

Asymptotic M5-brane entropy from S-duality

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ABSTRACT: We study M5-branes compactified on S^1 from the D0-D4 Witten index in the Coulomb phase. We first show that the prepotential of this index is S-dual, up to a simple anomalous part. This is an extension of the well-known S-duality of the 4d $\mathcal{N} = 4$ theory to the 6d (2,0) theory on finite T^2 . Using this anomalous S-duality, we find that the asymptotic free energy scales like N^3 when various temperature-like parameters are large. This shows that the number of 5d Kaluza-Klein fields for light D0-brane bound states is proportional to N^3 . We also compute some part of the asymptotic free energy from 6d chiral anomalies, which precisely agrees with our D0-D4 calculus.

KEYWORDS: Duality in Gauge Field Theories, Field Theories in Higher Dimensions, M-Theory, Supersymmetric Gauge Theory

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1 Introduction

Strong/weak-coupling duality, or S-duality, exists in a number of quantum systems. In 4d gauge theories, it takes the form of electromagnetic duality, which inverts the gauge coupling and exchanges the roles of elementary charged particles and magnetic monopoles [1]. It is realized in the simplest manner in maximally supersymmetric Yang-Mills (SYM) theory [2]. In this case, the spectrum of dyons in the Coulomb phase was shown to exhibit $SL(2, \mathbb{Z})$ duality [3], providing a robust evidence of S-duality. S-duality in quantum field theories has also been a cornerstone of developing string dualities [4]. In both QFT and string theory, S-duality provides valuable insights on the strongly coupled regions of the systems.

S-duality of maximal SYM has many implications. In particular, this duality is related to the existence of 6d superconformal field theories called (2, 0) theories [5]. 4d maximal SYM theories with gauge groups $U(N)$, $SO(2N)$, E_N are realized by compactifying 6d (2, 0) theories on small T^2 . The $SL(2, \mathbb{Z})$ duality originates from the modular transformation on T^2 . On one side, this relation highlights the far-reaching implications of the 6d CFTs to challenging lower dimensional systems. On the other hand, the 6d CFTs lack microscopic definitions, so that S-duality can provide useful clues to better understand the mysterious 6d CFTs. In this paper, we study the S-duality of the 6d (2, 0) theories compactified on finite T^2 , and use it to explore some interesting properties of these systems.

Our key observable of the 6d (2, 0) theory is the partition function of the D0-D4 system. More precisely, we study the Witten index of the quantum mechanical $U(k)$ gauge theory for k D0-branes bound to N separated D4-branes and fundamental open strings, and also study their generating function for arbitrary k . In M-theory, this system is made of N M5-branes wrapping S^1 . The D0-D4 systems describe the 6d (2, 0) CFT on M5-branes in

the sectors with nonzero Kaluza-Klein momenta. From the viewpoint of 5d super-Yang-Mills theory on D4-branes, this partition function is also known as Nekrasov's instanton partition function [6]. Although the 5d Yang-Mills description of its instanton solitons is UV incomplete, the D0-D4 system provides a UV complete description for computing the instanton partition function. The results in our paper rely only on the UV complete D0-D4 quantum mechanics.

The D0-D4 index was explored in [7]. From the 5d viewpoint, this is a partition function on $\mathbb{R}^4 \times S^1$, where S^1 is the temporal circle for the Witten index. However, with D0-branes (Yang-Mills instantons) providing the Kaluza-Klein (KK) modes of M-theory, there are evidences that this index (multiplied by a 5d perturbative part) is a partition function of the 6d (2, 0) SCFT on $\mathbb{R}^4 \times T^2$. See [7, 8] for the A_{N-1} theories, and [9] for the D_N theories.

Regarding the D0-D4 index as a 6d partition function on $\mathbb{R}^4 \times T^2$, one can naturally ask if it transforms in a simple manner under the S-duality acting on T^2 . In this paper, we establish the S-duality of the prepotential of this index, finding a simple anomaly of S-duality which we can naturally interpret with 6d chiral anomalies. Note that the prepotential F is the leading coefficient of the free energy $-\log Z \sim \frac{F(a,m)}{\epsilon_1 \epsilon_2}$ when the so-called Omega deformation is taken to zero, $\epsilon_1, \epsilon_2 \rightarrow 0$. a and m are Coulomb VEV and 5d $\mathcal{N} = 1^*$ mass parameter, explained in section 2. The anomaly of S-duality takes the following form. The prepotential F can be divided into two parts, $F(a, m) = F_{S\text{-dual}}(a, m) + F_{\text{anom}}(m)$, where $F_{S\text{-dual}}$ is related to its S-dual prepotential by a 6d generalization of the Legendre transformation. (See section 2 for details.) F_{anom} is a simple function which does not obey S-duality, thus named anomalous part. We find F_{anom} in a closed form in section 2, which in particular is independent of the Coulomb VEV a .

This finding has two major implications. Firstly, similar result was found for the prepotential F^{4d} of the 4d $\mathcal{N} = 2^*$ theory [10], related to our prepotential by taking the small T^2 limit. F^{4d} is also given by the sum $F_{S\text{-dual}}^{4d}(a, m) + F_{\text{anom}}^{4d}(m)$, where $F_{S\text{-dual}}^{4d}$ is self S-dual. Since F_{anom}^{4d} is independent of a , F^{4d} is S-dual in the Seiberg-Witten theory, which only sees a derivatives of F^{4d} . In our 6d uplift, F appearing in the Seiberg-Witten theory also does not see $F_{\text{anom}}(m)$ for the same reason.

Secondly, the partition function Z itself is a Witten index of the 6d theory on $\mathbb{R}^{4,1} \times S^1$. So the full prepotential $F = F_{S\text{-dual}} + F_{\text{anom}}$ including the a independent F_{anom} is physically meaningful, as the leading part of the free energy $-\log Z$ when ϵ_1, ϵ_2 are small. At this stage, we note our key discovery that F_{anom} contains a term proportional to N^3 in a suitable large N limit, to be explained in section 2. In particular, we further consider the limit in which the KK modes on the circle become light. This amounts to taking the chemical potential β conjugate to the KK momentum (D0-brane charge) to be much smaller than the inverse-radius of S^1 . The small β limit is the strong coupling limit of the 5d Yang-Mills theory, or the limit in which the sixth circle decompactifies. This is the regime in which 6d CFT physics should be visible. The prepotential in this limit can be computed from our anomalous S-duality, since it relates the small β (strong coupling) regime to the well-understood large β (weak coupling) regime. F_{anom} determines the small β limit of the free energy, and makes it scale like N^3 . We also show that the term in the asymptotic free

energy proportional to N^3 is related to the chiral anomaly of the 6d (2, 0) theory, using the methods of [11]. These findings show that the number of 5d KK fields for D0-brane bound states grows like N^3 , as we decompactify the system to 6d.

The rest of this paper is organized as follows. In sections 2.1 and 2.2, we develop the anomalous S-duality of the prepotential and test it either by expanding F in the 5d $\mathcal{N} = 1^*$ mass m , or by making the ‘M-string’ expansion [8]. In section 2.3, we discuss the 6d (2, 0) theories of D_N and E_N types. In section 3, we study the high temperature free energy and show that it scales like N^3 in a suitable large N limit. In section 3.1, we test our result for U(1) theory. In section 3.2, we account for the imaginary part of the asymptotic free energy from 6d chiral anomalies. Section 4 concludes with comments and future directions.

2 S-duality of 6d (2, 0) theories on $\mathbb{R}^4 \times T^2$

We shall study the Witten index of the D0-D4 quantum mechanics, consisting of k D0-branes and N D4-branes. This system is a quantum mechanical U(k) gauge theory with 8 supersymmetry and U(N) global symmetry. See, for instance, [7] for the details of this system. Here, we shall only explain some basic aspects. The bosonic variables consist of four Hermitian $k \times k$ matrices $a_m \sim a_{\alpha\dot{\beta}}$, two complex $k \times N$ matrices $q_{\dot{\alpha}}$, five Hermitian $k \times k$ matrices φ^I , and a quantum mechanical U(k) gauge field A_t . Here, $m = 1, 2, 3, 4$ is the vector index on \mathbb{R}^4 for the spatial worldvolume of the D4-branes. α and $\dot{\alpha}$ are doublet indices of SU(2)_l and SU(2)_r respectively, which form SO(4) rotation of \mathbb{R}^4 . $I = 1, \dots, 5$ is the vector index on \mathbb{R}^5 transverse to the D4-branes. When φ^I are all diagonal matrices, their eigenvalues are interpreted as D0-brane positions transverse to D4-branes. Similarly, when a_m are all diagonal, their eigenvalues are interpreted as D0-brane positions along D4-brane worldvolume. $q_{\dot{\alpha}}$ represent internal degrees of freedom. The bosonic potential energy is given by

$$V = \frac{1}{2} D^i D^i - \frac{1}{2} [\varphi^I, a_m]^2 + |\varphi^I q_{\dot{\alpha}}|^2 - \frac{1}{4} [\varphi^I, \varphi^J]^2, \tag{2.1}$$

where traces are assumed if necessary, $i = 1, 2, 3$ is the SU(2)_r triplet index. D^i are given by

$$D^i = (\tau^i)^{\dot{\alpha}\dot{\beta}} \left(q_{\dot{\alpha}} q^{\dot{\beta}} + \frac{1}{2} [a^{\dot{\beta}\alpha}, a_{\alpha\dot{\alpha}}] \right), \tag{2.2}$$

where τ^i are the Pauli matrices.

This system flows in IR to two branches. Classically, they are described by two branches of moduli space satisfying $V = 0$, or $D^i = 0$, $[\varphi^I, a_m] = 0$, $\varphi^I q_{\dot{\alpha}} = 0$ and $[\varphi^I, \varphi^J] = 0$. The first branch is obtained by taking $q_{\dot{\alpha}} = 0$, and a_m, φ^I to be diagonal matrices. The k sets of eigenvalues of (a_m, φ^I) represent the positions of k D0-branes on \mathbb{R}^9 , unbound to the D4-branes. The second branch is obtained by taking $\varphi^I = 0$, and $q_{\dot{\alpha}}, a_m$ to satisfy $D^i = 0$. After modding out by the U(k) gauge orbit, one can show that this branch is described by $4Nk$ real parameters. The two branches meet at $\varphi^I = 0, q_{\dot{\alpha}} = 0$. Far away from this intersection, each branch is described by a nonlinear sigma model (NLSM) on its moduli space. We are interested in the second branch, describing 6d CFT on M5-branes in

the sector with k units of KK momentum. The Witten index of the second branch can be computed easily by deforming the system by a Fayet-Iliopoulos (FI) parameter, shifting D^i in (2.2) by three constant ξ^i . After this deformation, the first branch becomes non-BPS, since $q_{\dot{\alpha}} = 0$ cannot solve $D^i = 0$ with $\xi^i \neq 0$. So the Witten index acquires contributions only from the second branch.

One can understand the second branch from the low energy field theory of D4-branes, the 5d maximal SYM theory. D0-branes are realized in Yang-Mills theory as instanton solitons, classically described by finite energy stationary solutions of the following BPS equation,

$$F_{mn} = \pm \frac{1}{2} \epsilon_{mnpq} F_{pq}, \quad m, n, p, q = 1, \dots, 4. \quad (2.3)$$

The finite energy solutions are labeled by the instanton number k , defined by

$$k \equiv \frac{1}{16\pi^2} \int_{\mathbb{R}^4} \text{tr}(F \wedge F) \in \mathbb{Z}. \quad (2.4)$$

We shall consider D0-branes rather than anti-D0-branes, with $k > 0$, equivalently (2.3) with + sign. k corresponds to the rank of the $U(k)$ gauge group of the quantum mechanics. The solutions of $D^i = 0$, modded out by the $U(k)$ gauge orbit, provides the moduli space of self-dual instantons. The NLSM on the second branch can be obtained by the moduli space approximation of the instanton solitons. However, this moduli space is known to be singular, having the so-called small instanton singularities. Due to these singularities, the NLSM description is incomplete, and needs a UV completion. This is naturally interpreted as inheriting the UV incompleteness of the 5d Yang-Mills description. Although we do not know how to UV complete the full 5d SYM, the NLSM can be UV completed to the $U(k)$ quantum mechanics.

We shall study the D0-D4 system in the Coulomb phase, with scalar vacuum expectation value (VEV). For $U(N)$ theories, the VEV is parametrized by constant $N \times N$ diagonal matrices $\Phi^I = \text{diag}(\Phi_1^I, \dots, \Phi_N^I)$. In the D0-D4 system, Φ^I deforms the bosonic potential V as

$$V = \frac{1}{2} D^i D^i - \frac{1}{2} [\varphi^I, a_m]^2 - \frac{1}{4} [\varphi^I, \varphi^J]^2 + |\varphi^I q_{\dot{\alpha}} - q_{\dot{\alpha}} \Phi^I|^2. \quad (2.5)$$

The N eigenvalues of Φ^I correspond to the positions of N D4-branes on \mathbb{R}^5 . We shall separate the D4-branes along a line, giving nonzero VEV to Φ^5 only. In this setting, we shall study the BPS bound states of the D0-branes and the fundamental open strings stretched along the Φ^5 direction, suspended between a pair of D4-branes. The bound states preserve 4 Hermitian supercharges. In 6d (2, 0) theory, we compactify a spatial direction on a circle with radius R' . The BPS states saturate the bound $E \geq \frac{P}{R'} + v_i q_i$, where E is the energy, and P is the quantized momentum on S^1 which is k in the D0-D4 system. v_i are the N eigenvalues of the scalar Φ^5 , and q_i 's are the $U(1)^N$ electric charges in the Coulomb branch, satisfying $q_1 + \dots + q_N = 0$. From the 6d viewpoint, they are the self-dual strings with charges q_i coming from open M2-branes, with P units of momenta on them. We also define $H \equiv R'(E - v_i q_i)$, which is the (dimensionless) energy on the self-dual strings.

The 6d index is defined by

$$Z(\tau, m, \epsilon_{1,2}, v) = \text{Tr} \left[(-1)^F e^{2\pi i \tau \frac{H+P}{2}} e^{-2\pi i \bar{\tau} \frac{H-P}{2}} e^{\epsilon_1 (J_1 + J_R) + \epsilon_2 (J_2 + J_R)} e^{2m J_L} e^{-v_i q_i} \right]. \quad (2.6)$$

Here, J_1, J_2 are two Cartans rotating the two 2-planes of \mathbb{R}^4 , J_L, J_R are the Cartans of $SU(2)_L \times SU(2)_R = SO(4) \subset SO(5)$ unbroken by the VEV of Φ^5 . The measure is chosen so that it commutes with 2 of the 4 Hermitian supercharges preserved by the BPS states, or a complex supercharge Q and its conjugate Q^\dagger . See [7] for the details. One also finds that $\frac{H-P}{2} \sim \{Q, Q^\dagger\}$. Since only the states saturating the BPS bound $H \geq P$ contribute to the index, Z is independent of $\bar{\tau}$. With $H = P$ understood, the factor $e^{2\pi i \tau \frac{H+P}{2}} \rightarrow e^{2\pi i \tau P}$ weights the BPS states with the momentum P along the circle. So Z can be written as

$$Z(\tau, m, \epsilon_{1,2}, v) = Z_{\text{pert}}(m, \epsilon_{1,2}, v) \sum_{k=0}^{\infty} q^k Z_k(m, \epsilon_{1,2}, v) \quad (2.7)$$

where $q \equiv e^{2\pi i \tau}$, and $Z_0 \equiv 1$ by definition. Z can be computed in the weakly coupled type IIA regime, in which D0-branes are much heavier than the stretched fundamental strings. Z_k is computed as the nonperturbative Witten index of the D0-D4 system with fixed k . Z_{pert} comes from the zero modes at $P = 0$, the perturbative open string modes on the D4-branes. This factor can also be understood as coming from the perturbative partition function of the 5d maximal SYM. Since we are in the weakly coupled regime, Z_{pert} can be computed unambiguously from the quadratic part of the Yang-Mills theory. Although we compute Z_{pert} and Z_k in this special regime, we naturally expect the result to be valid at general type IIA coupling, since this is a Witten index independent of the continuous coupling.

Z_k and Z_{pert} are known for classical gauge groups. For $U(N)$, Z_k is given by [7, 12–14]

$$Z_k = \sum_{Y_i; \sum_{i=1}^N |Y_i| = k} \prod_{i,j=1}^N \prod_{s \in Y_i} \frac{\sinh \frac{E_{ij}(s) + m - \epsilon_+}{2} \sinh \frac{E_{ij}(s) - m - \epsilon_+}{2}}{\sinh \frac{E_{ij}(s)}{2} \sinh \frac{E_{ij}(s) - 2\epsilon_+}{2}} \quad (2.8)$$

where $\epsilon_{\pm} \equiv \frac{\epsilon_1 \pm \epsilon_2}{2}$, and

$$E_{ij}(s) = v_i - v_j - \epsilon_1 h_i(s) + \epsilon_2 (v_j(s) + 1). \quad (2.9)$$

The summation is made over N Young diagrams Y_i with total number of boxes k , and s runs over all boxes of the Young diagram Y_i . $h_i(s)$ is the distance from s to the right end of the Young diagram Y_i , and $v_j(s)$ is the distance from s to the bottom end of the Young diagram Y_j . See [7] for the details. One often calls $Z_{\text{inst}} \equiv \sum_{k=0}^{\infty} q^k Z_k$ the instanton partition function.

Z_{pert} is given by [15, 16]

$$Z_{\text{pert}} = \prod_{\alpha \in \text{adj}} \left[\frac{\tilde{\Gamma}_3\left(\frac{\alpha(v) + \epsilon_+ + m}{2\pi i} \middle| \frac{\epsilon_1}{2\pi i}, \frac{\epsilon_2}{2\pi i}\right) \tilde{\Gamma}_3\left(\frac{\alpha(v) + \epsilon_+ - m}{2\pi i} \middle| \frac{\epsilon_1}{2\pi i}, \frac{\epsilon_2}{2\pi i}\right)}{\tilde{\Gamma}_3\left(\frac{\alpha(v)}{2\pi i} \middle| \frac{\epsilon_1}{2\pi i}, \frac{\epsilon_2}{2\pi i}\right) \tilde{\Gamma}_3\left(\frac{\alpha(v) + 2\epsilon_+}{2\pi i} \middle| \frac{\epsilon_1}{2\pi i}, \frac{\epsilon_2}{2\pi i}\right)} \right]^{\frac{1}{2}} \quad (2.10)$$

where $\tilde{\Gamma}_3(z|w_1, w_2) \equiv \Gamma_3(z|1, w_1, w_2) \Gamma_3(1-z|1, -w_1, -w_2)$, and $\Gamma_N(z|w_1, \dots, w_N)$ is the Barnes' Gamma function. As noted in [16], α in the adjoint representation includes Cartans, $\alpha = 0$, for which ' $\Gamma_3(0|\frac{\epsilon_1}{2\pi i}, \frac{\epsilon_2}{2\pi i})$ ' in the denominator would diverge. For these α ,

one replaces ‘ $\tilde{\Gamma}_3(0|w_1, w_2)$ ’ factors by $\tilde{\Gamma}'_3(0|w_1, w_2) \equiv \lim[z\tilde{\Gamma}_3(z|w_1, w_2)]$. See [16] for more details. For $t_1 \equiv e^{\epsilon_1} < 1$, $t_2 \equiv e^{\epsilon_2} < 1$, Z_{pert} can be rewritten as

$$Z_{\text{pert}} = e^{-\mathcal{F}} \prod_{\alpha \in \mathbf{adj}} \prod_{n_1, n_2 \geq 0} \left[\frac{(1 - e^{\alpha(v)} t_1^{n_1} t_2^{n_2})' (1 - e^{\alpha(v)} t_1 t_2 t_1^{n_1} t_2^{n_2})}{(1 - e^{\alpha(v) + \epsilon_+ \pm m} t_1^{n_1} t_2^{n_2})} \right]^{\frac{1}{2}}, \quad (2.11)$$

where prime here again means excluding the zero modes at $n_1 = n_2 = 0$ for the Cartans $\alpha = 0$. The overall factor \mathcal{F} is given for gauge group G by [15]

$$\begin{aligned} \mathcal{F} &= -\frac{\pi i}{2} \sum_{\alpha \in \mathbf{adj}} \left[\zeta_3 \left(0, \frac{\alpha(v)}{2\pi i} \middle| 1, \frac{\epsilon_1}{2\pi i}, \frac{\epsilon_2}{2\pi i} \right) + \zeta_3 \left(0, \frac{\alpha(v) + 2\epsilon_+}{2\pi i} \middle| 1, \frac{\epsilon_1}{2\pi i}, \frac{\epsilon_2}{2\pi i} \right) \right. \\ &\quad \left. - \zeta_3 \left(0, \frac{\alpha(v) + \epsilon_+ \pm m}{2\pi i} \middle| 1, \frac{\epsilon_1}{2\pi i}, \frac{\epsilon_2}{2\pi i} \right) \right] \\ &= -\frac{\pi i}{2} \sum_{\alpha(v) \in \mathbf{adj}} \frac{\epsilon_+^2 - m^2}{2\epsilon_1 \epsilon_2} (1 - 2\alpha(v)) = \frac{\pi i (m^2 - \epsilon_+^2)}{4\epsilon_1 \epsilon_2} |G|, \end{aligned} \quad (2.12)$$

where ζ_3 is the Barnes’ zeta function. When $t_1, t_2 < e^{\alpha(v)}$ for all $\alpha \in \mathbf{adj}$, Z_{pert} is rewritten as

$$Z_{\text{pert}}(v, \epsilon_{1,2}, m) = e^{-\mathcal{F}} PE \left[\frac{1}{2} \frac{\sinh \frac{m+\epsilon_+}{2} \sinh \frac{m-\epsilon_+}{2}}{\sinh \frac{\epsilon_1}{2} \sinh \frac{\epsilon_2}{2}} \chi_{\mathbf{adj}}(e^v) + \frac{r}{2} \right], \quad (2.13)$$

where $PE[f(x, y, z, \dots)] \equiv \exp \left[\sum_{n=1}^{\infty} \frac{1}{n} f(nx, ny, nz, \dots) \right]$, $\chi_{\mathbf{adj}} \equiv \sum_{\alpha \in \mathbf{adj}} e^{\alpha(v)}$, and r is the rank of gauge group which is $r = N$ for $U(N)$. The term $\frac{r}{2}$ in PE comes from excluding r fermionic zero modes for the Cartans.

One may multiply an alternative perturbative factor $\check{Z}_{\text{pert}} \equiv e^{-\epsilon_0} [Z_{\text{pert}}^{U(1)}]^N \hat{Z}_{\text{pert}}$ to Z_{inst} , where $[Z_{\text{pert}}^{U(1)}]^N$ is the perturbative partition function for the N Cartans, \hat{Z}_{pert} is defined by

$$\hat{Z}_{\text{pert}} = PE \left[\frac{\sinh \frac{m+\epsilon_+}{2} \sinh \frac{m-\epsilon_+}{2}}{\sinh \frac{\epsilon_1}{2} \sinh \frac{\epsilon_2}{2}} \sum_{\alpha > 0} e^{-\alpha(v)} \right], \quad (2.14)$$

and

$$\epsilon_0 = \frac{m^2 - \epsilon_+^2}{2\epsilon_1 \epsilon_2} \sum_{\alpha > 0} (\alpha(v) + \pi i). \quad (2.15)$$

Here all sums are over positive roots $\alpha > 0$. This expression is well defined when all $\alpha(v)$ are positive for positive roots and larger than $m, \epsilon_{1,2}$. This expression will be useful when studying S-duality from the M-string viewpoint, in section 2.2. Z_{pert} and \check{Z}_{pert} are different in subtle ways, which shall not affect the studies of prepotential in this paper but has implications on the S-duality of Z , which we comment on in section 2.2. (2.14) has a more natural interpretation as the Witten index of charged W-bosons in the Coulomb phase [7]. However, as an abstract partition function, Z_{pert} is more natural as it is manifestly Weyl-invariant.

It will also be useful to know the simple structures of the Abelian partition function, $Z^{U(1)} = Z_{\text{pert}}^{U(1)} Z_{\text{inst}}^{U(1)}$. Firstly, the perturbative U(1) partition function can be written as

$$\begin{aligned} Z_{\text{pert}}^{U(1)} &= e^{-\frac{\pi i(m^2 - \epsilon_+^2)}{4\epsilon_1 \epsilon_2}} PE \left[\frac{1}{2} \frac{\sinh \frac{m+\epsilon_+}{2} \sinh \frac{m-\epsilon_+}{2}}{\sinh \frac{\epsilon_1}{2} \sinh \frac{\epsilon_2}{2}} + \frac{1}{2} \right] \\ &= e^{-\frac{\pi i(m^2 - \epsilon_+^2)}{4\epsilon_1 \epsilon_2}} PE \left[\frac{1}{2} \frac{\sinh \frac{m+\epsilon_-}{2} \sinh \frac{m-\epsilon_-}{2}}{\sinh \frac{\epsilon_1}{2} \sinh \frac{\epsilon_2}{2}} \right] \end{aligned} \tag{2.16}$$

at $e^{\epsilon_1} < 1$, $e^{\epsilon_2} < 1$, by following the discussions till (2.13) for $N = 1$. The instanton part can be written as [17]

$$Z_{\text{inst}}^{U(1)} = \sum_{k=0}^{\infty} q^k Z_k = PE \left[\frac{\sinh \frac{m+\epsilon_-}{2} \sinh \frac{m-\epsilon_-}{2}}{\sinh \frac{\epsilon_1}{2} \sinh \frac{\epsilon_2}{2}} \frac{q}{1-q} \right], \tag{2.17}$$

after summing over all Young diagrams in (2.8).

Given $Z = Z_{\text{pert}} Z_{\text{inst}}$, or $\check{Z} = \check{Z}_{\text{pert}} Z_{\text{inst}}$, one can write this partition function as

$$Z = PE \left[\frac{f(\tau, m, \epsilon_{1,2}, v)}{2 \sinh \frac{\epsilon_1}{2} \cdot 2 \sinh \frac{\epsilon_2}{2}} \right] \equiv \exp \left[\sum_{n=1}^{\infty} \frac{1}{n} \frac{f(n\tau, nm, n\epsilon_{1,2}, nv)}{2 \sinh \frac{n\epsilon_1}{2} \cdot 2 \sinh \frac{n\epsilon_2}{2}} \right], \tag{2.18}$$

or a similar expression for \check{Z} using \check{f} . The expression appearing in PE is called the single particle index, containing all the information on the BPS bound states. The coefficients of f in fugacity expansion are also called Gopakumar-Vafa invariants [18, 19]. The factor $\frac{1}{2 \sinh \frac{\epsilon_1}{2} \cdot 2 \sinh \frac{\epsilon_2}{2}}$ comes from the center-of-mass zero modes of the particle on \mathbb{R}^4 , which would have caused the path integral for Z to diverge at $\epsilon_1 = \epsilon_2 = 0$. So $\epsilon_{1,2}$ also plays the role of IR regulators. $f(\tau, m, \epsilon_{1,2}, v)$ takes into account the relative degrees of freedom of the bound state, in which $\epsilon_{1,2}$ are just chemical potentials. In particular, $\epsilon_{1,2} \rightarrow 0$ limit is smooth in f .

In this paper, we shall mostly discuss the limit $\epsilon_1, \epsilon_2 \rightarrow 0$. In this limit, one finds

$$Z_{\text{pert}} \sim \exp \left[-\frac{F_{\text{pert}}(v, m)}{\epsilon_1 \epsilon_2} \right], \quad Z_{\text{inst}} \sim \exp \left[-\frac{F_{\text{inst}}(\tau, v, m)}{\epsilon_1 \epsilon_2} \right] \tag{2.19}$$

from (2.18). $F = F_{\text{pert}} + F_{\text{inst}}$ is the prepotential. F_{inst} can be obtained from (2.8) after a straightforward but tedious calculation. F_{pert} can be obtained from (2.13), which is given by

$$F_{\text{pert}}(v, m) = \frac{\pi i m^2}{4} |G| + \sum_{\alpha \in \text{adj}} \left(\text{Li}_3(e^{-\alpha(v)}) - \frac{1}{2} \text{Li}_3(e^{-(\alpha(v)+m)}) - \frac{1}{2} \text{Li}_3(e^{-(\alpha(v)-m)}) \right), \tag{2.20}$$

where $\text{Li}_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s}$ for $|x| < 1$, and can be continued to the complex x plane with a branch cut. The first term coming from \mathcal{F} will play no role in this paper. One way of obtaining (2.20) is to first take v, m to be purely imaginary, to guarantee convergence of the sum in (2.13), and take the limit $\epsilon_{1,2} \rightarrow 0$ to obtain (2.20). Then, (2.20) can be

analytically continued to complex v, m . One may alternatively start from \check{Z}_{pert} and obtain its prepotential,

$$\check{F}_{\text{pert}} = \frac{m^2}{2}(\pi i |\Delta_+| + \sum_{\alpha > 0} \alpha(v)) + \sum_{\alpha > 0} \left(2\text{Li}_3(e^{-\alpha(v)}) - \text{Li}_3(e^{-\alpha(v) \pm m}) \right) + r F_{\text{pert}}^{\text{U}(1)}. \quad (2.21)$$

Δ_+ is the set of positive roots. Here, from the identity

$$\text{Li}_n(e^{2\pi i x}) + (-1)^n (e^{-2\pi i x}) = -\frac{(2\pi i)^n}{n!} B_n(x) \quad (2.22)$$

for $0 < \text{Re}(x) \leq 1$ and $\text{Im}(x) < 0$, where $B_n(x)$'s are Bernoulli polynomials, one finds

$$\text{Li}_3(e^x) - \text{Li}_3(e^{-x}) = -\frac{(2\pi i)^3}{6} B_3\left(\frac{x}{2\pi i}\right) \quad (2.23)$$

for $\text{Re}(x) > 0$ and $0 < \text{Im}(x) \leq 2\pi$. So for simplicity, let us assume $\text{Re}(\alpha(v)) > \pm \text{Re}(m)$ for all positive roots α , and also $\text{Im}(\alpha(v))$ is chosen such that all $\text{Im}(\alpha(v) \pm m)$ are within the range $(0, 2\pi]$ for positive roots. Then one finds

$$\begin{aligned} F_{\text{pert}} - \check{F}_{\text{pert}} &= -\frac{m^2}{2} \sum_{\alpha > 0} \alpha(v) - \frac{(2\pi i)^3}{6} \sum_{\alpha > 0} \left[B_3\left(\frac{\alpha(v)}{2\pi i}\right) - \frac{1}{2} B_3\left(\frac{\alpha(v) \pm m}{2\pi i}\right) \right] \\ &= -\frac{\pi i m^2}{2} |\Delta_+| \end{aligned} \quad (2.24)$$

where we used $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$. So at least in this setting, F_{pert} and \check{F}_{pert} differ only by a trivial constant independent of v . The last constant will play no role in this paper.

It will be helpful to consider the prepotential of the U(1) theory separately. From (2.16) and (2.17), the prepotential $f_{\text{U}(1)} = F_{\text{pert}}^{\text{U}(1)} + F_{\text{inst}}^{\text{U}(1)}$ for the U(1) theory is given by

$$f_{\text{U}(1)} = \sum_{n=1}^{\infty} (2\text{Li}_3(q^n) - \text{Li}_3(e^m q^n) - \text{Li}_3(e^{-m} q^n)) + \frac{1}{2} (2\text{Li}_3(1) - \text{Li}_3(e^m) - \text{Li}_3(e^{-m})) + \frac{\pi i m^2}{4}. \quad (2.25)$$

For studying the S-duality of this prepotential, it will be useful to make an expansion of $f_{\text{U}(1)}$ in m . One first finds that the instanton part is given by

$$\begin{aligned} \sum_{n=1}^{\infty} (2\text{Li}_3(q^n) - \text{Li}_3(e^m q^n) - \text{Li}_3(e^{-m} q^n)) &= -m^2 \sum_{n=1}^{\infty} \text{Li}_1(q^n) - 2 \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^{2j+2}}{(2j+2)!} \text{Li}_{1-2j}(q^n) \\ &= m^2 \sum_{n=1}^{\infty} \log(1-q^n) - 2 \sum_{j,n,k=1}^{\infty} \frac{m^{2j+2}}{(2j+2)!} k^{2j-1} q^{nk} = m^2 \log \phi(\tau) - 2 \sum_{j,k=1}^{\infty} \frac{m^{2j+2}}{(2j+2)!} \frac{k^{2j-1} q^k}{1-q^k} \\ &= m^2 \log \phi(\tau) + \sum_{j=1}^{\infty} \frac{m^{2j+2}}{2j(2j+2)!} (E_{2j}(\tau) - 1), \end{aligned} \quad (2.26)$$

where $\phi(\tau) = \prod_{n=1}^{\infty} (1 - q^n) = q^{-\frac{1}{24}} \eta(\tau)$ is the Euler function, and we used the identity

$$\sum_{k=1}^{\infty} \frac{k^{2j-1} q^k}{1-q^k} = -\frac{B_{2j}}{4j} (E_{2j}(\tau) - 1) \quad (2.27)$$

for the Eisenstein series $E_{2n}(\tau)$. B_n are the Bernoulli numbers: $B_1 = \pm\frac{1}{2}$, $B_{2n+1} = 0$,

$$B_0 = 1, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad (2.28)$$

and so on. The perturbative prepotential can be expanded in m by using

$$\text{Li}_n(e^z) = \frac{z^{n-1}}{(n-1)!} (H_{n-1} - \log(-z)) + \sum_{k=0; k \neq n-1}^{\infty} \frac{\zeta(n-k)}{k!} z^k, \quad (2.29)$$

at small z , with $H_n = \sum_{p=1}^n \frac{1}{p}$. One finds

$$\frac{1}{2} (2\text{Li}_3(1) - \text{Li}_3(e^m) - \text{Li}_3(e^{-m})) = m^2 \left(\frac{1}{2} \log m - \frac{3}{4} + \frac{1}{4} \log(-1) \right) + \sum_{j=1}^{\infty} \frac{B_{2j} m^{2j+2}}{2j(2j+2)!} \quad (2.30)$$

Combining all, one obtains

$$f_{\text{U}(1)} = m^2 \left(\frac{1}{2} \log m - \frac{3}{4} + \frac{\pi i}{2} + \log \phi(\tau) \right) + \sum_{n=1}^{\infty} \frac{m^{2n+2} B_{2n}}{2n \cdot (2n+2)!} E_{2n}(\tau). \quad (2.31)$$

This will be useful later for understanding $N f_{\text{U}(1)}$, as a part of the $\text{U}(N)$ prepotential.

One can understand the chemical potentials from the viewpoint of the 4d effective action in the Coulomb branch. The dimensionless variables $m, \epsilon_{1,2}, v$ take the form of

$$m = RM, \quad \epsilon_{1,2} = R\epsilon_{1,2}, \quad v = Ra, \quad (2.32)$$

where R is the radius of the temporal circle of $\mathbb{R}^4 \times S^1$. M is the mass deformation parameter of the 4d $\mathcal{N} = 2^*$ Yang-Mills theory, or the 5d $\mathcal{N} = 1^*$ theory. (More precisely, M is 2π times the mass.) $\epsilon_{1,2}$ are the Omega deformation parameters which have dimensions of mass. a is the Coulomb VEV of the scalar field Φ^5 . τ is identified as

$$\tau = i \frac{R}{R'}, \quad (2.33)$$

where R' is the radius of the sixth circle. This is the inverse gauge coupling in 4d. τ can be complexified with a real part, given by the RR 1-form holonomy of type IIA theory on S^1 .

The 4 dimensional limit of the partition function is obtained by taking $R \rightarrow 0$ with fixed $\tau, M, \epsilon_{1,2}, a$. From (2.8), one finds that all sinh functions of $v, \epsilon_{1,2}, m$ are replaced by linear functions of $a, \epsilon_{1,2}, M$, and the R dependences cancel between numerator and denominator. As a result, the 4d limit Z_k^{4d} of the instanton partition function is given by a rational function of $M, \epsilon_{1,2}, a$ of degree 0. This makes Z_{inst}^{4d} and F_{inst}^{4d} to enjoy a simple scaling property,

$$Z_{\text{inst}}^{4d}(\tau, \lambda M, \lambda \epsilon_{1,2}, \lambda a) = Z_{\text{inst}}^{4d}(\tau, M, \epsilon_{1,2}, a), \quad F_{\text{inst}}^{4d}(\tau, \lambda M, \lambda a) = \lambda^2 F_{\text{inst}}^{4d}(\tau, M, a). \quad (2.34)$$

This will be used in section 2.1 to provide two interpretations of the 4d S-duality, and extend one version to 6d. As for the perturbative part F_{pert} , one can use (2.29) to obtain

$F_{\text{pert}}^{4d} \equiv \lim_{R \rightarrow 0} F_{\text{pert}}$. One finds

$$F_{\text{pert}}^{4d} = \sum_{\alpha \in \text{adj}} \left[M^2 \left(\log R - \frac{3}{4} \right) - \frac{\alpha(a)^2}{2} \log \alpha(a) + \frac{(\alpha(a) + M)^2}{4} \log(\alpha(a) + M) + \frac{(\alpha(a) - M)^2}{4} \log(\alpha(a) - M) \right] \quad (2.35)$$

where the first term independent of the Coulomb VEV is unphysical in the Seiberg-Witten theory. The perturbative prepotential satisfies the following pseudo-scaling property,

$$F_{\text{pert}}^{4d}(\lambda M, \lambda a) = \lambda^2 \left(F_{\text{pert}}^{4d}(M, a) + |G| \frac{M^2}{2} \log \lambda \right), \quad (2.36)$$

which is homogenous and degree 2 up to a Coulomb VEV independent shift.

Z_{inst} or F_{inst} are only known as q expansion when $q \ll 1$, or $\tau \rightarrow i\infty$. This is useful when the ‘temperature’ is much smaller than the Kaluz-Klein scale $\frac{1}{R}$, when the KK modes are ‘heavy.’ However, to study 6d SCFT, it is more interesting to explore the regime $q \rightarrow 1$, or $\tau \rightarrow i0^+$, in which case the circle effectively decompactifies. The two regimes are weakly coupled and strongly coupled regimes, respectively. So if there is S-duality for the partition function on $\mathbb{R}^4 \times T^2$, it will be helpful to study the interesting decompactifying regime from the well-understood region $\tau \rightarrow i\infty$. Developing the S-duality of the prepotential F is the goal of this section. (In section 2.2, we also comment on the S-duality of the full partition function.)

2.1 S-duality and its anomaly

Following [10], we review the basic set up for studying the S-duality of 4 dimensional prepotential, and extend it to the 6d theory on T^2 .

The prepotential F of general 4d $\mathcal{N} = 2$ gauge theory determines the effective action in the Coulomb branch. The magnetic dual description uses the dual Coulomb VEV $a_D(a)$ and the dual prepotential $F_D(a_D)$, defined by the following Legendre transformation,

$$a_D = \frac{1}{2\pi i} \frac{\partial F}{\partial a}, \quad F_D(a_D) = \mathcal{L}[F](a) \equiv F(a) - 2\pi i a_D a = F - a \frac{\partial F}{\partial a}. \quad (2.37)$$

For theories with higher rank $r > 1$, a has many components, a_i with $i = 1, \dots, r$. Expressions like $a \frac{\partial}{\partial a}$ should be understood with contracted i indices, i.e. $a \frac{\partial}{\partial a} \rightarrow \sum_{i=1}^r a_i \frac{\partial}{\partial a_i}$, whose sum structures will not be explicitly shown to make the notations simpler. For generic $\mathcal{N} = 2$ theories, F, F_D depend on other parameters like hypermultiplet masses and the coupling constant (or the dynamically generated scale Λ instead of the coupling).

For 4d $\mathcal{N} = 2^*$ theory, the prepotential F^{4d} (to be distinguished with the 6d prepotential F which we shall consider later) depends on the microscopic coupling constant τ and the adjoint hypermultiplet mass M . The prepotential can be divided into the classical, perturbative, and instanton contributions,

$$F^{4d} = F_{\text{cl}}(\tau, a) + F_{\text{pert}}^{4d}(a, M) + F_{\text{inst}}^{4d}(\tau, a, M) \equiv F_{\text{cl}}(\tau, a) + f^{4d}(\tau, a, M) \quad (2.38)$$

where $F_{\text{cl}}(\tau, a) = \pi i \tau a^2$, and $F_{\text{pert}}^{4\text{d}}$. $f^{4\text{d}} \equiv F_{\text{pert}}^{4\text{d}} + F_{\text{inst}}^{4\text{d}}$ is the quantum prepotential. To study self S-dual theories, it is convenient to define $F_D^{4\text{d}}$ as a function of the dual coupling $\tau_D = -\frac{1}{\tau}$. For the 4d $\mathcal{N} = 2^*$ theory, $F_D^{4\text{d}}$ is defined by

$$F_D^{4\text{d}}(\tau_D, a_D, M) = \mathcal{L}[F^{4\text{d}}](\tau, a, M) = F^{4\text{d}}(\tau, a, M) - a \frac{\partial F^{4\text{d}}}{\partial a}(\tau, a, M). \quad (2.39)$$

Then, self S-duality exists if $F_D^{4\text{d}}$ and $F^{4\text{d}}$ are same function, $F_D^{4\text{d}}(\tau, a, M) = F^{4\text{d}}(\tau, a, M)$. This S-duality has been tested in detail in [10]. More precisely, it was found that

$$F^{4\text{d}}(\tau, a, M) = F_{\text{S-dual}}^{4\text{d}}(\tau, a, M) + F_{\text{anom}}^{4\text{d}}(\tau, M), \quad (2.40)$$

where $F_{\text{S-dual}}^{4\text{d}}$ satisfies

$$F_{\text{S-dual}}^{4\text{d}}(\tau_D, a_D, M) = F_{\text{S-dual}}^{4\text{d}}(\tau, a, M) - a \frac{\partial F_{\text{S-dual}}^{4\text{d}}}{\partial a}(\tau, a, M), \quad (2.41)$$

and $F_{\text{anom}}^{4\text{d}}$ is an anomalous part of S-duality, depending on τ, M but is independent of the Coulomb VEV a [10]. Since the Coulomb branch effective action is obtained by taking a derivatives of $F^{4\text{d}}$, $F^{4\text{d}}$ and $F_{\text{S-dual}}^{4\text{d}}$ are identical in the Seiberg-Witten theory. This establishes the S-duality of the 4d $\mathcal{N} = 2^*$ theory in the Coulomb branch effective action.

Let us rephrase the 4d S-duality in a way that is suitable for 6d extension. $F_{\text{inst}}^{4\text{d}}$ satisfies the scaling property (2.34). Combining the perturbative part, one finds

$$F^{4\text{d}}(\tau, \lambda a, \lambda M) = \lambda^2 \left(F^{4\text{d}}(\tau, a, M) + |G| \frac{M^2}{2} \log \lambda \right). \quad (2.42)$$

Applying this to $F^{4\text{d}}(\tau_D, a_D, m)$, one obtains

$$F^{4\text{d}}(\tau_D, a_D/\tau, M/\tau) = \tau^{-2} F^{4\text{d}}(\tau_D, a_D, M) - \frac{M^2}{2\tau^2} \log \tau. \quad (2.43)$$

So the left hand side of (2.41) can be written as

$$\begin{aligned} F_{\text{S-dual}}^{4\text{d}}(\tau_D, a_D, M) &= \tau^2 F_{\text{S-dual}}^{4\text{d}}(\tau_D, a_D/\tau, M/\tau) \\ &+ \frac{|G|M^2}{2} \log \tau + \tau^2 F_{\text{anom}}^{4\text{d}}(\tau_D, M/\tau) - F_{\text{anom}}^{4\text{d}}(\tau_D, M). \end{aligned} \quad (2.44)$$

Let us consider the structure of $F_{\text{anom}}^{4\text{d}}$. Since the prepotential has mass dimension 2, one may think that its M dependence is simply M^2 . However, the perturbative part (2.35) shows that there is a term $\frac{rM^2}{2} \log M$ in $F^{4\text{d}}$ which scales in an odd manner. In the computational framework of [10], which we shall explain below in our 6d version, $F_{\text{S-dual}}^{4\text{d}}$ is by construction taken to be a series expansion in M^2 . This means that the odd term $\frac{rM^2}{2} \log M$ should have been put in $F_{\text{anom}}^{4\text{d}}$. Therefore, had one been doing the calculation of [10] using (2.35) as the perturbative part, one would have found that $F_{\text{anom}}^{4\text{d}} = \frac{rM^2}{2} \log M + M^2(\dots)$, where (\dots) only depends on τ . Using this structure, (2.44) can be rewritten as

$$F_{\text{S-dual}}^{4\text{d}}(\tau_D, a_D, M) = \tau^2 F_{\text{S-dual}}^{4\text{d}}(\tau_D, a_D/\tau, M/\tau) + (|G| - r) \frac{M^2}{2} \log \tau. \quad (2.45)$$

So defining

$$\tilde{F}_{\text{S-dual}}^{4d}(\tau, a, M) = F_{\text{S-dual}}^{4d}(\tau, a, M) - \frac{|G| - r}{2} M^2 \log M, \quad (2.46)$$

one finds that $\tilde{F}_{\text{S-dual}}^{4d}$ satisfies

$$\tau^2 \tilde{F}_{\text{S-dual}}^{4d}(\tau_D, a_D/\tau, M/\tau) = \tilde{F}_{\text{S-dual}}^{4d}(\tau, a, M) - a \frac{\partial \tilde{F}_{\text{S-dual}}^{4d}}{\partial a}(\tau, a, M), \quad (2.47)$$

instead of (2.41). To summarize, by trivially redefining $F_{\text{S-dual}}^{4d}$ and F_{anom}^{4d} by the last term of (2.46), one can reformulate the standard S-duality (2.41) as (2.47). Only (2.47) will naturally generalize to the S-duality on $\mathbb{R}^4 \times T^2$.

Now we seek for the S-duality of the 6d prepotential. Note that in 4d, (2.41) and (2.47) are equivalent by making a minor redefinition of F_{anom}^{4d} , using (2.42). In 6d, a property like (2.42) does not hold. Before making a quantitative study of the 6d S-duality, we first explain that (2.47) is more natural in 6d. To discuss the 6d prepotential, it is convenient to work with the dimensionless parameters $v, m, \epsilon_{1,2}$.

Firstly, in the 6d theory compactified on T^2 , the complex mass parameter m is simply the holonomy of the background gauge field for $\text{SU}(2)_L$ global symmetry, along the two sides of T^2 . Then after making an S-duality of the torus, exchanging two sides of T^2 , one naturally expects $m_D = \frac{m}{\tau}$. Let us briefly review this by taking a rectangular torus, for simplicity. In this case, the complex structure τ of the torus is purely imaginary. τ is related to the two radii of T^2 by

$$\tau = i \frac{R}{R'}, \quad (2.48)$$

where R' is the radius of the circle which compactifies the 6d theory to 5d SYM, and R is the radius of another circle which compactifies the 5d theory to 4d. The S-duality transformation exchanges $R \leftrightarrow R'$. So the dual complex structure is $\tau_D = i \frac{R'}{R} = -\frac{1}{\tau}$. More precisely, S-duality rotates the torus by 90 degrees on a plane. It also transforms the two $\text{SU}(2)_L$ holonomies along the two circles. Let $\text{Re}(M)$ be the holonomy on the circle with radius R , and $\text{Im}(M)$ that on the circle with radius R' . Under S-duality, one finds $\text{Re}(M_D) = \text{Im}(M)$, $\text{Im}(M_D) = -\text{Re}(M)$. So one finds $M_D = -iM$. In F , M appears in the dimensionless combination $m \equiv RM$, which transforms as

$$m_D = R' M_D = -i R' M = -i \frac{R'}{R} m = \frac{m}{\tau}. \quad (2.49)$$

The final result holds for complex τ . Similar property holds for $\epsilon_{1,2} \equiv R\epsilon_{1,2}$, i.e. $\epsilon_{1,2}^D = \frac{\epsilon_{1,2}}{\tau}$. This makes the appearance of $\frac{M}{\tau}$ to be more natural on the left hand side of (2.47).

Secondly, let us discuss how a should transform. In 4d, we already stated that

$$a_D = \tau a + \frac{1}{2\pi i} \frac{\partial f}{\partial a} \quad (2.50)$$

naturally appears on the left hand side of (2.41). For simplicity, let us discuss these variables in the limit of large Coulomb VEV, $v \equiv Ra \gg 1$, $a \gg m$. The second term can be ignored in this limit, yielding the semi-classical result $a_D = \tau a$. In this limit, we shall discuss what is the natural S-dual variable using the Abelian 6d (2, 0) theory. In 4d, $a_D = \tau a$ is a

natural aspect of S-duality being electromagnetic duality. Also, it makes sense to multiply a by a complex number τ , since a is a complex variable living on a plane. However, in 6d CFT on T^2 , a lives on a cylinder. The real part of a is the VEV of the real scalar in the 6d self-dual tensor multiplet, which is noncompact. On the other hand, the imaginary part of a comes from the holonomy of the 2-form tensor field B on T^2 , implying that it is a periodic variable. So it does not good make sense to rotate a living on a cylinder by complex τ . More precisely, the 6d scalar ϕ and the 5d scalar a are related by $a \sim R'\phi$. So one finds

$$a \sim R'(\phi + iB_{12}), \tag{2.51}$$

where 1 and 2 denote two directions of T^2 . Thus, $v = Ra \sim RR'(\phi + iB_{12})$ is invariant under $R \leftrightarrow R'$, meaning that it makes more sense to set $v_D \approx v$ in the limit $v \gg 1$. Using the dimensionful variables, This requires one to use $\frac{a_D}{\tau} \approx a$ as the dual variable, instead of $a_D \approx \tau a$. This does not rotate the variable a by a complex number, so makes better sense in 6d. Incidentally, we have already found the alternative (but equivalent) statement (2.47) of S-duality which uses $\frac{a_D}{\tau}$ as the dual variable, instead of a_D . Note that the usage of $\frac{a_D}{\tau} = a + \frac{1}{2\pi i\tau} \frac{\partial f^{4d}}{\partial a}$ is valid even beyond the semi-classical limit $a \gg M$. Thus, in the 6d uplift, it is natural and consistent to regard $v_D \equiv \frac{Ra_D}{\tau} = v + \frac{1}{2\pi i\tau} \frac{\partial f}{\partial v}$. as the dual variable.¹

So it appears natural to seek for a 6d generalization of (2.47) rather than (2.41). This is what we shall establish in the rest of this section. Namely, we shall find that the 6d prepotential is divided into two,

$$F = F_{\text{S-dual}}(\tau, v, m) + F_{\text{anom}}(\tau, m) \tag{2.52}$$

where $v = Ra$, $m = RM$, and F_{anom} is independent of the Coulomb VEV. $F_{\text{S-dual}}$ satisfies

$$\tau^2 F_{\text{S-dual}} \left(\tau_D = -\frac{1}{\tau}, v_D = v + \frac{1}{2\pi i\tau} \frac{\partial f}{\partial v}, \frac{m}{\tau} \right) = F_{\text{S-dual}}(\tau, v, m) - v \frac{\partial F_{\text{S-dual}}}{\partial v}(\tau, v, m). \tag{2.53}$$

We have some freedom to choose F_{anom} , by adding/subtracting v independent S-dual expressions to F_{anom} , $F_{\text{S-dual}}$. We shall explain that one can choose F_{anom} as

$$F_{\text{anom}} = N f_{\text{U}(1)}(\tau, m) + \frac{N^3 - N}{288} m^4 E_2(\tau) \tag{2.54}$$

where $f_{\text{U}(1)}$ is the U(1) prepotential (2.25). The first term $N f_{\text{U}(1)}$ comes from the N 6d Abelian tensor multiplets in $\text{U}(1)^N$, which has their own S-duality anomaly. The second term of F_{anom} is one of the key findings of this paper, which comes from the charged part of the partition function. After replacing $m = MR$, and multiplying $\frac{1}{R^2}$ to the above F_{anom} to get to the conventionally normalized prepotential (as noted in footnote 1), one can take the 4d limit of F_{anom} . The second term proportional to $N^3 - N$ vanishes in the 4d limit $R \rightarrow 0$, as it is proportional to $M^4 R^2$.

¹Here, one may wonder that f appearing on the right hand side should have been $R^2 f$. However, we shall define the prepotential as the coefficient of the dimensionless $\frac{1}{\epsilon_1 \epsilon_2}$, $-\log Z \sim \frac{f}{\epsilon_1 \epsilon_2}$, rather than $\frac{f^{4d}}{\epsilon_1 \epsilon_2}$ that is conventional in the Seiberg-Witten theory, making f dimensionless. Namely, f_{ours} in 6d is related to the conventionally normalized prepotential by $f_{\text{ours}} = R^2 f_{\text{conventional}}$.

With the motivations and results given, we now properly set up the calculation and show the claims made above. As in 4d, we decompose the 6d prepotential as

$$F(\tau, v, m) = F_{\text{cl}} + F_{\text{pert}} + F_{\text{inst}} \equiv F_{\text{cl}} + f, \quad (2.55)$$

where $F_{\text{cl}} \equiv \pi i \tau v^2$. The prepotential is S-dual if it satisfies

$$\tau^2 F \left(\tau_D = -\frac{1}{\tau}, v_D = v + \frac{1}{2\pi i \tau} \frac{\partial f}{\partial v}, m_D = \frac{m}{\tau} \right) = F(\tau, v, m) - v \frac{\partial F}{\partial v}(\tau, v, m). \quad (2.56)$$

We first study the structures of this equation, before showing that it is satisfied by our $F_{\text{S-dual}}$. Firstly, replacing F by F_{cl} , one can check that S-duality trivially holds at the classical level:

$$\tau^2 F_{\text{cl}}(\tau_D, v_D) = \tau^2 \left[-\frac{\pi i}{\tau} v^2 \right] = -\pi i \tau v^2 = F_{\text{cl}}(\tau, v) - v \frac{\partial F_{\text{cl}}}{\partial v}(\tau, v), \quad (2.57)$$

where v_D is replaced by its classical value $v_D = v$ (formally at $f = 0$). Now we subtract (2.56) by (2.57) to find the following condition for the quantum prepotential f :

$$\tau^2 f \left(-\frac{1}{\tau}, v + \frac{1}{2\pi i \tau} \frac{\partial f}{\partial v}, \frac{m}{\tau} \right) = f(\tau, v, m) + \frac{1}{4\pi i \tau} \left(\frac{\partial f}{\partial v}(\tau, v, m) \right)^2. \quad (2.58)$$

We are going to study the last equation. Note again that the effective action in the Coulomb branch only contains v derivatives of F , or f . Thus, in Seiberg-Witten theory, f is ambiguous by addition of v independent functions, possibly depending on τ and m . However, the S-duality requirement (2.58) is sensitive to the value of f , including the v independent part. So when one tries to establish the S-duality of the Coulomb branch effective action, one should have in mind that one may have to add suitable Coulomb VEV independent terms to f computed microscopically from Z .

Following [10], we shall establish the S-duality (2.58) and its anomaly (2.54) by expanding f in the mass m when it is small enough. We shall still get an exact statement (2.54), which we check for certain orders in m . One should however have in mind that the exact statement (2.54) may be valid only within a finite region of m, v in the complex planes. In section 2.2, we shed more lights on the exactness of (2.54), by making an M-string expansion [8].

As studied in the 4d limit [10], there is a natural way of achieving the S-duality requirement (2.58). This is to require that f is expanded in quasi-modular forms of suitable weights. To precisely explain its meaning, we first expand f in m as

$$f(\tau, v, m) = \sum_{n=1}^{\infty} m^{2n} f_n(\tau, v). \quad (2.59)$$

This series makes sense as follows. Firstly, the $m \rightarrow 0$ limit exhibits enhanced maximal supersymmetry. So at $m = 0$, the classical prepotential $F_{\text{cl}} = \pi i \tau v^2$ acquires no quantum corrections, meaning that f vanishes at $m = 0$. Also, the prepotential is an even function

of m , which restrict the expansion as above.² Then, following [10], we require that f_n is a quasi-modular form of weight $2n - 2$, which means the following. Quasi-modular forms are polynomials of the first three Eisenstein series E_2, E_4, E_6 , where each series has weight 2, 4, 6 respectively under S-duality in the following sense:

$$E_2(-1/\tau) = \tau^2 \left(E_2 + \frac{6}{\pi i \tau} \right), \quad E_4(-1/\tau) = \tau^4 E_4(\tau), \quad E_6(-1/\tau) = \tau^6 E_6(\tau). \quad (2.60)$$

More concretely, they are given by

$$E_2 = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}, \quad E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}, \quad E_6 = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}. \quad (2.61)$$

Higher Eisenstein series E_{2n} are polynomials of E_4, E_6 with weight $2n$. To study the quasi-modular property, it is helpful to decompose their dependence on τ into the dependence through E_2 and the dependence through E_4, E_6 . We thus write $f_n(\tau, v, E_2(\tau))$, where the τ dependence through E_2 is explicitly shown. A weight $2n - 2$ quasi-modular form f_n satisfies

$$f_n(-1/\tau, v, E_2(-1/\tau)) = \tau^{2n-2} f_n(\tau, v, E_2(\tau) + \delta), \quad (2.62)$$

where $\delta \equiv \frac{6}{\pi i \tau}$. In terms of f , this is equivalent to

$$\tau^2 f \left(-\frac{1}{\tau}, v, \frac{m}{\tau}, E_2(-1/\tau) \right) = f(\tau, v, m, E_2(\tau) + \delta). \quad (2.63)$$

We now investigate how quasi-modularity is related to the S-duality (2.58). One can make (2.63) to be equivalent to (2.58) by specifying the E_2 dependence of f , which we now turn to.

Let us first try to find the desired E_2 dependence, by requiring both (2.58) and (2.63). By applying (2.63) to $f(-\frac{1}{\tau}, v_D, \frac{m}{\tau}, E_2(-1/\tau))$, one obtains

$$\tau^2 f \left(-\frac{1}{\tau}, v + \frac{\delta}{12} \frac{\partial f}{\partial v}, \frac{m}{\tau}, E_2(-1/\tau) \right) = f \left(\tau, v + \frac{\delta}{12} \frac{\partial f}{\partial v}, m, E_2(\tau) + \delta \right), \quad (2.64)$$

where again recall that $\delta \equiv \frac{6}{\pi i \tau}$. Combining this with (2.58), one obtains

$$f \left(\tau, v + \frac{\delta}{12} \frac{\partial f}{\partial v}, m, E_2(\tau) + \delta \right) = f(\tau, v, m, E_2(\tau)) + \frac{\delta}{24} \left(\frac{\partial f}{\partial v}(\tau, v, m, E_2(\tau)) \right)^2. \quad (2.65)$$

We want to make this equation to hold, by specifying a particular E_2 dependence of f . [10] showed that the desired E_2 dependence is

$$\frac{\partial f}{\partial E_2} = -\frac{1}{24} \left(\frac{\partial f}{\partial v} \right)^2. \quad (2.66)$$

For the sake of completeness, we repeat the logics presented in [10] and expand it to make a proof. In fact, we shall make a stronger claim than needed. Namely, we need to find

²Strictly speaking, there is a term $\frac{rm^2}{2} \log m$ in the perturbative part, which is easiest to see from the 4d limit (2.35). However, we shall expand $f_{S\text{-dual}}$ as (2.59), while the term $\frac{rm^2}{2} \log m$ is moved to F_{anom} .

the E_2 dependence of f which guarantees (2.65) only at $\delta = \frac{6}{\pi i \tau}$. However, we shall show that (2.66) guarantees (2.65) for arbitrary independent parameter δ , and then set $\delta = \frac{6}{\pi i \tau}$ later.

As a warm-up, we follow [10] to make a series expansion of the left hand side of (2.65) in small δ , to see how (2.66) guarantees (2.65) at low orders. One finds that

$$(\text{l.h.s.}) = f + \frac{\delta}{12} \left(\frac{\partial f}{\partial v} \right)^2 + \delta \frac{\partial f}{\partial E_2} + \mathcal{O}(\delta^2). \quad (2.67)$$

So at δ^0 and δ^1 orders, one finds that it agrees with the right hand side if (2.66) is met.

Now assuming (2.66), we consider whether (2.65) is satisfied in full generality. To this end, we take δ derivative of both sides of (2.65), at fixed τ, v, E_2 ,

$$\frac{\partial}{\partial \delta}(\text{l.h.s.}) = \frac{1}{12} \frac{\partial f}{\partial v} \frac{\partial \tilde{f}}{\partial \tilde{v}} + \frac{\partial \tilde{f}}{\partial \tilde{E}_2} = \frac{1}{12} \frac{\partial f}{\partial v} \frac{\partial \tilde{f}}{\partial \tilde{v}} - \frac{1}{24} \left(\frac{\partial \tilde{f}}{\partial \tilde{v}} \right)^2, \quad \frac{\partial}{\partial \delta}(\text{r.h.s.}) = \frac{1}{24} \left(\frac{\partial f}{\partial v} \right)^2 \quad (2.68)$$

where for simplicity, we defined

$$\tilde{v} = v + \frac{\delta}{12} \frac{\partial f}{\partial v}, \quad \tilde{E}_2 = E_2 + \delta, \quad \tilde{f} = f(\tau, \tilde{v}, \tilde{E}_2). \quad (2.69)$$

Note that at the second step of the first equation in (2.68), we used $\frac{\partial \tilde{f}}{\partial \tilde{E}_2} = -\frac{1}{24} \left(\frac{\partial \tilde{f}}{\partial \tilde{v}} \right)^2$ which follows from (2.66). If (2.65) holds for general δ , its first derivative would yield

$$\frac{\partial \tilde{f}}{\partial \tilde{v}} = \frac{\partial f}{\partial v}, \quad (2.70)$$

which one can show by using (2.68). On the other hand, (2.70) together with the $\mathcal{O}(\delta^0)$ component of (2.65) is equivalent to (2.65), since the $\mathcal{O}(\delta^0)$ component is the only information lost by taking δ derivative. However, we have already shown around (2.67) that the $\mathcal{O}(\delta^0)$ component of (2.65) is satisfied. Therefore, showing (2.70) will be equivalent to showing (2.65). So will show (2.70) by assuming (2.66). We take δ derivative of $\frac{\partial \tilde{f}}{\partial \tilde{v}} - \frac{\partial f}{\partial v}$ at fixed v, E_2 . Again using (2.66), one obtains

$$\frac{\partial}{\partial \delta} \left(\frac{\partial \tilde{f}}{\partial \tilde{v}} - \frac{\partial f}{\partial v} \right) = -\frac{1}{12} \frac{\partial^2 \tilde{f}}{\partial \tilde{v}^2} \left(\frac{\partial \tilde{f}}{\partial \tilde{v}} - \frac{\partial f}{\partial v} \right). \quad (2.71)$$

So if $\frac{\partial \tilde{f}}{\partial \tilde{v}} - \frac{\partial f}{\partial v}$ is zero at a particular value of δ , (2.71) guarantees that it is zero at different values of δ . Since we already checked around (2.67) that (2.65) is true up to $\mathcal{O}(\delta^1)$, we have shown that (2.70) holds at $\mathcal{O}(\delta^0)$, or that $\frac{\partial \tilde{f}}{\partial \tilde{v}} - \frac{\partial f}{\partial v} = 0$ at $\delta = 0$. This establishes that (2.66) implies (2.70), and in turn that (2.66) implies (2.65). Finally, we insert $\delta = \frac{6}{\pi i \tau}$.

To summarize till here, (2.65) holds if f satisfies (2.66). But (2.65) and (2.63) implies the S-duality relation (2.58). Therefore, S-duality requirement (2.58) is satisfied if f satisfies the quasi-modular property (2.63) and the modular anomaly equation (2.66). In the rest of this subsection, we shall discuss the last two equations.

Following and extending [10], we show that the prepotential f obeys the two properties (2.63), (2.66), up to an anomalous part which is independent of the Coulomb VEV v .

Again following [10], our strategy is to first find a prepotential $f_{\text{S-dual}}$ in a series of m^2 which satisfies both (2.63) and (2.66). Then we show that $f - f_{\text{S-dual}}$ is independent of v .

We expand $f_{\text{S-dual}}$ like (2.59), $f_{\text{S-dual}} = \sum_{n=1}^{\infty} m^{2n} f_n(\tau, v)$. (2.66) is given in terms of f_n by

$$\frac{\partial f_n}{\partial E_2} = -\frac{1}{24} \sum_{m=1}^{n-1} \frac{\partial f_m}{\partial v} \frac{\partial f_{n-m}}{\partial v} \tag{2.72}$$

for $n \geq 2$. This equation can be used to recursively compute f_n . Namely, once we know f_m for $m = 1, \dots, n-1$, one can integrate the right hand side of (2.72) with E_2 to get f_n , up to an integration constant independent of E_2 . The integration constant is a polynomial of E_4 and E_6 with modular weight $2n-2$, whose coefficients depend only on v . These integration constants depending on v can be fixed once we know a few low order coefficients of f in q expansion. Also, to start the recursive construction, the first coefficient f_1 at m^2 should be known. It will turn out that this can be also fixed by the known perturbative part f_{pert} [10]. This way, one can recursively generate the coefficients of $f_{\text{S-dual}}$ from (2.63), (2.66) and the knowledge of the few low order coefficients of f in q expansion. We emphasize here that our purpose of making a recursive construction of $f_{\text{S-dual}}$ is to show that the Coulomb VEV dependent part of f is S-dual. Therefore, while fixing the integration constants and f_1 in $f_{\text{S-dual}}$ by using the low order q expansion coefficients of f , it suffices to use f up to the addition of any convenient expression independent of v . So for technical reasons, we shall fit these integration constants and f_1 by comparing $f_{\text{S-dual}}$ with

$$f(\tau, v, m) - N f_{\text{U}(1)}(\tau, m) \tag{2.73}$$

rather than f itself. Note that $N f_{\text{U}(1)}$ is the prepotential contribution from $\text{U}(1)^N$ Cartan part, coming from D0-branes bound to D4-branes but unbound to W-bosons which see v . One reason for comparing with $f - N f_{\text{U}(1)}$ is that $f_{\text{U}(1)}$ does not admit a power series expansion in m^2 like (2.59). The S-duality anomaly of $N f_{\text{U}(1)}$ can be calculated separately from (2.31).

With these understood, we start the recursive construction by determining f_1 . This can be fixed solely from the perturbative part of $(f - N f_{\text{U}(1)})_{\text{pert}}$ [10]. Namely, when instantons are bound to W-bosons, there are fermion zero modes which provide at least a factor of m^4 in f . This means that m^2 term f_1 should come from the perturbative part only. This fact can also be straightforwardly checked from the microscopic calculus. So one finds

$$f_1 = (f - N f_{\text{U}(1)})_{\text{pert}} \Big|_{m^2} = -\frac{1}{2} \sum_{\alpha \in \Delta} \text{Li}_1(e^{\alpha(v)}) = \frac{1}{2} \sum_{\alpha \in \Delta} \log(1 - e^{\alpha(v)}), \tag{2.74}$$

where Δ is the set of roots of $\text{U}(N)$.

One can then compute f_2 using (2.72) at $n = 2$,

$$\frac{\partial f_2}{\partial E_2} = -\frac{1}{24} \left(\frac{\partial f_1}{\partial v} \right)^2 = -\frac{1}{96} \sum_{\alpha, \beta \in \Delta} \frac{\alpha \cdot \beta}{(1 - e^{\alpha(v)})(1 - e^{\beta(v)})}. \tag{2.75}$$

One can integrate it with E_2 , to obtain

$$f_2 = -\frac{E_2(\tau)}{96} \sum_{\alpha, \beta \in \Delta} \frac{\alpha \cdot \beta}{(1 - e^{\alpha(v)})(1 - e^{\beta(v)})}. \tag{2.76}$$

There is no integration constant at weight 2. To proceed, we study the properties of the $U(N)$ roots. Δ consists of vectors of the form $e_i - e_j$, $i \neq j$, $i, j = 1, \dots, N$, where e_i are N orthogonal unit vectors. $\alpha \cdot \beta$ takes following possible values,

$$\begin{aligned} \alpha \cdot \beta &= \pm 2 \text{ if } \pm \beta = \alpha \\ \alpha \cdot \beta &= \pm 1 \text{ if } \pm \beta \in \Psi(\alpha) \\ \alpha \cdot \beta &= 0 \text{ otherwise,} \end{aligned} \tag{2.77}$$

where $\Psi(\alpha)$ is given for $\alpha = e_i - e_j$ by

$$\Psi(e_i - e_j) = \{k \neq i, j : e_i - e_k, e_k - e_j\}. \tag{2.78}$$

For a given α , there are $2(N - 2)$ elements of $\Psi(\alpha)$. Using this, one finds

$$\begin{aligned} f_2 &= -\frac{E_2(\tau)}{96} \left[\sum_{\alpha \in \Delta} \frac{2}{1 - e^{\alpha(v)}} \left(\frac{1}{1 - e^{\alpha(v)}} - \frac{1}{1 - e^{-\alpha(v)}} \right) + \sum_{\alpha \in \Delta} \sum_{\beta \in \Psi(\alpha)} \frac{1}{1 - e^{\alpha(v)}} \left(\frac{1}{1 - e^{\beta(v)}} - \frac{1}{1 - e^{-\beta(v)}} \right) \right] \\ &= -\frac{E_2(\tau)}{96} \left[\sum_{\alpha \in \Delta} \frac{4}{(1 - e^{\alpha(v)})^2} - \sum_{\alpha \in \Delta} \frac{2(N-1)}{1 - e^{\alpha(v)}} + \sum_{\alpha \in \Delta} \sum_{\beta \in \Psi(\alpha)} \frac{2}{(1 - e^{\alpha(v)})(1 - e^{\beta(v)})} \right] \end{aligned}$$

where we used $\frac{1}{1 - e^{\alpha(v)}} - \frac{1}{1 - e^{-\alpha(v)}} = \frac{2}{1 - e^{\alpha(v)}} - 1$. Using $\frac{1}{(1 - e^{\alpha(v)})^2} = \frac{1}{1 - e^{\alpha(v)}} + \frac{e^{\alpha(v)}}{(1 - e^{\alpha(v)})^2} = \frac{1}{1 - e^{\alpha(v)}} + \text{Li}_{-1}(e^{-\alpha(v)})$, the first term can be rewritten so that

$$f_2 = -\frac{E_2(\tau)}{96} \left[4 \sum_{\alpha \in \Delta} \text{Li}_{-1}(e^{\alpha(v)}) - \sum_{\alpha \in \Delta} \frac{2(N-3)}{1 - e^{\alpha(v)}} + \sum_{\alpha \in \Delta} \sum_{\beta \in \Psi(\alpha)} \frac{2}{(1 - e^{\alpha(v)})(1 - e^{\beta(v)})} \right]. \tag{2.79}$$

Here, one can simplify the second term by using

$$\sum_{\alpha \in \Delta} \frac{1}{1 - e^{\alpha(v)}} = \frac{1}{2} \sum_{\alpha \in \Delta} \left(\frac{1}{1 - e^{\alpha(v)}} + \frac{1}{1 - e^{-\alpha(v)}} \right) = \frac{1}{2} \sum_{\alpha \in \Delta} 1 = \frac{N(N-1)}{2}.$$

Also, using $\frac{1}{(1 - e^x)(1 - e^y)} + \frac{1}{(1 - e^{-x})(1 - e^{y-x})} + \frac{1}{(1 - e^{x-y})(1 - e^{-y})} = 1$, one finds

$$\begin{aligned} \sum_{\alpha \in \Delta} \sum_{\beta \in \Psi(\alpha)} \frac{1}{(1 - e^{\alpha(v)})(1 - e^{\beta(v)})} &= \sum_{i \neq j} \sum_{k \neq i, j} \left[\frac{1}{(1 - e^{v_i - v_j})(1 - e^{v_i - v_k})} + \frac{1}{(1 - e^{v_i - v_j})(1 - e^{v_k - v_j})} \right] \\ &= \frac{1}{3} \sum_{i \neq j \neq k \neq i} [(i, j, k) + (j, k, i) + (k, i, j)] = \sum_{i \neq j \neq k \neq i} \frac{2}{3} = \frac{2}{3} N(N-1)(N-2), \end{aligned} \tag{2.80}$$

where at the second step we symmetrized the summand by making a cyclic permutation of i, j, k . This simplifies the third term. One thus finds

$$f_2 = -\frac{E_2(\tau)}{24} \left[\sum_{\alpha \in \Delta} \text{Li}_{-1}(e^{\alpha(v)}) + \frac{N^3 - N}{12} \right] \tag{2.81}$$

at $\mathcal{O}(m^4)$ order.

Before proceeding to higher order coefficients f_n with $n \geq 3$, let us first discuss f_2 that we computed by requiring S-duality of $f_{\text{S-dual}}$. Note that at m^4 , we have obtained an all order result in the instanton expansion, coming from $E_2(\tau) = 1 - 24q - 72q^2 - 96q^3 - 168q^4 \dots$. So from the microscopic instanton calculus, one can expand $f(\tau, v, m)$ in small m , and we can compare f and $f_{\text{S-dual}}$ at m^4 order. We find that

$$(f - Nf_{\text{U}(1)}) - f_{\text{S-dual}} \Big|_{m^4} = \frac{N^3 - N}{288} m^4 E_2(\tau), \quad (2.82)$$

which we checked till q^2 order for general N , and till q^3 for $N = 2, 3$. Therefore, we find that the microscopic prepotential is compatible with S-duality at m^4 order, up to the addition of an ‘anomalous’ term on the right hand side independent of the Coulomb VEV.

One can make further recursive calculations of f_n for $n \geq 3$, using (2.72), and test the consistency of $f_{\text{S-dual}}$ with our microscopic f . The next recursion relation of (2.72) is

$$\frac{\partial f_3}{\partial E_2} = -\frac{1}{12} \frac{\partial f_1}{\partial v} \frac{\partial f_2}{\partial v}. \quad (2.83)$$

Knowing f_1, f_2 , one can integrate (2.83) to obtain

$$f_3 = -\frac{E_2(\tau)^2}{1152} \left[\sum_{\alpha \in \Delta} (2\text{Li}_{-3}(e^{\alpha(v)}) - 4\text{Li}_{-1}(e^{\alpha(v)})^2) + 2 \sum_{\alpha \in \Delta} \sum_{\beta \in \Psi(\alpha)} \text{Li}_0(e^{\alpha(v)}) \text{Li}_{-2}(e^{\beta(v)}) \right] + c_3(v) E_4(\tau). \quad (2.84)$$

The integration constant $c_3(v)$ can be determined by expanding f_3 in q , and comparing the q^0 order with the perturbative contribution $(f - Nf_{\text{U}(1)})_{\text{pert}}$ at m^6 order. One obtains

$$c_3(v) = \frac{1}{2880} \sum_{\alpha \in \Delta} \text{Li}_{-3}(e^{\alpha(v)}) - \frac{1}{288} \sum_{\alpha \in \Delta} \text{Li}_{-2}(e^{\alpha(v)})^2 + \frac{1}{576} \sum_{\alpha \in \Delta} \sum_{\beta \in \Psi(\alpha)} \text{Li}_0(e^{\alpha(v)}) \text{Li}_{-2}(e^{\beta(v)}). \quad (2.85)$$

Inserting this $c_3(v)$ in (2.84), one can further study the higher order coefficients of f_3 in q expansion, against the microscopic result f . We find that

$$(f - Nf_{\text{U}(1)}) - f_{\text{S-dual}} \Big|_{m^6} = 0, \quad (2.86)$$

which we checked till q^2 order for general N , and till q^3 order for $N = 2, 3$.

Integrating (2.72) to get higher f_n 's, the integration constants take the following form,

$$f_n(\tau, v) \leftarrow \sum_{4a+6b=2n-2, a \geq 0, b \geq 0} c_{a,b}(v) E_4(\tau)^a E_6(\tau)^b. \quad (2.87)$$

More concretely, one would get

$$f_4 \leftarrow c_{0,1} E_6, \quad f_5 \leftarrow c_{2,0} E_4^2, \quad f_6 \leftarrow c_{1,1} E_4 E_6, \quad f_7 \leftarrow c_{3,0} E_4^3 + c_{0,2} E_6^2, \quad f_8 \leftarrow c_{2,1} E_4^2 E_6, \dots \quad (2.88)$$

and so on. To fix the coefficients $c_{p,q}(v)$, one should use some low order data of $f - Nf_{\text{U}(1)}$. If there are $k + 1$ independent $c_{p,q}$'s, one should use up to k instanton coefficients of

$f - Nf_{U(1)}$ to fix them. Then from $k + 1$ or higher instantons, one can test f_n by comparing with $f - Nf_{U(1)}$. For general N , we tested $f_{S\text{-dual}}$ against $f - Nf_{U(1)}$ till m^6 and q^2 orders, as already reported above. For $U(2)$, we tested it till m^{14} (i.e. f_7) and q^3 orders. For $U(3)$, we tested it till m^8 (i.e. f_4) and q^3 orders. The result is that there is no further difference between $f_{S\text{-dual}}$ and $f - Nf_{U(1)}$ from m^6 and higher orders. Namely, we find that

$$f(\tau, v, m) = f_{S\text{-dual}}(\tau, v, m) + Nf_{U(1)}(\tau, m) + \frac{N^3 - N}{288} m^4 E_2(\tau). \quad (2.89)$$

The S-duality transformation of $f_{U(1)}$ can be separately derived from (2.31), which is

$$\begin{aligned} \tau^2 f_{U(1)}(-1/\tau, m/\tau) - f_{U(1)}(\tau, m) &= \frac{m^2}{2} \log \frac{\phi(-1/\tau)^2}{\tau \phi(\tau)^2} + \frac{m^4}{288} (\tau^{-2} E_2(-1/\tau) - E_2(\tau)) \\ &= \frac{m^2}{2} \left[\frac{\pi i \tau}{6} + \frac{\pi i}{6\tau} + \log(-i) \right] + \frac{m^4}{48\pi i \tau}, \end{aligned} \quad (2.90)$$

where $\eta(-1/\tau) = \sqrt{-i\tau} \eta(\tau)$. Although we checked (2.89) in a series expansion in small m , we believe it is an exact property, valid for finite m . In section 2.2, we shall provide another test of (2.89) by keeping m finite, but expanding f as a series of e^{-v} at positive Coulomb VEV v . Also, in section 2.3, we extend (2.89) to all ADE theories.

2.2 Derivation from M-strings

In this subsection, we provide another derivation of the S-duality of prepotential. The analysis here will provide more evidence for our S-duality anomaly at finite m . It may also provide strong hints towards the S-duality of the partition function $Z(\tau, v, m, \epsilon_{1,2})$ at finite $\epsilon_{1,2}$, but we postpone more extensive discussions on Z to a separate project.

We use an alternative partition function $\check{Z}(\tau, v, m, \epsilon_{1,2}) = \check{Z}_{\text{pert}} Z_{\text{inst}}$. This differs from Z by a different perturbative partition function \check{Z}_{pert} . For the purpose of studying the prepotential in the limit $\epsilon_{1,2} \rightarrow 0$, we already saw that they differ only by a trivial term, as shown in (2.24) in a suitable regime on the complex v plane. The last term of (2.24) is separately invariant under the S-duality transformation $(\tau, m, \epsilon_{1,2}) \rightarrow (-\frac{1}{\tau}, \frac{m}{\tau}, \frac{\epsilon_{1,2}}{\tau})$, and is independent of v , so it is completely fine to drop this constant factor when discussing the S-duality of prepotential. \check{Z} can be expanded in $e^{-\alpha_i(v)}$ when $\alpha_i(v)$ are positive and sufficiently large, where α_i ($i = 1, \dots, N-1$) are the simple roots of $SU(N)$. The expansion takes the form of [8]

$$\begin{aligned} Z(\tau, v, m, \epsilon_{1,2}) &= e^{-\epsilon_0} Z_{U(1)}(\tau, m, \epsilon_{1,2})^N \sum_{n_1, \dots, n_{N-1}=0}^{\infty} e^{-\sum_{i=1}^{N-1} n_i \alpha_i(v)} Z_{(n_i)}(\tau, m, \epsilon_{1,2}) \\ &\equiv e^{-\epsilon_0} Z_{U(1)}^N \hat{Z}. \end{aligned} \quad (2.91)$$

$Z_{U(1)}^N$ is for the D0-branes unbound to the W-bosons, just bound to one of the N D4-branes, which is this independent of $e^{-\alpha_i(v)}$. ϵ_0 is defined at the beginning of section 2 by

$$\epsilon_0 = \frac{m^2 - \epsilon_+^2}{2\epsilon_1 \epsilon_2} \sum_{\alpha > 0} (\alpha(v) + \pi i). \quad (2.92)$$

This factor has to be multiplied to guarantee (2.24). This expansion is reliable in a particular Weyl chamber of the Coulomb branch, with sufficiently large $\alpha_i(v)$ for all simple roots. The coefficient $Z_{(n_i)}(\tau, m, \epsilon_{1,2})$ at given self-dual string numbers n_i is computed from ‘M-strings,’ which are M2-branes suspended between separated M5-branes. $Z_{(n_i)}$ is the elliptic genus of the 2d quiver gauge theory with $U(n_1) \times \cdots \times U(n_{N-1})$ gauge group. See [20] for the details. $Z_{(n_i)}(\tau, m, \epsilon_{1,2})$ is given by [8] (we follow the notations of [21])

$$Z_{(n_i)} = \sum_{Y_1, \dots, Y_{N-1}; |Y_i|=n_i} \prod_{i=1}^N \prod_{s \in Y_i} \frac{\theta_1(\tau | \frac{E_{i,i+1}(s)-m+\epsilon_-}{2\pi i}) \theta_1(\tau | \frac{E_{i,i-1}(s)+m+\epsilon_-}{2\pi i})}{\theta_1(\tau | \frac{E_{i,i}(s)+\epsilon_1}{2\pi i}) \theta_1(\tau | \frac{E_{i,i}(s)-\epsilon_2}{2\pi i})}, \quad (2.93)$$

where $s = (a, b)$ denotes the position of each box in the Young diagram, and

$$E_{i,j}(s = (a, b)) = (Y_{i,a} - b)\epsilon_1 - (Y_{j,b}^T - a)\epsilon_2. \quad (2.94)$$

$Y_{i,a}$ is the length of the a 'th row of Y_i , and $Y_{j,b}^T$ is the length of the b 'th column of Y_j . Y_0 and Y_N are empty by definition.

We study the S-duality of \check{Z} in this setting. By using

$$\frac{\theta_1(-\frac{1}{\tau}, \frac{z}{\tau})}{\eta(-\frac{1}{\tau})} = e^{\frac{\pi iz^2}{\tau}} \frac{\theta_1(\tau, z)}{\eta(\tau)}, \quad (2.95)$$

one can show that $Z_{(n_i)}$ transforms as

$$Z_{(n_i)}\left(-\frac{1}{\tau}, \frac{m}{\tau}, \frac{\epsilon_{1,2}}{\tau}\right) = \exp\left[\frac{1}{4\pi i\tau} \left(\epsilon_1\epsilon_2 \sum_{i,j=1}^{N-1} A_{ij}n_i n_j + 2(m^2 - \epsilon_+^2) \sum_{i=1}^{N-1} n_i\right)\right] Z_{(n_i)}(\tau, m, \epsilon_{1,2}). \quad (2.96)$$

A_{ij} is the Cartan matrix for $SU(N)$, given by $A_{ii} = 2$, $A_{i,i+1} = A_{i,i-1} = -1$ and $A_{ij} = 0$ otherwise. This transformation can be expressed as a modular anomaly equation, as follows. First note that Jacobi's theta functions can be expressed as

$$\theta_1(\tau|z) = 2\pi iz \eta(\tau)^3 \exp\left[\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)(2k)!} E_{2k}(\tau)(2\pi iz)^{2k}\right]. \quad (2.97)$$

τ dependence of $Z_{(n_i)}$ can be understood as its dependence through $E_{2n}(\tau)$, since $\eta(\tau)^3$ factors cancel out in (2.93). As we did in section 2.1 for the prepotential, the τ dependence of $Z_{(n_i)}$ can again be decomposed into its dependence through $E_2(\tau)$, and the rest. To emphasize this, we write $Z_{(n_i)}(\tau, m, \epsilon_{1,2}, E_2)$. In this setting, the modular anomaly $e^{\frac{\pi iz^2}{\tau}}$ of (2.95) appears due to the dependence of θ_1 on $E_2(\tau)$, which is not modular. Therefore, we can rephrase (2.96) as

$$\frac{\partial}{\partial E_2} Z_{(n_i)}(\tau, m, \epsilon_{1,2}; E_2) = \frac{1}{24} \left(\epsilon_1\epsilon_2 \sum_{i,j=1}^{N-1} A_{ij}n_i n_j + 2(m^2 - \epsilon_+^2) \sum_{i=1}^{N-1} n_i\right) Z_{(n_i)}. \quad (2.98)$$

Let us consider $\hat{Z}(\tau, \lambda, m, \epsilon_{1,2}; E_2) = \sum_{n_i=0}^{\infty} e^{-n_i\lambda_i} Z_{(n_i)}(\tau, m, \epsilon_{1,2}; E_2)$, where we defined $\lambda_i \equiv v_i - v_{i+1} > 0$ with the choice of simple roots $\alpha_i = e_i - e_{i+1}$. Here we view \hat{Z} as a

function of λ 's, since it depends only on the differences of v 's. Now with the replacement $n_i \rightarrow -\frac{\partial}{\partial \lambda_i}$ in the Laplace transformation, one obtains

$$\frac{\partial}{\partial E_2} \hat{Z}(\tau, \lambda, m, \epsilon_{1,2}; E_2) = \frac{1}{24} \left(\epsilon_1 \epsilon_2 \sum_{i,j=1}^{N-1} A_{ij} \frac{\partial^2}{\partial \lambda_i \partial \lambda_j} - 2(m^2 - \epsilon_+^2) \sum_{i=1}^{N-1} \frac{\partial}{\partial \lambda_i} \right) \hat{Z}. \quad (2.99)$$

This is essentially the ‘holomorphic anomaly equation’, eq. (3.54) of [8]. In our context, we continue to study it as the modular anomaly equation.

We shall get better intuitions on the modular anomaly equation. Consider $\tilde{Z} \equiv e^{-\varepsilon_0} \hat{Z}$ with $\varepsilon_0 = \frac{m^2 - \epsilon_+^2}{2\epsilon_1 \epsilon_2} \sum_{\alpha > 0} (\alpha(v) + \pi i)$, still without $(Z_{U(1)})^N$ included. Since the v independent part of ε_0 is independent of E_2, λ , it does not affect (2.99). The v dependent part can be written as

$$\sum_{\alpha > 0} \alpha(v) = \sum_{i < j} (v_i - v_j) = \sum_{i=1}^N [(N-i) - (i-1)] v_i = \sum_{i=1}^N (N+1-2i) v_i. \quad (2.100)$$

Since $\frac{\partial \lambda_i(v)}{\partial v_j} = \delta_{i,j} - \delta_{i+1,j}$, $\frac{\partial}{\partial v_j}$ acting on a function of λ_i 's can be written as $\frac{\partial}{\partial v_j} = \frac{\partial \lambda_i(v)}{\partial v_j} \frac{\partial}{\partial \lambda_i}$,

$$\frac{\partial}{\partial v_1} = \frac{\partial}{\partial \lambda_1}, \quad \frac{\partial}{\partial v_j} = \frac{\partial}{\partial \lambda_j} - \frac{\partial}{\partial \lambda_{j-1}} \quad (j = 2, \dots, N-1), \quad \frac{\partial}{\partial v_N} = -\frac{\partial}{\partial \lambda_{N-1}}. \quad (2.101)$$

The Laplacian of v acting on a function of λ is given by

$$\nabla^2 \equiv \sum_{i=1}^N \frac{\partial^2}{\partial v_i^2} = A_{ij} \frac{\partial^2}{\partial \lambda_i \partial \lambda_j}. \quad (2.102)$$

One also finds that

$$\sum_{i=1}^{N-1} \frac{\partial}{\partial \lambda_i} = \sum_{i=1}^N (a-i) \partial_{v_i} \quad (2.103)$$

for any number a . Acting ∇^2 on $\tilde{Z} = e^{-\varepsilon_0} \hat{Z}$, one obtains

$$\begin{aligned} \nabla^2 \tilde{Z} &= e^{-\varepsilon_0} \nabla^2 \hat{Z} - 2 \frac{m^2 - \epsilon_+^2}{2\epsilon_1 \epsilon_2} e^{-\varepsilon_0} \sum_{i=1}^N (N+1-2i) \partial_{v_i} \hat{Z} + \left(\frac{m^2 - \epsilon_+^2}{2\epsilon_1 \epsilon_2} \right)^2 e^{-\varepsilon_0} \sum_{i=1}^N (2i-N-1)^2 \hat{Z} \\ &= e^{-\varepsilon_0} \left[\nabla^2 \hat{Z} - 2 \frac{m^2 - \epsilon_+^2}{\epsilon_1 \epsilon_2} \sum_{i=1}^{N-1} \frac{\partial}{\partial \lambda_i} \hat{Z} + \frac{N^3 - N}{3} \left(\frac{m^2 - \epsilon_+^2}{2\epsilon_1 \epsilon_2} \right)^2 \hat{Z} \right] \end{aligned} \quad (2.104)$$

where we used $\frac{\partial \varepsilon_0}{\partial v_i} = \frac{m^2 - \epsilon_+^2}{2\epsilon_1 \epsilon_2} (N+1-2i)$. Using (2.99), this can be rewritten as

$$\nabla^2 \tilde{Z} = e^{-\varepsilon_0} \left[\frac{24}{\epsilon_1 \epsilon_2} \frac{\partial}{\partial E_2} \hat{Z} + \frac{N^3 - N}{3} \left(\frac{m^2 - \epsilon_+^2}{2\epsilon_1 \epsilon_2} \right)^2 \hat{Z} \right]. \quad (2.105)$$

Thus, one finds that the partition function

$$Z_{\text{S-dual}} \equiv \exp \left[\frac{\epsilon_1 \epsilon_2}{24} \frac{N^3 - N}{12} \left(\frac{m^2 - \epsilon_+^2}{\epsilon_1 \epsilon_2} \right)^2 E_2 \right] \tilde{Z} = \exp \left[\frac{N^3 - N}{288} \frac{(m^2 - \epsilon_+^2)^2}{\epsilon_1 \epsilon_2} E_2 - \varepsilon_0 \right] \hat{Z} \quad (2.106)$$

satisfies the modular anomaly equation

$$\frac{\partial}{\partial E_2} Z_{\text{S-dual}}(\tau, v, m, \epsilon_{1,2}; E_2) = \frac{\epsilon_1 \epsilon_2}{24} \nabla^2 Z_{\text{S-dual}}. \quad (2.107)$$

Also, from the M-string expansion form of \hat{Z} , and the form of the prefactors we multiplied to define $Z_{\text{S-dual}}$, $Z_{\text{S-dual}}$ satisfies the following quasi-modularity condition,

$$Z_{\text{S-dual}}\left(-\frac{1}{\tau}, v, \frac{m}{\tau}, \frac{\epsilon_{1,2}}{\tau}; E_2\left(-\frac{1}{\tau}\right)\right) = Z_{\text{S-dual}}(\tau, v, m, \epsilon_{1,2}, E_2(\tau) + \delta) \quad (2.108)$$

where $\delta = \frac{6}{\pi i \tau}$. (2.107) and (2.108) are the two main properties of $Z_{\text{S-dual}}$.

Using (2.107) and (2.108), we would like to study the relation between $Z_{\text{S-dual}}(\tau, v, m, \epsilon_{1,2}, E_2)$ and $Z_{\text{S-dual}}(-\frac{1}{\tau}, v, \frac{m}{\tau}, \frac{\epsilon_{1,2}}{\tau}, E_2(-\frac{1}{\tau}))$. Since \tilde{Z} is related to $Z_{\text{S-dual}}$ in a simple manner, answering this question will tell us how \tilde{Z} transforms under S-duality. Then, since we already understand how the $U(1)^N$ part $Z_{U(1)}^N$ transforms under S-duality [38], we shall in turn know the S-duality transformation of \tilde{Z} . Using (2.108), we should understand how $Z_{\text{S-dual}}(\tau, v, m, \epsilon_{1,2}; E_2(\tau) + \delta)$ and $Z_{\text{S-dual}}(\tau, v, m, \epsilon_{1,2}; E_2)$ are related to each other, at same values of $\tau, v, m, \epsilon_{1,2}$ but with a shift of E_2 by $\delta = \frac{6}{\pi i \tau}$. Let us formally regard E_2 as time variable, and $Z_{\text{S-dual}}$ as a wavefunction on the space formed by v_i . Then (2.107) takes the form of heat equation, or Euclidean Schrödinger equation for a free particle. More precisely, taking τ to be purely imaginary for simplicity, δ is real and negative. So $-E_2$ plays the role of time, and we would like to understand the time evolution $Z_{\text{S-dual}}(E_2 + \delta)$ by $-\delta$ following the heat equation. The heat equation comes with proper sign when $\epsilon_1 \epsilon_2 < 0$. In this case, the time evolution is described by evolving $Z_{\text{S-dual}}(E_2)$ by the Gaussian heat kernel as

$$Z_{\text{S-dual}}(\tau, v, m, \epsilon_{1,2}; E_2(\tau) + \delta) = \int_{-\infty}^{\infty} \prod_{i=1}^N dv'_i K(v, v') Z_{\text{S-dual}}(\tau, v', m, \epsilon_{1,2}; E_2(\tau)), \quad (2.109)$$

where

$$K(v, v') = \left(\frac{i\tau}{\epsilon_1 \epsilon_2}\right)^{\frac{N}{2}} \exp\left[-\frac{\pi i \tau}{\epsilon_1 \epsilon_2} (v - v')^2\right] \quad (2.110)$$

is the heat kernel which approaches $K(v, v') \rightarrow \delta^{(N)}(v - v')$ when $\frac{i\tau}{\epsilon_1 \epsilon_2} \rightarrow 0$. When $\epsilon_1 \epsilon_2 > 0$, the ‘time evolution’ from $Z_{\text{S-dual}}(E_2)$ to $Z_{\text{S-dual}}(E_2 + \delta)$ is described by the time reversal of the heat equation. Therefore, one finds

$$Z_{\text{S-dual}}(\tau, v, m, \epsilon_{1,2}; E_2(\tau)) = \int_{-\infty}^{\infty} \prod_{i=1}^N dv'_i K_-(v, v') Z_{\text{S-dual}}(\tau, v', m, \epsilon_{1,2}; E_2(\tau) + \delta) \quad (2.111)$$

with $K_-(v, v') = \left(-\frac{i\tau}{\epsilon_1 \epsilon_2}\right)^{\frac{N}{2}} \exp\left[\frac{\pi i \tau}{\epsilon_1 \epsilon_2} (v - v')^2\right]$ for $\epsilon_1 \epsilon_2 > 0$.

The S-duality of $f_{\text{S-dual}}$ can be studied from (2.109) or (2.111) by a saddle point approximation of the v' integral at $\epsilon_{1,2} \rightarrow 0$. Using both equations yield identical results. One finds

$$Z_{\text{S-dual}} = e^{-\frac{f_{\text{S-dual}}}{\epsilon_1 \epsilon_2}} \sim \exp\left[-\frac{\hat{f}}{\epsilon_1 \epsilon_2} + \frac{N^3 - N}{288} \frac{m^4}{\epsilon_1 \epsilon_2} E_2(\tau) - \frac{m^2}{2\epsilon_1 \epsilon_2} \sum_{\alpha > 0} (\alpha(v) + \pi i)\right] \quad (2.112)$$

in the limit $\epsilon_{1,2} \rightarrow 0$, so that

$$f_{\text{S-dual}} = \hat{f} - \frac{N^3 - N}{288} m^4 E_2(\tau) + \frac{m^2}{2} \sum_{\alpha > 0} (\alpha(v) + \pi i). \quad (2.113)$$

We shall show that $f_{\text{S-dual}}$ defined this way is the same $f_{\text{S-dual}}$ defined and computed in section 2.1. Firstly, note that

$$\hat{f} = \check{f} - N f_{\text{U}(1)} - \frac{m^2}{2} \sum_{\alpha < 0} (\alpha(v) + \pi i) = f - N f_{\text{U}(1)} - \frac{m^2}{2} \sum_{\alpha < 0} (\alpha(v) + \pi i) - \frac{\pi i m^2 |\Delta_+|}{2} \quad (2.114)$$

from the relations $\check{Z} = e^{-\epsilon_0} [Z_{\text{U}(1)}]^N \hat{Z}$ and $\check{Z} \sim Z e^{\frac{\pi i m^2 |\Delta_+|}{2 \epsilon_1 \epsilon_2}}$. Inserting this in (2.113), one obtains

$$f_{\text{S-dual}} = f - N f_{\text{U}(1)} - \frac{N^3 - N}{288} m^4 E_2(\tau) - \frac{\pi i m^2 |\Delta_+|}{2}. \quad (2.115)$$

This is completely the same as the relation between f and $f_{\text{S-dual}}$ that we found in section 2.1, except the last term on the right hand side. However, we know that the last term comes from using slightly different perturbative partition function in \check{Z} , (2.24). Also this term can be completely ignored for studying S-duality since it is separately invariant under S-duality. Therefore, after discarding this last term, we find that $f_{\text{S-dual}}$ is the same as $f_{\text{S-dual}}$ defined in section 2.1. Now one can independently check that $f_{\text{S-dual}}$ is S-dual. Firstly, Since $Z_{\text{S-dual}}$ is quasi-modular, so is $f_{\text{S-dual}}$, i.e.

$$\tau^2 f_{\text{S-dual}} \left(-\frac{1}{\tau}, v, \frac{m}{\tau}, E_2(-1/\tau) \right) = f_{\text{S-dual}}(\tau, v, m, E_2(\tau) + \delta) \quad (2.116)$$

Secondly, inserting $Z_{\text{S-dual}} \sim \exp \left[-\frac{f_{\text{S-dual}}}{\epsilon_1 \epsilon_2} \right]$ into (2.107) and keeping the leading terms in the limit $\epsilon_{1,2} \rightarrow 0$, one finds $\frac{\partial f_{\text{S-dual}}}{\partial E_2} = -\frac{1}{24} \left(\frac{\partial f_{\text{S-dual}}}{\partial v} \right)^2$, the same modular anomaly equation that we studied in section 2.1 [22]. The last two equations guarantee the S-duality of $f_{\text{S-dual}}$, completing an alternative proof of S-duality based on M-strings.

While making an alternative derivation of the S-duality and its anomaly, we did not assume the smallness of m . On the other hand, around (2.24), we required $\text{Re}(\alpha(v)) > \pm \text{Re}(m)$ for the positive roots α , and that $\text{Im}(\alpha(v) \pm m)$ for positive roots be in the range $(0, 2\pi]$, to justify the uses of \check{F}_{pert} and \check{Z}_{pert} . So at least in this range, $F_{\text{anom}} = N f_{\text{U}(1)} + \frac{N^3 - N}{288} m^4 E_2(\tau)$ is exact at finite m . One can scan the whole complex planes of v and m , considering the multiple values of Li_3 function, to find the most general form of the S-duality anomaly when (2.24) is violated. We shall not do this exercise here.

Before closing this subsection, we comment on the nature of the S-duality transformation of $Z_{\text{S-dual}}$ or \check{Z} , and what it implies to the S-duality of Z . This issue is also related to the S-duality of the 4d limit of Z , which was studied in the context of AGT correspondence [23]. Taking the 4d limit $R \rightarrow 0$ with fixed $a, M, \epsilon_{1,2}, m, \tau$, (2.109) asserts that $Z^{4\text{d}}(\tau, a, m, \epsilon_{1,2})$ is related to its S-dual by a Gaussian S-duality kernel. This is because the R dependence K is given by

$$K(a, a') = R^{-N} \left(\frac{i\tau}{\epsilon_1 \epsilon_2} \right)^{\frac{N}{2}} \exp \left[-\frac{\pi i \tau}{\epsilon_1 \epsilon_2} (a - a')^2 \right], \quad (2.117)$$

and the overall R dependence is absorbed into the v' integration to be $\prod_i da'_i$. Expanding the exponent of the kernel, and inserting $a = \frac{a_D}{\tau}$, one obtains

$$\exp \left[-\frac{F_{\text{cl}}(a', \tau) - F_{\text{cl}}(a_D, -\frac{1}{\tau})}{\varepsilon_1 \varepsilon_2} \right] \exp \left[\frac{\pi i a_D a'}{\varepsilon_1 \varepsilon_2} \right], \tag{2.118}$$

where $F_{\text{cl}}(a, \tau) = \pi i \tau a^2$. The two F_{cl} 's can be absorbed into two Z^{4d} 's on the left and right hand sides of (2.109). Then, the 4d limit of (2.109) states that S-dualization is Fourier transformation. Our studies imply that the same result holds for $Z_{\text{S-dual}}$ in 6d.

In fact, the proper S-duality transformation of the 4d partition function is known *not* to be the Fourier transformation. Instead, based on the AGT correspondence, the S-duality kernel is asserted to be a nontrivial function given by the partition function of the 3d $T[\text{SU}(N)]$ theory on S^3 [24, 25]. On the other hand, it has been found that the S-duality of the Omega deformed partition function is the Fourier transformation at all perturbative orders in $\varepsilon_1, \varepsilon_2$ [15, 22, 26, 27]. An explanation of this was given in [26], which finds that the choice of proper ‘normalization factor’ independent of τ dressing the instanton partition function yields such a nonperturbative correction in $\varepsilon_{1,2}$. This is the ‘choice’ of $Z_{\text{pert}}^{\text{4d}}$, which was called $N(a)$ and $N_s(a)$ in [26]. Incidentally, what we find in 6d is analogous to the findings of [26]. Namely, Z was defined in section 2 with Z_{pert} which is manifestly invariant under Weyl symmetry. However, \check{Z} was defined with \check{Z}_{pert} which is not invariant under Weyl symmetry. In fact, we checked that the ratio $\frac{\check{Z}_{\text{pert}}}{Z_{\text{pert}}}$ at small $\varepsilon_{1,2}$ is nonperturbative in $\varepsilon_{1,2}$, which is qualitatively consistent with [26]. So along this line, it will be interesting to pursue the 6d extensions of [26]. We stress again that, all our findings in this subsection concerns the prepotential in the $\varepsilon_{1,2} \rightarrow 0$ limit, for which the distinction of Z_{pert} or \check{Z}_{pert} is irrelevant.

2.3 6d (2, 0) theories of D_N and E_N types

We generalize some studies we made for A_{N-1} type (2, 0) theories to D_N and E_N type theories. For D_N type theories, $f_{\text{S-dual}}$ can be compared with microscopic instanton calculus for the 5d $\text{SO}(2N)$ $\mathcal{N} = 1^*$ theory, or the D0-D4-O4 matrix quantum mechanics [9]. For E_N types, we make a prediction of the S-duality and our knowledge of 5d perturbative prepotential. All ADE results will be partly tested in section 3.2 from 6d chiral anomalies.

In the setting of section 2.1, the leading coefficient f_1 of $f_{\text{S-dual}} = \sum_{n=1}^{\infty} m^{2n} f_n$ is obtained from $(f - r f_{\text{U}(1)})_{\text{pert}}$, where r is the rank of the gauge group. The result is

$$f_1 = \frac{1}{2} \sum_{\alpha \in \Delta} \log(1 - e^{\alpha(v)}). \tag{2.119}$$

Then using (2.72) at $n = 2$, one finds

$$f_2 = -\frac{E_2(\tau)}{96} \sum_{\alpha, \beta \in \Delta} \frac{\alpha \cdot \beta}{(1 - e^{\alpha(v)})(1 - e^{\beta(v)})}. \tag{2.120}$$

To proceed, we classify the roots β depending on their norm with α . The possibilities are

$$\begin{aligned}
 (1) : \alpha \cdot \beta &= \pm 2 \text{ if } \pm \beta = \alpha \\
 (2) : \alpha \cdot \beta &= \pm 1 \text{ if } \pm \beta \in \Psi(\alpha) \\
 (3) : \alpha \cdot \beta &= 0 \text{ otherwise .}
 \end{aligned} \tag{2.121}$$

It is again important to understand the set $\Psi(\alpha)$ for ADE, which we explain now.

For $D_N = \text{SO}(2N)$, the $2N^2 - 2N$ roots in Δ are given by $\pm e_i \pm e_j$, where $i, j = 1, \dots, N$ and $i < j$. Elements of $\Psi(\alpha)$ are given for various α by

$$\begin{aligned}
 \alpha = e_i - e_j : \Psi(\alpha) &= \{k \neq i, j : e_i \pm e_k, \pm e_k - e_j\}, \quad 4(N-2) \text{ elements} \\
 \alpha = e_i + e_j : \Psi(\alpha) &= \{k \neq i, j : e_i \pm e_k, e_j \pm e_k\}, \quad 4(N-2) \text{ elements.}
 \end{aligned} \tag{2.122}$$

For E_6 , the number of roots is $|\Delta| = 72$. 40 roots take the form of $\pm e_i \pm e_j$ where $i \neq j$ and $i, j = 1, \dots, 5$, from the $\text{SO}(10)$ subalgebra. Additional 32 roots take the form of $\pm \frac{1}{2}(\pm e_1 \pm \dots \pm e_5 - e_6 - e_7 + e_8)$, where the total number of $-$ signs is even. The structure of $\Psi(\alpha)$ is given for various α as follows. Firstly, when $\alpha = e_i - e_j$, then

$$\Psi(\alpha) = \{k \neq i, j : e_i \pm e_k, \pm e_k - e_j\} \cup \left\{ \frac{1}{2}(e_i - e_j + \dots) \right\} \tag{2.123}$$

where \dots means that all possible signs are allowed in the 32 spinorial elements. Thus, one finds $12 + 8 = 20$ elements of $\Psi(\alpha)$ in this case. Similarly, for $\alpha = e_i + e_j$, one finds

$$\Psi(\alpha) = \{k \neq i, j : e_i \pm e_k, e_j \pm e_k\} \cup \left\{ \frac{1}{2}(e_i + e_j + \dots) \right\} \tag{2.124}$$

where \dots means the same. So again, one finds $|\Psi(\alpha)| = 12 + 8 = 20$. For $\alpha = -e_i - e_j$, one can do a similar analysis. Finally, α can be one of the 32 spinorial elements, $\alpha = \frac{s_0}{2}(s_1 e_1 + \dots + s_5 e_5 - e_6 - e_7 + e_8)$ with $s_0, \dots, s_5 = \pm 1$ and $s_1 \dots s_5 = 1$. Then,

$$\Psi(\alpha) = \{s_0(s_i e_i + s_j e_j)\} \cup \{\alpha - s_0(s_i e_i + s_j e_j)\}, \tag{2.125}$$

so $|\Psi(\alpha)| = {}_5C_2 + {}_5C_2 = 20$. For E_7 , $|\Delta| = 126$. 60 roots take the form of $\pm e_i \pm e_j$, $i, j = 1, \dots, 6$, from $\text{SO}(12)$ subalgebra. Additional 64 roots take the form of $\pm \frac{1}{2}(\pm e_1 \pm \dots \pm e_6 - e_7 + e_8)$, with total number of $-$ signs being even. Finally, 2 more roots are given by $\pm(e_7 - e_8)$. When $\alpha = \pm e_i \pm e_j$, $\Psi(\alpha)$ takes the same structure as that shown for E_6 . For instance, for $\alpha = e_i + e_j$, one finds $\Psi(\alpha) = \{k \neq i, j | e_i \pm e_k, e_j \pm e_k\} \cup \{\frac{1}{2}(e_i + e_j \dots)\}$ with $|\Psi(\alpha)| = 16 + 16 = 32$. When $\alpha = \frac{s_0}{2}(s_1 e_1 + \dots + s_6 e_6 - e_7 + e_8)$, with $s_1 \dots s_6 = 1$, one finds

$$\Psi(\alpha) = \{s_0(s_i e_i + s_j e_j)\} \cup \{\alpha - s_0(s_i e_i + s_j e_j)\} \cup \{s_0(e_8 - e_7), \alpha + s_0(e_7 - e_8)\} \tag{2.126}$$

with $|\Psi(\alpha)| = {}_6C_2 + {}_6C_2 + 2 = 32$. Finally, when $\alpha = e_7 - e_8$, one finds

$$\Psi(\alpha) = \left\{ s_1 \dots s_6 = -1 \mid \frac{1}{2}(s_1 e_1 + \dots + s_6 e_6 + e_7 - e_8) \right\}, \tag{2.127}$$

G	A_{N-1}	D_N	E_6	E_7	E_8
r	$N-1$	N	6	7	8
$ G $	N^2-1	$2N^2-N$	78	133	248
c_2	N	$2N-2$	12	18	30

Table 1. Data on simply laced Lie algebras.

with $|\Psi(\alpha)| = 32$. The case with $\alpha = e_8 - e_7$ is similar. For E_8 , $|\Delta| = 240$. 112 roots take the form of $\pm e_i \pm e_j$, $i, j = 1, \dots, 8$, from $\text{SO}(16)$ subalgebra. Additional 128 roots take the form of $\frac{1}{2}(\pm e_1 \pm \dots \pm e_8)$ with number of $-$ signs being even, forming the $\text{SO}(16)$ spinor representation. For $\alpha = e_i + e_j$, one finds $\Psi(\alpha) = \{k \neq i, j | e_i \pm e_k, e_j \pm e_k\} \cup \{\frac{1}{2}(e_i + e_j \dots)\}$, with $|\Psi(\alpha)| = 24 + 32 = 56$. Other cases with roots of the form $\alpha = \pm e_i \pm e_j$ can be studied similarly. For $\alpha = \frac{1}{2}(s_1 e_1 + \dots + s_8 e_8)$ with $s_1 \dots s_8 = 1$, one finds

$$\Psi(\alpha) = \{s_i e_i + s_j e_j\} \cup \{\alpha - (s_i e_i + s_j e_j)\} \quad (2.128)$$

with $|\Psi(\alpha)| = {}_8C_2 + {}_8C_2 = 56$. Including the $\text{SU}(N)$ case studied in section 2.1, one finds $|\Psi(\alpha)| = 2c_2 - 4$, where c_2 is the dual Coxeter number. See table 1. Another useful fact that can be checked with all $\Psi(\alpha)$ we listed above is that, if $\beta \in \Psi(\alpha)$, then $\alpha - \beta$ is also a root. One also finds $\alpha - \beta \in \Psi(\alpha)$, since $\alpha \cdot (\alpha - \beta) = 1$. So at given α , one finds

$$\sum_{\beta \in \Psi(\alpha)} f_{\alpha, \beta} = \sum_{\beta \in \Psi(\alpha)} f_{\alpha, \alpha - \beta} \quad (2.129)$$

for any expression $f_{\alpha, \beta}$.

By following the analysis for the $\text{U}(N)$ case, till (2.1), one finds

$$f_2 = -\frac{E_2}{96} \left[4 \sum_{\alpha \in \Delta} \text{Li}_{-1}(e^{\alpha(v)}) - (c_2 - 3)(|G| - r) + \sum_{\alpha \in \Delta} \sum_{\beta \in \Psi(\alpha)} \frac{2}{(1 - e^{\alpha(v)})(1 - e^{\beta(v)})} \right]. \quad (2.130)$$

Now we use the identity (2.129) to rewrite the last term in the parenthesis as

$$\frac{2}{3} \sum_{\alpha \in \Delta} \sum_{\beta \in \Psi(\alpha)} \left[\frac{1}{(1 - e^{\alpha(v)})(1 - e^{\beta(v)})} + \frac{1}{(1 - e^{-\alpha(v)})(1 - e^{\beta(v) - \alpha(v)})} + \frac{1}{(1 - e^{\alpha(v) - \beta(v)})(1 - e^{-\beta(v)})} \right]. \quad (2.131)$$

On the second term, we relabeled α into $-\alpha$ in the first sum, and then took $\beta - \alpha$ with $\beta \in \Psi(\alpha)$ as labeling the elements of $\Psi(-\alpha)$. The third term is simply the second term with renaming $\alpha \leftrightarrow \beta$. Using the identity $\frac{1}{(1 - e^x)(1 - e^y)} + \frac{1}{(1 - e^{-x})(1 - e^{y-x})} + \frac{1}{(1 - e^{x-y})(1 - e^{-y})} = 1$, (2.131) becomes $\frac{2}{3} \sum_{\alpha \in \Delta} \sum_{\beta \in \Psi(\alpha)} 1 = \frac{4}{3}(|G| - r)(c_2 - 2)$. Thus, one obtains

$$\begin{aligned} f_2 &= -\frac{E_2(\tau)}{24} \left[\sum_{\alpha \in \Delta} \text{Li}_{-1}(e^{\alpha(v)}) + \frac{1}{12}(c_2 + 1)(|G| - r) \right] \\ &= -\frac{E_2(\tau)}{24} \left[\sum_{\alpha \in \Delta} \text{Li}_{-1}(e^{\alpha(v)}) + \frac{c_2 |G|}{12} \right], \end{aligned} \quad (2.132)$$

where at the last step we used the identity $|G| = r(c_2 + 1)$ for simply-laced Lie algebra.

f_2 contains $E_2(\tau) = 1 - 24q + \dots$, so makes a prediction on the instanton corrections. For $G = \text{SO}(2N)$, one can compare this against microscopic instanton calculus for the 5d $\mathcal{N} = 1^*$ theory [9]. We compared the two results at 1 instanton level for $\text{SO}(8)$. Namely, (2.132) implies

$$f_2|_{q^1} = f_{\text{S-dual}}|_{m^4q^1} = \sum_{\alpha \in \Delta} \text{Li}_{-1}(e^{\alpha(v)}) + \frac{c_2|G|}{12}, \quad (2.133)$$

where $\text{Li}_{-1}(x) = \frac{x}{(1-x)^2}$. On the other hand, the single instanton partition function Z_1 for the $\text{SO}(2N)$ theory can be obtained by starting from the Witten index for the quantum mechanics describing an $\text{O}4^-$ plane, 2 D0-branes and $2N$ D4-branes (in the covering space). The index is a complicated residue sum. One should further subtract the contributions from D0-branes unbound to D4-O4, which was explained in [9]. Following this procedure, we checked that

$$f_{\text{S-dual}} - (f_{\text{SO}(8)} - 4f_{\text{U}(1)})|_{m^4q^1} = \frac{c_2|G|}{12}. \quad (2.134)$$

One can continue to generate higher order f_n 's, and also the microscopic instanton calculus for general D_N at higher order in q , and compare them. Here we simply conjecture

$$f(\tau, v, m) = f_{\text{S-dual}}(\tau, v, m) + r f_{\text{U}(1)}(\tau, m) + \frac{c_2|G|}{288} m^4 E_2(\tau) \quad (2.135)$$

for all $G = \text{SU}(N), \text{SO}(2N), E_N$, where r is the rank of G . For $G = \text{SU}(N)$, we have tested it extensively in section 2.1, after adding one free tensor multiplet to make it $\text{U}(N)$. For $G = \text{SO}(2N)$, we tested it till m^4, q^1 order only at $N = 4$, but in principle one can do all the calculus of section 2.1, following the methods of [9]. For E_N , this is just a prediction by assuming S-duality and 5d perturbative results. The last term proportional to $c_2|G|$ will be further tested in section 3, from the 6d chiral anomaly of $\text{SO}(5)$ R-symmetry.

3 High temperature limit of the index

In this section, we compute the asymptotic form of the prepotential at strong coupling, or high ‘temperature’ $\tau \rightarrow i0$. This is the limit in which the compactification radius R' of the sixth circle becomes large, or equivalently in which D0-branes become light. The key technique of computation will be the anomalous S-duality that we developed in section 2.

Our convention is that the strong coupling theory of our interest is the ‘S-dualized’ theory. So we take $\tau_D \rightarrow i0^+$, and $\tau = -\frac{1}{\tau_D} \rightarrow i\infty$. Recall $f_{\text{S-dual}}$ satisfies

$$\tau^2 f_{\text{S-dual}}(\tau_D, v_D, m_D) = f_{\text{S-dual}}(\tau, v, m) + \frac{1}{4\pi i \tau} \left(\frac{\partial f}{\partial v} \right)^2 \quad (3.1)$$

where $\tau_D = -\frac{1}{\tau}$, $v_D = v + \frac{1}{2\pi i \tau} \frac{\partial f}{\partial v}$, $m_D = \frac{m}{\tau}$. We replaced $f_{\text{S-dual}}$ by f when it appears with v derivatives, since f_{anom} is independent of v . Inserting $f_{\text{S-dual}} = f - f_{\text{anom}}$, one finds that

$$\tau^2 f(\tau_D, v_D, m_D) = f(\tau, v, m) + \frac{1}{4\pi i \tau} \left(\frac{\partial f}{\partial v} \right)^2 + \tau^2 f_{\text{anom}}(\tau_D, m_D) - f_{\text{anom}}(\tau, m). \quad (3.2)$$

Using (2.90) and $E_2(-1/\tau) = \tau^2 (E_2(\tau) + \frac{6}{\pi i \tau})$, one obtains

$$\tau^2 f_{\text{anom}}(\tau_D, m_D) - f_{\text{anom}}(\tau, m) = \frac{Nm^2}{2} \left(\log(-i) - \frac{\pi i \tau_D}{6} + \frac{\pi i \tau}{6} \right) + \frac{N^3 m^4}{48 \pi i \tau}. \quad (3.3)$$

Inserting this in (3.2), one obtains

$$f(\tau_D, v_D, m_D) = \tau^{-2} f(\tau, v, \tau m_D) + \frac{1}{4 \pi i \tau^3} \left(\frac{\partial f}{\partial v} \right)^2 + \frac{Nm_D^2}{2} \left(\log(-i) + \frac{\pi i}{6\tau} + \frac{\pi i \tau}{6} \right) + \frac{N^3 m_D^4}{48 \pi i} \tau. \quad (3.4)$$

We shall study it in the limit $\tau \rightarrow i\infty$ with m_D and v_D fixed.

The limit $\tau \rightarrow i\infty$ on the right hand side has to be understood with care, since $m = \tau m_D$ scales with τ . Also, we should study how v scales with $\tau \rightarrow i\infty$, at fixed v_D . Had v, m not scaled with τ , one would have naively expected that the instanton corrections in f would have been suppressed at $q \ll 1$, so that we could replace f on the right hand side by f_{pert} . Let us check when this is correct. This expectation is correct if $F_k(v, m)$ does not scale to be larger than q^k . From (2.8), F_k scales like $F_k \sim e^{\pm kNm}$ at $\pm \text{Re}(m) \gg 1$. For this factor to be smaller than q^k , one should require $|\text{Re}(\tau m_D)| < -\frac{2\pi i \tau}{N}$. Let us take τ to be purely imaginary for convenience (although most of our final results are valid for complex τ). Then, F_k can be ignored if

$$|\text{Im}(m_D)| < \frac{2\pi}{N}. \quad (3.5)$$

When $\text{Im}(m_D)$ reaches $\pm \frac{2\pi}{N}$, we encounter a phase transition, beyond which one should make a new q expansion on the right hand side. The correct nature of this phase transition will be commented on later. To make the simplest calculus at $\tau_D \rightarrow i0^+$, we take m_D to satisfy (3.5).

Let us also discuss how v should scale at fixed v_D . We shall first assume that v is finite at finite v_D , and then show that it is consistent with ignoring f_{inst} . If f_{inst} can be ignored, then the relation between v and v_D can be simplified as

$$v = v_D - \frac{1}{2\pi i \tau} \frac{\partial f_{\text{pert}}}{\partial v}(v, \tau m_D). \quad (3.6)$$

$\frac{\partial f_{\text{pert}}}{\partial v}$ is given by

$$\frac{\partial f_{\text{pert}}}{\partial v} = \sum_{\alpha \in \Delta} \alpha \left[\text{Li}_2(e^{\alpha v}) - \frac{1}{2} \text{Li}_2(e^{\alpha v \pm m}) \right]. \quad (3.7)$$

Since we assume that v is finite, the first term not containing m yields a subleading contribution, from the $\frac{1}{\tau} \rightarrow 0$ factor in (3.6). To be definite, we take $\text{Im}(m_D) < 0$ so that $\text{Re}(m) \gg 1$. Then,

$$\text{Li}_2(e^{\alpha(v)+m}) = -\text{Li}_2(e^{-\alpha(v)-m}) - \frac{\pi^2}{6} - \frac{1}{2} (\log(-1) + \alpha(v) + m)^2 \quad (3.8)$$

where we used $\text{Li}_2(e^x) + \text{Li}_2(e^{-x}) = -\frac{\pi^2}{6} - \frac{1}{2} (\log(-e^x))^2$ with the branch cut at $e^x \in (1, \infty)$. So one can approximate

$$\frac{\partial f_{\text{pert}}}{\partial v} \sim -\frac{1}{2} \sum_{\alpha \in \Delta} \alpha \left[-\frac{1}{2} (m + \alpha(v) + \log(-1))^2 \right], \quad (3.9)$$

where $-\text{Li}_2(e^{-\alpha(v)-m})$ can be ignored at $\text{Re}(m) \gg 1$. We ignored all the terms that vanish after summing over α , or are subleading in the $\frac{1}{\tau} \rightarrow 0$ limit. Expanding the square on the right hand side, the term proportional to $m^2 = m_D^2 \tau^2$ will vanish upon summing over α . The next term proportional to $m\alpha(v)$ will be the nonzero leading term. One obtains

$$v^i \approx v_D^i - \frac{m_D}{4\pi i} \sum_{\alpha \in \Delta} \alpha^i \alpha \cdot v = v_D^i - \frac{Nm_D}{2\pi i} (Pv)^i, \quad (3.10)$$

where we used

$$\sum_{\alpha \in \Delta} \alpha \otimes \alpha = \sum_{i \neq j} (e^i - e^j) \otimes (e^i - e^j) = 2(N-1)\mathbf{1}_{N \times N} - 2 \sum_{i \neq j} e^i \otimes e^j \equiv 2NP. \quad (3.11)$$

Here, P is the $N \times N$ projection to $\text{SU}(N)$. Decomposing $v = v_{\text{U}(1)} + v_{\text{SU}(N)}$, one finds that

$$v_{\text{U}(1)} = (v_{\text{U}(1)})_D, \quad v_{\text{SU}(N)} \approx \frac{1}{1 + \frac{Nm_D}{2\pi i}} (v_{\text{SU}(N)})_D \quad (3.12)$$

at $\tau \rightarrow i\infty$.³ Inserting this back to $\frac{\partial f_{\text{pert}}}{\partial v}$, one obtains

$$\frac{\partial f_{\text{pert}}}{\partial v^i} \approx -\frac{\tau m_D}{2} \sum_{\alpha \in \Delta} \alpha^i \alpha \cdot v = -N\tau m_D (v_{\text{SU}(N)})^i = -\frac{Nm_D}{1 + \frac{Nm_D}{2\pi i}} \tau (v_{\text{SU}(N)})_D^i. \quad (3.13)$$

Also f_{pert} itself is given by

$$f_{\text{pert}} = \sum_{\alpha \in \text{adj}} \left(\text{Li}_3(e^{\alpha(v)}) - \frac{1}{2} \text{Li}_3(e^{\alpha(v) \pm m}) \right) \approx \frac{1}{12} \sum_{\alpha \in \text{adj}} m^3 = \frac{N^2 \tau^3 m_D^3}{12} \quad (3.14)$$

where we used $\text{Li}_3(e^x) \approx -\frac{x^3}{6} - \frac{\pi i x^2}{2} + \frac{\pi^2 x}{3}$ if the real part of x is positive and large.

Therefore, the asymptotic prepotential is given by

$$\begin{aligned} f(\tau_D, v_D, m_D) &\rightarrow \tau^{-2} f(\tau, v, \tau m_D) + \frac{1}{4\pi i \tau^3} \left(\frac{\partial f}{\partial v} \right)^2 + \frac{\pi i N m_D^2 \tau}{12} + \frac{N^3 m_D^4 \tau}{48\pi i} \\ &\approx \frac{N^2 m_D^3 \tau}{12} + \frac{\pi i N m_D^2 \tau}{12} + \frac{N^3 m_D^4 \tau}{48\pi i}. \end{aligned} \quad (3.15)$$

In particular, one finds that the Coulomb VEV v_D does not appear in the asymptotic limit. This is natural since the Coulomb VEV is a dimensionful parameter, which should not be visible in the large momentum limit. This is a result for $-\frac{2\pi}{N} < \text{Im}(m_D) < 0$. When $0 < \text{Im}(m_D) < \frac{2\pi}{N}$, all the analysis above is same except the step of approximating $\frac{1}{\tau^2} f_{\text{pert}}(v, \tau m_D)$. In this case, $\frac{N^2 m_D^3}{12} \tau$ is replaced by $-\frac{N^2 m_D^3}{12} \tau$. Combining the two cases, one obtains

$$f_{\text{asympt}}^{(\pm)} = -\frac{\pi^3 i \tau}{3N} \left[\left(\frac{i N m_D}{2\pi} \right)^2 \pm 2 \left(\frac{i N m_D}{2\pi} \right)^3 + \left(\frac{i N m_D}{2\pi} \right)^4 \right], \quad (3.16)$$

³At $m_D = -\frac{2\pi i}{N}$, one finds that $v_{\text{SU}(N)}$ diverges. In this case, one has to approximate (3.6) by assuming that v can scale with τ . At $m_D = -\frac{2\pi i}{N}$, we checked for $N = 2, 3$ that $v_{\text{SU}(N)}$ scales like $\sqrt{\tau}$, which grows large but is much smaller than $m = \tau m_D$. Due to this fact, v does not affect the asymptotic free energy, and our final result for f_{asympt} below will be reliable even at $m_D = -\frac{2\pi i}{N}$.

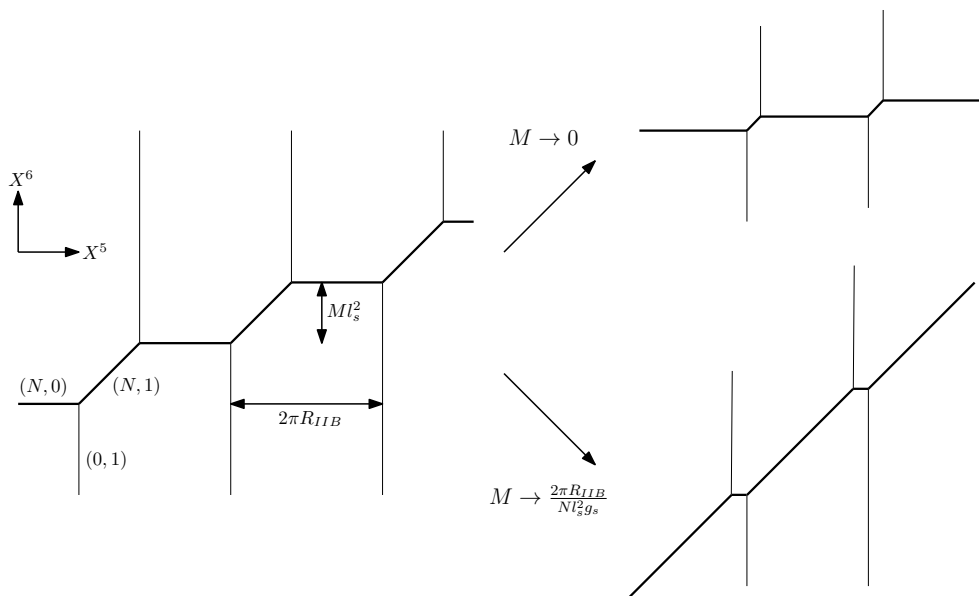


Figure 1. Type IIB 5-brane web for the 5d $\mathcal{N} = 1^*$ system.

where \pm signs are for $0 < \pm \text{Im}(m_D) < \frac{2\pi}{N}$, respectively. Finally, when $\text{Im}(m_D) = 0$, $\frac{1}{\tau^2} f_{\text{pert}}(v, \tau m_D)$ provides subleading contribution in τ so that one finds

$$f_{\text{asymp}}^{(0)} = -\frac{\pi^3 i \tau}{3N} \left[\left(\frac{iNm_D}{2\pi} \right)^2 + \left(\frac{iNm_D}{2\pi} \right)^4 \right], \quad (3.17)$$

where the superscript (0) means vanishing imaginary part of m_D . At this stage, we note that f_{asymp} at $\text{Im}(m_D) \neq 0$ can be written as the following holomorphic function with a branch cut,

$$f_{\text{asymp}} = -\frac{i\tau}{2\pi N} (2\text{Li}_4(1) - \text{Li}_4(e^{Nm_D}) - \text{Li}_4(e^{-Nm_D})). \quad (3.18)$$

This expression will be helpful later.

We first investigate f_{asymp} for purely imaginary $m_D \equiv ix$, at $-\frac{2\pi}{N} < x < \frac{2\pi}{N}$. One finds

$$f_{\text{asymp}} = -\frac{\pi^3 i \tau}{3N} \left[\left(\frac{Nx}{2\pi} \right)^2 - 2 \left| \frac{Nx}{2\pi} \right|^3 + \left(\frac{Nx}{2\pi} \right)^4 \right]. \quad (3.19)$$

The partition function undergoes a phase transition at $x = 0$, from certain perturbative particles being massless at $m_D \sim x = 0$. One may wonder how f_{asymp} behaves beyond $x = \pm \frac{2\pi}{N}$. At $x = \pm \frac{2\pi}{N}$, one finds from the S-dual picture that f_{inst} cannot be ignored, since $F_k q^k \sim (e^{\pm Nm} e^{2\pi i \tau})^k \sim \mathcal{O}(1)$ at $m = \tau m_D \rightarrow \mp 2\pi i \tau$. This means that particles with nonzero instanton number become light at these points. One can get some insights on these nonperturbative massless particles.

To see this, it is helpful to recall the type IIB 5-brane web realization of the 5d $\mathcal{N} = 1^*$ system. More precisely, we realize the ‘S-dualized’ setting at $\tau \rightarrow i\infty$, using weakly coupled type IIB string theory. The brane web first consists of N D5-branes and 1 NS5-brane, all

extended on 01234 directions, transverse to 789, and forming a web on the x^5 - x^6 plane. One makes a twisted compactification $(x^5, x^6) \sim (x^5 + 2\pi R_{\text{IIB}}, x^6 + M\ell_s^2)$. The D5-branes wrap x^5 direction, and form a web with the NS5-brane extended along x^6 , as shown in figure 1. The twisted compactification guarantees that the open strings with tension $\tau_{\text{F1}} = \frac{1}{2\pi\alpha'}$ (where $\alpha' = \ell_s^2$) suspended between D5-branes across the web have mass $\frac{M}{2\pi}$. D1-branes wrapping x^5 ending on NS5-brane are identified as Yang-Mills instantons. Unit instanton's mass is given by $\tau_{\text{D1}} \cdot 2\pi R_{\text{IIB}} = \frac{2\pi R_{\text{IIB}}}{2\pi\alpha' g_s}$, which should be identified with $\frac{1}{R'}$ in our M5-brane setting. So one obtains $2\pi R_{\text{IIB}} = \frac{2\pi\alpha' g_s}{R'}$. On the other hand, τ is given at zero axion by $\tau = \frac{i}{g_s}$, which should be identified in our M5-brane setting as $i\frac{R}{R'}$. So one finds $g_s = \frac{R'}{R}$. These provide the relations between the parameters R_{IIB}, g_s and the M5-brane parameters R, R' . The slope of the $(N, 1)$ 5-brane is $\frac{\Delta x^6}{\Delta x^5} = \frac{1}{g_s N}$, where Δx^5 and Δx^6 are the distances between the two ends of the segment on figure 1. We stated above that $\Delta x^6 = M\alpha'$, so one finds $\Delta x^5 = NMg_s\alpha'$.

In this setting, the segment of $(N, 1)$ 5-brane shrinks at $M = 0$. Here, the perturbative hypermultiplet particle becomes massless, corresponding to the fundamental strings connecting D5-branes across the NS5-brane. This causes the so-called flop phase transition. The singular term proportional to $|x|^3$ in (3.19) is caused by f_{pert} in the S-dual setting, from the particles becoming massless at $M = 0$. So we conclude that the cusp $\propto |x|^3$ is due to the flop transition.

As one increases positive M , the next transition happens when the $(N, 1)$ brane segment goes around the circle in x^5 direction, as shown on the bottom-right side of figure 1. This happens at $\Delta x^5 = NMg_s\alpha' = 2\pi R_{\text{IIB}}$,

$$NMg_s\alpha' = 2\pi R_{\text{IIB}} = \frac{2\pi\alpha' g_s}{R'} = \frac{2\pi\alpha'}{R}. \quad (3.20)$$

So one finds that the transition happens at

$$m_D = \frac{m}{\tau} = \frac{MR}{i/g_s} = -\frac{2\pi i}{N}, \quad (3.21)$$

precisely when F_{inst} cannot be ignored. Across $x = \pm \frac{2\pi}{N}$, i.e. $m = \mp \frac{2\pi i}{N}$, the N D5-brane segment shrinks. So across this value, another transition happens, with the D1-brane segment extended along the shrinking segment being massless.

As one continues to change M , transitions due to non-perturbative massless particles will happen at $x = \frac{2\pi n}{N}$ with n being integers. At $n = 1, 2, \dots, N-1$, the nature of this transition is hard to study. This is because the massless particles are nontrivial bound states of D1-branes. Also, studying the $\tau \rightarrow i\infty$ approximations around $x = 0$, not all massless particles were responsible for the cusp at $x = 0$. So it will be important to know which types of massless particles contribute to the cusp of f_{asympt} at $x = \frac{2\pi n}{N}$. However, if n is a multiple of N , one finds from the 5-brane web diagram that the transition is an $\text{SL}(2, \mathbb{Z})$ transformation of the transition at $x = 0$, so that the same type of cusp will happen. Indeed this has to be the case, since $x \sim x + 2\pi$ (or $m_D \sim m_D + 2\pi i$) is the periodicity of the instanton partition function.

Interestingly, if one takes the holomorphic extension (3.18) within $|\text{Im}(m_D)| < \frac{2\pi}{N}$ to the whole region of m_D , one gets a definite prediction on f_{asympt} as a function of real x ,

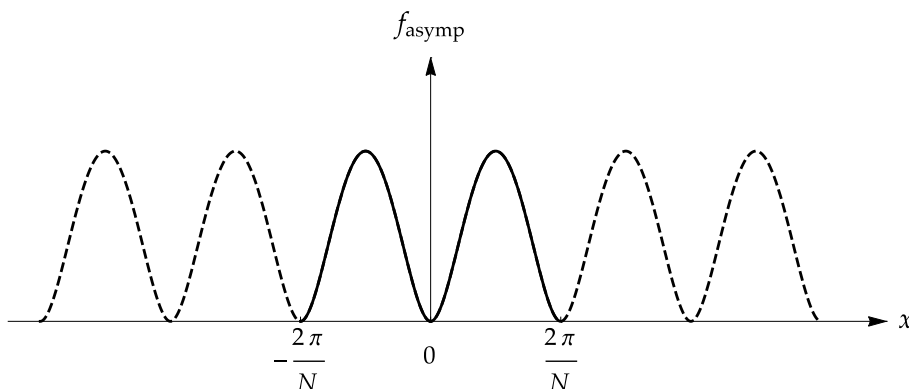


Figure 2. Continuation of the asymptotic free energy across flop transitions.

and also on the nature of phase transitions at all n . Plotting (3.18) for the entire real x , one finds figure 2. $f_{\text{asymp}}(x)$ for $\frac{2\pi n}{N} < x < \frac{2\pi(n+1)}{N}$ is given by simply translating the function in the range $0 < x < \frac{2\pi n}{N}$ by $\frac{2\pi n}{N}$. This means that all the cusp structures are completely the same at all n , at least in $f_{\text{asymp}}(x)$. It will be interesting to understand how the non-perturbative massless particles cause the same cusp in (3.18). Also, in (3.18) or in figure 2, f_{asymp} has a shorter period $x \sim x + \frac{2\pi}{N}$. It will be interesting to see if the reduced period has to do with multiple-wrapping of M5-branes on S^1 , analogous to the multiple-winding fundamental strings [28].

Now we study f_{asymp} for purely real m_D . The asymptotic free energy is given by

$$-\log Z \sim \frac{f_{\text{asymp}}^{(0)}}{\epsilon_1 \epsilon_2} = \frac{i}{3\epsilon_1 \epsilon_2 \tau_D} \left[\frac{N^3 m_D^4}{16\pi} - \frac{\pi N m_D^2}{4} \right]. \quad (3.22)$$

Holding real m_D fixed, and further taking the large N limit, one finds that the free energy is proportional to N^3 . Namely, one finds that the single particle index $f(\tau, \epsilon_1, \epsilon_2, m, v)$ in the limit $\epsilon_{1,2} \rightarrow 0$, $\tau \rightarrow i0^+$ is given by

$$\sum_{n=1}^{\infty} \frac{1}{n} f(n\tau, \epsilon_{1,2} = 0, nm, nv) \rightarrow -\frac{i}{3\epsilon_1 \epsilon_2 \tau} \left[\frac{N^3 m^4}{16\pi} - \frac{\pi N m^2}{4} \right], \quad (3.23)$$

where we dropped the D subscripts. This shows that the microscopic entropy (with minus sign for fermions) of light D0-branes bound to N D4-branes exhibit large number of bound states proportional to N^3 . The second term proportional to N clearly comes from N free tensor multiplets, as this comes from the S-dualization of $Nf_{U(1)}$. One can understand that the first term proportional to $N^3 m^4$ is a remnant of the cancelation between bosonic and fermionic states in the index, since this term vanishes at $m = 0$. It will be interesting to guess what kind of index $f(\tau, \epsilon_{1,2} = 0, m, v)$ would exhibit the above behavior in the high temperature limit. In particular, having the analytic expression (3.18) given in terms of Li_4 functions, with chemical potentials multiplied by N , it will be interesting to seek for an interpretation using multiple-wrapping of M5-branes, or instanton partons [29].

Finally, we comment that one can obtain the asymptotic free energy at $\tau_D \rightarrow i0^+$ for all ADE theories, starting from (2.135) and following the analysis of this section. To make

a similar calculation, one also needs to know the perturbative partition function, and the range of $\text{Im}(m_D)$ in which the instanton correction f_{inst} can be ignored on the right hand side. The perturbative prepotential is straightforward for all ADE. As for the instanton part, we should know when $F_k q^k$ is much smaller than 1 at $q \rightarrow 0$ for large real part of $m = \tau m_D$. The leading behavior of F_k for large real m can be easily inferred, by knowing the correct parameter scalings between the 5d $\mathcal{N} = 1^*$ theory and the pure $\mathcal{N} = 1$ theory. Namely, one finds

$$F_k q^k \sim e^{kc_2 m} q^k, \tag{3.24}$$

where c_2 is the dual Coxeter number of the gauge group G . This is because the pure 5d $\mathcal{N} = 1$ theory is obtained by taking the limit $m \rightarrow \infty$, $q \rightarrow 0$, with $\Lambda \sim e^{c_2 m} q$ held fixed. This means that one can ignore the instanton part in the region $-\frac{2\pi}{c_2} < \text{Im}(m_D) < \frac{2\pi}{c_2}$. By following the analysis for the $U(N)$ case, the asymptotic free energies of ADE theories are given by

$$f_{\text{asymp}}^{(\pm)} = -\frac{\pi^3 i \tau}{3} \left[r \left(\frac{i m_D}{2\pi} \right)^2 \pm 2|G| \left(\frac{i m_D}{2\pi} \right)^3 + (c_2 |G| + r) \left(\frac{i m_D}{2\pi} \right)^4 \right], \tag{3.25}$$

where \pm signs are for $0 < \pm \text{Im}(m_D) < \frac{2\pi}{c_2}$, and

$$f_{\text{asymp}}^{(0)} = -\frac{\pi^3 i \tau}{3} \left[r \left(\frac{i m_D}{2\pi} \right)^2 + (c_2 |G| + r) \left(\frac{i m_D}{2\pi} \right)^4 \right] \tag{3.26}$$

for $\text{Im}(m_D) = 0$.

3.1 Tests with $U(1)$ partition function

We provide a small consistency check of f_{asymp} for the $U(1)$ case. By this exercise, one can also get better intuitions on the true nature of the approximations and phase transitions, which perhaps may be a bit obscure in our S-duality based approach.

In the previous S-duality based approach, we first took $\epsilon_1, \epsilon_2 \rightarrow 0$ limit of the partition function, to focus on the prepotential only. Then using the S-duality, we extracted out the $\beta \rightarrow 0$ asymptotics of the prepotential, where $q = e^{2\pi i \tau} = e^{-\beta}$, at finite m and N . We reconsider the same limits directly with the $U(1)$ instanton partition function. The instanton partition function is given by

$$Z_{\text{inst}} = \exp \left[\sum_{n=1}^{\infty} \frac{1}{n} \frac{\sinh \frac{n(m \pm \epsilon_-)}{2}}{\sinh \frac{n\epsilon_1}{2} \sinh \frac{n\epsilon_2}{2}} \frac{e^{-n\beta}}{1 - e^{-n\beta}} \right] \sim \exp \left[\frac{4}{\epsilon_1 \epsilon_2} \sum_{n=1}^{\infty} \frac{\sinh^2 \frac{nm}{2}}{n^3} \frac{e^{-n\beta}}{1 - e^{-n\beta}} \right] \tag{3.27}$$

in the $\epsilon_1, \epsilon_2 \rightarrow 0$ limit. Now we take the $\beta \rightarrow 0$ limit at fixed m . This is somewhat tricky at real m , which we also take to be positive. This is because the above formula is valid for $m < \beta$ when m is real. Physically, this is because the partition function Z has poles at $m = n\beta$ for all positive integers n . So with fixed real m , one would hit many poles as one takes the $\beta \rightarrow 0$ limit. To deal with this situation more easily, we first continue m to be purely imaginary, $m = ix$, and continue back later to complex m .

Inserting $m = ix$ and taking $\beta \rightarrow 0$ limit, one obtains

$$\begin{aligned} Z &\sim \exp \left[-\frac{4}{\epsilon_1 \epsilon_2 \beta} \sum_{n=1}^{\infty} \frac{\sin^2 \frac{nx}{2}}{n^4} \right] = \exp \left[\frac{1}{\epsilon_1 \epsilon_2 \beta} \sum_{n=1}^{\infty} \frac{1}{n^4} (e^{inx} + e^{-inx} - 2) \right] \\ &= \exp \left[\frac{1}{\epsilon_1 \epsilon_2 \beta} (\text{Li}_4(e^{ix}) + \text{Li}_4(e^{-ix}) - 2\text{Li}_4(1)) \right]. \end{aligned} \quad (3.28)$$

The final expression can be continued to complex x . Here, we use the property

$$\text{Li}_n(e^{2\pi ix}) + (-1)^n \text{Li}_n(e^{-2\pi ix}) = -\frac{(2\pi i)^n}{n!} B_n(x), \quad (3.29)$$

where $0 \leq \text{Re}(x) < 1$ for $\text{Im}(x) \geq 0$. $B_n(x)$ are the Bernoulli polynomials, given by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \quad (3.30)$$

In particular, one finds $B_4(x) = \frac{1}{30}(-1 + 30x^2 - 60x^3 + 30x^4)$, so that

$$\text{Li}_4(e^{ix}) + \text{Li}_4(e^{-ix}) = -\frac{(2\pi)^4}{24} B_4(x/2\pi) = \frac{2\pi^4}{90} - \frac{2\pi^4}{3} \left[\left(\frac{x}{2\pi}\right)^2 - 2\left(\frac{x}{2\pi}\right)^3 + \left(\frac{x}{2\pi}\right)^4 \right] \quad (3.31)$$

for $0 \leq x < 2\pi$. This leads to the asymptotic formula

$$-\log Z \sim \frac{2\pi^4}{3\epsilon_1 \epsilon_2 \beta} \left[\left(\frac{x}{2\pi}\right)^2 - 2\left(\frac{x}{2\pi}\right)^3 + \left(\frac{x}{2\pi}\right)^4 \right] \quad (3.32)$$

for $0 \leq x < 2\pi$, which is in complete agreement with the S-duality-based result, (3.19), upon inserting $-i\tau \rightarrow \frac{2\pi}{\beta}$ and $N = 1$. When $-2\pi < x \leq 0$, we use a different identity of Li_4 function to find a similar expression, with the sign of the $\mathcal{O}(x^3)$ term flipped. This also shows that the continuation (3.18) beyond $-2\pi < x < 2\pi$ by the Li_4 functions to complex x is indeed correct.

3.2 6d chiral anomalies on Omega-deformed $\mathbb{R}^4 \times T^2$

In this subsection, we shall discuss the connection between the S-duality anomaly and the 6d chiral anomalies of global symmetries. In particular, we shall independently compute some part of our asymptotic free energy f_{asymp} based on chiral anomaly only. However, let us start by giving a general comment, on why one should naturally expect S-duality anomaly of the partition function if the system has chiral anomaly.

Consider a partition function of even dimensional chiral theories on T^2 , like 2d theories on T^2 or our system on $\mathbb{R}^4 \times T^2$. For a global symmetry, one turns on a background gauge field A . In particular, let us turn on the flat connection of A on T^2 . We shall only be interested in Abelian flat connections, characterized by the commuting holonomies along the two circles of T^2 . Large gauge transformations would have made both holonomies to be periodic, had there been no chiral anomalies. For simplicity, let us take a rectangular torus with two radii r_1, r_2 , respectively. Then the large gauge transformations would have given the periods $A_1 \sim A_1 + \frac{1}{r_1}$ and $A_2 \sim A_2 + \frac{1}{r_2}$. With matter fields having

integral charge q of this global symmetry, the modes of these fields would have frequencies $(\omega_1, \omega_2) = (\frac{n_1}{r_1} + qA_1, \frac{n_2}{r_2} + qA_2)$ on T^2 , with integral n_1, n_2 , which is invariant under the periodic shifts of A_1, A_2 . This is a consequence of these gauge symmetries. However, in quantum observables like the partition function on T^2 , these large gauge transformations may fail to be symmetries for theories with chiral anomalies. This is because one has to regularize the path integral over these modes, by regarding one of the two directions as temporal circle [30]. By this procedure, one of the two holonomies A_1, A_2 fail to be periodic in the partition functions. This is precisely what happen for the 2d elliptic genera [31]. We expect that similar things will happen to 6d chiral theories on $\mathbb{R}^4 \times T^2$, but we cannot make this expectation more precise here. We shall simply assume the failure of double periodicity of background holonomies due to chiral anomalies, and then explain that it forces the partition function to have S-duality anomaly, as we found in section 2 by nonzero F_{anom} .

Let us write the background holonomies as a complex number m . Had a free energy $F(\tau, m)$ on T^2 been exactly S-dual, then its exact S-duality $F(-1/\tau, m/\tau) = F(\tau, m)$ means that m has double period. This is because if the right hand side has period in one direction, say $F(\tau, m) = F(\tau, m + 1)$, the left hand side forces $F(-\frac{1}{\tau}, \frac{m}{\tau}) = F(-\frac{1}{\tau}, \frac{m}{\tau} + \frac{1}{\tau})$, and thus $F(\tau, m) = F(\tau, m - \tau)$, contradicting the obstruction of double periodicity from chiral anomaly. This comment applies to our 6d partition functions. So we naturally expect S-duality anomaly.

With these motivations in mind, rather than trying to elaborate on it, we shall make a concretely calculation which shows that a particular term in our asymptotic high temperature free energy dictated by F_{anom} can be computed using 6d chiral anomaly only.

Let us first explain the anomalies of the 6d (2, 0) theory of A_{N-1} type. More precisely, we shall consider the anomaly of the interacting A_{N-1} type theory times a decoupled free self-dual tensor multiplet theory. This corresponds to the system of N M5-branes including the decoupled center-of-mass multiplet. The anomaly polynomial 8-form is given by

$$I_8 = NI_8(1) + N(N^2 - 1)\frac{p_2(N)}{24} \tag{3.33}$$

where $I_8(1)$ is the anomaly of the single M5-brane theory, or one free (2, 0) tensor multiplet,

$$I_8(1) = \frac{1}{48} \left[p_2(N) - p_2(T) + \frac{1}{4} (p_1(T) - p_1(N))^2 \right]. \tag{3.34}$$

The Pontryagin classes are defined by

$$p_1 = -\frac{1}{2(2\pi)^2} \text{tr} R^2, \quad p_2 = \frac{1}{(2\pi)^4} \left[-\frac{1}{4} \text{tr} R^4 + \frac{1}{8} (\text{tr} R^2)^2 \right]. \tag{3.35}$$

Here, traces are acting on either 6×6 matrices for $\text{SO}(5, 1)$ tangent bundle T , or 5×5 matrices for $\text{SO}(5)$ normal bundle N . Taking their curvatures to be R and F , respectively, one finds

$$\begin{aligned} (2\pi)^4 I_8 = & \frac{N}{48} \left[-\frac{1}{4} \text{tr} F^4 + \frac{1}{8} (\text{tr} F^2)^2 + \frac{1}{4} \text{tr} R^4 - \frac{1}{8} (\text{tr} R^2)^2 + \frac{1}{16} (\text{tr} R^2 - \text{tr} F^2)^2 \right] \\ & + \frac{N^3 - N}{24} \left(-\frac{1}{4} \text{tr} F^4 + \frac{1}{8} (\text{tr} F^2)^2 \right). \end{aligned} \tag{3.36}$$

We shall restrict F to a Cartan part. In particular, since we shall be taking the Omega backgrounds to be small, the Cartan for $SU(2)_R$ will have much smaller background field than $SU(2)_L$, from $\epsilon_+ \ll m$. So we shall only turn on the background field for the Cartan in $SU(2)_L \subset SO(5)$, corresponding to our $\mathcal{N} = 1^*$ mass m . F is a 5×5 matrix-valued 2-form, whose components are $F^{ab} = -F^{ba}$ with $a, b = 1, \dots, 5$. The component corresponding to the Cartan of $SU(2)_L$ is obtained by keeping $F^{12} = -F^{21} = -F^{34} = F^{43} \equiv F$ only. With this restriction, one finds $\text{tr}(F^2) \rightarrow -4F^2$, $\text{tr}(F^4) \rightarrow 4F^4$. Inserting these, the $SO(5, 1)$ and $U(1) \subset SU(2)_L$ anomalies are given by

$$(2\pi)^4 I_8 \rightarrow \frac{N^3}{24} F^4 + \frac{N}{48} \left[\frac{1}{2} F^2 \text{tr} R^2 + \frac{1}{4} \text{tr} R^4 - \frac{1}{8} (\text{tr} R^2)^2 \right]. \quad (3.37)$$

Only the first term $\frac{N^3}{24} F^4$ will be relevant for the computations below.

Our goal is to compute some part of the asymptotic free energy at high temperature $\tau_D \rightarrow 0$, using 6d chiral anomalies. Recall that we found

$$S_{\text{eff}} = -\log Z \rightarrow \frac{f_{\text{asympt}}}{\epsilon_1 \epsilon_2} = \frac{i}{2^4 \cdot 3\pi \epsilon_1 \epsilon_2 \tau_D} [N^3 m^4 - 4\pi^2 N m^2 + \dots] \quad (3.38)$$

where \dots stands for the m^3 term which exists when m has imaginary component. The m^3 term will not be of our interest in this subsection. We obtained this expression at $\epsilon_{1,2} \ll 1$ and $\tau_D \rightarrow 0$, where $\tau_D \equiv \frac{\beta}{4\pi}(\mu + i)$ is the same τ_D used before. Often, we used purely imaginary τ_D with $\mu = i$, but we keep real μ in this subsection to see a clear relation to chiral anomalies. For a reason to be explained below, we would like to study the asymptotic free energy when all the chemical potentials $\epsilon_{1,2}, m$ are purely imaginary. So inserting $i\epsilon_{1,2}, im$ (with real $\epsilon_{1,2}, m$) in the places of $\epsilon_{1,2}, m$ in (3.38), one obtains $S_{\text{eff}} = -\frac{i}{2^4 \cdot 3\pi \epsilon_1 \epsilon_2 \tau_D} [N^3 m^4 + 4\pi^2 N m^2 + \mathcal{O}(m^3)]$. In this setting, we focus on the imaginary part of the effective action,

$$\text{Im}(S_{\text{eff}}) = -\frac{\mu}{12\epsilon_1 \epsilon_2 \beta (1 + \mu^2)} [N^3 m^4 + 4\pi^2 N m^2 + \mathcal{O}(m^3)], \quad (3.39)$$

and compute it from 6d chiral anomalies. Especially, we shall compute part of $\text{Im}(S_{\text{eff}})$ from the 5d effective action approach for the 6d theory on small temporal circle. 6d chiral anomaly determines a special class of terms in the 5d effective action. It turns out that, knowing the terms determined by anomaly, one can only compute the term proportional to m^4 . So we shall pay attention to the first term

$$\text{Im}(S_{\text{eff}}) \Big|_{m^4} = -\frac{\mu N^3 m^4}{12\epsilon_1 \epsilon_2 \beta (1 + \mu^2)}. \quad (3.40)$$

We shall argue below that this term is completely dictated by 6d chiral anomaly, and then we re-compute this term using chiral anomaly only. This will provide another strong test of our findings from the D0-D4 calculus. Then, since one naturally expects that supersymmetrization of (3.40) is holomorphic in τ_D , one can reconstruct the term $-\frac{iN^3 m^4}{2^4 \cdot 3\pi \epsilon_1 \epsilon_2 \tau_D}$ in (3.38).

We shall consider the 6d anomaly from the viewpoint of 5d effective action, obtained by compactification on a small circle of circumference $\beta \ll 1$, and discuss our asymptotic free energy f_{asymp} on $\mathbb{R}^4_{\epsilon_{1,2}} \times T^2$ in this setting. On T^2 , regarding one circle as the temporal circle, the partition function is an index of the form

$$Z(\tau, v, m, \epsilon_{1,2}) = \text{Tr} \left[(-1)^F e^{-\frac{\beta}{2}(H-i\mu P)} e^{\sum_{a=1}^2 \epsilon_a (J_a + J_R)} e^{2mJ_L} e^{-v_i q_i} \right]. \quad (3.41)$$

Real $\epsilon_{1,2}, m$ is consistent with the conventions for the partition function presented at the beginning of section 2. In this setting, the chemical potentials $\epsilon_{1,2}, m$ will twist the translation on the temporal circle in a way that the twisted time evolution is not unitary (simply because the factors in the trace are not unitary transformations). This would cause a complex deformation of the Euclidean action by twisting with chemical potentials.⁴ For a technical reason, it will be convenient to keep these twistings to preserve the reality of the action. So we replace

$$e^{\sum_{a=1}^2 \epsilon_a (J_a + J_R)} e^{2mJ_L} e^{-v_i q_i} \rightarrow e^{i \sum_{a=1}^2 \epsilon_a (J_a + J_R)} e^{2imJ_L} e^{-iv_i q_i}, \quad (3.42)$$

which will make real twists of the Euclidean action. This is equivalent to the insertions of $i\epsilon_{1,2}, im$ around (3.40). The factor $e^{-\frac{\beta}{2}H}$ demands us to consider a 6d Euclidean theory whose temporal coordinate y satisfies periodicity $y \sim y + \frac{\beta}{2}$. This forms a circle of the T^2 . Another circle factor is labeled by x , which we take to have periodicity $x \sim x + 2\pi$. Defining $\tau_D = \frac{\beta}{4\pi}(\mu + i)$, one obtains

$$e^{-\frac{\beta}{2}(H-i\mu P)} \equiv e^{2\pi i \tau_D \frac{H+P}{2}} e^{-2\pi i \bar{\tau}_D \frac{H-P}{2}} = e^{-2\pi \text{Im} \tau_D H + 2\pi i \text{Re} \tau_D P}. \quad (3.43)$$

So τ_D is the complex structure of T^2 . This torus is endowed with the metric

$$ds^2(T^2) = (dx - \mu dy)^2 + dy^2 \quad (3.44)$$

and periods $(x, y) \sim (x + 2\pi, y) \sim (x + \frac{\beta\mu}{2}, y + \frac{\beta}{2})$. Including the chemical potential ϵ_a , the metric of $\mathbb{R}^4 \times T^2$ is given by

$$ds^2(\mathbb{R}^4 \times T^2) = \sum_{a=1,2} \left| dz_a - \frac{2i\epsilon_a}{\beta} z_a dy \right|^2 + (dx - \mu dy)^2 + dy^2, \quad (3.45)$$

where z_a are complex coordinates of $\mathbb{C}^2 \sim \mathbb{R}^4$ with charges $J_a[z_b] = \delta_{ab}$. Finally, the chemical potential m is realized as the background gauge field $A = \frac{2m}{\beta} dy$ for $U(1) \subset SU(2)_L$. Also, $\frac{H-P}{2} \sim \{Q, \bar{Q}\}$, where Q is a supercharge preserved by the index. So Z is independent of $\bar{\tau}$.

⁴Strictly speaking, Lagrangian formulation is not known in 6d. So when we refer to a Lagrangian description, we mean a 5d Lagrangian after reducing on a small circle. See also comments in [11] concerning the conversion between twistings and background gauge fields in the presence of anomalies.

Following [11] (see also [32]), we shall make a KK reduction on the small circle along y , for small inverse-temperature $\beta \ll 1$. To this end, one rewrites the background in the form of

$$ds^2 = e^{2\phi}(dy + a)^2 + h_{ij}dx^i dx^j, \tag{3.46}$$

where h_{ij} with $i, j = 1, \dots, 5$ is the 5d metric, $e^{2\phi} = 1 + \mu^2 + \frac{4}{\beta^2} \sum_a \epsilon_a^2 |z_a|^2$ is the dilaton, and

$$a = \frac{1}{1 + \mu^2 + \frac{4\epsilon_a^2 |z_a|^2}{\beta^2}} \left(-\mu dx - \frac{2\epsilon_a |z_a|^2}{\beta} d\phi_a \right) \tag{3.47}$$

is the gravi-photon field, where $z_a = |z_a| e^{i\phi_a}$. The 6d background gauge field A for $U(1) \subset SU(2)_L$ is also rewritten in the form $A = A_6(dy + a) + \mathcal{A}$, where \mathcal{A} is the 5d background gauge field and A_6 is the 5d scalar. So one finds $A_6 = \frac{2m}{\beta}$ and $\mathcal{A} = -A_6 a$.

If the 6d theory compactified on a small circle has no 5d massless modes, one can express the thermal partition function in terms of a 5d local effective field theory of background fields, where the 5d derivative expansion corresponds to a β series expansion. As noted in [11], with massless modes in 5d, there could be nonlocal part of the effective action which is smooth in the $\beta \rightarrow 0$ limit. In our case, the non-local part comes from the 5d perturbative maximal SYM. There is additional difficulty in using the derivative expansion in our setting, since some of our background fields are proportional to β^{-1} , which may spoil the orderings provided by the derivative expansion. So it appears tricky to directly employ the formalism of [11, 32].

However, one can study the imaginary part (3.39) of our asymptotic free energy using the 5d approach. The imaginary part can be computed completely by knowing the 5d Chern-Simons like terms. To explain this, note first that we have been careful to set all our background fields to be real, e.g. by setting our chemical potentials to be imaginary. With real background fields turned on, suppose that we first reduce the 6d theory on a small circle to a general 5d *Lorentzian* spacetime. Then the 5d effective action is real, since Hermiticity is not broken in the Lorentzian theory. Now we Wick-rotate the ‘time’ direction in this 5d setting. Since all background fields are real, the only possible step which may cause complex effective action is the Wick rotation to Euclidean 5d space. Here, note that we are seeking for an effective action of the vectors a, \mathcal{A}, ω (spin connection), tensor h_{ij} , and scalars A_6, ϕ . To compute the imaginary part, one can focus on the local terms. This is because the nonlocal terms come from the determinant of 5d maximal SYM whose fields are covariantized by real background fields, which is real. Among the local terms obtained from scalar Lagrangian density, we should seek for terms containing the tensor ϵ^{ijklm} to obtain imaginary contribution after Wick rotation. It should be contracted with antisymmetric tensors formed by the background fields. There are many possibilities, arranged in derivative expansion. For instance, there could be complicated terms like $\sim da \wedge d\mathcal{A} \wedge d\phi f(\phi, A_6)$, and so on.

Although there are many terms, let us comment that there can be gauge invariant terms and gauge non-invariant terms in the imaginary action. The latter class should exist because the 5d effective action should realize 6d chiral anomalies. The coefficients of the terms in the latter class are thus completely determined by known 6d anomalies [11, 32].

Among the gauge invariant terms, there can be action coming from gauge invariant Lagrangian density, like the term that we illustrated in the last paragraph. Finally, there may be Chern-Simons terms in which Lagrangian densities are not gauge invariant but their integrals are. So the imaginary action takes the following structure,

$$\begin{aligned}
 S_{CS} &= S_{CS}^{(1)} + S_{CS}^{(2)} + S_{GI} \tag{3.48} \\
 S_{CS}^{(1)} &= \frac{i\kappa_1}{\beta^3} \int a \wedge da \wedge da + \frac{i\kappa_2}{\beta} \int \mathcal{A} \wedge d\mathcal{A} \wedge da + \frac{i\kappa_3}{\beta} \int a \wedge R \wedge R + i\kappa_4 \int \mathcal{A} \wedge d\mathcal{A} \wedge d\mathcal{A} + \dots \\
 S_{CS}^{(2)} &= -\frac{iDr_1}{96\pi^2} \int (A_6^4 a \wedge da \wedge da + 4A_6^3 \mathcal{A} \wedge da \wedge da + 6A_6^2 \mathcal{A} \wedge d\mathcal{A} \wedge da + 4A_6 \mathcal{A} \wedge d\mathcal{A} \wedge d\mathcal{A}) + \dots,
 \end{aligned}$$

where $r_1 = \frac{\beta}{4\pi}$ is the radius of the small sixth circle with circumference $\frac{\beta}{2}$. $S_{CS}^{(1)}$ consists of the gauge invariant Chern-Simons terms. $S_{CS}^{(2)}$ is part of the gauge non-invariant Chern-Simons terms that comes from $U(1) \subset SU(2)_L \subset SO(5)_R$ normal bundle anomaly in 6d, namely the first term $\sim \frac{N^3}{24} F^4$ of (3.37). Anomaly matching fixes $D = N^3$, as well as the relative coefficients as shown on the second line.⁵ The omitted terms \dots in $S_{CS}^{(1)}$ are other Chern-Simons terms containing ω , which we do not need here. The omitted terms in $S_{CS}^{(2)}$ can all be computed from mixed anomalies and gravitational anomalies of (3.37), which we do not work out here as we shall not need them. Finally, S_{GI} is the action containing ϵ^{ijklm} associated with gauge invariant Lagrangian density, e.g. $da \wedge da \wedge d\phi f(\phi, A_6)$, $d\mathcal{A} \wedge d\mathcal{A} \wedge d[(da)^{ij}(d\mathcal{A})_{ij}]g(\phi, A_6)$, and so on. One point we emphasize is that S_{GI} can come in infinite series of derivative expansion, while $S_{CS}^{(1)}$ and $S_{CS}^{(2)}$ consist of finite number of terms and can be completely classified.

The imaginary terms have rich possibilities. Here we consider the terms which are nonzero with our background, and also the leading terms in small $\epsilon_{1,2}$, proportional to $\frac{1}{\epsilon_1 \epsilon_2}$. $A_6 = \frac{2m}{\beta}$ is constant in our background. Also, $\mathcal{A} = -A_6 a$ is constant times the graviphoton. Plugging in these values, one obtains

$$(A_6)^n \epsilon^{ijklm} (\text{rank 5 antisymmetric tensor of } a, \phi, \omega, h). \tag{3.49}$$

The parenthesis consists of the fields reduced from 6d metric (3.45). Note that, after plugging in constant A_6 and $\mathcal{A} = -A_6 a$, all terms should be formally gauge invariant in the remaining fields. This is because the only possible gauge non-invariant terms $S_{CS}^{(2)}$, completely dictated by anomaly, also become gauge invariant like $A_6^4 a \wedge da \wedge da$ with constant A_6 .

Now we note the fact that, in the 6d metric, all z_a coordinates of \mathbb{R}^4 are multiplied by ϵ_a . So in the parenthesis of (3.49), the only z^a 's not associated with ϵ_a are derivatives. So one makes a formal derivative expansion of this term, assigning the ‘mass dimensions’ $[a] = 0$, $[\phi] = 0$, $[h] = 0$, $[\omega] = 1$. The lowest order term comes in two derivatives, and is proportional to $a \wedge da \wedge da$. There can be no other gauge-invariant terms at this order. This term indeed yields the desired $\frac{1}{\epsilon_1 \epsilon_2}$ scaling. Firstly, the integral $dx d^2 z_1 d^2 z_1$ can be scaled into $(\frac{\beta^2}{\epsilon_1 \epsilon_2})^2$ times a measure depending on $\frac{\epsilon_a z_a}{\beta}$. Also, two derivatives in $a \wedge da \wedge da$

⁵Following [11], we show the form of the action with constant value of A_6 , taking into account the covariant anomaly rather than the consistent anomaly. This is sufficient for our calculus of the free energy.

can also be scaled with $\frac{\epsilon_a}{\beta}$, yielding another overall factor $\frac{\epsilon_1 \epsilon_2}{\beta^2}$. z_a in the remaining integral appear in the combination $\frac{\epsilon_a z_a}{\beta}$, including the integration variable, so is independent of ϵ_a . So this term yields the right scaling $\sim \frac{1}{\epsilon_1 \epsilon_2}$. Therefore, to compute (3.40), we only need to consider those terms that reduce to

$$(A_6)^n a \wedge da \wedge da \tag{3.50}$$

upon plugging in our background. This implies that one does not have to consider S_{GI} of (3.48), since they are associated with local Lagrangian density and cannot provide terms like (3.50).

So we only consider $S_{\text{CS}}^{(1)}$ and $S_{\text{CS}}^{(2)}$ of (3.48). Unlike the coefficients of $S_{\text{CS}}^{(2)}$, coefficients of $S_{\text{CS}}^{(1)}$ cannot be determined with our limited knowledge of the 6d theory. So even after restricting our interest to the imaginary part (3.39) of the effective action, we cannot compute them all due to our ignorance on these coefficients. Since the second term of $S_{\text{CS}}^{(1)}$ is quadratic in \mathcal{A} , we cannot compute the $\mathcal{O}(m^2)$ term of (3.39). This is why we shall not need the mixed anomaly contributions in $S_{\text{CS}}^{(2)}$ coming from the term $\sim F^2 \text{tr} R^2$ in (3.37), which will also yield a contribution at $\mathcal{O}(m^2)$, since knowing them is incomplete to compute the whole $\mathcal{O}(m^2)$ contributions. Also, the $\mathcal{O}(m^3)$ term cannot be computed since we do not know κ_4 . However, the Chern-Simons terms that are quartic in \mathcal{A} and A_6 are completely dictated by 6d anomalies, as shown on the second line of (3.48). Note that quartic Chern-Simons term is allowed precisely because we allow gauge non-invariant Chern-Simons term, to match 6d anomalies which are fourth order in the fields. Thus, we can compute (3.40) from $S_{\text{CS}}^{(2)}$ of (3.48). Note also that, for imaginary chemical potentials, we have found earlier in this section that f_{asympt} undergoes phase transitions due to massless particles. This only changes $\mathcal{O}(m^3)$ or lower order terms, so that the m^4 order that we are going to compute is unaffected.

We also note in passing that, we can turn the logic around and use our D0-D4 results to constrain the 5d effective action. Namely, we know from our D0-D4 calculus the $\mathcal{O}(m^2)$ and $\mathcal{O}(m^3)$ coefficients of $\text{Im}(f_{\text{asympt}})$, and also the vanishing of the $\mathcal{O}(m^0)$ coefficient. This knowledge can be used to constrain $\kappa_1, \kappa_2, \kappa_3, \dots$ of (3.48). This information may be useful for studying other high temperature partition functions of the 6d (2, 0) theories.

Coming back to the computation of (3.40), we plug $\mathcal{A} = -A_6 a$ and $A_6 = \frac{2m}{\beta}$ into $S_{\text{CS}}^{(2)}$ of (3.48) to obtain

$$\frac{iN^3(A_6)^4 r_1}{96\pi^2} \int a \wedge da \wedge da. \tag{3.51}$$

To compute this, one should evaluate the gravi-photon Chern-Simons term,

$$\int a \wedge da \wedge da = \int \left(1 + \mu^2 + \frac{4\epsilon_a^2 |z_a|^2}{\beta^2}\right)^{-3} (-\mu dx) \wedge 2 \frac{4\epsilon_1 \epsilon_2}{\beta^2} 4dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \tag{3.52}$$

where $z_a \equiv x_a + iy_a$, with x_1, y_1, x_2, y_2 being the Cartesian coordinates of \mathbb{R}^4 . Since $\int dx = 2\pi$, $\int dx_a dy_a = \pi \int d(r_a^2)$, (3.52) becomes

$$-\frac{64\pi^3 \mu \epsilon_1 \epsilon_2}{\beta^2} \int_0^\infty \frac{d(r_1^2) d(r_2^2)}{\left(1 + \mu^2 + \frac{4\epsilon_a^2 r_a^2}{\beta^2}\right)^3} = -\frac{4\pi^3 \mu \beta^2}{\epsilon_1 \epsilon_2} \int_0^\infty \frac{dX dY}{(1 + \mu^2 + X + Y)^3} = -\frac{2\pi^3 \mu \beta^2}{(1 + \mu^2) \epsilon_1 \epsilon_2}, \tag{3.53}$$

where $X = \frac{4\epsilon_1^2 r_1^2}{\beta^2}$, $Y = \frac{4\epsilon_2^2 r_2^2}{\beta^2}$. So one obtains

$$\frac{iN^3(A_6)^4 r_1}{96\pi^2} \int a \wedge da \wedge da = -i \frac{N^3 \beta}{3 \cdot 2^7 \pi^3} \cdot \frac{16m^4}{\beta^4} \cdot \frac{2\pi^3 \mu \beta^2}{(1 + \mu^2)\epsilon_1 \epsilon_2} = -i \frac{N^3 m^4 \mu}{12\epsilon_1 \epsilon_2 \beta (1 + \mu^2)}, \quad (3.54)$$

where we plugged in $r_1 = \frac{\beta}{4\pi}$. This precisely agrees with (3.40), based on D0-D4 calculus.

Finally, let us comment that the same calculation can be done to test some part of (3.25) for all ADE theories. For ADE, (3.25) yields the imaginary part

$$\text{Im}(S_{\text{eff}}) \Big|_{m^4} = -\frac{\mu(c_2|G| + r)m^4}{12\epsilon_1 \epsilon_2 \beta (1 + \mu^2)}, \quad (3.55)$$

simply by changing the coefficient $N^3 \rightarrow c_2|G| + r$ from (3.25). On the other hand, the anomaly polynomial (3.33) is replaced by the following polynomial

$$I_8 = rI_8(1) + c_2|G| \frac{p_2(N)}{24} \quad (3.56)$$

for ADE. Again after restricting $\text{SO}(5)_R$ to $\text{U}(1) \subset \text{SU}(2)_L$, the term $\frac{N^3}{24}F^4$ of (3.37) is replaced by $\frac{c_2|G|+r}{24}F^4$. So the calculations of this subsection can be done by replacing all N^3 by $c_2|G| + r$, completely reproducing (3.55).

4 Conclusions and remarks

In this paper, we explored S-duality of the prepotential of the 6d (2, 0) theories compactified on T^2 , on the Coulomb branch. We found evidences of S-duality and its anomaly. Using this result, we computed the asymptotic free energy of this system compactified on S^1 (in the index version), when the Omega background parameters $\epsilon_{1,2}$ and the chemical potential β conjugate to the KK momentum are small. The asymptotic free energy is proportional to N^3 in a suitable large N limit, showing that the light KK fields exhibit the N^3 degrees of freedom. After suitably complexifying the chemical potentials, we showed that the imaginary part of the free energy proportional to N^3 is completely reproduced from the 6d chiral anomaly of the $\text{SO}(5)$ R-symmetry. Most results are generalized to the ADE class of (2, 0) theories.

In the literatures, the N^3 scalings of various observables of 6d (2, 0) theory have been found, using various approaches. Thermal entropy of black M5-branes [33] or various other quantities are computed from the gravity dual. Chiral anomalies are computed from the anomaly inflow mechanism [34]. The supersymmetric Casimir energy on S^5 was computed from the superconformal index [35–40]. Perhaps among these, the mysteries of 6d CFT may be most directly addressed from the thermal partition function calculus of [33]. So it would be desirable to have a microscopic view of this phenomenon by directly counting states of the 6d CFTs. As far as we are aware of, such a direct account for N^3 scaling of states has not been available from a microscopic quantum calculus yet. Our studies show the N^3 scalings of the microscopically counted degrees of freedom. More precisely, we compactified the 6d SCFT on S^1 , so N^3 degrees of freedom are absent at low energy.

However, at high temperature compared to the inverse-radius of the circle, we expect the 6d CFT physics to be visible, hopefully in our F . One subtlety is that fermionic states are counted with minus sign in the index, so there may be cancelation between bosons and fermions. Even after this possible cancelation, we find that the uncanceled free energy still exhibits N^3 scaling, which proves that the 6d CFT has N^3 degrees of freedom. We have provided an alternative study of the asymptotic free energy based on 6d chiral anomalies, which completely agrees with our D0-D4 calculus.

Our studies based on D0-D4 system also shows that the light D0-brane particles are responsible for the UV enhancement of degrees of freedom. Since D0-branes are the key objects which construct M-theory at strong coupling limit of the type IIA strings, it is natural to see that they are also responsible for the N^3 degrees of freedom of the 6d (2, 0) theory. It will be interesting to better understand the single particle index $f(\tau, \epsilon_{1,2}, m, v)$ which yields this behavior. In particular, conjectures on instanton partons [29] may be addressed in more detail.

The Coulomb branch partition function on $\mathbb{R}^4 \times T^2$ was used as building blocks of interesting CFT indices in the symmetric phase. We comment that our asymptotic free energy proportional to N^3 does not appear in these symmetric phase indices. Let us explain this with the 6d superconformal index, and the DLCQ index.

Firstly, it has been proposed that the D0-D4 partition function, or more precisely this partition function multiplied by the 5d perturbative part, is a building block for the 6d superconformal indices [35–38] on $S^5 \times S^1$. So one might wonder whether our finding $\log Z \propto \frac{N^3 m^4}{\epsilon_1 \epsilon_2 \beta}$ (with $\tau_D = \frac{i\beta}{2\pi}$) at high temperature has implications to the superconformal index. One can immediately see that the answer is negative. For this discussion, the relevant formula is presented in [38], which uses the product of 3 copies of Coulomb branch partition functions on $\mathbb{R}^4 \times T^2$ as the integrand. The angular momentum chemical potentials of $U(1)^2 \subset SO(6)$ on S^5 are labeled by three numbers a_1, a_2, a_3 satisfying $a_1 + a_2 + a_3 = 0$. In this setting, the 3 sets of Omega deformation parameters are given by $(\epsilon_1, \epsilon_2) = (a_2 - a_1, a_3 - a_1), (a_3 - a_2, a_1 - a_2), (a_1 - a_3, a_2 - a_3)$ respectively. Since the asymptotic formula for Z is obtained in the limit of small ϵ_1, ϵ_2 , one can study the superconformal index in the limit of small a_1, a_2, a_3 . In this limit, the most divergent part in $\epsilon_{1,2}$ is given by

$$\log Z_{S^5 \times S^1} \sim \frac{N^3 m^4}{\beta} \left[\frac{1}{(a_2 - a_1)(a_3 - a_1)} + \frac{1}{(a_3 - a_2)(a_1 - a_2)} + \frac{1}{(a_1 - a_3)(a_2 - a_3)} \right]. \quad (4.1)$$

It is an identity that the sum in the square bracket vanishes, so that the leading asymptotic part proportional to N^3 vanishes on $S^5 \times S^1$. So our f_{asymp} has no implication to the superconformal index. However, study of the subleading part $\mathcal{O}(\epsilon_{1,2})^0$ will be interesting, along the lines of our section 2.2. We hope to come back to this problem in the near future.

Secondly, the M5-brane theory compactified on a lightlike circle can be studied using the D0-D4 quantum mechanics [41, 42]. Its index at DLCQ momentum k can be computed by integrating the D0-D4 index in the Coulomb branch suitably with the Coulomb VEV v , as explained in [7]. So one finds (again with $\tau_D = \frac{i\beta}{2\pi} \rightarrow 0$)

$$Z_{\text{DLCQ}} \sim \exp \left[-\frac{N^3 m^4}{24 \epsilon_1 \epsilon_2 \beta} \right]. \quad (4.2)$$

Here, unlike the partition function on $\mathbb{R}^4 \times T^2$, where we have notion of multi-particles so that $\log Z$ itself is meaningful as the single particle index, the DLCQ index is defined with a confining harmonic potential on \mathbb{R}^4 [7]. Thus, the exponent cannot be physically meaningful separately. Also, the definition of Z_{DLCQ} is such that $\epsilon_+ = \frac{\epsilon_1 + \epsilon_2}{2}$ has to be real and bigger than other fugacities, as $e^{-\epsilon_+} < 1$ plays the role of main convergence parameter. So one has to set $\epsilon_1 \epsilon_2 > 0$. This implies that Z_{DLCQ} does not exhibit exponential growth, but is rather highly suppressed at small β , presumably due to boson/fermion cancelation.

From these observations on the superconformal index and the DLCQ index, one realizes that $Z_{\mathbb{R}^4 \times T^2}$ contains interesting dynamical information which may be wiped out in other observables.

Omega deformed partition functions can also be used to study 6d (1, 0) superconformal field theories. In fact, for many 6d (1, 0) systems, the index on $\mathbb{R}^4 \times T^2$ is known in the ‘self-dual string expansion,’ similar to the M-string expansion explained in our section 2.2. The coefficients like $Z_{(n_i)}$ of section 2.2 are elliptic genera of 2d CFTs for the 6d self-dual strings in the tensor branch. Those elliptic genera have been studied for various 6d (1, 0) theories [43–48]. The S-duality anomaly and the high temperature asymptotic free energies could be studied using the approaches explored in this paper. This may be an interesting approach to explore the rich physics of 6d CFTs and their compactifications to 5d/4d.

It would also be interesting to further study the S-duality of the full index of the (2, 0) theory, based on some ideas sketched in our section 2.2. Following [26], we find it interesting to study the Wilson/’t Hooft line defects uplifted to 6d surface operators. S-dualities of other defect operators should also be interesting.

Finally, one may ask if a suitable M2-brane partition function on $\mathbb{R}_\epsilon^2 \times S^1$ can exhibit $N^{\frac{3}{2}}$ scaling, where ϵ is the Omega deformation parameter. Although this scaling has been microscopically computed from the S^3 partition function, or the entanglement entropy, perhaps better physical intuitions can be obtained by directly accounting for where such degrees of freedom come from, like we did for 6d SCFTs on S^1 from D0-branes (instanton solitons).

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