

$\text{Pin}(d, d)$ covariance of pure spinor equations for supersymmetric vacua and non-Abelian T-duality

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ABSTRACT: In a supersymmetric compactification of Type II supergravity, preservation of $\mathcal{N} = 1$ supersymmetry in four dimensions requires that the structure group of the generalized tangent bundle $TM \oplus T^*M$ of the six dimensional internal manifold M is reduced from $\text{SO}(6, 6)$ to $\text{SU}(3) \times \text{SU}(3)$. This topological condition on the internal manifold implies existence of two globally defined compatible pure spinors Φ_1 and Φ_2 of non-vanishing norm. Furthermore, these pure spinors should satisfy certain first order differential equations. In this paper, we show that non-Abelian T-duality (NATD) is a solution generating transformation for these pure spinor equations. We first show that the pure spinor equations are covariant under $\text{Pin}(d, d)$ transformations. Then, we use the fact NATD is generated by a coordinate dependent $\text{Pin}(d, d)$ transformation. The key point is that the flux produced by this transformation is the same as the geometric flux associated with the isometry group, with respect to which one implements NATD. We demonstrate our method by studying NATD of certain solutions of Type IIB supergravity with $\text{SU}(2)$ isometry and $\text{SU}(3)$ structure.

KEYWORDS: String Duality, Superstring Vacua, Flux compactifications

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1 Introduction

Non-Abelian T-duality (NATD) is an extension of Abelian T-duality, which works well as a solution generating mechanism for string backgrounds with non-Abelian isometries. Although the rules for NATD for the metric, the B-field and the dilaton field has been known for almost thirty years [1, 2], those for the RR fields has been understood relatively recently [3]. Since then, NATD has been widely used to generate new supergravity solutions with interesting holographic duals, see for example [4]–[13].

Recently, NATD has been described as a coordinate dependent $O(d, d)$ transformation, [14–19]. The transformation under NATD of the metric, the B-field and the dilaton field is determined by a coordinate dependent $O(d, d)$ matrix, which we will be calling T_{NATD} , and the RR fields transform under the corresponding $\text{Pin}(d, d)$ transformation generated by S_{NATD} . Here $\rho(S_{\text{NATD}}) = T_{\text{NATD}}$, and ρ is the usual double covering homomorphism from $\text{Pin}(d, d)$ to $O(d, d)$. This approach makes it possible to view NATD as a solution generating transformation for Double Field Theory (DFT), a framework which provides an $O(d, d)$ covariant formulation for effective string actions [20–23] by introducing dual, winding type coordinates. Since T_{NATD} is not constant, it is not immediate that NATD should be a solution generating transformation for DFT. However, it is a special $O(d, d)$ matrix, determined by the structure constants C_{ij}^k of the isometry algebra of the original background, and viewed as a twist matrix within the framework of Gauged Double

Field Theory, it gives rise to geometric flux $f_{ij}^k = C_{ij}^k$. This is the key point in showing that NATD is a solution generating transformation in DFT, which then provides a unified framework to prove that it is a solution generating transformation for Type II (generalised) supergravity. For details, see [17]. For a similar approach, also see [18].

An important question is whether supersymmetry is preserved under NATD. This problem is addressed in various papers, notably in [11–13, 24–27]. In [11], the transformation under NATD of the gravitino and dilatino supersymmetry variations were shown to be the same provided that the Killing spinors did not depend on the isometry directions along which NATD was applied. Equivalently, supersymmetry was shown to be preserved (at least for a large class of backgrounds with $SU(2)$ isometry) if the Killing spinors had vanishing Kossmann-Lie derivative with respect to the Killing vector fields generating the isometry. In the papers [24–27] NATD is applied to backgrounds with $\mathcal{N} = 1$ supersymmetry. For such backgrounds, conditions for supersymmetry can be described by using tools from generalised geometry [28–30], as was first shown in [31]. In this case, equations coming from supersymmetry variations can be shown to be equivalent to a set of differential equations to be obeyed by two globally defined pure spinors. This will be discussed in detail in section 3. It is possible to apply NATD directly on these pure spinors and check whether the transformed pure spinors still satisfy the differential equations coming from supersymmetry. This was the approach taken in [24–27, 32], where various backgrounds with interesting holographic duals were examined. In each case, the geometry supports an $SU(3)$ structure with associated pure spinors, and it was checked by direct computation that the NAT dual of these pure spinors indeed satisfied the supersymmetry equations proving that NAT dual background also preserved at least $\mathcal{N} = 1$ supersymmetry. For a similar approach where one works with backgrounds supporting a G_2 structure, see [33]. It should be noted that the Kossmann derivative of the Killing spinors along the isometry directions vanish if and only if the Lie derivative of the pure spinors (constructed as bilinears of these Killing spinors) along these directions vanish. This condition was met by all the examples considered in the papers mentioned above.

In this paper, we describe the transformation of pure spinors under NATD viewed as a $\text{Pin}(d, d)$ transformation by utilizing the tools developed in [17]. This enables us to prove that NAT dual of pure spinors of $\mathcal{N} = 1$ vacua still satisfy these differential equations (and hence, the dual background will also preserve at least $\mathcal{N} = 1$ supersymmetry since Bianchi identities are also satisfied as shown in [17]), provided that they have vanishing Lie derivative along the isometry directions. To this end, we will first prove that the pure spinor equations for preserved $\mathcal{N} = 1$ supersymmetry are $\text{Pin}(d, d)$ covariant, by embedding them in DFT. Among other things, using the framework of DFT makes it easier to show that the action of the exterior derivative operator and the $\text{Pin}(d, d)$ transformation must (anti-)commute, which is the trickiest part in the proof. This is when the $\text{Pin}(d, d)$ matrix is constant. When it is not constant, as is the case with NATD, the pure spinor equations will not be left invariant. However, NATD is generated by a very special $\text{Pin}(d, d)$ transformation yielding geometric flux, as we discussed above and again, this becomes the key point in showing that solutions of pure spinor equations are still solutions after NATD, provided that the pure spinors have vanishing Lie derivative along the isometry

directions. Compared to the methods already present in the literature our method has various advantages. First of all, describing the dualisation of pure spinors as a $\text{Pin}(d, d)$ transformation makes the computations rather direct, as it is not needed to specify an ansatz for the seed background, as long as the isometry is respected by the whole geometry, the fields and the pure spinors. In particular, our proof is valid for any isometry group, not just $\text{SU}(2)$. We should also note that our method makes the determination of the G structure of the NAT dual background rather straightforward. In the particular examples we will study in section 4, the seed background will be assumed to support $\text{SU}(3)$ structure, and we will see directly how the associated pure spinors are transformed to pure spinors associated with an $\text{SU}(2)$ structure. More generally, starting with a pure spinor associated with a generic $\text{SU}(3) \times \text{SU}(3)$ structure, it is possible to work out the G -structure of the NAT dual background, as is done in [34] for Abelian T-duality. In this paper, we will focus on the invariance of the $\mathcal{N} = 1$ supersymmetry equations on pure spinors under NATD and will leave the discussion of the transformation of a generic $\text{SU}(3) \times \text{SU}(3)$ structure to future study.

The plan of this paper is as follows: in section 2, we review the methods developed in [17]. In section 3, we focus on the pure spinor equations, which were shown in [31, 35, 36] to be equivalent to the supersymmetry equations to be satisfied by Type II vacua with at least $\mathcal{N} = 1$ supersymmetry. We embed these equations in the framework of Double Field Theory, so that the covariance under a general constant $\text{Pin}(d, d)$ transformation becomes manifest. Then, in a separate subsection we discuss the case when the $\text{Pin}(d, d)$ transformation is coordinate dependent (as it happens for NATD), and show that whether the transformed pure spinors satisfy the differential equations or not is completely determined by the fluxes generated by the $\text{Pin}(d, d)$ transformation. Section 4 is devoted to explicit examples. This is the section where we consider the ansatz for Type IIB supergravity studied in [25]. This ansatz is general enough to cover many examples that are important in the context of AdS/CFT duality. We transform the pure spinors associated with the $\text{SU}(3)$ structure supported by the geometry by applying the $\text{Pin}(d, d)$ transformation generating the NATD and show that the resulting pure spinors (as well as the resulting metric, B field and the RR fields) are in agreement with the ones found in [25]. We end with a discussion of results and future directions in section 5.

2 Non-Abelian T-duality as an $\text{O}(d, d)$ transformation

The purpose of this preliminary section is to review the methods developed in [17], where it was shown that NATD of a given d dimensional Type II background with isometry G can be obtained through the action of a coordinate dependent $\text{O}(d, d)$ matrix (also called T_{NATD}) obtained by embedding the following $\text{O}(n, n)$ matrix:

$$T_{\text{NATD}} = \begin{pmatrix} 0 & 1_n \\ 1_n & \theta_{ij} \end{pmatrix}, \quad \theta_{ij} = \nu_k C_{ij}{}^k. \quad (2.1)$$

in $\text{O}(d, d)$ in the standard way (see 4.2.28–4.2.29 of [37]). Here, ν_k are coordinates of the NAT dual background, and $C_{ij}{}^k$ are the structure constants of the n dimensional Lie

algebra of the isometry group G , so $i, j, k = 1, \dots, n$. To keep things simpler, we will always assume that the isometry group acts without isotropy and $d \geq n$.¹

The presence of the Lie group $O(d, d)$, which is the global symmetry group of DFT, makes it possible to describe the transformation under NATD of the Type II supergravity fields as a transformation in DFT. More precisely, one rewrites the supergravity fields in terms of the DFT fields \mathcal{H}, d, χ , where \mathcal{H} is the generalized metric that encodes the metric and the B-field, d is the generalized dilaton field and χ is the spinor field that packages the modified RR fields of Type II supergravity in the democratic formulation. These fields, being solutions of Type II supergravity also solve the DFT equations in the supergravity frame.² As it is assumed that the isometry is respected by all the fields in the background, it is possible to go to a non-holonomic frame so that the DFT fields, when written with respect to such a frame, are independent of the isometry coordinates. In [17] we refer to such fields as *untwisted fields*. Plugging the initial DFT fields in the field equations of DFT (of both the NS-NS sector and RR sector of Type II supergravity), one sees that the untwisted DFT fields satisfy the field equations of Gauged Double Field Theory (GDFT),³ with geometric fluxes associated with isometry. It was shown in [17] that the NAT dual DFT fields \mathcal{H}', d', χ' are found by acting on the untwisted fields $\mathcal{H}(x), d(x), \chi(x)$,⁴ by the $O(d, d)$ matrix (2.1) as below:

$$\mathcal{H}'(x, \nu) = T_{\text{NATD}}(\nu)\mathcal{H}(x)(T_{\text{NATD}})^t(\nu) \tag{2.2}$$

$$\mathcal{K}'(x, \nu) = S_{\text{NATD}}(\nu)\mathcal{K}(x)(S_{\text{NATD}})^{-1}(\nu) \tag{2.3}$$

$$F'(x, \nu) = e^{-\sigma(\nu)}e^{-B'(x, \nu)}S_{\text{NATD}}(\nu)e^{B(x)}F(x) \tag{2.4}$$

$$d'(x, \nu) = d(x) + \sigma(\nu). \tag{2.5}$$

¹When the action of the isometry group G is not free, n might be bigger than d , in fact it may even exceed ten. However, the methods that are discussed here can be generalized as described in [59], where they assume that the action is not free but that there is a local symmetry group $H \subset G$. This is the case when the target space includes a coset space G/H . They show in [59] that NATD can be applied without fixing the local H invariance and that the dual action still inherits the H invariance. This can then be used to gauge away the $n - d$ redundant degrees of freedom after dualisation. Specifically, the background matrix $E = g + B$ in (3.7) is taken to be n dimensional but in such a way that only the part that projects onto the coset G/H is non-trivial. For example, if the background is S^3 and the isometry group used for dualisation is $SO(4)$ then the metric g is taken as the diagonal 6×6 matrix $\text{diag}(1, 1, 1, 0, 0, 0)$, see appendix B of [59]. After transforming under (2.1), which is in $O(6, 6)$, one ends up with a six dimensional background. As demonstrated in [59], 3 of the 6 dual coordinates ν_1, \dots, ν_6 of the dual background can be removed by using the local $SO(3)$ invariance that survives dualisation so that NAT dual of S^3 is also 3 dimensional, as it has to be. Although [59] does not describe NATD as an $O(d, d)$ transformation as we do here, their formulas can be easily converted to formulas involving $O(d, d)/\text{Pin}(d, d)$ transformations, as discussed in detail in [17].

²DFT is consistent only when one imposes the so called strong constraint, that effectively eliminates half of the doubled coordinates. This constraint is trivially satisfied when the DFT fields and gauge parameters are independent of the winding type coordinates. In this case, the DFT fields are said to belong to the supergravity frame, since the DFT action and field equations reduce to those of Type II supergravity in the democratic formulation.

³GDFT is a deformation of DFT, obtained from a Scherk-Schwarz reduction and the deformation is determined entirely by the fluxes associated with the Scherk-Schwarz twist matrix [38–41].

⁴Here, we call the spectator coordinates excluding the isometry directions collectively x and the doubled coordinates of the NAT dual background collectively ν .

Here, $\mathcal{K} = C_d^{-1}\mathbb{S}$, and C_d is given in (C.7) in appendix C. \mathbb{S} is the element in $\text{Spin}^-(d, d)$ that projects onto \mathcal{H} under the double covering homomorphism ρ between $\text{Pin}(d, d)$ and $\text{O}(d, d)$, that is $\rho(\mathbb{S}) = \mathcal{H}$. Similarly, $\rho(S_{\text{NATD}}) = T_{\text{NATD}}$ and up to a sign it is given as [17]

$$S_{\text{NATD}} = C_n S_\theta = S_\beta C_n. \tag{2.6}$$

The factors S_θ and S_β in S_{NATD} are the $\text{Spin}^+(10, 10)$ elements that projects onto the $\text{SO}^+(10, 10)$ matrix that generates the B -transformations and β -shifts with $\theta_{ij} = \nu_k C_{ij}^k$ and $\beta_{ij} = \nu_k C_{ij}^k$, respectively. $B'(x, \nu)$ that appears in (2.4) is read off from $\mathcal{H}'(x, \nu)$ in (2.2). The field $\sigma(\nu)$ in (2.5) and (2.4) is non-vanishing only when the isometry group is non-unimodular. For the purposes of this paper, it can be taken to be zero. The fact that the NAT dual fields can be written in terms of DFT fields as in (2.2)–(2.5) makes it straightforward to prove that NATD is a solution generating transformation for the field equations of Type II supergravity. In fact, all one has to do is to show that the fields in (2.2)–(2.5) solve the DFT equations, since the coordinates (x, ν) can be identified with the physical space-time coordinates, putting all the fields in the supergravity frame. Due to the special form of the fields, this amounts to showing that the untwisted fields $\mathcal{H}(x), d(x), F(x)$ appearing on the right hand side of (2.2)–(2.5) solve the field equations of GDFT, with fluxes generated by T_{NATD} . Now, the key point is that this is exactly the same as the geometric flux associated with the isometry group, that is, $f_{ij}^k = C_{ij}^k$, and we already know that the untwisted fields satisfy the GDFT equations with geometric flux. As a result, one concludes that NATD is a solution generating transformation for Type II supergravity, both in the NS-NS and the RR sector, simply owing to the fact that fluxes are preserved. The idea that preservation of flux should be a guiding principle in determining whether an $\text{O}(d, d)$ transformation is a solution generating transformation for supergravity has also been used in [42–45] and very recently in [46]. A similar approach was applied in [47] to find solution generating U-duality transformations within the framework of exceptional field theory. In the next section, we will see that the same principle also plays a key role in examining preservation of supersymmetry under NATD.

3 Covariance of pure spinor equations under $\text{Pin}(d, d)$

As was first shown in the seminal paper [35], the conditions to be obeyed by the internal space in a supersymmetric compactification of Type II supergravity can be neatly described within the framework of generalized complex geometry [28–30]. Demanding that the four dimensional solution preserves at least $\mathcal{N} = 1$ supersymmetry implies that the structure group of the generalized tangent bundle $TM \oplus T^*M$ of the six dimensional internal manifold M is reduced from $\text{SO}(6, 6)$ to $\text{SU}(3) \times \text{SU}(3)$. This topological condition on the internal manifold implies the existence of two globally defined compatible pure spinors Φ_1 and Φ_2 of non-vanishing norm. These $\text{Cliff}(6, 6)$ spinors can be constructed from the internal spinors arising from the 10 dimensional Killing spinors generating the supersymmetry transformations in 10 dimensions. A $\text{Cliff}(6, 6)$ spinor can be mapped to a non-homogenous differential form (a polyform) through the Clifford map. It was shown

in [31, 35, 36] (also see [48]) that the Killing spinor equations coming from supersymmetry variations is equivalent to the following differential equations for the two pure spinors:

$$d(e^{2A-\phi}e^B \wedge \Phi_1) = 0, \tag{3.1}$$

$$d(e^{2A-\phi}e^B \wedge \Phi_2) = e^{2A-\phi}dA \wedge e^B \wedge \bar{\Phi}_2 + \frac{i}{8}e^{3A}e^B \wedge \lambda(*_6F). \tag{3.2}$$

For computational details on derivation of these equations, see appendix A of [36] and appendix B of [48]. For the corresponding equations for general ten dimensional supersymmetric solutions which do not necessarily involve a four dimensional Minkowski space factor, see [49] (the equations above are discussed as a special case in their section 4.1). Note that for our purposes, we have presented the equations in a form where the B field appears explicitly, rather than writing them in terms of the differential operator $d_H = d + H \wedge$ as was originally done in [31, 35, 36].⁵

In the equations above, A is the warp factor that appears in the compactification ansatz

$$ds^2 = e^{2A(y)}dx_{3,1}^2 + g_{mn}dy^m dy^n, \quad m, n = 1, \dots, 6. \tag{3.3}$$

ϕ is the dilaton field and $*_6$ is the Hodge duality on the six dimensional internal manifold. F is related to the polyform $F^{(10)}$ that encodes the RR fields in the democratic formulation of supergravity [50] in the following way

$$F^{(10)} = F + \text{vol}_4 \wedge *_6(\lambda F). \tag{3.4}$$

Here, $F = F_0 + F_2 + F_4 + F_6$ for Type IIA and $F = F_1 + F_3 + F_5$ for Type IIB, and they are internal forms having components only along the six dimensional internal space. Also,

$$\lambda(A_n) \equiv (-1)^{Int[n/2]}A_n = (-1)^{n(n-1)/2}A_n \tag{3.5}$$

for an n -form A_n . As a $\text{Spin}(d, d)$ spinor F has positive chirality for Type IIA and is of negative chirality for Type IIB. The chirality of the pure spinor Φ_1 is the same as that of the RR fluxes and the pure spinor Φ_2 has opposite chirality.

In the next two subsections, we will show that these equations are covariant under $\text{Pin}(d, d)$ transformations.

3.1 Constant $\text{Pin}(d, d)$ transformation

Although (2.1) is non-constant, we start by considering the transformation of the pure spinor equations under a constant $O(d, d)$ matrix T and the corresponding $\text{Pin}(d, d)$ matrix P with $\rho(P) = T$, where ρ is the double covering homomorphism

$$\rho : \text{Pin}(d, d) \rightarrow O(d, d).$$

⁵In fact, it is more common in the literature to express these equation in terms of the operator $d_H = d - H \wedge$. This involves a field redefinition $H \rightarrow -H$ for Type IIA with respect to the conventions of [50]. In section 4 we will be looking at a IIA background with non-trivial B-field, so we prefer to agree with the conventions of [50] for Type IIA (since the conventions adopted in [17] agree with those of [50]), and this means we need the above field redefinition for Type IIB. This also means that our convention for the B-field is the opposite of that of [25]. Indeed, the B-field we find in (4.8) in section 4 has opposite sign compared to the B-field found in [25].

The transformation of the RR fluxes F under $P \in \text{Pin}(d, d)$ is [51]

$$F \rightarrow F' = P.F = e^{-B'} P e^B F. \quad (3.6)$$

Here, the transformation of the B-field is read off from the antisymmetric part of the transformed background matrix $E \equiv g + B$:

$$E \rightarrow E'(g', B') = T.E(g, B) = (aE + b)(cE + d)^{-1} \quad (3.7)$$

when the $O(d, d)$ matrix T is of the form

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (3.8)$$

Note that this is equivalent to the aforementioned transformation of the generalized metric [52]

$$\mathcal{H}'(g', B') = T\mathcal{H}(g, B)T^t. \quad (3.9)$$

It is known that the pure spinors transform under $\text{Pin}(d, d)$ in basically the same way as the RR fields transform. However, there is a slight change which makes sure that the norms of the pure spinors are kept invariant (up to a sign). The norm $\|\Phi\|$ of a pure spinor Φ is defined [31, 35, 36] via the Mukai pairing \langle, \rangle , which is an invariant bilinear form on spinors (see appendix C):

$$\langle \Phi, \bar{\Phi} \rangle = -i \|\Phi\|^2 \text{vol} \quad (3.10)$$

where vol is the volume form determined by the metric. As discussed in detail in appendix C, the Mukai pairing has the following transformation property under the action of certain elements P of $\text{Pin}(d, d)$:

$$\langle P\Phi_1, P\Phi_2 \rangle = \pm \langle \Phi_1, \Phi_2 \rangle, \quad (3.11)$$

where either $P \in \text{Spin}(d, d)$ or is of the form $P = C_n S$ or $P = S C_n$ with $S \in \text{Spin}(d, d)$ and C_n is as in (C.7) (Recall that the NATD matrix (2.6) is of this form). On the other hand the volume form $\text{vol} = \star_d 1$ transforms as

$$\star_d 1 = \sqrt{\det g} dy^1 \wedge \cdots \wedge dy^d \rightarrow \star'_d 1 = \sqrt{\det g'} dy^1 \wedge \cdots \wedge dy^d = G \sqrt{\det g} dy^1 \wedge \cdots \wedge dy^d = G \star_d 1,$$

where

$$G \equiv \frac{\sqrt{\det g'}}{\sqrt{\det g}} = \det(cE + d)^{-1}. \quad (3.12)$$

This follows immediately from the transformation of the metric g which can be read off from the symmetric part of E' in (3.7). Hence, the transformation of the pure spinors under $\text{Spin}(d, d)$ must be accompanied by a scale transformation with a factor of \sqrt{G} :⁶

$$\Phi \rightarrow \Phi' = \sqrt{G} P.\Phi = \sqrt{G} e^{-B'} P e^B \Phi \quad (3.13)$$

⁶In the framework of generalized geometry, the pure spinors Φ and $\sqrt{G}\Phi$ correspond to the same generalized complex structure, as they belong to the same pure spinor line sub-bundle of $\bigwedge^\bullet T^*$.

so that the norm (3.10) remains invariant up to a sign. Note that this extra factor of \sqrt{G} also ensures that

$$e^{2A'-\phi'} e^{B'} \wedge \Phi'_{1,2} = P \left(e^{2A-\phi} e^B \wedge \Phi_{1,2} \right) \quad (3.14)$$

since A is invariant and the dilaton field ϕ transforms exactly with the same \sqrt{G} factor:

$$e^{\phi'} = \sqrt{G} e^{\phi}. \quad (3.15)$$

The transformation rule (3.15) follows from the fact that the generalized dilaton field $e^{-2d} = \sqrt{\det g} e^{-2\phi}$ is invariant under $O(d, d)$ (consider the equation (2.5) with $\sigma = 0$), that is $e^{-2d'} = e^{-2d}$ so that:

$$e^{-2\phi'} \sqrt{\det g'} = e^{-2\phi} \sqrt{\det g}. \quad (3.16)$$

Now all we have to do is to figure out the transformation of the term involving Hodge duality on the right hand side of equation (3.2) and also to show that the action of P and the exterior derivative operator d on the $Clif(d, d)$ spinors $\Phi_{1,2}$ and F commutes.

For both purposes, we find it useful to embed these equations in Double Field Theory. Towards this we extend the exterior derivative operator $d = \frac{1}{2} \Gamma^i \partial_i$ to

$$d + \tilde{d} \equiv \frac{1}{2} \Gamma^M \partial_M = \frac{1}{2} (\Gamma^i \partial_i + \Gamma_i \tilde{\partial}^i) = \psi^i \partial_i + \psi_i \tilde{\partial}^i. \quad (3.17)$$

Here, the gamma matrices $\Gamma^M = (\Gamma_i, \Gamma^i)$ are the Clifford algebra elements satisfying the following Clifford product relations:

$$\{\Gamma_i, \Gamma^j\} = 2\delta_i^j, \quad \{\Gamma_i, \Gamma_j\} = 0, \quad \{\Gamma^i, \Gamma^j\} = 0, \quad (3.18)$$

For future reference we also defined in (3.17)

$$\psi^M \equiv \frac{1}{\sqrt{2}} \Gamma^M. \quad (3.19)$$

Also, we write $\star \lambda(F)$ as [21, 41]

$$\star \lambda(F) = -C_d^{-1} S_g^{-1} F, \quad (3.20)$$

where $S_g^{-1} = S_{g^{-1}}$ is the $\text{Spin}(d, d)$ element that projects onto the $\text{SO}(d, d)$ element

$$h_{g^{-1}} \equiv \begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix} \quad (3.21)$$

under the double covering homomorphism ρ that is, $\rho(S_{g^{-1}}) = h_{g^{-1}}$. Note that the equation (3.20) is valid in all even dimensions,⁷ in particular for \ast_6 with $d = 6$.

It is useful to write $C_d S_{g^{-1}}$ as $e^{-B} \mathcal{K}_d e^B$ where $\mathcal{K}_d = C_d^{-1} \mathbb{S}$ and $\mathbb{S} \equiv S_B^\dagger S_{g^{-1}} S_B$ is the $\text{Spin}(d, d)$ element that projects onto the generalized metric \mathcal{H} . Indeed,

$$e^{-B} \mathcal{K}_d e^B = e^{-B} C_d^{-1} \mathbb{S} e^B = e^{-B} C_d^{-1} S_B^\dagger S_{g^{-1}} S_B e^B \quad (3.22)$$

$$= e^{-B} C_d^{-1} C_d e^B C_d^{-1} S_{g^{-1}} e^{-B} e^B = C_d^{-1} S_{g^{-1}}. \quad (3.23)$$

⁷In odd dimensions, the definition of \mathcal{K} involves $(\psi^i + \psi_i)$, rather than the $(\psi^i - \psi_i)$ in (C.11) and (C.12). See [21] for more details.

where we have used that $S_B = e^{-B}$ and $S_B^\dagger = C_d S_{-B} C_d^{-1} = C_d e^B C_d^{-1}$. Rewriting (3.20) for $d = 6$ and in terms of \mathcal{K}_d , we have

$$*_6 \lambda(F) = -e^{-B} \mathcal{K}_6 e^B F. \quad (3.24)$$

Rewriting the equations (3.1) and (3.2) we get

$$\Gamma^M \partial_M (e^{2A-\phi} e^B \wedge \Phi_1) = 0, \quad (3.25)$$

$$\Gamma^M \partial_M (e^{2A-\phi} e^B \wedge \Phi_2) = e^{2A-\phi} \Gamma^M \partial_M A \wedge e^B \wedge \bar{\Phi}_2 \mp \frac{i}{8} e^{3A} \mathcal{K}_6 e^B F. \quad (3.26)$$

These equations reduce to equations (3.1) and (3.2) in the supergravity frame where fields do not depend on the winding type coordinates so that $\tilde{\partial}^i = 0$. The upper sign in the last term of (3.26) is for Type IIB and the lower sign is for Type IIA. This is because in six dimensions $*_6 \lambda = \lambda *_6$ for odd degree forms, whereas $*_6 \lambda = -\lambda *_6$ for even degree forms.

We know that $F^{(10)}$ in (3.4) transforms as in (3.6). Let us discuss what this implies for the transformation of the internal forms F . We have

$$\begin{aligned} F^{(10)'} &= e^{-B'} P e^B (F - \text{vol}_4 \wedge e^{-B} \mathcal{K}_6 e^B F) \\ &= e^{-B'} P e^B F - \text{vol}_4 \wedge e^{-B'} P \mathcal{K}_6 e^B F. \end{aligned} \quad (3.27)$$

where we have used (3.24) and the fact that vol_4 , being an even form, commutes with all elements of $\text{Pin}(d, d)$. To rewrite (3.27) in the form (3.4) we first define

$$F' \equiv e^{-B'} P e^B F, \quad (3.28)$$

which is again an internal form, as all the $\text{Pin}(d, d)$ operators on the left hand side have actions only on the internal space and then use the fact that under $P \in \text{Pin}(d, d)$ the field \mathcal{K}_d transforms as

$$\mathcal{K}_d \rightarrow P \mathcal{K}_d = \mathcal{K}'_d = P \mathcal{K}_d P^{-1}. \quad (3.29)$$

Inserting a $P^{-1} P$ after \mathcal{K} in the second term of the right hand side of (3.27) and using (3.24) and (3.28), we obtain

$$\begin{aligned} F^{(10)'} &= F' - \text{vol}_4 \wedge e^{-B'} \mathcal{K}'_6 e^{B'} F' \\ &= F' + \text{vol}_4 \wedge \tilde{*}_6 \lambda(F'). \end{aligned} \quad (3.30)$$

Note that F' has components only along the six dimensional deformed space and the Hodge duality is taken with respect to the metric after the $O(d, d)$ transformation. This shows us that not only the polyform $F^{(10)}$ that encodes the RR fields in the democratic formulation, but also the internal polyform F that appears in the pure spinor equations (3.1), (3.2) transform in the expected way as given in (3.28).

Using the transformation properties (3.29) and (3.28), (3.13), (3.15) and the fact that A is invariant under $\text{Pin}(d, d)$ we also see that

$$e^{3A'} \mathcal{K}'_d e^{B'} F' = P \left(e^{3A} \mathcal{K}_d e^B F \right). \quad (3.31)$$

In order to prove the covariance of the pure spinor equations under $\text{Pin}(d, d)$ we next discuss whether or not the generalized exterior derivative operator $\Gamma^M \partial_M$ commutes with the action of $\text{Pin}(d, d)$. We first start with $\text{Spin}(d, d)$ and show

$$\Gamma^M \partial_M (S \chi) = S (\Gamma^M \partial_M \chi), \quad S \in \text{Spin}(d, d) \quad (3.32)$$

for any spinor field χ . Using the relations

$$(h^{-1})^M_A \Gamma^A = S^{-1} \Gamma^M S, \quad (3.33)$$

where h is the $\text{SO}(d, d)$ element that satisfies $\rho(S^{-1}) = h$, we see that for constant $S \in \text{Spin}(d, d)$:

$$\Gamma^M \partial_M (S \chi) = \Gamma^M S \partial_M \chi = S \Gamma^A (h^{-1})^M_A \partial_M \chi. \quad (3.34)$$

Then, the commutation relation (3.32) holds, if we have

$$(h^{-1})^M_A \partial_M \chi = \partial_A \chi \quad (3.35)$$

Note that we would have in DFT,

$$(h^{-1})^M_A \partial_M \chi(hX) = \partial'_A \chi(X'). \quad (3.36)$$

since one also transforms $X \rightarrow X' = hX$. However in all the examples we will be looking at, the transformation generated by P will act only along the coordinates on which the pure spinors will not depend, so that we will always have $X' = X$ and hence $\partial'_A \chi = \partial_A \chi$. For example, if the background possesses d commuting isometries, it is possible to choose coordinates such that the fields depend on only $10 - d$ of the 10 coordinates. Associated with the d isometries, there is an $\text{O}(d, d)$ Abelian T-duality group acting on the background along these d coordinates. Since the coordinates have been chosen in such a way that none of the fields (including the global spinor fields) do not depend on these directions, we have $\partial'_A \varphi(X) = \partial_A \varphi(X)$, where φ denotes any field or gauge parameter in the theory and A runs through the $10 - d$ coordinates. To summarize, equation (3.32) holds as desired, as long as the condition (3.35) is satisfied. This immediately implies (using (3.13), (3.15), (3.29), (3.28) and the invariance of A) that the pure spinor equations (3.25), (3.26) are covariant under $\text{Spin}(d, d)$ transformations. Note that there is no sign flip in front of the last term on the right hand side of (3.26), since $\text{Spin}(d, d)$ transformations takes a solution of Type IIA/IIB to a solution also of Type IIA/IIB. However, a $\text{Pin}(d, d)$ transformation which involves odd number of reflections maps a solution of Type IIA to a solution of Type IIB and vice versa, and hence the sign of the aforementioned term in (3.26) flips after the transformation. Despite this, the pure spinor equations (3.25), (3.26) are still covariant, since for such P , the differential operator $d = \Gamma^M \partial_M$ and P anti-commutes, as we will now discuss.

Consider the $\text{Pin}(d, d)$ elements Λ_i given in (C.8). From the Clifford commutation relations (3.18) one can easily compute⁸

$$\Lambda_i \Gamma^M (\Lambda_i)^{-1} = \begin{cases} -\Gamma_i & \text{if } \Gamma^M = \Gamma^i \\ -\Gamma^i & \text{if } \Gamma^M = \Gamma_i \\ -\Gamma^M & \text{otherwise} \end{cases} \quad (3.38)$$

In this paper, we will be looking at the $\text{Pin}(d, d)$ elements that can be written as a product of $\text{Spin}(d, d)$ elements and Λ_i , simply because the NATD matrix is of this form. Our discussions here can be straightforwardly extended so as to include the $\text{Pin}(d, d)$ elements which also involve the elements Λ_i^+ given in (C.11), but we refrain from doing that in order to avoid equations cluttered with pluses and minuses.

Due to the relations (3.38), we see that

$$d(\Lambda_i \chi) = \Gamma^M \partial_M (\Lambda_i \chi) = -\Lambda_i \Gamma^M \partial_M \chi = -\Lambda_i d\chi, \quad (3.39)$$

provided that $M \neq i$ or $M \neq -i$, which then implies that the differential $d = \Gamma^M \partial_M$ and P commutes if P involves an even number of Λ_i s and they anti-commute otherwise. As discussed above, this condition is automatically satisfied for Abelian T-duality, due to the existence of d commuting isometries. This makes it possible to choose a coordinate system such that none of the fields depend on the coordinates along which the (constant) $\text{O}(d, d)/\text{Pin}(d, d)$ transformation acts, and hence the desired commutation or anti-commutation relations hold. Therefore, we conclude that the pure spinor equations are covariant under Abelian T-duality. As for NATD, (3.35) is also satisfied with a convenient choice of coordinates (again due to existence of isometries), but we still need to discuss the situation with non-constant P , since the NATD matrix (2.1) is not constant as has been assumed above. This discussion will be carried out in the next section.

Note that the covariance of the equations (3.1), (3.2) for certain cases has been discussed before, albeit in a different language. For example, in [34] the covariance of the pure spinor equations for backgrounds with $\text{U}(1)$ isometry was shown.⁹ Another example is the Lunin-Maldacena (LM) transformation (also called TsT transformation) which can be described as an $\text{O}(2, 2)$ transformation [53–55]. In [56] the transformation of the pure spinors corresponding to an $\text{SU}(3)$ structure under this $\text{O}(2, 2)$ transformation was discussed within the framework of generalized complex geometry, as we do here. That the transformed pure spinors (now corresponding to an $\text{SU}(2)$ structure) still satisfy the supersymmetry equations was also checked for this particular $\text{O}(2, 2)$ transformation.

⁸Note that (3.38) implies that $\rho(\Lambda_i) = h_i$, where

$$h_i = - \begin{pmatrix} 1 - E_i & E_i \\ E_i & 1 - E_i \end{pmatrix}, \quad (E_i)_{jk} = \delta_{ij} \delta_{ik}. \quad (3.37)$$

⁹More precisely, they studied the factorized duality for $d = 1$. See [37] for the discussion of how factorized duality, B-shifts and $\text{GL}(d)$ transformations are embedded in the T-duality group $\text{O}(d, d)$ for flat and curved backgrounds with d commuting isometries.

3.2 Non-constant $\text{Pin}(d, d)$ transformation

In this subsection, we extend the discussion in the previous subsection to the case where the $\text{Pin}(d, d)$ transformation (and hence the corresponding $\text{O}(d, d)$ transformation) depends on some of the internal coordinates. This is important, as the NATD transformation and the Yang-Baxter transformation are known to be generated by such coordinate dependent $\text{Pin}(d, d)$ transformations. The transformation properties summarized in (3.14), (3.31) are obviously still valid, even when $P \in \text{Pin}(d, d)$ is coordinate dependent. However, one has to be more careful in discussing the commutation of the exterior derivative operator d and the action of P , as now d also acts on P .

Let us first discuss the case when the $\text{Pin}(d, d)$ matrix does in fact lie in the subgroup $\text{Spin}^+(d, d)$, $P = S \in \text{Spin}^+(d, d)$ (so that we can use the useful identity (3.41)):

$$\begin{aligned} \Gamma^M \partial_M (S \chi(X)) &= \left\{ \Gamma^M S \partial_M + \Gamma^M S (S^{-1} \partial_M S) \right\} \chi(X) \\ &= S \Gamma^A (h^{-1})^M_A \left(\partial_M + S^{-1} \partial_M S \right) \chi(X), \end{aligned} \quad (3.40)$$

where $\rho(S^{-1}) = h$ and in passing to the second line, we have used (3.33). To calculate the second term in (3.40) we use an important identity that follows from the fact that the Lie algebras of $\text{SO}(d, d)$ and $\text{Spin}(d, d)$ are isomorphic:

$$\begin{aligned} \Gamma^A (h^{-1})^M_A S^{-1} \partial_M S &= \frac{1}{4} \Omega_{ABC} \Gamma^A \Gamma^B \Gamma^C \\ &= \frac{1}{12} f_{ABC} \Gamma^A \Gamma^B \Gamma^C \chi(X) - \frac{1}{2} f_B \Gamma^B \chi(X). \end{aligned} \quad (3.41)$$

Here, f_{ABC} are the fluxes associated with the matrix S (see [17, 41] for the definition).

Now, we again assume that the transformation matrix S is such that (3.35) is obeyed. We emphasize again that this condition is trivially satisfied if the field χ does not depend on the coordinates along which S and hence h acts nontrivially. This is indeed the case for NATD and is guaranteed by the fact that NATD acts along isometry directions. Then, under this assumption, we have

$$\Gamma^M \partial_M (S \chi) = S (\Gamma^A \nabla_A \chi), \quad (3.42)$$

where

$$\nabla_A = \partial_A + \frac{1}{12} f_{ABC} \Gamma^B \Gamma^C - \frac{1}{2} f_A. \quad (3.43)$$

Let us now discuss what happens when P involves odd number of Λ_i factors, so that P does not lie in the $\text{Spin}(d, d)$ subgroup (if the number of Λ_i factors is even, then S is still in $\text{Spin}(d, d)$, albeit not in the subgroup $\text{Spin}^+(d, d)$ connected to identity). For simplicity, we assume that P is of the form $P = C_n S$, where $S \in \text{Spin}^+(d, d)$ and C_n is as in (C.7) with n odd. Equation (3.33) is valid for all $\text{Pin}(d, d)$ elements, so we have

$$(h_1 \cdots h_n U)^M_A \Gamma^A = P^{-1} \Gamma^M P, \quad (3.44)$$

where U is the $\text{SO}^+(d, d)$ element that satisfies $\rho(S) = U$, and h_i satisfy $\rho(h_i) = \Lambda_i$ and are given in (3.37). When n is odd, it can be easily seen that $h_1 \cdots h_n = -J_n^d$, where J_n^d is

the $O(d, d)$ matrix obtained by embedding the $O(n, n)$ matrix

$$J_n = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}, \quad (3.45)$$

in $O(d, d)$ in the usual way (see 4.2.28–4.2.29 of [37]). Rewriting the first line of (3.40) for $P = C_n S$ and using (3.44) we have

$$\begin{aligned} \Gamma^M \partial_M (P \chi(X)) &= \left\{ \Gamma^M P \partial_M + \Gamma^M P (P^{-1} \partial_M S) \right\} \chi(X) \\ &= -P \Gamma^A (J_n^d U)_A^M \left(\partial_M + S^{-1} \partial_M S \right) \chi(X), \end{aligned} \quad (3.46)$$

where we have also used $P^{-1} \partial_M P = S^{-1} \partial_M S$ for $P = C_n S$. Using (3.41) again, one can see that we have

$$\Gamma^M \partial_M (P \chi) = -P (\Gamma^A \nabla_A \chi), \quad (3.47)$$

where ∇ is as in (3.43), now with fluxes $f'_{ABC} = (J_n^d)_A^D f_{DBC}$ with f being the fluxes associated with the $\text{Spin}^+(d, d)$ matrix S .

Collecting the results in (3.14), (3.31), (3.32), (3.42) and (3.47), we conclude that the fields *after* the transformation generated by the non-constant $P \in \text{Pin}(d, d)$ satisfy the supersymmetry equations (3.25), (3.26) if and only if the fields *before* the transformation satisfy the following equations, which can be regarded as a deformation of those in (3.25), (3.26) determined by the fluxes associated with P .

$$\Gamma^M \nabla_M (e^{2A-\phi} e^B \wedge \Phi_1) = 0, \quad (3.48)$$

$$\Gamma^M \nabla_M (e^{2A-\phi} e^B \wedge \Phi_2) = e^{2A-\phi} \Gamma^M \partial_M A \wedge e^B \wedge \bar{\Phi}_2 \mp (-1)^n \frac{i}{8} e^{3A} \mathcal{K}_6 e^B F. \quad (3.49)$$

Here, n is the number of Λ_i factors that appear in the definition of $P = C_n S, S \in \text{Spin}^+(d, d)$.

Before we move on to the next subsection, we would like note that the transformation of pure spinor equations under a non-constant $O(d, d)$ transformation was also studied in [57] and [58]. They called such transformations twist transformations and also used them as solution generating transformations in Type II theory.

3.3 Invariance under NATD

As we discussed in section 2, the transformation under NATD of the fields in the NS-NS sector can be performed via the action of the matrix T_{NATD} given in (2.1). Accordingly, the transformation of the RR fields can be performed via the projected element S_{NATD} under the double covering homomorphism between $\text{Pin}(d, d)$ and $O(d, d)$, see equations (2.2)–(2.5). An important point that should be stressed here is that S_{NATD} and T_{NATD} act on the so-called *untwisted fields* $g(X), B(X), \phi(X), \Phi(X)$ and $F(X)$. These untwisted fields depend only on $10 - \dim G$ coordinates, where $\dim G$ is the dimension of the non-Abelian isometry group G ,¹⁰ and is related to the background fields $g(X, \theta), B(X, \theta),$

¹⁰For simplicity, we assume here that the action of G is free. However, the whole argument can be extended to the case where isotropy group of the action of G is non-trivial, see [59].

$\phi(X, \theta), \Phi(X, \theta)$ and $F(X, \theta)$ exactly as in (2.2)–(2.5), where we replace the NATD coordinates ν with the space-time coordinates θ associated with the isometry directions¹¹ and the matrices T_{NATD} and S_{NATD} with L and S_L , respectively with,

$$L = \begin{pmatrix} l^T & 0 \\ 0 & l^{-1} \end{pmatrix}, \tag{3.50}$$

and $S_L \in \text{Spin}^+(d, d)$ is such that $\rho(S_L) = L$. Here, l is the $\text{GL}(10)$ matrix obtained by embedding the $\text{GL}(d)$ matrix l_d with components $(l_d)^I{}_i = l^I{}_i$ such that $(l_d)^I{}_m = l^a{}_i = 0$ and $(l_d)^a{}_m = \delta^a{}_m$. $l^I{}_i$ are components of the left invariant 1-forms $\sigma^I = l^I{}_i d\theta^i$ on G defined from the Maurer-Cartan form: $g^{-1}dg = \sigma^I T_I$ with T_I forming a basis for the Lie algebra \mathcal{G} of the isometry group G . For more details see [17]. We also assume that the pure spinors associated with the background respect the isometry so that (3.13) also holds for both pure spinors:¹²

$$\Phi(X, \theta) = \sqrt{G} S_L(\theta) \cdot \Phi(X) = \sqrt{\det l} e^{-B'(X, \theta)} S_L(\theta) e^{B(X)} \Phi. \tag{3.51}$$

Now suppose that the background we start with preserves at least $\mathcal{N} = 1$ supersymmetry so that the pure spinor equations (3.1), (3.2) are satisfied. According to the discussions in the previous subsection and the paragraph above, this means that the untwisted fields (which have no dependence on the isometry directions) satisfy the deformed pure spinor equations (3.48), (3.49), where the deformation is determined by the flux associated with the matrices L and S_L . But this is just geometric flux with $f_{ij}{}^k = C_{ij}{}^k$, see [17]. Now we act on these untwisted fields with the NATD matrices (2.1), (2.3) as in (2.2)–(2.5) to generate the NAT dual background. The resulting fields satisfy the field equations of Type II supergravity as was shown in [17] by embedding these equations in DFT. To check supersymmetry of the dual background we also transform the untwisted pure spinors of the initial background (that is, the pure spinors $\Phi(X)$ in (3.51) rather than $\Phi(X, \theta)$) as in (3.13) with $P = S_{\text{NATD}}$. Now we have to check whether these new pure spinors $\Phi(X, \nu)$ still satisfy the supersymmetry equations (3.1), (3.2). As discussed in the previous subsection, this is equivalent to checking whether the untwisted pure spinors satisfy the deformed supersymmetry equations (3.48), (3.49), where the deformation is determined by the flux associated with the NATD matrix S_{NATD} . As discussed above, due to the special form of the NATD matrix: $S_{\text{NATD}} = C_n S_\theta$, the associated flux can be computed by calculating the flux associated with S_θ first (which gives the H-flux) and then raising one index with J_n^6 . This yields geometric flux with $f_{ij}{}^k = C_{ij}{}^k$, and we already know that the untwisted pure spinors satisfy these deformed equations due to the existence of isometry respected by the initial background and the pure spinors associated with it. This completes the proof that a background that preserves $\mathcal{N} = 1$ supersymmetry will still be supersymmetric after NATD.

¹¹We choose a coordinate system adapted to the isometries so that the fields can be written in this separated form.

¹²Note that the second equality in (3.51) is valid due to the special form of L and S_L , see equations (4.22)–(4.23) in [17].

4 Examples

In this section, we will demonstrate how the NATD transformation formulas (2.2)–(2.5) and (3.13) work by looking at a specific class of Type IIB backgrounds, which were first studied in [25]. The topology of the background we will study is $R_{1,3} \times \mathcal{M}_3 \times S^3$ so that there is an $SU(2)$ isometry associated with S^3 , which can be utilized to perform NATD.

The ansatz for the metric and the 5-form flux is

$$ds^2 = e^{2A} dx_{1,3}^2 + ds^2(\mathcal{M}_3) + \sum_{i=1}^3 (e^i)^2, \quad (4.1)$$

$$\mathcal{F}_5 = \mathcal{F}_2 \wedge e^1 \wedge e^2 \wedge e^3$$

$$F_5 = (1 + \star)\mathcal{F}_5 = \mathcal{F}_2 \wedge e^1 \wedge e^2 \wedge e^3 - e^{4A} \star_3 \mathcal{F}_2 \wedge \text{Vol}_4$$

and $F_1 = F_3 = B = \phi = 0$. \mathcal{F}_2 is a 2-form, \star_3 is the Hodge star operator on \mathcal{M}_3 , and A is the warp factor. It is a function which has dependence only on the coordinates of \mathcal{M}_3 . S^3 is assumed to be fibered over \mathcal{M}_3 and hence the vielbeins e^i on S^3 have the form

$$e^i = \lambda_i(\sigma_i + \mathcal{A}_i). \quad (4.2)$$

Here, \mathcal{A}_i are 1-forms on \mathcal{M}_3 and λ_i are functions on \mathcal{M}_3 . The forms σ^i are left invariant 1-forms for the isometry group $SU(2)$ so that $d\sigma^i = \frac{1}{2}\epsilon_{jk}^i \sigma^j \wedge \sigma^k$. We denote the left invariant vector fields L_i , so $i_{L_i} \sigma^j = \delta_i^j$. We also define (as in [25]) a set of undetermined frame fields h^i so that

$$ds^2(\mathcal{M}_3) = \sum_{i=1}^3 (h^i)^2. \quad (4.3)$$

Another assumption that is made in [25] is that this geometry preserves at least $\mathcal{N} = 1$ supersymmetry in four dimensions in the form of an $SU(3)$ structure characterized by the following 2-form J and 3-form Ω which are given by means of a vielbein e^i and frame fields h^i :

$$J = h^3 \wedge e^3 + e^1 \wedge e^2 + h^1 \wedge h^2, \quad \Omega = (h^3 + ie^3) \wedge (e^1 + ie^2) \wedge (h^1 + ih^2). \quad (4.4)$$

As discussed in appendix A, $SU(3)$ structure can be regarded as a special case of $SU(3) \times SU(3)$ structure with associated pure spinors of the form (A.4). In our case, setting $\Phi_1 = \Phi_-$ and $\Phi_2 = \Phi_+$ we have

$$\Phi_+ = \frac{1}{8} e^{i\theta_+} e^A e^{-iJ}, \quad \Phi_- = -\frac{i}{8} e^{i\theta_-} e^A \Omega \quad (4.5)$$

Due to assumption of preservation of supersymmetry, these pure spinors must satisfy the equations (3.1), (3.2). As shown in [25], this forces $\theta_+ = \frac{\pi}{2}$ and $\mathcal{A}_1 = \mathcal{A}_2 = 0$. The possible values for θ_- for different geometries is given in appendix B of [25]. Comparing (4.5) with (A.4) one can see that it is of the general form of a general $SU(3)$ pure spinor with $a = e^{i\theta_-/2} e^{i\theta_+/2} e^{A/2}$ and $b = e^{i\theta_-/2} e^{-i\theta_+/2} e^{A/2}$, which satisfy $|a|^2 = |b|^2 = e^A$.

The ansatz (4.1) is general enough to cover many examples important for AdS/CFT duality, notably $AdS_5 \times T^{1,1}$, $AdS_5 \times Y^{p,q}$ and $AdS_5 \times S^5$. The detailed description of how

these backgrounds fall within this general ansatz can be found in appendix B of [25]. For example, for $T^{1,1}$ background the required values are as follows:

$$\begin{aligned} A &= \log r, & \mathcal{A}_3 &= \cos \theta d\varphi, & \theta_- &= 0, \\ \lambda_1 = \lambda_2 &= \frac{1}{\sqrt{6}}, & \lambda_3 &= \frac{1}{3}, & h^1 &= \frac{1}{\sqrt{6}} \sin \theta d\varphi, & h^2 &= \frac{1}{\sqrt{6}} d\theta, & h^3 &= \frac{dr}{r}. \end{aligned}$$

On the other hand, the required values for the $AdS_5 \times S^5$ background are:

$$\begin{aligned} A &= \log 2R, & \mathcal{A}_3 &= 0, & \theta_- &= \beta, & \lambda_1 = \lambda_2 = \lambda_3 &= \cos \alpha, \\ h^1 &= 2 \frac{R \cos \alpha d\alpha + \sin \alpha dR}{R}, & h^2 &= 2 \sin \alpha d\beta, & h^3 &= 2 \frac{\cos \alpha dR - R \sin \alpha d\alpha}{R}. \end{aligned}$$

Now, we perform the NATD transformation of the background described by the ansatz (4.1). We begin with the transformation of the metric and the B-field. For this we use (3.7) where T is obtained by embedding T_{NATD} in (2.1) in $O(6,6)$ in the usual way. We will call this $O(6,6)$ matrix also T_{NATD} .¹³ Then we read off the transformed metric and the transformed B-field from the symmetric and antisymmetric parts of E' , respectively:

$$E'(g', B') = T_{\text{NATD}} \cdot E(g, B) \quad (4.6)$$

$$g' = \frac{E' + E'^t}{2}, \quad B' = \frac{E' - E'^t}{2} \quad (4.7)$$

As mentioned before, this transformation is equivalent to what is given in (2.2). We refer to [17] for details. This gives

$$\begin{aligned} ds'^2 &= e^{2A} dx_{1,3}^2 + ds^2(\mathcal{M}_3) + \frac{1}{\Delta} \left((\nu_i \nu_j + \frac{\lambda_1^2 \lambda_2^2 \lambda_3^2}{\lambda_{(i)}^2} \delta_{(i)j}) d\nu_i d\nu_j - 2\lambda_3^2 \lambda_2^2 \nu_2 d\nu_1 \mathcal{A}_3 \right. \\ &\quad \left. + 2\lambda_3^2 \lambda_1^2 \nu_1 d\nu_2 \mathcal{A}_3 + (\lambda_3^2 \Delta - 4\lambda_3^4 (\lambda_1^2 \lambda_2^2 + \nu_3^2)) \mathcal{A}_3 \mathcal{A}_3 \right) \\ B' &= -\frac{1}{\Delta} \left(\frac{1}{2} \epsilon_{ijk} \nu_i \lambda_i^2 d\nu_j \wedge d\nu_k + \lambda_3^2 \nu_3 \nu_1 d\nu_1 \wedge \mathcal{A}_3 \right. \\ &\quad \left. + \lambda_3^2 \nu_3 \nu_2 d\nu_2 \wedge \mathcal{A}_3 + (\lambda_3^2 \nu_3^2 + \lambda_1^2 \lambda_2^2 \lambda_3^2) d\nu_3 \wedge \mathcal{A}_3 \right), \\ \Delta &= G^{-1} = \lambda_1^2 \lambda_2^2 \lambda_3^2 + \lambda_1^2 \nu_1^2 + \lambda_2^2 \nu_2^2 + \lambda_3^2 \nu_3^2 \\ e^{-2\phi'} &= \Delta \end{aligned} \quad (4.8)$$

These are the same as the results obtained in [25] (except for a sign difference in the B-field, see footnote 5).

Next, we perform the NATD transformation of the RR flux F_5 from the transformation rule (2.4), with S_{NATD} (2.6). To this end, it is convenient to write the spinor field F that packages the RR fluxes as a non-homogeneous differential form as follows (see [17, 41]):

$$F = \sum_p \left(F^{(p)} + F_i^{(p-1)} \sigma^i + \frac{1}{2} F_{ij}^{(p-2)} \sigma^i \wedge \sigma^j + F^{(p-3)} \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \right), \quad (4.9)$$

¹³Note that, since the isometry group $SU(2)$ is three dimensional, the matrix (2.1) is in $O(3,3)$.

where each p -form is decomposed according to how many legs it has along the $SU(2)$ directions. This non-homogeneous differential form maps to a Clifford algebra element in the usual way where we identify the element σ^i with the Clifford algebra element ψ^i , for $i = 1, 2, 3$, see (3.19). It has the following form:

$$F = \sum_p \left(F^{(p)} + F_i^{(p-1)} \psi^i + \frac{1}{2} F_{ij}^{(p-2)} \psi^i \psi^j + F^{(p-3)} \psi^1 \psi^2 \psi^3 \right) \quad (4.10)$$

Then, the spinorial action of ψ^i on F is given by wedge product, whereas the spinorial action of ψ_i is given by contraction [17, 41]:

$$\psi^i \cdot F = \psi^i \wedge F, \quad \psi_i \cdot F = i_{\psi_i} F. \quad (4.11)$$

Since there is no B-field, we will first calculate the action of CS_θ on differential forms then apply $e^{-B'}$. The action of S_θ in (2.6) on a non-homogeneous differential form α is as follows:

$$\begin{aligned} S_\theta \cdot \alpha &= e^{-\theta} \wedge \alpha = \alpha + \nu_k \epsilon_{ij}^k \psi^i \wedge \psi^j \wedge \alpha \\ &= \alpha + \nu_1 \psi^2 \wedge \psi^3 \wedge \alpha + \nu_2 \psi^3 \wedge \psi^2 \wedge \alpha + \nu_3 \psi^1 \wedge \psi^2 \wedge \alpha \end{aligned} \quad (4.12)$$

On the other hand, the action of C given in (C.7) can be calculated by using (4.11). We calculate the following NATD transformed RR flux.

$$\begin{aligned} F'_5 &= e^{-B'} CS_\theta F_5 \\ &= \lambda_1 \lambda_2 \lambda_3 \mathcal{F}_2 - \lambda_1 \lambda_2 \lambda_3 B' \wedge \mathcal{F}_2 - \lambda_1 \lambda_2 \lambda_3 \psi^3 \wedge \mathcal{A}_3 \wedge \mathcal{F}_2 + e^{4A} \text{Vol}_4 \wedge \nu^i \psi^i \wedge \star_3 \mathcal{F}_2 \\ &\quad - e^{4A} \text{Vol}_4 \wedge \psi^1 \wedge \psi^2 \wedge \psi^3 \wedge \star_3 \mathcal{F}_2 - B' \wedge e^{4A} \text{Vol}_4 \wedge \nu^i \psi^i \wedge \star_3 \mathcal{F}_2 \end{aligned} \quad (4.13)$$

where the Hodge duality \star_3 is taken with respect to the transformed metric. This polyform packages all the RR fluxes of the NAT dual background, which we read off (after identifying ψ^i with $d\nu_i$) to be:

$$F'_2 = \lambda_1 \lambda_2 \lambda_3 \mathcal{F}_2, \quad (4.14)$$

$$F'_4 = (-B' + \mathcal{A}_3 \wedge d\nu_3) \wedge F'_2, \quad (4.15)$$

$$F'_6 = \star_{10} F'_4 = e^{4A} \text{Vol}_4 \wedge \nu_i d\nu_i \wedge \star_3 \mathcal{F}_2, \quad (4.16)$$

$$F'_8 = -\star_{10} F'_2 = -B' \wedge F'_6 + e^{4A} \text{Vol}_4 \wedge \star_3 \mathcal{F}_2 \wedge d\nu_1 \wedge d\nu_2 \wedge d\nu_3. \quad (4.17)$$

These agree with the results obtained in [25] (up to sign differences in B' , and the 6- and 8-forms due to differences in conventions, see footnote 5).

Finally, we will apply the NATD transformation rule (3.13) (with $P = S_{\text{NATD}}$) to the $SU(3)$ pure spinors given in (4.5) (which are known to satisfy the supersymmetry equations (3.1), (3.2)) and obtain the NAT-dual pure spinors Φ'_+ and Φ'_- . The explicit form of the transformed pure spinors are presented in appendix B. In obtaining the results there, we first calculate $S_\theta \Phi_-$:

$$\begin{aligned} S_\theta \Phi_- &= \Phi_- + \nu_1 \psi^2 \wedge \psi^3 \wedge \Phi_- + \nu_2 \psi^3 \wedge \psi^2 \wedge \Phi_- + \nu_3 \psi^1 \wedge \psi^2 \wedge \Phi_- \\ &= \Phi_- - \frac{i}{8} e^{i\theta} e^A \{ \lambda_1 \nu_1 \psi^2 \wedge \psi^3 \wedge h^3 \wedge \psi^1 \wedge h^1 + i \lambda_1 \nu_1 \psi^2 \wedge \psi^3 \wedge h^3 \wedge \psi^1 \wedge h^2 \\ &\quad - \lambda_1 \lambda_3 \nu_1 \psi^2 \wedge \psi^3 \wedge \mathcal{A}_3 \wedge \psi^1 \wedge h^2 + i \lambda_2 \nu_2 \psi^3 \wedge \psi^1 \wedge h^3 \wedge \psi^2 \wedge h^1 \\ &\quad - \lambda_2 \nu_2 \psi^3 \wedge \psi^1 \wedge h^3 \wedge \psi^2 \wedge h^2 - i \lambda_2 \lambda_3 \nu_2 \psi^3 \wedge \psi^1 \wedge \mathcal{A}_3 \wedge \psi^2 \wedge h^2 \} \end{aligned}$$

Applying $\sqrt{G} e^{-B'} C$ to $S_\theta \Phi_-$ above, we obtain Φ'_- , whose explicit form is given in (B.2).

Now we calculate $S_\theta \Phi_+$:

$$\begin{aligned}
S_\theta \Phi_+ &= \Phi_+ + \nu_1 \psi^2 \wedge \psi^3 \wedge \Phi_+ + \nu_2 \psi^3 \wedge \psi^2 \wedge \Phi_+ + \nu_3 \psi^1 \wedge \psi^2 \wedge \Phi_+ \\
&= \Phi_+ + \frac{1}{8} e^{i\theta_+} e^A \{ \nu_1 \psi^2 \wedge \psi^3 - i \lambda_3 \nu_1 \psi^2 \wedge \psi^3 \wedge h^3 \wedge \mathcal{A}_3 - i \nu_1 \psi^2 \wedge \psi^3 \wedge h^1 \wedge h^2 \\
&\quad + \nu_2 \psi^3 \wedge \psi^1 - i \lambda_3 \nu_2 \psi^3 \wedge \psi^1 \wedge h^3 \wedge \mathcal{A}_3 - i \nu_2 \psi^3 \wedge \psi^1 \wedge h^1 \wedge h^2 \\
&\quad + \nu_3 \psi^1 \wedge \psi^2 - i \lambda_3 \nu_3 \psi^1 \wedge \psi^2 \wedge h^3 \wedge \mathcal{A}_3 - i \lambda_3 \nu_3 \psi^1 \wedge \psi^2 \wedge h^3 \wedge \psi^3 \\
&\quad - i \nu_3 \psi^1 \wedge \psi^2 \wedge h^1 \wedge h^2 + \lambda_3 \nu_3 \psi^1 \wedge \psi^2 \wedge \psi^3 \wedge h^1 \wedge h^2 \wedge h^3 \}
\end{aligned}$$

Applying $\sqrt{G} e^{-B'} C$ to $S_\theta \Phi_+$ we obtain Φ'_+ , whose explicit form is given in (B.1).

One can check by direct computation that the transformed pure spinors Φ'_- and Φ'_+ can be written in the following form:

$$\Phi'_- = -\frac{i}{8} e^A e^{i\theta_-} e^{\frac{1}{2}z \wedge \bar{z}} \wedge \omega \tag{4.18}$$

$$\Phi'_+ = -\frac{1}{8} e^{i\theta_+} e^A e^{-ij} \wedge z, \tag{4.19}$$

where the complex 1-form $z = v + iw$, and the real and complex 2-forms j and ω are as given below

$$z = -\frac{1}{\sqrt{\Delta}} \left((\lambda_1 \lambda_2 \lambda_3 + i \lambda_3 \nu_3) h^3 - \nu_1 d\nu_1 - \nu_2 d\nu_2 - (\nu_3 - i \lambda_1 \lambda_2) d\nu_3 \right) \tag{4.20}$$

$$\begin{aligned}
j &= \frac{1}{\Delta} \left(\Delta h^1 \wedge h^2 + \lambda_1 \lambda_2 \lambda_3^2 d\nu_1 \wedge d\nu_2 + \lambda_1 \lambda_2 \lambda_3^2 \nu_1 d\nu_1 \wedge \mathcal{A}_3 + \lambda_2^2 \lambda_3 \nu_2 d\nu_1 \wedge h^3 \right. \\
&\quad \left. - \lambda_1 \lambda_2 \lambda_3^2 \nu_2 \mathcal{A}_3 \wedge d\nu_2 + \lambda_1^2 \lambda_3 \nu_1 h^3 \wedge d\nu_2 - (\lambda_2^2 \lambda_3 \nu_2^2 + \lambda_1^2 \lambda_3 \nu_1^2) \mathcal{A}_3 \wedge h^3 \right)
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
\omega &= \frac{1}{\sqrt{\Delta}} \left(\lambda_2 \lambda_3 h^1 \wedge d\nu_1 + i \lambda_1 \lambda_3 h^1 \wedge d\nu_2 + (\nu_1 \lambda_1 + i \nu_2 \lambda_2) h^1 \wedge h^3 + (i \nu_1 \lambda_1 - \nu_2 \lambda_2) h^2 \wedge h^3 \right. \\
&\quad \left. + i \lambda_2 \lambda_3 h^2 \wedge d\nu_1 - \lambda_1 \lambda_3 h^2 \wedge d\nu_2 - (i \lambda_2 \lambda_3 \nu_2 + \lambda_1 \lambda_3 \nu_1) h^2 \wedge \mathcal{A}_3 \right).
\end{aligned} \tag{4.22}$$

Comparing (4.18), (4.19) with (A.6) one can see that they define an SU(2) structure, as can be seen by taking $a = e^{i\theta_-/2} e^{i\theta_+/2} e^{A/2}$ and $b = e^{-i\theta_-/2} e^{i\theta_+/2} e^{A/2}$ in (A.6). Note that $|a|^2 = |b|^2 = e^A$, as needed. So, under NATD, a background with SU(3) structure is transformed to a background with SU(2) structure, as has been demonstrated many times in the literature previously, in particular in [25, 26].

The results we present in (4.18)–(4.22) are in agreement with those obtained in [25].¹⁴ Whether these transformed pure spinors satisfy the supersymmetry equations (3.1), (3.2) was checked in [25] by direct computation. The results we obtained in section 3.3 make such a calculation redundant. Indeed, the pure spinors (4.18), (4.19) are obtained through the action of S_{NATD} and we have proved that this transformation maps solutions of (3.1), (3.2) to new solutions.

¹⁴To be more precise, the results presented in equation (5.4) of [25] differ from our results in (4.18), (4.19) with an extra -1 factor in Φ'_+ and with a $-i$ factor in Φ'_- although the differential forms (5.6), (5.7) in [25] and ours in (4.20)–(4.22) are exactly the same. However, we checked that the pure spinors (4.18), (4.19) satisfy the pure spinor equations (3.25), (3.26).

5 Conclusions and outlook

In this paper, we studied how the pure spinor equations (3.1), (3.2) transform under NATD. These are equations to be satisfied for preservation of $\mathcal{N} = 1$ supersymmetry in compactifications of Type II string theory to four dimensions. Our approach in analyzing supersymmetry under NATD is different from those in the literature in that we exploit the recently discovered fact that NATD can be described as an $O(d, d)/\text{Pin}(d, d)$ (in the NS-NS/RR sectors) transformation. Although this is a coordinate dependent transformation we start in section 3.1 by considering constant $\text{Pin}(d, d)$ transformations. Writing the equations (3.1), (3.2) in terms of DFT fields makes it easy to show that they are $\text{Pin}(d, d)$ covariant. This then means that solutions of these equations will be mapped to new solutions under $\text{Pin}(d, d)$. This analysis can be regarded as a generalization of those carried out in [34] and [56], where the behavior of pure spinor equations under Abelian T-duality (a certain type of $O(1, 1)/\text{Pin}(1, 1)$ transformation) and LM deformations (a certain type of $O(2, 2)/\text{Pin}(2, 2)$ transformation) was studied, respectively.

Since the NATD matrix is coordinate-dependent, further analysis is needed to see whether solutions are mapped to solutions under NATD. This is done in section 3.2. We show in that section that this is indeed the case, due to the simple fact that the fluxes generated by the NATD matrix (regarded as a twist matrix within the formalism of GDFT) is the same as the geometric flux associated with the isometry group that is used to perform NATD. This idea of ‘preservation of flux’ has been used before to analyze field equations of supergravity under NATD in [17], under YB deformations in [42–46] and under U-duality transformations in [47].

As we emphasized before, our approach here in analyzing supersymmetry equations under NATD is novel, as it implements NATD as an $O(d, d)/\text{Pin}(d, d)$ transformation. We believe that this starting point is quite useful, as it has been realized in various works recently that there are other interesting $O(d, d)$ transformations that can be utilized to generate new supergravity backgrounds, notably related with integrable deformations of string sigma models [15, 43, 45], [60]–[63]. The approach taken here would also be useful to analyze supersymmetry of such backgrounds. Also, viewing NATD as a $\text{Pin}(d, d)$ transformation makes it easier to apply it to other backgrounds, which fall outside the ansatz considered in section 4 with different isometry groups and supporting a generic $SU(3) \times SU(3)$ structure [36, 64, 65]. The methods we employed here are well suited to analyze the supersymmetry and structure group of the resulting backgrounds. We plan to consider these issues in future work.

A $SU(3)$ and $SU(2)$ structures and pure spinors

The structure group of the generalized tangent bundle $TM \oplus T^*M$ of the six dimensional internal manifold M reduces to $SU(3) \times SU(3)$ if there exists two globally defined $SU(3) \times SU(3)$ pure spinors Φ_1 and Φ_2 of non-vanishing norm, [35, 36]. Adopting the conventions

of [56], the explicit form of these pure spinors can be given as below:

$$\Phi_+ = \frac{1}{8} \left[c_1 \bar{c}_3 e^{-ij} + c_2 \bar{c}_4 e^{ij} - i(c_1 \bar{c}_4 \omega + \bar{c}_3 c_2 \bar{\omega}) \right] \wedge e^{z\bar{z}/2}, \quad (\text{A.1})$$

$$\Phi_- = \frac{1}{8} \left[i(c_2 c_4 \bar{\omega} - c_1 c_3 \omega) + (c_2 c_3 e^{ij} - c_1 c_4 e^{-ij}) \right] \wedge z. \quad (\text{A.2})$$

where c_1, c_2, c_3, c_4 are complex functions on M . For a background of the form (3.3) requirement of existence of supersymmetric branes imposes that $|c_1|^2 + |c_2|^2 = |c_3|^2 + |c_4|^2 = e^A$ [56].¹⁵ Here $z = v + iw$ is a complex 1-form, j is a real 2-form and ω is a complex 2-form.

Reduction of the structure group of the tangent bundle TM to $SU(3)$ is equivalent to existence on M of an invariant real 2-form J and a complex 3-form Ω satisfying the following compatibility conditions, [66–69].

$$\frac{i}{8} \Omega \wedge \bar{\Omega} = \frac{1}{3!} J \wedge J \wedge J, \quad J \wedge \Omega = 0. \quad (\text{A.3})$$

$SU(3)$ structure can be regarded as a special case of $SU(3) \times SU(3)$ structure [28–30] and the form of the pure spinor describing the $SU(3)$ structure is a special case of (A.1) and (A.2) [35, 36, 56]

$$SU(3): \quad \Phi_- = -i \frac{ab}{8} \Omega, \quad \Phi_+ = \frac{a\bar{b}}{8} e^{-iJ} \quad (\text{A.4})$$

Comparing to (A.1) and (A.2) we have $J = j + v \wedge w$, $\Omega = \omega \wedge (v + iw)$ and $c_1 = a$, $c_3 = b$, $c_2 = c_4 = 0$. Due to the condition $|c_1|^2 + |c_2|^2 = |c_3|^2 + |c_4|^2 = e^A$ we need $|a|^2 = |b|^2 = e^A$. In (4.5) we had $a = e^{i\theta_-/2} e^{i\theta_+/2} e^{A/2}$ and $b = e^{i\theta_-/2} e^{-i\theta_+/2} e^{A/2}$.

On the other hand, $SU(2)$ structure on M is characterized by the existence of a complex 1-form $z = v + iw$, a real 2-form j and a complex 2-form ω satisfying the following compatibility conditions [66–69]:

$$\begin{aligned} \omega \wedge j &= 0, & (\text{A.5}) \\ i_z j &= i_z \omega = 0, \\ \omega \wedge \bar{\omega} &= 2j \wedge j. \end{aligned}$$

Again, $SU(2)$ structure can be regarded as a special case of $SU(3) \times SU(3)$ structure [28–30] and the form of the pure spinor describing the $SU(2)$ structure is a special case of (A.1) and (A.2). The corresponding pure spinors are [35, 36, 56]:

$$SU(2): \quad \Phi_- = -\frac{ab}{8} e^{-ij} \wedge (v + iw), \quad \Phi_+ = -i \frac{a\bar{b}}{8} e^{-iv \wedge w} \wedge \omega. \quad (\text{A.6})$$

Comparing to (A.1) and (A.2) we have $c_2 = c_3 = 0$ and $c_1 = a$, $c_4 = b$, again with $|a|^2 = |b|^2 = e^A$. In (4.18), (4.19) we had $a = e^{i\theta_-/2} e^{i\theta_+/2} e^{A/2}$ and $b = e^{-i\theta_-/2} e^{i\theta_+/2} e^{A/2}$.

¹⁵In fact only then the pure spinor equations whose general form is given in equation (2.17) of [56] reduce to those in (3.1), (3.2). See section 2 of [56] for more details.

B NAT-dual pure spinors

Transformation of the SU(3) pure spinors given in (4.5) under the NATD transformation yields the pure spinors presented below:

$$\begin{aligned}
 \Phi'_+ = & -\frac{1}{8\sqrt{\Delta}} e^{i\theta_+} e^A \left(\nu_1 d\nu_1 + \nu_2 d\nu_2 + (\nu_3 - i\lambda_1\lambda_2) d\nu_3 - (\lambda_1\lambda_2\lambda_3 + i\lambda_3\nu_3) h^3 \right. \\
 & - (\lambda_1^2\lambda_2^2\lambda_3^2 + i\lambda_1\lambda_2\lambda_3^2\nu_3) d\nu_1 \wedge d\nu_2 \wedge d\nu_3 + (i\lambda_1\lambda_2\lambda_3 - \lambda_3\nu_3) h^1 \wedge h^2 \wedge h^3 \\
 & - i\nu_1 h^1 \wedge h^2 \wedge d\nu_1 - i\nu_2 h^1 \wedge h^2 \wedge d\nu_2 - (\lambda_1\lambda_2 + i\nu_3) h^1 \wedge h^2 \wedge d\nu_3 \\
 & - \frac{1}{\Delta} (i\lambda_3^3\nu_3^2 + \lambda_1\lambda_2\lambda_3^3\nu_3 - i\lambda_3\Delta) d\nu_1 \wedge d\nu_2 \wedge h^3 \\
 & - \frac{1}{\Delta} (i\lambda_2^2\lambda_3\nu_2\nu_3 + \lambda_1\lambda_2^3\lambda_3\nu_2) d\nu_1 \wedge h^3 \wedge d\nu_3 \\
 & - \frac{1}{\Delta} (i\lambda_1^2\lambda_3\nu_1\nu_3 + \lambda_1^3\lambda_2\lambda_3\nu_1) h^3 \wedge d\nu_2 \wedge d\nu_3 \\
 & + \frac{1}{\Delta} (i\lambda_1\lambda_2\lambda_3^2\nu_1\nu_3 + \lambda_1^2\lambda_2^2\lambda_3^2\nu_1) d\nu_1 \wedge d\nu_3 \wedge \mathcal{A}_3 \\
 & + \frac{1}{\Delta} (i\lambda_1\lambda_2\lambda_3^2\nu_2\nu_3 + \lambda_1^2\lambda_2^2\lambda_3^2\nu_2) d\nu_2 \wedge d\nu_3 \wedge \mathcal{A}_3 \\
 & + \frac{1}{\Delta} (\lambda_1\lambda_2\lambda_3^2\nu_1\nu_3 - i\lambda_1^2\lambda_2^2\lambda_3^3\nu_1 - i\lambda_2^2\lambda_3\nu_1\nu_2^2 - i\lambda_1^2\lambda_3\nu_1^3) d\nu_1 \wedge h^3 \wedge \mathcal{A}_3 \\
 & + \frac{1}{\Delta} (i\lambda_1^2\lambda_2^2\lambda_3^3\nu_2 - \lambda_1\lambda_2\lambda_3^3\nu_2\nu_3 - i\lambda_2^2\lambda_3\nu_2^3 - \lambda_1^2\lambda_3\nu_2\nu_1^2) d\nu_2 \wedge h^3 \wedge \mathcal{A}_3 \\
 & - \frac{1}{\Delta} (\lambda_1\lambda_2^3\lambda_3\nu_2^2 + \lambda_3\lambda_2\lambda_1^3\nu_1^2 + i\lambda_2^2\lambda_3\nu_2^2\nu_3 + i\lambda_1^2\lambda_3\nu_3\nu_1^2) d\nu_3 \wedge h^3 \wedge \mathcal{A}_3 \\
 & + \frac{1}{\Delta} (i\lambda_1\lambda_2\lambda_3^3\nu_3 - \lambda_3^3\nu_3^2 + \lambda_3\Delta) h^1 \wedge h^2 \wedge h^3 \wedge d\nu_1 \wedge d\nu_2 \\
 & + \frac{1}{\Delta} (\lambda_2^2\lambda_3\nu_2\nu_3 - i\lambda_1\lambda_2^3\nu_2\nu_3) h^1 \wedge h^2 \wedge h^3 \wedge d\nu_1 \wedge d\nu_3 \\
 & + \frac{1}{\Delta} (i\lambda_1^3\lambda_2\lambda_3\nu_1 - \lambda_1^2\lambda_3\nu_1\nu_3) h^1 \wedge h^2 \wedge h^3 \wedge d\nu_2 \wedge d\nu_3 \\
 & \left. + \frac{1}{\Delta} (i\lambda_1^2\lambda_2^2\lambda_3^2 - \lambda_1\lambda_2\lambda_3^2\nu_3) h^1 \wedge h^2 \wedge d\nu_1 \wedge d\nu_2 \wedge d\nu_3 \right) \quad (\text{B.1})
 \end{aligned}$$

Similarly, we transform Φ_- under NATD and obtain

$$\begin{aligned}
 \Phi'_- = & \frac{-i}{8\sqrt{\Delta}} e^{i\theta_-} e^A \left(-\lambda_2\lambda_3 d\nu_1 \wedge h^1 - i\lambda_1\lambda_3 d\nu_2 \wedge h^1 \right. \\
 & + (\nu_1\lambda_1 + i\nu_2\lambda_2) h^1 \wedge h^3 + i\lambda_2\lambda_3 h^2 \wedge d\nu_1 \\
 & - \lambda_1\lambda_3 h^2 \wedge d\nu_2 - (i\nu_1\lambda_1 - \nu_2\lambda_2) h^2 \wedge h^3 - (i\lambda_2\lambda_3\nu_2 + \lambda_1\lambda_3\nu_1) h^2 \wedge \mathcal{A}_3 \\
 & + \frac{1}{\Delta} (\lambda_1\lambda_3^2\nu_1\nu_3 + i\lambda_2\lambda_3^2\nu_2\nu_3) h^3 \wedge d\nu_1 \wedge h^1 \wedge d\nu_2 \\
 & + \frac{1}{\Delta} (i\lambda_1\lambda_3^2\nu_1\nu_3 - \lambda_2\lambda_3^2\nu_2\nu_3) h^3 \wedge d\nu_1 \wedge h^2 \wedge d\nu_2 \\
 & + \frac{1}{\Delta} (i\lambda_2^3\nu_2^2 + \lambda_1\lambda_2^2\nu_2\nu_1 - i\lambda_2\Delta) h^3 \wedge d\nu_3 \wedge h^1 \wedge d\nu_1 \\
 & \left. + \frac{1}{\Delta} (\lambda_1\Delta - i\lambda_1\lambda_2\nu_2\nu_1 - \lambda_1^3\nu_1^2) h^3 \wedge d\nu_3 \wedge h^1 \wedge d\nu_2 \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{\Delta} (\lambda_2 \Delta + i \lambda_1 \lambda_2^2 \nu_2 \nu_1 - \lambda_2^3 \nu_2^2) h^3 \wedge d\nu_3 \wedge h^2 \wedge d\nu_1 \\
 & + \frac{1}{\Delta} (i \lambda_1 \Delta + \lambda_2 \lambda_1^2 \nu_2 \nu_1 - i \lambda_1^3 \nu_1^2) h^3 \wedge d\nu_3 \wedge h^2 \wedge d\nu_2 \\
 & - \frac{1}{\Delta} \lambda_1 \lambda_2 \lambda_3 (\lambda_1 \nu_1 + i \lambda_2 \nu_2) d\nu_1 \wedge d\nu_2 \wedge d\nu_3 \wedge h^1 \\
 & + \frac{1}{\Delta} \lambda_1 \lambda_2 \lambda_3 (\lambda_2 \nu_2 - i \lambda_1 \nu_1) d\nu_1 \wedge d\nu_2 \wedge d\nu_3 \wedge h^2 \\
 & + \frac{1}{\Delta} (i \lambda_1 \lambda_3^2 \nu_1 \nu_2 \nu_3 - \lambda_2 \lambda_3^2 \nu_2^2 \nu_3) h^3 \wedge d\nu_2 \wedge h^2 \wedge \mathcal{A}_3 \\
 & + \frac{1}{\Delta} (i \lambda_1 \lambda_3^2 \nu_1^2 \nu_3 - \lambda_2 \lambda_3^2 \nu_1 \nu_2 \nu_3) h^3 \wedge d\nu_1 \wedge h^2 \wedge \mathcal{A}_3 \\
 & - \frac{1}{\Delta} (\lambda_1^2 \lambda_2^3 \lambda_3^2 \nu_2 + \lambda_2 \lambda_3^2 \nu_2 \nu_3^2 - i \lambda_2^2 \lambda_1^3 \lambda_3^2 \nu_1 - i \lambda_1 \lambda_3^2 \nu_1 \nu_3^2) h^3 \wedge d\nu_3 \wedge h^2 \wedge \mathcal{A}_3 \\
 & + \frac{1}{\Delta} (\lambda_1 \lambda_2^2 \lambda_3 \nu_1 \nu_2 - i \lambda_1^2 \lambda_2 \lambda_3 \nu_1^2) d\nu_1 \wedge d\nu_3 \wedge h^2 \wedge \mathcal{A}_3 \\
 & + \frac{i}{\Delta} (\lambda_1 \lambda_2^2 \lambda_3 \nu_2^2 - \lambda_1^2 \lambda_2 \lambda_3 \nu_1 \nu_2) d\nu_2 \wedge d\nu_3 \wedge h^2 \wedge \mathcal{A}_3
 \end{aligned} \tag{B.2}$$

Note that after the transformation ψ^i are identified with $d\nu_i$. It can be checked by direct computation that the spinors Φ'_+ and Φ'_- can be written as in (4.18)–(4.22).

C Mukai pairing

Mukai pairing is the natural inner product on the Clifford module $\wedge^\bullet T^*$ and described as follows.

$$\langle , \rangle : S \otimes S \rightarrow \wedge^n T^*:$$

$$\langle \chi_1, \chi_2 \rangle = (\tau(\chi_1) \wedge \chi_2)_{\text{top}} = \sum_j (-1)^j (\chi_1^{2j} \wedge \chi_2^{n-2j} + \chi_1^{2j+1} \wedge \chi_2^{n-2j-1}), \quad \chi_1, \chi_2 \in \wedge^\bullet T^*, \tag{C.1}$$

here $(\)_{\text{top}}$ denotes the top degree component of the form and the superscript k denotes the k -form component of the form. This is equivalent to

$$\langle \chi_1, \chi_2 \rangle = (\chi_1 \wedge \lambda(\chi_2))_{\text{top}}, \tag{C.2}$$

where λ is the natural linear extension of λ in (3.5) to a non-homogeneous differential form.

Mukai pairing is symmetric in dimensions $n \equiv 0, 1 \pmod{4}$ and is skew-symmetric otherwise:

$$\langle \chi_1, \chi_2 \rangle = (-1)^{n(n-1)/2} \langle \chi_2, \chi_1 \rangle. \tag{C.3}$$

See [41] for details.

Mukai pairing has an important property related to the action of the Spin group, [30]:

$$\langle S\chi_1, S\chi_2 \rangle = \pm \langle \chi_1, \chi_2 \rangle, \quad S \in \text{Spin}(d, d). \tag{C.4}$$

This follows from

$$\langle P\chi_1, P\chi_2 \rangle = (P, P) \langle \chi_1, \chi_2 \rangle, \tag{C.5}$$

where $(,)$ is the natural indefinite inner product defined as

$$(X + \xi, X + \xi) = i_X \xi = \xi(X), \quad X + \xi \in T \oplus T^*.$$

Since $(P, P) = \pm 1$, when $P \in \text{Spin}(d, d)$, (C.5) implies (C.4). In the special case when $S \in \text{Spin}^+(d, d)$ we have $(S, S) = +1$, so Mukai pairing is invariant under the connected component to identity, $\text{Spin}^+(d, d)$. See [30] for further details.

The NATD matrix is not an element of $\text{Spin}^+(d, d)$. However, due its special form given in (2.6) we still have

$$\langle S_{\text{NATD}} \chi_1, S_{\text{NATD}} \chi_2 \rangle = - \langle \chi_1, \chi_2 \rangle. \quad (\text{C.6})$$

This can be seen as follows: as discussed in detail in [21, 41] the matrix that appears in the definition of (2.6) (and also of \mathcal{K} with $n = d$) is

$$C_n = \Lambda_1 \cdots \Lambda_n, \quad (\text{C.7})$$

where

$$\Lambda_i = (\psi^i - \psi_i). \quad (\text{C.8})$$

Here, ψ_i, ψ^i are elements of the Clifford algebra $\text{Cliff}(d, d)$ given in (3.19), and hence obey the commutation relations following from (3.18). Since $(\Lambda_i, \Lambda_i) = -i\psi_i \psi^i = -1$, repeated use of (C.5) gives

$$\langle C_n \chi_1, C_n \chi_2 \rangle = (-1)^n \langle \chi_1, \chi_2 \rangle \quad (\text{C.9})$$

From this it follows (again using (C.5) repeatedly)

$$\langle S_{\text{NATD}} \chi_1, S_{\text{NATD}} \chi_2 \rangle = \langle S_\beta C_3 \chi_1, S_\beta C_3 \chi_2 \rangle = \langle C_3 \chi_1, C_3 \chi_2 \rangle = - \langle \chi_1, \chi_2 \rangle, \quad (\text{C.10})$$

as claimed. Note that in the second equality we used the fact that $S_\beta \in \text{Spin}^+(6, 6)$.

In addition to the elements Λ_i defined in (C.8), it is also useful to define the elements

$$\Lambda_i^+ = (\psi^i + \psi_i), \quad (\text{C.11})$$

and

$$C_n^+ = \Lambda_1^+ \cdots \Lambda_n^+. \quad (\text{C.12})$$

From the Clifford commutation relations (3.18) one can easily compute

$$\Lambda_i^+ \cdot \Gamma^M \cdot (\Lambda_i^+)^{-1} = \begin{cases} \Gamma_i & \text{if } \Gamma^M = \Gamma^i \\ \Gamma^i & \text{if } \Gamma^M = \Gamma_i \\ -\Gamma^M & \text{otherwise} \end{cases} \quad (\text{C.13})$$

This then means that $\rho(\Lambda_i^+) = h_i^+$, where

$$h_i^+ = - \begin{pmatrix} 1 - E_i & -E_i \\ -E_i & 1 - E_i \end{pmatrix}, \quad (E_i)_{jk} = \delta_{ij} \delta_{ik}. \quad (\text{C.14})$$

The charge conjugation matrix which appears in the definition of $\mathcal{K} \in \text{Spin}(d, d)$ is C_d for even d , whereas it is C_d^+ for odd d , as explained in [21].

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References

- [1] X.C. de la Ossa and F. Quevedo, *Duality symmetries from nonAbelian isometries in string theory*, *Nucl. Phys. B* **403** (1993) 377 [[hep-th/9210021](#)] [[INSPIRE](#)].
- [2] E. Alvarez, L. Álvarez-Gaumé and Y. Lozano, *On nonAbelian duality*, *Nucl. Phys. B* **424** (1994) 155 [[hep-th/9403155](#)] [[INSPIRE](#)].
- [3] K. Sfetsos and D.C. Thompson, *On non-abelian T-dual geometries with Ramond fluxes*, *Nucl. Phys. B* **846** (2011) 21 [[arXiv:1012.1320](#)] [[INSPIRE](#)].
- [4] Y. Lozano, E. O Colgain, K. Sfetsos and D.C. Thompson, *Non-abelian T-duality, Ramond Fields and Coset Geometries*, *JHEP* **06** (2011) 106 [[arXiv:1104.5196](#)] [[INSPIRE](#)].
- [5] G. Itsios, Y. Lozano, E. O Colgain and K. Sfetsos, *Non-Abelian T-duality and consistent truncations in type-II supergravity*, *JHEP* **08** (2012) 132 [[arXiv:1205.2274](#)] [[INSPIRE](#)].
- [6] G. Itsios, C. Núñez, K. Sfetsos and D.C. Thompson, *Non-Abelian T-duality and the AdS/CFT correspondence:new $N = 1$ backgrounds*, *Nucl. Phys. B* **873** (2013) 1 [[arXiv:1301.6755](#)] [[INSPIRE](#)].
- [7] J. Jeong, O. Kelekci and E. O Colgain, *An alternative IIB embedding of $F(4)$ gauged supergravity*, *JHEP* **05** (2013) 079 [[arXiv:1302.2105](#)] [[INSPIRE](#)].
- [8] K. Sfetsos and D.C. Thompson, *New $\mathcal{N} = 1$ supersymmetric AdS_5 backgrounds in Type IIA supergravity*, *JHEP* **11** (2014) 006 [[arXiv:1408.6545](#)] [[INSPIRE](#)].
- [9] E. Caceres, N.T. Macpherson and C. Núñez, *New Type IIB Backgrounds and Aspects of Their Field Theory Duals*, *JHEP* **08** (2014) 107 [[arXiv:1402.3294](#)] [[INSPIRE](#)].
- [10] N.T. Macpherson, C. Núñez, L.A. Pando Zayas, V.G.J. Rodgers and C.A. Whiting, *Type IIB supergravity solutions with AdS_5 from Abelian and non-Abelian T dualities*, *JHEP* **02** (2015) 040 [[arXiv:1410.2650](#)] [[INSPIRE](#)].
- [11] O. Kelekci, Y. Lozano, N.T. Macpherson and E.O. Colgáin, *Supersymmetry and non-Abelian T-duality in type-II supergravity*, *Class. Quant. Grav.* **32** (2015) 035014 [[arXiv:1409.7406](#)] [[INSPIRE](#)].
- [12] L.A. Pando Zayas, V.G.J. Rodgers and C.A. Whiting, *Supergravity solutions with AdS_4 from non-Abelian T-dualities*, *JHEP* **02** (2016) 061 [[arXiv:1511.05991](#)] [[INSPIRE](#)].
- [13] G. Itsios, Y. Lozano, J. Montero and C. Núñez, *The AdS_5 non-Abelian T-dual of Klebanov-Witten as a $\mathcal{N} = 1$ linear quiver from M5-branes*, *JHEP* **09** (2017) 038 [[arXiv:1705.09661](#)] [[INSPIRE](#)].
- [14] F. Hassler, *Poisson-Lie T-duality in Double Field Theory*, *Phys. Lett. B* **807** (2020) 135455 [[arXiv:1707.08624](#)] [[INSPIRE](#)].

- [15] D. Lüst and D. Osten, *Generalised fluxes, Yang-Baxter deformations and the $O(d, d)$ structure of non-abelian T-duality*, *JHEP* **05** (2018) 165 [[arXiv:1803.03971](#)] [[INSPIRE](#)].
- [16] S. Demulder, F. Hassler and D.C. Thompson, *Doubled aspects of generalised dualities and integrable deformations*, *JHEP* **02** (2019) 189 [[arXiv:1810.11446](#)] [[INSPIRE](#)].
- [17] A. Catal-Ozer, *Non-Abelian T-duality as a Transformation in Double Field Theory*, *JHEP* **08** (2019) 115 [[arXiv:1904.00362](#)] [[INSPIRE](#)].
- [18] Y. Sakatani, *Type II DFT solutions from Poisson-Lie T-duality/plurality*, [arXiv:1903.12175](#) [[INSPIRE](#)].
- [19] M. Bugden, *Non-abelian T-folds*, *JHEP* **03** (2019) 189 [[arXiv:1901.03782](#)] [[INSPIRE](#)].
- [20] A.A. Tseytlin, *Duality symmetric closed string theory and interacting chiral scalars*, *Nucl. Phys. B* **350** (1991) 395 [[INSPIRE](#)].
- [21] O. Hohm, S.K. Kwak and B. Zwiebach, *Double Field Theory of Type II Strings*, *JHEP* **09** (2011) 013 [[arXiv:1107.0008](#)] [[INSPIRE](#)].
- [22] C. Hull and B. Zwiebach, *Double Field Theory*, *JHEP* **09** (2009) 099 [[arXiv:0904.4664](#)] [[INSPIRE](#)].
- [23] C. Hull and B. Zwiebach, *The Gauge algebra of double field theory and Courant brackets*, *JHEP* **09** (2009) 090 [[arXiv:0908.1792](#)] [[INSPIRE](#)].
- [24] K. Sfetsos and D.C. Thompson, *New $\mathcal{N} = 1$ supersymmetric AdS_5 backgrounds in Type IIA supergravity*, *JHEP* **11** (2014) 006 [[arXiv:1408.6545](#)] [[INSPIRE](#)].
- [25] N.T. Macpherson, C. Núñez, D.C. Thompson and S. Zacarias, *Holographic Flows in non-Abelian T-dual Geometries*, *JHEP* **11** (2015) 212 [[arXiv:1509.04286](#)] [[INSPIRE](#)].
- [26] A. Barranco, J. Gaillard, N.T. Macpherson, C. Núñez and D.C. Thompson, *G-structures and Flavouring non-Abelian T-duality*, *JHEP* **08** (2013) 018 [[arXiv:1305.7229](#)] [[INSPIRE](#)].
- [27] J. Gaillard, N.T. Macpherson, C. Núñez and D.C. Thompson, *Dualising the Baryonic Branch: Dynamic $SU(2)$ and confining backgrounds in IIA*, *Nucl. Phys. B* **884** (2014) 696 [[arXiv:1312.4945](#)] [[INSPIRE](#)].
- [28] N. Hitchin, *Lectures on generalized geometry*, [arXiv:1008.0973](#) [[INSPIRE](#)].
- [29] N. Hitchin, *Generalized Calabi-Yau manifolds*, *Quart. J. Math.* **54** (2003) 281 [[math/0209099](#)] [[INSPIRE](#)].
- [30] M. Gualtieri, *Generalized complex geometry*, Ph.D. Thesis, Oxford University, Oxford U.K. (2003) [[math/0401221](#)] [[INSPIRE](#)].
- [31] M. Graña, R. Minasian, M. Petrini and A. Tomasiello, *Type II strings and generalized Calabi-Yau manifolds*, *Comptes Rendus Physique* **5** (2004) 979 [[hep-th/0409176](#)] [[INSPIRE](#)].
- [32] E. Caceres, N.T. Macpherson and C. Núñez, *New Type IIB Backgrounds and Aspects of Their Field Theory Duals*, *JHEP* **08** (2014) 107 [[arXiv:1402.3294](#)] [[INSPIRE](#)].
- [33] N.T. Macpherson, *Non-Abelian T-duality, G_2 -structure rotation and holographic duals of $N = 1$ Chern-Simons theories*, *JHEP* **11** (2013) 137 [[arXiv:1310.1609](#)] [[INSPIRE](#)].
- [34] M. Graña, R. Minasian, M. Petrini and D. Waldram, *T-duality, Generalized Geometry and Non-Geometric Backgrounds*, *JHEP* **04** (2009) 075 [[arXiv:0807.4527](#)] [[INSPIRE](#)].
- [35] M. Graña, R. Minasian, M. Petrini and A. Tomasiello, *Generalized structures of $N = 1$ vacua*, *JHEP* **11** (2005) 020 [[hep-th/0505212](#)] [[INSPIRE](#)].

- [36] M. Graña, R. Minasian, M. Petrini and A. Tomasiello, *A Scan for new $N = 1$ vacua on twisted tori*, *JHEP* **05** (2007) 031 [[hep-th/0609124](#)] [[INSPIRE](#)].
- [37] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, *Phys. Rept.* **244** (1994) 77 [[hep-th/9401139](#)] [[INSPIRE](#)].
- [38] M. Graña and D. Marques, *Gauged Double Field Theory*, *JHEP* **04** (2012) 020 [[arXiv:1201.2924](#)] [[INSPIRE](#)].
- [39] D. Geissbuhler, *Double Field Theory and $N = 4$ Gauged Supergravity*, *JHEP* **11** (2011) 116 [[arXiv:1109.4280](#)] [[INSPIRE](#)].
- [40] G. Aldazabal, W. Baron, D. Marques and C. Núñez, *The effective action of Double Field Theory*, *JHEP* **11** (2011) 052 [Erratum *ibid.* **11** (2011) 109] [[arXiv:1109.0290](#)] [[INSPIRE](#)].
- [41] A. Catal-Ozer, *Duality Twisted Reductions of Double Field Theory of Type II Strings*, *JHEP* **09** (2017) 044 [[arXiv:1705.08181](#)] [[INSPIRE](#)].
- [42] R. Borsato, A. Vilar López and L. Wulff, *The first α' -correction to homogeneous Yang-Baxter deformations using $O(d, d)$* , *JHEP* **07** (2020) 103 [[arXiv:2003.05867](#)] [[INSPIRE](#)].
- [43] R. Borsato and L. Wulff, *Quantum Correction to Generalized T Dualities*, *Phys. Rev. Lett.* **125** (2020) 201603 [[arXiv:2007.07902](#)] [[INSPIRE](#)].
- [44] F. Hassler and T. Rochais, *α' -Corrected Poisson-Lie T-duality*, *Fortsch. Phys.* **68** (2020) 2000063 [[arXiv:2007.07897](#)] [[INSPIRE](#)].
- [45] R. Borsato and S. Driezen, *Supergravity solution-generating techniques and canonical transformations of σ -models from $O(D, D)$* , *JHEP* **05** (2021) 180 [[arXiv:2102.04498](#)] [[INSPIRE](#)].
- [46] R. Borsato, S. Driezen and F. Hassler, *An algebraic classification of solution generating techniques*, *Phys. Lett. B* **823** (2021) 136771 [[arXiv:2109.06185](#)] [[INSPIRE](#)].
- [47] K. Gubarev and E.T. Musaev, *Polyvector deformations in eleven-dimensional supergravity*, *Phys. Rev. D* **103** (2021) 066021 [[arXiv:2011.11424](#)] [[INSPIRE](#)].
- [48] L. Martucci and P. Smyth, *Supersymmetric D-branes and calibrations on general $N = 1$ backgrounds*, *JHEP* **11** (2005) 048 [[hep-th/0507099](#)] [[INSPIRE](#)].
- [49] A. Tomasiello, *Generalized structures of ten-dimensional supersymmetric solutions*, *JHEP* **03** (2012) 073 [[arXiv:1109.2603](#)] [[INSPIRE](#)].
- [50] E. Bergshoeff, R. Kallosh, T. Ortín, D. Roest and A. Van Proeyen, *New formulations of $D = 10$ supersymmetry and D8-O8 domain walls*, *Class. Quant. Grav.* **18** (2001) 3359 [[hep-th/0103233](#)] [[INSPIRE](#)].
- [51] M. Fukuma, T. Oota and H. Tanaka, *Comments on T dualities of Ramond-Ramond potentials on tori*, *Prog. Theor. Phys.* **103** (2000) 425 [[hep-th/9907132](#)] [[INSPIRE](#)].
- [52] B. Zwiebach, *Double Field Theory, T-duality, and Courant Brackets*, *Lect. Notes Phys.* **851** (2012) 265 [[arXiv:1109.1782](#)] [[INSPIRE](#)].
- [53] O. Lunin and J.M. Maldacena, *Deforming field theories with $U(1) \times U(1)$ global symmetry and their gravity duals*, *JHEP* **05** (2005) 033 [[hep-th/0502086](#)] [[INSPIRE](#)].
- [54] L.F. Alday, G. Arutyunov and S. Frolov, *Green-Schwarz strings in TsT-transformed backgrounds*, *JHEP* **06** (2006) 018 [[hep-th/0512253](#)] [[INSPIRE](#)].

- [55] A. Catal-Ozer, *Lunin-Maldacena deformations with three parameters*, *JHEP* **02** (2006) 026 [[hep-th/0512290](#)] [[INSPIRE](#)].
- [56] R. Minasian, M. Petrini and A. Zaffaroni, *Gravity duals to deformed SYM theories and Generalized Complex Geometry*, *JHEP* **12** (2006) 055 [[hep-th/0606257](#)] [[INSPIRE](#)].
- [57] D. Andriot, R. Minasian and M. Petrini, *Flux backgrounds from Twists*, *JHEP* **12** (2009) 028 [[arXiv:0903.0633](#)] [[INSPIRE](#)].
- [58] D. Andriot, E. Goi, R. Minasian and M. Petrini, *Supersymmetry breaking branes on solvmanifolds and de Sitter vacua in string theory*, *JHEP* **05** (2011) 028 [[arXiv:1003.3774](#)] [[INSPIRE](#)].
- [59] R. Borsato and L. Wulff, *Non-abelian T-duality and Yang-Baxter deformations of Green-Schwarz strings*, *JHEP* **08** (2018) 027 [[arXiv:1806.04083](#)] [[INSPIRE](#)].
- [60] J.-i. Sakamoto, Y. Sakatani and K. Yoshida, *Homogeneous Yang-Baxter deformations as generalized diffeomorphisms*, *J. Phys. A* **50** (2017) 415401 [[arXiv:1705.07116](#)] [[INSPIRE](#)].
- [61] A. Çatal Özer and S. Tunali, *Yang-Baxter Deformation as an $O(d, d)$ Transformation*, *Class. Quant. Grav.* **37** (2020) 075003 [[arXiv:1906.09053](#)] [[INSPIRE](#)].
- [62] D. Orlando, S. Reffert, Y. Sekiguchi and K. Yoshida, *$O(d, d)$ transformations preserve classical integrability*, *Nucl. Phys. B* **950** (2020) 114880 [[arXiv:1907.03759](#)] [[INSPIRE](#)].
- [63] T. Codina and D. Marques, *Generalized Dualities and Higher Derivatives*, *JHEP* **10** (2020) 002 [[arXiv:2007.09494](#)] [[INSPIRE](#)].
- [64] P. Koerber and D. Tsimpis, *Supersymmetric sources, integrability and generalized-structure compactifications*, *JHEP* **08** (2007) 082 [[arXiv:0706.1244](#)] [[INSPIRE](#)].
- [65] D. Andriot, *New supersymmetric flux vacua with intermediate $SU(2)$ structure*, *JHEP* **08** (2008) 096 [[arXiv:0804.1769](#)] [[INSPIRE](#)].
- [66] S. Salamon, *Pitman Research Notes in Mathematics. Vol. 201: Riemannian Geometry and Holonomy Groups*, Longman, Harlow U.K. (1989).
- [67] J.P. Gauntlett, D. Martelli and D. Waldram, *Superstrings with intrinsic torsion*, *Phys. Rev. D* **69** (2004) 086002 [[hep-th/0302158](#)] [[INSPIRE](#)].
- [68] G. Dall'Agata and N. Prezas, *$N = 1$ geometries for M-theory and type IIA strings with fluxes*, *Phys. Rev. D* **69** (2004) 066004 [[hep-th/0311146](#)] [[INSPIRE](#)].
- [69] G. Dall'Agata, *On supersymmetric solutions of type IIB supergravity with general fluxes*, *Nucl. Phys. B* **695** (2004) 243 [[hep-th/0403220](#)] [[INSPIRE](#)].