eta}_8 + h.c. = \text{const. } \pi \pi \dots \pi \underbrace{\psi \cdots \psi}_4 \underbrace{\eta \eta \cdots \eta}_8 + h.c., \quad (3.34)$$

with any number of pions, four ψ 's and eight η 's. It is invariant under the full UV symmetries $SU(N)_c \times SU(8)_f \times \tilde{U}(1) \times U(1)_{\psi\chi}$, and *a fortiori* with the unbroken symmetries $\prod_{\ell=1}^{N-1} U(1)_\ell \times SU(8)_f \times \tilde{U}(1)$ as well as some discrete symmetries (see below), surviving in the infrared. Such a Lagrangian is not invariant under the anomalous $U(1)_{an}$.

The structure of (3.34) may be understood from the original multi-fermion 't Hooft effective instanton potential, e.g.,

$$\epsilon_{a_1 a_2 \dots a_8} \underbrace{(\psi\chi)_j^i (\psi\chi)_k^j \cdots (\psi\chi)_i^p}_{N-2} \underbrace{(\psi\eta^{a_1} \eta^{a_2}) \cdots (\psi\eta^{a_7} \eta^{a_8})}_4, \quad (3.35)$$

where a possible (certainly not unique) way to contract the color $SU(N)$ and the flavor $SU(8)$ indices in an invariant way is shown. The idea is to realize these symmetry properties in terms of the infrared degrees of freedom. In particular, by replacing the condensate $\psi\chi$ with a slowly varying fields $U(x)$ of (3.31), one arrives at (3.34).

The Yukawa interactions among π, ψ, η are forbidden by the unbroken symmetries, see table 2. The only possible interactions among them are those arising from the instanton-induced amplitude such as (3.34).

The $\psi\chi\eta$ system has three global U(1) symmetries. $\tilde{U}(1)$ symmetry is nonanomalous and remains unbroken. It is a manifest symmetry of the low-energy effective action. The consequences of the nonanomalous but spontaneously broken $U(1)_{\psi\chi}$ symmetry and anomalous but not-spontaneously-broken symmetry $U(1)_{an}$ are a little subtler.

$U(1)_{\psi\chi}$ is spontaneously broken by the $\psi\chi$ condensate. It is a nonanomalous symmetry in the UV: the fermion charges are such that the $U(1)_{\psi\chi}$ anomalies due to the $SU(N)$ gauge interactions cancel. In the IR it is spontaneously broken by the $\psi\chi$ condensate, a NG boson (π) is produced by the current from the vacuum, and at the same time $SU(N)$ is dynamically broken to $\prod_{\ell} U(1)_{\ell}$.

Now there seems to be a paradox. In the underlying (UV) theory the global $U(1)_{\psi\chi}$ symmetry acts on the fermions as:

$$\psi \rightarrow e^{i\frac{N-2}{N^*}\beta}\psi, \quad \chi \rightarrow e^{-i\frac{N+2}{N^*}\beta}\chi, \quad \eta \rightarrow \eta. \quad (3.36)$$

The 't Hooft effective instanton potential (3.35) is indeed invariant under this. It is important to note however that such an invariance is not invalidated by the $SU(N)$ anomalies as the contributions from the ψ and χ fermions cancel, see (3.24).

Now in the infrared, it is spontaneously broken and $U(1)_{\psi\chi}$ symmetry is realized partially nonlinearly, as

$$\pi(x) \rightarrow \pi(x) - \frac{4F}{N^*}\beta, \quad U(x) \rightarrow e^{-\frac{4i\beta}{N^*}}U(x), \quad (3.37)$$

see (3.31) and (3.36). If we assume that the fermions remaining massless in the infrared, see table 2, in particular ψ^{ii} , $i = 1, 2, \dots, N$, transform under $U(1)_{\psi\chi}$ as in (3.36):

$$\psi^{ii} \rightarrow e^{i\frac{N-2}{N^*}\beta}\psi^{ii}, \quad (3.38)$$

the effective potential in the infrared, the first term of (3.34), is indeed invariant. This shows that $U(1)_{\psi\chi}$ symmetry is realized partially nonlinearly (3.37) and partially linearly (3.38) at low energies.

But now the anomaly due to the ψ^{ii} loops,

$$\Delta\mathcal{L}^{\text{eff}} = \frac{N-2}{N^*}\beta \frac{1}{16\pi^2} \sum_{j=1}^{N-1} e_j^2 F_{\mu\nu}^{(j)} \tilde{F}^{(j)\mu\nu} \quad (3.39)$$

cannot be cancelled, as there are no other massless fermions left in the infrared theory.

Clearly such an argument is too naïve, and neglects the fact that the $U(1)_{\psi\chi}$ symmetry in the infrared does not only involve the massless fermion, but also the pion, transforming inhomogeneously as in (3.37). The answer to this apparent puzzle is that the low-energy effective Lagrangian (3.30) actually contains an axion-like term

$$\mathcal{L}(\pi, A_{\mu}^{(i)}) = \pi(x) \frac{N-2}{4F} \frac{1}{16\pi^2} \sum_{j=1}^{N-1} e_j^2 F_{\mu\nu}^{(j)} \tilde{F}^{(j)\mu\nu} \quad (3.40)$$

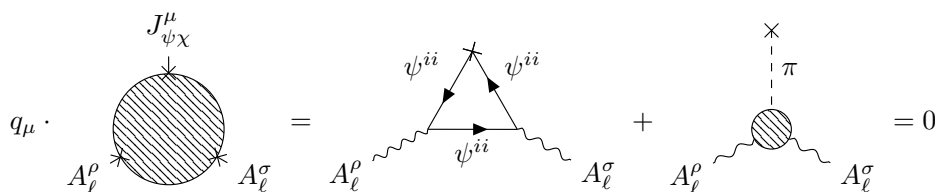


Figure 1. In UV, the absence of the gauge anomaly in the $U(1)_{\psi\chi}$ symmetry is due to the cancellation of anomalies from the ψ and χ fermions (see (3.24)). At low energies it is secured by the cancellation between the contribution of the ψ^{ii} loops ($i = 1, 2, \dots, N - 1$) and the pion pole term. The requirement that the sum of the two cancel, is nothing but the $\pi \rightarrow A_\ell A_\ell$ amplitude given by (3.40).

which transforms under (3.37) as

$$\Delta\mathcal{L}(\pi, A_\mu^{(i)}) = -\frac{N-2}{N^*}\beta\frac{1}{16\pi^2}\sum_{j=1}^{N-1}e_j^2F_{\mu\nu}^{(j)}\tilde{F}^{(j)\mu\nu} \tag{3.41}$$

cancelling exactly the anomaly due to the ψ^{ii} loops, (3.39), ensuring the $U(1)_{\psi\chi}$ invariance of the system.

The conclusion is that the effective low-energy Lagrangian contains an axion-like term, (3.40), besides the standard terms, explicit in (3.34). Another, equivalent way to reach the same conclusion is to consider various three-point functions,

$$\int d^4x e^{iq\cdot x}\langle 0|T\{J_{\psi\chi}^\mu(x)A_\ell^\nu(y)A_\ell^\lambda(0)\}|0\rangle, \quad \ell = 1, 2, \dots, N - 1, \tag{3.42}$$

multiplying it by q_μ and taking the limit $q_\mu \rightarrow 0$. See figure 1

Let us now consider the anomalous, but unbroken $U(1)_{an}$ symmetry. As we noted already $\mathcal{L}^{(eff)}$ is not invariant under it. The effect of the $U(1)_{an}$ anomaly however is not exhausted in the explicit breaking of $U(1)_{an}$ symmetry in $\mathcal{V}(x)$. As the $U(1)_{an}$ charge of the low-energy, massless fermions is well defined, see (3.26) and table 2, it manifests itself also through the massless ψ fermion loops,

$$J_{an}^\mu = i\sum_{i=1}^N\bar{\psi}_{ii}\bar{\sigma}^\mu\psi^{ii}, \quad \partial_\mu J_{an}^\mu = \frac{1}{16\pi^2}\sum_{j=1}^{N-1}e_j^2F_{\mu\nu}^{(j)}\tilde{F}^{(j)\mu\nu}. \tag{3.43}$$

Such an anomaly has a natural interpretation as a remnant of the original strong anomaly (3.26) in the UV theory. The original strong anomaly divergence equation has turned into the anomalous divergences due to the weak $U(1)^{N-1}$ gauge interactions of the low-energy theory.

To summarize, the symmetry realization pattern of various $U(1)$ symmetries in the $\psi\chi\eta$ model is subtly different from the one in QCD (the $U(1)_A$ problem and the massive η meson) [44]–[50], in QCD with the electromagnetic interactions ($\pi_0 \rightarrow 2\gamma$ decay through the ABJ anomaly), or in QCD with Peccei-Quinn symmetry [51–53] (with an extra scalar

or heavy quarks, giving rise to the axion and its coupling to the topological density of QCD), even though, here and there, we see some analogous features.

4 The generalized anomalies

The assumption of dynamical Abelianization and its possible consequences studied in section 3 are certainly consistent with the conventional 't Hooft anomaly matching requirement, as reviewed above. We now check such physics scenario in the $\psi\chi\eta$ model, against more stringent consistency requirements arising from the mixed anomalies involving some higher symmetries. Let us recapitulate the symmetry of the system,

$$\frac{\text{SU}(N) \times \tilde{U}(1) \times \text{U}(1)_{\psi\chi} \times \text{SU}(8)}{\mathbb{Z}_N \times \mathbb{Z}_{8/N^*}}, \quad (4.1)$$

where

$$\begin{aligned} \text{U}(1)_{\psi\chi} : & \quad \psi \rightarrow e^{i\frac{N-2}{N^*}\beta}\psi, & \quad \chi \rightarrow e^{-i\frac{N+2}{N^*}\beta}\chi, \\ \tilde{U}(1) : & \quad \psi \rightarrow e^{2i\gamma}\psi, & \quad \chi \rightarrow e^{-2i\gamma}\chi, & \quad \eta \rightarrow e^{-i\gamma}\eta, \end{aligned} \quad (4.2)$$

and

$$\begin{aligned} \mathbb{Z}_{8/N^*} &= \text{SU}(8) \cap (\tilde{U}(1) \times \text{U}(1)_{\psi\chi}), \\ \mathbb{Z}_N &= \text{SU}(N) \cap \tilde{U}(1). \end{aligned} \quad (4.3)$$

We wish now to find out if new, stronger consistency conditions on the realization of these symmetries arise by making full use of the global structure of the symmetry group, eq. (4.1), i.e., by gauging some 1-form center symmetries, such as \mathbb{Z}_N and/or \mathbb{Z}_{8/N^*} . To do so, however, it is necessary to make use of symmetries which are not broken by the color $\text{SU}(N)$ or the weak $\text{SU}(8)$ gauge interactions, including the nonperturbative effects (instantons). Gauging the 1-form \mathbb{Z}_{8/N^*} symmetry involves necessarily gauging the $\tilde{U}(1)$ symmetry, which is already broken by the $\text{SU}(8)$ instantons, see eq. (4.2). It would not be a simple task to disentangle the effects of the new anomalies due to the gauging of a discrete center symmetry, from the conventional anomalies due to the $\text{SU}(8)$ instantons. Such a precaution appears to be relevant, because we are here interested in possible new mixed anomalies on continuous symmetries $\tilde{U}(1)$ and $\text{U}(1)_{\psi\chi}$ or on some of their discrete subgroups.

These considerations lead us to preclude the idea of gauging the 1-form \mathbb{Z}_{8/N^*} symmetry, and below we shall focus on the color-flavor locked 1-form \mathbb{Z}_N center symmetry, gauge it in conjunction with some 0-form $\text{U}(1)$ symmetries of the model, and examine whether such a simultaneous gauging would suffer from some topological obstructions (generalized 't Hooft anomalies).

4.1 Calculation of the mixed anomalies

The global structure of the symmetry of the gauge groups reproduced in (4.1) ~ (4.3) means that it should be possible to introduce a more faithful way the redundancies in the summation over the gauge field configurations are eliminated. In our context, this brings

us to consider effectively a projective $SU(N)/\mathbb{Z}_N$ group as the strong gauge group, and with its 2-form $B_c^{(2)}$ fields carrying a fractional 't Hooft fluxes

$$\frac{1}{8\pi^2} \int_{\Sigma_4} (B_c^{(2)})^2 \in \frac{\mathbb{Z}}{N^2} . \tag{4.4}$$

The way this is done concretely has been explained in [20]-[36], by introducing the 1-form \mathbb{Z}_N gauge field, and imposing the condition (4.5) for a \mathbb{Z}_N gauge field. As our \mathbb{Z}_N center symmetry is a color-flavor locked symmetry, to render it properly a 1-form symmetry one must accompany the $SU(N)$ Wilson loop with a $\tilde{U}(1)$ holonomy (Aharonov-Bohm) loop for the fermions. See also the related discussion (vii) in section 4.2 below.

Taking these points into account now we introduce the gauge fields as

- a_c : the $SU(N)$ color gauge field;
- a_f : the $SU(8)$ flavor gauge field;
- \tilde{A} : the gauge field for $\tilde{U}(1)$;
- $A_{\psi\chi}$: the gauge field for $U(1)_{\psi\chi}$;
- $B_c^{(1)}, B_c^{(2)}$: \mathbb{Z}_N gauge field.

The pairs of gauge fields $(B_c^{(1)}, B_c^{(2)})$ for the \mathbb{Z}_N 1-form symmetry satisfy the constraints⁹

$$N B_c^{(2)} = dB_c^{(1)} . \tag{4.5}$$

Following the by now well-understood procedure for gauging a 1-form discrete symmetry, one also introduces redundant $U(N)$ gauge fields

- \tilde{a}_c : the $U(N)$ color gauge field;

where

$$\tilde{a}_c = a_c + \frac{1}{N} B_c^{(1)} . \tag{4.6}$$

The central idea is that one then imposes the invariance under the 1-form gauge transformations

$$\begin{aligned} B_c^{(2)} &\rightarrow B_c^{(2)} + d\lambda_c , & B_c^{(1)} &\rightarrow B_c^{(1)} + N\lambda_c , \\ \tilde{a}_c &\rightarrow \tilde{a}_c + \lambda_c , \end{aligned} \tag{4.7}$$

where λ_c is a 1-form $U(1)$ gauge function.¹⁰ The $\tilde{U}(1)$ and $U(1)_{\psi\chi}$ gauge fields \tilde{A} and $A_{\psi\chi}$ transform under these as (see (2.12), (2.5))

$$\tilde{A} \rightarrow \tilde{A} - \lambda_c , \quad A_{\psi\chi} \rightarrow A_{\psi\chi} . \tag{4.8}$$

⁹The suffices are to indicate the 1-form or 2-form nature of these gauge fields, e.g., $B^{(1)} = B_\mu dx^\mu$, etc.

¹⁰In the standard gauging of a 0-form $U(1)$ symmetry, $\psi \rightarrow e^{i\lambda}\psi$; $A_\mu \rightarrow A_\mu + \frac{1}{e}\partial_\mu\lambda$, $\lambda(x)$ is a 0-form gauge function.

The requirement of the invariance under the 1-form gauge transformations (4.7)–(4.8) realizes the elimination of the redundancies, (4.3).

The (1-form) gauge invariant Dirac operators are accordingly

$$d + \mathcal{R}_S \left(\tilde{a}_c - \frac{1}{N} B_c^{(1)} \right) + 2 \left(\tilde{A} + \frac{1}{N} B_c^{(1)} \right) + \frac{N-2}{N^*} A_{\psi\chi}, \quad (4.9)$$

acting on ψ ,

$$d + \mathcal{R}_{A^*} \left(\tilde{a}_c - \frac{1}{N} B_c^{(1)} \right) - 2 \left(\tilde{A} + \frac{1}{N} B_c^{(1)} \right) - \frac{N+2}{N^*} A_{\psi\chi}, \quad (4.10)$$

acting on χ , and

$$d - \left(\tilde{a}_c - \frac{1}{N} B_c^{(1)} \right) + a_f - \left(\tilde{A} + \frac{1}{N} B_c^{(1)} \right), \quad (4.11)$$

acting on η . Note that written this way the expression inside each bracket is invariant under (4.7)–(4.8). In the above we have introduced a (hopefully) self-evident notation for $SU(N)$ algebras in symmetric and anti-symmetric representations adequate for the ψ and χ fields. By construction the combination $\tilde{a}_c - \frac{1}{N} B_c^{(1)}$ belongs to the $SU(N)$ algebra.

Before proceeding, it is useful to record the relation between \tilde{A} , $A_{\psi\chi}$ and the straightforward A_ψ , A_χ , A_η gauge fields associated with the $U(1)_\psi$, $U(1)_\chi$, $U(1)_\eta$ fermion number symmetries:

$$\psi \rightarrow e^{i\alpha_\psi} \psi, \quad \chi \rightarrow e^{i\alpha_\chi} \chi, \quad \eta \rightarrow e^{i\alpha_\eta} \eta. \quad (4.12)$$

They can be read off from (4.9)–(4.11):

$$A_\psi = 2\tilde{A} + \frac{N-2}{N^*} A_{\psi\chi}, \quad A_\chi = -2\tilde{A} - \frac{N+2}{N^*} A_{\psi\chi}, \quad A_\eta = -\tilde{A}. \quad (4.13)$$

The gauge field tensors felt by the fermions corresponding to (4.9)–(4.11) are:

$$\begin{aligned} \mathcal{R}_S(F(\tilde{a}) - B_c^{(2)}) + 2(d\tilde{A} + B_c^{(2)}) + \frac{N-2}{N^*} dA_{\psi\chi}, \\ \mathcal{R}_{A^*}(F(\tilde{a}) - B_c^{(2)}) - 2(d\tilde{A} + B_c^{(2)}) - \frac{N+2}{N^*} dA_{\psi\chi}, \\ \mathcal{R}_{F^*}(F(\tilde{a}) - B_c^{(2)}) + F_f(a_f) - (d\tilde{A} + B_c^{(2)}). \end{aligned} \quad (4.14)$$

The anomalies are compactly expressed by a 6D anomaly functional

$$\begin{aligned} \mathcal{A}^{6D} = \int \frac{2\pi}{3!(2\pi)^3} \left\{ \text{tr}_c \left(\mathcal{R}_S(\tilde{F}_c - B_c^{(2)}) + 2(d\tilde{A} + B_c^{(2)}) + \frac{N-2}{N^*} dA_{\psi\chi} \right)^3 \right. \\ \left. + \text{tr}_c \left(\mathcal{R}_{A^*}(\tilde{F}_c - B_c^{(2)}) - 2(d\tilde{A} + B_c^{(2)}) - \frac{N+2}{N^*} dA_{\psi\chi} \right)^3 \right. \\ \left. + \text{tr}_{c,8} \left(-(\tilde{F}_c - B_c^{(2)}) + F_f(a_f) - (d\tilde{A} + B_c^{(2)}) \right)^3 \right\}. \end{aligned} \quad (4.15)$$

Expanding the 6D anomaly functional (4.15), one finds

$$\begin{aligned}
& \frac{2\pi}{3!(2\pi)^3} \int \{[(N+4) - (N-4) - 8] \text{tr}_c(\tilde{F}_c - B_c^{(2)})^3\} \\
& + \frac{2\pi N}{3!(2\pi)^3} \int \text{tr}_8(F_{\mathbb{F}}(a_f))^3 \\
& + \frac{1}{8\pi^2} \int \text{tr}_c(\tilde{F}_c - B_c^{(2)})^2 \left\{ (N+2) \left[2(d\tilde{A} + B_c^{(2)}) + \frac{N-2}{N^*} dA_{\psi\chi} \right] \right. \\
& \quad \left. + (N-2) \left[-2(d\tilde{A} + B_c^{(2)}) - \frac{N+2}{N^*} dA_{\psi\chi} \right] + 8[-(d\tilde{A} + B_c^{(2)})] \right\} \\
& + N \frac{1}{8\pi^2} \int \text{tr}_8(F_{\mathbb{F}}(a_f))^2 [-(d\tilde{A} + B_c^{(2)})] \\
& + \frac{1}{24\pi^2} \int \left\{ \frac{N(N+1)}{2} \left[2(d\tilde{A} + B_c^{(2)}) + \frac{N-2}{N^*} dA_{\psi\chi} \right]^3 \right. \\
& \quad + \frac{N(N-1)}{2} \left[-2(d\tilde{A} + B_c^{(2)}) - \frac{N+2}{N^*} dA_{\psi\chi} \right]^3 \\
& \quad \left. + 8N[-(d\tilde{A} + B_c^{(2)})]^3 \right\}, \tag{4.16}
\end{aligned}$$

by making use of the known formulas for the traces of quadratic and cubic forms in different representations. The terms in the first line, proportional to $\text{tr}_c(\tilde{F}_c - B_c^{(2)})^3$ trivially cancel out, reflecting the anomaly-free nature of the $SU(N)$ color group. Note also that the third and fourth lines, namely the terms containing $\text{tr}_c(\tilde{F}_c - B_c^{(2)})^2$, completely cancel each other, due to the fact that only anomaly-free combinations ($\tilde{U}(1)$ and $U(1)_{\psi\chi}$) of the $U(1)$ symmetries are being considered. A further gauging of the 1-form center symmetry (by the introduction of $B_c^{(2)}$) obviously does not affect this. Taking into account these cancellations one arrives at

$$\begin{aligned}
& \frac{2\pi N}{3!(2\pi)^3} \int \text{tr}_8(F_{\mathbb{F}}(a_f))^3 + N \frac{1}{8\pi^2} \int \text{tr}_8(F_{\mathbb{F}}(a_f))^2 [-(d\tilde{A} + B_c^{(2)})] \\
& + \frac{1}{24\pi^2} \int \left\{ \frac{N(N+1)}{2} \left[2(d\tilde{A} + B_c^{(2)}) + \frac{N-2}{N^*} dA_{\psi\chi} \right]^3 \right. \\
& \quad + \frac{N(N-1)}{2} \left[-2(d\tilde{A} + B_c^{(2)}) - \frac{N+2}{N^*} dA_{\psi\chi} \right]^3 \\
& \quad \left. + 8N[-(d\tilde{A} + B_c^{(2)})]^3 \right\}. \tag{4.17}
\end{aligned}$$

4.2 Observations

The mixed anomalies involving the 0-form $\tilde{U}(1)$ and $U(1)_{\psi\chi}$ symmetries and the 1-form discrete center symmetry \mathbb{Z}_N can now be found by studying the terms

$$\propto B_c^{(2)} \tilde{A}, \quad \propto B_c^{(2)} A_{\psi\chi}, \tag{4.18}$$

in the 5D WZW action, and considering the variations

$$\delta\tilde{A} = d\delta\tilde{A}^{(0)}, \quad \delta A_{\psi\chi} = d\delta A_{\psi\chi}^{(0)}, \tag{4.19}$$

which correspond to the phase transformations of the fermions (4.2), with

$$\delta\tilde{A}^{(0)} = \gamma, \quad \delta A_{\psi\chi}^{(0)} = \beta, \quad (4.20)$$

to give the anomalous variations of the partition function in the boundary 4D action (the anomaly inflow). It turns out that the anomaly expression (4.17) contains quite a remarkable set of interesting physics implications.

- (i) The first line of (4.17) simply represents the $SU(8)^3$ and $\tilde{U}(1) - [SU(8)]^2$ anomalies, dressed by the 2-form gauge field $B_c^{(2)}$. The associated matching of the conventional anomalies in the UV and IR has already been discussed in section 3.1. As noted in [17], once the standard 't Hooft anomaly matching equations are satisfied for *continuous* symmetries, the 1-form gauging (introduction of the $B_c^{(2)}$ fields and their fractional fluxes) does not affect the UV-IR anomaly matching.
- (ii) The terms proportional to $(d\tilde{A} + B_c^{(2)})^3$ in (4.17) cancel each other completely. This means that the mixed anomaly of the form

$$d\tilde{A}(B_c^{(2)})^2 \quad (4.21)$$

is absent. There is no obstruction in gauging the $\tilde{U}(1)$ symmetry together with the color-flavor locked \mathbb{Z}_N center symmetry. The $\tilde{U}(1)$ symmetry may well remain unbroken.

- (iii) The mixed anomaly of the form $dA_{\psi\chi}(B_c^{(2)})^2$ is present: it is given by

$$-\frac{4N^2}{N^*} dA_{\psi\chi}(B_c^{(2)})^2, \quad (4.22)$$

which is equal to

$$N^2(dA_\psi + dA_\chi)(B_c^{(2)})^2, \quad (4.23)$$

in view of eq. (4.13). In other words, $U(1)_{\psi\chi}$ symmetry cannot be gauged consistently, when the 1-form color \mathbb{Z}_N symmetry is gauged.

To the best of our knowledge, the last phenomenon is new. In all studies on generalized 't Hooft anomaly matching studied so far [20]-[33], nontrivial mixed anomalies concerned the possible breaking of some discrete symmetry. Here, we find that a *continuous* $U(1)$ symmetry is affected by a mixed 0-form-1-form anomaly. It appears that this is a typical, rather than exceptional, phenomenon in chiral gauge theories.

- (iv) The mixed anomaly (4.23) shows also that $(\mathbb{Z}_{N+2})_\psi$, $(\mathbb{Z}_{N-2})_\chi$, are both broken by the 1-form gauging of \mathbb{Z}_N . The action of a $(\mathbb{Z}_{N+2})_\psi$ transformation, for instance, is described by the variation in the 5D action

$$\delta A_\psi = d\delta A_\psi^{(0)}, \quad \delta A_\psi^{(0)} = \frac{2\pi k}{N+2}; \quad k \in \mathbb{Z} \quad (4.24)$$

which then yields the anomalous variation of the 4D action

$$\delta S = \frac{1}{8\pi^2} \int_{\Sigma_4} N^2 (B_c^{(2)})^2 \frac{2\pi k}{N+2} = \frac{2\pi k}{N+2} \mathbb{Z}, \quad (4.25)$$

under

$$\psi \rightarrow e^{\frac{2\pi i k}{N+2}} \psi, \tag{4.26}$$

where the fractional 't Hooft flux (4.4) of our $SU(N)/\mathbb{Z}_N$ theory has been taken into account.

Note that even though a generic $(\mathbb{Z}_{N+2})_\psi$ transformation changes the partition function, the effect of $k = N + 2$ transformation is found to be trivial. This is as it should be. By definition a $(\mathbb{Z}_{N+2})_\psi$ “transformation” with $k = N + 2$ means $\psi \rightarrow \psi$: it is not a transformation at all. In other words, the coefficient N^2 found above, (4.23), is significant.

Similarly for $(\mathbb{Z}_{N-2})_\chi$.

(v) However, a particular subgroup,

$$\mathbb{Z}_{N^*} \subset (\mathbb{Z}_{N+2})_\psi \times (\mathbb{Z}_{N-2})_\chi, \tag{4.27}$$

remains unaffected by the mixed anomaly involving the color-flavor locked 1-form \mathbb{Z}_N symmetry. This can be seen as follows. The condition imposed for the conservation by (4.23) is that

$$\frac{2\pi k}{N+2} + \frac{2\pi \ell}{N-2} = 2\pi \times \text{integer}. \tag{4.28}$$

Such a condition can be solved by

$$k = \frac{N+2}{N^*} p, \quad \ell = \frac{N-2}{N^*} (N^* - p), \quad p = 1, 2, \dots, N^*, \tag{4.29}$$

that is

$$\psi \rightarrow e^{2\pi i p/N^*} \psi; \quad \chi \rightarrow e^{-2\pi i p/N^*} \chi, \quad p \in \mathbb{Z}_{N^*}, \quad N^* = \text{GCD}(N+2, N-2). \tag{4.30}$$

Note that the conservation of this \mathbb{Z}_{N^*} subgroup is perfectly consistent with the assumption that condensation $\langle \psi \chi \rangle \neq 0$ forms in the infrared, as was noted in section 3.3, see (3.19).

(vi) We saw above (the point (iii)) that $U(1)_{\psi\chi}$ symmetry cannot be gauged consistently, when the 1-form color \mathbb{Z}_N symmetry is gauged. However the form of the mixed anomaly (4.22) shows that a global $U(1)_{\psi\chi}$ transformation gives a trivial phase to the partition function, for its discrete subgroup, $\mathbb{Z}_{4/N^*} \subset U(1)_{\psi\chi}$, as defined in (3.17), (3.18). This is perfectly consistent with the dynamical Abelianization, as $\langle \psi \chi \rangle \neq 0$ implies $U(1)_{\psi\chi} \rightarrow \mathbb{Z}_{4/N^*}$, thus no residual anomaly matching condition has to be satisfied.¹¹

¹¹We recall that $\mathbb{Z}_{4/N^*} \subset U(1)_{\psi\chi}$ is also a subgroup of $SU(8) \times \tilde{U}(1)$, thus it is naturally included in the global IR group as written in (3.14).

(vii) To understand better the situation, it is useful to see how this anomaly arises from the fractionalization of the $\tilde{U}(1)$ fluxes and the more mundane $U(1)_{\psi\chi} - [\tilde{U}(1)]^2$ anomaly. As one gauges $\mathbb{Z}_N^{(1)}$ (i.e., “1-form \mathbb{Z}_N symmetry”), i.e. one considers $\frac{SU(N)_c \times \tilde{U}(1)}{\mathbb{Z}_N} = U(N)$ gauge bundles that are not $SU(N)_c \times \tilde{U}(1)$ gauge bundles, both the instanton number and the fluxes of $\tilde{U}(1)$ are fractionalized. The $U(1)_{\psi\chi} - [SU(N)_c]^2$ (strong) anomaly vanishes identically, therefore the fractionalization of the $SU(N)_c$ instanton number has no consequences on $U(1)_{\psi\chi}$. However, the $U(1)_{\psi\chi} - [\tilde{U}(1)]^2$ anomaly does not vanish. In particular, by gauging $\tilde{U}(1)$ but not $\mathbb{Z}_N^{(1)}$, one sees from the $U(1)_{\psi\chi} - [\tilde{U}(1)]^2$ anomaly that the partition function gets a phase,

$$\mathcal{Z} \rightarrow e^{-i\beta \frac{4N^2}{N^*} \int d\tilde{A} \wedge d\tilde{A}} \mathcal{Z}, \tag{4.31}$$

under $e^{i\beta} \in U(1)_{\psi\chi}$. This phase is trivial ($2\pi\mathbb{Z}$) for $\mathbb{Z}_{4N^2/N^*} \subset U(1)_{\psi\chi}$. By gauging also $\mathbb{Z}_N^{(1)}$, as the $\tilde{U}(1)$ fluxes fractionalizes,

$$\frac{1}{2\pi} \int d\tilde{A} = \mathbb{Z} \rightarrow \frac{1}{2\pi} \int d\tilde{A} + B_c^{(2)} = \frac{1}{N} \mathbb{Z}, \tag{4.32}$$

the 't Hooft anomaly free subgroup of $U(1)_{\psi\chi}$ is further reduced to \mathbb{Z}_{4/N^*} .

(viii) The fact that \mathbb{Z}_{4/N^*} is free of mixed anomalies is, by itself, an interesting consistency check. This is because, being $\mathbb{Z}_{4/N^*} \subseteq SU(8) \times \tilde{U}(1)$, as $SU(8) \times \tilde{U}(1)$ does not suffer any mixed 't Hooft anomaly with $\mathbb{Z}_N^{(1)}$, also \mathbb{Z}_{4/N^*} must be free of such anomaly. Instead, from the calculation above, the fact that \mathbb{Z}_{4/N^*} is free of mixed anomaly is nontrivial: if the coefficient of the $dA_{\psi\chi} (B_c^{(2)})^2$ term in (4.17) were different from $-\frac{4N^2}{N^*}$, it would not hold.

(ix) $(\mathbb{Z}_8)_\eta$ and $SU(8)$ itself, are neither broken by the standard instantons nor in the presence of the 1-form gauge fields $(B_c^{(2)}, B_c^{(1)})$.

5 Summary and discussion

In this work, we revisited the infrared dynamics of the chiral $\psi\chi\eta$ theory, assuming dynamical Abelianization caused by bifermion condensate in the adjoint representation of the $SU(N)$ gauge group. In the first part, the symmetries of the system are studied and the working of the conventional 't Hooft anomaly matching has been briefly reviewed, and the possible form of the effective low-energy action is studied, by taking also into account also of the strong anomaly.

In the second part of the work, we have checked these ideas against more stringent constraints following the mixed-anomaly involving certain 0-form $U(1)$ symmetries and 1-form color-flavor locked \mathbb{Z}_N center symmetry. The results of the analysis, summarized in section 4.2, tell us that the proposed infrared physics, characterized by dynamical Abelianization, is consistent with the implications of the mixed anomalies and, perhaps, implied by them. The comparison between the implications of the mixed anomalies and those expected

	$\tilde{U}(1)$	$U(1)_{\psi\chi}$	$(\mathbb{Z}_{N+2})_{\psi}$	$(\mathbb{Z}_{N-2})_{\chi}$	$SU(8)_{\eta}$	\mathbb{Z}_{N^*}	\mathbb{Z}_{4/N^*}
Mixed Anomalies	✓	X	X	X	✓	✓	✓
Dyn. Abel.	✓	X	X	X	✓	✓	✓

Table 3. Dynamical Abelianization postulate of the present work is confronted with the implications of the mixed anomalies. ✓ for a conserved symmetry, X for a broken symmetry. The discrete \mathbb{Z}_{N^*} symmetry is defined in (3.19), or in (4.27)–(4.30). \mathbb{Z}_{4/N^*} is defined in (3.17).

from the assumption of the bifermion adjoint condensate and dynamical Abelianization, is shown in table 3. It is seen that the pattern of the symmetry realization (breaking) in the infrared, suggested by the mixed anomalies involving the gauged 1-form \mathbb{Z}_N symmetry, are well reproduced by the dynamical Abelianization proposed in this work.

In this work we have examined the consistency of the hypothesis of dynamical Abelianization, that a bifermion condensate forms in the infrared, of the form, (3.1), (3.2). It is possible that a bifermion condensate in the adjoint representation forms, but with a different symmetry breaking pattern, e.g.,

$$\langle (\psi\chi)_j^i \rangle = c \begin{cases} (N-m) \delta_j^i, & i, j = 1, \dots, m \\ -m \delta_j^i, & i, j = m+1, \dots, N \end{cases} \quad c \sim \mathcal{O}(\Lambda_0^3). \quad (5.1)$$

In this case, the strong gauge group would be broken as

$$SU(N)_c \rightarrow SU(m)_c \times SU(N-m)_{c'} \times U(1)_e, \quad (5.2)$$

where $U(1)_e$ is generated by $T \propto \text{diag}(\underbrace{N-m}_n, \underbrace{-m}_{N-m})$. A quick look at the massless spectrum expected from such a symmetry breaking shows that the system below the scale Λ_0 , is basically a pair of $\psi\chi\eta$ models with $SU(m)$ and $SU(N-m)$ gauge groups, respectively. The system is asymptotically free and continues to evolve towards the infrared. We shall not pursue further such a tumbling-like scenario, but it is possible that at the end the system flows into the full dynamical Abelianization, studied in section 3.

Even though we have focused our attention in this work on the $\psi\chi\eta$ theory for definiteness, there are other chiral gauge theories in which a bifermion condensate in the adjoint representation might occur and in which dynamical Abelianization might be decisive in determining the infrared physics. Possible examples are

- (i) $SU(N)$ theory (with N even), with odd number of fermions in the self-adjoint antisymmetric order $N/2$ tensor representation, studied in [17, 27];
- (ii) A generalization of the $SU(N)$ $\psi\chi\eta$ model with a matter fermion content,

$$\psi^{\{ij\}, m}, \quad \chi_{[ij]}, \quad \eta_j^B, \quad m = 1, 2, \quad B = 1, 2, \dots, N+12, \quad (5.3)$$

or

$$2 \square\square + \begin{array}{|c|} \hline \bar{\square} \\ \hline \square \\ \hline \end{array} + (N+12) \bar{\square}. \quad (5.4)$$

studied in [16], and

(iii) $SU(N)$ theories with fermions in the complex representation, $\frac{N-4}{k} \psi^{\{ij\}}$'s and $\frac{N+4}{k} \bar{\chi}_{[ij]}$,

$$\frac{N-4}{k} \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \oplus \frac{N+4}{k} \begin{array}{|c|} \hline \square \\ \hline \end{array}, \quad (5.5)$$

(k being a common divisor of $N-4$ and $N+4$) studied recently [17, 36].

In all of them, the conventional 't Hooft anomaly matching analysis is consistent with dynamical Abelianization hypothesis, and in some cases, the preliminary analysis involving the generalized symmetries and the mixed anomalies appears to give further support [17, 36] for it. Still, in some of this class of models, the symmetry breaking pattern may be different from dynamical Abelianization, allowing for more general types of infrared gauge theories. We will come back to the discussion of these models in a separate investigation.

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A Chiral Ward-Takahashi identities and NG bosons

Let us briefly review the fate of a continuous, global symmetry, say G_f , when a condensate forms in the infrared which is not invariant under it. Let the associated conserved current be J_μ and charge Q . The field ϕ (elementary or composite) condenses and breaks G_f . The field $\tilde{\phi}$ (elementary or composite) is such that it is transformed by the G_f transformation into ϕ :

$$Q \equiv \int d^3x J_0, \quad [Q, \tilde{\phi}] = \phi, \quad \langle \phi \rangle \neq 0. \quad (A.1)$$

The Ward-Takahashi like identity

$$\begin{aligned} \lim_{q_\mu \rightarrow 0} i q^\mu \int d^4x e^{-iq \cdot x} \langle 0 | T \{ J_\mu(x) \tilde{\phi}(0) \} | 0 \rangle &= \lim_{q_\mu \rightarrow 0} \int d^4x e^{-iq \cdot x} \partial_\mu \langle 0 | T \{ J_\mu(x) \tilde{\phi}(0) \} | 0 \rangle = \\ &= \int d^3x \langle 0 | [J_0(x), \tilde{\phi}(0)] | 0 \rangle = \langle 0 | [Q, \tilde{\phi}(0)] | 0 \rangle = \langle 0 | \phi(0) | 0 \rangle \neq 0. \end{aligned} \quad (A.2)$$

implies that the two-point function

$$\int d^4x e^{-iq \cdot x} \langle 0 | T \{ J_\mu(x) \tilde{\phi}(0) \} | 0 \rangle \quad (A.3)$$

is singular at $q^\mu \rightarrow 0$. Under the assumption that the G_f symmetry is broken spontaneously, such a singularity is due to a massless scalar particle in the spectrum. This particle, known as Nambu-Goldstone (NG) boson (a “pion”, symbolically) must be produced from the vacuum by the broken current J^μ :

$$\langle 0 | J_\mu(q) | \pi \rangle = i q_\mu F_\pi, \quad \langle \pi | \tilde{\phi} | 0 \rangle \neq 0. \quad (A.4)$$

such that the two point function (A.3), when contracted by q^μ , behaves as

$$\lim_{q^\mu \rightarrow 0} q^\mu \cdot q_\mu \frac{F_\pi \langle \pi | \tilde{\phi} | 0 \rangle}{q^2} \sim \text{const} . \quad (\text{A.5})$$

The constant F_π represents the amplitude for the broken current to produce the pion from the vacuum (the pion decay constant). The field $\tilde{\phi}$ is known as the pion interpolating field.

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