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Determinantal Calabi-Yau varieties in Grassmannians and the Givental I-functions

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Abstract: We examine a class of Calabi-Yau varieties of the determinantal type in Grassmannians and clarify what kind of examples can be constructed explicitly. We also demonstrate how to compute their genus-0 Gromov-Witten invariants from the analysis of the Givental I-functions. By constructing I-functions from the supersymmetric localization formula for the two dimensional gauged linear sigma models, we describe an algorithm to evaluate the genus-0 A-model correlation functions appropriately. We also check that our results for the Gromov-Witten invariants are consistent with previous results for known examples included in our construction.

Keywords: Topological Strings, Differential and Algebraic Geometry, Sigma Models

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1 Introduction

String compactifications to lower dimensions preserving supersymmetry motivate us to study Calabi-Yau varieties as initiated in [1]. To analyze various properties of Calabi-Yau backgrounds, it is useful to consider the two dimensional gauged linear sigma model (GLSM) [2]. This model corresponds to the UV description of the non-linear sigma models on Calabi-Yau backgrounds and has provided a powerful technique to compute the correlation functions exactly (see for example [3]). Utilizing the duality property of the model, a physical understanding about mirror symmetry has also been developed [4] (see also [5, 6] for several recent developments).

On the other hand, by using the supersymmetric localization techniques [7], exact formula for the GLSM partition functions [8, 9] (see also [10, 11]) and correlation functions [12, 13] on the 2-sphere backgrounds has been clarified. This means that one can evaluate the genus-0 Gromov-Witten invariants from the GLSM calculation in a direct fashion. This methodology has also been applied to the GLSMs with non-abelian gauge groups and the study of the complete intersection Calabi-Yau varieties in Grassmannians has progressed in the last few years [14, 15]. Several notable aspects of the GLSM correlation functions have also been clarified in [16, 17].

While the toric complete intersection varieties described by abelian GLSMs have been thoroughly investigated in various contexts, a comprehensive understanding about the non-complete intersection varieties requires further efforts. In [18], as an example of a class of non-complete intersections, the determinantal varieties [19, 20] and the associated non-abelian GLSMs have been investigated. The aim of our work is to make advances in the study of the determinantal varieties and provides a further step toward the comprehensive understanding of general Calabi-Yau backgrounds with non-abelian GLSM descriptions.

In this paper we explicitly clarify what kind of determinantal Calabi-Yau varieties in Grassmannians can be constructed while satisfying several requirements. We mainly focus on the 3-fold examples and the analysis for the determinantal Calabi-Yau 2-folds and 4-folds is summarized in appendix. We will also study the quantum aspects of the GLSM associated with determinantal varieties. Our method is based on the analysis of the so-called Givental I-functions [21–23] which can be extracted from the localization formula for the GLSM on a supersymmetric 2-sphere. In particular, we will compute the genus-0 Gromov-Witten invariants of determinantal Calabi-Yau varieties by using a conjectural handy formula, and check that our results coincide with previous results for known examples included in our classification.

This paper is organized as follows. First we examine a class of determinantal Calabi-Yau varieties in Grassmannians satisfying several requirements and specify the possible examples in section 2. In section 3, we briefly review the computation of the genus-0 invariants of complete intersection Calabi-Yau varieties in complex projective spaces and Grassmannians utilizing the I-functions, and provide a conjectural formula for Grassmannian Calabi-Yau varieties. In section 4, we compute the genus-0 invariants of determinantal Calabi-Yau varieties with GLSM realizations by using the algorithm described in section 3. Section 5 is devoted to conclusions and discussions. In appendix A we summarize our results of the analysis for determinantal Calabi-Yau 2-folds and 4-folds. In appendix B, we take a brief look at the computation of Hodge numbers using the Koszul complex and demonstrate several computations explicitly. In appendix C we summarize the data of the genus-0 invariants of several determinantal Calabi-Yau 4-folds.

2 Determinantal Calabi-Yau varieties in Grassmannians

In this section, we first review a minimal ingredient of determinantal varieties, following [18] (see also [19, 20]). Afterwards we classify a class of determinantal Calabi-Yau 3-folds in Grassmannians satisfying appropriate conditions.

2.1 Definitions

Let V be a compact algebraic variety, and $A : \mathcal{E}_p \to \mathcal{F}_q$ be a linear map from a rank p vector bundle $\mathcal{E}_p$ on V to a rank q vector bundle $\mathcal{F}_q$ on V. Here we assume that the linear map A is a global holomorphic section of the rank pq-bundle $\mathrm{Hom}(\mathcal{E}_p, \mathcal{F}_q) \cong \mathcal{E}_p^* \otimes \mathcal{F}_q$ with maximal rank at a generic point of V. By representing the linear map A locally as a $q \times p$ matrix $A(\phi)$ of the holomorphic sections, a determinantal variety is defined as

$$Z(A, \ell) = \{ \ \phi \in V \ | \ \mathrm{rank}\ A(\phi) \leq \ell \ \}, \quad 0 \leq \ell < \min(p, q), \tag{2.1}$$

where ϕ denotes the homogeneous coordinates on V. The complex codimension of $Z(A, \ell)$ in V is given by

$$\mathrm{codim}\, Z(A, \ell) = (p - \ell)(q - \ell). \tag{2.2}$$

Here $(\ell + 1) \times (\ell + 1)$ minors of $A(\phi)$ generate the ideal $I(Z(A, \ell))$. Since $\mathrm{codim}\, Z(A, \ell) = (p - \ell)(q - \ell) < \binom{p}{\ell+1}\binom{q}{\ell+1}$ for $\ell \geq 1$, the ideal $I(Z(A, \ell))$ has non-trivial relations called syzygies and the determinantal variety $Z(A, \ell)$ for $\ell \geq 1$ is not a complete intersection. As argued in [18], a simple analysis of the Jacobian matrix implies that $Z(A, \ell)$ for $\ell \geq 1$ has singular loci along $Z(A, \ell - 1) \subset Z(A, \ell)$ only. One can resolve these singularities by the so-called *incidence correspondence* [18, 19],

$$X_A^V = \{ \ (\phi, x) \in V_{\mathcal{E}_p, p-\ell} \ | \ A(\phi)x = 0 \ \} \ \longrightarrow \ Z(A, \ell), \tag{2.3}$$

where $V_{\mathcal{E}_p,p-\ell}$ denotes the fibration

$$G(p-\ell,\mathcal{E}_p) \longrightarrow V_{\mathcal{E}_p,p-\ell} \xrightarrow{\pi} V, \tag{2.4}$$

with Grassmannian fibers $G(p-\ell,\mathcal{E}_p)$ of $(p-\ell)$-planes with respect to the p-dimensional fibers of $\mathcal{E}_p$. It is worth noting that the codimension of the singular loci in V is $\operatorname{codim} Z(A,\ell-1) = \operatorname{codim} Z(A,\ell) + p + q - 2\ell + 1$, and then the determinantal variety $Z(A,\ell)$ with the dimension less than $p+q-2\ell+1$ does not have singular loci [18].[1]

Remark 2.1 ([18])**.** Since $\ell < \min(p,q)$, the determinantal varieties with dimension less than 2 do not have singular loci. The determinantal 3-folds have singular points only when $p=q=\ell+1$. The determinantal 4-folds have singular lines only when $p=q=\ell+1$, and have singular points only when $(p,q)=(\ell+1,\ell+2)$ or $(p,q)=(\ell+2,\ell+1)$.

In this paper we only consider the square ($p=q$) determinantal varieties with

$$V = G(k,n), \qquad \mathcal{E}_p = \mathcal{O}_V^{\oplus p}, \qquad \operatorname{rank} \mathcal{F}_p = p, \tag{2.5}$$

where $G(k,n)$ is the complex Grassmannian defined by the set of k-planes in $\mathbb{C}^n$, and $\mathcal{O}_V$ is the structure sheaf of V. Then the variety $V_{\mathcal{E}_p,p-\ell}$ can be described by a product variety $V_{\mathcal{E}_p,p-\ell} \cong G(k,n) \times G(p-\ell,p)$ and the incidence correspondence (2.3) becomes

$$X_A := X_A^{G(k,n)} = \left\{ \ (\phi,x) \in G(k,n) \times G(\ell_p^\vee,p) \ | \ A(\phi)x = 0 \ \right\}, \qquad \ell_p^\vee := p-\ell. \tag{2.6}$$

In addition we require $n = \ell_p^\vee \mathfrak{c}_1(\mathcal{F}_p)$ derived from Calabi-Yau condition [18]. Here

$$c_1(\mathcal{F}_p) = \mathfrak{c}_1(\mathcal{F}_p)\,\sigma_1$$

is the first Chern class of $\mathcal{F}_p$ and $\sigma_1 = c_1(\mathcal{Q})$ is the Schubert class of $G(k,n)$. $\mathcal{Q}$ is the universal quotient bundle on $G(k,n)$. The dimension of X_A is given by $\dim X_A = k(n-k) - (p-\ell)^2 = \ell_p^\vee \left(k\mathfrak{c}_1(\mathcal{F}_p) - \ell_p^\vee\right) - k^2$. By taking the duality $G(k,n) \cong G(n-k,n)$ into consideration, here we only consider the case with $2k \le n$. Furthermore, the rank condition $0 \le \ell < p$ can be rephrased as $0 < \ell_p^\vee \le p = \operatorname{rank} \mathcal{F}_p$.

In summary, we have seen that the following conditions[2] must be satisfied in order to realize the determinantal varieties appropriately.

1.	Dimensional condition:	$\ell_p^\vee \left(k\mathfrak{c}_1(\mathcal{F}_p) - \ell_p^\vee\right) = k^2 + \dim X_A.$	(2.7)
2.	Calabi-Yau condition:	$n = \ell_p^\vee \mathfrak{c}_1(\mathcal{F}_p).$	(2.8)
3.	Duality condition:	$2k \le n.$	(2.9)
4.	Rank condition:	$0 < \ell_p^\vee \le p = \operatorname{rank} \mathcal{F}_p.$	(2.10)

In the following, we will classify the determinantal Calabi-Yau 3-folds satisfying the above four conditions. Although we consider the desingularized determinantal varieties X_A, the following analysis also gives a classification of $Z(A,\ell)$.[3]

[1] As noted in [18], one can also use the incidence correspondence (2.3) to describe the determinantal varieties without singular loci.

[2] Note that we do not impose *irreducibility* or the conditions $H^i(X_A,\mathcal{O}_{X_A}) = 0$ in our analysis.

[3] See appendix A for the analysis of determinantal Calabi-Yau 2-folds and 4-folds.

2.2 General dimensions

Before moving on to the discussion about the determinantal Calabi-Yau 3-folds, let us consider general implications of the above requirements. In general dimensions, obviously the following two ansatz always satisfy the dimensional condition (2.7).

$$\text{Ansatz (I)}: \qquad \left(\ell_p^\vee, k\mathfrak{c}_1(\mathcal{F}_p)\right) = \left(1, \dim X_A + k^2 + 1\right), \tag{2.11}$$

$$\text{Ansatz (II)}: \qquad \left(\ell_p^\vee, k\mathfrak{c}_1(\mathcal{F}_p)\right) = \left(\dim X_A + k^2, \dim X_A + k^2 + 1\right). \tag{2.12}$$

In the following, we will illustrate what kind of setups for $\left(k, n; \ell_p^\vee, \mathfrak{c}_1(\mathcal{F}_p)\right)$ satisfy all the above requirements if we start from the Ansatz (I) or (II).[4]

2.2.1 Ansatz (I)

In this case, the Calabi-Yau condition (2.8) becomes

$$n = \frac{1}{k}\left(\dim X_A + k^2 + 1\right). \tag{2.13}$$

Then the duality condition (2.9) implies

$$k^2 \leq \dim X_A + 1, \tag{2.14}$$

which means that examples with $k \geq 3$ provide determinantal varieties with $\dim X_A \geq 8$.

When $k = 1$, V is given by $G(1, n) \cong \mathbb{P}^{n-1}$ and one obtains the solutions with

$$\left(k, n; \ell_p^\vee, \mathfrak{c}_1(\mathcal{F}_p)\right) = (1, \dim X_A + 2; 1, \dim X_A + 2). \tag{2.15}$$

In this case the rank condition (2.10) is trivially satisfied, and appropriate $\mathcal{F}_p$ on V are given by the following vector bundles associated with the integer partitions of $\dim X_A + 2$:

$$\mathcal{F}_p = \oplus_{i=1}^r \mathcal{O}_V(p_i), \qquad p_1 \geq p_2 \geq \cdots \geq p_r > 0, \quad \sum_{i=1}^r p_i = \dim X_A + 2. \tag{2.16}$$

When $k = 2$, V becomes $G(2, n)$ and one finds the solutions with

$$\left(k, n; \ell_p^\vee, \mathfrak{c}_1(\mathcal{F}_p)\right) = (2, (\dim X_A + 5)/2; 1, (\dim X_A + 5)/2). \tag{2.17}$$

Since n has to be an integer, this type of solution can exist only when the dimension of X_A is odd. Moreover, (2.14) requires $\dim X_A \geq 3$.

2.2.2 Ansatz (II)

In this case, the Calabi-Yau condition (2.8) becomes

$$n = \frac{1}{k}\left(\dim X_A + k^2\right)\left(\dim X_A + k^2 + 1\right), \tag{2.18}$$

[4]Of course there exist other solutions which do not belong to the Ansatz (I) or (II). In section 2.3 we have also taken into account this kind of solutions and checked, by Mathematica and Maple, that our result exhausted all the possible solutions up to $k = 50$.

and the duality condition (2.9) is trivially satisfied. Thus we only need to consider the rank condition given by

$$\dim X_A + k^2 \leq \text{rank } \mathcal{F}_p. \tag{2.19}$$

For example, when $k = 1$, we obtain the following solutions

$$\begin{aligned}&\left(k, n; \ell_p^\vee, \mathfrak{c}_1(\mathcal{F}_p)\right) = (1, (\dim X_A + 1)(\dim X_A + 2); \dim X_A + 1, \dim X_A + 2)\\ &\text{with } \mathcal{F}_p = \mathcal{O}_V(1)^{\oplus \dim X_A} \oplus \mathcal{O}_V(2),\ \mathcal{O}_V(1)^{\oplus(\dim X_A+2)}.\end{aligned} \tag{2.20}$$

Note that the rank condition (2.19) strongly constrain the possible vector bundles.

2.3 Determinantal Calabi-Yau 3-folds

Here we will focus on the square determinantal Calabi-Yau 3-folds and determine what kind of setups satisfy the above four conditions. Let us start with the dimensional condition given by

$$\ell_p^\vee \left(k\mathfrak{c}_1(\mathcal{F}_p) - \ell_p^\vee\right) = k^2 + 3. \tag{2.21}$$

Then we will find out which type of choices for $\left(k, n; \ell_p^\vee, \mathfrak{c}_1(\mathcal{F}_p)\right)$ can be possible while changing the parameter k.

2.3.1 $k = 1$

In this case we have $V = G(1, n) \cong \mathbb{P}^{n-1}$. From (2.15) and (2.16) one finds that there exists a "quintic family" (see for example [24]) given by

$$\left(k, n; \ell_p^\vee, \mathfrak{c}_1(\mathcal{F}_p)\right) = (1, 5; 1, 5) \text{ with } \mathcal{F}_p = \oplus_{i=1}^r \mathcal{O}_V(p_i),\ p_1 \geq p_2 \geq \cdots \geq p_r > 0,\ \sum_{i=1}^r p_i = 5. \tag{2.22}$$

The example constructed from $\mathcal{F}_p = \mathcal{O}_V(5)$ with $p = 1$ (i.e. $\ell = 0$) is the well-known quintic Calabi-Yau 3-fold, which is the zero locus of a holomorphic section of $\mathcal{O}_{\mathbb{P}^4}(5)$.

Since $\ell_p^\vee = 1$ (i.e. $p = \ell + 1$), according to the Remark 2.1, generically the above quintic families have singular points. The determinantal Calabi-Yau 3-folds in this class are connected by the deformations of complex structures, and it is known that the desingularized 3-folds are related by the so-called extremal transitions.[5]

Apart from the above quintic family, one can also find the following solutions

$$\begin{aligned}&\left(k, n; \ell_p^\vee, \mathfrak{c}_1(\mathcal{F}_p)\right) = (1, 8; 2, 4) \text{ with}\\ &\qquad \mathcal{F}_p = \mathcal{O}_V(1) \oplus \mathcal{O}_V(3),\ \mathcal{O}_V(2)^{\oplus 2},\ \mathcal{O}_V(1)^{\oplus 2} \oplus \mathcal{O}_V(2),\ \mathcal{O}_V(1)^{\oplus 4}.\end{aligned} \tag{2.23}$$

Here the first two examples with $p = 2$ (i.e. $\ell = 0$) in (2.23) can be identified with the well-known complete intersection Calabi-Yau 3-folds as

$$X_A \text{ with } \mathcal{O}_V(1) \oplus \mathcal{O}_V(3) \longleftrightarrow X_{3,3} \subset \mathbb{P}^5, \qquad X_A \text{ with } \mathcal{O}_V(2)^{\oplus 2} \longleftrightarrow X_{2,2,2,2} \subset \mathbb{P}^7,$$

[5] The comparison of topological invariants in section 4.3.1 makes this point clearly understandable.

where $X_{d_1,\ldots,d_r} \subset \mathbb{P}^{n-1}$ denotes the complete intersection variety defined by the zero locus of a holomorphic section of the vector bundle $\oplus_{a=1}^{r}\mathcal{O}_{\mathbb{P}^{n-1}}(d_a)$. The last two examples in (2.23) are Gulliksen-Negård type 3-folds studied in [25].

Moreover, (2.20) provides another type of solutions given by

$$\left(k,n;\ell_p^\vee,\mathfrak{c}_1(\mathcal{F}_p)\right) = (1,20;4,5) \text{ with } \mathcal{F}_p = \mathcal{O}_V(1)^{\oplus 3}\oplus\mathcal{O}_V(2),\ \mathcal{O}_V(1)^{\oplus 5}. \tag{2.24}$$

The Calabi-Yau 3-fold X_A in (2.24) constructed from $\mathcal{F}_p = \mathcal{O}_V(1)^{\oplus 3}\oplus\mathcal{O}_V(2)$ with $p=4$ (i.e. $\ell = 0$) has the following isomorphism:

$$X_A \text{ with } \mathcal{O}_V(1)^{\oplus 3}\oplus\mathcal{O}_V(2) \longleftrightarrow X_{2,2,2,2}\subset\mathbb{P}^7.$$

The other example constructed from $\mathcal{F}_p = \mathcal{O}_V(1)^{\oplus 5}$ has been studied in [18].

2.3.2 $k = 2$

In this case, V becomes the Grassmannians $V = G(2,n)$. Compared with the complex projective spaces, there exist additional components for the vector bundles on the Grassmannians, as explained in the followings.

When $k \geq 2$, beside the line bundle $\mathcal{O}_V(d)$ on $V = G(k,n)$, one can also consider vector bundles with rank greater than one denoted by

$$\mathcal{S}^* \quad\text{and}\quad \mathcal{Q}.$$

These are known as the dual of the universal subbundle and the universal quotient bundle on $G(k,n)$, respectively. Note that they fulfill the relation $\wedge^k\mathcal{S}^*\cong\mathcal{O}_V(1)$ and satisfy

$$\operatorname{rank}\mathcal{S}^* = k,\quad \mathfrak{c}_1(\mathcal{S}^*) = 1,\quad \operatorname{rank}\mathcal{Q} = n-k,\quad \mathfrak{c}_1(\mathcal{Q}) = 1.$$

Accordingly, general irreducible vector bundles can be constructed as

$$\begin{aligned}
\operatorname{Sym}^m\mathcal{S}^*(d) &:= \operatorname{Sym}^m\mathcal{S}^*\otimes\mathcal{O}_V(d), &\qquad \wedge^m\mathcal{S}^*(d) &:= \wedge^m\mathcal{S}^*\otimes\mathcal{O}_V(d),\\
\operatorname{Sym}^m\mathcal{Q}(d) &:= \operatorname{Sym}^m\mathcal{Q}^*\otimes\mathcal{O}_V(d), &\qquad \wedge^m\mathcal{Q}(d) &:= \wedge^m\mathcal{Q}^*\otimes\mathcal{O}_V(d),
\end{aligned}$$

where

$$\begin{aligned}
\operatorname{rank}\operatorname{Sym}^m\mathcal{S}^*(d) &= \binom{k+m-1}{m}, &\qquad \mathfrak{c}_1\left(\operatorname{Sym}^m\mathcal{S}^*(d)\right) &= \binom{k+m-1}{k}+d\binom{k+m-1}{m},\\
\operatorname{rank}\wedge^m\mathcal{S}^*(d) &= \binom{k}{m}, &\qquad \mathfrak{c}_1\left(\wedge^m\mathcal{S}^*(d)\right) &= \binom{k-1}{m-1}+d\binom{k}{m},\\
\operatorname{rank}\operatorname{Sym}^m\mathcal{Q}(d) &= \binom{n-k+m-1}{m}, &&\\
\mathfrak{c}_1\left(\operatorname{Sym}^m\mathcal{Q}(d)\right) &= \binom{n-k+m-1}{n-k}+d\binom{n-k+m-1}{m}, &&\\
\operatorname{rank}\wedge^m\mathcal{Q}(d) &= \binom{n-k}{m}, &\qquad \mathfrak{c}_1\left(\wedge^m\mathcal{Q}(d)\right) &= \binom{n-k-1}{m-1}+d\binom{n-k}{m}.
\end{aligned}$$

Returning to the main subject of the classification, (2.17) with the rank condition (2.10) implies that the following setups are possible

$$\begin{aligned}&\big(k,n;\ell_p^\vee,\mathfrak{c}_1(\mathcal{F}_p)\big)=(2,4;1,4)\text{ with}\\&\mathcal{F}_p=\mathcal{O}_V(4),\ \mathcal{O}_V(1)\oplus\mathcal{O}_V(3),\ \mathcal{O}_V(2)^{\oplus 2},\ \mathcal{O}_V(1)^{\oplus 2}\oplus\mathcal{O}_V(2),\ \mathcal{S}^*\oplus\mathcal{O}_V(3),\\&\qquad\mathcal{S}^*(1)\oplus\mathcal{O}_V(1),\ \mathcal{O}_V(1)^{\oplus 4},\ \mathcal{S}^*\oplus\mathcal{O}_V(1)\oplus\mathcal{O}_V(2),\ \mathrm{Sym}^2\mathcal{S}^*\oplus\mathcal{O}_V(1),\\&\qquad\mathcal{S}^*\oplus\mathcal{O}_V(1)^{\oplus 3},\ (\mathcal{S}^*)^{\oplus 2}\oplus\mathcal{O}_V(2),\ (\mathcal{S}^*)^{\oplus 2}\oplus\mathcal{O}_V(1)^{\oplus 2},\ (\mathcal{S}^*)^{\oplus 3}\oplus\mathcal{O}_V(1),\ (\mathcal{S}^*)^{\oplus 4}.\end{aligned}\tag{2.25}$$

Note that $\mathcal{S}^*\cong\mathcal{Q}$ on $G(2,4)$ and $\wedge^2\mathcal{S}^*\cong\mathcal{O}_V(1)$ when $k=2$. The example constructed from $\mathcal{F}_p=\mathcal{O}_V(4)$ with $p=1$ (i.e. $\ell=0$) is the complete intersection Grassmannian Calabi-Yau 3-fold in $G(2,4)$ with the vector bundle $\mathcal{O}_{G(2,4)}(4)$. Since $\ell_p^\vee=1$ (i.e. $p=\ell+1$), as discussed in the case for the quintic family (2.22), these determinantal Calabi-Yau 3-folds generically have singular points and the desingularized 3-folds are connected through the extremal transitions (see also section 4.3.3).

Another class of solutions can be obtained by the ansatz (II) in (2.12) and the result is

$$\begin{aligned}&\big(k,n;\ell_p^\vee,\mathfrak{c}_1(\mathcal{F}_p)\big)=(2,28;7,4)\text{ with}\\&\mathcal{F}_p=(\mathcal{S}^*)^{\oplus 3}\oplus\mathcal{O}_V(1),\ (\mathcal{S}^*)^{\oplus 4},\ \mathcal{Q}\oplus\mathcal{O}_V(3),\ \mathcal{Q}\oplus\mathcal{O}_V(1)\oplus\mathcal{O}_V(2),\ \mathcal{Q}\oplus\mathcal{O}_V(1)^{\oplus 3},\\&\qquad\mathcal{Q}^{\oplus 2}\oplus\mathcal{O}_V(2),\ \mathcal{Q}^{\oplus 2}\oplus\mathcal{O}_V(1)^{\oplus 2},\ \mathcal{Q}^{\oplus 3}\oplus\mathcal{O}_V(1),\ \mathcal{Q}^{\oplus 4}.\end{aligned}\tag{2.26}$$

Here the Calabi-Yau 3-fold constructed from $\mathcal{F}_p=(\mathcal{S}^*)^{\oplus 3}\oplus\mathcal{O}_V(1)$ with $p=7$ (i.e. $\ell=0$) can be identified with the complete intersection Grassmannian Calabi-Yau 3-fold in $G(2,7)$ with the vector bundle $\mathcal{O}_{G(2,7)}(1)^{\oplus 7}$.

2.3.3 $k\geq 3$

In this case, we have $V=G(k,n)$. Interestingly, there exist four "infinite families" given by

$$\begin{aligned}&\big(k,n;\ell_p^\vee,\mathfrak{c}_1(\mathcal{F}_p)\big)=(k_i,5\ell_i;\ell_i,5),\quad i\in\mathbb{N}\\&\text{with }k_1=4,\ \ell_1=19,\ k_{i+1}=\ell_i,\ \ell_{i+1}=-k_i+5\ell_i:\ \mathcal{F}_p=(\mathcal{S}^*)^{\oplus 5},\ \mathcal{Q}^{\oplus 5},\\&\big(k,n;\ell_p^\vee,\mathfrak{c}_1(\mathcal{F}_p)\big)=(k_i,4\ell_i;\ell_i,4),\quad i\in\mathbb{N}\\&\text{with }k_1=7,\ \ell_1=26,\ k_{i+1}=\ell_i,\ \ell_{i+1}=-k_i+4\ell_i:\ \mathcal{F}_p=(\mathcal{S}^*)^{\oplus 4},\ \mathcal{Q}^{\oplus 4}.\end{aligned}\tag{2.27}$$

By using mathematical induction, one can check that the duality condition (2.9), the rank condition (2.10), and in particular $\ell_p^\vee<\operatorname{rank}\mathcal{F}_p$, are maintained. Since we do not impose the irreducibility condition in our analysis, it is still possible that the above infinite families can be reduced to other trivial or non-trivial examples. In any case, it is required to thoroughly investigate various topological invariants of these higher rank examples, and this would require a considerable effort and we leave this issue as an open problem.

3 I-functions and Gromov-Witten invariants

In this section, we briefly overview the computation of genus-0 Gromov-Witten invariants using the Givental I-functions [21–23] (see also [26]). We will also provide a handy formula for the computations of the Gromov-Witten invariants of Grassmannian Calabi-Yau varieties, which is also applicable to the determinantal varieties.

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3.1 Building blocks of I-functions

When a Fano or a Calabi-Yau variety X has a GLSM realization with gauge group G, one can easily construct the Givental I-function of X by using the supersymmetric localization formula (see [14–17, 27]). Here we clarify the building blocks of the I-function of X associated with such a GLSM on the Ω-deformed 2-sphere $S^2_\hbar$ which has a vector multiplet and chiral matter multiplets with R-charge 0 or 2 under $\mathrm{U}(1)_R$. The deformation parameter $\hbar$ is identified with an equivariant parameter.

Let $\mathbf{x} = (x_1, \ldots, x_{\mathrm{rk}(\mathfrak{g})}) \in \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}$ be Coulomb branch parameters and $\mathbf{q} = (q_1, \ldots, q_{\mathrm{rk}(\mathfrak{g})}) \in \mathbb{Z}^{\mathrm{rk}(\mathfrak{g})} \subset i\mathfrak{h}$ be magnetic charges for Cartan subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ associated with G, where $\mathrm{rk}(\mathfrak{g})$ denotes the rank of $\mathfrak{g}$. Here the parameters $\mathbf{x}$ are identified with the Chern roots of X which give the total Chern class of X as

$$c(X) = \prod_{i=1}^{\mathrm{rk}(\mathfrak{g})} (1 + x_i)\,. \tag{3.1}$$

To construct the I-function of X, first we need a "classical block" associated with the subgroup $\mathrm{U}(1)^{\mathsf{c}} \subset G$, where c is the number of the central. The Fayet-Iliopoulos (FI) parameters ξ_a and theta angles θ_a, $a = 1, \ldots, \mathsf{c}$, are associated with each $\mathrm{U}(1)^{\mathsf{c}}$ factor, and the classical block of the I-function is given by

$$I^{\mathrm{c}}_{\mathbf{q}}(\mathbf{z}; \mathbf{x}; \hbar) = \mathrm{e}^{2\pi\sqrt{-1}\,\boldsymbol{\tau}(\mathbf{x}/\hbar + \mathbf{q})}, \qquad \boldsymbol{\tau} = \{\tau_a\} := \sqrt{-1}\,\xi_a + \frac{1}{2\pi}\theta_a. \tag{3.2}$$

Here the parameters $\mathbf{z} = \{z_a\} = \mathrm{e}^{2\pi\sqrt{-1}\,\tau_a}$ represent the exponentiated Kähler moduli of X, and the canonical pairing $\boldsymbol{\tau}(*)$ is defined by embedding $\boldsymbol{\tau}$ into $\mathfrak{h}^* \otimes_{\mathbb{R}} \mathbb{C}$.

Other contributions come from the 1-loop determinants of multiplets of the GLSM. The vector multiplet provides a block given by

$$I^{\mathrm{vec}}_{\mathbf{q}}(\mathbf{x}; \hbar) = \prod_{\alpha \in \Delta_+} (-1)^{\alpha(\mathbf{q})} \frac{\alpha(\mathbf{x}) + \alpha(\mathbf{q})\hbar}{\alpha(\mathbf{x})}, \tag{3.3}$$

where Δ_+ is the set of positive roots of $\mathfrak{g}$. In general, the GLSM also has chiral matter multiplets Φ with R-charge 0 and P with R-charge 2 in a certain representation $\mathbf{R}$. Note that one can turn on a twisted mass parameter λ while preserving supersymmetry, which is identified with an equivariant parameter. Their contributions are given as follows:

$$I^{\Phi}_{\mathbf{q}}(\mathbf{x}, \lambda; \hbar) = \begin{cases} \prod_{\rho \in \mathbf{R}} \prod_{p=1}^{\rho(\mathbf{q})} (\rho(\mathbf{x}) + \lambda + p\hbar)^{-1}\,, & \text{for } \ \rho(\mathbf{q}) \geq 1, \\ 1, & \text{for } \ \rho(\mathbf{q}) = 0, \\ \prod_{\rho \in \mathbf{R}} \prod_{p=0}^{-\rho(\mathbf{q})-1} (\rho(\mathbf{x}) + \lambda - p\hbar)\,, & \text{for } \ \rho(\mathbf{q}) \leq -1, \end{cases} \tag{3.4}$$

and

$$I^{P}_{\mathbf{q}}(\mathbf{x}, \lambda; \hbar) = \begin{cases} \prod_{\rho \in \mathbf{R}} \prod_{p=1}^{-\rho(\mathbf{q})} (-\rho(\mathbf{x}) - \lambda + p\hbar)\,, & \text{for } \ \rho(\mathbf{q}) \leq -1, \\ 1, & \text{for } \ \rho(\mathbf{q}) = 0, \\ \prod_{\rho \in \mathbf{R}} \prod_{p=0}^{\rho(\mathbf{q})-1} (-\rho(\mathbf{x}) - \lambda - p\hbar)^{-1}\,, & \text{for } \ \rho(\mathbf{q}) \geq 1, \end{cases} \tag{3.5}$$

Field	U(1)	twisted mass	$\mathrm{U}(1)_R$
Φ_i	$+1$	$-w_i$	0
P_a	$-d_a$	λ_a	2

Table 1. Matter content of the U(1) GLSM for the complete intersection variety X_1 in $\mathbb{P}^{n-1}$. Here $i = 0, \ldots, n-1$ and $a = 1, \ldots, r$.

where ρ denotes the weight of $\mathbf{R}$. Note that the products $\alpha(*)$ and $\rho(*)$ are defined by the canonical pairing.

Combining all the above building blocks (3.2), (3.3), (3.4), and (3.5), after taking a sum over the magnetic charges $\mathbf{q}$, one can construct the Givental I-function as (3.7), (3.19), and (4.3). As we will see next, the genus-0 Gromov-Witten invariants can be extracted from this function.

3.2 Examples

Here we will demonstrate how to compute genus-0 Gromov-Witten invariants via the I-functions for well-studied examples, and find out a useful formula for treating Grassmannian Calabi-Yau varieties.

3.2.1 Complete intersections in $\mathbb{P}^{n-1}$

Let us consider a complete intersection variety $X_1 = X_{d_1,\ldots,d_r} \subset \mathbb{P}^{n-1}$ defined by the zero locus of a holomorphic section of a vector bundle $\mathcal{E} = \oplus_{a=1}^r \mathcal{O}_V(d_a)$ on $V = \mathbb{P}^{n-1}$ satisfying Fano or Calabi-Yau condition $\sum_{a=1}^r d_a \leq n$. Note that $\operatorname{rank}\mathcal{E} = r$ and $\mathfrak{c}_1(\mathcal{E}) = \sum_{a=1}^r d_a$. This variety has a complex dimension

$$\dim X_1 = n - r - 1, \tag{3.6}$$

and is described by a U(1) GLSM whose matter content is shown in table 1. This model has a superpotential $W = \sum_{a=1}^r P_a G_a(\Phi)$ where $G_a(\Phi)$ are homogeneous degree d_a polynomials of the chiral matter multiplets Φ_i.

For each matter multiplet we assign twisted masses and U(1) R-charges as described in table 1. Combining the building blocks (3.2), (3.4) and (3.5) with the assignment in table 1, the I-function in the geometric large volume phase with FI parameter $\xi > 0$ is constructed as [21–23]

$$\begin{aligned} I_{X_1}^{\{w_i\},\{\lambda_a\}}(z;x;\hbar) &= \sum_{q=0}^{\infty} I_q^{\mathrm{c}}(z;x;\hbar) \left(\prod_{i=0}^{n-1} I_q^{\Phi_i}(x,w_i;\hbar) \right) \left(\prod_{a=1}^{r} I_q^{P_a}(x,\lambda_a;\hbar) \right) \\ &= z^{x/\hbar} \sum_{q=0}^{\infty} \frac{\prod_{a=1}^r \prod_{p=1}^{d_a q} (d_a x - \lambda_a + p\hbar)}{\prod_{i=0}^{n-1} \prod_{p=1}^{q} (x - w_i + p\hbar)} z^q. \end{aligned} \tag{3.7}$$

Geometrically $z = \mathrm{e}^{-2\pi\xi + \sqrt{-1}\theta}$ provides the Kähler moduli parameter of X_1, and x is identified with the equivariant second cohomology element of X_1 satisfying $\prod_{i=0}^{n-1}(x - w_i) =$

0, where the twisted masses w_i give the equivariant parameters acting on $\mathbb{P}^{n-1}$. The twisted masses λ_a correspond to the equivariant parameters acting on $\mathcal{E} = \oplus_{a=1}^r \mathcal{O}_V(d_a)$.

Then it can be shown that the I-function (3.7) obeys the ordinary differential equation

$$\left[\prod_{i=0}^{n-1}(\hbar\Theta_z - w_i) - z\prod_{a=1}^{r}\prod_{p=1}^{d_a}(\hbar d_a\Theta_z - \lambda_a + p\hbar)\right] I_{X_1}^{\{w_i\},\{\lambda_a\}}(z;x;\hbar) = 0, \quad \Theta_z := z\frac{d}{dz}. \tag{3.8}$$

In the Calabi-Yau case $\sum_{a=1}^r d_a = n$ with vanishing equivariant parameters $w_i = \lambda_a = 0$, the differential equation (3.8) yields the Picard-Fuchs equation for the periods of the holomorphic $(n-r-1)$-form on X_1 given by [28–31]

$$\Theta_z^r\left[\Theta_z^{n-r} - \left(\prod_{a=1}^r d_a\right) z \prod_{a=1}^{r}\prod_{p=1}^{d_a-1}(d_a\Theta_z + p)\right] I_{X_1}(z;x;\hbar) = 0, \tag{3.9}$$

where $I_{X_1}(z;x;\hbar) := I_{X_1}^{\{\mathbf{0}\},\{\mathbf{0}\}}(z;x;\hbar)$. If we expand the I-function around $\hbar = \infty$ as

$$I_{X_1}(z;x;\hbar) = \sum_{k=0}^{n-r-1} I_k(z)\left(\frac{x}{\hbar}\right)^k, \tag{3.10}$$

the coefficients $I_k(z)$ precisely give the solutions to the Picard-Fuchs equation. One can also obtain the flat coordinate q on the Kähler moduli space of X_1 through the relation

$$\log q = \frac{I_1(z)}{I_0(z)} = \log z + O(z), \tag{3.11}$$

called the mirror map. It has been shown in [32] that the genus-0 3-point A-model correlators $\langle \mathcal{O}_h \mathcal{O}_{h^k} \mathcal{O}_{h^{n-r-k-2}} \rangle_{\mathbb{P}^1}$, $k = 1, \ldots, n-r-1$, which enumerate the number of rational curves are given by

$$\begin{aligned}\langle \mathcal{O}_h \mathcal{O}_{h^k} \mathcal{O}_{h^{n-r-k-2}} \rangle_{\mathbb{P}^1} &= \kappa\,\frac{\widehat{I}_{k+1}(z(q))}{\widehat{I}_1(z(q))} \\ &= \kappa + \sum_{d=1}^{\infty} n_d(h^1, h^k, h^{n-r-k-2})\,\frac{q^d}{1-q^d},\end{aligned} \tag{3.12}$$

where $\widehat{I}_k(z)$ are inductively constructed from the I-functions as

$$\begin{aligned}&\widehat{I}_0(z) = I_0(z), \\ &\widehat{I}_k(z) = \Theta_z \frac{1}{\widehat{I}_{k-1}(z)} \Theta_z \frac{1}{\widehat{I}_{k-2}(z)} \cdots \Theta_z \frac{1}{\widehat{I}_1(z)} \Theta_z \frac{I_k(z)}{\widehat{I}_0(z)}, \qquad k = 1, \ldots, n-r-1.\end{aligned} \tag{3.13}$$

Here the observable $\mathcal{O}_{h^p}$ is associated with the hyperplane class $h \in H^{1,1}(X_1)$, and

$$\kappa = \int_{X_1} h^{n-r-1} = \left(\prod_{a=1}^r d_a\right)\int_{\mathbb{P}^{n-1}} h^{n-1} = \prod_{a=1}^r d_a \tag{3.14}$$

Field	U(k)	twisted mass	$U(1)_R$
Φ_I	$\mathbf{k}$	$-w_I$	0
P_a	$\det^{-d_a}$	λ_a	2

Table 2. Matter content of the U(k) GLSM for the complete intersection variety X_2 in $G(k,n)$. Here $I = 1,\ldots,n$ and $a = 1,\ldots,r$.

is the classical intersection number of X_1. $\widehat{I}_k(z)$ have relations

$$\widehat{I}_k(z) = \widehat{I}_{n-r-k}(z), \qquad k = 1,\ldots,n-r-1. \tag{3.15}$$

Note that there is a selection rule $\sum_{i=1}^{n} p_i = \dim X_1 + n - 3$ to realize non-trivial genus-0 n-point correlators $\langle \mathcal{O}_{h^{p_1}} \cdots \mathcal{O}_{h^{p_n}} \rangle_{\mathbb{P}^1}$ arising from the index theorem. The number $n_d(h^1, h^k, h^{n-r-k-2})$ in (3.12) is an integer and enumerates the number of degree d holomorphic maps intersecting with the cycles dual to h, h^k, and $h^{n-r-k-2}$ [30, 31, 33] (see also [34, 35]).

In a special case with $k = 1$, the relation $\Theta_z = \widehat{I}_1(z)\Theta_q$ and the so-called divisor equation $\langle \mathcal{O}_h \cdots \rangle_{\mathbb{P}^1} = \Theta_q \langle \cdots \rangle_{\mathbb{P}^1}$ imply that

$$\langle \mathcal{O}_{h^{n-r-3}} \rangle_{\mathbb{P}^1} = \kappa \frac{I_2(z(q))}{I_0(z(q))} = \frac{\kappa}{2} (\log q)^2 + \sum_{d=1}^{\infty} n_d(h^{n-r-3}) \,\mathrm{Li}_2(q^d), \tag{3.16}$$

where $\mathrm{Li}_p(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^p}$. Here $n_d(h^{n-r-3}) = n_d(h,h,h^{n-r-3})/d^2$ is an integer which enumerates the number of degree d holomorphic maps intersecting with the cycle dual to h^{n-r-3}. When $n - r = 4$ (i.e. $\dim X_1 = 3$), by the divisor equation, (3.16) yields [36, 37]

$$\langle * \rangle_{\mathbb{P}^1} = \kappa \int^q \frac{I_2(z(q'))}{I_0(z(q'))} \frac{dq'}{q'} = \frac{\kappa}{3!} (\log q)^3 + \sum_{d=1}^{\infty} n_d \,\mathrm{Li}_3(q^d), \tag{3.17}$$

where the number $n_d = n_d(h)/d$ is a genus-0 integer invariant.

3.2.2 Complete intersections in $G(k,n)$

Let us consider a complete intersection variety X_2 defined by the zero locus of a holomorphic section of $\mathcal{E} = \oplus_{a=1}^{r} \mathcal{O}_V(d_a)$ on Grassmannian $V = G(k,n)$ satisfying Fano or Calabi-Yau condition $\sum_{a=1}^{r} d_a \le n$. Note that $\mathrm{rank}\,\mathcal{E} = r$ and $\mathfrak{c}_1(\mathcal{E}) = \sum_{a=1}^{r} d_a$. The variety X_2 has a complex dimension

$$\dim X_2 = k(n-k) - r, \tag{3.18}$$

and can be described by a U(k) GLSM whose matter content is given in table 2. This model has a superpotential $W = \sum_{a=1}^{r} P_a G_a(B)$ where $G_a(B)$ are homogeneous degree d_a polynomials of the baryonic variables $B_{I_1\ldots I_k} = \epsilon_{i_1\ldots i_k} \Phi_{I_1}^{i_1} \cdots \Phi_{I_k}^{i_k}$ called the Plücker coordinates [38].

For each matter multiplet we assign twisted masses and U(1) R-charges as described in table 2. Combining the associated building blocks (3.2), (3.3), (3.4) and (3.5) for the

U(k) vector multiplet and the chiral multiples in table 2, we can construct the I-function for X_2 in the geometric phase with large FI parameter $\xi > 0$ as [39]

$$
\begin{aligned}
I_{X_2}^{\{w_I\},\{\lambda_a\}}(z;\mathbf{x};\hbar) &= \sum_{\mathbf{q}\in(\mathbb{Z}_{\geq 0})^k} I_{\mathbf{q}}^{\mathrm{c}}(z;\mathbf{x};\hbar)\, I_{\mathbf{q}}^{\mathrm{vec}}(\mathbf{x};\hbar) \left(\prod_{I=1}^{n} I_{\mathbf{q}}^{\Phi_I}(\mathbf{x},w_I;\hbar)\right)\left(\prod_{a=1}^{r} I_{\mathbf{q}}^{P_a}(\mathbf{x},\lambda_a;\hbar)\right) \\
&= z^{\sum_{i=1}^k x_i/\hbar} \sum_{\mathbf{q}\in(\mathbb{Z}_{\geq 0})^k} \left((-1)^{k-1} z\right)^{\sum_{i=1}^k q_i} \prod_{1\leq i<j\leq k} \frac{x_i - x_j + (q_i - q_j)\hbar}{x_i - x_j} \\
&\quad \times \frac{\prod_{a=1}^r \prod_{p=1}^{d_a \sum_{i=1}^k q_i} \left(d_a \sum_{i=1}^k x_i - \lambda_a + p\hbar\right)}{\prod_{I=1}^n \prod_{i=1}^k \prod_{p=1}^{q_i} (x_i - w_I + p\hbar)}.
\end{aligned}
\tag{3.19}
$$

Geometrically $z = \mathrm{e}^{-2\pi\xi + \sqrt{-1}\theta}$ provides the Kähler moduli parameter of X_2, and x_i are identified with the degree 2 elements in the equivariant cohomology of X_2. The twisted masses w_I and λ_a correspond to the equivariant parameters acting on $G(k,n)$ and $\mathcal{E} = \oplus_{a=1}^r \mathcal{O}_V(d_a)$, respectively.

Remark 3.1. For $w_I = 0$, the cohomology ring of $G(k,n)$ is given by [40] (see [41] for the equivariant quantum cohomology ring):

$$
H^*\big(G(k,n)\big) \cong \mathbb{C}[x_1,\dots,x_k]^{S_k} / \big(h_{n-k+1}(\mathbf{x}),\dots,h_n(\mathbf{x})\big), \tag{3.20}
$$

where S_k is the symmetric group on k elements and

$$
h_p(\mathbf{x}) = \sum_{1\leq i_1\leq i_2\leq\ldots\leq i_p\leq k} x_{i_1} x_{i_2}\cdots x_{i_p}
$$

are the complete symmetric polynomials.

For the Calabi-Yau case $\sum_{a=1}^r d_a = n$ with vanishing equivariant parameters $w_I = \lambda_a = 0$, the I-function $I_{X_2}(z;\mathbf{x};\hbar) = I_{X_2}^{\{\mathbf{0}\},\{\mathbf{0}\}}(z;\mathbf{x};\hbar)$ can be expanded around $\hbar = \infty$ as

$$
I_{X_2}(z;\mathbf{x};\hbar) = \sum_{|P|=0}^{\infty} I_P(z)\, \frac{s_P(\mathbf{x})}{\hbar^{|P|}}, \qquad |P| = \sum_{i=1}^k p_i, \tag{3.21}
$$

where $s_P(\mathbf{x}) = s_P(x_1,\dots,x_k)$ is the Schur polynomial for a partition $P = \{p_1,\dots,p_k\}$, and note that $s_{p,0,\dots,0}(\mathbf{x}) = h_p(\mathbf{x})$ and $s_{1,1,\dots,1}(\mathbf{x}) = e_p(\mathbf{x}) = \sum_{1\leq i_1<i_2<\ldots<i_p\leq k} x_{i_1}x_{i_2}\cdots x_{i_p}$. The flat coordinate which provides the mirror map is given by

$$
\log q = \frac{I_1(z)}{I_0(z)} = \log z + O(z). \tag{3.22}
$$

As a non-abelian generalization of the formula (3.16), here we conjecture that the genus-0 1-point A-model correlator

$$
\langle \mathcal{O}_H \rangle_{\mathbb{P}^1} = \frac{\kappa_H}{2}\,(\log q)^2 + \sum_{d=1}^{\infty} n_d(H)\, \mathrm{Li}_2(q^d), \tag{3.23}
$$

for the Grassmannian Calabi-Yau variety X_2 is given by

$$\langle \mathcal{O}_H \rangle_{\mathbb{P}^1} = \int_{X_2} H \left(\frac{I_2\left(z(q)\right)}{I_0\left(z(q)\right)} \sigma_2 + \frac{I_{1,1}\left(z(q)\right)}{I_0\left(z(q)\right)} \sigma_{1,1} \right). \tag{3.24}$$

Here

$$\kappa_H = \int_{X_2} H\ (\sigma_2 + \sigma_{1,1}) = \int_{X_2} H\, \sigma_1^2 \tag{3.25}$$

is the classical intersection number associated with the Poincaré dual H of a codimension $\dim X_2 - 2$ cycle in X_2, where σ_P denotes the Poincaré dual of a Schubert cycle of codimension $|P|$ in $G(k,n)$ [42]. The numbers $n_d(H)$ are integer invariants associated with H which are related to Gromov-Witten invariants of X_2 [11, 43–45].

The Giambelli's formula and the definition of Schur polynomials yield

$$\sigma_{1,1} = \sigma_1^2 - \sigma_2, \qquad s_{1,1}(\mathbf{x}) = \sum_{1\le i<j\le k} x_i x_j = s_1(\mathbf{x})^2 - s_2(\mathbf{x}).$$

Then one can reformulate the expression in (3.24) in terms of the classes σ_1 and σ_2 as

$$\begin{aligned} \frac{I_2(z)}{I_0(z)}\sigma_2 + \frac{I_{1,1}(z)}{I_0(z)}\sigma_{1,1} &= \frac{I_{1,1}(z)}{I_0(z)}\sigma_1^2 + \frac{I_2(z)-I_{1,1}(z)}{I_0(z)}\sigma_2 \\ &= \frac{-I_{X_2}[x_1^2] + I_{X_2}[x_1x_2]}{I_{X_2}[1]}\sigma_1^2 + \frac{2I_{X_2}[x_1^2] - I_{X_2}[x_1x_2]}{I_{X_2}[1]}\sigma_2, \end{aligned} \tag{3.26}$$

where $I_{X_2}[t]$ denotes the coefficient of t at $\hbar = 1$ in the expansion (3.21). Then, it is obvious that the first term in (3.23) is determined from the classical block. One can also see that for $k = 1$ the relations $I_{X_2}[x_1x_2] = 0$ and $\sigma_2 = \sigma_1^2$ imply that the formula (3.24) reduces to (3.16).

Now we claim that the conjectural formula (3.24) is also applicable not only for complete intersection Grassmannian Calabi-Yau varieties but also for the determinantal Calabi-Yau varieties, as we will see in the next section.

Remark 3.2. Instead of Pieri's formula for Schubert cycles, the intersection numbers of Grassmannian $G(k,n)$ can also be computed by Martin's formula [40]:

$$\int_{G(k,n)} \prod_{|R|=k(n-k)} \sigma_R = \frac{(-1)^{\frac{1}{2}k(k-1)}}{k!} \left(\prod_{i=1}^{k} \oint_{x_i=0} \frac{dx_i}{2\pi\sqrt{-1}} \right) \frac{\prod_{1\le i<j\le k}(x_i-x_j)^2}{\prod_{i=1}^k x_i^n} \prod_{|R|=k(n-k)} s_R(\mathbf{x}). \tag{3.27}$$

Utilizing this formula, one can then compute the intersection numbers of complete intersection Grassmannian Calabi-Yau varieties by considering the top Chern class of their normal bundles in $G(k,n)$. For example, (3.25) for X_2 is computed as

$$\kappa_H = \int_{X_2} H\, \sigma_1^2 = \int_{G(k,n)} H\, \sigma_1^2 \prod_{a=1}^{r} d_a \sigma_1.$$

Generic case can also be treated with a slight modification. Suppose that a Grassmannian Calabi-Yau variety X, defined by the zero locus of a holomorphic section of a vector bundle on $G(k,n)$, has a GLSM realization with a massless matter multiplet P in a representation $\overline{\mathbf{R}}$ of U(k) with R-charge 2. Then one obtains

$$\int_X \prod_{|R|=\dim X} \sigma_R = \frac{(-1)^{\frac{1}{2}k(k-1)}}{k!} \left(\prod_{i=1}^{k} \oint_{x_i=0} \frac{dx_i}{2\pi\sqrt{-1}} \right) \frac{\prod_{1\le i<j\le k}(x_i-x_j)^2}{\prod_{i=1}^{k} x_i^n} I^P(\mathbf{x}) \prod_{|R|=\dim X} s_R(\mathbf{x}), \tag{3.28}$$

where

$$I^P(\mathbf{x}) = \prod_{\rho\in\mathbf{R}} \rho(\mathbf{x}).$$

Let us consider the dual of the universal subbundle $\mathcal{E} = \mathcal{S}^*$ on $G(k,n)$. A Grassmannian Calabi-Yau variety defined by the zero locus of a holomorphic section of $\mathcal{E} = \mathcal{S}^*$ is described by a U(k) GLSM with a superpotential $W = P_i G(\Phi)^i$. Here $G(\Phi)$ is a homogeneous degree 1 polynomial of Φ_I $(I = 1,\ldots,n)$ in $\mathbf{k}$ of the U(k) gauge group with R-charge 0, and P in $\overline{\mathbf{k}}$ with R-charge 2. For the matter multiplet P with twisted mass λ, (3.5) becomes

$$I^P_{\mathbf{q}}(\mathbf{x},\lambda;\hbar) = \prod_{i=1}^{k}\prod_{p=1}^{q_i}(x_i-\lambda+p\hbar)\,, \quad \text{for } P \text{ in } \overline{\mathbf{k}} \text{ with } R\text{-charge 2}. \tag{3.29}$$

Similarly, for instance, for vector bundles $\mathcal{E} = \mathrm{Sym}^m\,\mathcal{S}^*(d)$ and $\mathcal{E} = \wedge^m \mathcal{S}^*(d)$ we get

$$I^P_{\mathbf{q}}(\mathbf{x},\lambda;\hbar) = \prod_{1\le i_1\le\cdots\le i_m\le k} \; \prod_{p=1}^{\sum_{j=1}^{m} q_{i_j} + d\sum_{i=1}^{k} q_i} \left(\sum_{j=1}^{m} x_{i_j} + d\sum_{i=1}^{k} x_i - \lambda + p\hbar \right), \tag{3.30}$$

with P in $\mathrm{Sym}^m\,\overline{\mathbf{k}} \otimes \det^{-d}$ and

$$I^P_{\mathbf{q}}(\mathbf{x},\lambda;\hbar) = \prod_{1\le i_1<\cdots<i_m\le k} \; \prod_{p=1}^{\sum_{j=1}^{m} q_{i_j} + d\sum_{i=1}^{k} q_i} \left(\sum_{j=1}^{m} x_{i_j} + d\sum_{i=1}^{k} x_i - \lambda + p\hbar \right), \tag{3.31}$$

with P in $\wedge^m\,\overline{\mathbf{k}} \otimes \det^{-d}$, respectively. Using these building blocks with the help of our formula (3.24), one can obtain the genus-0 Gromov-Witten invariants of Grassmannian Calabi-Yau varieties computed e.g. in [14].

To consider a Grassmannian Calabi-Yau variety associated with the universal quotient bundle $\mathcal{E} = \mathcal{Q}$ on $G(k,n)$, a little ingenuity is needed. By tensoring $\mathcal{O}_{G(k,n)}(d)$ with the short exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}_{G(k,n)}^{\oplus n} \longrightarrow \mathcal{Q} \longrightarrow 0, \tag{3.32}$$

as

$$0 \longrightarrow \mathcal{S}(d) \xrightarrow{f} \mathcal{O}_{G(k,n)}(d)^{\oplus n} \xrightarrow{g} \mathcal{Q}(d) \longrightarrow 0, \tag{3.33}$$

Field	U(k)	U($\ell_p^\vee$)	U(1)$_R$
Φ_a	$\mathbf{k}$	$\mathbf{1}$	0
X_i	$\mathbf{1}$	$\overline{\boldsymbol{\ell}}_p^\vee$	0
P	$\overline{\mathbf{R}}_p$	$\boldsymbol{\ell}_p^\vee$	2

$\overset{\text{dual}}{\longleftrightarrow}$

Field	U(k)	U(ℓ)	U(1)$_R$
Φ_a	$\mathbf{k}$	$\mathbf{1}$	0
$\widetilde{X}_i$	$\mathbf{1}$	$\boldsymbol{\ell}$	0
$\widetilde{Y}$	$\mathbf{R}_p$	$\overline{\boldsymbol{\ell}}$	0
$\widetilde{P}_i$	$\overline{\mathbf{R}}_p$	$\mathbf{1}$	2

Table 3. The left table describes matter content of the PAX model for the desingularized determinantal Calabi-Yau variety in $G(k,n)$, where $a = 1,\dots,n$, $i = 1,\dots,p$, and $\mathbf{R}_p$ is a representation in the gauge group U(k) which describes the vector bundle $\mathcal{F}_p$. U(1)$_R$ denotes an R-charge which is assigned to the matter content. The right table describes matter content of the PAXY model which is a dual GLSM of the PAX model.

one can realize a corresponding GLSM for the vector bundle $\mathcal{E} = \mathcal{Q}(d)$ from the viewpoint of a quotient $\mathcal{O}_{G(k,n)}(d)^{\oplus n}/\mathrm{im}(f)$. The resulting model consists of n matter multiplets P_i in $\det^{-d}$, $i = 1,\dots,n$ of U(k) with R-charge 2, and a single matter multiplet Y in $\overline{\mathbf{k}} \otimes \det^d$ of U(k) with R-charge 0 [46]. The associated building block of the I-function without twisted mass is then given by

$$I_{\mathbf{q}}^{\{P_i\}/Y}(\mathbf{x};\hbar) = \frac{\prod_{p=1}^{d\sum_{i=1}^k q_i}\left(d\sum_{i=1}^k x_i + p\hbar\right)^n}{\prod_{i=1}^k \prod_{p=1}^{-q_i+d\sum_{j=1}^k q_j}\left(-x_i + d\sum_{j=1}^k x_j + p\hbar\right)}. \tag{3.34}$$

4 I-functions of determinantal Calabi-Yau varieties

In this section, we describe how to utilize our formula (3.24) to compute genus-0 Gromov-Witten invariants of the determinantal Calabi-Yau varieties. Here we focus on the desingularized determinantal Calabi-Yau variety X_A in (2.6) which can be described by a U(k) $\times$ U($\ell_p^\vee$) GLSM with matter content in the left of table 3. This GLSM is called a PAX model and has a superpotential [18]

$$W_{PAX} = \sum_{i,j=1}^{p}\sum_{\alpha=1}^{\ell_p^\vee} P_{\alpha i} A(\Phi)_{ij} X_{j\alpha}. \tag{4.1}$$

The PAX model has several distinct phases. Let ξ_1 and ξ_2 be the FI parameters associated with the central U(1) factors of U(k) and U($\ell_p^\vee$), respectively. For example, a geometric phase called a "X_A phase" with $\xi_1 > 0$ and $\xi_2 < 0$ of the PAX model in the IR describes the variety X_A in (2.6), and another geometric phase "X_{A^T} phase" with $k\xi_1 + \ell_p^\vee \xi_2 > 0$ and $\xi_2 > 0$ corresponds to an incidence correspondence constructed from the transposed matrix $A(\phi)^T$.

Remark 4.1. Alternatively, one can consider Seiberg-like dual [38] of the PAX model as follows. The chiral matter multiplet P in the fundamental representation $\boldsymbol{\ell}_p^\vee$ under the U($\ell_p^\vee$) factor corresponds to the vector bundle $\mathcal{S}^*$ on $G(\ell_p^\vee, p)$. By taking the Seiberg-like duality with respect to the gauge group U($\ell_p^\vee$), $\mathcal{S}^*$ is mapped to $\mathcal{Q}$ on $G(\ell,p)$ and as indicated by the short exact sequence (3.32), the chiral matter multiplet P is mapped to the dual chiral matter multiplets $\widetilde{Y}$ and $\widetilde{P}_i$ in the right of table 3. This dualized GLSM is called a PAXY model with gauge group $\mathrm{U}(k)\times\mathrm{U}(\ell)$ and has a superpotential given by [18]

$$W_{PAXY}=\sum_{i,j=1}^{p}\widetilde{P}_{ij}\Big(A(\Phi)_{ij}-\sum_{\beta=1}^{\ell}\widetilde{Y}_{i\beta}\widetilde{X}_{\beta j}\Big). \tag{4.2}$$

4.1 I-functions and A-model correlators

Let us consider the PAX model with massless matter multiplets shown in table 3. From the building blocks (3.2), (3.3), (3.4) and (3.5), the I-function of this model in the X_A phase with FI parameters $\xi_1>0$ and $\xi_2<0$ is given by

$$\begin{aligned}I_{X_A}(z,w;\mathbf{x},\mathbf{y};\hbar)=&\sum_{(\mathbf{q},\mathbf{r})\in(\mathbb{Z}_{\geq 0})^k\times(\mathbb{Z}_{\geq 0})^{\ell_p^\vee}} I^{\mathrm{c}}_{\mathbf{q}}(z,w;\mathbf{x},\mathbf{y};\hbar)\, I^{\mathrm{vec}}_{\mathbf{q},\mathbf{r}}(\mathbf{x},\mathbf{y};\hbar)\\&\times\big(I^{\Phi}_{\mathbf{q}}(\mathbf{x};\hbar)\big)^n\big(I^{X}_{\mathbf{r}}(\mathbf{y};\hbar)\big)^p\, I^{P}_{\mathbf{q},\mathbf{r}}(\mathbf{x},\mathbf{y};\hbar),\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}I^{\mathrm{c}}_{\mathbf{q}}(z,w;\mathbf{x},\mathbf{y};\hbar)&=z^{\sum_{i=1}^k x_i/\hbar}\Big((-1)^{k-1}z\Big)^{\sum_{i=1}^k q_i}\,w^{\sum_{i=1}^{\ell_p^\vee} y_i/\hbar}\Big((-1)^{\ell_p^\vee-1}w\Big)^{\sum_{i=1}^{\ell_p^\vee} r_i},\\ I^{\mathrm{vec}}_{\mathbf{q},\mathbf{r}}(\mathbf{x},\mathbf{y};\hbar)&=\Bigg(\prod_{1\leq i<j\leq k}\frac{x_i-x_j+(q_i-q_j)\hbar}{x_i-x_j}\Bigg)\Bigg(\prod_{1\leq i<j\leq \ell_p^\vee}\frac{y_i-y_j+(r_i-r_j)\hbar}{y_i-y_j}\Bigg),\\ I^{\Phi}_{\mathbf{q}}(\mathbf{x};\hbar)&=\frac{1}{\prod_{i=1}^k\prod_{d=1}^{q_i}(x_i+d\hbar)},\qquad I^{X}_{\mathbf{r}}(\mathbf{y};\hbar)=\frac{1}{\prod_{i=1}^{\ell_p^\vee}\prod_{d=1}^{r_i}(y_i+d\hbar)},\\ I^{P}_{\mathbf{q},\mathbf{r}}(\mathbf{x},\mathbf{y};\hbar)&=\prod_{\rho\in\mathbf{R}_p}\prod_{i=1}^{\ell_p^\vee}\prod_{d=1}^{\rho(\mathbf{q})+r_i}(\rho(\mathbf{x})+y_i+d\hbar)\,.\end{aligned} \tag{4.4}$$

Here $z=\mathrm{e}^{-2\pi\xi_1+\sqrt{-1}\theta_1}$ and $w=\mathrm{e}^{2\pi\xi_2-\sqrt{-1}\theta_2}$ are moduli parameters associated with the central $\mathrm{U}(1)^2\subset\mathrm{U}(k)\times\mathrm{U}(\ell_p^\vee)$, and in particular w parametrizes the blowing up in (2.3). $\mathbf{x}$ (*resp.* $\mathbf{y}$) are identified with the degree 2 elements in the cohomology of $G(k,n)$ (*resp.* $G(\ell_p^\vee,p)$).

As performed in (3.21), the I-function (4.3) can be expanded around $\hbar=\infty$ in terms of Schur polynomials as

$$I_{X_A}(z,w;\mathbf{x},\mathbf{y};\hbar)=\sum_{|Q|,|R|=0}^{\infty}I_{Q;R}(z,w)\,\frac{s_Q(\mathbf{x})\,s_R(\mathbf{y})}{\hbar^{|Q|+|R|}}. \tag{4.5}$$

The flat coordinates q_z and q_w, which provide the exponentiated Kähler moduli parameters of X_A, are given by

$$\log q_z=\frac{I_{1;0}(z,w)}{I_{0;0}(z,w)}=\log z+O(z,w),\qquad \log q_w=\frac{I_{0;1}(z,w)}{I_{0;0}(z,w)}=\log w+O(z,w). \tag{4.6}$$

From our conjectural formula (3.24), one can deduce a formula for the genus-0 1-point A-model correlator $\langle \mathcal{O}_H \rangle_{\mathbb{P}^1}$ in the X_A phase as

$$\begin{aligned}\langle \mathcal{O}_H \rangle_{\mathbb{P}^1} &= \int_{X_A} H \left(\frac{I_{2;0}(z,w)}{I_{0;0}(z,w)}\sigma_2 + \frac{I_{1,1;0}(z,w)}{I_{0;0}(z,w)}\sigma_{1,1} \right. \\ &\quad \left. + \frac{I_{1;1}(z,w)}{I_{0;0}(z,w)}\sigma_1\tau_1 + \frac{I_{0;2}(z,w)}{I_{0;0}(z,w)}\tau_2 + \frac{I_{0;1,1}(z,w)}{I_{0;0}(z,w)}\tau_{1,1}\right) \\ &= \frac{\kappa_H^\sigma}{2}(\log q_z)^2 + \kappa_H^{\sigma\tau}\log q_z \log q_w + \frac{\kappa_H^\tau}{2}(\log q_w)^2 + \sum_{d_1,d_2=1}^{\infty} n_{d_1,d_2}(H)\,\mathrm{Li}_2(q_z^{d_1}q_w^{d_2}).\end{aligned} \tag{4.7}$$

Here

$$\kappa_H^\sigma = \int_{X_A} H\,\sigma_1^2, \qquad \kappa_H^{\sigma\tau} = \int_{X_A} H\,\sigma_1\tau_1, \qquad \kappa_H^\tau = \int_{X_A} H\,\tau_1^2, \tag{4.8}$$

are the classical intersection numbers associated with the Poincaré dual H of a codimension $\dim X_A - 2$ cycle in X_A, where σ_P (*resp.* τ_P) is the Poincaré dual of a Schubert cycle of codimension $|P|$ in $G(k,n)$ (*resp.* $G(\ell_p^\vee, p)$). The genus-0 invariants $n_{d_1,d_2}(H)$ associated with H, which are related to Gromov-Witten invariants, are conjecturally integers.

Remark 4.2. When $\dim X_A = 3$, analogous to the formula (3.17), the above result (4.7) yields

$$\begin{aligned}\langle * \rangle_{\mathbb{P}^1} &= \frac{\kappa_{\sigma_1}^\sigma}{3!}(\log q_z)^3 + \frac{\kappa_{\sigma_1}^{\sigma\tau}}{2}(\log q_z)^2\log q_w + \frac{\kappa_{\tau_1}^{\sigma\tau}}{2}\log q_z(\log q_w)^2 + \frac{\kappa_{\tau_1}^\tau}{3!}(\log q_w)^3 \\ &\quad + \sum_{d_1,d_2=1}^{\infty} n_{d_1,d_2}\,\mathrm{Li}_3(q_z^{d_1}q_w^{d_2}),\end{aligned} \tag{4.9}$$

where $n_{d_1,d_2} = n_{d_1,d_2}(\sigma_1)/d_1$ (for $d_1 \neq 0$) and $n_{d_1,d_2} = n_{d_1,d_2}(\tau_1)/d_2$ (for $d_2 \neq 0$) each provide genus-0 integer invariants.

4.2 An algorithm to compute genus-0 invariants

In a similar fashion to the computation (3.26), by taking the classes σ_1, σ_2, τ_1 and τ_2 for the special Schubert cycles, the 1-point correlator (4.7) can be evaluated with

$$\begin{aligned}&\frac{I_{2;0}(z,w)}{I_{0;0}(z,w)}\sigma_2 + \frac{I_{1,1;0}(z,w)}{I_{0;0}(z,w)}\sigma_{1,1} + \frac{I_{1;1}(z,w)}{I_{0;0}(z,w)}\sigma_1\tau_1 + \frac{I_{0;2}(z,w)}{I_{0;0}(z,w)}\tau_2 + \frac{I_{0;1,1}(z,w)}{I_{0;0}(z,w)}\tau_{1,1} \\ &= \frac{-I_{X_A}[x_1^2] + I_{X_A}[x_1x_2]}{I_{X_A}[1]}\sigma_1^2 + \frac{2I_{X_A}[x_1^2] - I_{X_A}[x_1x_2]}{I_{X_2}[1]}\sigma_2 + \frac{I_{X_A}[x_1y_1]}{I_{X_A}[1]}\sigma_1\tau_1 \\ &\quad + \frac{-I_{X_A}[y_1^2] + I_{X_A}[y_1y_2]}{I_{X_A}[1]}\tau_1^2 + \frac{2I_{X_A}[y_1^2] - I_{X_A}[y_1y_2]}{I_{X_2}[1]}\tau_2,\end{aligned} \tag{4.10}$$

where $I_{X_A}[t]$ denotes the coefficient of t at $\hbar = 1$ in the expansion (4.5). From the coefficients $I_{X_A}[t]$ and the classical intersection numbers

$$\int_{X_A} H\,\sigma_1^2, \quad \int_{X_A} H\,\sigma_2, \quad \int_{X_A} H\,\sigma_1\tau_1, \quad \int_{X_A} H\,\tau_1^2, \quad \int_{X_A} H\,\tau_2, \tag{4.11}$$

one can compute the 1-point A-model correlator (4.7) and obtain the integer invariants. The classical intersection numbers can be computed by Martin's formula (3.28) as

$$\begin{aligned}\int_{X_A}\prod_{|Q|+|R|=\dim X_A}\sigma_Q\,\tau_R=&\frac{(-1)^{\frac{1}{2}k(k-1)+\frac{1}{2}\ell_p^\vee(\ell_p^\vee-1)}}{k!\,\ell_p^\vee!}\left(\prod_{i=1}^{k}\oint_{x_i=0}\frac{dx_i}{2\pi\sqrt{-1}}\right)\left(\prod_{i=1}^{\ell_p^\vee}\oint_{y_i=0}\frac{dy_i}{2\pi\sqrt{-1}}\right)\\&\times\frac{\prod_{1\le i<j\le k}(x_i-x_j)^2}{\prod_{i=1}^{k}x_i^n}\frac{\prod_{1\le i<j\le\ell_p^\vee}(y_i-y_j)^2}{\prod_{i=1}^{\ell_p^\vee}y_i^p}I^P(\mathbf{x},\mathbf{y})\\&\times\prod_{|Q|+|R|=\dim X_A}s_Q(\mathbf{x})\,s_R(\mathbf{y}),\end{aligned}\tag{4.12}$$

where

$$I^P(\mathbf{x},\mathbf{y})=\prod_{\rho\in\mathbf{R}_p}\prod_{i=1}^{\ell_p^\vee}(\rho(\mathbf{x})+y_i)\,.$$

4.3 Illustrative examples of the computations

Here we will consider several examples of the desingularized determinantal Calabi-Yau 3-folds investigated in section 2.3 and compute their genus-0 invariants n_{d_1,d_2} defined in (4.9).[6]

4.3.1 Quintic family

The determinantal Calabi-Yau 3-folds in (2.22) are connected with the famous quintic Calabi-Yau 3-fold with $(h^{1,1},h^{2,1})=(1,101)$ which can be described as a "trivial" determinantal 3-fold with $\mathcal{F}_p=\mathcal{O}_V(5)$ on $V=\mathbb{P}^4$. In terms of the parameters in section 3.2.1, the quintic 3-fold is characterized as X_1 with $n=5$, $r=1$ and $d_1=5$. The classical intersection number (3.14) of X_1 is given by $\kappa=5$ and the genus-0 invariants n_d in (3.17) are well-known to be [36]

$$\begin{aligned}&n_1=2875,\ n_2=609250,\ n_3=317206375,\ n_4=242467530000,\\&n_5=229305888887625,\ldots.\end{aligned}\tag{4.13}$$

The quintic family can be described by GLSMs with $\mathrm{U}(1)\times\mathrm{U}(1)$ gauge group. Following section 4.2 and appendix B, one can compute topological invariants of the quintic family as summarized in table 4, which is consistent with the previous works. Here one can also check that $h^{1,0}=0$. By comparing (4.13) with the entries n_{d_1,d_2} in table 4 of each determinantal 3-fold, we see that they exhibit a behavior of the extremal transition [47] (see also [10]):

$$n_d=\sum_{d_2=0}^{N}n_{d,d_2},\tag{4.14}$$

where N is a certain finite positive integer.

[6] We only focus on the determinantal varieties described by $\mathrm{U}(k)\times\mathrm{U}(\ell_p^\vee)$ PAX models with $k\le 2$, $\ell_p^\vee\le 2$. In appendix C we summarize our computational results for several determinantal Calabi-Yau 4-folds.

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$\mathcal{F}_p = \mathcal{O}_V(1) \oplus \mathcal{O}_V(4)$: $(h^{1,1}, h^{2,1}) = (2, 86)$							
Intersection numbers		$\sigma_1^3 = 5$, $\sigma_1^2\tau_1 = 4$, $\sigma_1\tau_1^2 = 0$, $\tau_1^3 = 0$					
n_{d_1,d_2}	$d_1 = 0$	1	2	3	4	5	6
$d_2 = 0$		640	10032	288384	10979984	495269504	24945542832
1	16	2144	231888	23953120	2388434784	232460466048	22229609118768
2	0	120	356368	144785584	36512550816	7251261673320	1242876017216016
3	0	-32	14608	144051072	115675981232	50833652046112	16156774167471792
4	0	3	-4920	5273880	85456640608	106397389165188	69178537204963920
5	0	0	1680	-1505472	3018009984	62800738246496	107220234702633360
6	0	0	-480	512136	-748922304	2196615443648	52910679981204144

$\mathcal{F}_p = \mathcal{O}_V(2) \oplus \mathcal{O}_V(3)$: $(h^{1,1}, h^{2,1}) = (2, 66)$							
Intersection numbers		$\sigma_1^3 = 5$, $\sigma_1^2\tau_1 = 6$, $\sigma_1\tau_1^2 = 0$, $\tau_1^3 = 0$					
n_{d_1,d_2}	$d_1 = 0$	1	2	3	4	5	6
$d_2 = 0$		366	2670	35500	606264	12210702	273649804
1	36	1584	73728	3286224	142523712	6060689280	253954899504
2	0	909	255960	34736049	3387935304	273906849222	19594379113848
3	0	16	231336	106245024	23702767680	3623779411776	436922554063224
4	0	0	45216	119474748	66922830504	19938817169442	4093759996324344
5	0	0	360	48046176	85607985132	53346064121712	19206910967576760
6	0	0	-20	5357838	49765200024	74247746393898	49456242071288532

$\mathcal{F}_p = \mathcal{O}_V(1) \oplus \mathcal{O}_V(2)^{\oplus 2}$: $(h^{1,1}, h^{2,1}) = (2, 58)$							
Intersection numbers		$\sigma_1^3 = 5$, $\sigma_1^2\tau_1 = 8$, $\sigma_1\tau_1^2 = 4$, $\tau_1^3 = 0$					
n_{d_1,d_2}	$d_1 = 0$	1	2	3	4	5	6
$d_2 = 0$		144	140	144	112	144	140
1	44	1120	13520	107264	645048	3190528	13669600
2	0	1354	113916	3627224	68006448	901242596	9287483360
3	0	256	258840	29390080	1463601384	44141205824	937689927488
4	0	1	183690	89360780	11490671144	741564140238	30303625673624
5	0	0	37896	115185728	41359928372	5682155162688	434288936956304
6	0	0	1248	64102328	74832601592	22827028536708	3267218929443668

$\mathcal{F}_p = \mathcal{O}_V(1)^{\oplus 2} \oplus \mathcal{O}_V(3)$: $(h^{1,1}, h^{2,1}) = (2, 68)$							
Intersection numbers		$\sigma_1^3 = 5$, $\sigma_1^2\tau_1 = 7$, $\sigma_1\tau_1^2 = 3$, $\tau_1^3 = 0$					
n_{d_1,d_2}	$d_1 = 0$	1	2	3	4	5	6
$d_2 = 0$		204	204	132	204	204	132
1	34	1348	26843	338016	3050972	21359344	123786248
2	0	1290	179490	9621696	299056816	6401442680	103385827082
3	0	35	292557	59496360	5101530190	260050051116	9166825459347
4	0	-2	108312	127400436	29874798664	3367972159714	235659178171360
5	0	0	1909	97863426	75032773743	18958650256980	2557795024380895
6	0	0	-68	22115268	84738954674	52879556793440	13984934136290076

$\mathcal{F}_p = \mathcal{O}_V(1)^{\oplus 3} \oplus \mathcal{O}_V(2)$: $(h^{1,1}, h^{2,1}) = (2, 56)$							
Intersection numbers		$\sigma_1^3 = 5$, $\sigma_1^2\tau_1 = 9$, $\sigma_1\tau_1^2 = 7$, $\tau_1^3 = 2$					
n_{d_1,d_2}	$d_1 = 0$	1	2	3	4	5	6
$d_2 = 0$		84	10	0	0	0	0
1	46	865	4461	9380	9380	4461	865
2	0	1478	60360	760580	4423324	14207450	27724124
3	0	438	211547	10517154	200833886	1987023580	11758507011
4	0	10	238798	51571964	2762153102	67275586298	926085646998
5	0	0	86203	107216585	16493768487	916157767777	26171128616181
6	0	0	7826	99623760	48905658096	6224190580040	353098716104028

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$\mathcal{F}_p = \mathcal{O}_V(1)^{\oplus 5}$: $(h^{1,1}, h^{2,1}) = (2, 52)$ [24, 48, 49]							
Intersection numbers		$\sigma_1^3 = 5$, $\sigma_1^2\tau_1 = 10$, $\sigma_1\tau_1^2 = 10$, $\tau_1^3 = 5$					
n_{d_1,d_2}	$d_1 = 0$	1	2	3	4	5	6
$d_2 = 0$		50	0	0	0	0	0
1	50	650	1475	650	50	0	0
2	0	1475	29350	148525	250550	148525	29350
3	0	650	148525	3270050	24162125	75885200	110273275
4	0	50	250550	24162125	545403950	5048036025	22945154050
5	0	0	148525	75885200	5048036025	114678709000	1231494256550
6	0	0	29350	110273275	22945154050	1231494256550	27995704239850

Table 4. Genus-0 invariants of determinantal 3-folds in (2.22) with $V = \mathbb{P}^4$.

4.3.2 Determinantal Calabi-Yau 3-folds in (2.23)

Next, let us consider the determinantal Calabi-Yau 3-folds described in (2.23) with $p \neq 2$. These examples can be described by GLSMs with $U(1) \times U(2)$ gauge group. Using the methodology we established in section 4.2, one can obtain the genus-0 invariants summarized in table 5. Here one can also check that $h^{1,0} = 0$.

4.3.3 Determinantal Calabi-Yau 3-folds in (2.25)

In a similar spirit to the quintic family discussed above, the determinantal Calabi-Yau 3-folds in (2.25) are connected with the complete intersection Calabi-Yau 3-fold with $(h^{1,1}, h^{2,1}) = (1, 89)$ corresponding to the "trivial" determinantal Calabi-Yau 3-fold with $\mathcal{F}_p = \mathcal{O}_V(4)$ on $V = G(2,4)$, namely X_2 with $k = 2$, $n = 4$, $r = 1$ and $d_1 = 4$ in the language of section 3.2.2. This family is described by GLSMs with $U(2) \times U(1)$ gauge group.

The classical intersection numbers (3.25) of X_2 are given by $\sigma_1^3 = 8$, $\sigma_1\sigma_2 = 4$, and the genus-0 invariants $n_d = n_d(\sigma_1)/d$ in (3.24) are

$$n_1 = 1280, \ n_2 = 92288, \ n_3 = 15655168, \ n_4 = 3883902528, \ n_5 = 1190923282176, \ldots. \tag{4.15}$$

The genus-0 invariants for other determinantal Calabi-Yau 3-folds in (2.25) are summarized in table 6, where one can check that they exhibit the behavior (4.14) of the extremal transition and $h^{1,0} = 0$.

Note that, via the incidence correspondence (2.6), a geometric phase of the determinantal Calabi-Yau variety with $\mathcal{F}_p = \mathcal{O}_V(1)^{\oplus 4}$ on $V = \mathbb{P}^7$ in (2.23) can be identified with a geometric phase of the variety with $\mathcal{F}_p = (\mathcal{S}^*)^{\oplus 4}$ on $V = G(2,4)$ in (2.25) [18]. Indeed, by taking $d_1 \leftrightarrow d_2$, the genus-0 invariants n_{d_1,d_2} of the former coincide with the genus-0 invariants of the latter [10].

$\mathcal{F}_p = \mathcal{O}_V(1)^{\oplus 2} \oplus \mathcal{O}_V(2)$: $(h^{1,1}, h^{2,1}) = (2, 58)$ (Gulliksen-Negård Calabi-Yau 3-fold [18, 25])								
Intersection numbers	$\sigma_1^3 = 17$, $\sigma_1^2\tau_1 = 10$, $\sigma_1\tau_1^2 = 4$, $\sigma_1\tau_2 = 0$, $\tau_1^3 = 0$, $\tau_1\tau_2 = 0$							
n_{d_1,d_2}	$d_1 = 0$	1	2	3	4	5	6	7
$d_2 = 0$		156	116	156	112	156	116	156
1	0	256	6656	63232	415232	2159360	9583104	37772288
2	0	1	1248	193678	8278144	172114785	2326878112	23641531470
3	0	0	0	10496	5211136	592671744	28906081792	822717728768
4	0	0	0	0	111712	136564760	31768995672	2999009092032
5	0	0	0	0	0	1394944	3522539520	1444421355520
6	0	0	0	0	0	0	19318752	89779792749
$\mathcal{F}_p = \mathcal{O}_V(1)^{\oplus 4}$: $(h^{1,1}, h^{2,1}) = (2, 34)$ (Gulliksen-Negård Calabi-Yau 3-fold [10, 25, 50, 51])								
Intersection numbers	$\sigma_1^3 = 20$, $\sigma_1^2\tau_1 = 20$, $\sigma_1\tau_1^2 = 16$, $\sigma_1\tau_2 = 6$, $\tau_1^3 = 8$, $\tau_1\tau_2 = 4$							
n_{d_1,d_2}	$d_1 = 0$	1	2	3	4	5	6	7
$d_2 = 0$		56	0	0	0	0	0	0
1	0	192	896	192	0	0	0	0
2	0	56	2544	23016	41056	23016	2544	56
3	0	0	896	52928	813568	3814144	6292096	3814144
4	0	0	0	23016	1680576	35857016	284749056	933789504
5	0	0	0	192	813568	66781440	1784024064	20090433088
6	0	0	0	0	41056	35857016	3074369392	96591652016

Table 5. Genus-0 invariants of determinantal 3-folds in (2.23) with $V = \mathbb{P}^7$.

$\mathcal{F}_p = \mathcal{O}_V(1) \oplus \mathcal{O}_V(3)$: $(h^{1,1}, h^{2,1}) = (2, 72)$								
Intersection numbers	$\sigma_1^3 = 8$, $\sigma_1\sigma_2 = 4$, $\sigma_1^2\tau_1 = 6$, $\sigma_1\tau_1^2 = 0$, $\sigma_2\tau_1 = 3$, $\tau_1^3 = 0$							
n_{d_1,d_2}	$d_1 = 0$	1	2	3	4	5	6	7
$d_2 = 0$		348	2706	35416	606516	12209820	273653140	6617946300
1	18	900	41778	1871784	81468792	3473471196	145835134092	6050552127264
2	0	36	46548	8009712	864795636	74041264872	5497197606864	370175324505012
3	0	-4	1512	5604204	1928672640	363480492960	49681240379520	5528217639011448
4	0	0	-306	153936	985016556	530436671676	148552854522624	28868137556536800
5	0	0	54	-24768	25990110	214272257040	159209292083400	60303976799146560
6	0	0	-4	5940	-3264792	5674351788	53439787982532	50841527973755064
$\mathcal{F}_p = \mathcal{O}_V(2)^{\oplus 2}$: $(h^{1,1}, h^{2,1}) = (2, 58)$								
Intersection numbers	$\sigma_1^3 = 8$, $\sigma_1\sigma_2 = 4$, $\sigma_1^2\tau_1 = 8$, $\sigma_1\tau_1^2 = 0$, $\sigma_2\tau_1 = 4$, $\tau_1^3 = 0$							
n_{d_1,d_2}	$d_1 = 0$	1	2	3	4	5	6	7
$d_2 = 0$		256	1248	10496	111712	1394944	19318752	288338176
1	32	768	21888	591872	15653568	406723584	10427720448	264554741760
2	0	256	46016	3851264	229545472	11320801792	494003913216	19776092919808
3	0	0	21888	6747904	952111808	90236788736	6690341483648	419279237824512
4	0	0	1248	3851264	1489057408	286163875840	36930089276288	3663867073538048
5	0	0	0	591872	952111808	414664112384	97746565623808	15741994226581504
6	0	0	0	10496	229545472	286163875840	134131710670016	36555466071304192
$\mathcal{F}_p = \mathcal{O}_V(1)^{\oplus 2} \oplus \mathcal{O}_V(2)$: $(h^{1,1}, h^{2,1}) = (2, 56)$								

Intersection numbers	$\sigma_1^3 = 8, \quad \sigma_1\sigma_2 = 4, \quad \sigma_1^2\tau_1 = 10, \quad \sigma_1\tau_1^2 = 4, \quad \sigma_2\tau_1 = 5, \quad \tau_1^3 = 0$							
n_{d_1,d_2}	$d_1 = 0$	1	2	3	4	5	6	7
$d_2 = 0$		140	152	140	108	140	152	140
1	34	692	8310	67644	424226	2179788	9628540	37862432
2	0	436	37266	1201096	23129444	318263924	3423444286	30397041864
3	0	12	38424	4809332	251071058	7882006668	174584679336	2978341361748
4	0	0	8072	6408160	936362724	64838871368	2796104549608	85836804179264
5	0	0	66	2838032	1449869614	230006825996	19186188980224	1035385789366608
6	0	0	-2	329036	956057192	393389988300	65626229819274	6246121752675024

$\mathcal{F}_p = \mathcal{S}^* \oplus \mathcal{O}_V(3)$: $(h^{1,1}, h^{2,1}) = (2, 77)$								
Intersection numbers	$\sigma_1^3 = 8, \quad \sigma_1\sigma_2 = 4, \quad \sigma_1^2\tau_1 = 7, \quad \sigma_1\tau_1^2 = 3, \quad \sigma_2\tau_1 = 3, \quad \tau_1^3 = 0$							
n_{d_1,d_2}	$d_1 = 0$	1	2	3	4	5	6	7
$d_2 = 0$		195	195	150	195	195	150	195
1	13	1030	24479	330960	3035018	21301930	123660710	622928364
2	0	78	65007	5464206	213740347	5220791429	91165319219	1233231670475
3	0	-26	3822	9502026	1561721228	114639515100	5115340545693	159519319143362
4	0	3	-1820	503243	2028893885	514721709028	58258127191937	3983242948904679
5	0	0	858	-215410	103906805	535733030960	185997625577552	29104035511228470
6	0	0	-312	111267	-38991863	27312140744	162043340071962	71687188824610803

$\mathcal{F}_p = \mathcal{S}^*(1) \oplus \mathcal{O}_V(1)$: $(h^{1,1}, h^{2,1}) = (2, 49)$								
Intersection numbers	$\sigma_1^3 = 8, \quad \sigma_1\sigma_2 = 4, \quad \sigma_1^2\tau_1 = 11, \quad \sigma_1\tau_1^2 = 5, \quad \sigma_2\tau_1 = 5, \quad \tau_1^3 = 0$							
n_{d_1,d_2}	$d_1 = 0$	1	2	3	4	5	6	7
$d_2 = 0$		110	113	113	110	94	110	113
1	41	632	5449	32522	155463	628866	2256445	7358644
2	0	486	29680	672004	9213931	91886539	730644383	4890880851
3	0	52	40521	3389134	122021518	2682580356	42201281320	518612135254
4	0	0	15206	6089576	584124117	27553828341	823110963896	17728177368851
5	0	0	1318	4251622	1230515498	127127937012	7158680727853	265381636196294
6	0	0	1	1125074	1223539121	297185353890	32148039886801	2051420839803630

$\mathcal{F}_p = \mathcal{O}_V(1)^{\oplus 4}$: $(h^{1,1}, h^{2,1}) = (2, 50)$								
Intersection numbers	$\sigma_1^3 = 8, \quad \sigma_1\sigma_2 = 4, \quad \sigma_1^2\tau_1 = 12, \quad \sigma_1\tau_1^2 = 8, \quad \sigma_2\tau_1 = 6, \quad \tau_1^3 = 2$							
n_{d_1,d_2}	$d_1 = 0$	1	2	3	4	5	6	7
$d_2 = 0$		80	20	0	0	0	0	0
1	40	560	2800	6800	9104	6800	2800	560
2	0	560	22220	274784	1695200	6283360	15291620	25650640
3	0	80	42208	2102160	40381840	417187840	2708790480	12060977392
4	0	0	22220	5443840	299074880	7435705920	106637235608	1000779043760
5	0	0	2800	5443840	929117120	53663104048	1580847225600	28485500761200
6	0	0	20	2102160	1343346240	187910411760	11181575861220	371398929912800

$\mathcal{F}_p = \mathcal{S}^* \oplus \mathcal{O}_V(1) \oplus \mathcal{O}_V(2)$: $(h^{1,1}, h^{2,1}) = (2, 57)$								
Intersection numbers	$\sigma_1^3 = 8, \quad \sigma_1\sigma_2 = 4, \quad \sigma_1^2\tau_1 = 11, \quad \sigma_1\tau_1^2 = 7, \quad \sigma_2\tau_1 = 5, \quad \tau_1^3 = 2$							
n_{d_1,d_2}	$d_1 = 0$	1	2	3	4	5	6	7
$d_2 = 0$		88	5	0	0	0	0	0
1	33	634	4048	10037	10037	4048	634	33
2	0	548	28997	466086	3406527	13159772	28776716	37157620
3	0	10	45347	3181936	80480431	1027716204	7647030133	35755062323
4	0	0	13736	6686966	522689207	17306100970	313954566036	3528240156238

5	0	0	165	4524366	1312841562	108179999795	4280740941876	99081868036162
6	0	0	-10	780282	1358341003	305921060292	25988397030539	1167267498525808

$\mathcal{F}_p = \mathrm{Sym}^2\,\mathcal{S}^* \oplus \mathcal{O}_V(1)$: $(h^{1,1}, h^{2,1}) = (2, 32)$								
Intersection numbers	$\sigma_1^3 = 8$, $\sigma_1\sigma_2 = 4$, $\sigma_1^2\tau_1 = 14$, $\sigma_1\tau_1^2 = 12$, $\sigma_2\tau_1 = 5$, $\tau_1^3 = 4$							
n_{d_1,d_2}	$d_1 = 0$	1	2	3	4	5	6	7
$d_2 = 0$		20	22	0	0	0	0	0
1	58	348	870	1160	870	348	58	0
2	0	844	9460	42320	115744	200724	244280	200724
3	0	68	35968	541140	3646870	14883488	41436000	83496920
4	0	0	34722	2839040	47787096	402821800	2153902504	8105770980
5	0	0	11050	5898656	298453714	5287652400	51848056504	335849637824
6	0	0	196	4822716	908058576	37135584632	678692927028	7409928380632

$\mathcal{F}_p = \mathcal{S}^* \oplus \mathcal{O}_V(1)^{\oplus 3}$: $(h^{1,1}, h^{2,1}) = (2, 49)$								
Intersection numbers	$\sigma_1^3 = 8$, $\sigma_1\sigma_2 = 4$, $\sigma_1^2\tau_1 = 13$, $\sigma_1\tau_1^2 = 11$, $\sigma_2\tau_1 = 6$, $\tau_1^3 = 5$							
n_{d_1,d_2}	$d_1 = 0$	1	2	3	4	5	6	7
$d_2 = 0$		52	1	0	0	0	0	0
1	41	486	1318	917	113	0	0	0
2	0	632	15206	94206	216954	202196	72260	7686
3	0	110	40521	1125074	10519903	43910603	91555625	99039844
4	0	0	29680	4251622	124486831	1484582184	8931510318	29965206018
5	0	0	5449	6089576	579108969	17340098333	242953144372	1875605165389
6	0	0	113	3389134	1223539121	92586552714	2802737114627	44031493485406

$\mathcal{F}_p = (\mathcal{S}^*)^{\oplus 2} \oplus \mathcal{O}_V(2)$: $(h^{1,1}, h^{2,1}) = (2, 56)$								
Intersection numbers	$\sigma_1^3 = 8$, $\sigma_1\sigma_2 = 4$, $\sigma_1^2\tau_1 = 12$, $\sigma_1\tau_1^2 = 10$, $\sigma_2\tau_1 = 5$, $\tau_1^3 = 5$							
n_{d_1,d_2}	$d_1 = 0$	1	2	3	4	5	6	7
$d_2 = 0$		48	-2	0	0	0	0	0
1	34	544	1719	544	34	0	0	0
2	0	688	19704	138352	291762	138352	19704	688
3	0	0	48165	1682784	18006204	75544928	126642213	75544928
4	0	0	22206	5807280	214145556	2945951712	18597811286	57190487824
5	0	0	561	6279840	910538594	34261029504	557526592367	4630265286624
6	0	0	-68	1729152	1545311902	167569246816	6356737689516	116628229665712

$\mathcal{F}_p = (\mathcal{S}^*)^{\oplus 2} \oplus \mathcal{O}_V(1)^{\oplus 2}$: $(h^{1,1}, h^{2,1}) = (2, 46)$								
Intersection numbers	$\sigma_1^3 = 8$, $\sigma_1\sigma_2 = 4$, $\sigma_1^2\tau_1 = 14$, $\sigma_1\tau_1^2 = 14$, $\sigma_2\tau_1 = 6$, $\tau_1^3 = 9$							
n_{d_1,d_2}	$d_1 = 0$	1	2	3	4	5	6	7
$d_2 = 0$		28	0	0	0	0	0	0
1	44	404	579	28	-2	0	0	0
2	0	708	9486	26276	15912	432	-2	0
3	0	140	35891	511640	2079058	2757236	1011037	29956
4	0	0	36284	2887060	41253512	209384984	432768018	355277816
5	0	0	9641	5964836	295048376	4172615020	24615473481	67966106564
6	0	0	406	4749072	923105328	35005695588	489248692862	3213917918364

$\mathcal{F}_p = (\mathcal{S}^*)^{\oplus 3} \oplus \mathcal{O}_V(1)$: $(h^{1,1}, h^{2,1}) = (2, 41)$								
Intersection numbers	$\sigma_1^3 = 8$, $\sigma_1\sigma_2 = 4$, $\sigma_1^2\tau_1 = 15$, $\sigma_1\tau_1^2 = 17$, $\sigma_2\tau_1 = 6$, $\tau_1^3 = 14$							
n_{d_1,d_2}	$d_1 = 0$	1	2	3	4	5	6	7
$d_2 = 0$		10	0	0	0	0	0	0
1	49	308	231	0	0	0	0	0
2	0	794	5349	5729	231	0	0	0
3	0	168	29491	190382	287583	76182	78	0

4	0	0	40547	1681790	10332969	18880381	9662787	760431
5	0	0	15540	5118106	119727638	699168640	1461234039	1090271882
6	0	0	1120	5820116	572514233	9758035439	54759243098	126157897721

$\mathcal{F}_p = (\mathcal{S}^*)^{\oplus 4}$: $(h^{1,1}, h^{2,1}) = (2, 34)$ [10]								
Intersection numbers			$\sigma_1^3 = 8$,	$\sigma_1\sigma_2 = 4$,	$\sigma_1^2\tau_1 = 16$,	$\sigma_1\tau_1^2 = 20$,	$\sigma_2\tau_1 = 6$,	$\tau_1^3 = 20$
n_{d_1,d_2}	$d_1 = 0$	1	2	3	4	5	6	7
$d_2 = 0$		0	0	0	0	0	0	0
1	56	192	56	0	0	0	0	0
2	0	896	2544	896	0	0	0	0
3	0	192	23016	52928	23016	192	0	0
4	0	0	41056	813568	1680576	813568	41056	0
5	0	0	23016	3814144	35857016	66781440	35857016	3814144
6	0	0	2544	6292096	284749056	1784024064	3074369392	1784024064

Table 6. Genus-0 invariants of determinantal 3-folds in (2.25) with $V = G(2, 4)$.

5 Conclusions

In this paper we have examined a class of square determinantal Calabi-Yau varieties in Grassmannians satisfying appropriate conditions about dimension, a Calabi-Yau definition, duality $G(k, n) \cong G(n-k, n)$, and rank of the vector bundles. We found that infinite families of examples associated with non-abelian quiver GLSMs might be possible. Furthermore, we explicitly demonstrated how to compute genus-0 integer invariants of the determinantal Calabi-Yau varieties via the Givental I-functions. By constructing the I-functions from the supersymmetric localization formula for the GLSM on a supersymmetric 2-sphere, we provided a guideline for the evaluation of the genus-0 A-model correlators. We also found the handy formula for the 1-point correlators for Grassmannian Calabi-Yau varieties, which turned out to be generalized into the cases with the determinantal varieties. We hope that our results would give a clue to understand various properties of the less studied GLSMs with non-abelian gauge groups.

Finally, we comment on possible future research directions.

- Since we have not imposed irreducibility as a requirement, to make our classification more rigorous, a comprehensive study of topological invariants such as Hodge numbers and Gromov-Witten invariants for the infinite families in (2.27), (A.4), and (A.10) is indispensable to check whether they are appropriate irreducible Calabi-Yau varieties.

- We have classified the square determinantal varieties based on the requirement (2.5). It would be interesting to examine determinantal varieties with general vector bundles $\mathcal{E}_p$ such as $\mathcal{E}_p = L \otimes \mathcal{O}_V^{\oplus p}$, where L is a line bundle, as studied in [25].

- We conjectured the formula (3.24) for the genus-0 1-point A-model correlators for Grassmannian Calabi-Yau varieties, which generalizes the formula (3.16). It would be interesting to find out the 3-point extension of our formula (3.24) as a natural generalization of the formula (3.12) studied in [32], and give a proof of it.

- In [52], GLSM realizations of the so-called Veronese embeddings and the Segre embeddings were proposed, and it opened up the possibility of more broad class of Calabi-Yau varieties. Various exotic Calabi-Yau examples including the constructions in [53] have also been discussed, and it would be interesting to consider these examples and discuss their I-functions.

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A Determinantal Calabi-Yau 2-folds and 4-folds

In section 2.3, we have focused on the realization of a class of determinantal Calabi-Yau 3-folds of square type. In a similar spirit, here we discuss the classification of determinantal Calabi-Yau 2-folds and 4-folds satisfying the requirements (2.7)–(2.10).

A.1 Determinantal Calabi-Yau 2-folds

When $\dim X_A = 2$, the dimensional condition (2.7) becomes

$$\ell_p^\vee \left(k\mathfrak{c}_1(\mathcal{F}_p) - \ell_p^\vee\right) = k^2 + 2. \tag{A.1}$$

Note that, as mentioned in Remark 2.1, the generic determinantal Calabi-Yau 2-folds X_A do not have the singular loci. In the following, we clarify which type of choices for $\left(k, n; \ell_p^\vee, \mathfrak{c}_1(\mathcal{F}_p)\right)$ can be possible while changing the parameter k.

A.1.1 $k = 1$

When $k = 1$ we have $V = G(1, n) \cong \mathbb{P}^{n-1}$, and the generic solution (2.15) with (2.16) provides a "quartic family" given by

$$\left(k, n; \ell_p^\vee, \mathfrak{c}_1(\mathcal{F}_p)\right) = (1, 4; 1, 4) \ \text{ with } \ \mathcal{F}_p = \oplus_{i=1}^r \mathcal{O}_V(p_i), \ p_1 \geq p_2 \geq \cdots \geq p_r > 0, \ \sum_{i=1}^r p_i = 4. \tag{A.2}$$

In addition, (2.20) provides another class of solutions

$$\left(k, n; \ell_p^\vee, \mathfrak{c}_1(\mathcal{F}_p)\right) = (1, 12; 3, 4) \ \text{ with } \ \mathcal{F}_p = \mathcal{O}_V(1)^{\oplus 2} \oplus \mathcal{O}_V(2), \ \mathcal{O}_V(1)^{\oplus 4}. \tag{A.3}$$

Here the Calabi-Yau 2-fold X_A constructed from $\mathcal{F}_p = \mathcal{O}_V(1)^{\oplus 2} \oplus \mathcal{O}_V(2)$ with $p = 3$ (i.e. $\ell = 0$) can be described by the complete intersection Calabi-Yau 2-fold in $\mathbb{P}^5$ with the vector bundle $\mathcal{O}_{\mathbb{P}^5}(2)^{\oplus 3}$ as

$$X_A \text{ with } \mathcal{O}_V(1)^{\oplus 2} \oplus \mathcal{O}_V(2) \ \longleftrightarrow \ X_{2,2,2} \subset \mathbb{P}^5.$$

A.1.2 $k \geq 2$

When $k \geq 2$ we have $V = G(k,n)$. In this case, there exist two "infinite families" of solutions as determinantal 2-folds satisfying all the requirements (A.1), (2.8), (2.9), and (2.10).

$$\begin{aligned}&\left(k,n;\ell_p^\vee,\mathfrak{c}_1(\mathcal{F}_p)\right) = (k_i, 4\ell_i; \ell_i, 4), \quad i \in \mathbb{N}\\ &\text{where } k_1 = 3,\ \ell_1 = 11,\ k_{i+1} = \ell_i,\ \ell_{i+1} = -k_i + 4\ell_i \ \text{ with } \ \mathcal{F}_p = (\mathcal{S}^*)^{\oplus 4},\ \mathcal{Q}^{\oplus 4}.\end{aligned} \tag{A.4}$$

A.2 Determinantal Calabi-Yau 4-folds

When $\dim X_A = 4$, the dimensional condition (2.7) becomes

$$\ell_p^\vee \left(k\mathfrak{c}_1(\mathcal{F}_p) - \ell_p^\vee\right) = k^2 + 4. \tag{A.5}$$

A.2.1 $k = 1$

When $k = 1$ we have $V = G(1,n) \cong \mathbb{P}^{n-1}$, and the generic solution (2.15) with (2.16) provides a "sextic family" given by

$$\left(k,n;\ell_p^\vee,\mathfrak{c}_1(\mathcal{F}_p)\right) = (1,6;1,6) \ \text{ with } \ \mathcal{F}_p = \oplus_{i=1}^r \mathcal{O}_V(p_i),\ p_1 \geq p_2 \geq \cdots \geq p_r > 0,\ \sum_{i=1}^r p_i = 6. \tag{A.6}$$

Via the incidence correspondence (2.6), the sextic family is connected each other through the extremal transitions (see appendix C.1).

In addition, (2.20) provides another class of solutions

$$\left(k,n;\ell_p^\vee,\mathfrak{c}_1(\mathcal{F}_p)\right) = (1,30;5,6) \ \text{ with } \ \mathcal{F}_p = \mathcal{O}_V(1)^{\oplus 4} \oplus \mathcal{O}_V(2),\ \mathcal{O}_V(1)^{\oplus 6}. \tag{A.7}$$

Here the Calabi-Yau 4-fold X_A constructed from $\mathcal{F}_p = \mathcal{O}_V(1)^{\oplus 4} \oplus \mathcal{O}_V(2)$ with $p = 5$ (i.e. $\ell = 0$) can be described by the complete intersection Calabi-Yau 4-fold in $\mathbb{P}^9$ with the vector bundle $\mathcal{O}_{\mathbb{P}^5}(2)^{\oplus 5}$ as

$$X_A \text{ with } \mathcal{O}_V(1)^{\oplus 4} \oplus \mathcal{O}_V(2) \ \longleftrightarrow \ X_{2,2,2,2,2} \subset \mathbb{P}^9.$$

A.2.2 $k = 2$

When $k = 2$ we have $V = G(2,n)$. In this case there exists a class of determinantal 4-folds which satisfy the conditions (A.5), (2.8), (2.9), and (2.10) given by[7]

$$\begin{aligned}&\left(k,n;\ell_p^\vee,\mathfrak{c}_1(\mathcal{F}_p)\right) = (2,6;2,3) \ \text{ with}\\ &\mathcal{F}_p = \mathcal{O}_V(1) \oplus \mathcal{O}_V(2),\ \mathcal{S}^*(1),\ \mathcal{O}_V(1)^{\oplus 3},\ \mathcal{S}^* \oplus \mathcal{O}_V(2),\ \mathrm{Sym}^2\mathcal{S}^*,\ \mathcal{S}^* \oplus \mathcal{O}_V(1)^{\oplus 2},\\ &\qquad (\mathcal{S}^*)^{\oplus 2} \oplus \mathcal{O}_V(1),\ (\mathcal{S}^*)^{\oplus 3},\ \mathcal{Q} \oplus \mathcal{O}_V(2),\ \mathcal{Q} \oplus \mathcal{O}_V(1)^{\oplus 2},\ \mathcal{Q}^{\oplus 2} \oplus \mathcal{O}_V(1),\ \mathcal{Q}^{\oplus 3},\\ &\qquad \wedge^2\mathcal{Q},\ \wedge^3\mathcal{Q},\ \mathcal{S}^* \oplus \mathcal{Q} \oplus \mathcal{O}_V(1),\ (\mathcal{S}^*)^{\oplus 2} \oplus \mathcal{Q},\ \mathcal{S}^* \oplus \mathcal{Q}^{\oplus 2}.\end{aligned} \tag{A.8}$$

[7] In appendix C.2, we see that the determinantal 4-fold associated with $\mathcal{F}_p = \mathrm{Sym}^2\mathcal{S}^*$ is not a Calabi-Yau variety with $(h^{1,0}, h^{2,0}) = (0,0)$ but an irreducible holomorphic symplectic variety with $(h^{1,0}, h^{2,0}) = (0,1)$.

Here the Calabi-Yau 4-fold X_A constructed from $\mathcal{F}_p = \mathcal{O}_V(1) \oplus \mathcal{O}_V(2)$ (*resp.* $\mathcal{S}^*(1)$) with $p = 2$ (i.e. $\ell = 0$) can be described by the complete intersection Grassmannian Calabi-Yau 4-fold in $G(2,6)$ with the vector bundle $\mathcal{O}_{G(2,6)}(1)^{\oplus 2} \oplus \mathcal{O}_{G(2,6)}(2)^{\oplus 2}$ (*resp.* $\mathcal{S}^*(1)^{\oplus 2}$ on $G(2,6)$).

We find that there exist another type of solutions satisfying all the requirements (A.5), (2.8), (2.9), and (2.10) given by

$$\begin{aligned}&\big(k,n;\ell_p^\vee,\mathfrak{c}_1(\mathcal{F}_p)\big) = (2,12;4,3) \quad \text{with} \\ &\mathcal{F}_p = \mathcal{S}^* \oplus \mathcal{O}_V(1)^{\oplus 2},\ (\mathcal{S}^*)^{\oplus 2} \oplus \mathcal{O}_V(1),\ (\mathcal{S}^*)^{\oplus 3},\ \mathcal{Q} \oplus \mathcal{O}_V(2), \\ &\qquad \mathcal{Q} \oplus \mathcal{O}_V(1)^{\oplus 2},\ \mathcal{Q}^{\oplus 2} \oplus \mathcal{O}_V(1),\ \mathcal{Q}^{\oplus 3}.\end{aligned} \tag{A.9}$$

Here the Calabi-Yau 4-fold X_A constructed from $\mathcal{F}_p = \mathcal{S}^* \oplus \mathcal{O}_V(1)^{\oplus 2}$ with $p = 4$ (i.e. $\ell = 0$) can be described by the complete intersection Grassmannian Calabi-Yau 4-fold in $G(2,8)$ with the vector bundle $\mathcal{O}_{G(2,8)}(1)^{\oplus 8}$.

A.2.3 $k \geq 3$

When $k \geq 3$ we have $V = G(k,n)$, and there exist six "infinite families" of determinantal 4-folds given by

$$\begin{aligned}&\big(k,n;\ell_p^\vee,\mathfrak{c}_1(\mathcal{F}_p)\big) = (k_i,6\ell_i;\ell_i,6), \quad i \in \mathbb{N} \\ &\text{with} \quad k_1 = 5,\ \ell_1 = 29,\ k_{i+1} = \ell_i,\ \ell_{i+1} = -k_i + 6\ell_i:\ \mathcal{F}_p = (\mathcal{S}^*)^{\oplus 6},\ \mathcal{Q}^{\oplus 6}, \\ &\big(k,n;\ell_p^\vee,\mathfrak{c}_1(\mathcal{F}_p)\big) = (k_i,3\ell_i;\ell_i,3), \quad i \in \mathbb{N} \\ &\text{with} \quad k_1 = 4,\ \ell_1 = 10,\ k_{i+1} = -3k_i + 8\ell_i,\ \ell_{i+1} = -8k_i + 21\ell_i:\ \mathcal{F}_p = (\mathcal{S}^*)^{\oplus 3},\ \mathcal{Q}^{\oplus 3}, \\ &\big(k,n;\ell_p^\vee,\mathfrak{c}_1(\mathcal{F}_p)\big) = (k_i,3\ell_i;\ell_i,3), \quad i \in \mathbb{N} \\ &\text{with} \quad k_1 = 10,\ \ell_1 = 26,\ k_{i+1} = -3k_i + 8\ell_i,\ \ell_{i+1} = -8k_i + 21\ell_i:\mathcal{F}_p = (\mathcal{S}^*)^{\oplus 3},\ \mathcal{Q}^{\oplus 3},\end{aligned} \tag{A.10}$$

where the third (*resp.* fourth) and the fifth (*resp.* sixth) families of 4-folds associated with $\mathcal{F}_p = (\mathcal{S}^*)^{\oplus 3}$ (*resp.* $\mathcal{Q}^{\oplus 3}$) are given by the same recurrence relation with the different initial conditions. By using mathematical induction, one can check that the duality condition (2.9), the rank condition (2.10), and in particular $\ell_p^\vee < \operatorname{rank} \mathcal{F}_p$, are maintained.

B Hodge number calculations via the Koszul complex

Following [54] (see also e.g. [55–58]), here we briefly review how to compute cohomologies and Hodge numbers of Calabi-Yau varieties via the Koszul complex. We will demonstrate the explicit computations for several examples.

B.1 General algorithm

Let V be a complex manifold, $\mathcal{E}_p$ be a rank p vector bundle over V and consider the locus $X \subset V$ as a holomorphic section of $\mathcal{E}_p$. In this appendix we describe how to compute the cohomologies

$$H^{\dim X-1,i}(X) = H^i(X,TX), \quad \text{or} \quad H^{1,i}(X) = H^i(X,T^*X), \quad i = 0,1,\dots,\dim X, \tag{B.1}$$

via the Koszul complex.

B.1.1 Step 1: computation of (B.2)

First we describe a method to compute bundle-valued cohomologies of X,

$$H^i(X, \mathcal{F}_V|_X), \tag{B.2}$$

by using the *Koszul exact sequence.*

The Koszul exact sequence gives the resolution of $\mathcal{O}_X$ over V as

$$0 \longrightarrow \wedge^p \mathcal{E}_p^* \longrightarrow \cdots \longrightarrow \wedge^2 \mathcal{E}_p^* \longrightarrow \mathcal{E}_p^* \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_X \longrightarrow 0. \tag{B.3}$$

For the Koszul exact sequence, the *Koszul spectral sequence* (see e.g. [42]),

$$\left\{E_r^{i,q}\right\}, \quad i = 0, 1, \ldots, \dim V, \quad q = 0, 1, \ldots, p, \quad r = 0, 1, 2, \ldots,$$

can be associated as follows. Starting from

$$E_0^{i,q} = H^i(V, \wedge^q \mathcal{E}_p^*), \tag{B.4}$$

define d_r-cohomology recursively as

$$E_{r+1}^{i,q} = \frac{\ker\left(d_r : \ E_r^{i,q} \ \longrightarrow \ E_r^{i-r,q-r-1}\right)}{\mathrm{im}\left(d_r : \ E_r^{i+r,q+r+1} \ \longrightarrow \ E_r^{i,q}\right)}, \tag{B.5}$$

which is associated with *differentials*

$$d_r \ : \ E_r^{i,q} \ \longrightarrow \ E_r^{i-r,q-r-1}, \tag{B.6}$$

with $d_r \circ d_r = 0$. Here $E_r^{i,q} = 0$ for $i, q < 0$, $i > \dim V$, and $q > p$. At finite $r = r_0$, $E_r^{i,q}$ converges to $E_{r_0}^{i,q} = E_{r_0+1}^{i,q} = \ldots = E_\infty^{i,q}$ and we obtain a cohomology of X as

$$\sum_{q=0}^{p} E_\infty^{i+q,q} \implies H^i(X, \mathcal{O}_X) = H^{0,i}(X), \tag{B.7}$$

where the summation represents a formal sum.

Remark B.1. By tensoring the Koszul exact sequence (B.3) with a vector bundle $\mathcal{F}_V$ (e.g. $\mathcal{E}_p$, $\mathcal{E}_p^*$, TV, T^*V, etc.) over V, one obtains the resolution of $\mathcal{F}_V|_X$ over V as

$$0 \longrightarrow \wedge^p \mathcal{E}_p^* \otimes \mathcal{F}_V \longrightarrow \cdots \longrightarrow \wedge^2 \mathcal{E}_p^* \otimes \mathcal{F}_V \longrightarrow \mathcal{E}_p^* \otimes \mathcal{F}_V \longrightarrow \mathcal{F}_V \longrightarrow \mathcal{F}_V|_X \longrightarrow 0. \tag{B.8}$$

Then, by considering the Koszul spectral sequence associated with (B.8), one can obtain the bundle-valued cohomologies $H^i(X, \mathcal{F}_V|_X)$ in (B.2).

Therefore, by using the Koszul spectral sequence, the cohomologies $H^i(X, \mathcal{F}_V|_X)$ can be computed from the cohomologies $H^i(V, \wedge^q \mathcal{E}_p^* \otimes \mathcal{F}_V)$. For computing these quantities, the Bott-Borel-Weil theorem B.5 is quite useful. To state the theorem, consider a flag manifold

$$V = \frac{\mathrm{U}(N)}{\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_F)}, \qquad N = \sum_{i=1}^{F} n_i. \tag{B.9}$$

A holomorphic homogeneous vector bundle $\mathcal{F}_V$ over V can be described by a representation of $\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_F)$, where a representation of each $\mathrm{U}(n)$ is described by a Young diagram which is given by a monotonically increasing sequence with length n of integers as $(a_1, \ldots, a_n)$, $a_i \leq a_{i+1}$. Then, a vector bundle $\mathcal{F}_V$ is described by a representation of $\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_F)$ as

$$\mathcal{F}_V \sim (a_1, \ldots, a_{n_1} | b_1, \ldots, b_{n_2} | \cdots | r_1, \ldots, r_{n_F}). \tag{B.10}$$

Example B.2. For $V = \mathbb{P}^{n-1} = \mathrm{U}(n)/[\mathrm{U}(1) \times \mathrm{U}(n-1)]$, in terms of the representations of $\mathrm{U}(1) \times \mathrm{U}(n-1)$, one can describe e.g.,

$$\begin{aligned} \mathcal{S} &\sim (1|0, \ldots, 0), & \mathcal{S}^* &\sim (-1|0, \ldots, 0), \\ \mathcal{O}_V(p) = (\mathcal{S}^*)^{\otimes p} &\sim (-p|0, \ldots, 0), & \mathcal{O}_V(p)^* = \mathcal{S}^{\otimes p} &\sim (p|0, \ldots, 0), \\ TV &\sim (-1|0, \ldots, 0, 1), & T^*V &\sim (1|-1, 0, \ldots, 0), \end{aligned} \tag{B.11}$$

where $\mathcal{S}$ is the universal subbundle on $V = \mathbb{P}^{n-1}$. The representations of tensor product and wedge product are obtained as e.g.,

$$\mathcal{O}_V(p)^* \otimes T^*V \sim (p+1|-1, 0, \ldots, 0), \qquad \wedge^2 TV \sim (-2|0, \ldots, 0, 1, 1).$$

Example B.3. For $V = G(k, n) = \mathrm{U}(n)/[\mathrm{U}(k) \times \mathrm{U}(n-k)]$, in terms of the representations of $\mathrm{U}(k) \times \mathrm{U}(n-k)$, one can describe e.g.,

$$\begin{aligned} \mathcal{S} &\sim (0, \ldots, 0, 1|0, \ldots, 0), \qquad \mathcal{S}^* \sim (-1, 0, \ldots, 0|0, \ldots, 0), \\ \mathcal{O}_V(p) = (\det \mathcal{S}^*)^{\otimes p} &\sim (-p, \ldots, -p|0, \ldots, 0), \\ \mathcal{O}_V(p)^* = (\det \mathcal{S})^{\otimes p} &\sim (p, \ldots, p|0, \ldots, 0), \\ TV &\sim (-1, 0, \ldots, 0|0, \ldots, 0, 1), \qquad T^*V \sim (0, \ldots, 0, 1|-1, 0, \ldots, 0), \end{aligned} \tag{B.12}$$

where $\mathcal{S}$ is the rank k universal subbundle on $V = G(k, n)$. The representations of tensor product and wedge product are obtained e.g. for $V = G(2, 5)$ as

$$\begin{aligned} \mathcal{S}(-1) = \mathcal{S} \otimes \mathcal{O}_V(1)^* &\sim (1, 2|0, \ldots, 0), & \wedge^2 \mathcal{S}(-1) &\sim (3, 3|0, \ldots, 0, 0), \\ \mathcal{S}(-1) \wedge \mathcal{O}_V(2)^* &\sim (3, 4|0, \ldots, 0, 0), & \big(\wedge^2 \mathcal{S}(-1)\big) \wedge \mathcal{O}_V(2)^* &\sim (5, 5|0, \ldots, 0, 0). \end{aligned}$$

Example B.4. For product manifold $V = \mathbb{P}^2 \times G(2, 4)$, one obtains e.g.,

$$\begin{aligned} \mathcal{S}(0, -1) = \mathcal{S} \otimes \mathcal{O}_V(0, 1)^* &\sim \left(\begin{array}{cc|cc} & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{array} \right), & \wedge^2 \mathcal{S}(0, -1) &\sim \left(\begin{array}{cc|cc} & 2 & 0 & 0 \\ 3 & 3 & 0 & 0 \end{array} \right), \\ \mathcal{S}(0, -1) \wedge \mathcal{O}_V(1, 1)^* &\sim \left(\begin{array}{cc|cc} & 2 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{array} \right), & \big(\wedge^2 \mathcal{S}(0, -1)\big) \wedge \mathcal{O}_V(1, 1)^* &\sim \left(\begin{array}{cc|cc} & 3 & 0 & 0 \\ 4 & 4 & 0 & 0 \end{array} \right), \end{aligned}$$

where $\mathcal{S}$ is the rank $(1, 2)$ universal subbundle on V.

Using the above representations (B.10) for vector bundles, the Bott-Borel-Weil theorem is stated as follows.

Theorem B.5 (Bott-Borel-Weil)**.** *Let $\mathcal{F}_V$ be a holomorphic homogeneous vector bundle, represented as (B.10), over a flag manifold V in (B.9). Then at most only one of the cohomologies $H^i(V, \mathcal{F}_V)$ is non-trivial ($\cong \mathbb{C}^D$), and D is given by the dimension of an irreducible representation $(y_1, \dots, y_N)$ of $\mathrm{U}(N)$ determined as follows:*

1. *For the sequence (B.10), add the sequence $(1, 2, \dots, N)$ as*

$$(a_1 + 1, a_2 + 2, \dots, a_{n_1} + n_1, b_1 + n_1 + 1, \dots, b_{n_2} + n_1 + n_2, \dots, r_{n_F} + N).$$

2. *If the above sequence contains any same number, the cohomologies $H^i(V, \mathcal{F}_V)$ are trivial, if not;*

3. *Minimally swap the above sequence, with the minimal swapping number I, so that the resulting sequence gives a strictly increasing sequence $(y'_1, \dots, y'_N)$, $y'_i < y'_{i+1}$.*

4. *For the above swapped sequence, subtracting the sequence $(1, 2, \dots, N)$ as*

$$(y_1, y_2, \dots, y_N) = (y'_1 - 1, y'_2 - 2, \dots, y'_N - N),$$

one obtains a representation $(y_1, \dots, y_N)$ of $\mathrm{U}(N)$ which gives $H^I(V, \mathcal{F}_V)$.

Remark B.6. The dimension D of a representation $(y_1, \dots, y_N)$ of $\mathrm{U}(N)$, which is given by a Young diagram Y with length y_i for the i-th row, is computed by

$$D = \prod_{s \in Y} \frac{F_Y(s)}{H_Y(s)}, \tag{B.13}$$

where $H_Y(s)$ is the hook length of s in Y, and $F_Y(s) = N - i + j$ for $s = (i, j)$ (the box of i-th row and j-th column).

Remark B.7. When V is a product manifold $V = V_1 \times V_2$ of two flag manifolds V_1 and V_2, for computing the cohomologies of V one can use the *Künneth formula*

$$H^i(V, \mathcal{F}_V) = \bigoplus_{i_1 + i_2 = i} H^{i_1}(V_1, \mathcal{F}_V|_{V_1}) \otimes H^{i_2}(V_2, \mathcal{F}_V|_{V_2}), \tag{B.14}$$

where $\mathcal{F}_V$ is a vector bundle over V and each $\mathcal{F}_V|_{V_i}$ is the restricted vector bundle over V_i.

B.1.2 Step 2: computation of (B.1)

By Step 1 one can compute, in particular, $H^{0,i}(X) = H^i(X, \mathcal{O}_X)$, $H^i(X, TV|_X)$, $H^i(X, \mathcal{E}_p|_X)$, $H^i(X, T^*V|_X)$, and $H^i(X, \mathcal{E}_p^*|_X)$. Now, using these results one can compute the cohomologies (B.1) via the short exact sequence (adjunction formula)

$$0 \longrightarrow TX \longrightarrow TV|_X \longrightarrow \mathcal{E}_p|_X \longrightarrow 0, \tag{B.15}$$

or its dual

$$0 \longrightarrow \mathcal{E}_p^*|_X \longrightarrow T^*V|_X \longrightarrow T^*X \longrightarrow 0. \tag{B.16}$$

The former and the latter in (B.1) are related by the Hodge dual, and then in the following we only describe the computation of the latter by using the exact sequence (B.16). The exact sequence (B.16) induces the following long exact sequence:

$$\begin{array}{rcccc} 0 \longrightarrow & H^0(X, \mathcal{E}_p^*|_X) \longrightarrow & H^0(X, T^*V|_X) \longrightarrow & H^0(X, T^*X) \longrightarrow & \\ \longrightarrow & H^1(X, \mathcal{E}_p^*|_X) \longrightarrow & H^1(X, T^*V|_X) \longrightarrow & H^1(X, T^*X) \longrightarrow \ldots & \\ \ldots \longrightarrow & H^{\dim X}(X, \mathcal{E}_p^*|_X) \longrightarrow & H^{\dim X}(X, T^*V|_X) \longrightarrow & H^{\dim X}(X, T^*X) \longrightarrow 0. & \end{array} \tag{B.17}$$

Then, using the above exact sequence, from the cohomologies $H^i(X, T^*V|_X)$ and $H^i(X, \mathcal{E}_p^*|_X)$ obtained in Step 1, one can compute the cohomologies $H^i(X, T^*X)$.

Remark B.8. To find other cohomologies, one can use the following well-known relations

$$\begin{aligned} \textit{Complex conjugate:} \quad & H^{i,j}(X) \cong H^{j,i}(X), \\ \textit{Hodge duality:} \quad & H^{i,j}(X) \cong H^{\dim X - i, \dim X - j}(X). \end{aligned}$$

Furthermore, if X admits the unique holomorphic 3-form in $H^{\dim X,0}(X) = \mathbb{C}$, *Serre duality* gives $H^{0,i}(X) \cong H^{\dim X,i}(X)$.

Remark B.9. To compute the Hodge numbers of X, one can also use a formula for the *Hirzebruch χ_y-genus* (see [59] for the explicit formulae of $\dim X = 2, 3, 4$ written in terms of the Chern classes of X):

$$\chi_y = \sum_{i,j=0}^{\dim X} (-1)^j \dim H^j(X, \wedge^i T^*X)\, y^i = \int_X \prod_{k=1}^{p} \frac{(1 + y\mathrm{e}^{-x_k})\, x_k}{1 - \mathrm{e}^{-x_k}}, \tag{B.18}$$

which is derived from *Hirzebruch-Riemann-Roch index theorem*. Here x_k, $k = 1, \ldots, p$, are the Chern roots of X in (3.1).

B.2 Examples

We demonstrate the explicit computations of cohomologies for some examples based on the strategy in appendix B.1.

B.2.1 Quintic Calabi-Yau 3-fold: $\mathcal{E}_1 = \mathcal{O}_V(5)$ on $V = \mathbb{P}^4$

As a famous example, consider the quintic Calabi-Yau 3-fold X defined by the zero locus of a holomorphic section of $\mathcal{E}_1 = \mathcal{O}_V(5)$ on $V = \mathbb{P}^4$ [36]. Using the representations in Example B.2, the Koszul exact sequence (B.3) is given by

$$0 \longrightarrow (5|0,0,0,0) \longrightarrow (0|0,0,0,0) \longrightarrow \mathcal{O}_X \longrightarrow 0. \tag{B.19}$$

By (B.7) and the Bott-Borel-Weil theorem B.5, one finds that

$$H^0(V, \mathcal{O}_V) = \mathbb{C} \implies H^{0,0}(X) = \mathbb{C}, \quad H^4(V, \mathcal{O}_V(5)^*) = \mathbb{C} \implies H^{0,3}(X) = \mathbb{C}, \tag{B.20}$$

and $H^{0,1}(X)$ and $H^{0,2}(X)$ are trivial, i.e. $H^{0,1}(X) = H^{0,2}(X) = 0$.

For $\mathcal{F}_V = \mathcal{O}_V(5)^*$ and $\mathcal{F}_V = T^*V$ the exact sequence (B.8) yields

$$0 \longrightarrow (10|0,0,0,0) \longrightarrow (5|0,0,0,0) \longrightarrow \mathcal{O}_V(5)^*|_X \longrightarrow 0, \tag{B.21}$$
$$0 \longrightarrow (6|-1,0,0,0) \longrightarrow (1|-1,0,0,0) \longrightarrow T^*V|_X \longrightarrow 0. \tag{B.22}$$

From (B.21) one finds that

$$\begin{aligned}\ker\big(d_0:\ H^4(V,\mathcal{O}_V(5)^*\otimes\mathcal{O}_V(5)^*) = \mathbb{C}^{126} &\longrightarrow H^4(V,\mathcal{O}_V(5)^*) = \mathbb{C}\big)\\ \Longrightarrow\ H^3(X,\mathcal{O}_V(5)^*|_X) = \mathbb{C}^{125},\end{aligned} \tag{B.23}$$

and $H^i(X,\mathcal{O}_V(5)^*|_X)$, $i=0,1,2$, are trivial. From (B.22) one finds that

$$\begin{aligned} H^1(V,T^*V) = \mathbb{C} &\implies H^1(X,T^*V|_X) = \mathbb{C},\\ H^4(V,\mathcal{O}_V(5)^*\otimes T^*V) = \mathbb{C}^{24} &\implies H^3(X,T^*V|_X) = \mathbb{C}^{24},\end{aligned} \tag{B.24}$$

and $H^0(X,T^*V|_X)$ and $H^2(X,T^*V|_X)$ are trivial. Using these results, the exact sequence (B.17) yields the following exact sequences:

$$\begin{aligned} &0 \longrightarrow H^0(X,T^*X) \longrightarrow 0,\\ &0 \longrightarrow \mathbb{C} \longrightarrow H^1(X,T^*X) \longrightarrow 0,\\ &0 \longrightarrow H^2(X,T^*X) \longrightarrow \mathbb{C}^{125} \xrightarrow{f} \mathbb{C}^{24} \longrightarrow H^3(X,T^*X) \longrightarrow 0.\end{aligned} \tag{B.25}$$

Then one obtains

$$\begin{aligned} H^{1,0}(X) &= H^0(X,T^*X) = 0, & H^{1,1}(X) &= H^1(X,T^*X) = \mathbb{C},\\ H^{1,2}(X) &= H^2(X,T^*X) = \ker(f) = \mathbb{C}^{101}, & H^{1,3}(X) &= H^3(X,T^*X) = \mathrm{coker}(f) = 0.\end{aligned} \tag{B.26}$$

As a result, the Hodge diamond is obtained as

$$\begin{matrix} &&&h^{0,0}&&& \\ &&h^{1,0}&&h^{0,1}&& \\ &h^{2,0}&&h^{1,1}&&h^{0,2}& \\ h^{3,0}&&h^{2,1}&&h^{1,2}&&h^{0,3} \\ &h^{3,1}&&h^{2,2}&&h^{1,3}& \\ &&h^{3,2}&&h^{2,3}&& \\ &&&h^{3,3}&&& \end{matrix} = \begin{matrix} &&&1&&& \\ &&0&&0&& \\ &0&&1&&0& \\ 1&&101&&101&&1 \\ &0&&1&&0& \\ &&0&&0&& \\ &&&1&&& \end{matrix}.$$

B.2.2 Grassmannian Calabi-Yau 3-fold: $\mathcal{E}_3 = \mathcal{S}^*(1) \oplus \mathcal{O}_V(2)$ on $V = G(2,5)$

As a non-abelian example, consider a Grassmannian Calabi-Yau 3-fold X defined by the zero locus of a holomorphic section of $\mathcal{E}_3 = \mathcal{S}^*(1) \oplus \mathcal{O}_V(2)$ on $V = G(2,5)$ [60, 61]. Using the representations in Example B.3, the Koszul exact sequence (B.3) is given by

$$0 \to (5,5|0,0,0) \to \begin{matrix}(3,3|0,0,0)\\ \oplus\\ (3,4|0,0,0)\end{matrix} \to \begin{matrix}(1,2|0,0,0)\\ \oplus\\ (2,2|0,0,0)\end{matrix} \to (0,0|0,0,0) \to \mathcal{O}_X \to 0. \tag{B.27}$$

By (B.7) and the Bott-Borel-Weil theorem B.5, one finds that

$$H^0(V,\mathcal{O}_V)=\mathbb{C} \implies H^{0,0}(X)=\mathbb{C}, \quad H^6(V,\wedge^3\mathcal{E}_3^*)=\mathbb{C} \implies H^{0,3}(X)=\mathbb{C}, \tag{B.28}$$

and $H^{0,1}(X)$ and $H^{0,2}(X)$ are trivial, i.e. $H^{0,1}(X)=H^{0,2}(X)=0$.

For $\mathcal{F}_V=\mathcal{S}(1)$, $\mathcal{O}_V(2)^*$, and T^*V the exact sequence (B.8) yields respectively,

$$0\to(6,7|0,0,0)\to \begin{matrix}(4,5|0,0,0)\\ \oplus\\ (5,5|0,0,0)\\ \oplus\\ (4,6|0,0,0)\end{matrix} \to \begin{matrix}(2,4|0,0,0)\\ \oplus\\ (3,3|0,0,0)\\ \oplus\\ (3,4|0,0,0)\end{matrix} \to(1,2|0,0,0)\to\mathcal{S}(1)|_X\to 0, \tag{B.29}$$

$$0\to(7,7|0,0,0)\to \begin{matrix}(5,5|0,0,0)\\ \oplus\\ (5,6|0,0,0)\end{matrix} \to \begin{matrix}(3,4|0,0,0)\\ \oplus\\ (4,4|0,0,0)\end{matrix} \to(2,2|0,0,0)\to\mathcal{O}(2)^*|_X\to 0, \tag{B.30}$$

and

$$0\to(5,6|-1,0,0)\to \begin{matrix}(3,4|-1,0,0)\\ \oplus\\ (3,5|-1,0,0)\\ \oplus\\ (4,4|-1,0,0)\end{matrix} \to \begin{matrix}(1,3|-1,0,0)\\ \oplus\\ (2,2|-1,0,0)\\ \oplus\\ (2,3|-1,0,0)\end{matrix} \to(0,1|-1,0,0)\to T^*V|_X\to 0. \tag{B.31}$$

From (B.29) one finds that

$$\begin{aligned}\ker\big(d_0:\ H^6(V,\wedge^3\mathcal{E}_3^*\otimes\mathcal{S}(1))=\mathbb{C}^{40}\ &\longrightarrow\ H^6(V,\wedge^2\mathcal{E}_3^*\otimes\mathcal{S}(1))=\mathbb{C}\big)\\ \implies H^3(X,\mathcal{S}(1)|_X)=\mathbb{C}^{39},&\end{aligned} \tag{B.32}$$

and $H^i(X,\mathcal{S}(1)|_X)$, $i=0,1,2$, are trivial. From (B.30) one finds that

$$\begin{aligned}\ker\big(d_0:\ H^6(V,\wedge^3\mathcal{E}_3^*\otimes\mathcal{O}_V(2)^*)=\mathbb{C}^{50}\ &\longrightarrow\ H^6(V,\wedge^2\mathcal{E}_3^*\otimes\mathcal{O}_V(2)^*)=\mathbb{C}^6\big)\\ \implies H^3(X,\mathcal{O}_V(2)^*|_X)=\mathbb{C}^{44},&\end{aligned} \tag{B.33}$$

and $H^i(X,\mathcal{O}_V(2)^*|_X)$, $i=0,1,2$, are trivial. Then one gets $H^3(X,\mathcal{E}_3^*|_X)=\mathbb{C}^{83}$ by (B.32) and (B.33). From (B.31) one finds that

$$\begin{aligned}H^1(V,T^*V)=\mathbb{C}\ &\implies\ H^1(X,T^*V|_X)=\mathbb{C},\\ H^6(V,\wedge^3\mathcal{E}_3^*\otimes T^*V)=\mathbb{C}^{24}\ &\implies\ H^3(X,T^*V|_X)=\mathbb{C}^{24},\end{aligned} \tag{B.34}$$

and $H^0(X,T^*V|_X)$ and $H^2(X,T^*V|_X)$ are trivial. Using these results, the exact sequence (B.17) yields the following exact sequences:

$$\begin{aligned}&0\longrightarrow H^0(X,T^*X)\longrightarrow 0,\\ &0\longrightarrow \mathbb{C}\longrightarrow H^1(X,T^*X)\longrightarrow 0,\\ &0\longrightarrow H^2(X,T^*X)\longrightarrow \mathbb{C}^{83}\xrightarrow{f}\mathbb{C}^{24}\longrightarrow H^3(X,T^*X)\longrightarrow 0.\end{aligned} \tag{B.35}$$

Then one obtains

$$H^{1,0}(X) = H^0(X, T^*X) = 0, \qquad H^{1,1}(X) = H^1(X, T^*X) = \mathbb{C},$$
$$H^{1,2}(X) = H^2(X, T^*X) = \ker(f) = \mathbb{C}^{59}, \qquad H^{1,3}(X) = H^3(X, T^*X) = \mathrm{coker}(f) = 0. \tag{B.36}$$

As a result, the Hodge diamond is obtained as

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 59 & & 59 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}.$$

B.2.3 Determinantal Calabi-Yau 3-fold in (2.25) with $\mathcal{F}_3 = \mathcal{S}^*(1) \oplus \mathcal{O}_V(1)$

Consider a determinantal Calabi-Yau 3-fold in (2.25) with $\mathcal{F}_3 = \mathcal{S}^*(1) \oplus \mathcal{O}_V(1)$. We especially consider a geometric phase, and then the desingularized determinantal Calabi-Yau 3-fold X is defined by the locus of a holomorphic section of $\mathcal{E}_3 = \mathcal{E}_3^{(2)} \oplus \mathcal{E}_3^{(1)}$ on $V' = G(2,4) \times \mathbb{P}^2$, where $\mathcal{E}_3^{(2)} = \mathcal{S}^* \otimes \mathcal{O}_{V'}(1,0)$ and $\mathcal{E}_3^{(1)} = \mathcal{O}_{V'}(1,1)$. Using the representations in Example B.4, the Koszul exact sequence (B.3) is given by

$$0 \to \left(\begin{array}{cc|cc} 4 & 4 & 0 & 0 \\ & 3 & 0 & 0 \end{array}\right) \to \begin{array}{c} \left(\begin{array}{cc|cc} 3 & 3 & 0 & 0 \\ & 2 & 0 & 0 \end{array}\right) \\ \oplus \\ \left(\begin{array}{cc|cc} 2 & 3 & 0 & 0 \\ & 2 & 0 & 0 \end{array}\right) \end{array} \to \begin{array}{c} \left(\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ & 1 & 0 & 0 \end{array}\right) \\ \oplus \\ \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ & 1 & 0 & 0 \end{array}\right) \end{array} \to \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \end{array}\right) \to \mathcal{O}_X \to 0. \tag{B.37}$$

By (B.7) and the Bott-Borel-Weil theorem B.5, one finds that

$$H^0(V', \mathcal{O}_{V'}) = \mathbb{C} \implies H^{0,0}(X) = \mathbb{C}, \quad H^6(V', \wedge^3 \mathcal{E}_3^*) = \mathbb{C} \implies H^{0,3}(X) = \mathbb{C}, \tag{B.38}$$

and $H^{0,1}(X)$ and $H^{0,2}(X)$ are trivial, i.e. $H^{0,1}(X) = H^{0,2}(X) = 0$.

The exact sequence (B.8) gives, for $\mathcal{F}_{V'} = \mathcal{E}_3^{(2)\,*}$,

$$0 \to \left(\begin{array}{cc|cc} 5 & 6 & 0 & 0 \\ & 4 & 0 & 0 \end{array}\right) \to \begin{array}{c} \left(\begin{array}{cc|cc} 4 & 5 & 0 & 0 \\ & 3 & 0 & 0 \end{array}\right) \\ \oplus \\ \left(\begin{array}{cc|cc} 3 & 5 & 0 & 0 \\ & 3 & 0 & 0 \end{array}\right) \\ \oplus \\ \left(\begin{array}{cc|cc} 4 & 4 & 0 & 0 \\ & 3 & 0 & 0 \end{array}\right) \end{array} \to \begin{array}{c} \left(\begin{array}{cc|cc} 2 & 4 & 0 & 0 \\ & 2 & 0 & 0 \end{array}\right) \\ \oplus \\ \left(\begin{array}{cc|cc} 3 & 3 & 0 & 0 \\ & 2 & 0 & 0 \end{array}\right) \\ \oplus \\ \left(\begin{array}{cc|cc} 2 & 3 & 0 & 0 \\ & 2 & 0 & 0 \end{array}\right) \end{array} \to \left(\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ & 1 & 0 & 0 \end{array}\right) \to \mathcal{E}_3^{(2)\,*}|_X \to 0, \tag{B.39}$$

for $\mathcal{F}_{V'} = \mathcal{E}_3^{(1)\,*}$,

$$0 \to \left(\begin{array}{cc|cc} 5 & 5 & 0 & 0 \\ & 4 & 0 & 0 \end{array}\right) \to \begin{array}{c} \left(\begin{array}{cc|cc} 4 & 4 & 0 & 0 \\ & 3 & 0 & 0 \end{array}\right) \\ \oplus \\ \left(\begin{array}{cc|cc} 3 & 4 & 0 & 0 \\ & 3 & 0 & 0 \end{array}\right) \end{array} \to \begin{array}{c} \left(\begin{array}{cc|cc} 2 & 3 & 0 & 0 \\ & 2 & 0 & 0 \end{array}\right) \\ \oplus \\ \left(\begin{array}{cc|cc} 2 & 2 & 0 & 0 \\ & 2 & 0 & 0 \end{array}\right) \end{array} \to \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ & 1 & 0 & 0 \end{array}\right) \to \mathcal{E}_3^{(1)\,*}|_X \to 0, \tag{B.40}$$

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for $\mathcal{F}_{V'} = T^*G(2,4)$,

$$0 \to \left(\begin{array}{cc|cc} 4 & 5 & -1 & 0 \\ & 3 & 0 & 0 \end{array}\right) \to \begin{array}{c} \left(\begin{array}{cc|cc} 3 & 4 & -1 & 0 \\ & 2 & 0 & 0 \end{array}\right) \\ \oplus \\ \left(\begin{array}{cc|cc} 2 & 4 & -1 & 0 \\ & 2 & 0 & 0 \end{array}\right) \\ \oplus \\ \left(\begin{array}{cc|cc} 3 & 3 & -1 & 0 \\ & 2 & 0 & 0 \end{array}\right) \end{array} \to \begin{array}{c} \left(\begin{array}{cc|cc} 1 & 3 & -1 & 0 \\ & 1 & 0 & 0 \end{array}\right) \\ \oplus \\ \left(\begin{array}{cc|cc} 2 & 2 & -1 & 0 \\ & 1 & 0 & 0 \end{array}\right) \\ \oplus \\ \left(\begin{array}{cc|cc} 1 & 2 & -1 & 0 \\ & 1 & 0 & 0 \end{array}\right) \end{array} \to \left(\begin{array}{cc|cc} 0 & 1 & -1 & 0 \\ & 0 & 0 & 0 \end{array}\right) \to T^*G(2,4)|_X \to 0, \tag{B.41}$$

and for $\mathcal{F}_{V'} = T^*\mathbb{P}^2$,

$$0 \to \left(\begin{array}{cc|cc} 4 & 4 & 0 & 0 \\ & 4 & 0 & 0 \end{array}\right) \to \begin{array}{c} \left(\begin{array}{cc|cc} 3 & 3 & 0 & 0 \\ & 3 & -1 & 0 \end{array}\right) \\ \oplus \\ \left(\begin{array}{cc|cc} 2 & 3 & 0 & 0 \\ & 3 & -1 & 0 \end{array}\right) \end{array} \to \begin{array}{c} \left(\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ & 2 & -1 & 0 \end{array}\right) \\ \oplus \\ \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ & 2 & -1 & 0 \end{array}\right) \end{array} \to \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ & 1 & -1 & 0 \end{array}\right) \to T^*\mathbb{P}^2|_X \to 0. \tag{B.42}$$

From (B.39) one finds that

$$\begin{aligned} &\ker\big(d_0 :\ H^6(V', \wedge^3\mathcal{E}_3^* \otimes \mathcal{E}_3^{(2)\,*}) = \mathbb{C}^{60} \ \longrightarrow\ H^6(V', \wedge^2\mathcal{E}_3^* \otimes \mathcal{E}_3^{(2)\,*}) = \mathbb{C}^5\big) \\ &\qquad \Longrightarrow\ H^3(X, \mathcal{E}_3^{(2)\,*}|_X) = \mathbb{C}^{55}, \end{aligned} \tag{B.43}$$

and $H^i(X, \mathcal{E}_3^{(2)\,*}|_X)$, $i = 0, 1, 2$, are trivial. From (B.40) one finds that

$$\begin{aligned} &\ker\big(d_0 :\ H^6(V', \wedge^3\mathcal{E}_3^* \otimes \mathcal{E}_3^{(1)\,*}) = \mathbb{C}^{18} \ \longrightarrow\ H^6(V', \wedge^2\mathcal{E}_3^* \otimes \mathcal{E}_3^{(1)\,*}) = \mathbb{C}\big) \\ &\qquad \Longrightarrow\ H^3(X, \mathcal{E}_3^{(1)\,*}|_X) = \mathbb{C}^{17}, \end{aligned} \tag{B.44}$$

and $H^i(X, \mathcal{E}_3^{(1)\,*}|_X)$, $i = 0, 1, 2$, are trivial. Then one gets $H^3(X, \mathcal{E}_3^*|_X) = \mathbb{C}^{72}$ by (B.43) and (B.44). From (B.41) one finds that

$$\begin{aligned} H^1(V', T^*G(2,4)) = \mathbb{C} \ &\Longrightarrow\ H^1(X, T^*G(2,4)|_X) = \mathbb{C}, \\ H^6(V', \wedge^3\mathcal{E}_3^* \otimes T^*G(2,4)) = \mathbb{C}^{15} \ &\Longrightarrow\ H^3(X, T^*G(2,4)|_X) = \mathbb{C}^{15}, \end{aligned} \tag{B.45}$$

and $H^0(X, T^*G(2,4)|_X)$ and $H^2(X, T^*G(2,4)|_X)$ are trivial. From (B.42) one finds that

$$\begin{aligned} H^1(V', T^*\mathbb{P}^2) = \mathbb{C} \ &\Longrightarrow\ H^1(X, T^*\mathbb{P}^2|_X) = \mathbb{C}, \\ H^6(V', \wedge^3\mathcal{E}_3^* \otimes T^*\mathbb{P}^2) = \mathbb{C}^8 \ &\Longrightarrow\ H^3(X, T^*\mathbb{P}^2|_X) = \mathbb{C}^8, \end{aligned} \tag{B.46}$$

and $H^0(X, T^*\mathbb{P}^2|_X)$ and $H^2(X, T^*\mathbb{P}^2|_X)$ are trivial. Then one gets $H^1(X, T^*V'|_X) = \mathbb{C}^2$ and $H^3(X, T^*V'|_X) = \mathbb{C}^{23}$ by (B.45) and (B.46). Using these results, the exact sequence (B.17) yields the following exact sequences:

$$\begin{aligned} &0 \longrightarrow H^0(X, T^*X) \longrightarrow 0, \\ &0 \longrightarrow \mathbb{C}^2 \longrightarrow H^1(X, T^*X) \longrightarrow 0, \\ &0 \longrightarrow H^2(X, T^*X) \longrightarrow \mathbb{C}^{72} \xrightarrow{\ f\ } \mathbb{C}^{23} \longrightarrow H^3(X, T^*X) \longrightarrow 0. \end{aligned} \tag{B.47}$$

Then one obtains

$$H^{1,0}(X) = H^0(X, T^*X) = 0, \qquad H^{1,1}(X) = H^1(X, T^*X) = \mathbb{C}^2,$$
$$H^{1,2}(X) = H^2(X, T^*X) = \ker(f) = \mathbb{C}^{49}, \quad H^{1,3}(X) = H^3(X, T^*X) = \mathrm{coker}(f) = 0. \tag{B.48}$$

As a result, the Hodge diamond is obtained as

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 2 & & 0 & \\ 1 & & 49 & & 49 & & 1 \\ & 0 & & 2 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}.$$

C Genus-0 invariants of determinantal Calabi-Yau 4-folds

Following the techniques introduced in section 4.2, here we consider genus-0 invariants of the desingularized determinantal Calabi-Yau 4-folds clarified in appendix A.2 which are described by $\mathrm{U}(k) \times \mathrm{U}(\ell_p^\vee)$ PAX models with $k, \ell_p^\vee = 1, 2$, while ignoring the examples with the universal quotient bundle $\mathcal{Q}$ in $\mathcal{F}_p$. For the computation, we need to evaluate the classical intersection numbers (4.11) simply denoted by

$$H\sigma_1^2, \quad H\sigma_2, \quad H\sigma_1\tau_1, \quad H\tau_1^2, \quad H\tau_2. \tag{C.1}$$

We also compute their Hodge numbers by analysing the Koszul complex introduced in appendix B. Note that here the genus-0 invariants $n_{d_1,d_2}(H)$ in (4.7) are denoted as

$$\begin{aligned} &n_{d_1,d_2,11} = n_{d_1,d_2}(\sigma_1^2), \quad n_{d_1,d_2,\sigma} = n_{d_1,d_2}(\sigma_2), \quad n_{d_1,d_2,12} = n_{d_1,d_2}(\sigma_1\tau_1), \\ &n_{d_1,d_2,22} = n_{d_1,d_2}(\tau_1^2), \quad n_{d_1,d_2,\tau} = n_{d_1,d_2}(\tau_2). \end{aligned} \tag{C.2}$$

C.1 Sextic family

As a higher dimensional analogue of the quintic family discussed in section 4.3.1, we consider the determinantal Calabi-Yau 4-folds in (A.6) which can be described by $\mathrm{U}(1)\times\mathrm{U}(1)$ GLSMs. The "trivial" determinantal Calabi-Yau 4-fold with $\mathcal{F}_p = \mathcal{O}_V(6)$ on $V = \mathbb{P}^5$ corresponds to the sextic Calabi-Yau 4-fold X_1 with $n = 6$, $r = 1$, $d_1 = 6$ in section 3.2.1. The Hodge numbers are $(h^{1,1}, h^{2,1}, h^{2,2}, h^{3,1}) = (1, 0, 1752, 426)$. The classical intersection number (3.14) and the genus-0 invariants (3.16) of the sextic X_1 are given by $\kappa = 6$ and

$$\begin{aligned} &n_{1,11} = 60480, \ n_{2,11} = 440884080, \ n_{3,11} = 6255156277440, \\ &n_{4,11} = 117715791990353760, \ldots, \end{aligned} \tag{C.3}$$

respectively [30].

Several genus-0 invariants of the sextic family (A.6) are summarized in table 7, where one can check that $(h^{1,0}, h^{2,0}) = (0,0)$ and there is a relation originated from the extremal transition:

$$n_{d,11} = \sum_{d_2=0}^{N} n_{d,d_2,11}, \tag{C.4}$$

where N is a certain finite positive integer.

$\mathcal{F}_p = \mathcal{O}_V(1) \oplus \mathcal{O}_V(5)$: $(h^{1,1}, h^{2,1}, h^{2,2}, h^{3,1}) = (2, 0, 1452, 350)$, $(n_{d_1,d_2,22} = 0)$						
Intersection numbers		$\sigma_1^4 = 6$,	$\sigma_1^3\tau_1 = 5$,	$\sigma_1^2\tau_1^2 = 0$,	$\sigma_1\tau_1^3 = 0$,	$\tau_1^4 = 0$
$n_{d_1,d_2,11}$	$d_1 = 0$	1	2	3	4	5
$d_2 = 0$		11100	5974850	5337637100	5961261947000	7549696778037500
1	0	47800	139595300	341903160900	781526104500800	1722498037214056500
2	0	2300	288301400	2474705048600	12772788325116200	51691531760557694400
3	0	-900	10709800	3363595465000	51229393390313200	425107528698920155100
4	0	200	-5618400	103567454100	51958819718170400	1158355364337024993600
5	0	-20	2835300	-51911590000	1403818415592500	938149531037521616000
$n_{d_1,d_2,12}$	$d_1 = 0$	1	2	3	4	5
$d_2 = 0$		2875	1218500	951619125	969870120000	1146529444438125
1	25	43025	80799950	156102470525	304442819735350	596487343049391900
2	0	7075	268094350	1716513933050	7342810580729600	25898280425210696100
3	0	-3325	27921700	3182702667725	38694830186103150	274863504753902753625
4	0	850	-16827350	244126695475	49743335407652800	920672555667202043750
5	0	-100	9475875	-139333207500	3049558752331250	905924121779310315625

$\mathcal{F}_p = \mathcal{O}_V(2) \oplus \mathcal{O}_V(4)$: $(h^{1,1}, h^{2,1}, h^{2,2}, h^{3,1}) = (2, 0, 984, 233)$, $(n_{d_1,d_2,22} = 0)$						
Intersection numbers		$\sigma_1^4 = 6$,	$\sigma_1^3\tau_1 = 8$,	$\sigma_1^2\tau_1^2 = 0$,	$\sigma_1\tau_1^3 = 0$,	$\tau_1^4 = 0$
$n_{d_1,d_2,11}$	$d_1 = 0$	1	2	3	4	5
$d_2 = 0$		5152	933968	274818272	100238592192	41343866067168
1	0	30464	30631168	24983703040	18823860029184	13607365845297920
2	0	24384	148136832	344078200064	540875866571264	689210122091722112
3	0	512	196648704	1484112439552	5090782137162496	11687877699117056512
4	0	-32	63767008	2480102598912	20463045985953792	88125209117491672192
5	0	0	833024	1597722995712	39020950213890816	338009904921145655552
$n_{d_1,d_2,12}$	$d_1 = 0$	1	2	3	4	5
$d_2 = 0$		1280	184576	46965504	15535610112	5954616410880
1	64	32000	19441024	12147379712	7666954166848	4863492485707008
2	0	45696	155375232	264748238336	338696685934592	370700658349715200
3	0	1792	288667776	1555786807552	4265241324428224	8297783461923275008
4	0	-128	121767232	3295974933504	21444381835426304	77477798899226917376
5	0	0	2638208	2582769371136	49128078374461248	354147270418259729152

$\mathcal{F}_p = \mathcal{O}_V(3)^{\oplus 2}$: $(h^{1,1}, h^{2,1}, h^{2,2}, h^{3,1}) = (2, 0, 780, 182)$, $(n_{d_1,d_2,22} = 0)$						
Intersection numbers		$\sigma_1^4 = 6$,	$\sigma_1^3\tau_1 = 9$,	$\sigma_1^2\tau_1^2 = 0$,	$\sigma_1\tau_1^3 = 0$,	$\tau_1^4 = 0$
$n_{d_1,d_2,11}$	$d_1 = 0$	1	2	3	4	5
$d_2 = 0$		3996	528012	111620808	29176888824	8616413173572
1	0	26244	18834444	10994448492	5928552658692	3066843382569540
2	0	26244	107617896	174298692024	193364991313056	174792376622296872
3	0	3996	186923376	916489110132	2154915386009316	3451110870456005940
4	0	0	107617896	2025684267264	10761703969222224	31389410244093246936
5	0	0	18834444	2025684267264	27258432537609648	151496100969322358520

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$n_{d_1,d_2,12}$	$d_1=0$	1	2	3	4	5
$d_2=0$		1053	105624	19272978	4557793536	1248939462915
1	81	27945	12158991	5429580417	2449467003132	1110716561847951
2	0	50787	114975450	136362364218	122998311431838	95426230272412266
3	0	10935	280385064	978583930209	1836173061024606	2489535822420553203
4	0	0	207878238	2745855802998	11484979618590612	28070822227226686614
5	0	0	44344341	3331196998794	34978559566645782	161618974169879654460

$\mathcal{F}_p = \mathcal{O}_V(2)^{\oplus 3}$: $(h^{1,1}, h^{2,1}, h^{2,2}, h^{3,1}) = (2, 0, 600, 137)$

Intersection numbers: $\sigma_1^4 = 6, \quad \sigma_1^3\tau_1 = 12, \quad \sigma_1^2\tau_1^2 = 8, \quad \sigma_1\tau_1^3 = 0, \quad \tau_1^4 = 0$

$n_{d_1,d_2,11}$	$d_1=0$	1	2	3	4	5
$d_2=0$		1104	13464	196848	3102144	52343184
1	0	14016	1708224	120184704	6485946432	298466405568
2	0	30240	24792672	6198463872	882834675456	89849127844224
3	0	14016	107708352	89554908096	31167666411840	6550759785428544
4	0	1104	172438656	529075766016	444446786009856	187337558146915488
5	0	0	107708352	1473092769024	3125024856532800	2640802670799084864

$n_{d_1,d_2,12}$	$d_1=0$	1	2	3	4	5
$d_2=0$		384	3744	47232	670272	10462080
1	96	16128	1240896	68822016	3180759840	130766711040
2	0	60480	28308096	5320032000	629362595328	55970237060352
3	0	39936	168950976	102040729344	28860224852448	5218075942288896
4	0	4032	344877312	753418660608	505820047097088	181160699096012928
5	0	0	261882432	2521023722496	4224146727978144	3003266065628143872

$n_{d_1,d_2,22}$	$d_1=0$	1	2	3	4	5
$d_2=0$		0	0	0	0	0
1	96	7296	358080	14528256	528892320	17932977792
2	0	38592	13180992	1930140672	186374277120	13980674976768
3	0	31104	97081920	47789472384	11376521292384	1772136040869504
4	0	3648	226720032	411451649280	237591482788608	74578392862419072
5	0	0	190013376	1533694993920	2239140736880928	1413152060075199360

$\mathcal{F}_p = \mathcal{O}_V(1) \oplus \mathcal{O}_V(2) \oplus \mathcal{O}_V(3)$: $(h^{1,1}, h^{2,1}, h^{2,2}, h^{3,1}) = (2, 0, 732, 170)$

Intersection numbers: $\sigma_1^4 = 6, \quad \sigma_1^3\tau_1 = 11, \quad \sigma_1^2\tau_1^2 = 6, \quad \sigma_1\tau_1^3 = 0, \quad \tau_1^4 = 0$

$n_{d_1,d_2,11}$	$d_1=0$	1	2	3	4	5
$d_2=0$		1547	29197	664966	16655276	449773471
1	0	17415	3387141	390771675	34733397231	2631133285191
2	0	31234	40960891	16736425757	3941526901738	666261117547152
3	0	10138	143247860	196780670876	113318694822063	39594013933183682
4	0	147	172936880	924949591952	1299468913703502	913025985733713523
5	0	-1	72200686	1974502966996	7224038700165558	10275054838460143332

$n_{d_1,d_2,12}$	$d_1=0$	1	2	3	4	5
$d_2=0$		546	8022	159708	3637752	91579530
1	85	19550	2414660	220345310	16823027671	1141797701894
2	0	61679	45928018	14111157767	2763493715428	408754388583466
3	0	28503	221344804	220417729691	103132562736885	31012895322615811
4	0	607	341620521	1297053558298	1454510648416542	868094813403547075
5	0	-5	173699427	3333597269546	9614835008030966	11495407244589530600

$n_{d_1,d_2,22}$	$d_1=0$	1	2	3	4	5
$d_2=0$		0	0	0	0	0
1	66	7944	665916	45550248	2770233906	155976830808

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2	0	33774	19165296	4739973306	774617782512	98109047877348
3	0	18270	110465976	92373759582	37217080568574	9812878163478678
4	0	498	190515834	619593755556	610866205882020	325210132883359866
5	0	-6	104930694	1742691953364	4476709752274068	4833888549187725312

$\mathcal{F}_p = \mathcal{O}_V(1)^{\oplus 2} \oplus \mathcal{O}_V(4)$: $(h^{1,1}, h^{2,1}, h^{2,2}, h^{3,1}) = (2, 0, 1068, 254)$						
Intersection numbers		$\sigma_1^4 = 6, \quad \sigma_1^3\tau_1 = 9, \quad \sigma_1^2\tau_1^2 = 4, \quad \sigma_1\tau_1^3 = 0, \quad \tau_1^4 = 0$				
$n_{d_1,d_2,11}$	$d_1 = 0$	1	2	3	4	5
$d_2 = 0$		2796	111420	5415876	297906744	17836490652
1	0	25368	11074512	2923919928	597405875232	103952142270864
2	0	31452	96756426	92450088216	50890408972608	20094224706004404
3	0	1020	215383416	746257722660	1032988712010672	857190260480341476
4	0	-168	115603488	2184801219792	7902484863731640	13590303106004642712
5	0	12	2269284	2399750411904	27079947685057788	100137122291641121868
$n_{d_1,d_2,12}$	$d_1 = 0$	1	2	3	4	5
$d_2 = 0$		960	30096	1297728	65879904	3714521280
1	57	27438	7644177	1605192702	282936273240	44282085901044
2	0	3657	104754642	75519972306	34661964976188	12004979836835943
3	0	-681	320554332	807765491547	910677818722380	651630952247594127
4	0	57	222052140	2956519017480	8551880283800160	12515007949041304332
5	0	0	7374009	3919508517492	34811597561726766	108347819585126752047
$n_{d_1,d_2,22}$	$d_1 = 0$	1	2	3	4	5
$d_2 = 0$		0	0	0	0	0
1	36	10104	2000580	323373624	46020698976	6018914291280
2	0	28260	39402408	23483405064	9172938062064	2759901496567260
3	0	2340	140674032	304708034988	301404502100016	191986430652018684
4	0	-420	108023472	1249191338400	3231021483376128	4278942916449123696
5	0	36	4447332	1793863512336	14384155887545592	40973480285079165564

$\mathcal{F}_p = \mathcal{O}_V(1)^{\oplus 2} \oplus \mathcal{O}_V(2)^{\oplus 2}$: $(h^{1,1}, h^{2,1}, h^{2,2}, h^{3,1}) = (2, 0, 588, 134)$						
Intersection numbers		$\sigma_1^4 = 6, \quad \sigma_1^3\tau_1 = 13, \quad \sigma_1^2\tau_1^2 = 12, \quad \sigma_1\tau_1^3 = 4, \quad \tau_1^4 = 0$				
$n_{d_1,d_2,11}$	$d_1 = 0$	1	2	3	4	5
$d_2 = 0$		624	1512	2448	2352	3696
1	0	10552	560168	12419664	167237208	1611465704
2	0	29420	12216604	1270882648	63182183032	1933845747256
3	0	17892	74261296	28590861496	4065328815272	310191664249228
4	0	1988	162756004	245615224064	91643735242360	15636111812478864
5	0	4	139994444	965140012064	959192743343064	351280339787531780
$n_{d_1,d_2,12}$	$d_1 = 0$	1	2	3	4	5
$d_2 = 0$		256	504	768	784	1280
1	97	12412	431558	7712632	90771993	798269252
2	0	59597	14304660	1139364456	47795706040	1296194947220
3	0	51377	118235744	33425512482	3910275895911	259643765174029
4	0	7379	329186320	355897081724	107040862884600	15662234678584878
5	0	19	343523755	1673793068984	1321513374907908	410052719920052585
$n_{d_1,d_2,22}$	$d_1 = 0$	1	2	3	4	5
$d_2 = 0$		0	0	0	0	0
1	132	7696	167672	2165664	19882116	143012720
2	0	51428	8977216	553197376	18798577216	427186381936
3	0	53076	91577920	21040910824	2061819081980	117330402886948

4	0	8732	290650960	261518705648	67487950758592	8623613017337400
5	0	28	332988380	1368920496992	941519514353136	258784559873434676

$\mathcal{F}_p = \mathcal{O}_V(1)^{\oplus 3} \oplus \mathcal{O}_V(3)$: $(h^{1,1}, h^{2,1}, h^{2,2}, h^{3,1}) = (2, 0, 744, 173)$						
Intersection numbers		$\sigma_1^4 = 6$, $\sigma_1^3\tau_1 = 12$, $\sigma_1^2\tau_1^2 = 10$, $\sigma_1\tau_1^3 = 3$, $\tau_1^4 = 0$				
$n_{d_1,d_2,11}$	$d_1 = 0$	1	2	3	4	5
$d_2 = 0$		876	2754	2340	4506	6384
1	0	13224	1111680	40000284	854694672	12548849748
2	0	31692	20814618	3502913724	285343689342	14296324624164
3	0	14388	105558792	66167398224	15392391723972	1933287656228904
4	0	312	182631774	469032117948	288248187932658	80990413510330656
5	0	-12	111190416	1474197621264	2474334577233804	1501801964420459808
$n_{d_1,d_2,12}$	$d_1 = 0$	1	2	3	4	5
$d_2 = 0$		363	726	702	1452	1815
1	84	15267	846654	24454455	451560702	5971621263
2	0	63390	24022410	3100366203	213030667314	9432564106950
3	0	40614	165803226	76310133777	14614556335185	1597429012014378
4	0	1383	364788126	670587309870	332222425569714	80079498824062290
5	0	-57	269640918	2524188530034	3364141372570932	1730133151583505285
$n_{d_1,d_2,22}$	$d_1 = 0$	1	2	3	4	5
$d_2 = 0$		0	0	0	0	0
1	102	8976	325302	6898368	99988848	1084024620
2	0	51372	14396598	1462589520	82497883614	3089416732524
3	0	38808	121109886	45893323332	7447857934212	704572082465418
4	0	1716	301039440	466225271184	200138738462880	42499979609169498
5	0	-72	242665410	1939341503028	2272289621718792	1043316736192230672

$\mathcal{F}_p = \mathcal{O}_V(1)^{\oplus 4} \oplus \mathcal{O}_V(2)$: $(h^{1,1}, h^{2,1}, h^{2,2}, h^{3,1}) = (2, 0, 552, 125)$						
Intersection numbers		$\sigma_1^4 = 6$, $\sigma_1^3\tau_1 = 14$, $\sigma_1^2\tau_1^2 = 16$, $\sigma_1\tau_1^3 = 9$, $\tau_1^4 = 2$				
$n_{d_1,d_2,11}$	$d_1 = 0$	1	2	3	4	5
$d_2 = 0$		360	110	0	0	0
1	0	7780	181660	1182060	3226100	4267500
2	0	27260	5703230	247915040	4199645570	35730530600
3	0	21540	47022520	8368536780	487011519380	13321298891500
4	0	3500	137972490	101219111200	16761773920870	1158268337627600
5	0	40	161154360	546681379380	252762644451260	40097079604396400
$n_{d_1,d_2,12}$	$d_1 = 0$	1	2	3	4	5
$d_2 = 0$		165	40	0	0	0
1	100	9345	147740	785690	1894775	2320375
2	0	55990	6844020	231492010	3350412730	25503215825
3	0	62310	76006130	10030946845	485539930370	11668962937750
4	0	13125	282277260	149233093550	20076931277380	1199570758766725
5	0	185	399094290	960929261945	354879437832140	48004138643512850
$n_{d_1,d_2,22}$	$d_1 = 0$	1	2	3	4	5
$d_2 = 0$		0	0	0	0	0
1	170	6860	65700	250120	468350	468350
2	0	58020	5042710	129717540	1502417400	9512031410
3	0	77400	70019970	7403781260	296605169360	6050035785700
4	0	18700	298541880	129860240700	14833108148590	766859594261640
5	0	300	465232050	936876897400	298655994540650	35488204982996150

$\mathcal{F}_p = \mathcal{O}_V(1)^{\oplus 6}$: $(h^{1,1}, h^{2,1}, h^{2,2}, h^{3,1}) = (2, 0, 492, 110)$, $(n_{d_1,d_2,22} = n_{d_2,d_1,11})$ [11]

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Intersection numbers		$\sigma_1^4=6,\quad \sigma_1^3\tau_1=15,\quad \sigma_1^2\tau_1^2=20,\quad \sigma_1\tau_1^3=15,\quad \tau_1^4=6$				
$n_{d_1,d_2,11}$	$d_1=0$	1	2	3	4	5
$d_2=0$		210	0	0	0	0
1	0	5670	59430	100170	34650	1680
2	0	24360	2579640	47382930	264433680	546221760
3	0	24360	28015260	2324403900	55841697870	539959428960
4	0	5670	107096220	38404166850	2848564316640	80315543697900
5	0	210	165382980	277070715810	60035324018880	4163431890254700
$n_{d_1,d_2,12}$	$d_1=0$	1	2	3	4	5
$d_2=0$		105	0	0	0	0
1	105	6930	50715	71085	21420	945
2	0	50715	3166800	45928155	221593050	413457450
3	0	71085	45928155	2851172100	57546197940	493317415605
4	0	21420	221593050	57546197940	3492450469200	85788539294850
5	0	945	413457450	493317415605	85788539294850	5102793274479600

Table 7. Genus-0 invariants of determinantal 4-folds in (A.6) with $V=\mathbb{P}^5$.

C.2 Determinantal Calabi-Yau 4-folds in (A.8)

Finally, we consider the determinantal Calabi-Yau 4-folds with $p\neq 2$ in (A.8) which are described by $\mathrm{U}(2)\times\mathrm{U}(2)$ GLSMs, while ignoring the examples with the universal quotient bundle $\mathcal{Q}$ in $\mathcal{F}_p$. We summarized the genus-0 invariants of (C.2) in table 8, where one can check that $(h^{1,0},h^{2,0})=(0,0)$. For the Calabi-Yau 4-folds with $\mathcal{F}_p=\mathcal{S}^*\oplus\mathcal{O}_V(1)^{\oplus 2}$, $(\mathcal{S}^*)^{\oplus 2}\oplus\mathcal{O}_V(1)$, $(\mathcal{S}^*)^{\oplus 3}$, just due to a technical complexity, instead of the Hodge numbers we give the χ_y-genera $\chi_i=\sum_{j=0}^4(-1)^j h^{i,j}$, $i=0,1,2$ obtained by the formula (B.18).

Note that the determinantal 4-fold with $\mathcal{F}_p=\mathrm{Sym}^2\,\mathcal{S}^*$ in (A.8) is an irreducible holomorphic symplectic variety with $(h^{1,0},h^{2,0})=(0,1)$, and all the genus-0 invariants vanished. This property is known to be a general phenomenon for irreducible holomorphic symplectic varieties (see e.g. [62]).

$\mathcal{F}_p=\mathcal{O}_V(1)^{\oplus 3}$: $(h^{1,1},h^{2,1},h^{2,2},h^{3,1})=(2,0,384,83)$, $(n_{d_1,d_2,\tau}=0)$							
Intersection numbers		$\sigma_1^4=84,\quad \sigma_1^2\sigma_2=54,\quad \sigma_1^3\tau_1=42,\quad \sigma_1^2\tau_1^2=14,\quad \sigma_1^2\tau_2=0,\quad \sigma_2^2=36,$					
$\sigma_1\sigma_2\tau_1=27,\quad \sigma_2\tau_1^2=9,\quad \sigma_2\tau_2=0,\quad \sigma_1\tau_1^3=0,\quad \sigma_1\tau_1\tau_2=0,\quad \tau_1^4=0,\quad \tau_1^2\tau_2=0,\quad \tau_2^2=0$							
$n_{d_1,d_2,11}$	$d_1=0$	1	2	3	4	5	6
$d_2=0$		966	6258	40194	313992	2465694	20471724
1	0	966	79464	2850624	73342920	1577557254	30264388560
2	0	0	6258	2850624	353216472	23351152860	1075419836442
3	0	0	0	40194	73342920	23351152860	3280923722160
4	0	0	0	0	313992	1577557254	1075419836442
5	0	0	0	0	0	2465694	30264388560
$n_{d_1,d_2,\sigma}$	$d_1=0$	1	2	3	4	5	6
$d_2=0$		639	3987	25857	201888	1584999	13160502
1	0	639	51804	1846260	47378196	1017817191	19510365672
2	0	0	3987	1846260	229064418	15125263182	695710713879
3	0	0	0	25857	47378196	15125263182	2125753214616
4	0	0	0	0	201888	1017817191	695710713879
5	0	0	0	0	0	1584999	19510365672

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$n_{d_1,d_2,12}$	$d_1=0$	1	2	3	4	5	6
$d_2=0$		210	1218	7182	50400	369810	2894220
1	0	756	39732	1136940	25336080	491193528	8693374452
2	0	0	5040	1713684	176608236	10236028824	426294634584
3	0	0	0	33012	48006840	13115124036	1640461861080
4	0	0	0	0	263592	1086363726	649125201858
5	0	0	0	0	0	2095884	21571014108

$n_{d_1,d_2,22}$	$d_1=0$	1	2	3	4	5	6
$d_2=0$		0	0	0	0	0	0
1	0	546	13692	260484	4432344	69555990	1036360500
2	0	0	3822	837228	64074948	2961523404	102768656970
3	0	0	0	25830	27103104	5840618616	604275131880
4	0	0	0	0	213192	664726188	325599224244
5	0	0	0	0	0	1726074	13914000156

$\mathcal{F}_p = \mathcal{S}^* \oplus \mathcal{O}_V(2)$: $(h^{1,1}, h^{2,1}, h^{2,2}, h^{3,1}) = (2, 0, 636, 146)$, $(n_{d_1,d_2,\tau} = 0)$

Intersection numbers: $\sigma_1^4 = 48$, $\sigma_1^2\sigma_2 = 31$, $\sigma_1^3\tau_1 = 24$, $\sigma_1^2\tau_1^2 = 8$, $\sigma_1^2\tau_2 = 0$, $\sigma_2^2 = 21$, $\sigma_1\sigma_2\tau_1 = 14$, $\sigma_2\tau_1^2 = 4$, $\sigma_2\tau_2 = 0$, $\sigma_1\tau_1^3 = 0$, $\sigma_1\tau_1\tau_2 = 0$, $\tau_1^4 = 0$, $\tau_1^2\tau_2 = 0$, $\tau_2^2 = 0$

$n_{d_1,d_2,11}$	$d_1=0$	1	2	3	4	5	6
$d_2=0$		1536	17280	244224	3772608	62805504	1099018368
1	0	1536	288768	23427072	1353129984	64724846592	2742129192960
2	0	0	17280	23427072	7040804352	1086348288000	114929521132032
3	0	0	0	244224	1353129984	1086348288000	366458865408000
4	0	0	0	0	3772608	64724846592	114929521132032
5	0	0	0	0	0	62805504	2742129192960

$n_{d_1,d_2,\sigma}$	$d_1=0$	1	2	3	4	5	6
$d_2=0$		864	9576	133920	2053872	34018272	592976376
1	0	1152	186624	14379264	808399872	38013725952	1591301187072
2	0	0	13248	16001280	4557463488	680963318784	70502641947072
3	0	0	0	190080	952611840	726901318656	237300376059648
4	0	0	0	0	2965896	46471290624	78693300038592
5	0	0	0	0	0	49723776	1996675868160

$n_{d_1,d_2,12}$	$d_1=0$	1	2	3	4	5	6
$d_2=0$		384	3744	47232	670272	10462080	173868768
1	0	1152	144384	9550848	484282368	21085670400	830565021696
2	0	0	13536	13876224	3520402176	481923477504	46543883110656
3	0	0	0	196992	868847616	604424810496	183229432704000
4	0	0	0	0	3102336	43639176192	68385638021376
5	0	0	0	0	0	52343424	1911564171264

$n_{d_1,d_2,22}$	$d_1=0$	1	2	3	4	5	6
$d_2=0$		0	0	0	0	0	0
1	0	768	49152	2224128	86114304	3039080448	100762681344
2	0	0	9792	6549504	1255686144	138466443264	11215284827136
3	0	0	0	149760	470679552	260967776256	66108681437184
4	0	0	0	0	2432064	25592586240	33057039737856
5	0	0	0	0	0	41881344	1181761830912

$\mathcal{F}_p = \mathcal{S}^* \oplus \mathcal{O}_V(1)^{\oplus 2}$: $(\chi_0, \chi_1, \chi_2) = (2, -80, 364)$

Intersection numbers: $\sigma_1^4 = 86$, $\sigma_1^2\sigma_2 = 55$, $\sigma_1^3\tau_1 = 66$, $\sigma_1^2\tau_1^2 = 42$, $\sigma_1^2\tau_2 = 17$, $\sigma_2^2 = 37$, $\sigma_1\sigma_2\tau_1 = 41$, $\sigma_2\tau_1^2 = 25$, $\sigma_2\tau_2 = 10$, $\sigma_1\tau_1^3 = 18$, $\sigma_1\tau_1\tau_2 = 9$, $\tau_1^4 = 4$, $\tau_1^2\tau_2 = 2$, $\tau_2^2 = 2$

$n_{d_1,d_2,11}$	$d_1=0$	1	2	3	4	5	6

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$d_2 = 0$		496	244	0	0	0	0
1	0	1312	38880	238740	605136	785832	546816
2	0	44	41500	3158040	65572008	586267944	2862291270
3	0	0	760	1641368	253291560	11296776072	221197911448
4	0	0	0	23284	88370324	21272865380	1596400086708
5	0	0	0	-592	863776	5440233652	1834451480648
$n_{d_1,d_2,\sigma}$	$d_1 = 0$	1	2	3	4	5	6
$d_2 = 0$		292	140	0	0	0	0
1	0	880	24204	144470	361496	466204	323180
2	0	34	27450	2002612	40636882	358061288	1731003173
3	0	0	556	1087240	162344808	7095866952	137011548860
4	0	0	0	16830	58584756	13723707354	1011627956132
5	0	0	0	-440	622184	3609161614	1188798184500
$n_{d_1,d_2,12}$	$d_1 = 0$	1	2	3	4	5	6
$d_2 = 0$		168	72	0	0	0	0
1	0	1176	23688	119628	271752	328080	216216
2	0	72	37524	2229240	39561432	317016480	1428444174
3	0	0	1056	1492992	191488464	7501134312	133180541856
4	0	0	0	31128	80807544	16742371056	1125010922424
5	0	0	0	-864	1140240	4994099532	1482866868960
$n_{d_1,d_2,22}$	$d_1 = 0$	1	2	3	4	5	6
$d_2 = 0$		0	0	0	0	0	0
1	0	1040	10516	37316	66648	66648	37316
2	0	112	31924	1296048	17835110	117630928	452438518
3	0	0	1356	1271728	125327752	4009228688	60469656020
4	0	0	0	37768	69214312	11674274648	663812607488
5	0	0	0	-1184	1367852	4298534956	1076178498476
$n_{d_1,d_2,\tau}$	$d_1 = 0$	1	2	3	4	5	6
$d_2 = 0$		0	0	0	0	0	0
1	0	320	4618	18418	34404	34404	18418
2	0	28	10858	522792	7879519	55031576	219730439
3	0	0	382	432616	47989060	1663887680	26568226946
4	0	0	0	11040	23512884	4349537480	265076269952
5	0	0	0	-320	402214	1457274582	393989709022

$\mathcal{F}_p = (\mathcal{S}^*)^{\oplus 2} \oplus \mathcal{O}_V(1)$: $(\chi_0, \chi_1, \chi_2) = (2, -62, 292)$

Intersection numbers: $\sigma_1^4 = 92$, $\sigma_1^2\sigma_2 = 58$, $\sigma_1^3\tau_1 = 92$, $\sigma_1^2\tau_1^2 = 80$, $\sigma_1^2\tau_2 = 19$, $\sigma_2^2 = 40$, $\sigma_1\sigma_2\tau_1 = 56$, $\sigma_2\tau_1^2 = 46$, $\sigma_2\tau_2 = 22$, $\sigma_1\tau_1^3 = 56$, $\sigma_1\tau_1\tau_2 = 32$, $\tau_1^4 = 32$, $\tau_1^2\tau_2 = 19$, $\tau_2^2 = 13$

$n_{d_1,d_2,11}$	$d_1 = 0$	1	2	3	4	5	6
$d_2 = 0$		180	0	0	0	0	0
1	0	1280	11520	0	0	0	0
2	0	180	39420	725760	1285920	110180	-36660
3	0	0	11520	1981440	54604800	288737280	294952960
4	0	0	0	725760	127668480	4632572700	44638440480
5	0	0	0	0	54604800	9651020800	425483704320
$n_{d_1,d_2,\sigma}$	$d_1 = 0$	1	2	3	4	5	6
$d_2 = 0$		90	0	0	0	0	0
1	0	800	6560	0	0	0	0
2	0	130	24730	428000	729020	59850	-19710
3	0	0	7840	1244480	32729280	167506560	166957440

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4	0	0	0	482400	80204960	2803401590	26273744220
5	0	0	0	0	35821760	6063510400	259122088640
$n_{d_1,d_2,12}$	$d_1=0$	1	2	3	4	5	6
$d_2=0$		60	0	0	0	0	0
1	0	1280	7680	0	0	0	0
2	0	300	39420	562560	846120	58940	-18660
3	0	0	15360	1981440	45327360	209679360	192616960
4	0	0	0	888960	127668480	4000335300	34440454440
5	0	0	0	0	63882240	9651020800	376969835520
$n_{d_1,d_2,22}$	$d_1=0$	1	2	3	4	5	6
$d_2=0$		0	0	0	0	0	0
1	0	1280	3840	0	0	0	0
2	0	480	37200	360960	439440	19040	-5520
3	0	0	19200	1835520	32509440	125291520	98252800
4	0	0	0	1013760	117776640	3046100640	22457270160
5	0	0	0	0	69619200	8892083200	297725928960
$n_{d_1,d_2,\tau}$	$d_1=0$	1	2	3	4	5	6
$d_2=0$		0	0	0	0	0	0
1	0	560	2480	0	0	0	0
2	0	160	17200	207920	270080	14280	-4140
3	0	0	7600	863840	17630880	73506240	60984000
4	0	0	0	425520	55648880	1598550680	12693468720
5	0	0	0	0	30000800	4206510400	153166332320

$\mathcal{F}_p = (\mathcal{S}^*)^{\oplus 3}$: $(\chi_0, \chi_1, \chi_2) = (2, -22, 132)$, $(n_{d_1,d_2,22} = n_{d_2,d_1,11},\ n_{d_1,d_2,\tau} = n_{d_2,d_1,\sigma})$

Intersection numbers: $\sigma_1^4 = 102$, $\sigma_1^2\sigma_2 = 63$, $\sigma_1^3\tau_1 = 120$, $\sigma_1^2\tau_1^2 = 128$, $\sigma_1^2\tau_2 = 72$, $\sigma_2^2 = 45$, $\sigma_1\sigma_2\tau_1 = 72$, $\sigma_2\tau_1^2 = 72$, $\sigma_2\tau_2 = 36$, $\sigma_1\tau_1^3 = 120$, $\sigma_1\tau_1\tau_2 = 72$, $\tau_1^4 = 102$, $\tau_1^2\tau_2 = 63$, $\tau_2^2 = 45$

$n_{d_1,d_2,11}$	$d_1=0$	1	2	3	4	5	6
$d_2=0$		0	0	0	0	0	0
1	0	960	1200	0	0	0	0
2	0	420	20160	42300	600	0	0
3	0	0	22800	668160	1867200	197520	0
4	0	0	210	1206540	28032600	94238940	29975670
5	0	0	0	91440	67122240	1368583200	5269901040
$n_{d_1,d_2,\sigma}$	$d_1=0$	1	2	3	4	5	6
$d_2=0$		0	0	0	0	0	0
1	0	540	540	0	0	0	0
2	0	270	11340	21330	270	0	0
3	0	0	14580	382320	984960	95580	0
4	0	0	135	756270	16113600	50752710	15091785
5	0	0	0	58860	41366160	787977720	2873723940
$n_{d_1,d_2,12}$	$d_1=0$	1	2	3	4	5	6
$d_2=0$		0	0	0	0	0	0
1	0	960	720	0	0	0	0
2	0	720	20160	31920	360	0	0
3	0	0	31920	679680	1547760	137280	0
4	0	0	360	1547760	28646400	81883440	22591320
5	0	0	0	137280	81883440	1400849280	4715787120

Table 8. Genus-0 invariants of determinantal 4-folds in (A.8) with $V = G(2,6)$.

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