

The Kerr-Schild double copy in curved spacetime

Nadia Bahjat-Abbas,^a Andrés Luna^b and Chris D. White^a

^aCentre for Research in String Theory, School of Physics and Astronomy,
Queen Mary University of London, 327 Mile End Road, London E1 4NS, U.K.

^bSchool of Physics and Astronomy, University of Glasgow,
Glasgow G12 8QQ, Scotland, U.K.

E-mail: n.bahjat-abbas@qmul.ac.uk, a.luna-godoy.1@research.gla.ac.uk,
christopher.white@qmul.ac.uk

ABSTRACT: The double copy is a much-studied relationship between scattering amplitudes in gauge and gravity theories, that has subsequently been extended to classical field solutions. In nearly all previous examples, the graviton field is defined around Minkowski space. Recently, it has been suggested that one may set up a double copy for gravitons defined around a non-trivial background. We investigate this idea from the point of view of the classical double copy. First, we use Kerr-Schild spacetimes to construct graviton solutions in curved space, as double copies of gauge fields on non-zero gauge backgrounds. Next, we find that we can reinterpret such cases in terms of a graviton on a non-Minkowski background, whose single copy is a gauge field in the same background spacetime. The latter type of double copy persists even when the background is not of Kerr-Schild form, and we provide examples involving conformally flat metrics. Our results will be useful in extending the remit of the double copy, including to possible cosmological applications.

KEYWORDS: Gauge-gravity correspondence, Scattering Amplitudes

ARXIV EPRINT: [1710.01953](https://arxiv.org/abs/1710.01953)

Contents

1	Introduction	1
2	The classical double copy	3
3	Kerr-Schild solutions in curved space	7
3.1	Single Kerr-Schild solutions	7
3.2	Multiple Kerr-Schild solutions	10
4	Conformally flat background metrics	15
5	Discussion	17
A	The type B single copy in d dimensions	19

1 Introduction

The structure of gauge and gravity theories, and the relationships between them, continue to be the subjects of active research. An important example of such a relationship is the *double copy* [1–3], which states that perturbative scattering amplitudes in gravity theories both with and without supersymmetry can be obtained from their counterparts in a non-abelian gauge theory by exchanging the couplings, and also replacing colour information with kinematic information in a prescribed way. This relies on a certain colour-kinematics interplay — *BCJ duality* — being possible in the gauge theory [1]. Both BCJ duality and the double copy are proven at tree-level [3–11], where the latter is equivalent to the known KLT relations [12] between gauge and gravity amplitudes, that arise from string theory. At loop level these properties remain conjectural, although highly non-trivial evidence exists at multiloop level in a variety of theories [2, 13–34]. All-order evidence is possible in certain special kinematic limits [35–42], and other related studies can be found in [43–67].

It remains unclear whether or not the double copy is an accident of perturbative scattering amplitudes, or represents a much deeper relationship between gauge, gravity and related theories. A number of recent studies have therefore looked at whether or not other quantities can be matched up and, if so, whether the relevant relationship is related to the double copy for amplitudes. Reference [68] considered a family of exact classical solutions in gravity, *stationary Kerr-Schild metrics*, and found that these could indeed be single-copied to Yang-Mills theory. Well-known gravitational objects such as the Schwarzschild and Kerr black holes emerge as special cases. The Kerr-Schild framework can also be generalised to include (time-dependent) plane waves, and known properties of amplitudes in the self-dual sector of Yang-Mills theory and gravity [10]. The case of an

arbitrarily accelerating particle was considered in ref. [69]. This also has a Kerr-Schild form, which has the effect of forcing the radiation to appear as a source term on the right-hand side of the Yang-Mills and Einstein equations. This source term could be related to known amplitudes for the Bremsstrahlung of photon and graviton radiation, thus more tightly establishing the link between the classical and amplitude double copies.

Reference [70] went beyond the simple Kerr-Schild form used in ref. [10], in considering the Taub-NUT solution. This has a double Kerr-Schild form, yet nevertheless can be single copied to a gauge theory dyon, whose magnetic monopole charge maps to the NUT charge in the gravity theory. Furthermore, ref. [70] already hinted at the possibility of constructing a double copy around a non-trivial background metric (namely the de Sitter metric), and we will return to this in what follows. Further work relating to the classical double copy has investigated whether or not the source terms in the field equations are physically meaningful [71], and whether the copy can be extended to solutions involving inverse powers of the coupling [72, 73]. An alternative body of work has focused on constructing gravity fields from convolutions of gauge fields, in a variety of theories [59, 74–78]. The double copy has also been applied to classical solutions generated order-by-order in perturbation theory [79–81].

Recently, ref. [82] considered generalising the double copy for amplitudes to include a non-trivial background metric in the gravity theory. Let us define the graviton $h_{\mu\nu}$ according to

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}, \quad (1.1)$$

where $g_{\mu\nu}$ is the full metric, and $\kappa = \sqrt{32\pi G_N}$, with G_N Newton's constant. In the standard BCJ double copy for amplitudes [1–3], one identifies the background metric $\bar{g}_{\mu\nu}$ with the Minkowski metric $\eta_{\mu\nu}$, so that zero graviton corresponds to the complete absence of gravity. One may instead consider a different choice, and indeed there may be good practical reasons for doing so: in a variety of astrophysical and / or cosmological applications, one must analyse perturbations around a non-trivial background metric. Reference [82] considered the particular case of so-called *sandwich plane waves*, namely plane wave solutions whose deviation from Minkowski space has a finite extent in space and time.¹ One may consider such waves in either gauge theory or gravity, and the authors demonstrate explicitly that a three-point amplitude for a graviton defined as the deviation from a gravitational sandwich wave can be obtained as the double copy of a gauge theory three-point function, where the gauge field is defined around a gauge theory sandwich wave. They further note that this procedure is obtainable from ambitwistor string theory [83] (see also [84, 85]), which would in principle provide a general framework for formulating a similar procedure for different types of background.

Given the previously observed links between the Minkowski space amplitude double copy of refs. [1–3] and the classical double copy of refs. [68–70], the results of ref. [82] suggest that some analogue of the curved space amplitude double copy should also be possible for classical solutions. The aim of this paper is to study this issue, and we will

¹More precisely, such waves are confined to a finite region of the lightcone coordinate $u = z - t$, for a wave travelling in the $+z$ direction.

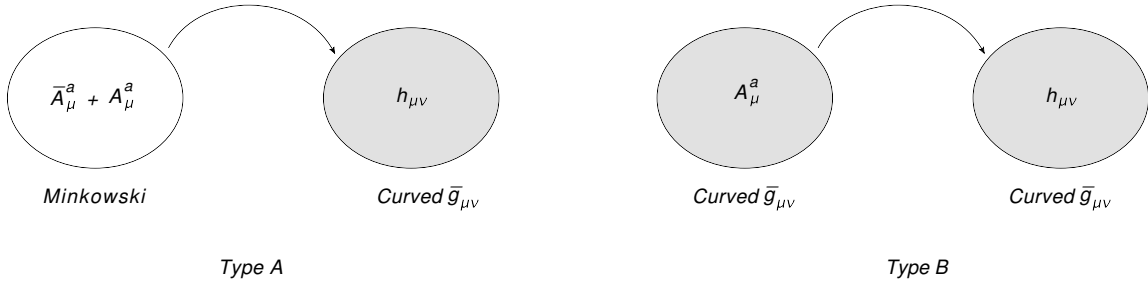


Figure 1. Two possible interpretations of a double copy in curved space: in *type A*, a gauge field has a non-trivial background field \bar{A}_μ^a in Minkowski space, and copies to a graviton defined on a curved background $\bar{g}_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ and \bar{A}_μ^a are themselves related by a double copy relationship. In *type B*, a gauge field on a non-dynamical curved background $\bar{g}_{\mu\nu}$ double copies to a graviton defined around the same background.

present a number of examples. Firstly, we will construct Kerr-Schild solutions on a curved background by trivially rewriting single Kerr-Schild solutions. We will be able to interpret such solutions as double copies of gauge fields with non-trivial backgrounds, and we will call this relationship a *type A* curved space double copy. We will also find an alternative interpretation, namely that one may regard the graviton as being the double copy of a gauge field living on a non-dynamical curved spacetime background, which we will refer to as *type B*. The difference between these two double copies is shown schematically in figure 1, and the second of these is perhaps at odds with what one normally means by the double copy, which relates entire gravity solutions to gauge theory counterparts in flat space. It is then presumably the case that the type B map is not fully general, but exists only in special cases. That does not however, reduce its usefulness, where it applies.

After examining simple Kerr-Schild examples, we will generalise our findings to multiple Kerr-Schild solutions, including a reexamination of the Taub-NUT spacetime considered in ref. [70]. Finally, we will show a family of non-trivial examples of the type B double copy map, in which the background spacetime is conformally flat, without a Kerr-Schild form. This illustrates that this second type of double copy map may be more applicable than naïvely thought, and can also provide a double copy in cases in which it is not known how to construct a double copy of type A.

The structure of the paper is as follows. In section 2, we briefly review relevant details regarding the classical double copy that will be needed in what follows. In section 3, we study Kerr-Schild solutions from the viewpoint of a curved space double copy. In section 4, we consider the example of Kerr-Schild solutions built upon conformally flat background metrics. Finally, in section 5, we discuss our results and conclude. Technical details are contained in the appendix.

2 The classical double copy

Here, we briefly review the double copy for classical solutions of refs. [68–70]. Given that this will be the focus of our paper, we will not discuss in detail the corresponding story

for amplitudes (see e.g. [86–88] for pedagogical reviews). Our starting point is to consider *Kerr-Schild* metrics, in which the graviton, defined in eq. (1.1), is given by

$$h_{\mu\nu} = \frac{\kappa}{2} \phi k_\mu k_\nu. \quad (2.1)$$

Here $\phi(x^\mu)$ is a scalar function, and k_μ is a 4-vector that is null and geodesic both with respect to the background and the whole metric. That is

$$g^{\mu\nu} k_\mu k_\nu = \bar{g}^{\mu\nu} k_\mu k_\nu = 0, \quad k^\mu D_\mu k_\nu = 0, \quad (2.2)$$

where D_μ is the covariant derivative associated with the background metric, i.e. $D_\rho(\bar{g}_{\mu\nu}) = 0$. The null property implies that the index of the Kerr-Schild vector k_μ can be raised with either $\bar{g}^{\mu\nu}$ or $g^{\mu\nu}$, where the inverse metric takes the simple form

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \frac{\kappa^2}{2} k^\mu k^\nu \phi. \quad (2.3)$$

Upon substituting the ansatz of eq. (1.1) into the Einstein equations, one finds a linear form for the Ricci tensor with the particular index placement shown below:

$$R^\mu{}_\nu = \bar{R}^\mu{}_\nu - \kappa h^\mu{}_\rho \bar{R}^\rho{}_\nu + \frac{\kappa}{2} D_\rho (D_\nu h^{\mu\rho} + D^\mu h^\rho{}_\nu - D^\rho h^\mu{}_\nu), \quad (2.4)$$

where $\bar{R}_{\mu\nu}$ is the Ricci tensor associated with $\bar{g}_{\mu\nu}$. Thus, if one finds a field ϕ and Kerr-Schild vector k_μ such that the Einstein equations (vacuum or otherwise) are solved, this constitutes an exact solution i.e. the graviton receives no higher order corrections.

Equation (2.4) simplifies in the case that the background space is taken to be Minkowski

$$R^\mu{}_\nu \xrightarrow{\bar{g}_{\mu\nu} \rightarrow \eta_{\mu\nu}} \frac{\kappa}{2} \partial_\rho (\partial_\nu h^{\mu\rho} + \partial^\mu h^\rho{}_\nu - \partial^\rho h^\mu{}_\nu). \quad (2.5)$$

For a given field ϕ and Kerr-Schild vector k^μ , one may define a (non-abelian) gauge field according to

$$A_\mu^a = c^a \phi k_\mu, \quad (2.6)$$

where c^a is an arbitrary constant colour vector. For all stationary Kerr-Schild metrics (i.e. those not depending explicitly on time), this gauge field solves the linearised Yang-Mills equations:

$$\partial^\mu F_{\mu\nu}^a = 0, \quad (2.7)$$

where $F_{\mu\nu}^a$ is the abelian-like field strength tensor²

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a. \quad (2.8)$$

To see this, note that eqs. (2.1), (2.5) imply

$$R^0{}_i \xrightarrow{\bar{g}_{\mu\nu} \rightarrow \eta_{\mu\nu}} \frac{\kappa}{2} \partial_\rho [\partial_i(\phi k^0 k^\rho) - \partial^\rho(\phi k^0 k_i)] \quad (2.9)$$

²The non-linear term that is usually present in the non-abelian field strength vanishes upon substituting the ansatz of eq. (2.6).

where we use latin indices to denote spatial components. Without loss of generality, one may choose a coordinate system such that $k^0 = 1$, such that the result of eq. (2.7) follows. As in gravity, the linearisation of the field equations is exact, so that the gauge field receives no higher order corrections.

The gauge field of eq. (2.6) is referred to as the *single copy* of its corresponding Kerr-Schild graviton $h_{\mu\nu}$, by analogy with the BCJ double copy for amplitudes [1–3]. In fact, the two double copies are related to each other. One way to see this is take the *zeroth copy* of eq. (2.6), which involves replacing the remaining copy of the vector k_μ with a second colour vector:

$$\Phi^{aa'} = c^a \tilde{c}^{a'} \phi. \tag{2.10}$$

This is a solution of a *biadjoint scalar* field theory, whose equation is

$$\partial^2 \Phi^{aa'} - y f^{abc} \tilde{f}^{a'b'c'} \Phi^{bb'} \Phi^{cc'} = 0, \tag{2.11}$$

where f^{abc} and $\tilde{f}^{a'b'c'}$ are structure constants associated with two (potentially different) Lie groups. Indeed, eq. (2.10) is such that the nonlinear term in eq. (2.11) vanishes, leaving the simpler equation

$$\partial^2 \Phi^{aa'} = 0. \tag{2.12}$$

When sources are present on the right-hand side, we can then interpret ϕ as a scalar propagator, integrated over the source distribution. The fact that one does not modify the function ϕ upon taking the single or zeroth copies of eqs. (2.1), (2.6) is similar to the fact that denominators of amplitudes (themselves interpretable as scalar propagators) remain the same in biadjoint scalar, gauge and gravity theories. As mentioned above, stronger evidence for the connection between the classical and amplitude double copies comes from considering the accelerated particle, where the Kerr-Schild description recovers known amplitudes for the emission of soft photons and gravitons [69].

It is not known how to fully generalise the classical double copy to exact solutions which are not of Kerr-Schild form. One way to make progress is to construct solutions order-by-order in a perturbation expansion in κ , as explored in refs. [79–81]. One may also explore known generalisations of the Kerr-Schild ansatz. An example is the *double Kerr-Schild* ansatz, in which the graviton has the form

$$h_{\mu\nu} = \frac{\kappa}{2} \left[\phi_1 k_\mu k_\nu + \phi_2 l_\mu l_\nu \right] \tag{2.13}$$

There are now two scalar fields ϕ_i , and two separate Kerr-Schild vectors k^μ and l^μ , each of which satisfies the null and geodesic requirements of eq. (2.2), as well as the mutual orthogonality condition

$$g^{\mu\nu} k_\mu l_\nu = \bar{g}^{\mu\nu} k_\mu l_\nu = 0. \tag{2.14}$$

Unlike in the single Kerr-Schild case, this ansatz is not guaranteed to linearise the Einstein equations: one finds a correction term to eq. (2.4), whose explicit form (in the present notation) may be found in ref. [70]. A special case where linearity indeed occurs is the Taub-NUT solution [89, 90], whose Kerr-Schild form was first presented in ref. [91]. The two terms in eq. (2.13) contain, respectively, a Schwarzschild-like point mass M at the

origin, and a *NUT charge* N , where the latter gives rise to a rotational character in the gravitational field at spatial infinity. Reference [70] single copied this solution by analogy with eq. (2.6):

$$A_\mu^a = c^a \left[\phi_1 k_\mu + \phi_2 l_\mu \right]. \tag{2.15}$$

Note that the single copy is taken term-by-term, analogous to the BCJ double copy for amplitudes. The gauge theory solution was found to be a dyon. The electric charge in the gauge theory maps to the Schwarzschild mass, as it must do for consistency with the pure Schwarzschild case. The magnetic monopole charge of the gauge theory solution maps to the NUT charge, thus making precise the statement that the Taub-NUT solution can be thought of as magnetic-monopole-like.

Of particular interest for the present study is the fact that ref. [70] considered the Taub-NUT solution on a de Sitter background, as well as Minkowski space. The corresponding gauge field was then found to satisfy the *curved space* Maxwell equations

$$D^\mu F_{\mu\nu}^a = 0, \tag{2.16}$$

where D^μ is the covariant derivative associated with de Sitter space, and

$$F_{\mu\nu}^a \equiv D_\mu A_\nu^a - D_\nu A_\mu^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a \tag{2.17}$$

is the curved space field strength tensor.³ This is already an example of the type B double copy illustrated in figure 1, in which a graviton defined around a non-Minkowski background is identified with a gauge field living on the same (non-dynamical) background. Given the fact that this map is not what one ordinarily associates with the double copy, it is not clear what, if at all, the zeroth copy of the gauge field corresponds to. For the Taub-NUT example of ref. [70], the biadjoint field

$$\Phi^{aa'} = c^a \tilde{c}^{a'} (\phi_1 + \phi_2) \tag{2.18}$$

was found to satisfy the equation

$$D^2 \Phi^{aa'} \equiv D^\mu D_\mu \Phi^{aa'} = -2\lambda \Phi^{aa'}, \tag{2.19}$$

where λ is the cosmological constant. It was speculated that this was a solution of the biadjoint theory conformally coupled to gravity, with Lagrangian

$$\mathcal{L} = \frac{1}{2} (D^\mu \Phi^{aa'}) (D_\mu \Phi^{aa'}) - \frac{y}{6} f^{abc} \tilde{f}^{a'b'c'} \Phi^{aa'} \Phi^{bb'} \Phi^{cc'} - \frac{\mathcal{R}}{12} \Phi^{aa'} \Phi^{aa'}, \tag{2.20}$$

where \mathcal{R} is the Ricci scalar, a property making use of the fact that $\mathcal{R} \propto \lambda$ for de Sitter spacetime. The constant of proportionality is precisely such as to make eq. (2.19) follow from eq. (2.20), in four spacetime dimensions.

Having reviewed all necessary details regarding the Kerr-Schild double copy, we now turn to the investigation of other curved space examples.

³The second equality in eq. (2.17) follows from the fact that terms involving the Christoffel symbol vanish upon forming the antisymmetric combination of covariant derivatives.

3 Kerr-Schild solutions in curved space

As stated in the introduction, our examination of curved space instances of the classical double copy is motivated by the results of ref. [82], concerning a double copy of type A. This associates a gauge theory amplitude in the presence of a non-trivial background field, with a gravity amplitude defined with respect to a non-Minkowski background metric, where the gauge and gravity background fields should be related. In this section, we will see that Kerr-Schild solutions indeed provide a natural framework for constructing such double copies for exact field solutions, rather than perturbative amplitudes.

3.1 Single Kerr-Schild solutions

The simplest such examples can be constructed, albeit rather artificially, by starting with single Kerr-Schild metrics around Minkowski space. We may split up such solutions according to

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + \phi k_\mu k_\nu \\ &= \eta_{\mu\nu} + \phi_1 k_\mu k_\nu + \phi_2 k_\mu k_\nu, \end{aligned} \tag{3.1}$$

where we have introduced

$$\phi_1 = \xi\phi, \quad \phi_2 = (1 - \xi)\phi, \quad 0 \leq \xi \leq 1. \tag{3.2}$$

Thus, as is well-known (see e.g. [92]), any given single Kerr-Schild metric can always be thought of as a double Kerr-Schild metric. Following the discussion of section 2, it is straightforward to single copy eq. (3.1) term-by-term, resulting in the gauge field

$$A_\mu^a = c^a \left[\phi_1 k_\mu + \phi_2 k_\mu \right]. \tag{3.3}$$

This is itself a rewriting of eq. (2.6), that is ultimately possible due to the linearity of the field equations in the Kerr-Schild double copy. However, we can reinterpret eqs. (3.1), (3.3) as follows. By defining

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \phi_1 k_\mu k_\nu, \tag{3.4}$$

we may rewrite eq. (3.1) as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \tilde{h}_{\mu\nu}, \quad \tilde{h}_{\mu\nu} = \phi_2 k_\mu k_\nu, \tag{3.5}$$

so that the solution of eq. (3.1) may be regarded as containing a graviton field involving only the field ϕ_2 , defined with respect to the non-Minkowski background $\bar{g}_{\mu\nu}$. On the gauge theory side, we can define

$$\bar{A}_\mu^a = c^a \phi_1 k_\mu, \tag{3.6}$$

so that the solution of eq. (3.3) becomes

$$A_\mu^a = \bar{A}_\mu^a + \tilde{A}_\mu^a, \quad \tilde{A}_\mu^a = c^a \phi_2 k_\mu. \tag{3.7}$$

This is thus our first example of a type A curved space double copy, for classical solutions rather than amplitudes. A gauge field defined with respect to a non-trivial background

field copies to a graviton field with a non-trivial background, where the two backgrounds are themselves related (i.e. they are themselves Kerr-Schild, so we know how to double copy them).

As indicated in figure 1, there is another way to consider double copies in curved space (type B). Namely, it may be possible to single copy a graviton defined with respect to a non-Minkowski background, to a gauge field living on the same background. To this end, one may consider the graviton field $\tilde{h}_{\mu\nu}$ of eq. (3.5), which single copies to the field \tilde{A}_μ^a of eq. (3.7). On the gauge theory side, one may impose the same background $\bar{g}_{\mu\nu}$, and examine the curved space Maxwell equations

$$D^\mu \tilde{F}_{\mu\nu}^a = j_\nu, \tag{3.8}$$

where $\tilde{F}_{\mu\nu}^a$ is the field strength tensor formed from the gauge field \tilde{A}_μ^a . For a consistent double copy of type B, one requires that the source current is somehow related to the energy-momentum tensor in a recognisable way, so that the two solutions are related. Let us give two examples. Firstly, one may consider the Schwarzschild metric, for which

$$\phi = \frac{2M}{r}, \quad k^\mu = (1, 1, 0, 0), \tag{3.9}$$

where we adopt spherical polar coordinates (t, r, θ, ϕ) . Writing the graviton as

$$h_{\mu\nu} = \frac{2M_1}{r} k_\mu k_\nu + \frac{2M_2}{r} k_\mu k_\nu, \quad M_1 + M_2 = M, \tag{3.10}$$

we may define the background field

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}, \quad \bar{h}_{\mu\nu} = \frac{2M_1}{r} k_\mu k_\nu, \tag{3.11}$$

and then single copy the graviton

$$\tilde{h}_{\mu\nu} = \frac{2M_2}{r} k_\mu k_\nu \tag{3.12}$$

to get a gauge field

$$\tilde{A}_\mu^a = \frac{c^a}{r} k_\mu. \tag{3.13}$$

The curved space Maxwell equations of eq. (3.8) then yield⁴

$$j_\mu^a = 0, \tag{3.14}$$

which is indeed consistent: the Schwarzschild metric is a vacuum solution in General Relativity. Here we find that its curved space single copy is also a (gauge theory) vacuum solution, on the curved space defined by $\bar{g}_{\mu\nu}$.

A second example is given by de Sitter spacetime, which has the Kerr-Schild form

$$\phi = \lambda r^2, \quad k_\mu = (-1, 1, 0, 0), \tag{3.15}$$

⁴Here, we do not include the delta function source at the origin, corresponding to the point charge (mass) sourcing the field \tilde{A}_μ^a ($h_{\mu\nu}$).

where λ is the cosmological constant. Splitting this similarly to eq. (3.10) gives

$$h_{\mu\nu} = \lambda_1 r^2 k_\mu k_\nu + \lambda_2 r^2 k_\mu k_\nu, \quad \lambda_1 + \lambda_2 = \lambda. \quad (3.16)$$

We can then define the graviton

$$\tilde{h}_{\mu\nu} = \lambda_2 r^2 k_\mu k_\nu, \quad (3.17)$$

whose single copy gauge field

$$\tilde{A}_\mu^a = c^a \lambda_2 r^2 k_\mu \quad (3.18)$$

satisfies the curved space Maxwell equation with

$$j_\nu^a = (6\lambda_2, 0, 0, 0). \quad (3.19)$$

Again this makes sense: the graviton is sourced by a constant energy density filling all space which, in the gauge theory, becomes a constant charge density. The single copy has thus turned momentum degrees of freedom into colour degrees of freedom, precisely as in the flat space case examined in ref. [70] (and also the single copy for amplitudes [1–3]).

We have not been able to prove in general that the curved space Maxwell equations are satisfied for arbitrary single Kerr-Schild solutions that are rewritten in the form of eq. (3.5). However, we have at least shown for some special — and, indeed, astrophysically relevant — cases, a type B double copy map is possible. The question then arises of how general this map is. The conventional double copy, in its simplest form, relates a gauge theory to a gravity theory. A gauge theory on a curved background (even if this is non-dynamical) would appear to involve gravity, and thus this type of double copy map seems to relate a coupled Einstein-gauge theory system to itself. One does not then expect this map to be fully general, or to apply to supersymmetric generalisations that are known to work in flat space.

Evidence towards this viewpoint can be gleaned by examining the zeroth copy. As discussed in section 2, the Kerr-Schild field ϕ is found to satisfy the linearised biadjoint scalar field equation, and can be interpreted as a scalar propagator. In the type II double copy, we can take the zeroth copy of the gauge field \tilde{A}_μ^a to generate a scalar field

$$\tilde{\Phi}^{aa'} = c^a \tilde{c}^{a'} \phi_2, \quad (3.20)$$

and consider the curved space linearised biadjoint equation

$$D^\mu D_\mu \Phi^{aa'} = c^a \tilde{c}^{a'} \xi, \quad (3.21)$$

which defines ξ . For the Schwarzschild and de Sitter examples, we find

$$\xi_{\text{SWC}} = -\frac{4M_1 M_2}{r^4}, \quad \xi_{\text{dS}} = 6\lambda_2 - 10r^2 \lambda_1 \lambda_2 \quad (3.22)$$

respectively, which we can not straightforwardly interpret as being related to the source current in the gauge theory. It thus seems that the type B double copy can indeed associate a gauge theory solution in curved space with a gravity counterpart, at the expense of not having a consistent zeroth copy. This also sheds light on the speculation of ref. [70], that the zeroth copy for a curved background may result in a biadjoint scalar theory conformally coupled to gravity (eq. (2.20)). The results of eq. (3.22) provide a simple counter-example to this conjecture, showing that the situation is more complex than previously thought.

3.2 Multiple Kerr-Schild solutions

In the previous section, we used single Kerr-Schild solutions to provide some first examples of curved space double copies, of both type A and type B. Here, we study whether such conclusions also apply to more complicated solutions. As a first generalisation, we may consider multiple Kerr-Schild solutions in Minkowski space, namely those of form

$$g_{\mu\nu} = \eta_{\mu\nu} + \sum_i \phi_i k_\mu^{(i)} k_\nu^{(i)}, \quad (3.23)$$

where each vector $k_\mu^{(i)}$ is null and geodesic with respect to both the Minkowski and full metric, and the set of Kerr-Schild vectors obeys the mutual orthogonality relations

$$\eta^{\mu\nu} k_\mu^{(i)} k_\nu^{(j)} = g^{\mu\nu} k_\mu^{(i)} k_\nu^{(j)} = 0, \quad \forall i, j. \quad (3.24)$$

In certain cases, this ansatz linearises the mixed Ricci tensor R^μ_ν , and thus provides an exact solution of the Einstein equations.⁵ We further consider the general class of multi-Kerr-Schild solutions in which each term in the graviton is itself a solution of the linearised Einstein equations. In the stationary case, we may then single copy eq. (3.23) to produce a gauge field

$$A_\mu^a = c^a \sum_i \phi_i k_\mu^{(i)}. \quad (3.25)$$

Given that each term in the graviton constitutes a stationary Kerr-Schild solution, the results of ref. [68] immediately imply that each term in eq. (3.25) satisfies the linearised Yang-Mills equations. Linearity then implies that the complete field of eq. (3.25) is also a solution, and thus a well-defined single copy of the gravity result.

As for the solution of eq. (3.1), we can use any multi-Kerr-Schild solution of the form of eqs. (3.23), (3.25) to construct a type A curved space double copy. To do this, one may partition the terms in eq. (3.23) into two sets Γ_1 and Γ_2 , before defining

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \sum_{i \in \Gamma_1} \phi_i k_\mu^{(i)} k_\nu^{(i)}, \quad \bar{A}_\mu^a = c^a \sum_{i \in \Gamma_1} \phi_i k_\mu^{(i)}, \quad (3.26)$$

and

$$\tilde{h}_{\mu\nu} = \eta_{\mu\nu} + \sum_{i \in \Gamma_2} \phi_i k_\mu^{(i)} k_\nu^{(i)}, \quad \tilde{A}_\mu^a = c^a \sum_{i \in \Gamma_2} \phi_i k_\mu^{(i)}. \quad (3.27)$$

The full gravity and gauge fields may now be written as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \tilde{h}_{\mu\nu}, \quad A_\mu^a = \bar{A}_\mu^a + \tilde{A}_\mu^a. \quad (3.28)$$

This is indeed an example of the type A double copy shown in figure 1: the gauge field \tilde{A}_μ^a defined with respect to the background field \bar{A}_μ^a double copies to the graviton $\tilde{h}_{\mu\nu}$, defined with respect to the background metric $\bar{g}_{\mu\nu}$.

⁵Examples involving more than two terms are the higher dimensional Taub-NUT-like solutions of refs. [93, 94].

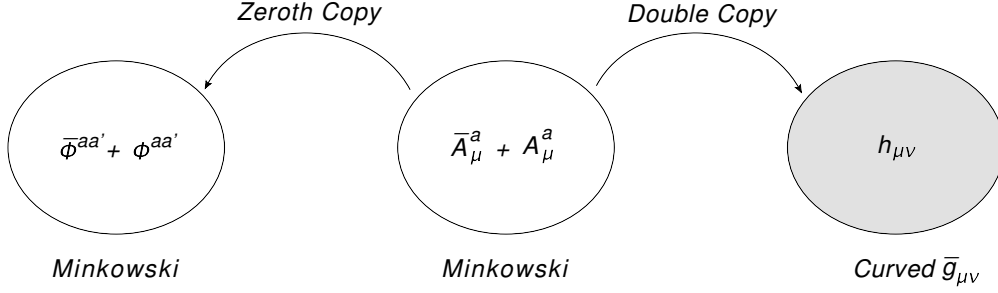


Figure 2. Generalisation of the type A double copy of figure 1 to include the zeroth copy, which relates the gauge field defined with a non-trivial background to similar solutions in a biadjoint scalar theory.

Furthermore, the zeroth copy is also well-defined, as for the flat space classical double copy: from eq. (3.25), we may define the biadjoint field

$$\Phi^{aa'} = c^a \tilde{c}^{a'} \sum_i \phi_i. \quad (3.29)$$

The fact that each term in the gauge field satisfies the linearised Yang-Mills equations implies, again from ref. [68], that each term in eq. (3.29) satisfies the linearised biadjoint scalar equation. Similarly to eq. (3.26), we may then define

$$\bar{\Phi}^{aa'} = c^a \tilde{c}^{a'} \sum_{i \in \Gamma_1} \phi_i, \quad \tilde{\Phi}^{aa'} = c^a \tilde{c}^{a'} \sum_{i \in \Gamma_2} \phi_i, \quad (3.30)$$

so that the full biadjoint field can be written

$$\Phi^{aa'} = \bar{\Phi}^{aa'} + \tilde{\Phi}^{aa'}. \quad (3.31)$$

This is a direct analogue of the type A double copy between gauge theory and gravity: a classical field defined with respect to a background copies between biadjoint scalar and gauge theory. The relationship between the three theories is shown in figure 2. Given that we will always be talking about solutions of the linearised Yang-Mills equations from now on, we will omit colour indices and vectors in what follows.

We may also examine whether or not it is possible to construct a type B double copy for multi-Kerr-Schild solutions, by considering specific examples. In section 3.1, we saw that this was indeed possible for the Schwarzschild and de Sitter solutions, split according to eqs. (3.1), (3.2). More generally, we can take either of these gravitons as part of the background metric $\bar{g}_{\mu\nu}$, and allow either of them to be the perturbation $\tilde{h}_{\mu\nu}$. The full list of possibilities is enumerated in table 1, where the full metric is given by eq. (3.1), with $k^\mu = (1, 1, 0, 0)$ in spherical polar coordinates. The first two rows contain the pure Schwarzschild (SWC) and de Sitter (dS) metrics, and the third / fourth rows the cases already considered in the previous section. Finally, the fifth and sixth rows contain the metric formed by perturbing the Schwarzschild solution with a de Sitter Kerr-Schild graviton, and vice versa. For each metric, we give an expression for the timelike component j^t of the source current

Metric	ϕ_1	ϕ_2	j^t	ξ
SWC	0	$2m_2/r$	0	0
dS	0	$\lambda_2 r^2$	$6\lambda_2$	$6\lambda_2$
SWC+SWC	$2m_1/r$	$2m_2/r$	0	$-4m_1m_2/r^4$
dS+dS	$\lambda_1 r^2$	$\lambda_2 r^2$	$6\lambda_2$	$6\lambda_2 - 10r^2\lambda_1\lambda_2$
SWC+dS	$2m_1/r$	$\lambda_2 r^2$	$6\lambda_2$	$6\lambda_2 - 8m_1\lambda_2/r$
dS+SWC	$\lambda_1 r^2$	$2m_2/r$	0	$4\lambda_1m_2/r$

Table 1. Table of type B single and zeroth copies of Kerr-Schild metrics of the form of eq. (3.1), where ϕ_1 and ϕ_2 are allowed to be different. Here A+B denotes a Kerr-Schild graviton for metric B considered as a perturbation on background metric A, where SWC and dS represent the Schwarzschild and de Sitter gravitons respectively.

that appears in the curved space Maxwell equation of eq. (3.8) (the spacelike components are found to vanish in all cases), as well as the quantity ξ that appears on the right-hand side of the curved space linearised biadjoint equation (eq. (3.21)).

In all cases, the type B single copy indeed holds. That is, the gauge theory contains a source current consistent with the perturbation term in the gauge field: zero in the Schwarzschild case,⁶ and a uniform charge density in the de Sitter case, whose counterpart in gravity is the cosmological constant. There are no terms in the source current which are sensitive to the field ϕ_1 , which would invalidate the picture of figure 1. The zeroth copy holds only in the cases of a pure single Kerr-Schild solution (i.e. the cases considered in the original classical double copy of refs. [68, 70]). For all of the double Kerr-Schild solutions, the source includes a position-dependent term that has no immediately evident counterpart in the gauge or gravity theory.

In the above examples, the full metric contains two Kerr-Schild terms, each of which has the same vector k^μ , corresponding to a spherically symmetric system. We can then ask what the most general results for j^t and ξ are, for unspecified functions $\phi_1(r)$ and $\phi_2(r)$. The results are

$$j^t = \frac{2\phi_2'(r)}{r} + \phi_2''(r) = \nabla_M^2 \phi_2, \quad \xi = \nabla^2 \phi_2 = j^t(1 - \phi_1(r)) - \phi_1'(r)\phi_2'(r). \quad (3.32)$$

Here ∇^2 is the Laplacian operator associated with the full background metric, and ∇_M^2 the corresponding operator in Minkowski space. We thus conclude that if ϕ_2 is associated with a vacuum solution in Minkowski space, the type II single copy is well-defined, in that it is also a vacuum solution. However, the source for the zeroth copy involves the background field ϕ_1 and thus does not seem to have a meaningful interpretation. Of course, the fields ϕ_1 and ϕ_2 in eq. (3.32) are not arbitrary, but must be fixed by the Einstein equations. For the case of spherically symmetric (and stationary) vacuum solutions up to the presence of a cosmological constant, the only possible solutions are the Schwarzschild and de Sitter cases examined already in table 1. Nevertheless, the general form of the current in eq. (3.32) does not rule out that there may be non-trivial solutions with extended sources, such that

⁶As earlier, we do not bother showing the delta function source at the origin.

one may still find a consistent single copy interpretation. It is not known even in the flat space case how to construct such maps (see e.g. refs. [69, 71] for discussions of source terms in various contexts).

Above we have discussed cases in which the background scalar field ϕ_1 is spherically symmetric. Our results are more general than this, however. We have explicitly checked that our conclusion that the type B single copy is a vacuum solution if ϕ_2 is associated with a vacuum solution in Minkowski space, holds true even if ϕ_1 has an arbitrary spatial and temporal dependence.

It is furthermore useful to note that, as in the flat space cases considered in ref. [68], one may transform the type B single copy gauge field into a more recognisable form. Starting with the gauge field in spherical polar coordinates,

$$A_\mu = \phi_2(-1, 1, 0, 0),$$

one may perform a gauge transformation

$$A_\mu \rightarrow A'_\mu = A_\mu + D_\mu\chi(r) = A_\mu + \partial_\mu\chi(r), \tag{3.33}$$

where

$$\chi(r) = - \int^r dr' \phi_2(r'), \tag{3.34}$$

so that eq. (3.33) implies

$$A'_\mu = (-\phi_2, 0, 0, 0). \tag{3.35}$$

Thus, ϕ_2 indeed has the interpretation of an electrostatic potential.

As implied above by the above results, the type B double copy is not necessarily expected to be a fully general map between exact solutions in gauge and gravity theories in curved space. However, it is interesting to examine whether or not it shares the property of the type A (and amplitude) double copies, in being independent of the number of spacetime dimensions d . Indeed, one may show that for a d -dimensional background metric $\bar{g}_{\mu\nu}$ of the form of eq. (3.4), the gauge field \tilde{A}_μ of eq. (3.7) satisfies the Maxwell equations, with a current density given by⁷

$$j^\mu = (\nabla_M^2 \phi_2, 0, 0, \dots, 0), \tag{3.36}$$

where the Minkowski-space Laplacian on the right-hand side is in $(d-1)$ space dimensions. Thus, our above discussion generalises for any d . We present a proof of these statements in appendix A.

Having examined multiple Kerr-Schild solutions where each term contains the same Kerr-Schild vector k^μ , it is instructive to instead consider an example in which these vectors can be different. One such example is the Taub-NUT solution, for which the metric takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu + \psi l_\mu l_\nu. \tag{3.37}$$

⁷Equation (3.36) also turns out to be true when the field ϕ_1 depends on time and the non-radial spatial coordinates.

The Minkowski line element can be written as

$$ds^2 = -dt^2 + \frac{\rho^2}{a^2 + r^2} dr^2 + \rho^2 d\theta^2 + (r^2 + a^2) \sin^2 \theta d\varphi^2 \quad (3.38)$$

in spheroidal coordinates, where

$$\rho^2 \equiv r^2 + a^2 \cos^2 \theta. \quad (3.39)$$

The vectors k_μ and l_μ are defined by

$$l_\mu dx^\mu = dt + \frac{\rho^2}{a^2 + r^2} dr - a \sin^2 \theta d\varphi \quad (3.40)$$

$$k_\mu dx^\mu = dt - \frac{i\rho^2}{a \sin \theta} d\theta + \frac{r^2 + a^2}{a} d\varphi, \quad (3.41)$$

while the scalar functions ϕ and ψ are given by

$$\psi = \frac{2mr}{\rho^2}, \quad \phi = \frac{2la \cos \theta}{\rho^2}. \quad (3.42)$$

As for the various metrics considered in table 1, in considering the type B single copy, we can take either of the Kerr-Schild terms to be part of the background metric, resulting in two possibilities:

$$\begin{aligned} \text{Case 1:} \quad & g_{\mu\nu} = \bar{g}_{\mu\nu} + \phi k_\mu k_\nu, & \bar{g}_{\mu\nu} &= \eta_{\mu\nu} + \psi l_\mu l_\nu, \\ \text{Case 2:} \quad & g_{\mu\nu} = \bar{g}_{\mu\nu} + \psi l_\mu l_\nu, & \bar{g}_{\mu\nu} &= \eta_{\mu\nu} + \phi k_\mu k_\nu. \end{aligned}$$

The gauge fields obtained from the single copy of the perturbation term in both cases satisfy homogeneous Maxwell equations

$$j^\nu = 0, \quad (3.43)$$

so that the single copy is indeed consistent (n.b. the Taub-NUT solution is a vacuum solution). The zeroth copy factor is given in both cases by

$$\xi = \frac{4\phi\psi(\rho^2 - 2r^2)}{\rho^4}, \quad (3.44)$$

so that, consistently with our previous results, the type B single copy does not appear to be meaningful.

Let us summarise the results of this section. We have examined whether it is possible to construct a double copy for classical solutions that mimics the result found for amplitudes in ref. [82]. We have indeed found such a procedure, based on the same Kerr-Schild solutions that were used to formulate a flat space double copy in refs. [68–70]. In this picture, a gauge field defined with respect to a non-trivial background field copies to a graviton defined with respect to a background metric, where the background fields in the two theories are related. We were able to relate the background fields due to the fact that they obeyed the original Kerr-Schild double copy by themselves. Furthermore, there is a well-defined zeroth copy,

in which the resulting biadjoint field is also defined with respect to a background, where the latter is the zeroth copy of the background gauge field. We call this procedure a *type A curved space double copy*, to distinguish it from an alternative procedure (*type B*) in which the gauge field lives on a non-dynamical curved spacetime, and copies to a graviton field defined with respect to the same spacetime. In this picture, the zeroth copy does not appear to be meaningful, in that the biadjoint field appears not to be physically related to its gauge theory counterpart, due to the presence of unwanted source terms.

In all of the above cases, we knew how to construct a type A double copy due to the fact that we could relate the background gauge field with its gravitational counterpart. The type B double copy, however, does not require such a relationship, as the same curved metric appears in both the gauge and gravity theories. It is then interesting to look for examples of this relationship in which the background metric is *not* of Kerr-Schild form, and thus cannot be single-copied according to the procedure of refs. [68–70]. We have indeed found such examples, which we describe in the following section.

4 Conformally flat background metrics

In this section, we consider conformally flat spacetimes. More specifically, we consider spacetimes whose metrics can be written (in some coordinate system) as a conformal transformation of Minkowski space:

$$\bar{g}_{\mu\nu} = \Omega^2(x^\mu)\eta_{\mu\nu}. \tag{4.1}$$

As the bar notation on the left-hand side already suggests, we will use such metrics as background metrics for Kerr-Schild solutions. This will work for any conformally flat metric, given that if k^μ is null and geodesic with respect to the Minkowski metric, it is straightforward to show that it is also null and geodesic with respect to $\bar{g}_{\mu\nu}$.

As a warm-up, let us examine the case where the background is the well-known Einstein static universe. For convenience, we will adopt the coordinates and conventions of ref. [95], such that the line element is

$$d\bar{s}^2 = -dt^2 + dr^2 - 2a \sin^2 \theta d\varphi dr + \frac{|\beta|^2}{\mathcal{D}^2} d\theta^2 + (|\beta|^2 + a^2 \sin^2 \theta) \sin^2 \theta d\varphi^2, \tag{4.2}$$

where

$$\begin{aligned} \mathcal{D} &= 1 - (a^2/R_0^2) \sin^2 \theta, \\ \beta &= (R_0^2 - a^2)^{1/2} \sin \frac{r}{R_0} + ia \cos \theta. \end{aligned}$$

The Ricci tensor and scalar for this metric take the particularly simple forms

$$\bar{R}_{\mu\nu} = \frac{2}{R_0^2}(\bar{g}_{\mu\nu} + \bar{u}_\mu \bar{u}_\nu), \quad \bar{R} = \frac{6}{R_0^2}, \tag{4.3}$$

respectively, where u_μ is the unit timelike vector given by

$$\bar{u}_\mu = (1, 0, 0, 0), \quad \bar{u}^\mu = (-1, 0, 0, 0). \tag{4.4}$$

We can construct a solution

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + 2Hk_\mu k_\nu \quad (4.5)$$

of single Kerr-Schild form, where for the Kerr-Schild term we adopt the notation of ref. [95] for ease of comparison. The Kerr-Schild vector k_μ is defined by

$$\sqrt{2}k_\mu = (-1, -1, 0, a \sin^2 \theta), \quad (4.6)$$

while the scalar function

$$H = mD_\mu k^\mu, \quad (4.7)$$

with D_μ the covariant derivative associated with $\bar{g}_{\mu\nu}$. The solution defined by eqs. (4.5)–(4.7) corresponds to a rotating black hole over the Einstein static universe. In order to further examine the effect of this perturbation, we note that the mixed-index Ricci tensor of the full metric can be recast in the form

$$R^\mu{}_\nu = -\frac{2}{R_0^2}(1-H)(\delta^\mu{}_\nu + u^\mu u_\nu), \quad (4.8)$$

where we have introduced the vectors

$$u^\mu = \frac{\bar{u}^\mu}{\sqrt{1-H}}, \quad u_\mu = \frac{1}{\sqrt{1-H}}(\bar{u}_\mu + \sqrt{2}Hk_\mu). \quad (4.9)$$

The Einstein equations then become

$$R^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu R = -8\pi((\rho + p)u^\mu u_\nu + p\delta^\mu{}_\nu) + \Lambda\delta^\mu{}_\nu. \quad (4.10)$$

That is, the matter content of the theory is that of a perfect fluid, whose energy density ρ and pressure p are given in this case by

$$8\pi\rho = \frac{3}{R_0^2}(1-H) - \Lambda, \quad (4.11)$$

$$8\pi p = -\frac{1}{R_0^2}(1-H) + \Lambda. \quad (4.12)$$

We see that the presence of the Kerr-Schild term acts to redefine the parameters associated with the background metric, reminiscent of the split Kerr-Schild metrics we considered in section 3.1. A number of other such solutions are presented in ref. [95].

We may single copy the graviton appearing in eq. (4.5) by defining the gauge field

$$A_\mu^a = c^a H k_\mu, \quad (4.13)$$

which we find satisfies the homogeneous linearised Yang-Mills equation

$$D_\mu F^{\mu\nu} = 0, \quad (4.14)$$

where D_μ , as above, is the covariant derivative for the Einstein static universe. This provides an example of the type B double copy of figure 1: on the gravity side, a fluid is

needed to source the background metric. There is no corresponding source current in the gauge theory, as there is no background gauge field, unlike in the type A double copy.

In the previous examples of the type B double copy, we saw that the zeroth copy did not appear to have a meaningful interpretation. Interestingly, in the present example the field H satisfies the homogeneous linearised biadjoint scalar equation

$$D^2 H = 0, \tag{4.15}$$

which indeed leads to a well-defined zeroth copy for this case.

Having seen a particular example of the type B single copy for non-Kerr-Schild backgrounds, let us now consider the general case of background metrics of the form of eq. (4.1), where the Minkowski metric is given in spherical polar coordinates (t, r, θ, ϕ) , so that the conformally transformed metric takes the form

$$\bar{g}_{\mu\nu} = \Omega^2(x^\mu) \text{diag}(-1, 1, r^2, r^2 \sin^2 \theta). \tag{4.16}$$

Upon constructing the gauge field

$$A_\mu = k_\mu \phi_2(r), \quad k_\mu = (-1, 1, 0, 0), \tag{4.17}$$

we find that this satisfies the curved space Maxwell equation (in the spacetime whose metric is $\bar{g}_{\mu\nu}$)

$$D_\mu F^{\mu\nu} = j^\nu, \quad j^\nu = \left(\frac{\nabla_M^2 \phi_2}{\Omega^4(x^\mu)}, 0, 0, 0 \right), \tag{4.18}$$

where ∇_M^2 is the Minkowski space Laplacian operator. Note that this result does not require the conformal factor Ω to have spherical symmetry — it may be a general function of (t, r, θ, ϕ) . From eq. (4.18), we see that if the gauge field of eq. (4.17) satisfies a vacuum Maxwell equation in Minkowski space, it also does so in the conformally transformed metric, analogous to the double Kerr-Schild examples considered in the previous section. Thus, the Minkowski space single copy extends to a type B curved space double copy, even though the background metric $\bar{g}_{\mu\nu}$ does not have a Kerr-Schild form, and thus is not immediately amenable to a type A single copy. We may also examine the type B zeroth copy, and one finds the curved space linearised biadjoint equation

$$D^\mu D_\mu \phi_2 = \frac{\nabla_M^2 \phi_2}{\Omega^2(x^\mu)} + \frac{2\phi_2'(r)\partial_r \Omega(x^\mu)}{\Omega^3(x^\mu)}. \tag{4.19}$$

The second term on the right-hand side involves a spatial derivative of the conformal factor, which is not present in the gauge theory source. Thus, it does not seem possible to interpret the zeroth copy in general, in line with our previous conclusions for the type B procedure.

5 Discussion

The aim of this paper has been to examine whether or not the classical double copy of refs. [68–70] can yield gravitons defined around a non-Minkowski background metric. Our motivation is the recent study of [82], which constructs such a procedure for amplitudes.

They consider gauge fields corresponding to perturbations around a plane-wave solution, whose amplitudes double copy to amplitudes for gravitons defined with respect to a gravitational plane wave background. We call this an example of a *type A* curved space double copy, depicted schematically in figure 1, and in which a gauge field defined with respect to a non-trivial background copies to a graviton defined with respect to a non-Minkowski metric, where the background fields in both cases are related. We have shown that one can indeed construct such a double copy for classical solutions, based on Kerr-Schild solutions, which underlie the classical double copy of refs. [68–70] in flat space. Furthermore, there is a well-defined zeroth copy, which maps the gauge field to a biadjoint scalar field, satisfying a linearised equation of motion. This is itself interesting for the curved space amplitude double copy of ref. [82]. There, the authors considered three-point amplitudes, which do not contain propagator factors. For the flat space double copy, the fact that the zeroth copy exists for classical solutions is related to how one deals with propagator factors in amplitudes. Thus, the fact that the zeroth copy also works for the type A curved space double copy suggests that the results of ref. [82] can indeed be generalised to higher-point amplitudes.

We also saw that it was possible to interpret the type A classical double copy in an alternative way, namely that one can associate a graviton defined with respect to a non-Minkowski background metric, with a gauge field living on the same spacetime. We named this a *type B* double copy, and presented a number of non-trivial examples. In almost all of the cases studied here, the zeroth copy does not have a meaningful interpretation, suggesting that the type B double copy is not a fully general relationship between gauge and gravity theories that is rooted in first principles, but rather a map that applies in certain special cases. Nevertheless, it could be very useful to have such a map, particularly when the background spacetimes (e.g. the de Sitter metric) are cosmologically relevant. Furthermore, a type B double copy may exist even when it is not known how to formulate a type A copy, due to e.g. having a background metric that is not of Kerr-Schild form — we have here given the explicit example of conformally flat metrics, including the Einstein static universe. One may prove in general that for stationary spherically symmetric gauge fields on a conformally flat metric, a vacuum Maxwell equation in Minkowski space implies a vacuum solution on the curved space, and thus a meaningful type B single copy.

Note that, as for the classical double copy in flat space, our results are tied to a particular choice of coordinates on the gravity side, and thus a particular gauge. For amplitudes, the double copy is also gauge-dependent, in that the kinematic numerators do not satisfy kinematic Jacobi relations in arbitrary gauges. Nevertheless, there are gauge-independent *BCJ relations* between partial amplitudes associated with different colour structures, and it is interesting to ponder whether this has a counterpart in the classical double copy considered here. Unfortunately the answer is no, given that Kerr-Schild solutions (and their gauge theory counterparts) linearise the equations of motion. In principle, one could obtain the classical field solutions from (off-shell) Feynman diagrams, if one knew how to construct a graviton propagator in the Kerr-Schild coordinate system, and a corresponding gluon propagator in the gauge theory. At linear order, there is only a single Feynman diagram, and thus no counterpart of the BCJ relations. This does not mean, however, that they have no analogue in classical solutions that do not linearise the field equations (such

as those considered recently in refs. [79–81, 96]). In that case, there are multiple colour structures at higher orders in perturbation theory, and thus indeed a potential classical realisation of the BCJ relations. This issue certainly deserves further investigation.

Our results provide a way of extending the classical double copy of refs. [68–70], and will prove useful in further investigations of the double copy, and its applications. A number of avenues for further work suggest themselves. Firstly, it would be interesting to know whether a type A double copy can be set up for background metrics that are not Kerr-Schild. Secondly, it would be useful to systematically determine the circumstances in which the type B double copy applies, including the possible addition of non-trivial source terms. Finally, one may investigate whether the type A or type B double copies allow for new insights or calculations relevant for astrophysics and cosmology.

Acknowledgments

We thank Tim Adamo, Ricardo Monteiro and Donal O’Connell for discussions and / or comments on the manuscript. CDW and NBA are supported by the U.K. Science and Technology Facilities Council (STFC). AL is funded by a Conacyt studentship, and thanks Queen Mary University of London for hospitality.

A The type B single copy in d dimensions

In this appendix, we consider the Maxwell equation for the gauge field

$$A_\mu = \phi_2(r)k_\mu, \quad k_\mu = (-1, 1, 0, \dots, 0)$$

in a d -dimensional spacetime with spherical polar coordinates $(t, r, \theta_1, \dots, \theta_{d-2})$ whose metric is given by

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \phi_1(x^\rho)k_\mu k_\nu.$$

Note that, for the sake of generality, we allow ϕ_1 to depend on all coordinates. One may then construct the field strength tensor

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

where the second equality follows in the absence of torsion. It is straightforward to show that the only non-zero components of this tensor are

$$F_{tr} = -F_{rt} = -\phi_2'(r). \tag{A.1}$$

The curved space Maxwell equation is given by

$$D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \bar{\Gamma}_{\mu\alpha}^\mu F^{\alpha\nu} + \bar{\Gamma}_{\mu\alpha}^\nu F^{\mu\alpha} = j^\nu, \tag{A.2}$$

where $\bar{\Gamma}_{\beta\gamma}^\alpha$ is the Christoffel symbol associated with $\bar{g}_{\mu\nu}$. From the standard result

$$\bar{\Gamma}_{\beta\gamma}^\alpha = \frac{1}{2}\bar{g}^{\alpha\sigma}(\partial_\beta\bar{g}_{\gamma\sigma} + \partial_\gamma\bar{g}_{\beta\sigma} - \partial_\sigma\bar{g}_{\beta\gamma}),$$

one finds that the only non-zero Christoffel symbols are given by

$$\begin{aligned}\bar{\Gamma}_{tr}^t &= -\bar{\Gamma}_{rr}^r = -\frac{1}{2}(1 + \phi_1)\partial_r\phi_1 + \frac{1}{2}\bar{g}^{tr}\partial_t\bar{g}_{rr} \\ \bar{\Gamma}_{tt}^t &= -\bar{\Gamma}_{rt}^r = \frac{1}{2}(-1 + \phi_1)\partial_t\phi_1 + \frac{1}{2}\partial_r\phi_1 \\ \bar{\Gamma}_{\theta_i r}^{\theta_i} &= \frac{1}{r}.\end{aligned}\tag{A.3}$$

The current in eq. (A.2) is then found to be

$$j^\mu = (\nabla_M^2\phi_2, 0, 0, \dots, 0),\tag{A.4}$$

where

$$\nabla_M^2 f(r) = \frac{1}{r^{d-2}}\partial_r\left(r^{d-2}\partial_r f(r)\right)\tag{A.5}$$

is the Minkowski-space Laplacian of a spherically symmetric function in $(d - 1)$ spatial dimensions.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] Z. Bern, J.J.M. Carrasco and H. Johansson, *New Relations for Gauge-Theory Amplitudes*, *Phys. Rev. D* **78** (2008) 085011 [[arXiv:0805.3993](https://arxiv.org/abs/0805.3993)] [[INSPIRE](#)].
- [2] Z. Bern, J.J.M. Carrasco and H. Johansson, *Perturbative Quantum Gravity as a Double Copy of Gauge Theory*, *Phys. Rev. Lett.* **105** (2010) 061602 [[arXiv:1004.0476](https://arxiv.org/abs/1004.0476)] [[INSPIRE](#)].
- [3] Z. Bern, T. Dennen, Y.-t. Huang and M. Kiermaier, *Gravity as the Square of Gauge Theory*, *Phys. Rev. D* **82** (2010) 065003 [[arXiv:1004.0693](https://arxiv.org/abs/1004.0693)] [[INSPIRE](#)].
- [4] N.E.J. Bjerrum-Bohr, P.H. Damgaard and P. Vanhove, *Minimal Basis for Gauge Theory Amplitudes*, *Phys. Rev. Lett.* **103** (2009) 161602 [[arXiv:0907.1425](https://arxiv.org/abs/0907.1425)] [[INSPIRE](#)].
- [5] S. Stieberger, *Open and Closed vs. Pure Open String Disk Amplitudes*, [arXiv:0907.2211](https://arxiv.org/abs/0907.2211) [[INSPIRE](#)].
- [6] N.E.J. Bjerrum-Bohr, P.H. Damgaard, T. Sondergaard and P. Vanhove, *Monodromy and Jacobi-like Relations for Color-Ordered Amplitudes*, *JHEP* **06** (2010) 003 [[arXiv:1003.2403](https://arxiv.org/abs/1003.2403)] [[INSPIRE](#)].
- [7] B. Feng, R. Huang and Y. Jia, *Gauge Amplitude Identities by On-shell Recursion Relation in S-matrix Program*, *Phys. Lett. B* **695** (2011) 350 [[arXiv:1004.3417](https://arxiv.org/abs/1004.3417)] [[INSPIRE](#)].
- [8] S.H. Henry Tye and Y. Zhang, *Dual Identities inside the Gluon and the Graviton Scattering Amplitudes*, *JHEP* **06** (2010) 071 [*Erratum ibid.* **04** (2011) 114] [[arXiv:1003.1732](https://arxiv.org/abs/1003.1732)] [[INSPIRE](#)].
- [9] C.R. Mafra, O. Schlotterer and S. Stieberger, *Explicit BCJ Numerators from Pure Spinors*, *JHEP* **07** (2011) 092 [[arXiv:1104.5224](https://arxiv.org/abs/1104.5224)] [[INSPIRE](#)].

- [10] R. Monteiro and D. O’Connell, *The Kinematic Algebra From the Self-Dual Sector*, *JHEP* **07** (2011) 007 [[arXiv:1105.2565](#)] [[INSPIRE](#)].
- [11] N.E.J. Bjerrum-Bohr, P.H. Damgaard, R. Monteiro and D. O’Connell, *Algebras for Amplitudes*, *JHEP* **06** (2012) 061 [[arXiv:1203.0944](#)] [[INSPIRE](#)].
- [12] H. Kawai, D.C. Lewellen and S.H.H. Tye, *A Relation Between Tree Amplitudes of Closed and Open Strings*, *Nucl. Phys. B* **269** (1986) 1 [[INSPIRE](#)].
- [13] Z. Bern, L.J. Dixon, D.C. Dunbar, M. Perelstein and J.S. Rozowsky, *On the relationship between Yang-Mills theory and gravity and its implication for ultraviolet divergences*, *Nucl. Phys. B* **530** (1998) 401 [[hep-th/9802162](#)] [[INSPIRE](#)].
- [14] M.B. Green, J.H. Schwarz and L. Brink, *$N = 4$ Yang-Mills and $N = 8$ Supergravity as Limits of String Theories*, *Nucl. Phys. B* **198** (1982) 474 [[INSPIRE](#)].
- [15] Z. Bern, J.S. Rozowsky and B. Yan, *Two loop four gluon amplitudes in $N = 4$ super Yang-Mills*, *Phys. Lett. B* **401** (1997) 273 [[hep-ph/9702424](#)] [[INSPIRE](#)].
- [16] J.J. Carrasco and H. Johansson, *Five-Point Amplitudes in $N = 4$ super-Yang-Mills Theory and $N = 8$ Supergravity*, *Phys. Rev. D* **85** (2012) 025006 [[arXiv:1106.4711](#)] [[INSPIRE](#)].
- [17] J.J.M. Carrasco, M. Chiodaroli, M. Günaydin and R. Roiban, *One-loop four-point amplitudes in pure and matter-coupled $N \leq 4$ supergravity*, *JHEP* **03** (2013) 056 [[arXiv:1212.1146](#)] [[INSPIRE](#)].
- [18] C.R. Mafra and O. Schlotterer, *The Structure of n -Point One-Loop Open Superstring Amplitudes*, *JHEP* **08** (2014) 099 [[arXiv:1203.6215](#)] [[INSPIRE](#)].
- [19] R.H. Boels, R.S. Isermann, R. Monteiro and D. O’Connell, *Colour-Kinematics Duality for One-Loop Rational Amplitudes*, *JHEP* **04** (2013) 107 [[arXiv:1301.4165](#)] [[INSPIRE](#)].
- [20] N.E.J. Bjerrum-Bohr, T. Dennen, R. Monteiro and D. O’Connell, *Integrand Oxidation and One-Loop Colour-Dual Numerators in $N = 4$ Gauge Theory*, *JHEP* **07** (2013) 092 [[arXiv:1303.2913](#)] [[INSPIRE](#)].
- [21] Z. Bern, S. Davies, T. Dennen, Y.-t. Huang and J. Nohle, *Color-Kinematics Duality for Pure Yang-Mills and Gravity at One and Two Loops*, *Phys. Rev. D* **92** (2015) 045041 [[arXiv:1303.6605](#)] [[INSPIRE](#)].
- [22] Z. Bern, S. Davies and T. Dennen, *The Ultraviolet Structure of Half-Maximal Supergravity with Matter Multiplets at Two and Three Loops*, *Phys. Rev. D* **88** (2013) 065007 [[arXiv:1305.4876](#)] [[INSPIRE](#)].
- [23] J. Nohle, *Color-Kinematics Duality in One-Loop Four-Gluon Amplitudes with Matter*, *Phys. Rev. D* **90** (2014) 025020 [[arXiv:1309.7416](#)] [[INSPIRE](#)].
- [24] Z. Bern, S. Davies, T. Dennen, A.V. Smirnov and V.A. Smirnov, *Ultraviolet Properties of $N = 4$ Supergravity at Four Loops*, *Phys. Rev. Lett.* **111** (2013) 231302 [[arXiv:1309.2498](#)] [[INSPIRE](#)].
- [25] S.G. Naculich, H. Nastase and H.J. Schnitzer, *All-loop infrared-divergent behavior of most-subleading-color gauge-theory amplitudes*, *JHEP* **04** (2013) 114 [[arXiv:1301.2234](#)] [[INSPIRE](#)].
- [26] Y.-J. Du, B. Feng and C.-H. Fu, *Dual-color decompositions at one-loop level in Yang-Mills theory*, *JHEP* **06** (2014) 157 [[arXiv:1402.6805](#)] [[INSPIRE](#)].

- [27] C.R. Mafra and O. Schlotterer, *Towards one-loop SYM amplitudes from the pure spinor BRST cohomology*, *Fortsch. Phys.* **63** (2015) 105 [[arXiv:1410.0668](#)] [[INSPIRE](#)].
- [28] Z. Bern, S. Davies and T. Dennen, *Enhanced ultraviolet cancellations in $\mathcal{N} = 5$ supergravity at four loops*, *Phys. Rev. D* **90** (2014) 105011 [[arXiv:1409.3089](#)] [[INSPIRE](#)].
- [29] C.R. Mafra and O. Schlotterer, *Two-loop five-point amplitudes of super Yang-Mills and supergravity in pure spinor superspace*, *JHEP* **10** (2015) 124 [[arXiv:1505.02746](#)] [[INSPIRE](#)].
- [30] S. He, R. Monteiro and O. Schlotterer, *String-inspired BCJ numerators for one-loop MHV amplitudes*, *JHEP* **01** (2016) 171 [[arXiv:1507.06288](#)] [[INSPIRE](#)].
- [31] Z. Bern, S. Davies and J. Nohle, *Double-Copy Constructions and Unitarity Cuts*, *Phys. Rev. D* **93** (2016) 105015 [[arXiv:1510.03448](#)] [[INSPIRE](#)].
- [32] G. Mogull and D. O’Connell, *Overcoming Obstacles to Colour-Kinematics Duality at Two Loops*, *JHEP* **12** (2015) 135 [[arXiv:1511.06652](#)] [[INSPIRE](#)].
- [33] M. Chiodaroli, M. Günaydin, H. Johansson and R. Roiban, *Spontaneously Broken Yang-Mills-Einstein Supergravities as Double Copies*, *JHEP* **06** (2017) 064 [[arXiv:1511.01740](#)] [[INSPIRE](#)].
- [34] Z. Bern, J.J.M. Carrasco, W.-M. Chen, H. Johansson, R. Roiban and M. Zeng, *The Five-Loop Four-Point Integrand of $N = 8$ Supergravity as a Generalized Double Copy*, [arXiv:1708.06807](#) [[INSPIRE](#)].
- [35] S. Oxburgh and C.D. White, *BCJ duality and the double copy in the soft limit*, *JHEP* **02** (2013) 127 [[arXiv:1210.1110](#)] [[INSPIRE](#)].
- [36] C.D. White, *Factorization Properties of Soft Graviton Amplitudes*, *JHEP* **05** (2011) 060 [[arXiv:1103.2981](#)] [[INSPIRE](#)].
- [37] S. Melville, S.G. Naculich, H.J. Schnitzer and C.D. White, *Wilson line approach to gravity in the high energy limit*, *Phys. Rev. D* **89** (2014) 025009 [[arXiv:1306.6019](#)] [[INSPIRE](#)].
- [38] A. Luna, S. Melville, S.G. Naculich and C.D. White, *Next-to-soft corrections to high energy scattering in QCD and gravity*, *JHEP* **01** (2017) 052 [[arXiv:1611.02172](#)] [[INSPIRE](#)].
- [39] R. Saotome and R. Akhoury, *Relationship Between Gravity and Gauge Scattering in the High Energy Limit*, *JHEP* **01** (2013) 123 [[arXiv:1210.8111](#)] [[INSPIRE](#)].
- [40] A. Sabio Vera, E. Serna Campillo and M.A. Vazquez-Mozo, *Color-Kinematics Duality and the Regge Limit of Inelastic Amplitudes*, *JHEP* **04** (2013) 086 [[arXiv:1212.5103](#)] [[INSPIRE](#)].
- [41] H. Johansson, A. Sabio Vera, E. Serna Campillo and M.Á. Vázquez-Mozo, *Color-Kinematics Duality in Multi-Regge Kinematics and Dimensional Reduction*, *JHEP* **10** (2013) 215 [[arXiv:1307.3106](#)] [[INSPIRE](#)].
- [42] H. Johansson, A. Sabio Vera, E. Serna Campillo and M.A. Vazquez-Mozo, *Color-kinematics duality and dimensional reduction for graviton emission in Regge limit*, [arXiv:1310.1680](#) [[INSPIRE](#)].
- [43] R. Monteiro and D. O’Connell, *The Kinematic Algebras from the Scattering Equations*, *JHEP* **03** (2014) 110 [[arXiv:1311.1151](#)] [[INSPIRE](#)].
- [44] M. Tolotti and S. Weinzierl, *Construction of an effective Yang-Mills Lagrangian with manifest BCJ duality*, *JHEP* **07** (2013) 111 [[arXiv:1306.2975](#)] [[INSPIRE](#)].
- [45] C.-H. Fu, Y.-J. Du and B. Feng, *Note on Construction of Dual-trace Factor in Yang-Mills Theory*, *JHEP* **10** (2013) 069 [[arXiv:1305.2996](#)] [[INSPIRE](#)].

- [46] Y.-J. Du, B. Feng and C.-H. Fu, *The Construction of Dual-trace Factor in Yang-Mills Theory*, *JHEP* **07** (2013) 057 [[arXiv:1304.2978](#)] [[INSPIRE](#)].
- [47] C.-H. Fu, Y.-J. Du and B. Feng, *An algebraic approach to BCJ numerators*, *JHEP* **03** (2013) 050 [[arXiv:1212.6168](#)] [[INSPIRE](#)].
- [48] S.G. Naculich, *Scattering equations and BCJ relations for gauge and gravitational amplitudes with massive scalar particles*, *JHEP* **09** (2014) 029 [[arXiv:1407.7836](#)] [[INSPIRE](#)].
- [49] S.G. Naculich, *Scattering equations and virtuous kinematic numerators and dual-trace functions*, *JHEP* **07** (2014) 143 [[arXiv:1404.7141](#)] [[INSPIRE](#)].
- [50] M. Chiodaroli, M. Günaydin, H. Johansson and R. Roiban, *Scattering amplitudes in $\mathcal{N} = 2$ Maxwell-Einstein and Yang-Mills/Einstein supergravity*, *JHEP* **01** (2015) 081 [[arXiv:1408.0764](#)] [[INSPIRE](#)].
- [51] J.J.M. Carrasco, R. Kallosh, R. Roiban and A.A. Tseytlin, *On the U(1) duality anomaly and the S-matrix of $N = 4$ supergravity*, *JHEP* **07** (2013) 029 [[arXiv:1303.6219](#)] [[INSPIRE](#)].
- [52] S. Litsey and J. Stankowicz, *Kinematic numerators and a double-copy formula for $N = 4$ super-Yang-Mills residues*, *Phys. Rev. D* **90** (2014) 025013 [[arXiv:1309.7681](#)] [[INSPIRE](#)].
- [53] S. Nagy, *Chiral Squaring*, *JHEP* **07** (2016) 142 [[arXiv:1412.4750](#)] [[INSPIRE](#)].
- [54] A. Anastasiou, L. Borsten, M.J. Hughes and S. Nagy, *Global symmetries of Yang-Mills squared in various dimensions*, *JHEP* **01** (2016) 148 [[arXiv:1502.05359](#)] [[INSPIRE](#)].
- [55] H. Johansson and A. Ochirov, *Color-Kinematics Duality for QCD Amplitudes*, *JHEP* **01** (2016) 170 [[arXiv:1507.00332](#)] [[INSPIRE](#)].
- [56] S. Lee, C.R. Mafra and O. Schlotterer, *Non-linear gauge transformations in $D = 10$ SYM theory and the BCJ duality*, *JHEP* **03** (2016) 090 [[arXiv:1510.08843](#)] [[INSPIRE](#)].
- [57] S.M. Barnett, *Maxwellian theory of gravitational waves and their mechanical properties*, *New J. Phys.* **16** (2014) 023027 [[INSPIRE](#)].
- [58] M. Chiodaroli, *Simplifying amplitudes in Maxwell-Einstein and Yang-Mills-Einstein supergravities*, [arXiv:1607.04129](#) [[INSPIRE](#)].
- [59] G.L. Cardoso, S. Nagy and S. Nampuri, *A double copy for $\mathcal{N} = 2$ supergravity: a linearised tale told on-shell*, *JHEP* **10** (2016) 127 [[arXiv:1609.05022](#)] [[INSPIRE](#)].
- [60] S. Mizera, *Inverse of the String Theory KLT Kernel*, *JHEP* **06** (2017) 084 [[arXiv:1610.04230](#)] [[INSPIRE](#)].
- [61] Z. Bern, J.J. Carrasco, W.-M. Chen, H. Johansson and R. Roiban, *Gravity Amplitudes as Generalized Double Copies of Gauge-Theory Amplitudes*, *Phys. Rev. Lett.* **118** (2017) 181602 [[arXiv:1701.02519](#)] [[INSPIRE](#)].
- [62] M. Chiodaroli, M. Günaydin, H. Johansson and R. Roiban, *Explicit Formulae for Yang-Mills-Einstein Amplitudes from the Double Copy*, *JHEP* **07** (2017) 002 [[arXiv:1703.00421](#)] [[INSPIRE](#)].
- [63] H. Johansson, G. Kälin and G. Mogull, *Two-loop supersymmetric QCD and half-maximal supergravity amplitudes*, *JHEP* **09** (2017) 019 [[arXiv:1706.09381](#)] [[INSPIRE](#)].
- [64] H. Johansson and J. Nohle, *Conformal Gravity from Gauge Theory*, [arXiv:1707.02965](#) [[INSPIRE](#)].

- [65] C. Cheung and G.N. Remmen, *Twofold Symmetries of the Pure Gravity Action*, *JHEP* **01** (2017) 104 [[arXiv:1612.03927](#)] [[INSPIRE](#)].
- [66] C. Cheung and G.N. Remmen, *Hidden Simplicity of the Gravity Action*, *JHEP* **09** (2017) 002 [[arXiv:1705.00626](#)] [[INSPIRE](#)].
- [67] C. Cheung, C.-H. Shen and C. Wen, *Unifying Relations for Scattering Amplitudes*, [arXiv:1705.03025](#) [[INSPIRE](#)].
- [68] R. Monteiro, D. O’Connell and C.D. White, *Black holes and the double copy*, *JHEP* **12** (2014) 056 [[arXiv:1410.0239](#)] [[INSPIRE](#)].
- [69] A. Luna, R. Monteiro, I. Nicholson, D. O’Connell and C.D. White, *The double copy: Bremsstrahlung and accelerating black holes*, *JHEP* **06** (2016) 023 [[arXiv:1603.05737](#)] [[INSPIRE](#)].
- [70] A. Luna, R. Monteiro, D. O’Connell and C.D. White, *The classical double copy for Taub-NUT spacetime*, *Phys. Lett. B* **750** (2015) 272 [[arXiv:1507.01869](#)] [[INSPIRE](#)].
- [71] A.K. Ridgway and M.B. Wise, *Static Spherically Symmetric Kerr-Schild Metrics and Implications for the Classical Double Copy*, *Phys. Rev. D* **94** (2016) 044023 [[arXiv:1512.02243](#)] [[INSPIRE](#)].
- [72] C.D. White, *Exact solutions for the biadjoint scalar field*, *Phys. Lett. B* **763** (2016) 365 [[arXiv:1606.04724](#)] [[INSPIRE](#)].
- [73] P.-J. De Smet and C.D. White, *Extended solutions for the biadjoint scalar field*, *Phys. Lett. B* **775** (2017) 163 [[arXiv:1708.01103](#)] [[INSPIRE](#)].
- [74] A. Anastasiou, L. Borsten, M.J. Duff, L.J. Hughes and S. Nagy, *Yang-Mills origin of gravitational symmetries*, *Phys. Rev. Lett.* **113** (2014) 231606 [[arXiv:1408.4434](#)] [[INSPIRE](#)].
- [75] L. Borsten and M.J. Duff, *Gravity as the square of Yang-Mills?*, *Phys. Scripta* **90** (2015) 108012 [[arXiv:1602.08267](#)] [[INSPIRE](#)].
- [76] A. Anastasiou et al., *Twin supergravities from Yang-Mills theory squared*, *Phys. Rev. D* **96** (2017) 026013 [[arXiv:1610.07192](#)] [[INSPIRE](#)].
- [77] A. Anastasiou, L. Borsten, M.J. Duff, A. Marrani, S. Nagy and M. Zoccali, *Are all supergravity theories Yang-Mills squared?*, [arXiv:1707.03234](#) [[INSPIRE](#)].
- [78] L. Borsten, *On $D = 6$, $\mathcal{N} = (2, 0)$ and $\mathcal{N} = (4, 0)$ theories*, [arXiv:1708.02573](#) [[INSPIRE](#)].
- [79] W.D. Goldberger and A.K. Ridgway, *Radiation and the classical double copy for color charges*, *Phys. Rev. D* **95** (2017) 125010 [[arXiv:1611.03493](#)] [[INSPIRE](#)].
- [80] W.D. Goldberger, S.G. Prabhu and J.O. Thompson, *Classical gluon and graviton radiation from the bi-adjoint scalar double copy*, *Phys. Rev. D* **96** (2017) 065009 [[arXiv:1705.09263](#)] [[INSPIRE](#)].
- [81] A. Luna et al., *Perturbative spacetimes from Yang-Mills theory*, *JHEP* **04** (2017) 069 [[arXiv:1611.07508](#)] [[INSPIRE](#)].
- [82] T. Adamo, E. Casali, L. Mason and S. Nekoar, *Scattering on plane waves and the double copy*, [arXiv:1706.08925](#) [[INSPIRE](#)].
- [83] T. Adamo, E. Casali, L. Mason and S. Nekoar, *Amplitudes on plane waves from ambitwistor strings*, *JHEP* **11** (2017) 160 [[arXiv:1708.09249](#)] [[INSPIRE](#)].

- [84] L. Mason and D. Skinner, *Ambitwistor strings and the scattering equations*, *JHEP* **07** (2014) 048 [[arXiv:1311.2564](#)] [[INSPIRE](#)].
- [85] T. Adamo, E. Casali and D. Skinner, *A Worldsheet Theory for Supergravity*, *JHEP* **02** (2015) 116 [[arXiv:1409.5656](#)] [[INSPIRE](#)].
- [86] J.J.M. Carrasco, *Gauge and Gravity Amplitude Relations*, [arXiv:1506.00974](#) [[INSPIRE](#)].
- [87] C. Cheung, *TASI Lectures on Scattering Amplitudes*, [arXiv:1708.03872](#) [[INSPIRE](#)].
- [88] C.D. White, *The double copy: gravity from gluons*, [arXiv:1708.07056](#) [[INSPIRE](#)].
- [89] A.H. Taub, *Empty space-times admitting a three parameter group of motions*, *Annals Math.* **53** (1951) 472.
- [90] E. Newman, L. Tamubrino and T. Unti, *Empty space generalization of the Schwarzschild metric*, *J. Math. Phys.* **4** (1963) 915 [[INSPIRE](#)].
- [91] Z.W. Chong, G.W. Gibbons, H. Lü and C.N. Pope, *Separability and Killing tensors in Kerr-Taub-NUT-de Sitter metrics in higher dimensions*, *Phys. Lett. B* **609** (2005) 124 [[hep-th/0405061](#)] [[INSPIRE](#)].
- [92] H. Stephani, D. Kramer, M.A. MacCallum, C. Hoenselaers and E. Herlt, *Exact solutions of Einstein's field equations*, Cambridge University Press, (2003).
- [93] W. Chen, H. Lü and C.N. Pope, *General Kerr-NUT-AdS metrics in all dimensions*, *Class. Quant. Grav.* **23** (2006) 5323 [[hep-th/0604125](#)] [[INSPIRE](#)].
- [94] W. Chen and H. Lü, *Kerr-Schild structure and harmonic 2-forms on (A)dS-Kerr-NUT metrics*, *Phys. Lett. B* **658** (2008) 158 [[arXiv:0705.4471](#)] [[INSPIRE](#)].
- [95] A.H. Taub, *Generalised Kerr-Schild space-times.*, *Annals Phys.* **134** (1981) 326.
- [96] A. Luna, I. Nicholson, D. O'Connell and C.D. White, *Inelastic Black Hole Scattering from Charged Scalar Amplitudes*, [arXiv:1711.03901](#) [[INSPIRE](#)].