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Tri-direct CP in the Littlest Seesaw playground

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Abstract: We discuss spontaneously broken CP symmetry in two right-handed neutrino models based on the idea of having a *different residual flavour symmetry*, together with a *different residual CP symmetry*, associated with each of the two right-handed neutrinos. The charged lepton sector also has a *different residual flavour symmetry*. In such a *tri-direct CP approach*, we show that the combination of the three residual flavour and two residual CP symmetries provides a new way of fixing the parameters. To illustrate the approach, we revisit the Littlest Seesaw (LSS) model based on S_4 and then propose new variants which have not so far appeared in the literature, with different predictions for each variant. We analyse numerically the predictions of the new variants, and then propose an explicit model which can realise one of the successful benchmark points, based on the atmospheric flavon vacuum alignment $(1,\omega^2,\omega)$ and the solar flavon vacuum alignment $(1,-7/2,-7/2)$.

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Contents

1 Introduction

Following the discovery of neutrino mass and mixing, we are now firmly in the precision era of measurements. In the standard parametrisation of the lepton mixing matrix [1], all the three lepton mixing angles θ_{12}, θ_{13} and θ_{23} and the mass squared differences $\Delta m^2_{21} \equiv m^2_2 - m^2_1$ and $\Delta m^2_{31} \equiv m^2_3 - m^2_1$ has been precisely measured in a large number of neutrino oscillation experiments. At present the 3σ ranges of these mixing parameters are determined to be [2–4]

$$\begin{gathered} 0.272 \le \sin^2\theta_{12} \le 0.346, \quad 0.01981 \le \sin^2\theta_{13} \le 0.02436, \quad 0.418 \le \sin^2\theta_{23} \le 0.613, \\ 6.80\times 10^{-5}\mathrm{eV}^2 \le \Delta m^2_{21} \le 8.02\times 10^{-5}\mathrm{eV}^2, \quad 2.399\times 10^{-3}\mathrm{eV}^2 \le \Delta m^2_{31} \le 2.593\times 10^{-3}\mathrm{eV}^2\,, \end{gathered} \tag{1.1}$$

where these results are as quoted in [4] for normal ordering (NO) neutrino mass spectrum, and similar results are obtained for inverted ordering (IO) spectrum except that the sign of Δm^2_{31} is reversed.

Non-Abelian discrete finite family symmetry groups G_f have been widely used to explain the lepton mixing angles as well as CP violating phases, see refs. [5–10] for reviews. One of the most successful and popular model independent approaches is to impose a discrete family symmetry G_f together with a non-commuting CP symmetry H_{CP}. In the *semi-direct CP approach*, the $G_f \rtimes H_{\mathrm{CP}}$ symmetry is subsequently spontaneously broken, leaving residual symmetries $G_\nu \rtimes H^\nu_{\rm CP}$ in the neutrino sector and $G_l \rtimes H^l_{\rm CP}$ in the charged lepton sector, leading to mixing angle and CP phase predictions. In the present paper we shall generalise the *semi-direct CP approach* to a *tri-direct CP approach*, based on the two right-handed neutrino seesaw mechanism, as we now discuss.

The most appealing possibility for the origin of neutrino mass seems to be the seesaw mechanism which, in its original formulation, involves heavy right-handed Majorana neutrinos [11–16]. The most minimal version of the seesaw mechanism involves one [17] or two right-handed neutrinos [18]. In order to reduce the number of free parameters still further to the smallest number possible, and hence increase predictivity, various approaches to the two right-handed neutrino seesaw model have been suggested, such as postulating one [19] or two [20] texture zeroes, however such two texture zero models are now phenomenologically excluded [21] for the case of a normal neutrino mass hierarchy.

The minimal successful seesaw scheme with normal hierarchy is called the Littlest Seesaw (LSS) model [22–24]. The LSS model corresponds to a two right-handed neutrino models with a particularly simple pattern of Dirac mass matrix elements in the basis where both the charged lepton mass matrix and the two-right-handed neutrino mass matrix are diagonal. The Dirac mass matrix involves only one texture zero, but the number of parameters is reduced dramatically since each column of this matrix is controlled by a single parameter. In practice this is achieved by introducing a Non-Abelian discrete family symmetry, which is spontaneously broken by flavon fields with particular vacuum alignments governed by remnant subgroups of the family symmetry. This leads to a highly constrained model which is remarkably consistent with current data, but which can be tested in forthcoming neutrino experiments [25]. The LSS approach may also be incorporated into grand unified models [26–29].

Originally the LSS model was formulated in the *indirect approach* based on a family symmetry to give the required vacuum alignments, but without any residual symmetry in the neutrino or charged lepton sector [22]. Later it was realised that it preserves a *different residual flavour symmetry for each flavon*, in the diagonal mass basis of two right-handed neutrinos, leading to a highly predictive set of possible alignments [23]. Most recently it was understood that the LSS model may arise from a *semi-direct symmetry* approach corresponding to a *different residual flavour symmetry for each charge sector*, where a particular residual flavour symmetry may be assumed in each of the neutrino and charged lepton sectors. To be precise, in the *semi-direct symmetry* approach, it was shown that there is an SU subgroup of S_4 in the neutrino sector and the T subgroup of S_4 in the charged lepton sector, leading to a constrained form of TM1 mixing [24] in which the first column of the tri-bimaximal mixing matrix is preserved, but with the reactor angle and CP phases fixed by the same two parameters which fix the neutrino masses.

The LSS model is a general and predictive framework for explaining neutrino masses and lepton mixing, and it is not confined to TM1. For instance, the golden LSS is an another viable class of LSS models [30], the flavor symmetry group is A_5 and it is spontaneously broken to different residual subgroups in the charged lepton, atmospheric neutrino and solar neutrino sectors. The golden LSS predicts the lepton mixing is of GR1 form where the first column of the golden ratio mixing matrix is preserved [30]. In both the original LSS

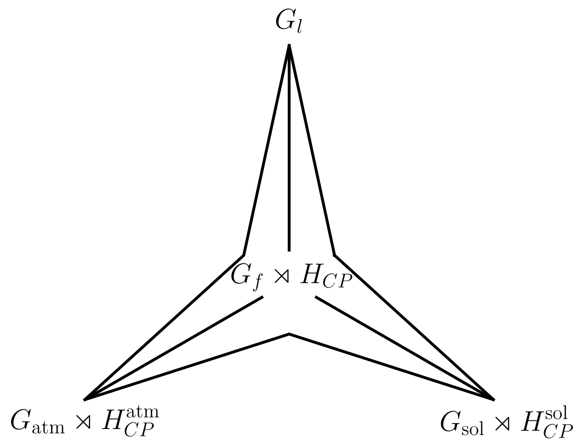

Figure 1. A sketch of the *tri-direct CP approach* for two right-handed neutrino models, where the high energy family and CP symmetry $G_f \rtimes H_{\rm CP}$ is spontaneously broken down to $G_{\rm atm} \rtimes H_{\rm CP}^{\rm atm}$ in the sector of one of the right-handed neutrinos, and $G_{\rm sol} \rtimes H_{\rm CP}^{\rm sol}$ in the sector of the other right-handed neutrino, with the charged lepton sector having a *different residual flavour symmetry* G_l.

and golden LSS models, it was always assumed that there is a high energy CP symmetry which is completely broken in each of the sectors, with no residual CP symmetry.

In this paper we propose a new *tri-direct CP approach* for two right-handed neutrino models based on the idea of spontaneously broken family and CP symmetry, leaving a *different residual flavour symmetry*, together with a *different residual CP symmetry*, in each of the two right-handed neutrino sectors. In other words, the high energy family and CP symmetry $G_f \rtimes H_{\rm CP}$ is spontaneously broken down to $G_{\rm atm} \rtimes H_{\rm CP}^{\rm atm}$ in the sector of one of the right-handed neutrinos, and $G_{\rm sol} \rtimes H_{\rm CP}^{\rm sol}$ in the sector of the other right-handed neutrino, with the charged lepton sector having a *different residual flavour symmetry* G_l, as schematically illustrated in figure 1. The tri-direct CP approach is a hybrid of the direct and indirect approaches. The common residual symmetry of the neutrino sector in the direct model is splitted into two branches: the residual symmetries associated with the atmospheric and solar neutrinos. In comparison with the indirect model, the alignments associated with each right-handed neutrino are enforced by residual symmetry. In such a *tri-direct CP approach* the combination of the three residual symmetries provides a new way of fixing the parameters. To illustrate the approach, we revisit the Littlest Seesaw (LSS) model based on S_4 and show that the *tri-direct CP approach* uniquely fixes some parameters of the model.[1] Following the *tri-direct CP approach*, we also propose new variants of the LSS model which have not so far appeared in the literature, with different predictions for each variant. We analyse numerically the predictions of the new variants, and then propose an explicit model which can realise one of the successful benchmark points, based on the atmospheric flavon vacuum alignment $(1, \omega^2, \omega)$ and the solar flavon vacuum alignment $(1, -7/2, -7/2)$.

[1]This is similar to having separate residual symmetries for each right-handed neutrino arising from S_4, as in [23], but here we also impose separate residual CP symmetries.

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The layout of this paper is as follows. In section 2 we propose the tri-direct CP approach for two right-handed neutrino models. In section 3 we apply the tri-direct CP approach to the Littlest Seesaw model and see that it reproduces the usual neutrino mass matrices arising from uniquely fixed vacuum alignments. In section 4 we show how new variants of the Littlest Seesaw emerge from the tri-direct CP approach, and we perform a comprehensive numerical analysis of a selection of benchmark points within the LSS variants arising from S_4, in order to determine their viability and predictions. In section 5 the tri-direct CP approach is extended to three right-handed neutrino models. In section 6 we propose an explicit model which can realise one of the successful benchmark points, based on the atmospheric flavon vacuum alignment $(1, \omega^2, \omega)$ and the solar flavon vacuum alignment $(1, -7/2, -7/2)$. Section 7 concludes the paper. The appendix A describes the diagonalization of a general subdiagonal neutrino mass matrix, note that the neutrino mass matrix predicted in the tri-direct CP approach can always be reduced to a subdiagonal one by performing a unitary transformation.

2 The tri-direct CP approach

In a two right-handed neutrino model, the light neutrino masses are generated through the seesaw mechanism, and only two right-handed neutrinos are introduced, denoted here as N^c_{atm} and N^c_{sol}. In the right-handed neutrino diagonal basis, the most general Lagrangian can be written as

$$\begin{aligned}\mathcal{L} = &-y_l L\phi_l E^c - y_{\text{atm}} L\phi_{\text{atm}} N^c_{\text{atm}} - y_{\text{sol}} L\phi_{\text{sol}} N^c_{\text{sol}} \\ &- \frac{1}{2} x_{\text{atm}} \xi_{\text{atm}} N^c_{\text{atm}} N^c_{\text{atm}} - \frac{1}{2} x_{\text{sol}} \xi_{\text{sol}} N^c_{\text{sol}} N^c_{\text{sol}} + \text{h.c.}\,,\end{aligned} \tag{2.1}$$

where we use a two-component notation for the fermion fields to keep the formula compact. The fields L and $E^c \equiv (e^c, \mu^c, \tau^c)^T$ stand for the left-handed lepton doublets and the right-handed charged leptons respectively, the flavons ϕ_l, ϕ_{sol} and ϕ_{atm} can be either Higgs fields or combinations of the electroweak Higgs doublet together with flavons, and the Majoron flavons ξ_{atm} and ξ_{sol} are standard model singlets.

In order to predict both neutrino masses and lepton mixing parameters, a non-abelian discrete flavor symmetry G_f and generalized CP symmetry H_{CP} are imposed on the model. Both flavor symmetry and the CP symmetry act on the flavor space in a non-trivial way. The flavor symmetry G_f and CP symmetry H_{CP} should be compatible with each other in the theory, and consequently the following consistency condition has to be satisfied [31–33]

$$X_{\mathbf{r}} \rho^*_{\mathbf{r}}(g) X^\dagger_{\mathbf{r}} = \rho_{\mathbf{r}}(g'), \quad g, g' \in G_f\,, \tag{2.2}$$

where $\rho_{\mathbf{r}}(g)$ is the representation matrix of the element g in the irreducible representation $\mathbf{r}$ of G_f, and $X_{\mathbf{r}}$ is the generalized CP transformation matrix of H_{CP}. The elements g and g' in eq. (2.2) are in general different. Hence the mathematical structure of the full symmetry at high energy scale is in general a semi-direct product of flavor symmetry G_f and generalized CP symmetry H_{CP} [31]. Moreover, it has been shown that physical CP transformations always have to be class-inverting automorphisms of G_f [33], namely for any $g \in G_f$, there

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should always exists an element h such that $g' = hg^{-1}h^{-1}$. We assign the three generations of left-handed leptons L to a faithful irreducible three-dimensional representation of G_f, the flavons ϕ_l, $\phi_{\rm atm}$ and $\phi_{\rm sol}$ transform as triplets under the flavor symmetry G_f while $N^c_{\rm atm}$ and $N^c_{\rm sol}$ are singlets of G_f. The flavons $\xi_{\rm atm}$ and $\xi_{\rm sol}$ are also singlets under G_f, nevertheless they could transform differently under the shaping symmetry. In the present work, we assume the parent symmetry $G_f \rtimes H_{\rm CP}$ is broken down to G_l, $G_{\rm atm} \rtimes H^{\rm atm}_{\rm CP}$ and $G_{\rm sol} \rtimes H^{\rm sol}_{\rm CP}$ in the charged lepton, atmospheric neutrino and solar neutrino sectors, respectively. This paradigm is schematically illustrated in figure 1, and we call it as tri-direct CP approach. Notice that the tri-direct CP approach has three branches of residual symmetries, and it is a generalization of the so called direct approach in which the flavor symmetry is broken to two distinct abelian subgroups G_l and G_ν in the changed lepton and neutrino sectors respectively. Here we require that the residual flavor symmetry G_l is capable of distinguishing the three generations, i.e., the order of G_l can not be smaller than 3. The invariance of the charged lepton mass term under the action of G_l leads to

$$g^\dagger_l m^\dagger_l m_l g_l = m^\dagger_l m_l, \quad g_l \in G_l\,, \tag{2.3}$$

which implies

$$[g_l, m^\dagger_l m_l] = 0\,, \tag{2.4}$$

where for simplicity we have used the same notations for the abstract elements of G_f and the representation matrices in the triplet representation to which the lepton doublets L are assigned. In addition, the charged lepton mass matrix m_l defined in the right-left basis $l^c m_l l$. The representation matrix g_l can be diagonalized by a unitary transformation U_l, i.e.,

$$U^\dagger_l g_l U_l = \text{diag}(e^{i\alpha_e}, e^{i\alpha_\mu}, e^{i\alpha_\tau})\,, \tag{2.5}$$

where $e^{i\alpha_{e,\mu,\tau}}$ are all roots of unity since g_l is an element of the discrete flavor symmetry group G_f and it is of finite order, and U_l is determined up to exchange of its column and possible overall phases of the single columns. From eq. (2.4), it follows that the hermitian combination $m^\dagger_l m_l$ is diagonalized by U_l as well.

As regards the atmospheric and solar neutrino sectors, the residual CP symmetry should be compatible with the residual flavor symmetry. As a consequence, the following constrained consistency conditions have to be fulfilled,

$$X^{\rm atm}_{\mathbf{r}} \rho^*_{\mathbf{r}}(g^{\rm atm}_i)(X^{\rm atm}_{\mathbf{r}})^{-1} = \rho_{\mathbf{r}}(g^{\rm atm}_j), \qquad g^{\rm atm}_i, g^{\rm atm}_j \in G_{\rm atm}\,, \tag{2.6a}$$

$$X^{\rm sol}_{\mathbf{r}} \rho^*_{\mathbf{r}}(g^{\rm sol}_i)(X^{\rm sol}_{\mathbf{r}})^{-1} = \rho_{\mathbf{r}}(g^{\rm sol}_j), \qquad g^{\rm sol}_i, g^{\rm sol}_j \in G_{\rm sol}\,. \tag{2.6b}$$

This implies that the mathematical structure of the subgroup comprising the residual flavor and CP symmetries is in general a semi-direct product. The semi-direct product structure will reduce to a direct product if $g^{\rm atm}_i = g^{\rm atm}_j$ and $g^{\rm sol}_i = g^{\rm sol}_j$. In particular, we note that the reduction of the semidirect product structure to direct product always holds true for a generic residual Z_2 flavor symmetry. For given residual flavor symmetries $G_{\rm atm}$ and $G_{\rm sol}$, the residual CP symmetries $H^{\rm atm}_{\rm CP}$ and $H^{\rm sol}_{\rm CP}$ can be easily obtained by solving the constraints in eqs. (2.6a) and (2.6b). The requirement that the subgroup $G_{\rm atm} \rtimes H^{\rm atm}_{\rm CP}$ is conserved entails that the vacuum expectation value (VEV) of $\phi_{\rm atm}$ should be invariant under the symmetry $G_{\rm atm} \rtimes H^{\rm atm}_{\rm CP}$, i.e.

$$\begin{aligned} g_{\rm atm}\langle\phi_{\rm atm}\rangle &= \langle\phi_{\rm atm}\rangle, \qquad & g_{\rm atm} \in G_{\rm atm}\,, \\ X_{\rm atm}\langle\phi_{\rm atm}\rangle^* &= \langle\phi_{\rm atm}\rangle, \qquad & X_{\rm atm} \in H^{\rm atm}_{\rm CP}\,, \end{aligned} \tag{2.7}$$

which allow us to fix the alignment of $\langle\phi_{\rm atm}\rangle$. In the same fashion, for the symmetry $G_{\rm sol} \rtimes H^{\rm sol}_{\rm CP}$ to hold, the VEV $\langle\phi_{\rm sol}\rangle$ has to be invariant under $G_{\rm sol} \rtimes H^{\rm sol}_{\rm CP}$,

$$\begin{aligned} g_{\rm sol}\langle\phi_{\rm sol}\rangle &= \langle\phi_{\rm sol}\rangle, \qquad & g_{\rm sol} \in G_{\rm sol}\,, \\ X_{\rm sol}\langle\phi_{\rm sol}\rangle^* &= \langle\phi_{\rm sol}\rangle, \qquad & X_{\rm sol} \in H^{\rm sol}_{\rm CP}\,. \end{aligned} \tag{2.8}$$

After the electroweak and flavor symmetry breaking, the flavon VEVs $\langle\phi_l\rangle$, $\langle\phi_{\rm atm}\rangle$, $\langle\phi_{\rm sol}\rangle$, $\langle\xi_{\rm atm}\rangle$ and $\langle\xi_{\rm sol}\rangle$ are non-vanishing. Then we can read out the neutrino Dirac mass matrix and the heavy Majorana mass matrix of $N^c_{\rm atm}$ and $N^c_{\rm sol}$,

$$m_D = \begin{pmatrix} y_{\rm atm}\langle\phi_{\rm atm}\rangle, & y_{\rm sol}\langle\phi_{\rm sol}\rangle \end{pmatrix}, \qquad m_N = \begin{pmatrix} x_{\rm atm}\langle\xi_{\rm atm}\rangle & 0 \\ 0 & x_{\rm sol}\langle\xi_{\rm sol}\rangle \end{pmatrix}, \tag{2.9}$$

where we omit the relevant Clebsch-Gordan coefficients which appear in both contractions $y_{\rm atm}L\phi_{\rm atm}N^c_{\rm atm}$ and $y_{\rm sol}L\phi_{\rm sol}N^c_{\rm sol}$ in order to form invariants under G_f. Using the seesaw formula, we can obtain the low energy effective light neutrino mass matrix

$$m_\nu = -\frac{y^2_{\rm atm}}{x_{\rm atm}}\frac{\langle\phi_{\rm atm}\rangle\langle\phi_{\rm atm}\rangle^T}{\langle\xi_{\rm atm}\rangle} - \frac{y^2_{\rm sol}}{x_{\rm sol}}\frac{\langle\phi_{\rm sol}\rangle\langle\phi_{\rm sol}\rangle^T}{\langle\xi_{\rm sol}\rangle}\,. \tag{2.10}$$

Since two right-handed neutrinos are introduced in this approach, the lightest neutrino must be massless. Indeed we find the light neutrino mass matrix satisfy

$$m_\nu\left[\langle\phi_{\rm atm}\rangle \times \langle\phi_{\rm sol}\rangle\right] = (0,0,0)^T\,, \tag{2.11}$$

where $\langle\phi_{\rm atm}\rangle \times \langle\phi_{\rm sol}\rangle$ denotes the cross product of $\langle\phi_{\rm atm}\rangle$ and $\langle\phi_{\rm sol}\rangle$. This means that the column vector $\langle\phi_{\rm atm}\rangle \times \langle\phi_{\rm sol}\rangle$ is an eigenvector of m_ν with zero eigenvalue, depending on the modulus of each entry, it can be either the first column or the third column of the neutrino unitary transformation matrix U_ν which diagonalizes m_ν. Accordingly the neutrino mass spectrum can be normal order with $m_1 = 0$ or inverted ordering with $m_3 = 0$. The other two remaining columns of U_ν are orthogonal to $\langle\phi_{\rm atm}\rangle \times \langle\phi_{\rm sol}\rangle$, and their exact forms can be determined for given residual symmetry in a model, as explicitly shown in the following two sections. We can thus determine the lepton mixing matrix $U_{PMNS} = U^\dagger_l U_\nu$. Furthermore, we notice that the two columns of the Dirac mass matrix m_D would be exchanged if the roles of $G_{\rm atm} \rtimes H^{\rm atm}_{\rm CP}$ and $G_{\rm sol} \rtimes H^{\rm sol}_{\rm CP}$ are switched. Thus the same neutrino mass matrix would be obtained if one interchanges $y_{\rm atm}$ with $y_{\rm sol}$ and $x_{\rm atm}$ with $x_{\rm sol}$. The tri-direct CP approach provides new opportunity for model building, it allows us to construct quite predictive neutrino mass models as simple as the LSS model.

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3 Littlest Seesaw in the tri-direct CP approach

In this section, we shall show that the original Littlest seesaw model [23, 24] can be reproduced from the above tri-direct approach. The flavor symmetry group is chosen to be S_4 which has been extensively studied in the literature, see [34, 35] for the group theory of S_4. In the present work, we shall adopt the conventions of [35], and S_4 can be generated by three generators S, T and U, which obey the relations

$$S^2 = T^3 = U^2 = (ST)^3 = (SU)^2 = (TU)^2 = (STU)^4 = 1\,. \tag{3.1}$$

The S_4 group has five irreducible representations: $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{3}'$. The one-dimensional unitary representations are given by,

$$\begin{aligned} \mathbf{1} \;&:\quad S=1, \qquad T=1, \qquad U=1\,, \\ \mathbf{1}' \;&:\quad S=1, \qquad T=1, \qquad U=-1\,. \end{aligned} \tag{3.2}$$

In the double representation, we have

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.3}$$

with $\omega = e^{2\pi i/3}$. For the triplet representation $\mathbf{3}$, in a basis where the element T is diagonal, the generators are

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad U = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.4}$$

In another triplet representation $\mathbf{3}'$, the generator U is simply opposite in sign with to that in the $\mathbf{3}$. We assume that both lepton doublet L and the atmospheric flavon $\phi_{\rm atm}$ are assigned to S_4 triplet $\mathbf{3}$, the solar flavon $\phi_{\rm sol}$ transforms as $\mathbf{3}'$ while the right-handed Majorana neutrino $N^c_{\rm atm}$ is $\mathbf{1}$ and $N^c_{\rm sol}$ is $\mathbf{1}'$ of S_4. The residual flavor symmetry of the charged lepton sector is taken to be $G_l = Z_3^T$. The most general hermitian matrix $m_l^\dagger m_l$ invariant under Z_3^T is diagonal with generic entries, where m_l is the charged lepton mass matrix. As a consequence, the unitary transformation U_l is a unit matrix up to permutations and rephasing of column vectors. The atmospheric and solar residual symmetries are $G_{\rm atm} = Z_2^U$ and $G_{\rm sol} = Z_2^{SU}$ with $X_{\rm atm} = X_{\rm sol} = 1$, and they uniquely fix the vacuum alignments of $\langle\phi_{\rm atm}\rangle$ and $\langle\phi_{\rm sol}\rangle$ as follows

$$\langle\phi_{\rm atm}\rangle = v_{\rm atm}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \langle\phi_{\rm sol}\rangle = v_{\rm sol}\begin{pmatrix} 1 \\ n \\ 2-n \end{pmatrix}, \tag{3.5}$$

where both vacuum expectation values $v_{\rm atm}$ and $v_{\rm sol}$ are real, and n is a real parameter.

It is interesting to compare the two different residual symmetries here, $G_{\rm atm} = Z_2^U$ and $G_{\rm sol} = Z_2^{SU}$ of the tri-direct CP approach, to the semi-direct approach of [24] where there was a common residual symmetry in both the atmospheric and solar neutrino sectors, namely Z_2^{SU}. In the semi-direct approach [24], the atmospheric alignment $\langle\phi_{\rm atm}\rangle \propto (0,1,-1)^T$ could not be uniquely fixed by the residual symmetry Z_2^{SU}, and was achieved through the dynamical terms in the potential of a concrete model. As regards the solar alignment, Z_2^{SU} enforces $\langle\phi_{\rm sol}\rangle \propto (1,n,2-n)^T$ with n being generally complex in [24] while n is a real parameter because of the CP symmetry in the tri-direct CP approach.

Applying the seesaw formula and multiplying L_3 by a minus sign, we obtain the effective light neutrino mass matrix[2]

$$m_\nu = m_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + m_s e^{i\eta} \begin{pmatrix} 1 & n-2 & n \\ n-2 & (n-2)^2 & n(n-2) \\ n & n(n-2) & n^2 \end{pmatrix}. \tag{3.6}$$

As shown in [24, 25], the benchmark values of $n=3$ and $\eta = 2\pi/3$ can give a phenomenologically successful description of neutrino masses and lepton mixing parameters, e.g.

$$\begin{aligned} &m_a = 26.691\,\text{meV}, \quad m_s = 2.673\,\text{meV}, \quad \sin^2\theta_{13} = 0.0223\,, \\ &\sin^2\theta_{12} = 0.318, \quad \sin^2\theta_{23} = 0.488, \quad \delta_{CP} = -0.517\pi, \quad \beta = -0.402\pi\,, \\ &m_1 = 0\,\text{meV}, \quad m_2 = 8.563\,\text{meV}, \quad m_3 = 49.993\,\text{meV}, \quad m_{ee} = 2.673\,\text{meV}\,, \end{aligned} \tag{3.7}$$

where m_{ee} is the effective mass in neutrinoless double beta decay. Notice that the predictions in eq. (3.7) generally receive subleading corrections in a concrete model. Since the charged lepton masses are not constrained in this approach, the lepton mixing matrix is determined up to all possible row permutations. In other words, the neutrino mass matrix m_ν in eq. (3.6) can be multiplied by any permutation matrix from the left and right sides simultaneously. If the second and third rows and columns of m_ν are exchanged [24, 25], we obtain a second form of the Littlest seesaw model with

$$m_\nu = m_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + m_s e^{i\eta} \begin{pmatrix} 1 & n & n-2 \\ n & n^2 & n(n-2) \\ n-2 & n(n-2) & (n-2)^2 \end{pmatrix}. \tag{3.8}$$

The phase value $\eta = -2\pi/3$ is preferred in the case of $n=3$. Excellent agreement with experimental data can be achieved, and a numerical benchmark is

$$\begin{aligned} &m_a = 26.688\,\text{meV}, \quad m_s = 2.674\,\text{meV}, \quad \sin^2\theta_{13} = 0.0223\,, \\ &\sin^2\theta_{12} = 0.318, \quad \sin^2\theta_{23} = 0.512, \quad \delta_{CP} = -0.483\pi, \quad \beta = 0.402\pi\,, \\ &m_1 = 0\,\text{meV}, \quad m_2 = 8.565\,\text{meV}, \quad m_3 = 49.991\,\text{meV}, \quad m_{ee} = 2.674\,\text{meV}\,. \end{aligned} \tag{3.9}$$

For both versions of Littlest seesaw, the neutrino mass matrix leads to the trimaximal TM1 mixing [23–25], in which the first column of the mixing matrix is fixed to be that of the tri-bimaximal mixing matrix. See [23, 24] for exact expressions of light neutrino masses and lepton mixing parameters.

[2]The effective light neutrino Majorana mass matrix given by the Lagrangian $\mathcal{L}_{\rm eff} = -\frac{1}{2}\nu_{L_i}(m_\nu)_{ij}\nu_{L_j}$ + h.c. in two-component notation.

4 Littlest Seesaw variants in the tri-direct CP approach

Following the framework of tri-direct CP approach presented in section 2, we find that many new mixing patterns compatible with experimental data can obtained from the S_4 flavor symmetry group and CP [36]. In order to illustrate how the neutrino mass and mixing parameters are predicted in the tri-direct CP approach, as an example, we shall show a new Littlest seesaw model which can be achieved from the S_4 flavor symmetry in combination with the generalized CP symmetry $H_{\rm CP}$, where $H_{\rm CP}$ is the collection of the generalized CP transformations $X_{\mathbf{r}}$. In our working basis, the generalized CP transformation compatible with S_4 turns out to be of the same form as flavor symmetry transformation [35, 37], i.e.

$$X_{\mathbf{r}} = \rho_{\mathbf{r}}(h), \qquad h \in S_4\,, \tag{4.1}$$

where h can be any element of S_4. The three left-handed leptons L are assigned to S_4 triplet $\mathbf{3}$, the atmospheric flavon $\phi_{\rm atm}$ and solar flavon $\phi_{\rm sol}$ transform as $\mathbf{3}$ and $\mathbf{3}'$ respectively, and the two right-handed neutrinos $N^c_{\rm atm}$ and $N^c_{\rm sol}$ are S_4 singlets $\mathbf{1}$ and $\mathbf{1}'$ respectively. We assume that the S_4 and CP symmetries are broken down to Z_3^T, $Z_2^{TST^2}\times H^{\rm atm}_{\rm CP}$ and $Z_2^U \times H^{\rm sol}_{\rm CP}$ in the charged lepton, atmospheric neutrino and solar neutrino sectors, respectively. Since the representation matrix T is diagonal in the working basis, the residual symmetry $G_l = Z_3^T$ would determine that the hermitian combination $m_l^\dagger m_l$ is diagonalized by a unit matrix up to permutations and phases of its column vectors. Notice that no new constraint is obtained even if the possible residual CP symmetry in the charged lepton sector is further taken into account [38].

The residual symmetry $Z_2^{TST^2}\times H^{\rm atm}_{\rm CP}$ in the atmospheric neutrino sector should be well defined such that the constrained consistency condition in eq. (2.6a) has to be satisfied, i.e.

$$X^{\rm atm}_{\mathbf{r}}\rho^*_{\mathbf{r}}(TST^2)(X^{\rm atm}_{\mathbf{r}})^{-1} = \rho_{\mathbf{r}}(TST^2)\,. \tag{4.2}$$

It is easy to check that $X^{\rm atm}_{\mathbf{r}}$ can only take the following eight possible values,

$$H^{\rm atm}_{\rm CP} = \{\rho_{\mathbf{r}}(SU), \rho_{\mathbf{r}}(T^2), \rho_{\mathbf{r}}(ST^2S), \rho_{\mathbf{r}}(T^2STU), \rho_{\mathbf{r}}(U), \rho_{\mathbf{r}}(ST^2), \rho_{\mathbf{r}}(T^2S), \rho_{\mathbf{r}}(TST^2U)\}\,. \tag{4.3}$$

For the first four CP transformations of $H^{\rm atm}_{\rm CP}$, the vacuum alignment $\langle\phi_{\rm atm}\rangle$ which preserves $Z_2^{TST^2}\times H^{\rm atm}_{\rm CP}$ is of the following form

$$\langle\phi_{\rm atm}\rangle = v_{\rm atm}\left(1,\omega^2,\omega\right)^T\,, \tag{4.4}$$

where the parameter $v_{\rm atm}$ is real. If $X^{\rm atm}_{\mathbf{r}}$ is one of the last four CP transformations in $H^{\rm atm}_{\rm CP}$ (e.g. $X_{\rm atm} = U$), the VEV $\langle\phi_{\rm atm}\rangle$ is enforced to be the form

$$\langle\phi_{\rm atm}\rangle = iv_{\rm atm}\left(1,\omega^2,\omega\right)^T\,, \tag{4.5}$$

with $v_{\rm atm}$ being real. The above two vacuum configurations in eq. (4.4) and eq. (4.5) differ from each other in the overall factor i which can be compensated by shifting the sign of the coupling $x_{\rm atm}$. Without loss of generality, in the following we shall take the atmospheric vacuum in eq. (4.4) and the corresponding residual CP is $X_{\rm atm} = SU$. As regards the solar neutrino sector. the residual CP transformation $X_{\mathbf{r}}^{\rm sol}$ of $H_{\rm CP}^{\rm sol}$ is determined by the consistency condition

$$X_{\mathbf{r}}^{\rm sol}\rho_{\mathbf{r}}^{*}(U)(X_{\mathbf{r}}^{\rm sol})^{-1}=\rho_{\mathbf{r}}(U)\,. \tag{4.6}$$

We find the allowed values of $H_{\rm CP}^{\rm sol}$ are

$$H_{\rm CP}^{\rm sol}=\{\rho_{\mathbf{r}}(1),\rho_{\mathbf{r}}(U),\rho_{\mathbf{r}}(S),\rho_{\mathbf{r}}(SU)\}\,. \tag{4.7}$$

For the case of $X_{\mathbf{r}}^{\rm sol}=\rho_{\mathbf{r}}(1),\rho_{\mathbf{r}}(U)$, the vacuum configuration $\langle\phi_{\rm sol}\rangle$ invariant under $Z_2^U\times H_{\rm CP}^{\rm sol}$ is of the form

$$\langle\phi_{\rm sol}\rangle=v_{\rm sol}\,(1,x,x)^T\;, \tag{4.8}$$

where the VEV $v_{\rm sol}$ is real and x is in general a dimensionless real number. For the remaining choices $X_{\mathbf{r}}^{\rm sol}=\rho_{\mathbf{r}}(S),\rho_{\mathbf{r}}(SU)$, the subgroup $Z_2^U\times H_{\rm CP}^{\rm sol}$ is preserved only if $\langle\phi_{\rm sol}\rangle$ is aligned along the following direction,

$$\langle\phi_{\rm sol}\rangle=v_{\rm sol}\,(1+2ix,1-ix,1-ix)^T\;, \tag{4.9}$$

with both parameters $v_{\rm sol}$ and x being real. However, detailed numerical analysis reveals that the experimentally favored neutrino masses and mixing angles can not be obtained for the solar flavon VEV shown in eq. (4.9). Hence we shall focus on the residual CP transformation $X_{\rm sol}=U$ and the resulting vacuum alignment $\langle\phi_{\rm sol}\rangle\propto(1,x,x)$ in this work.

It is useful to remind the S_4 singlet contraction rules for $\mathbf{3}\otimes\mathbf{3}\to\mathbf{1}$ and $\mathbf{3}\otimes\mathbf{3'}\to\mathbf{1'}$ [35],

$$\begin{aligned}(\alpha\beta)_{\mathbf{1}}&=\alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2\,,\\(\alpha\gamma)_{\mathbf{1'}}&=\alpha_1\gamma_1+\alpha_2\gamma_3+\alpha_3\gamma_2\,,\end{aligned} \tag{4.10}$$

where $\alpha=(\alpha_1,\alpha_2,\alpha_3)^T$ and $\beta=(\beta_1,\beta_2,\beta_3)^T$ are S_4 triplet $\mathbf{3}$, and $\gamma=(\gamma_1,\gamma_2,\gamma_3)^T$ transforms as $\mathbf{3'}$. When the flavor fields $\phi_{\rm atm}$ and $\phi_{\rm sol}$ acquire a non-vanishing VEV as shown in eq. (4.4) and eq. (4.8) respectively, from the Lagrangian of eq. (2.1), we can read out the Dirac neutrino mass matrix m_D as well as a diagonal right-handed neutrino mass matrix m_N

$$m_D=\begin{pmatrix}y_{\rm atm}v_{\rm atm}&y_{\rm sol}v_{\rm sol}\\\omega y_{\rm atm}v_{\rm atm}&xy_{\rm sol}v_{\rm sol}\\\omega^2y_{\rm atm}v_{\rm atm}&xy_{\rm sol}v_{\rm sol}\end{pmatrix},\qquad m_N=\begin{pmatrix}M_{\rm atm}&0\\0&M_{\rm sol}\end{pmatrix}, \tag{4.11}$$

with $M_{\rm atm}=x_{\rm atm}\langle\xi_{\rm atm}\rangle$ and $M_{\rm sol}=x_{\rm sol}\langle\xi_{\rm sol}\rangle$. The light neutrino mass matrix is given by the seesaw formula,

$$m_\nu=-m_Dm_N^{-1}m_D^T=m_a\begin{pmatrix}1&\omega&\omega^2\\\omega&\omega^2&1\\\omega^2&1&\omega\end{pmatrix}+e^{i\eta}m_s\begin{pmatrix}1&x&x\\x&x^2&x^2\\x&x^2&x^2\end{pmatrix}, \tag{4.12}$$

where a physically irrelevant overall phase factor is dropped, $m_a=|y_{\rm atm}^2v_{\rm atm}^2/M_{\rm atm}|$, $m_s=|y_{\rm sol}^2v_{\rm sol}^2/M_{\rm sol}|$ and the relative phase $\eta=2\arg(y_{\rm sol}v_{\rm sol})-2\arg(y_{\rm atm}v_{\rm atm})+\arg(M_{\rm atm})-\arg(M_{\rm sol})$. We see that m_ν depends on four parameters m_a, m_s, η and x to describe the neutrino masses and mixing parameters, consequently this model is quite predictive. Moreover, we shall fix x and η to certain values in a concrete models. It is easy to check that the above neutrino mass matrix m_ν fulfills

$$m_\nu \begin{pmatrix} -i\sqrt{3}x \\ x-\omega^2 \\ -x+\omega \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} . \tag{4.13}$$

This implies that the column vector $(-i\sqrt{3}x, x-\omega^2, -x+\omega)^T$ is an eigenvector of m_ν with zero eigenvalue. As a consequence, the neutrino mass matrix m_ν can be block diagonalized by the following unitary transformation

$$U_{\nu 1} = \begin{pmatrix} -\frac{i\sqrt{3}x}{\sqrt{5x^2+2x+2}} & \sqrt{\frac{2(x^2+x+1)}{5x^2+2x+2}} & 0 \\ \frac{x-\omega^2}{\sqrt{5x^2+2x+2}} & -\frac{i\sqrt{3}x\left(x-\omega^2\right)}{\sqrt{2(x^2+x+1)(5x^2+2x+2)}} & \frac{x-\omega^2}{\sqrt{2(x^2+x+1)}} \\ \frac{\omega-x}{\sqrt{5x^2+2x+2}} & \frac{i\sqrt{3}x(x-\omega)}{\sqrt{2(x^2+x+1)(5x^2+2x+2)}} & \frac{x-\omega}{\sqrt{2(x^2+x+1)}} \end{pmatrix} . \tag{4.14}$$

Then m_ν becomes

$$m'_\nu = U^T_{\nu 1} m_\nu U_{\nu 1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y & z \\ 0 & z & w \end{pmatrix} , \tag{4.15}$$

with

$$\begin{aligned} y &= \frac{5x^2+2x+2}{2\left(x^2+x+1\right)}(m_a + e^{i\eta} m_s) \equiv |y| e^{i\phi_y} , \\ z &= -\frac{\sqrt{5x^2+2x+2}}{2\left(x^2+x+1\right)} \left[(x+2)m_a - x(2x+1)e^{i\eta} m_s\right] \equiv |z| e^{i\phi_z} , \\ w &= \frac{1}{2(x^2+x+1)} \left[(x+2)^2 m_a + x^2\left(2x+1\right)^2 e^{i\eta} m_s\right] \equiv |w| e^{i\phi_w} . \end{aligned} \tag{4.16}$$

Since m'_ν is essentially a two by two complex symmetric matrix, it can be exactly diagonalized, as shown in detail in [39],

$$U^T_{\nu 2} m'_\nu U_{\nu 2} = \text{diag}(0, m_2, m_3) . \tag{4.17}$$

The procedure for diagonalization of the neutrino mass matrix m'_ν is given in appendix A, and the explicit forms of $U_{\nu 2}$, m_2 and m_3 can be found there. Because the charged lepton mass matrix $m_l^\dagger m_l$ is a diagonal, the lepton mixing completely arises from the neutrino sector. Thus the lepton mixing matrix is derived as

$$\begin{aligned} U_{PMNS} &= U_{\nu 1} U_{\nu 2} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\sqrt{6}x}{\sqrt{5x^2+2x+2}} & 2i\sqrt{\frac{x^2+x+1}{5x^2+2x+2}}\cos\theta & 2i\sqrt{\frac{x^2+x+1}{5x^2+2x+2}} e^{i\psi}\sin\theta \\ \sqrt{\frac{2(x^2+x+1)}{5x^2+2x+2}} & -e^{-i\psi}\sin\theta - \frac{i\sqrt{3}x\cos\theta}{\sqrt{5x^2+2x+2}} & \cos\theta - \frac{i\sqrt{3}xe^{i\psi}\sin\theta}{\sqrt{5x^2+2x+2}} \\ \sqrt{\frac{2(x^2+x+1)}{5x^2+2x+2}} & e^{-i\psi}\sin\theta - \frac{i\sqrt{3}x\cos\theta}{\sqrt{5x^2+2x+2}} & -\cos\theta - \frac{i\sqrt{3}xe^{i\psi}\sin\theta}{\sqrt{5x^2+2x+2}} \end{pmatrix} P_\nu , \end{aligned} \tag{4.18}$$

up to possible row permutations, where $P_\nu = \mathrm{diag}(1, e^{i(\psi+\rho)/2}, e^{i(-\psi+\sigma)/2})$ is a diagonal phase matrix, and an overall phase of each row has been absorbed by the charged lepton fields. The expressions for the parameters θ, ψ, ρ and σ are given in appendix A. Comparing with the standard parametrization of the lepton mixing matrix [1],

$$U_{PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{\rm CP}} \\ -s_{12}c_{23}-c_{12}s_{13}s_{23}e^{i\delta_{\rm CP}} & c_{12}c_{23}-s_{12}s_{13}s_{23}e^{i\delta_{\rm CP}} & c_{13}s_{23} \\ s_{12}s_{23}-c_{12}s_{13}c_{23}e^{i\delta_{\rm CP}} & -c_{12}s_{23}-s_{12}s_{13}c_{23}e^{i\delta_{\rm CP}} & c_{13}c_{23} \end{pmatrix} \mathrm{diag}(e^{i\frac{\alpha}{2}}, e^{i\frac{\beta}{2}}, 1)\,, \tag{4.19}$$

where $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$, and the Majorana phase α is unphysical since $m_1 = 0$, we can extract the results for the lepton mixing angles and find

$$\begin{aligned} \sin^2\theta_{13} &= \frac{2\left(x^2+x+1\right)\sin^2\theta}{5x^2+2x+2}\,, \\ \sin^2\theta_{12} &= 1 - \frac{3x^2}{3x^2+2\left(x^2+x+1\right)\cos^2\theta}\,, \\ \sin^2\theta_{23} &= \frac{1}{2} + \frac{x\sqrt{3\left(5x^2+2x+2\right)}\sin 2\theta\sin\psi}{2\left[3x^2+2\left(x^2+x+1\right)\cos^2\theta\right]}\,, \end{aligned} \tag{4.20}$$

and for the CP invariants we obtain

$$J_{\rm CP} = \frac{\sqrt{3}x\left(x^2+x+1\right)\sin 2\theta\cos\psi}{2\left(5x^2+2x+2\right)^{3/2}}\,, \qquad I_1 = \frac{\left(x^2+x+1\right)^2\sin^2 2\theta\sin(\rho-\sigma)}{\left(5x^2+2x+2\right)^2}\,. \tag{4.21}$$

The Jarlskog invariant $J_{\rm CP}$ [40] and the Majorana invariant I_1 [41–44] related to the Majorana phase β are defined in the usual way

$$\begin{aligned} J_{\rm CP} &= \Im(U_{PMNS,11}U_{PMNS,33}U^*_{PMNS,13}U^*_{PMNS,31}) = \frac{1}{8}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}\cos\theta_{13}\sin\delta_{\rm CP}\,, \\ I_1 &= \Im(U^2_{PMNS,12}U^{*2}_{PMNS,13}) = \frac{1}{4}\sin^2\theta_{12}\sin^2 2\theta_{13}\sin(\beta+2\delta_{\rm CP})\,. \end{aligned} \tag{4.22}$$

From the expressions of mixing angles in eq. (4.20), we easily see that the solar mixing angle θ_{12} and the reactor mixing angle θ_{13} fulfill the following sum rules

$$\cos^2\theta_{12}\cos^2\theta_{13} = \frac{3x^2}{5x^2+2x+2}\,. \tag{4.23}$$

This implies that θ_{13} and θ_{12} are strongly correlated with each other, as shown in figure 2. Inserting the 3σ allowed regions $0.272 \leq \sin^2\theta_{12} \leq 0.346$ and $0.01981 \leq \sin^2\theta_{13} \leq 0.02436$ [4], we find that the parameter x should be in the interval $-5.481 \leq x \leq -1.223$. Notice that the value of x is also subject to the constraints from the measured values of θ_{23} and neutrino mass squared differences. If all the four input parameters m_a, m_s, η and x are treated as free parameters and both lepton mixing angles and the mass splittings Δm^2_{21} and Δm^2_{31} are required to be in the experimentally favored 3σ ranges, we find the solar mixing angle is allowed in a narrow region $0.329 \leq \sin^2\theta_{12} \leq 0.346$ which is represented by the orange in figure 2. The forthcoming JUNO experiment will be capable of reducing the error of

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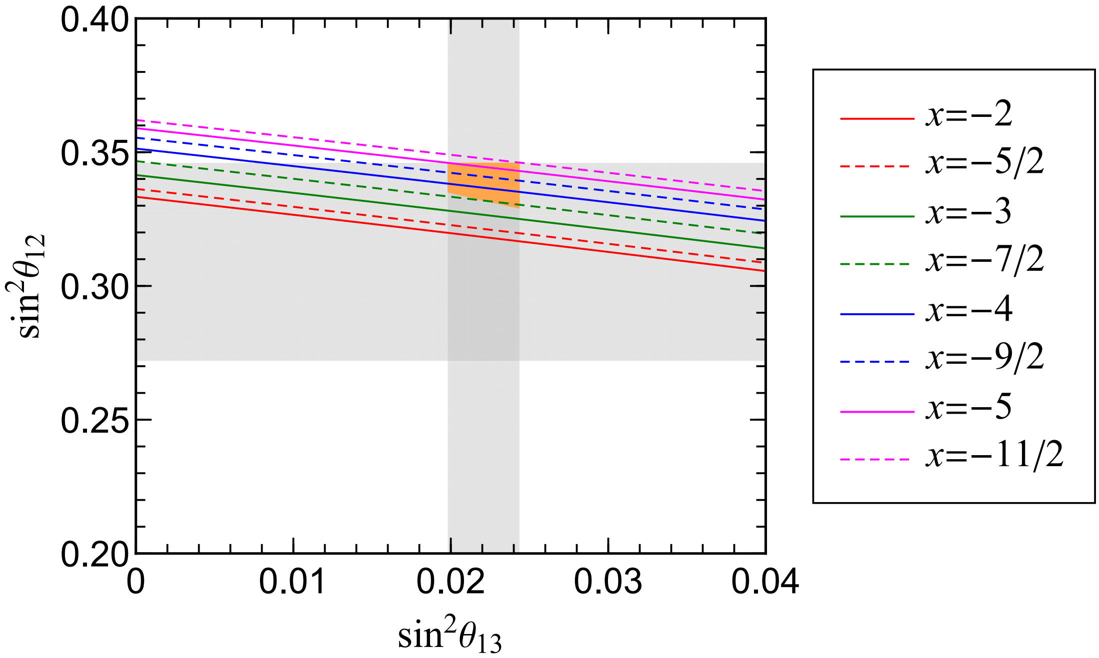

Figure 2. Correlation between $\sin^2\theta_{12}$ and $\sin^2\theta_{13}$ given by eq. (4.23) for various values of x. The gray bands represent the experimentally preferred 3σ ranges of $\sin^2\theta_{13}$ and $\sin^2\theta_{12}$ adapted from [4]. The orange area denotes the most generally allowed regions of $\sin^2\theta_{13}$ and $\sin^2\theta_{12}$ in the new Littlest seesaw model, where the four input parameter m_a, m_s, η and x are randomly chosen and the resulting mixing angles and mass squared differences are required to lie in the experimentally preferred 3σ regions [4].

$\sin^2\theta_{12}$ to about 0.1° or around 0.3% [45]. Therefore we can expect JUNO to identify with considerable confidence if the present model is compatible with experimental data.

It is notable that all the three mixing angles and Jarlskog invariant $J_{\rm CP}$ depend on only two parameters θ and ψ. As a consequence, we can express the Dirac CP phase $\delta_{\rm CP}$ in terms of the mixing angles,

$$\cos\delta_{\rm CP} = \frac{\cot 2\theta_{23}\left[3x^2-\left(4x^2+x+1\right)\cos^2\theta_{13}\right]}{\sqrt{3}\,|x|\sin\theta_{13}\sqrt{(5x^2+2x+2)\cos^2\theta_{13}-3x^2}}\,,$$
$$\sin\delta_{\rm CP} = \text{sign}(x\cos\psi)\csc 2\theta_{23}\sqrt{1+\frac{\left(x^2+x+1\right)^2\cot^2\theta_{13}\cos^2 2\theta_{23}}{3x^2\left[3x^2\tan^2\theta_{13}-2\left(x^2+x+1\right)\right]}}\,. \tag{4.24}$$

For maximal atmospheric mixing angle $\theta_{23}=\pi/4$, we have $\cos 2\theta_{23}=0$ and $\csc 2\theta_{23}=1$. Then this sum rule gives $\cos\delta_{\rm CP}=0$ and $\sin\delta_{\rm CP}=\pm1$ which implies maximal Dirac phase $\delta_{\rm CP}=\pm\pi/2$. We show the contour plot of $\delta_{\rm CP}/\pi$ in the plane $\sin^2\theta_{23}$ versus $\sin^2\theta_{13}$ in figure 3 for $x=-7/2,-4,-9/2,-5$. It is remarkable that the Dirac CP violation phase $\delta_{\rm CP}$ is predicted to lie in a narrow range around -0.5π.

Furthermore we perform a comprehensive numerical analysis. The input parameters x, $r=m_s/m_a$ and η are treated as random real numbers in the ranges $x\in[-6,-1]$, $r\in[0,10]$ and $\eta\in[0,2\pi]$, then we calculate the values of mixing angles $\sin^2\theta_{ij}$, the CP

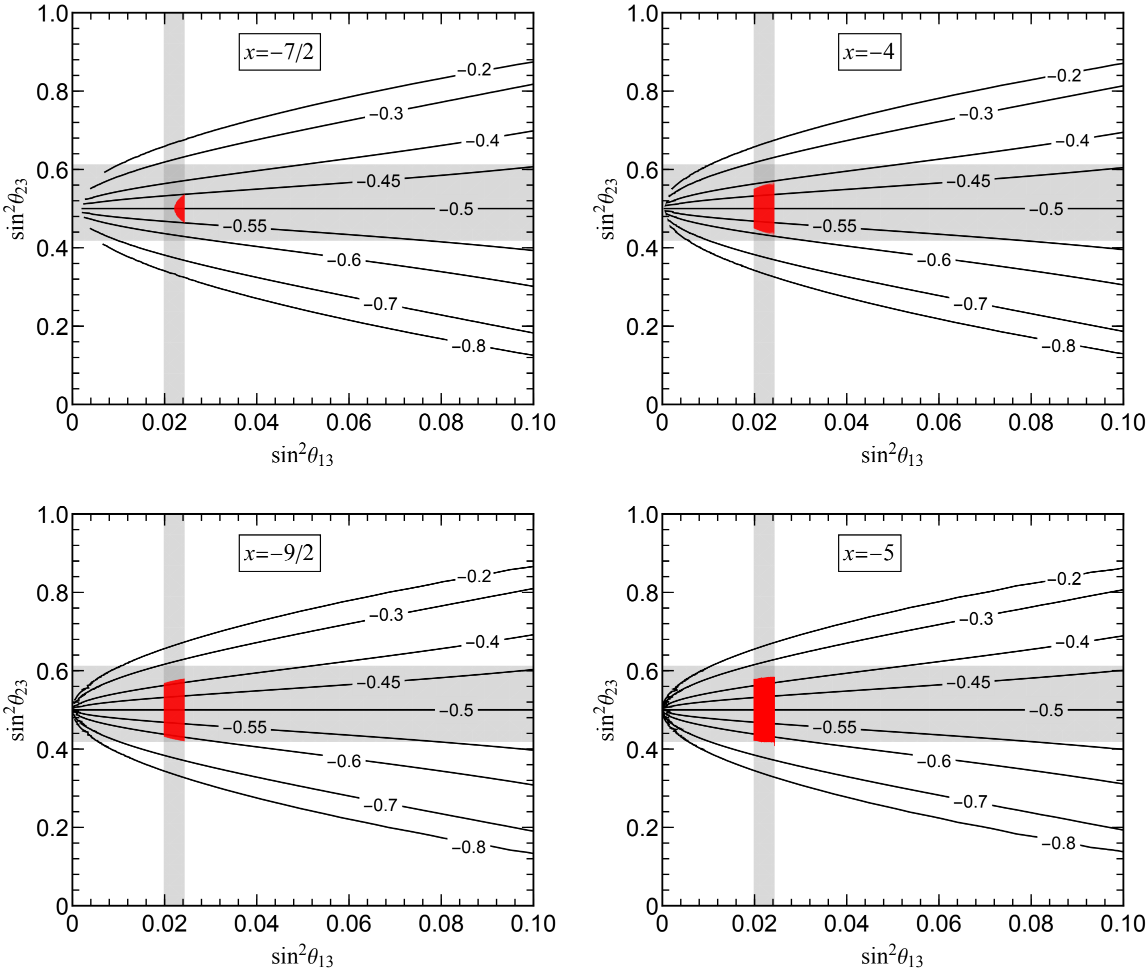

Figure 3. Contour plot of $\delta_{\rm CP}/\pi$ in the $\sin^2\theta_{13} - \sin^2\theta_{23}$ plane for the benchmark values of $x = -7/2, -4, -9/2, -5$. The contour lines are obtained by using the sum rule in eq. (4.24). The gray bands represent the experimentally preferred 3σ ranges of the mixing angles adapted from [4]. The red areas in the plane are the most generally allowed regions of $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$ for given value of x, where the four input parameter m_a, m_s and η are randomly chosen and the resulting mixing angles and mass squared differences are required to lie in the experimentally preferred 3σ regions [4].

violation phases $\delta_{\rm CP}$ and β and the mass ratio m_2^2/m_3^2 for each value of the input parameters x, r and η. We require $\sin^2\theta_{ij}$ and m_2^2/m_3^2 to lie in their 3σ regions obtained in the global analysis of neutrino oscillation data [4]. In order to accommodate the present experimental data, we find the allowed region of the parameter x is $-5.475 \leq x \leq -3.370$. Regarding the predictions for the mixing angles, we find that all values of $\sin^2\theta_{13}$ in its 3σ range are allowed, $\sin^2\theta_{23}$ is constrained to lie in the interval $0.418 \leq \sin^2\theta_{23} \leq 0.584$, and the solar angle is found to lie in a narrow interval around its 3σ upper bound $0.329 \leq \sin^2\theta_{12} \leq 0.346$. What concerns the CP phases, the values of $\delta_{\rm CP}$ lie around $-\pi/2$, in the range $-0.629\pi \leq \delta_{\rm CP} \leq -0.371\pi$, and the allowed range of the Majorana phase is $-0.571\pi \leq \beta \leq 0.571\pi$. These predictions may be tested at future long baseline experiments, as discussed in [25].

In order to quantitatively measure how well the present model can describe the experimental data, we define a χ^2 function to estimate the goodness-of-fit of a chosen values of the input parameters m_a, r, η and x,

$$\chi^2 = \sum_{i=1}^{5} \left(\frac{P_i(m_a, r, \eta, x) - O_i}{\sigma_i} \right)^2 , \tag{4.25}$$

where $O_i \in \{\sin^2\theta_{12}, \sin^2\theta_{23}, \sin^2\theta_{13}, \Delta m^2_{21}, \Delta m^2_{31}\}$ are the global best fit values of the observable quantities, and σ_i denote the 1σ deviations of the corresponding quantities. The values of O_i and σ_i are taken from the global data analysis [4]. P_i is the theoretical predictions for the mixing angles $\sin^2\theta_{ij}$ and the mass splittings Δm^2_{21} and Δm^2_{31} as complex nonlinear functions of the free parameters of the model. Notice that the value of Dirac phase $\delta_{\rm CP}$ is less constrained at present, consequently its contribution is not included in the χ^2 function. Since the number of free parameters is less than the number of observables, it is not completely evident that the model can successfully fit the data. For each value of the input parameters, we can obtain the predicted values P_i and the corresponding χ^2, and the optimum input parameters yield the lowest χ^2. We have carried out the χ^2 minimization, we find the minimum of χ^2 is $\chi^2_{\rm min} = 3.957$, and values of the input parameters at the $\chi^2_{\rm min}$ read

$$m_a = 3.709\,\text{meV}, \quad r = 0.537, \quad \eta = 1.055\pi, \quad x = -3.556\,. \tag{4.26}$$

The predictions for various observable quantities obtained at the best fit point are

$$\begin{aligned} \sin^2\theta_{13} &= 0.0222, & \sin^2\theta_{12} &= 0.332, & \sin^2\theta_{23} &= 0.515, \\ \delta_{\rm CP} &= -0.478\pi, & \beta &= 0.0574\pi, & m_1 &= 0\,\text{meV}\,, \\ m_2 &= 8.604\,\text{meV}, & m_3 &= 49.938\,\text{meV}, & m_{ee} &= 1.778\,\text{meV}\,. \end{aligned} \tag{4.27}$$

We plot the best fit values of χ^2 as a function of η in figure 4, where five typical values of $x = -7/2, -4, -9/2, -5, -11/2$ are chosen for illustration. We notice that low χ^2 values such as $\chi^2 < 10$ can be achieved. It is obvious that the values of χ^2 is quite sensitive to the phase η, and the model can give very good fits to the leptonic mixing angles and the neutrino masses for certain values of η. We see that the experimental data can be described very well for $x = -7/2$ and η around π. In table 1, we show the best fit values of the mixing parameters and neutrino masses for some benchmark values of x and η. Once the values of x and η are fixed, the light neutrino mass matrix m_ν would depend on only two free parameters m_s and m_a whose values can be determined by the neutrino mass squared differences Δm^2_{21} and Δm^2_{31}, then three lepton mixing angles and CP violation phases $\delta_{\rm CP}$ and β can be predicted. We see that the effective Majorana mass m_{ee} is in the range of 1 and 3 meV such that it is impossible to be measured in foreseeable future. An particularly interesting example is the case of $x = -7/2$ and $\eta = \pi$, it predicts maximal atmospheric mixing angle $\theta_{23} = \pi/4$ and maximal Dirac phase $\delta_{\rm CP} = -\pi/2$ which are favored by the present data from T2K and NOνA [46, 47]. The reason is because the general neutrino mass m_ν shown in eq. (4.12) has a accidental $\mu\tau$ reflection symmetry in the case of $\eta = \pi$ [48].

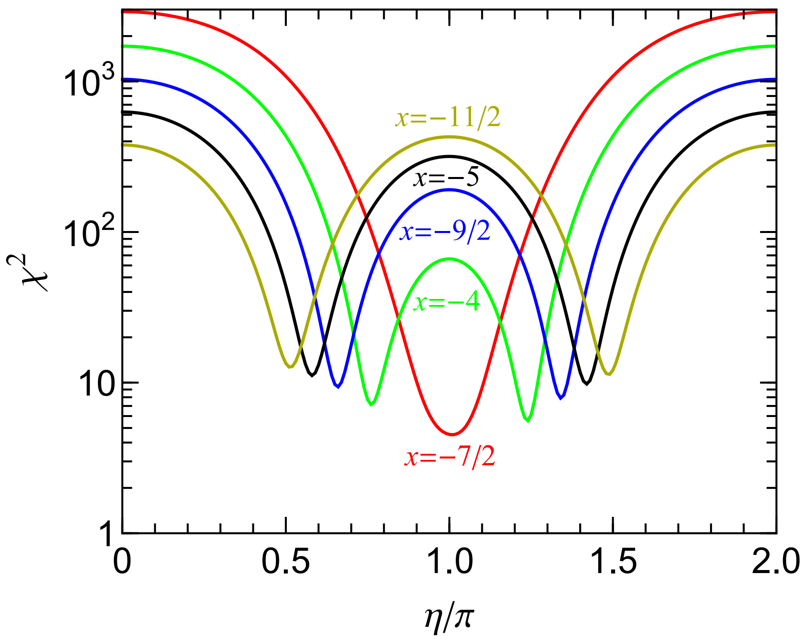

Figure 4. Variation of χ^2 with respect to the phase η for the typical values of $x = -7/2$, -4, $-9/2$, -5, $-11/2$.

It is noteworthy that the cases predicting inverted neutrino masses can also be achieved from the tri-direct CP approach, although the neutrino mass spectrum is determined to be normal ordering in the present Littlest Seesaw variants, as shown in table 1. In the following, we shall present an example giving inverted neutrino masses, for more other examples see [36]. Analogous to the above Littlest Seesaw variants, both lepton doublet L and the atmospheric flavon $\phi_{\rm atm}$ are assigned to S_4 triplet $\mathbf{3}$, the solar flavon $\phi_{\rm sol}$ transforms as $\mathbf{3'}$ while the right-handed neutrino $N^c_{\rm atm}$ is $\mathbf{1}$ and $N^c_{\rm sol}$ is $\mathbf{1'}$ of S_4. The residual flavor symmetry of the charged lepton sector is taken to be $G_l = Z_3^T$. The residual symmetries of the atmospheric neutrino and the solar neutrino sectors are $Z_2^U \times H^{\rm atm}_{\rm CP}$ and $Z_2^{TU} \times H^{\rm sol}_{\rm CP}$ respectively with $H^{\rm atm}_{\rm CP} = \{1, U\}$ and $H^{\rm sol}_{\rm CP} = \{U, T\}$. Then it is easy to check that the alignments of $\phi_{\rm atm}$ and $\phi_{\rm sol}$ are $\langle\phi_{\rm atm}\rangle \propto (0, 1, -1)^T$ and $\langle\phi_{\rm sol}\rangle \propto \left(1, x\omega, x\omega^2\right)^T$ respectively, where x is a real parameter. As a consequence, the neutrino mass matrix is of the form

$$m_\nu = m_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + m_s e^{i\eta} \begin{pmatrix} 1 & x\omega^2 & x\omega \\ x\omega^2 & x^2\omega & x^2 \\ x\omega & x^2 & x^2\omega^2 \end{pmatrix}, \tag{4.28}$$

where m_a and m_s are real parameters. We find the lepton mixing matrix is given by

$$U_{PMNS} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{2e^{-i\psi}\sin\theta}{\sqrt{2+x^2}} & \frac{2\cos\theta}{\sqrt{2+x^2}} & -\frac{\sqrt{2}x}{\sqrt{2+x^2}} \\ -\cos\theta - \frac{xe^{-i\psi}\sin\theta}{\sqrt{2+x^2}} & e^{i\psi}\sin\theta - \frac{x\cos\theta}{\sqrt{2+x^2}} & -\frac{\sqrt{2}}{\sqrt{2+x^2}} \\ \cos\theta - \frac{xe^{-i\psi}\sin\theta}{\sqrt{2+x^2}} & -e^{i\psi}\sin\theta - \frac{x\cos\theta}{\sqrt{2+x^2}} & -\frac{\sqrt{2}}{\sqrt{2+x^2}} \end{pmatrix} P_\nu\,, \tag{4.29}$$

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x	η	m_a(meV)	r	$\chi^2_{\rm min}$	$\sin^2\theta_{13}$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	$\delta_{\rm CP}/\pi$	β/π	m_2(meV)	m_3(meV)	m_{ee}(meV)
-4	$\frac{3\pi}{4}$	3.723	0.419	7.703	0.0226	0.336	0.440	-0.588	-0.264	8.603	49.939	2.843
	$\frac{5\pi}{4}$	3.723	0.419	6.130	0.0226	0.336	0.560	-0.412	0.264	8.603	49.939	2.843
	$\frac{4\pi}{5}$	3.690	0.425	12.206	0.0204	0.338	0.450	-0.577	-0.211	8.577	49.974	2.591
	$\frac{6\pi}{5}$	3.691	0.425	10.716	0.0204	0.338	0.550	-0.423	0.211	8.578	49.973	2.591
	$\frac{7\pi}{9}$	3.706	0.422	8.135	0.0213	0.337	0.446	-0.583	-0.234	8.592	49.954	2.701
	$\frac{11\pi}{9}$	3.706	0.422	6.588	0.0213	0.337	0.555	-0.417	0.234	8.593	49.953	2.701
-5	$\frac{3\pi}{5}$	3.737	0.264	13.077	0.0211	0.345	0.423	-0.622	-0.423	8.625	49.914	3.558
	$\frac{7\pi}{5}$	3.736	0.264	11.693	0.0211	0.345	0.577	-0.378	0.422	8.623	49.916	3.557
	$\frac{4\pi}{7}$	3.738	0.262	11.547	0.0226	0.344	0.422	-0.619	-0.452	8.586	49.962	3.647
	$\frac{10\pi}{7}$	3.737	0.263	10.201	0.0226	0.344	0.578	-0.381	0.452	8.585	49.964	3.646
	$\frac{5\pi}{9}$	3.736	0.262	14.269	0.0234	0.344	0.421	-0.618	-0.469	8.559	49.999	3.694
	$\frac{13\pi}{9}$	3.735	0.262	12.937	0.0234	0.344	0.579	-0.382	0.469	8.557	50.001	3.693
$-\frac{7}{2}$	π	3.720	0.553	4.528	0.0227	0.332	0.5	-0.5	0	8.622	49.918	1.663
	$\frac{8\pi}{9}$	3.750	0.546	12.130	0.0241	0.331	0.470	-0.542	-0.117	8.663	49.871	1.955
	$\frac{10\pi}{9}$	3.750	0.546	11.158	0.0241	0.331	0.530	-0.458	0.117	8.663	49.870	1.955
	$\frac{9\pi}{10}$	3.744	0.547	10.125	0.0238	0.331	0.473	-0.538	-0.105	8.656	49.878	1.903
	$\frac{11\pi}{10}$	3.745	0.547	9.248	0.0238	0.331	0.527	-0.462	0.105	8.657	49.878	1.904
$-\frac{9}{2}$	$\frac{2\pi}{3}$	3.724	0.329	9.712	0.0217	0.341	0.429	-0.610	-0.352	8.605	49.937	3.288
	$\frac{4\pi}{3}$	3.724	0.329	8.224	0.0217	0.341	0.571	-0.390	0.352	8.604	49.938	3.287
	$\frac{5\pi}{8}$	3.732	0.326	15.464	0.0239	0.340	0.424	-0.610	-0.396	8.566	49.990	3.454
	$\frac{11\pi}{8}$	3.731	0.326	14.061	0.0239	0.340	0.576	-0.390	0.396	8.564	49.992	3.454
	$\frac{7\pi}{10}$	3.708	0.332	17.788	0.0199	0.342	0.433	-0.607	-0.317	8.610	49.931	3.147
	$\frac{13\pi}{10}$	3.708	0.332	16.242	0.0199	0.342	0.567	-0.393	0.317	8.610	49.932	3.147

Table 1. The best fit values of the lepton mixing angles, CP violation phases $\delta_{\rm CP}$ and β, and the neutrino masses m_2 and m_3 for some typical values of x and η in new variants of Littlest seesaw model arising from the tri-direct CP approach with S_4. The predictions for the effective Majorana mass m_{ee} are listed in the last column. We would like to remind that the lightest neutrino is massless $m_1 = 0$ for each case.

with

$$P_\nu = \text{diag}\left(e^{i(\psi+\rho)/2}, e^{i(-\psi+\sigma)/2}, 1\right)\,. \tag{4.30}$$

The three lepton mixing angles can be read off as,

$$\sin^2\theta_{13} = \frac{x^2}{2+x^2}\,, \quad \sin^2\theta_{12} = \cos^2\theta\,, \quad \sin^2\theta_{23} = \frac{1}{2}\,. \tag{4.31}$$

It is notable that the atmospheric mixing angle θ_{23} is exactly maximal. Moreover, the two CP rephasing invariants are given by

$$J_{\rm CP} = -\frac{x\sin 2\theta\sin\psi}{2\,(2+x^2)^{3/2}}\,, \qquad I_1 = -\frac{\sin^2 2\theta\sin(\rho-\sigma)}{(2+x^2)^2}\,. \tag{4.32}$$

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The experimentally measured values of lepton mixing angles and neutrino masses can be accommodated well in this case, e.g.

$$\begin{aligned}
&x=-0.213, && \eta=-0.0171\pi, && m_a=25.670\,\text{meV}, && r=1.823\,,\\
&\sin^2\theta_{13}=0.0223, && \sin^2\theta_{12}=0.307, && \sin^2\theta_{23}=0.5,\ \delta_{CP}/\pi=0.975, && \beta/\pi=-0.174,\\
&m_1=49.193\,\text{meV}, && m_2=49.940\,\text{meV}, && m_3=0, && m_{ee}=46.579\,\text{meV}\,.
\end{aligned} \tag{4.33}$$

From the examples of Littlest Seesaw model and its variants studied above, we see that the light neutrino mass matrix m_ν is generally predicted to depend on four parameters m_a, m_s, x and η in the tri-direct CP approach, and m_ν would depend on only two parameters m_a and m_s once x and η are fixed to certain simple regular values by explicit superpotential alignment terms in a concrete model. Nevertheless, the neutrino mass matrix m_ν generally involve six complex parameters in the original two right-handed neutrino model. Obviously the tri-direct CP model is rather predictive beyond all doubt.

5 Extending the tri-direct CP approach to three right-handed neutrino models

Motivated by the principle of minimality and the idea of constrained sequential dominance, in section 2 we have assumed that there are only two right-handed neutrinos and the third one is approximately decoupled. In fact, the tri-direct CP approach is a general paradigm of neutrino mass model building, and it is not mandatory to have two right-handed neutrinos. This approach can be straightforwardly extended to the conventional seesaw model with three right-handed neutrinos denote as $N^c_{\rm atm}$, $N^c_{\rm sol}$ and $N^c_{\rm dec}$. Then the Lagrangian for the charged lepton and neutrino masses takes the form

$$\begin{aligned}
\mathcal{L} = &-y_l L\phi_l E^c - y_{\rm atm}L\phi_{\rm atm}N^c_{\rm atm} - y_{\rm sol}L\phi_{\rm sol}N^c_{\rm sol} - y_{\rm dec}L\phi_{\rm dec}N^c_{\rm dec} - \frac{1}{2}x_{\rm atm}\xi_{\rm atm}N^c_{\rm atm}N^c_{\rm atm}\\
&-\frac{1}{2}x_{\rm sol}\xi_{\rm sol}N^c_{\rm sol}N^c_{\rm sol} - \frac{1}{2}x_{\rm dec}\xi_{\rm dec}N^c_{\rm sol}N^c_{\rm dec} + \text{h.c.}\,,
\end{aligned} \tag{5.1}$$

where two additional terms related to $N^c_{\rm dec}$ appear in comparison with eq. (2.1). The field $\phi_{\rm dec}$ can be either Higgs fields or combination of the electroweak Higgs doublet together with flavons, and it transform as triplet under the flavor symmetry G_f. The Majoron flavon $\xi_{\rm dec}$ are standard model and G_f singlet. Similar to section 2, we assume that the residual subgroups G_l, $G_{\rm atm}\rtimes H^{\rm atm}_{\rm CP}$, $G_{\rm sol}\rtimes H^{\rm sol}_{\rm CP}$ and $G_{\rm dec}\rtimes H^{\rm dec}_{\rm CP}$ are preserved by the Yukawa interaction terms of the charged leptons E^c, the atmospheric neutrino $N^c_{\rm atm}$, the solar neutrino $N^c_{\rm sol}$ and the decoupled neutrino $N^c_{\rm dec}$ respectively. The vacuum alignments $\langle\phi_{\rm atm}\rangle$, $\langle\phi_{\rm sol}\rangle$ and $\langle\phi_{\rm dec}\rangle$ are dictated by the residual flavor and CP symmetries, and they constitute three columns of the Dirac neutrino mass matrix,

$$\begin{aligned}
m_D &= \Big(y_{\rm atm}\langle\phi_{\rm atm}\rangle,\quad y_{\rm sol}\langle\phi_{\rm sol}\rangle,\quad y_{\rm dec}\langle\phi_{\rm dec}\rangle\Big)\,,\\
m_N &= \begin{pmatrix} x_{\rm atm}\langle\xi_{\rm atm}\rangle & 0 & 0\\ 0 & x_{\rm sol}\langle\xi_{\rm sol}\rangle & 0\\ 0 & 0 & x_{\rm dec}\langle\xi_{\rm dec}\rangle\end{pmatrix}.
\end{aligned} \tag{5.2}$$

The light neutrino mass matrix given by the seesaw formula is

$$m_\nu = -\frac{y_{\rm atm}^2}{x_{\rm atm}}\frac{\langle\phi_{\rm atm}\rangle\langle\phi_{\rm atm}\rangle^T}{\langle\xi_{\rm atm}\rangle} - \frac{y_{\rm sol}^2}{x_{\rm sol}}\frac{\langle\phi_{\rm sol}\rangle\langle\phi_{\rm sol}\rangle^T}{\langle\xi_{\rm sol}\rangle} - \frac{y_{\rm dec}^2}{x_{\rm dec}}\frac{\langle\phi_{\rm dec}\rangle\langle\phi_{\rm dec}\rangle^T}{\langle\xi_{\rm dec}\rangle}\,. \tag{5.3}$$

In the limit $\langle\xi_{\rm dec}\rangle \gg \langle\xi_{\rm atm}\rangle, \langle\xi_{\rm sol}\rangle$, it reduces to the setup discussed in section 2. Here we shall give an example for illustration. The lepton doublet L, the atmospheric flavon $\phi_{\rm atm}$ and the flavon $\phi_{\rm dec}$ are assumed to transforms as $\mathbf{3}$ under S_4, the solar flavon $\phi_{\rm sol}$ transforms as $\mathbf{3}'$ while the right-handed neutrinos $N^c_{\rm atm}$ and $N^c_{\rm dec}$ are S_4 singlet $\mathbf{1}$ and $N^c_{\rm sol}$ is $\mathbf{1}'$ under S_4. The flavor group S_4 and CP symmetry are broken to $G_l = Z_3^T$ in the charged lepton sector. The residual symmetries of the atmospheric neutrino, the solar neutrino and the decoupled neutrino sectors are $Z_2^U \times H_{\rm CP}^{\rm atm}$, $Z_2^{SU} \times H_{\rm CP}^{\rm sol}$ and $Z_2^{TST^2} \times H_{\rm CP}^{\rm dec}$ respectively, where the residual CP symmetries are given by $H_{\rm CP}^{\rm atm} = \{1, U\}$, $H_{\rm CP}^{\rm sol} = \{1, SU\}$ and $H_{\rm CP}^{\rm dec} = \{SU, T^2STU\}$. The vacuum alignments of $\phi_{\rm atm}$, $\phi_{\rm sol}$ and $\phi_{\rm dec}$ are constrained by the residual symmetry to take the following form

$$\langle\phi_{\rm atm}\rangle \propto (0,1,-1)^T\,, \quad \langle\phi_{\rm dec}\rangle \propto \left(1,\omega^2,\omega\right)^T\,, \quad \langle\phi_{\rm sol}\rangle \propto (1,3,-1)^T\,. \tag{5.4}$$

Then we can read out the light neutrino mass matrix

$$m_\nu = m_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + m_a r_1 e^{i\eta_1} \begin{pmatrix} 1 & -1 & 3 \\ -1 & 1 & -3 \\ 3 & -3 & 9 \end{pmatrix} + m_a r_2 e^{i\eta_2} \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{pmatrix}\,, \tag{5.5}$$

where m_a, r_1, r_2, η_1 and η_2 are real. Excellent agreement with experimental data can be achieved in this scenario, and a numerical benchmark is

$$\begin{aligned} &m_a = 26.707\,\text{meV}, \quad r_1 = 0.101, \quad r_2 = 0.00300, \quad \eta_1 = 2\pi/3, \quad \eta_2 = 0\,, \\ &\sin^2\theta_{13} = 0.0221, \quad \sin^2\theta_{12} = 0.318, \quad \sin^2\theta_{23} = 0.486\,, \\ &\delta_{\rm CP} = -0.519\pi, \quad \alpha_{21} = -0.533\pi, \quad \alpha_{31} = -0.135\pi\,, \\ &m_1 = 0.120\,\text{meV}, \quad m_2 = 8.601\,\text{meV}, \quad m_3 = 49.942\,\text{meV}, \quad m_{ee} = 2.648\,\text{meV}\,. \end{aligned} \tag{5.6}$$

It is interesting to perform a comprehensive study of lepton mixing patterns which can be obtained from S_4 and other flavor groups from the tri-direct CP approach with three right-handed neutrinos. The phenomenological implications and model building aspects of such a scheme also deserve further investigation. These topics goes far beyond the scope of this paper since they deserve a dedicated full work of its own.

Before closing this section, we would like to compare the tri-direct CP approach with other flavor symmetry model building schemes such as direct approach, semidirect approach and indirect approach. In common with the indirect approach [7], each column of the Dirac neutrino mass matrix is effectively promoted to a flavon field which transforms as a triplet under the flavor symmetry G_f. The tri-direct CP approach generalizes the indirect approach to constrain the vacuum alignment by residual flavor and CP symmetries. As regards the direct [7] and semidirect models [31, 35], a residual symmetry $Z_2\times Z_2$ or $Z_2\times CP$ is preserved by the neutrino mass matrix, the lepton mixing matrix is fully determined by residual symmetry or determined in terms of a single real parameter θ while neutrino masses are not constrained [7, 31, 35]. In direct and semidirect models, only the structure of flavor symmetry group and the remnant symmetries are assumed, the neutrino mass generation mechanism and the breaking mechanism of flavor and CP symmetries are irrelevant such that one can analyze the predictions for lepton mixing parameters in a model-independent way [7, 31, 35]. Similar to the semidirect models with residual symmetry $Z_2 \times CP$, the assumed residual symmetry of tri-direct CP approach fixes one column of the neutrino mixing matrix to be $\langle\phi_{\rm atm}\rangle \times \langle\phi_{\rm sol}\rangle$ which could depend on a real parameter x. However, there is no common residual symmetry of the light neutrino mass matrix in the tri-direc CP approach since the residual symmetries $G_{\rm atm} \rtimes H^{\rm atm}_{\rm CP}$ and $G_{\rm sol} \rtimes H^{\rm sol}_{\rm CP}$ of the atmospheric and solar neutrino sectors are generally different. In contrast with direct and semidirect approaches, the tri-direct CP approach can predict neutrino masses. As shown in sections 3 and 4, the light neutrino mass matrix m_ν generally depends on only four real parameters m_a, m_s, x and η such that lepton mixing angles, CP phases and neutrino masses are strongly correlated with each other in tri-direct CP models. We remark that the tri-direct CP approach assumes type-I seesaw mechanism for neutrino mass generation, consequently it is not applicable in radiative neutrino mass models and so on.

6 A concrete model

In this section, we shall construct an explicit model based on the model independent analysis in section 4. The flavor symmetry S_4 together with CP symmetry is imposed in the model. The auxiliary symmetry is taken to be $Z_5 \times Z_6 \times Z_8$ to ensure the needed vacuum alignment and to forbid unwanted couplings. The auxiliary symmetry Z_6 is helpful to reproduce the observed charged lepton mass hierarchies, and it imposes different powers of flavon fields for the electron, muon and tauon terms. The shaping symmetry Z_8 disentangles the charged lepton sector from the neutrino sector, and Z_5 further distinguishes the atmospheric neutrino sector from the solar neutrino sector. Moreover, it is straightforward to show that such a symmetry is sufficient to suppress higher dimensional terms. The spontaneous breaking of S_4 and CP symmetries to the residual symmetries Z_3^T, $Z_2^{TST^2} \times X_{\rm atm}$ and $Z_2^U \times X_{\rm sol}$ in the charged lepton, atmospheric neutrino and solar neutrino sectors are achieved in the model, where the residual CP transformations $X_{\rm atm} = SU$ and $X_{\rm sol} = U$. As a consequence, the desired vacuum configurations in eqs. (4.4), (4.8) are naturally produced. The solar alignment parameter is fixed to be $x = -7/2$ with $\eta = \pi$ through the dynamical terms in the potential. We formulate our model in the framework of supersymmetry since the minimization of the scalar potential would be considerably simplified. The three generations of left-handed lepton doublets L are embedded into a triplet $\mathbf{3}$, while the right-handed charged leptons e^c, μ^c and τ^c all transform as $\mathbf{1}$ yet they carry different charges of shaping symmetry. The two right-handed neutrinos $\nu^c_{\rm atm}$ and $\nu^c_{\rm sol}$ are assigned to $\mathbf{1}$ and $\mathbf{1}'$ of S_4 respectively. The S_4 flavor symmetry is then broken by suitable flavons which are singlets under the standard model gauge group. The fields of the model and their classification under the symmetry are summarized in table 2. Similar to other flavor models in the literature [5–10], additional flavon fields besides the necessary flavons ϕ_a,

	L	e^c	μ^c	τ^c	$\nu^c_{\rm atm}$	$\nu^c_{\rm sol}$	$H_{u,d}$	η_l	ϕ_l	ξ_a	ϕ_a	ξ_s	ζ_s	η_s	χ_s	ψ_s	φ_s	ϕ_s	ξ^0_l	ϕ^0_l	ξ^0_a	ϕ^0_a	κ^0	ρ^0	σ^0	η^0	χ^0	ϕ^0_s	ξ^0_s
S_4	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}'$	$\mathbf{3}'$	$\mathbf{3}'$	$\mathbf{3}'$	$\mathbf{1}$	$\mathbf{3}'$	$\mathbf{1}$	$\mathbf{3}'$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{3}'$	$\mathbf{3}'$	$\mathbf{1}$
Z_5	1	1	1	1	1	ω_5^4	1	1	1	1	1	ω_5^2	ω_5	ω_5	ω_5^2	ω_5^4	ω_5^3	ω_5	1	1	1	1	ω_5^4	ω_5	ω_5^2	ω_5^3	ω_5	ω_5^3	ω_5^3
Z_6	1	ω_6^3	ω_6^4	ω_6^5	ω_6	1	1	ω_6	ω_6	ω_6^4	ω_6^5	1	1	1	1	1	1	1	ω_6^4	ω_6^4	ω_6^2	ω_6^2	1	1	1	1	1	1	1
Z_8	1	1	1	1	ω_8	ω_8^2	1	1	1	ω_8^6	ω_8^7	ω_8^4	ω_8^4	ω_8	ω_8	ω_8	ω_8	ω_8^6	1	1	ω_8^2	ω_8^2	ω_8^6	ω_8^6	ω_8^6	ω_8^6	ω_8^6	ω_8^6	ω_8^4

Table 2. Fields and their transformation properties under the flavor symmetry $S_4 \times Z_5 \times Z_6 \times Z_8$, where the phases are $\omega_5 = e^{2\pi i/5}$, $\omega_6 = e^{\pi i/3}$ and $\omega_8 = e^{\pi i/4}$.

ϕ_s, ξ_a and ξ_s are required in order to achieve the desired vacuum configuration. Their transformation properties are shown in table 2. We would like to remind the readers that we adopt the convention of [35] for the S_4 group, and all the Clebsch-Gordan coefficients have been listed in the appendix of [35].

6.1 Vacuum alignment

We will use the supersymmetric F-term alignment mechanism to generate the flavon VEVs in eqs. (4.4) and (4.8). A $\mathrm{U}(1)_R$ symmetry related to R-parity and the presence of driving fields in the flavon superpotential are common features of this mechanism. The driving fields indicated with the superscript "0" and the symmetry assignments are collected in table 2. As usual, the VEVs of the driving fields are assumed to be vanishing. In the supersymmetric limit, the F-terms of the driving fields have to vanish such that the vacuum of the flavons gets aligned. The minimization equation of the scalar potential for the charged lepton, atmospheric neutrino and solar neutrino sectors are separated from each other at the renormalizable level.

At leading order, the most general driving superpotential w_d invariant under $S_4 \times Z_5 \times Z_6 \times Z_8$ is given by

$$w_d = w_d^l + w_d^{\rm atm} + w_d^{\rm sol}\,, \tag{6.1}$$

where the three parts w_d^l, $w_d^{\rm atm}$ and $w_d^{\rm sol}$ read

$$\begin{aligned}
w_d^l &= g_1\xi^0_l\left(\eta_l\eta_l\right)_{\mathbf{1}} + g_2\xi^0_l\left(\phi_l\phi_l\right)_{\mathbf{1}} + g_3\left(\phi^0_l\left(\eta_l\phi_l\right)_{\mathbf{3}'}\right)_{\mathbf{1}} + g_4\left(\phi^0_l\left(\phi_l\phi_l\right)_{\mathbf{3}'}\right)_{\mathbf{1}}\,,\\
w_d^{\rm atm} &= M_{\xi_a}\xi^0_a\xi_a + h_1\xi^0_a\left(\phi_a\phi_a\right)_{\mathbf{1}} + h_2\left(\phi^0_a\left(\phi_a\phi_a\right)_{\mathbf{3}'}\right)_{\mathbf{1}}\,,\\
w_d^{\rm sol} &= f_1\kappa^0\left(\chi_s\psi_s\right)_{\mathbf{1}} + f_2\kappa^0\left(\varphi_s\varphi_s\right)_{\mathbf{1}} + f_3\left(\rho^0\left(\chi_s\chi_s\right)_{\mathbf{2}}\right)_{\mathbf{1}} + f_4\left(\sigma^0\left(\psi_s\psi_s\right)_{\mathbf{2}}\right)_{\mathbf{1}} + f_5\left(\eta^0\left(\eta_s\eta_s\right)_{\mathbf{2}}\right)_{\mathbf{1}}\\
&\quad + f_6\left(\eta^0\left(\varphi_s\psi_s\right)_{\mathbf{2}}\right)_{\mathbf{1}} + f_7\left(\chi^0\left(\chi_s\chi_s\right)_{\mathbf{3}'}\right)_{\mathbf{1}} + f_8\left(\chi^0\left(\eta_s\varphi_s\right)_{\mathbf{3}'}\right)_{\mathbf{1}} + f_9\left(\phi^0_s\left(\psi_s\varphi_s\right)_{\mathbf{3}'}\right)_{\mathbf{1}}\\
&\quad + f_{10}\zeta_s\left(\phi^0_s\phi_s\right)_{\mathbf{1}} + M_{\xi_s}\xi^0_s\xi_s + f_{11}\xi^0_s\left(\phi_s\phi_s\right)_{\mathbf{1}}\,,
\end{aligned} \tag{6.2}$$

where $(\ldots)_{\mathbf{r}}$ refers to a contraction of the S_4 indices into the representation $\mathbf{r}$. All the coupling constants are real parameters since the theory is required to be invariant under the generalized CP transformations. The driving superpotential w_d^l is responsible for the alignment of η_l and ϕ_l. The equations for the vanishing of the derivatives of w_d^l with respect to each component of the driving fields ξ_l^0 and ϕ_l^0 are

$$\begin{aligned}
\frac{\partial w_d^l}{\partial \xi_l^0} &= 2g_1\eta_{l_1}\eta_{l_2} + g_2\left(\phi_{l_1}^2 + 2\phi_{l_2}\phi_{l_3}\right) = 0\,,\\
\frac{\partial w_d^l}{\partial \phi_{l_1}^0} &= g_3(\eta_{l_1}\phi_{l_2} - \eta_{l_2}\phi_{l_3}) + 2g_4\left(\phi_{l_1}^2 - \phi_{l_2}\phi_{l_3}\right) = 0\,,\\
\frac{\partial w_d^l}{\partial \phi_{l_2}^0} &= g_3(\eta_{l_1}\phi_{l_1} - \eta_{l_2}\phi_{l_2}) + 2g_4\left(\phi_{l_2}^2 - \phi_{l_1}\phi_{l_3}\right) = 0\,,\\
\frac{\partial w_d^l}{\partial \phi_{l_3}^0} &= g_3(\eta_{l_1}\phi_{l_3} - \eta_{l_2}\phi_{l_1}) + 2g_4\left(\phi_{l_3}^2 - \phi_{l_1}\phi_{l_2}\right) = 0\,.
\end{aligned} \tag{6.3}$$

These equations are satisfied by the alignment

$$\langle\eta_l\rangle = (0, v_{\eta_l})^T\,, \qquad \langle\phi_l\rangle = (0, v_{\phi_l}, 0)^T\,, \quad \text{with} \quad v_{\phi_l} = \frac{g_3}{2g_4}v_{\eta_l}\,, \tag{6.4}$$

where v_{η_l} is undetermined. In the atmospheric neutrino sector, the F-flatness condition of the driving fields ξ_a^0 and ϕ_a^0 leads to

$$\begin{aligned}
\frac{\partial w_d^{\text{atm}}}{\partial \xi_a^0} &= M_{\xi_a}\xi_a + h_1\left(\phi_{a_1}^2 + 2\phi_{a_2}\phi_{a_3}\right) = 0\,,\\
\frac{\partial w_d^{\text{atm}}}{\partial \phi_{a_1}^0} &= h_2\left(2\phi_{a_1}^2 - 2\phi_{a_2}\phi_{a_3}\right) = 0\,,\\
\frac{\partial w_d^{\text{atm}}}{\partial \phi_{a_2}^0} &= h_2\left(2\phi_{a_2}^2 - 2\phi_{a_1}\phi_{a_3}\right) = 0\,,\\
\frac{\partial w_d^{\text{atm}}}{\partial \phi_{a_3}^0} &= h_2\left(2\phi_{a_3}^2 - 2\phi_{a_1}\phi_{a_2}\right) = 0\,,
\end{aligned} \tag{6.5}$$

from which we can extract the vacuum expectation values of ξ_a and ϕ_a,[3]

$$\langle\xi_a\rangle = v_{\xi_a}, \quad \langle\phi_a\rangle = v_{\phi_a}\left(1, \omega^2, \omega\right)^T, \quad \text{with} \quad v_{\phi_a}^2 = -\frac{M_{\xi_a}}{3h_1}v_{\xi_a}\,. \tag{6.6}$$

We see that the desired atmospheric flavon alignment of ϕ_a in eq. (4.4) is realized, and it preserves the subgroup $Z_2^{TST^2}$. Moreover, the ratio $v_{\phi_a}^2/v_{\xi_a} = -\frac{M_{\xi_a}}{3h_1}$ contributing to the parameter m_a is determined to be real. Then we proceed to derive the solar flavon alignment in a short sequence of steps. The F-term conditions of the driving field ρ^0 and σ^0 read

$$\begin{aligned}
\frac{\partial w_d^{\text{sol}}}{\partial \rho_1^0} &= f_3\left(2\chi_{s_1}\chi_{s_2} + \chi_{s_3}^2\right) = 0\,,\\
\frac{\partial w_d^{\text{sol}}}{\partial \rho_2^0} &= f_3\left(2\chi_{s_1}\chi_{s_3} + \chi_{s_2}^2\right) = 0\,,\\
\frac{\partial w_d^{\text{sol}}}{\partial \sigma_1^0} &= f_4\left(2\psi_{s_1}\psi_{s_2} + \psi_{s_3}^2\right) = 0\,,\\
\frac{\partial w_d^{\text{sol}}}{\partial \sigma_2^0} &= f_4\left(2\psi_{s_1}\psi_{s_3} + \psi_{s_2}^2\right) = 0\,.
\end{aligned} \tag{6.7}$$

[3] The alignment of ϕ_a can also be along the direction $(1,1,1)^T$ and it is related to the chosen one in eq. (6.6) by a T transformation.

A solution to this set of equations is given by

$$\langle \chi_s \rangle = v_{\chi_s}(1,0,0)^T, \qquad \langle \psi_s \rangle = v_{\psi_s}\,(1,-2,-2)^T\ . \tag{6.8}$$

Subsequently the F-flatness of the driving field η^0 and χ^0 leads to

$$\begin{aligned}
\frac{\partial w_d^{\rm sol}}{\partial \eta_1^0} &= f_5\eta_{s_1}^2 + f_6(\varphi_{s_1}\psi_{s_2}+\varphi_{s_2}\psi_{s_1}+\varphi_{s_3}\psi_{s_3}) = 0\,,\\
\frac{\partial w_d^{\rm sol}}{\partial \eta_2^0} &= f_5\eta_{s_2}^2 + f_6(\varphi_{s_1}\psi_{s_3}+\varphi_{s_2}\psi_{s_2}+\varphi_{s_3}\psi_{s_1}) = 0\,,\\
\frac{\partial w_d^{\rm sol}}{\partial \chi_1^0} &= 2f_7\left(\chi_{s_1}^2-\chi_{s_2}\chi_{s_3}\right) + f_8(\eta_{s_1}\varphi_{s_2}+\eta_{s_2}\varphi_{s_3}) = 0\,,\\
\frac{\partial w_d^{\rm sol}}{\partial \chi_2^0} &= 2f_7\left(\chi_{s_2}^2-\chi_{s_1}\chi_{s_3}\right) + f_8(\eta_{s_1}\varphi_{s_1}+\eta_{s_2}\varphi_{s_2}) = 0\,,\\
\frac{\partial w_d^{\rm sol}}{\partial \chi_3^0} &= 2f_7\left(\chi_{s_3}^2-\chi_{s_1}\chi_{s_2}\right) + f_8(\eta_{s_1}\varphi_{s_3}+\eta_{s_2}\varphi_{s_1}) = 0\,,
\end{aligned} \tag{6.9}$$

which generate the alignment

$$\langle \eta_s \rangle = v_{\eta_s}\,(1,1)^T\,, \qquad \langle \varphi_s \rangle = v_{\varphi_s}\,(1,-1,-1)^T\ , \tag{6.10}$$

with

$$v_{\varphi_s} = \frac{f_5 v_{\eta_s}^2}{f_6 v_{\psi_s}}, \qquad v_{\chi_s}^2 = \frac{f_5 f_8 v_{\eta_s}^3}{f_6 f_7 v_{\psi_s}}\,. \tag{6.11}$$

Furthermore, we notice that the contraction of $\langle\psi_s\rangle$ and $\langle\varphi_s\rangle$ to an S_4 triplet $\mathbf{3'}$ is of the form

$$(\langle\psi_s\rangle\langle\varphi_s\rangle)_{\mathbf{3'}} \propto \begin{pmatrix} -2\\ 7\\ 7\end{pmatrix}\ . \tag{6.12}$$

Here we have used the S_4 contraction rule for $\mathbf{3'}\otimes\mathbf{3'}\to\mathbf{3'}$: $(\alpha\beta)_{\mathbf{3'}} = (2\alpha_1\beta_1-\alpha_2\beta_3-\alpha_3\beta_2, 2\alpha_3\beta_3-\alpha_1\beta_2-\alpha_2\beta_1, 2\alpha_2\beta_2-\alpha_1\beta_3-\alpha_3\beta_1)^T$, where $\alpha=(\alpha_1,\alpha_2,\alpha_3)^T$ and $\beta=(\beta_1,\beta_2,\beta_3)^T$ transform as $\mathbf{3'}$ [35]. Therefore the solar flavon alignment arises from the ϕ_s^0 dependent driving terms $f_9\left(\phi_s^0\,(\psi_s\varphi_s)_{\mathbf{3'}}\right)_{\mathbf{1}} + f_{10}\zeta_s\left(\phi_s^0\phi_s\right)_{\mathbf{1}}$, and accordingly the minimization equations of the scalar potential are given by

$$\begin{aligned}
\frac{\partial w_d^{\rm sol}}{\partial \phi_{s_1}^0} &= f_{10}\zeta_s\phi_{s_1} + f_9(2\varphi_{s_1}\psi_{s_1}-\varphi_{s_2}\psi_{s_3}-\varphi_{s_3}\psi_{s_2}) = 0\,,\\
\frac{\partial w_d^{\rm sol}}{\partial \phi_{s_2}^0} &= f_{10}\zeta_s\phi_{s_3} + f_9(-\varphi_{s_1}\psi_{s_3}+2\varphi_{s_2}\psi_{s_2}-\varphi_{s_3}\psi_{s_1}) = 0\,,\\
\frac{\partial w_d^{\rm sol}}{\partial \phi_{s_3}^0} &= f_{10}\zeta_s\phi_{s_2} + f_9(-\varphi_{s_1}\psi_{s_2}-\varphi_{s_2}\psi_{s_1}+2\varphi_{s_3}\psi_{s_3}) = 0\,,
\end{aligned} \tag{6.13}$$

which uniquely determine the solar alignment,

$$\langle\zeta_s\rangle = v_{\zeta_s}, \quad \langle\phi_s\rangle = v_{\phi_s}\,(1,-7/2,-7/2)^T\ , \quad \text{with} \quad v_{\phi_s} = \frac{2f_9 v_{\varphi_s} v_{\psi_s}}{f_{10} v_{\zeta_s}}\,. \tag{6.14}$$

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Finally the F-term conditions of κ^0 and ξ_s^0 are

$$\begin{aligned}\frac{\partial w_d^{\rm sol}}{\partial \kappa^0} &= f_1(\chi_{s_1}\psi_{s_1}+\chi_{s_2}\psi_{s_3}+\chi_{s_3}\psi_{s_2})+f_2(\varphi_{s_1}^2+2\varphi_{s_2}\varphi_{s_3})=0\,,\\ \frac{\partial w_d^{\rm sol}}{\partial \xi_s^0} &= M_{\xi_s}\xi_s+f_{11}\left(\phi_{s_1}^2+2\phi_{s_2}\phi_{s_3}\right)=0\,,\end{aligned} \tag{6.15}$$

which together with eq. (6.11) lead to the following relations among the VEVs v_{ξ_s}, v_{η_s}, v_{χ_s}, v_{ψ_s}, v_{φ_s} and v_{ϕ_s}

$$v_{\chi_s}^5=-\frac{f_1f_5f_8^3}{3f_2f_6f_7^3}v_{\eta_s}^5,\quad v_{\varphi_s}=\frac{f_7v_{\chi_s}^2}{f_8v_{\eta_s}},\quad v_{\psi_s}=\frac{f_5f_8v_{\eta_s}^3}{f_6f_7v_{\chi_s}^2},\quad v_{\phi_s}^2=-\frac{2M_{\xi_s}v_{\xi_s}}{51f_{11}}\,. \tag{6.16}$$

Notice that the vacuum configurations of η_s, χ_s, ψ_s, φ_s and ϕ_s preserve the subgroup Z_2^U. In addition, the ratio $v_{\phi_s}^2/v_{\xi_s}=-\frac{2M_{\xi_s}}{51f_{11}}$ which contributes to the parameter m_s is real because of the CP symmetry.

As regards the higher order corrections to the driving superpotential w_d, we note the operators comprising one driving field and three flavons are forbidden for the assignments in table 1. The subleading contributions to w_d^l which can shift the vacuum of η_l and ϕ_l in eq. (6.4) contain five flavons, and they are highly suppressed by $1/\Lambda^3$ with respect to the renormalizable terms. The subleading terms of $w_d^{\rm atm}$ and $w_d^{\rm sol}$ involve four flavon fields,[4] the resulting corrections are suppressed by $1/\Lambda^2$ compared to the contribution from the leading order terms and therefore can be safely neglected.

6.2 The structure of the model

The lowest dimensional Yukawa operators invariant under the family symmetry $S_4\times Z_5\times Z_6\times Z_8$, responsible for the charged lepton masses, are given by

$$\begin{aligned} w_l=&\frac{y_\tau}{\Lambda}(L\phi_l)_{\mathbf 1}\tau^cH_d+\frac{y_{\mu_1}}{\Lambda^2}(L\,(\eta_l\phi_l)_{\mathbf 3})_{\mathbf 1}\mu^cH_d+\frac{y_{\mu_2}}{\Lambda^2}(L\,(\phi_l\phi_l)_{\mathbf 3})_{\mathbf 1}\mu^cH_d\\ &+\frac{y_{e_1}}{\Lambda^3}(L\phi_l)_{\mathbf 1}(\eta_l\eta_l)_{\mathbf 1}e^cH_d+\frac{y_{e_2}}{\Lambda^3}((L\phi_l)_{\mathbf 2}(\eta_l\eta_l)_{\mathbf 2})_{\mathbf 1}e^cH_d+\frac{y_{e_3}}{\Lambda^3}((L\eta_l)_{\mathbf 3}(\phi_l\phi_l)_{\mathbf 3})_{\mathbf 1}e^cH_d\\ &+\frac{y_{e_4}}{\Lambda^3}((L\eta_l)_{\mathbf 3'}(\phi_l\phi_l)_{\mathbf 3'})_{\mathbf 1}e^cH_d+\frac{y_{e_5}}{\Lambda^3}(L\phi_l)_{\mathbf 1}(\phi_l\phi_l)_{\mathbf 1}e^cH_d+\frac{y_{e_6}}{\Lambda^3}((L\phi_l)_{\mathbf 2}(\phi_l\phi_l)_{\mathbf 2})_{\mathbf 1}e^cH_d\\ &+\frac{y_{e_7}}{\Lambda^3}((L\phi_l)_{\mathbf 3}(\phi_l\phi_l)_{\mathbf 3})_{\mathbf 1}e^cH_d+\frac{y_{e8}}{\Lambda^3}((L\phi_l)_{\mathbf 3'}(\phi_l\phi_l)_{\mathbf 3'})_{\mathbf 1}e^cH_d\,,\end{aligned} \tag{6.17}$$

where all couplings are real since generalized CP symmetry is imposed on the model. Because the contraction $(\phi_l\phi_l)_{\mathbf 3}$ vanishes due to the antisymmetry of the associated Clebsch-Gordan coefficients, the terms proportional to y_{μ_2}, y_{e_3} and y_{e_7} give null contributions. Plugging the VEVs of η_l and ϕ_l in eq. (6.4) into the above superpotential w_l, we find the charged lepton mass matrix is diagonal and the three charged lepton masses are

$$\begin{aligned} m_e&=\left|\left(y_{e_6}-2y_{e_8}-2y_{e_4}v_{\eta_l}/v_{\phi_l}+y_{e_2}v_{\eta_l}^2/v_{\phi_l}^2\right)\frac{v_{\phi_l}^3}{\Lambda^3}\right|v_d,\\ m_\mu&=\left|y_{\mu_1}\frac{v_{\eta_l}v_{\phi_l}}{\Lambda^2}\right|v_d,\qquad m_\tau=\left|y_\tau\frac{v_{\phi_l}}{\Lambda}\right|v_d\,,\end{aligned} \tag{6.18}$$

[4]A single operator $(\sigma^0\phi_s^3)_{\mathbf 1}$ is allowed by the symmetry of the model at order $\mathcal{O}(1/\Lambda)$. However, this term vanishes exactly when the vacuum alignment of ϕ_s in eq. (6.14) is inserted.

where $v_d = \langle H_d \rangle$. Note that auxiliary symmetry $Z_5 \times Z_6 \times Z_8$ imposes different powers of η_l and ϕ_l for the electron, muon and tau lepton mass terms. As a result, the electron, muon and tau masses arise at order $(\langle \Phi_l \rangle / \Lambda)^3$, $(\langle \Phi_l \rangle / \Lambda)^2$ and $\langle \Phi_l \rangle / \Lambda$ respectively, where Φ_l is either ϕ_l or η_l. The realistic mass hierarchy can be reproduced if $\langle \Phi_l \rangle / \Lambda$ is of order λ^2, where $\lambda \simeq 0.23$ denotes the Cabibbo angle. Moreover, the subleading operators related to e^c, μ^c and τ^c comprise five flavons and consequently are suppressed by $1/\Lambda^5$. Such corrections have a minor impact on the results for the charged lepton masses and lepton mixing parameters and can be neglected.

In the neutrino sector, the leading order operators contributing to the neutrino masses are

$$w_\nu = \frac{y_a}{\Lambda} (L\phi_a)_{\mathbf{1}} H_u \nu^c_{\text{atm}} + \frac{y_s}{\Lambda} (L\phi_s)_{\mathbf{1}'} H_u \nu^c_{\text{sol}} + x_a \nu^c_{\text{atm}} \nu^c_{\text{atm}} \xi_a + x_s \nu^c_{\text{sol}} \nu^c_{\text{sol}} \xi_s \,, \tag{6.19}$$

where the coupling constants y_a, y_s, x_a and x_s are real parameters because of the imposed CP symmetry. The neutrino Dirac mass matrix m_D arises from the first two terms in eq. (6.19). With the vacuum alignments of ϕ_a and ϕ_s given in eq. (6.6) and eq. (6.14), we find m_D takes the following form

$$m_D = \begin{pmatrix} y_a v_{\phi_a} & y_s v_{\phi_s} \\ \omega y_a v_{\phi_a} & -\frac{7}{2} y_s v_{\phi_s} \\ \omega^2 y_a v_{\phi_a} & -\frac{7}{2} y_s v_{\phi_s} \end{pmatrix} \frac{v_u}{\Lambda} \,, \tag{6.20}$$

where $v_u = \langle H_u \rangle$. When the singlet flavons ξ_a and ξ_s obtain VEVs, the last two terms of w_ν lead to a diagonal right-handed neutrino mass matrix

$$m_N = \begin{pmatrix} x_a v_{\xi_a} & 0 \\ 0 & x_s v_{\xi_s} \end{pmatrix} . \tag{6.21}$$

Using the seesaw relation $m_\nu = -m_D m_N^{-1} m_D^T$, we can read off the light neutrino Majorana mass matrix

$$m_\nu = m_a \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{pmatrix} + m_s e^{i\eta} \begin{pmatrix} 1 & -7/2 & -7/2 \\ -7/2 & 49/4 & 49/4 \\ -7/2 & 49/4 & 49/4 \end{pmatrix} , \tag{6.22}$$

with

$$m_a = -\frac{y_a^2 v_{\phi_a}^2}{x_a v_{\xi_a}} \frac{v_u^2}{\Lambda^2}, \qquad m_s e^{i\eta} = -\frac{y_s^2 v_{\phi_s}^2}{x_s v_{\xi_s}} \frac{v_u^2}{\Lambda^2} . \tag{6.23}$$

We see that the resulting neutrino mass matrix in eq. (6.22) is of the same form as eq. (4.12) but with fixed value $x = -7/2$. Moreover, both ratios $v_{\phi_a}^2 / v_{\xi_a}$ and $v_{\phi_s}^2 / v_{\xi_s}$ can be expressed in terms of the parameters of the driving superpotential and thus they are real, as shown in section 6.1. As a consequence, the relative phase η is either 0 or π. The desired value $\eta = \pi$ can be achieved for $h_1 f_{11} x_a x_s M_{\xi_a} M_{\xi_s} < 0$.

Following the procedure outline in section 4, we find the lepton mixing matrix for $\eta = \pi$ takes the following form

$$U_{PMNS} = \frac{1}{5\sqrt{6}} \begin{pmatrix} 7\sqrt{2} & -2\sqrt{13}\, i \cos\theta & 2\sqrt{13} \sin\theta \\ \sqrt{26} & 7i\cos\theta - 5\sqrt{3}\sin\theta & -7\sin\theta + 5\sqrt{3}\, i\cos\theta \\ \sqrt{26} & 7i\cos\theta + 5\sqrt{3}\sin\theta & -7\sin\theta - 5\sqrt{3}\, i\cos\theta \end{pmatrix} , \tag{6.24}$$

with

$$\sin 2\theta = \frac{10|14r-1|}{13\sqrt{289r^2+32r+4}}, \quad \cos 2\theta = \frac{3\,(57r+8)}{13\sqrt{289r^2+32r+4}}, \tag{6.25}$$

where $r = m_s/m_a$. Then the analytical expressions for the lepton mixing angles can be extracted,

$$\sin^2\theta_{13} = \frac{26}{75}\sin^2\theta, \quad \sin^2\theta_{12} = \frac{26\cos^2\theta}{62+13\cos 2\theta}, \quad \sin^2\theta_{23} = \frac{1}{2}\,. \tag{6.26}$$

We see that the atmospheric angle θ_{23} is maximal, the solar and the reactor mixing angles fulfill the sum rule

$$\cos^2\theta_{12}\cos^2\theta_{13} = \frac{49}{75}\,. \tag{6.27}$$

Using the 3σ interval $0.01981 \leq \sin^2\theta_{13} \leq 0.02436$ [4], from eq. (6.26) we find the allowed range of the parameter r is $0.448 \leq r \leq 0.634$ and the exact sum rule in eq. (6.27) gives $0.330 \leq \sin^2\theta_{12} \leq 0.333$ which can be tested by JUNO in near future [45]. For the CP invariants, we get

$$J_{\rm CP} = -\frac{91}{750\sqrt{3}}\sin 2\theta, \quad I_1 = 0\,. \tag{6.28}$$

Therefore the Dirac phase $\delta_{\rm CP}$ is maximal and the Majorana phase β is trivial with

$$\delta_{\rm CP} = -0.5\pi, \quad \beta = 0\,. \tag{6.29}$$

Furthermore, we report the exact results for the neutrino masses

$$\begin{aligned} m_2^2 &= \frac{9}{8}m_a^2\left(289r^2 - 18r + 4 - |2-17r|\sqrt{289r^2+32r+4}\right),\\ m_3^2 &= \frac{9}{8}m_a^2\left(289r^2 - 18r + 4 + |2-17r|\sqrt{289r^2+32r+4}\right), \end{aligned} \tag{6.30}$$

with the lightest neutrino massless $m_1 = 0$. For the best fit values of $m_a = 3.720\,\text{meV}$ and $r = 0.553$, the neutrino masses m_2 and m_3 are

$$m_2 = 8.622\,\text{meV}, \qquad m_3 = 49.918\,\text{meV}\,, \tag{6.31}$$

as given in table 1. We note that the next-to-leading operators of w_ν are $\nu^c_{\rm atm}\nu^c_{\rm atm}(\phi_a^2)_{\mathbf{1}}$, $\nu^c_{\rm sol}\nu^c_{\rm sol}(\phi_s^2)_{\mathbf{1}}$ and $\nu^c_{\rm atm}\nu^c_{\rm sol}(\phi_a\phi_s)_{\mathbf{1}'}$. The contributions of the first two terms can be absorbed via a redefinition of the parameters x_a and x_s since both $v^2_{\phi_a}/v_{\xi_a}$ and $v^2_{\phi_s}/v_{\xi_s}$ are real. The last term will generate off-diagonal elements of the right-handed neutrino mass matrix. The corresponding corrections to the leading order results for the mixing parameters are of relative order λ^2. In summary, we have reproduced the benchmark model of the new LSS variants highlighted in table 1.

7 Conclusion

In this paper we have proposed a new *tri-direct CP approach* for two right-handed neutrino models based on the idea that the high energy family and CP symmetry $G_f \rtimes H_{\rm CP}$ is spontaneously broken down to $G_{\rm atm} \rtimes H^{\rm atm}_{\rm CP}$ in the sector of one of the right-handed neutrinos, and $G_{\rm sol} \rtimes H_{\rm CP}^{\rm sol}$ in the sector of the other right-handed neutrino, with the charged lepton sector having a residual flavour symmetry G_l, as illustrated in figure 1.

In such a *tri-direct CP approach* we have shown that the combination of the three residual symmetries provides a new way of fixing the parameters. In particular it can lead to vacuum alignments in the neutrino sector which are uniquely fixed by symmetry, unlike the *semi-direct CP approach* where not all such vacuum alignments are uniquely fixed. To illustrate the approach, we have revisited the Littlest Seesaw model based on S_4 and shown that the *tri-direct CP approach* based on $G_{\rm atm} = Z_2^U$ and $G_{\rm sol} = Z_2^{SU}$ uniquely fixes alignments which are not uniquely fixed in the *semi-direct CP approach* based on a common residual symmetry $G_\nu = Z_2^{SU}$ in the neutrino sector.

Following the *tri-direct CP approach*, we have also proposed new variants of the Littlest Seesaw model which have not so far appeared in the literature, with different predictions for each variant. We have performed a comprehensive numerical analysis of a selection of benchmark points within the LSS variants arising from S_4, in order to determine their viability and predictions. Although the benchmarks within most of the variants have a larger χ^2 than the original LSS model, which provides an excellent agreement with experimental data, one of the benchmarks has a relatively low $\chi^2 \approx 4.5$. We have proposed an explicit model which can realise this successful benchmark point, based on the atmospheric flavon vacuum alignment $(1, \omega^2, \omega)$ and the solar flavon vacuum alignment $(1, -7/2, -7/2)$. The model has exact accidental $\mu\tau$ reflection symmetry [48] and hence predicts maximal atmospheric mixing and maximal Dirac CP violation.

Although the flavor symmetry and CP symmetry are completely broken in the whole neutrino sector, the tri-direct CP models are rather predictive. The light neutrino mass matrix is generally predicted to depend on only four parameters m_a, m_s, x and η, and the last two parameters x and η can be fixed to certain benchmark values by explicit superpotential alignment terms in a concrete model. The cross product $\langle\phi_{\rm atm}\rangle \times \langle\phi_{\rm sol}\rangle$ is an eigenvector of m_ν with zero eigenvalue, and neutrino mass spectrum can be either normal ordering or inverted ordering. Finally we note that the tri-direct CP approach can be extended to the case of three right-handed neutrinos, then the flavons associated with each right-handed neutrino in the Yukawa couplings preserve different residual symmetries. Such a scenario can also lead to phenomenologically viable lepton mixing angles and neutrino masses.

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A Diagonalization of the neutrino mass matrix m'_ν

In this appendix, we will present the results for the diagonalization of m'_ν. From eqs. (4.15) and (4.16), we find the neutrino mass matrix m'_ν can be written as

$$m'_\nu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & |y|e^{i\phi_y} & |z|e^{i\phi_z} \\ 0 & |z|e^{i\phi_z} & |w|e^{i\phi_w} \end{pmatrix}, \tag{A.1}$$

where the parameters y, z, w, ϕ_y, ϕ_z and ϕ_w are given in eq. (4.16). The neutrino mass matrix m'_ν can be exactly diagonalized by a unitary matrix $U_{\nu 2}$ [39],

$$U^T_{\nu 2} m'_\nu U_{\nu 2} = \mathrm{diag}(0, m_2, m_3)\,. \tag{A.2}$$

The light neutrino masses m_2 and m_3 are given by

$$m_2^2 = \frac{1}{2}\left[|y|^2 + |w|^2 + 2|z|^2 - \frac{|w|^2 - |y|^2}{\cos 2\theta}\right], \quad m_3^2 = \frac{1}{2}\left[|y|^2 + |w|^2 + 2|z|^2 + \frac{|w|^2 - |y|^2}{\cos 2\theta}\right], \tag{A.3}$$

where the rotation angle θ is specified by

$$\begin{aligned} \sin 2\theta &= \frac{2|z|\sqrt{|y|^2 + |w|^2 + 2|y||w|\cos(\phi_y + \phi_w - 2\phi_z)}}{\sqrt{(|w|^2 - |y|^2)^2 + 4|z|^2\left[|y|^2 + |w|^2 + 2|y||w|\cos(\phi_y + \phi_w - 2\phi_z)\right]}}, \\ \cos 2\theta &= \frac{|w|^2 - |y|^2}{\sqrt{(|w|^2 - |y|^2)^2 + 4|z|^2\left[|y|^2 + |w|^2 + 2|y||w|\cos(\phi_y + \phi_w - 2\phi_z)\right]}}\,. \end{aligned} \tag{A.4}$$

The unitary matrix $U_{\nu 2}$ in eq. (A.2) takes the following form,

$$U_{\nu 2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta\, e^{i(\psi+\rho)/2} & \sin\theta\, e^{i(\psi+\sigma)/2} \\ 0 & -\sin\theta\, e^{i(-\psi+\rho)/2} & \cos\theta\, e^{i(-\psi+\sigma)/2} \end{pmatrix}, \tag{A.5}$$

where the phases ψ, ρ and σ are expressed in terms of model parameters as

$$\begin{aligned} \sin\psi &= \frac{-|y|\sin(\phi_y - \phi_z) + |w|\sin(\phi_w - \phi_z)}{\sqrt{|y|^2 + |w|^2 + 2|y||w|\cos(\phi_y + \phi_w - 2\phi_z)}}, \\ \cos\psi &= \frac{|y|\cos(\phi_y - \phi_z) + |w|\cos(\phi_w - \phi_z)}{\sqrt{|y|^2 + |w|^2 + 2|y||w|\cos(\phi_y + \phi_w - 2\phi_z)}}, \\ \sin\rho &= -\frac{(m_2^2 - |z|^2)\sin\phi_z + |y||w|\sin(\phi_y + \phi_w - \phi_z)}{m_2\sqrt{|y|^2 + |w|^2 + 2|y||w|\cos(\phi_y + \phi_w - 2\phi_z)}}, \\ \cos\rho &= \frac{(m_2^2 - |z|^2)\cos\phi_z + |y||w|\cos(\phi_y + \phi_w - \phi_z)}{m_2\sqrt{|y|^2 + |w|^2 + 2|y||w|\cos(\phi_y + \phi_w - 2\phi_z)}}, \\ \sin\sigma &= -\frac{(m_3^2 - |z|^2)\sin\phi_z + |y||w|\sin(\phi_y + \phi_w - \phi_z)}{m_3\sqrt{|y|^2 + |w|^2 + 2|y||w|\cos(\phi_y + \phi_w - 2\phi_z)}}, \\ \cos\sigma &= \frac{(m_3^2 - |z|^2)\cos\phi_z + |y||w|\cos(\phi_y + \phi_w - \phi_z)}{m_3\sqrt{|y|^2 + |w|^2 + 2|y||w|\cos(\phi_y + \phi_w - 2\phi_z)}}\,. \end{aligned} \tag{A.6}$$

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