Quantum entanglement of locally excited states in Maxwell theory

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ABSTRACT: In 4 dimensional Maxwell gauge theory, we study the changes of (Rényi) entanglement entropy which are defined by subtracting the entropy for the ground state from the one for the locally excited states, generated by acting with gauge invariant local operators on the state. The changes for the operators which we consider in this paper reflect the electric-magnetic duality. The late-time value of changes can be interpreted in terms of electromagnetic quasi-particles. When the operator constructed of both electric and magnetic fields acts on the ground state, it shows that the operator acts on the late-time structure of quantum entanglement differently from free scalar fields.

Contents

1 Introduction and summary

Quantum entanglement significantly distinguishes quantum states from classical states. It can characterize conformal field theories [1–4] and topological phases [5–7]. In *Gauge/Gravity* correspondence [8–10], the structure of quantum entanglement in quantum field theories (QFTs) living on the boundary is expected to be related to the gravity in the bulk [11–14]. There has been a lot of work done to reveal how the structure of quantum entanglement on the boundary corresponds to the geometry in the bulk [15–25]. Therefore it is important to uncover the fundamental features which quantum entanglement possesses. (Rényi) entanglement entropy is one of the useful quantities to investigate them.

However the definition of (Rényi) entanglement entropy in gauge theories has subtleties [26–41]. In gauge theories, physical states have to be gauge invariant. They obey constraints which guarantee their gauge invariance. They make it difficult to divide the Hilbert space into subsystems A and B because the physical degrees of freedom in A depends on the freedom in B due to the constraints. Their boundary is ∂A. Then the definition of (Rényi) entanglement entropy needs a precise method of dividing Hilbert space

and defining the reduced density matrix ρ_A which is given by tracing out the degrees of freedom in B,

$$\rho_A = \mathrm{tr}_B \rho. \tag{1.1}$$

On the other hand, the entropy in QFTs depends on a UV cutoff (ultraviolet cutoff) δ because by definition it has the UV divergence. It is given by a series expansion in conformal field theories. The physical degrees of freedom around ∂A have the significant effect on the terms which depend on δ. The method of dividing the Hilbert space is expect to affect the degrees of freedom around ∂A in the direct fashion. In the present paper, we study the changes of (Rényi) entanglement entropy $\Delta S_A^{(n)}$ which is defined by subtracting the entropy for the ground state from the one for the locally excited state, which is defined by acting with a local operator on the ground state. Here we assume that the operator is located far from ∂A. We will explain it more in the next section. As in [42–48], their changes do not possess the UV divergence. More precisely, they measure how the local operator changes the structure of quantum entanglement. Therefore, they are expected to avoid the subtleties which (Rényi) entanglement entropy has.

In this paper we study $\Delta S_A^{(n)}$ in $4d$ Maxwell gauge theory, which is a free CFT [49]. The previous works [44–46] show the time evolution of $\Delta S_A^{(n)}$ can be interpreted in terms of relativistic propagation of entangled quasi-particles which are created by local operators. In the free theories, the late-time value of $\Delta S_A^{(n)}$ is given by a constant, which depends on the operators. It comes from the quantum entanglement between quasi-particles. As in [46], the late-time entanglement structure depends on the kind of quasi-particles. The authors in [47] show that in the specific $2d$ CFTs, it is related to the quantum dimension of the operator which acts on the ground state. The authors in [48, 50] have shown that in holographic theories the late time value of $\Delta S_A^{(n)}$ logarithmically increases similarly to the behavior of entanglement entropy for the local quenches [51, 52]. $\Delta S_A^{(n)}$ in the finite temperature system was investigated by the authors in [53]. Many works have been done to study the fundamental properties of $\Delta S_A^{(n)}$ [54–59]. The time evolution and late-time value of $\Delta S_A^{(n)}$ depend on theories and the quasi-particles created by the local operator. Then we study how the structure of quantum entanglement is changed by gauge invariant local operators such as electric and magnetic fields. In particular, we study how the late-time structure of quantum entanglement depends on them. More precisely, we study the effect of quasi-particles on the structure. We also study whether $\Delta S_A^{(n)}$ for a gauge invariant locally excited state reflects electric-magnetic duality.

Summary. Here, we briefly summarize our results in this work. We study how the various local gauge invariant operators change the structure of quantum entanglement by measuring the time evolution of $\Delta S_A^{(n)}$ for various gauge invariant local operators. We also study whether $\Delta S_A^{(n)}$ is invariant under the electric-magnetic duality transformation.

Electric-magnetic duality. As it will be explained later, $\Delta S_A^{(n)}$ for locally excited states is invariant under the transformation $E_i \to -B_i$ and $B_i \to E_i$ where E and B are electric and magnetic fields, respectively.

E_i or B_i. If only E_i or B_i acts on the ground state, the time evolution of $\Delta S_A^{(n)}$ depends on the one which acts on the ground state. Because $\Delta S_A^{(n)}$ reflects the electric-magnetic duality, the entropy for B_i is equal to that for E_i, which is the electric field along the same direction as that of magnetic field. $\Delta S_A^{(n)}$ for the electric and magnetic fields along the direction vertical to the entangling surface increases slower than those for fields along the directions parallel to the surface. However there are no differences between the effects of electromagnetic field and that of scalar one on the entanglement structure at the late time.[1]

Composite operators. If the composite operator such as $\mathbf{B}^2$ acts on the ground state, they lead to the late-time structure of quantum entanglement in the same manner as a specific scalar operator. Then the late-time value of $\Delta S_A^{(n)}$ for that can be interpreted in terms of quasi-particles created by the scalar operator. However $\Delta S_A^{(n)}$ for some specific operators (e.g, $E_2 B_3$) constructed of both electric and magnetic fields can not be interpreted in terms of the scalar quasi-particles but of electromagnetic ones, which is explained in section 4. Here B_3 (E_3) and B_2 (E_2) are the magnetic (electric) fields along the direction perpendicular to ∂A as we will explain it later.

Late-time algebra. We interpret the late-time values of $\Delta S_A^{(n)}$ in terms of electromagnetic quasi-particles created by an electromagnetic field, and derive a late-time algebra which they obey. There are commutation relations between the particles of the same kinds of fields . As we will show later, there are also additional relations between E_2 (E_3) and B_3 (B_2), which are parallel to the entangling surface. They make the effect of electromagnetic fields different from that of scalar fields on the late-time structure of quantum entanglement.

Organization. This paper is organized as follows. In section 2, we will explain locally excited states and how to compute $\Delta S_A^{(n)}$ in the replica trick. We study the time evolution and late-time value of $\Delta S_A^{(n)}$ for various gauge invariant local operators in section 3. We interpret the late-time value of $\Delta S_A^{(n)}$ in terms of entangled quasi-particles in section 4. We study how they have the effect on the late-time structure of quantum entanglement. We finish with the conclusion, future problems and the detail of propagators is included in appendices.

2 How to compute excesses of (Rényi) entanglement entropy

By measuring the excess of (Rényi) entanglement entropies $\Delta S_A^{(n)}$, we study how local gauge-invariant operators changes the structure of quantum entanglement in the $4d$ Maxwell gauge theory:

$$S = -\frac{1}{4} \int d^4 x F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma}, \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $g^{\mu\nu} = \mathrm{diag}\,(-1, 1, 1, 1)$.

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[1] When only a component of the electric or magnetic one acts on the ground state, we do not consider the linear combination of them in this paper.
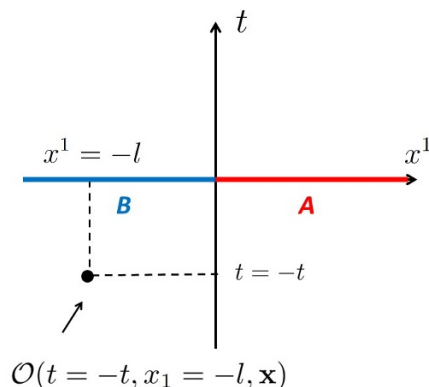
Figure 1. The location of local gauge invariant operator in Minkowski spacetime.

In this section, we explain the definition of locally excited state and how to compute $\Delta S_A^{(n)}$ in the replica method.

The definition of locally excited states. The locally excited state is defined by acting with a gauge invariant local operator $\mathcal{O}$ such as $F^{\mu\nu}$ on the ground state:

$$|\Psi\rangle = \mathcal{N}\mathcal{O}(-t, -l, \mathbf{x}) |0\rangle . \tag{2.2}$$

where $\mathcal{N}$ is a normalization constant and $|0\rangle$ is a gauge invariant state. As in figure 1, $\mathcal{O}$ is located at $t = -t$, $x^1 = -l$ and $\mathbf{x} = (x^2, x^3)$.

Subsystem. As in the previous works [44–47], the subsystem A is defined by ($t = 0$, $x^1 \geq 0$) as in figure 1. In free theories $\Delta S_A^{(n)}$ approaches to a constant, which comes from quantum entanglement between entangled quasi-particles. In this paper we would like to study how the constant depends on gauge invariant operators. Therefore the region in figure 1 is chosen as A.

Excesses of (Rényi) entanglement entropy. Here we explain more about the definition of $\Delta S_A^{(n)}$. (Rényi) entanglement entropy for the ground state is a static quantity, which does not depend on time. Then we define the excesses of (Rényi) entanglement entropy $\Delta S_A^{(n)}$ by subtracting $S_A^{(n)}$ for the ground state from those for locally excited states,

$$\Delta S_A^{(n)} = S_A^{(n),EX} - S_A^{(n),G}, \tag{2.3}$$

where $S_A^{(n),EX}$ and $S_A^{(n),G}$, are (Rényi) entanglement entropies for the excited states in (2.2) and the ground state $|0\rangle$, respectively. In the sense that $\Delta S_A^{(n)}$ does not depend on δ, it is a "renormalized" (Rényi) entanglement entropy.

2.1 The replica trick

We would like to study the time evolution of $\Delta S_A^{(n)}$ in 4d Minkowski spacetime. However in this paper we do not directly study the changes of entanglement structure in the spacetime. Instead, we compute $\Delta S_A^{(n)}$ in Euclidean space by the replica trick. After that we perform

the analytic continuation, which we will explain later. Then we compute the real time evolution of $\Delta S_A^{(n)}$.

As in [44–47], a reduced density matrix in Euclidean space is given by

$$\rho = \tilde{\mathcal{N}}^2 \mathcal{O}(\tau_e, -l, \mathbf{x}) |0\rangle \langle 0| \mathcal{O}^\dagger(\tau_l, -l, \mathbf{x}), \tag{2.4}$$

where τ is Euclidean time. By introducing a polar coordinate, $(\tau_{l,e}, -l)$ is mapped to $(r_{1,2}, \theta_{1,2})$ as in figure 2.

The reduced density matrix of ρ introduced above, which is our interest now, can be written in the form of path integral

$$[\rho_A]_{\Phi_1,\Phi_2} = \left(Z_1^{EX}\right)^{-1} \int D\Phi \mathcal{O}^\dagger(x_1)\mathcal{O}(x_2) e^{-S[\Phi]} \delta(\Phi(-\Delta, x^i) - \Phi_1(x^i)) \cdot \delta(\Phi(\Delta, x^i) - \Phi_2(x^i)) \tag{2.5}$$

where the indices i, μ is $i = 1, 2, 3$, $\mu = 0, 1, 2, 3$, and x^i runs over the region A, which is in this case $x^1 > 0$ and $x^2, x^3 \in \mathbf{R}$. The $\Phi_1 = \Phi_1(x^i)$ and $\Phi_2 = \Phi_2(x^i)$ introduced in the path integral inside the δ-function with a positive value $\Delta \ll 1$ (which is taken to the zero limit in the end), represent the boundary condition at region A. Z_1^{EX} is defined as

$$Z_1^{EX} = \int D\Phi \mathcal{O}^\dagger(x_1)\mathcal{O}(x_2) e^{-S[\Phi]}. \tag{2.6}$$

The n-th power of the reduced density matrix is written as

$$[\rho_A^n]_{\Phi_1,\Phi_2} = \left(Z_1^{EX}\right)^{-n} \int D\Phi \left(\prod_{l=1}^n \mathcal{O}^\dagger(x_1^l)\mathcal{O}(x_2^l)\right) e^{-S_n[\Phi]} \delta(\Phi(-\Delta, x^i) - \Phi_1(x^i))$$
$$\cdot \delta(\Phi(\Delta, x^i) - \Phi_2(x^i)). \tag{2.7}$$

Note that since the path integral is performed over an n-sheeted Riemann surface, the action is now represented as $S_n[\Phi]$. x_1^l, x_2^l represent the operator insertion points on the l-th sheet, and the boundary condition Φ_1, Φ_2 is now at the 1-st and n-th Riemann sheet, respectively.

Thus, in the replica trick (Rényi) entanglement entropies for (2.4) and the ground state are respectively given by taking the trace log of (2.7),[2]

$$S_A^{(n),EX} = \frac{1}{1-n} \log \left[\frac{\int D\Phi \mathcal{O}^\dagger(r_1, \theta_1^n)\mathcal{O}(r_2, \theta_2^n) \cdots \mathcal{O}(r_1, \theta_1^1)^\dagger \mathcal{O}(r_2, \theta_2^1) e^{-S_n[\Phi]}}{\left(\int D\Phi \mathcal{O}(r_1, \theta_1^1)^\dagger \mathcal{O}(r_2, \theta_2^1) e^{-S_1[\Phi]}\right)^n} \right],$$
$$S_A^{(n),G} = \frac{1}{1-n} \log \left[\frac{\int D\Phi e^{-S_n[\Phi]}}{\left(\int D\Phi e^{-S_1[\Phi]}\right)^n} \right], \tag{2.8}$$

where $\theta_{1,2}^k = \theta_{1,2} + 2(k-1)\pi$. The actions S_n and S_1 are defined on the n-sheeted geometry Σ_n (see figure 3) and the flat space Σ_1, respectively. By substituting (Rényi) entanglement entropies in (2.8) into (2.3), $\Delta S_A^{(n)}$ is given by

$$\Delta S_A^{(n)} = \frac{1}{1-n} \log \left[\frac{\langle \mathcal{O}^\dagger(r_1, \theta_1^n)\mathcal{O}(r_2, \theta_2^n) \cdots \mathcal{O}^\dagger(r_1, \theta_1^1)\mathcal{O}(r_2, \theta_2^1)\rangle_{\Sigma_n}}{\langle \mathcal{O}^\dagger(r_1, \theta_1^1)\mathcal{O}(r_2, \theta_2^1)\rangle_{\Sigma_1}^n} \right] \tag{2.9}$$
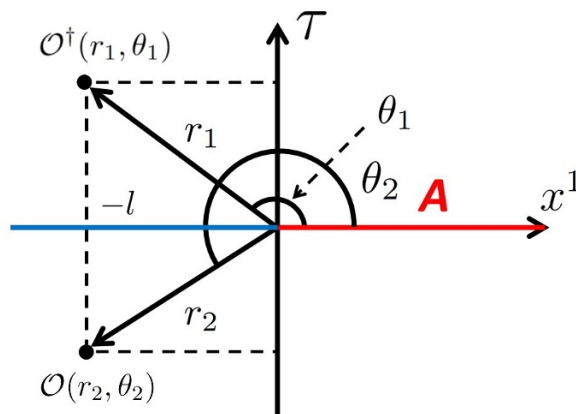
Figure 2. The location of local gauge invariant operator in Euclidean space.

We only need to compute propagators on Σ_n in order to compute $\Delta S_A^{(n)}$ in free field theories. The two point function of gauge fields A_a is defined by $-\langle A_a(r,\theta,\mathbf{x})A_b(r',\theta',\mathbf{x}')\rangle = G_{ab}(r,r',\theta-\theta',\mathbf{x}-\mathbf{x}')$. If we choose a specific gauge,[3] their Green's functions obey the same equation of motion as that for $4d$ free massless scalar field theory,

$$\partial_r^2 G_b^a(r,r',\theta-\theta',\mathbf{x}-\mathbf{x}')+\frac{1}{r}\partial_r G_b^a(r,r',\theta-\theta',\mathbf{x}-\mathbf{x}') \tag{2.10}$$

$$+\frac{1}{r^2}\partial_\theta^2 G_b^a(r,r',\theta-\theta',\mathbf{x}-\mathbf{x}')+\partial_\mathbf{x}^2 G_b^a(r,r',\theta-\theta',\mathbf{x}-\mathbf{x}')=-\frac{\delta_b^a\delta(r-r')\delta(\theta-\theta')\delta^2(\mathbf{x}-\mathbf{x}')}{r},$$

where $a=\{\tau,x^1,x^2,x^3\}$.[4]

The solution of the equation is given by

$$G_{ab}\left(r,r,'\theta-\theta',\mathbf{x}-\mathbf{x}'\right)=\frac{\delta_{ab}\sinh\left(\frac{t_0}{n}\right)}{8n\pi^2 rr'\sinh t_0\left(\cosh\left(\frac{t_0}{n}\right)-\cos\left(\frac{\theta-\theta'}{n}\right)\right)}, \tag{2.11}$$

where t_0 is defined by

$$\cosh t_0=\frac{r^2+r'^2+(x^2-x'^2)^2+(x^3-x'^3)^2}{2rr'}. \tag{2.12}$$

(2.11) has been obtained by the authors in [44–46, 48, 60, 61].

Analytic continuation. After computing Green's functions on Σ_n in Euclidean space, we perform the following analytic continuation,

$$A_\tau=iA_t,\ \partial_\tau=i\partial_t,\ \tau_l=\epsilon-it,\ \tau_e=-\epsilon-it, \tag{2.13}$$

where ϵ is a smearing parameter which is introduced to keep the norm of the excited state finite. Analytic-continued Green's functions depend on ϵ. We are interested in the behavior of $\Delta S_A^{(n)}$ in the limit $\epsilon\to 0$. Their leading behavior ($\sim\mathcal{O}\left(\frac{1}{\epsilon^4}\right)$) are summarized in appendix A.

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[2]This formula was introduced in [42, 43] . For detail of computation, see also [44–47].

[3]The chosen gauge corresponds to Feynman gauge in Minkowski spacetime.

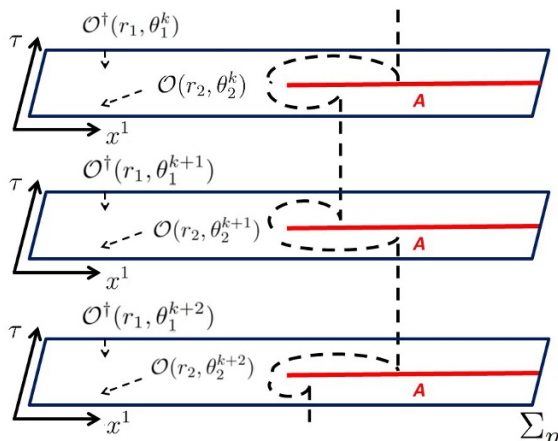[4]$G_b^a\equiv\delta^{ac}G_{cb}$ where $\delta_{ac}=\mathrm{diag}(1,1,1,1)$.

Figure 3. A picture of n-sheeted geometry Σ_n.

Gauge independence. The quantity $\Delta S_A^{(n)}$ does not depend on the choice of gauge fixing or boundary condition, since $\Delta S_A^{(n)}$ is the difference of the Rényi EE of ground state and excited state. More precisely, when we decompose the Hilbert space into two components, as discussed in [26], we need to specify a precise boundary condition. Here, we assume that this boundary condition can be imposed by an insertion of a corresponding local operator at the boundary. On the other hand, as is discussed in [45], the operator at the boundary does not contribute to the result since $\frac{l}{\epsilon} \gg 1$. Therefore, through out the computation of $\Delta S_A^{(n)}$, the result does not depend on the boundary condition. Moreover, we are allowed to choose an arbitrary boundary condition. Thus, we can set a gauge invariant boundary condition for the Green's functions of gauge invariant operators, so that the result will be gauge invariant.

3   Excesses of (Rényi) entanglement entropy

In this section, we study the time evolution and the late time value of $\Delta S_A^{(n)}$ in the following three cases.

(i) Only one electric or magnetic field $\mathcal{O} = E_i, B_i$ acts on the ground state. The time evolution of $\Delta S_A^{(n)}$ depends on the operator which acts on the ground state. If electromagnetic fields are changed by $F_{\mu\nu} \to \tilde{F_{\mu\nu}} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, $\Delta S_A^{(n)}$ does not change. Here $\epsilon_{\mu\nu\rho\sigma}$ is an antisymmetric tensor. The late-time value of $\Delta S_A^{(n)}$ does not depend on the operator. It can be interpreted in terms of the quasi-particle created by a scalar operator ϕ.

(ii) Composite operators which act on the ground state are constructed of only electric or magnetic fields such as $\mathbf{E}^2$ and $\mathbf{B}^2$. $\Delta S_A^{(n)}$ for $\mathbf{E}^2$ is equivalent to the entropy for $\mathbf{B}^2$. Thus $\Delta S_A^{(n)}$ for them are invariant under the electric-magnetic duality transformation. There are no differences between their effect on the (Rényi) entanglement entropy. Its late-time values can be interpreted in terms of quasi-particles, which are created

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by the operator constructed of massless free scalar fields $\sum_{a=1}^{3}(\phi^a)^2$. Here a denotes the kinds of fields.

(iii) Local operators are constructed of both electric and magnetic fields such as $E_1^2 + B_1^2$, $B_3 E_2$ and $\mathbf{B} \cdot \mathbf{E}$. The late-time value of $\Delta S_A^{(n)}$ shows that there is a significant difference between the effect of E_1 (B_1) and $E_{2,3}$ $(B_{2,3})$ on the late-time entanglement structure. Here E_1 (B_1) is the electric (magnetic) field along the direction vertical to the entangling surface. On the other hand, $E_{2,3}$ $(B_{2,3})$ is the electric (magnetic) field along the direction parallel to the entangling surface. As it will be explained in the next section, the difference comes from the commutation relation between electromagnetic quasi-particles created by E_2 (E_3) and the particles created by B_3 (B_2).

3.1 $\mathcal{O} = E_i$ or B_i

Here locally excited states are defined by acting with only E_i or B_i on the ground state. $\Delta S_A^{(n)}$ is given by (2.9) in the replica method with Euclidean signature. After performing the analytic continuation in (2.13) and taking the limit $\epsilon \to 0$, their time evolution is given as follows. $\Delta S_A^{(n)}$ vanishes before $t = l(> 0)$, but after $t = l$, it increases. The detail of their time evolution is summarized in table 1. After taking the late time limit $(0 < l \ll t)$, they are given by

$$\Delta S_A^{(n)} \sim \log 2. \tag{3.1}$$

Their late time value is the same as that for ϕ in free massless scalar field theories with any spacetime dimensions. It can be interpreted as (Rényi) entanglement entropy for maximally entangled state in 2-qubit system. Therefore they do not show the difference between the effect of electromagnetic fields and that of free scalar one on the late-time structure of quantum entanglement. However, the time evolution of $\Delta S_A^{(n)}$ depends on the local operator which acts on the ground state as in figure 4. Even at $t \sim l$, time evolution of $\Delta S_A^{(n)}$ depends on the one which acts on the ground state. If $\mathcal{O} = B_{2,3}$ or $E_{2,3}$ acts on the ground state, $\Delta S_A^{(n \geq 2)}$ is given by

$$\Delta S_A^{(n)} \sim \frac{n}{n-1} \left(\frac{3(t-l)}{4l} \right) + \cdots, \tag{3.2}$$

where $\cdots$ are contributions from the higher order $\mathcal{O}\left(\left(\frac{t-l}{l} \right)^2 \right)$. $\Delta S_A^{(n \geq 2)}$ for $\mathcal{O} = E_1$ or B_1 at $t \sim l$ is given by

$$\Delta S_A^{(n)} \sim \frac{n}{n-1} \left(\frac{3(t-l)^2}{4l^2} \right) + \cdots. \tag{3.3}$$

$\cdots$ are contributions from the higher order $\mathcal{O}\left(\left(\frac{t-l}{l} \right)^3 \right)$. Their time evolution shows that quasi-particles created by $E_{2,3}$ $(B_{2,3})$ enter the region A faster than those generated by E_1 (B_1). These behaviors seem to be natural since particles created by E_1 (B_1) do not propagate along the direction parallel to x^1.

$\Delta S_A^{(n)}$ in table 1 shows that they are invariant under the transformation,

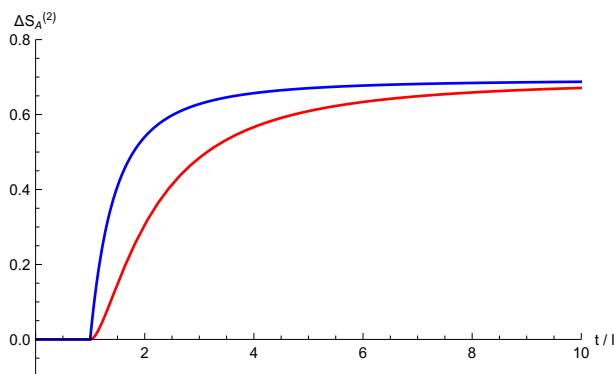$$F_{\mu\nu} \to \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \tag{3.4}$$

Figure 4. The time evolution of $\Delta S_A^{(2)}$ for E_1 (B_1) and $E_{2,3}$ ($B_{2,3}$). The horizontal and vertical axes correspond to time t/l and $\Delta S_A^{(2)}$, respectively. The red and blue lines correspond to $\Delta S_A^{(2)}$ for E_1 (B_1) and $E_{2,3}$ ($B_{2,3}$), respectively.

$\mathcal{O}$	$\Delta S_A^{(n)}$
E_1 or B_1	$\frac{1}{1-n} \log \left(\left(-\frac{(l+t)^2(l-2t)}{4t^3} \right)^n + \left(\frac{(t-l)^2(l+2t)}{4t^3} \right)^n \right)$
$E_{2,3}$	$\frac{1}{1-n} \log \left(\left(\frac{-l^3-3lt^2+4t^3}{8t^3} \right)^n + \left(\frac{l^3+3lt^2+4t^3}{8t^3} \right)^n \right)$

Table 1. $\Delta S_A^{(n)}$ for $\mathcal{O}$ in the region $0 < l \leq t$.

where $\epsilon_{\mu\nu\rho\sigma}$ is an anti-symmetric tensor. Under the transformation in (3.4), the local operator E_i (B_i) changes to $-B_i$ (E_i). Therefore, this duality changes a locally excited state to a different one.

3.2 Composite operators constructed of only electric or magnetic fields

The excited states which we consider here are generated by acting with the following operators: (a) $E_i E_j$ or $B_i B_j$, (b) $\mathbf{E}^2$ or $\mathbf{B}^2$. We study the time evolution and the late-time value of $\Delta S_A^{(n)}$ for them.

3.2.1 $\mathcal{O} = E_i E_j$ or $B_i B_j$

First we consider $\Delta S_A^{(n)}$ for the excited states generated by acting with $E_i E_j$ or $B_i B_j$ on the ground state. When $i = j$, the late time-value of $\Delta S_A^{(n)}$ is given by

$$\Delta S_A^{(n)} = -\frac{1}{1-n} \log \frac{4^n}{2^n + 2}. \tag{3.5}$$

It is the same as that of $\Delta S_A^{(n)}$ for ϕ^2 in the massless free scalar field theories as in [44, 45]. Therefore, the late-time value of $\Delta S_A^{(n)}$ ((Rényi) entanglement entropy of operator) can be interpreted in terms of entangled quasi-particles created by ϕ^2.

$\mathcal{O}$	D_1	N_1	N_2	N_3
E_1^2 or B_1^2	$\left(2\left(\frac{1}{16}\right)^2\right)^n$	$\left(2\,(f_1)^2\right)^n$	$\left(2\,(f_2)^2\right)^n$	$2^{2n}\,(f_1 f_2)^n$
$E_{2,3}^2$ or $B_{2,3}^2$	$\left(2\left(\frac{1}{16}\right)^2\right)^n$	$\left(2\,(f_3)^2\right)^n$	$\left(2\,(f_4)^2\right)^n$	$2^{2n}\,(f_3 f_4)^n$
$E_1 E_{2,3}$ or $E_1 B_{2,3}$ or $B_1 E_{2,3}$ or $B_1 B_{2,3}$	$\left(\frac{1}{16}\right)^{2n}$	$(f_1)^n\,(f_3)^n$	$(f_2)^n\,(f_4)^n$	$(f_2)^n\,(f_3)^n + (f_1)^n\,(f_4)^n$
$E_2 E_3$ or $B_2 B_3$	$\left(\frac{1}{16}\right)^{2n}$	$(f_3)^{2n}$	$(f_4)^{2n}$	$2\,(f_3)^n\,(f_4)^n$
$E_2 B_2$ or $E_3 B_3$	$\left(\left(\frac{1}{16}\right)^2\right)^n$	$(f_3)^{2n}$	$(f_4)^{2n}$	$2\,(f_3)^n\,(f_4)^n$
$E_1 B_1$	$\left(\frac{1}{16}\right)^{2n}$	$(f_1)^{2n}$	$(f_2)^{2n}$	$2\,(f_1)^n\,(f_2)^n$
Functions	$f_1 = -\frac{(l-2t)(l+t)^2}{64t^3}$	$f_2 = \frac{(l+2t)(l-t)^2}{64t^3}$	$f_3 = \frac{l^3+3lt^2+4t^3}{128t^3}$	$f_4 = \frac{-l^3-3lt^2+4t^3}{128t^3}$

Table 2. $\Delta S_A^{(n)} = \frac{1}{1-n}\log\left[\frac{N_1+N_2+N_3}{D_1}\right]$ for $\mathcal{O}$ in the region $0 < l < t$.
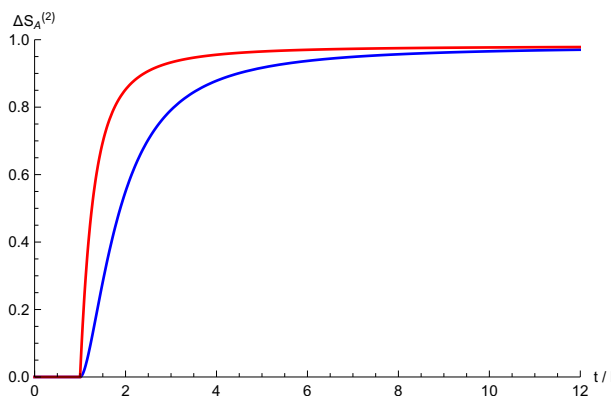

Figure 5. The time evolution of $\Delta S_A^{(2)}$ for $E_1 E_1$ and $E_2 E_2$. The horizontal and vertical axes correspond to time t/l and the excess of $\Delta S_A^{(2)}$, respectively. The blue line represents the evolution for $E_1 E_1$, and the red line for $E_2 E_2$. In the limit $t \to \infty$, they are all $\log \frac{8}{3}$.

When $i \neq j$, $\Delta S_A^{(n)}$ at the late time is given by

$$\Delta S_A^{(n)} = \log 4, \tag{3.6}$$

which can be interpreted as maximum (Rényi) entanglement entropy for $\rho_A = \frac{1}{4}\mathrm{diag}(1,1,1,1)$. It is the same as $\Delta S_A^{(n)}$ for the excited state given by acting with the operator $\phi^a \phi^b$ on the ground state. Here a,b denote the kind of scalar fields, and $a \neq b$. There are two kinds of massless free scalar fields. The time evolution of $\Delta S_A^{(n)}$ for them is summarized in table 2. Table 2 shows $\Delta S_A^{(n)}$ is invariant under the transformation in (3.4).

The time dependence of each excitation is displayed in figures 5 and 6. The evolutions without taking the limit $t \to \infty$ is different. However, amazingly the $\Delta S_A^{(2)}$ agree in the limit $t \to \infty$, for each cases.

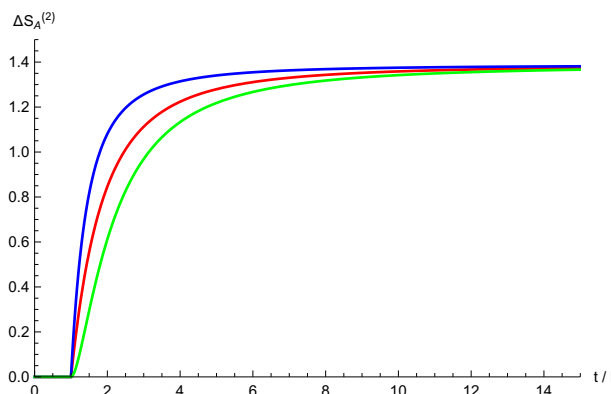**Figure 6**. The time evolution of $\Delta S_A^{(2)}$ for $E_1 E_2$, $E_2 E_3$ ($E_2 B_2$) and $E_1 B_1$. The horizontal and vertical axes correspond to time t/l and the excess of $\Delta S_A^{(2)}$, respectively. The red line represents the evolution for $E_1 E_2$, the blue line for $E_2 E_3$ ($E_2 B_2$) and the green line for $E_1 B_1$. In the limit $t \to \infty$, they are all $2 \log 2$.

3.2.2 $\mathcal{O} = \mathbf{E}^2$ or $\mathbf{B}^2$

In order to study whether E_1 acts on the late-time structure of quantum entanglement differently from $E_{2,3}$,[5] we study the late-time value of $\Delta S_A^{(n)}$ for the given locally excited state:

$$|\Psi\rangle = \mathcal{N} \mathbf{E}^2(-t, -l, \mathbf{x}) |0\rangle. \tag{3.7}$$

Before studying its late-time value, we comment on its time evolution. Before $t = l$, $\Delta S_A^{(n)}$ for the state in (3.7) vanishes and after $t = l$, it increases. Its time evolution is summarized in table 3.

After $t = l$, as in table 3, $\Delta S_A^{(n)}$ is given by

$$\Delta S_A^{(n)} = \frac{1}{1-n} \log \left[\frac{N_1 + N_2 + P_1 + P_2 + P_3}{D} \right], \tag{3.8}$$

where D, N_i and P_i are defined in table 3. If we take the late time limit $(0 < l \ll t)$, the ratios of P_i and N_i to D reduce to constant numbers [62],

$$\left(\frac{N_1}{D} \right)^{\frac{1}{n}} = \left(\frac{N_2}{D} \right)^{\frac{1}{n}} = 4^{-1}, \quad \left(\frac{P_1}{D} \right)^{\frac{1}{n}} = \left(\frac{P_2}{D} \right)^{\frac{1}{n}} = \left(\frac{P_3}{D} \right)^{\frac{1}{n}} = \left(\frac{1}{6} \right), \tag{3.9}$$

where we ignore the higher order contribution $\mathcal{O}\left(\frac{l}{t}\right)$. Amazingly, the sum of them is 1. Therefore if the effective reduced density matrix is defined by

$$\Delta S_A^{(n)} = \frac{1}{1-n} \log \left[\mathrm{tr}_A \left(\rho_A^e \right)^n \right], \tag{3.10}$$

then the matrix is given by

$$\rho_A^e = \frac{1}{24} \mathrm{diag}(6, 4, 4, 4, 6). \tag{3.11}$$

---

[5]The effect of E_1 can be different from that of $E_{2,3}$ on the structure since we choose $t = 0$, $x^1 \geq 0$ as the subsystem A.

$\mathcal{O}$	D_1	N_1	N_2	P_1	P_2	P_3
$\mathbf{B}^2 or \mathbf{E}^2$	$\left(2\cdot 3\left(\frac{1}{16}\right)^2\right)^n$	$\left(2f_1^2 + 2\cdot 2f_3^2\right)^n$	$\left(2f_2^2 + 2\cdot 2f_4^2\right)^n$	$2^{2n} f_1^n f_2^n$	$2^{2n} f_3^n f_4^n$	$2^{2n} f_3^n f_4^n$
$B_1^2 + E_1^2$	$\left(2\cdot 2\left(\frac{1}{16}\right)^2\right)^n$	$\left(2\cdot 2f_1^2\right)^n$	$\left(2\cdot 2f_2^2\right)^n$	$2^{2n} f_1^n f_2^n$	$2^{2n} f_1^n f_2^n$	0
$E_2 B_3$ or $E_3 B_2$	$\left(2^2\left(\frac{1}{4\cdot 8}\right)^2\right)^n$	$\left(2(g_1)^2 + 2(g_3)^2\right)^n$	$\left(2(g_2)^2 + 2(g_4)^2\right)^n$	$2^{2n}(g_2)^n(g_3)^n$	$2^{2n}(g_1)^n(g_4)^n$	0
$F^{\mu\nu}F_{\mu\nu}$ or $\mathbf{B}\cdot\mathbf{E}$	$\left(2\cdot 2\left(\frac{1}{16}\right)^2 + 2\cdot 4^2\left(\frac{1}{4\cdot 8}\right)^2\right)^n$	$\left(2\cdot 2f_1^2 + 2\cdot 4^2 g_1\cdot g_3\right)^n$	$\left(2\cdot 2f_2^2 + 2\cdot 4^2 g_2\cdot g_4\right)^n$	$2\cdot 2^{2n}(f_1)^n(f_2)^n$	$2\cdot 4^{2n}(g_1)^n(g_2)^n$	$2\cdot 4^{2n}(g_3)^n(g_4)^n$
$B_2 E_3 - B_3 E_2$	$\left(4\cdot 2\left(\frac{1}{4\cdot 8}\right)^2\right)^n$	$\left(2\cdot 2g_3^2 + 2\cdot 2g_1^2\right)^n$	$\left(2\cdot 2g_4^2 + 2\cdot 2g_2^2\right)^n$	$2^{2n}(g_3)^n(g_2)^n$	$2^{2n}(g_1)^n(g_4)^n$	0
Functions	$g_1 = \frac{(l+t)^3}{4\cdot 64 t^3}$	$g_2 = \frac{(t-l)^3}{4\cdot 64 t^3}$	$g_3 = \frac{(l+t)\left(l^2-4lt+7t^2\right)}{4\cdot 64 t^3}$	$g_4 = \frac{(t-l)\left(l^2+4lt+7t^2\right)}{4\cdot 64 t^3}$		

Table 3. $\Delta S_A^{(n)} = \frac{1}{1-n}\log\left[\frac{N_1+N_2+P_1+P_2+P_3}{D_1}\right]$ for $\mathcal{O}$ in the region $0 < l < t$.

The excess of $n-$th (Rényi) entanglement entropy, entanglement entropy and Min entropy are respectively given by

$$\Delta S_A^{(n)} = \frac{1}{n-1}\log\left(\frac{12^n}{3\cdot 2^n + 2\cdot 3^n}\right),$$
$$\Delta S_A = \frac{\log(24)}{2}, \tag{3.12}$$
$$\Delta S_A^{(\infty)} = \log 4,$$

which can be interpreted in terms of quasi-particles created by $\left(\phi^1\right)^2 + \left(\phi^2\right)^2 + \left(\phi^3\right)^2$, which is constructed of three kinds of free scalar fields. Therefore, there are no differences between the effect of E_1 and that of $E_{2,3}$ on the late-time structure of quantum entanglement. As in the table 2, $\Delta S_A^{(n)}$ for $\mathbf{E}^2$ is equivalent to that for $\mathbf{B}^2$. Therefore, they are electric-magnetic duality invariants.

3.3 Composite operators constructed of both electric and magnetic fields

In the previous two subsections we study how the entanglement structure changes at the late time if either electric or magnetic fields act on the ground state. However, we do not uncover how it changes at the late time when both of them act on the ground state. Here we study $\Delta S_A^{(n)}$ for (a) $E_1^2 + B_1^2$, (b) $E_i B_j$ and (c) $F^{\mu\nu}F_{\mu\nu}$ and $\mathbf{B}\cdot\mathbf{E}$, which can show that E_i and B_i act on the late-time structure of quantum entanglement differently from scalar fields such as ϕ^a.

3.3.1 $E_1^2 + B_1^2$

Here in order to study whether there are differences between the effects of electric and magnetic fields on the late-time structure of quantum entanglement, we study $\Delta S_A^{(n)}$ for the following excited state:

$$|\Psi\rangle = \mathcal{N}\left(E_1^2 + B_1^2\right)(-t, -l, \mathbf{x})|0\rangle. \tag{3.13}$$

Before investing the late time value of $\Delta S_A^{(n)}$, let's study the time evolution of $\Delta S_A^{(n)}$. $\Delta S_A^{(n)}$ vanishes before $t = l$. After $t = l$, its time evolution is summarized in table 3. If

we take the late time limit $t \to \infty$, the late time value of $\Delta S_A^{(n)}$ reduces to the (Rényi) entanglement entropy whose effective reduced density matrix is given by

$$\rho_A^e = \frac{1}{4}\text{diag}\,(1,1,1,1)\,. \tag{3.14}$$

Its entropies are given by

$$\Delta S_A^{(n)} = \Delta S_A = \Delta S_A^{(\infty)} = \log 4\,. \tag{3.15}$$

It shows there are no differences between the effects of electric and magnetic fields on the late-time structure.

3.3.2 $E_i B_j$

Here let's find out how the operators constructed of both electric and magnetic fields, $E_i B_j$, affect the late-time structure of quantum entanglement. The late-time values of $\Delta S_A^{(n)}$ for $E_i B_j$ except for $E_2 B_3$ and $E_3 B_2$ are the same as (3.6). Their time evolution is summarized in table 2.

On the other hand, after $t = l$ the time evolution of $\Delta S_A^{(n)}$ for $E_2 B_3$ or $E_3 B_2$ is summarized in table 3. We can see that it has the electric-magnetic duality from the table 3. The late-time value of $\Delta S_A^{(n)}$ is given by (Rényi) entanglement entropy whose reduced density matrix is given by

$$\rho_A^e = \frac{1}{64}\text{diag}(25,7,7,25)\,. \tag{3.16}$$

Its entropies are given by

$$\begin{aligned}
\Delta S_A^{(n)} &= -\frac{\log\left(2^{1-6n}\left(7^n + 25^n\right)\right)}{n-1}, \\
\Delta S_A &= \log\left(\frac{64}{5 \cdot 5^{9/16} 7^{7/32}}\right), \\
\Delta S_A^{(\infty)} &= 2\log\left(\frac{8}{5}\right).
\end{aligned} \tag{3.17}$$

It shows how different the effect of E_1 (B_1) is from that of $E_{2,3}$ ($B_{2,3}$) on the structure. The value can not be interpreted in terms of quasi-particles created by scalar fields such as $\phi^a \phi^b$. As we will explain later, in the entangled quasi-particle interpretation, there is a commutation relation between the quasi-particle created by E_2 (B_2) and that by B_3 (E_3).

3.3.3 $\mathbf{B} \cdot \mathbf{E}$ and $F_{\mu\nu} F^{\mu\nu}$ and $B_2 E_3 - B_3 E_2$

We finally study $\Delta S_A^{(n)}$ for more complicated operators, $\mathbf{B} \cdot \mathbf{E}$, $F_{\mu\nu} F^{\mu\nu}$ and $B_2 E_3 - B_3 E_2$. Before $t = l$, $\Delta S_A^{(n)}$ for them vanishes, but after $t = l$, it increases. Their details are summarized in table 3.[6] It shows that $\Delta S_A^{(n)}$ for $\mathbf{B} \cdot \mathbf{E}$ is the same as that for $F^{\mu\nu} F_{\mu\nu}$.

---

[6] $\Delta S_A^{(n)}$ is commuted by the Green's functions in appendix B.

The effective reduced density matrices for $\mathbf{B} \cdot \mathbf{E}$ ($F^{\mu\nu}F_{\mu\nu}$) and $B_2 E_3 - B_3 E_2$ are given by

$$\rho_A^e = \frac{1}{192} \text{diag} \left(30, 30, 16, 16, 49, 49, 1, 1\right), \qquad \text{for} \quad \mathcal{O} = \mathbf{B} \cdot \mathbf{E} \quad (F^{\mu\nu}F_{\mu\nu}),$$
$$\rho_A^e = \frac{1}{128} \text{diag} \left(50, 50, 7, 7, 7, 7\right), \qquad \text{for} \quad \mathcal{O} = B_2 E_3 - B_3 E_2. \tag{3.18}$$

Entropies are respectively given by

$$\Delta S_A^{(n)} = \frac{1}{n-1} \log \left(\frac{2^{6n-1} 3^n}{16^n + 30^n + 49^n + 1} \right),$$
$$\Delta S_A = \log \left(\frac{32 \cdot 3^{11/16} \sqrt[48]{\frac{2}{7}}}{7 \cdot 5^{5/16}} \right), \tag{3.19}$$
$$\Delta S_A^{(\infty)} = \log \left(\frac{192}{49} \right).$$

which are for $\mathcal{O} = \mathbf{B} \cdot \mathbf{E}$ (or $F^{\mu\nu}F_{\mu\nu}$), and

$$\Delta S_A^{(n)} = -\frac{\log \left(2^{1-7n} \left(2 \cdot 7^n + 50^n \right) \right)}{n-1},$$
$$\Delta S_A = \log \left(\frac{64 \left(\frac{2}{7} \right)^{7/32}}{5 \cdot 5^{9/16}} \right), \tag{3.20}$$
$$\Delta S_A^{(\infty)} = 2 \log \left(\frac{8}{5} \right),$$

which are for $\mathcal{O} = B_2 E_3 - B_3 E_2$. As we will explain in the next section, they can be reproduced by using a late-time algebra which electromagnetic quasi-particles obey.

The time dependence of $\mathbf{E}^2$, $F^{\mu\nu}F_{\mu\nu}$ and $B_2 E_3 - B_3 E_2$ are shown in figure 7. We can see that their $\Delta S_A^{(2)}$ increases diffently and reach different finite values in the limit $t \to \infty$. As we see in the following section, we find significant differences in the late time algebra between $\mathbf{E}^2$ and the others.

4 A late-time algebra

We interpret the late-time value of $\Delta S_A^{(n)}$ in terms of quasi-particles. More precisely, let's interpret the effective reduced density matrix in (3.10) in terms of quasi-particles. The effective reduced density matrix for the excited state generated by a composite operator $\mathcal{O}(-t, -l, \mathbf{x})$ is defined by

$$\Delta S_A^{(n)} = \frac{1}{1-n} \log \left[\text{tr}_A \left(\rho_A^e \right)^n \right] = \frac{1}{1-n} \log \left[\text{tr}_A \left(\hat{\mathcal{N}}^2 \mathcal{O} |0\rangle \langle 0| \mathcal{O}^\dagger \right)^n \right], \tag{4.1}$$

where $\hat{\mathcal{N}}$ is a normalization constant. The operator $\mathcal{O}$ is assumed to be constructed of electric and magnetic fields.[7] As in [44–46, 48], these fields can be decomposed into left

---

[7]Here ρ_A^e is not the same as the reduced density matrix for the locally excited state. It is for a "effective" state $\hat{\mathcal{N}}\mathcal{O} |0\rangle$. It is different form the "original" locally excited state.
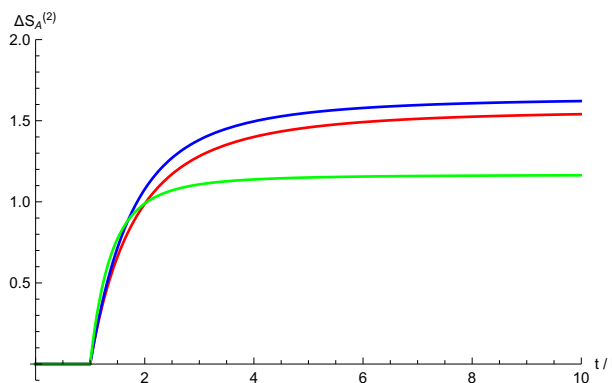
Figure 7. The time evolution of $\mathbf{E}^2$, $F^{\mu\nu}F_{\mu\nu}$ and $B_2 E_3 - B_3 E_2$. The horizontal and vertical axes correspond to time t/l and the excess of $\Delta S_A^{(2)}$ respectively. The red line represents the evolution for $\mathbf{E}^2$, the blue line for $F^{\mu\nu}F_{\mu\nu}$ and the green line for $B_2 E_3 - B_3 E_2$.

moving and right moving electromagnetic quasi-particles as follows,

$$E_i = E_i^{L\dagger} + E_i^{R\dagger} + E_i^L + E_i^R,$$
$$B_i = B_i^{L\dagger} + B_i^{R\dagger} + B_i^L + B_i^R, \tag{4.2}$$

where since we take $x^1 \geq 0$ as A in this paper, left-moving and right-moving quasi-particles correspond to particles included in B and A at late time, respectively. The ground state for them is defined by

$$E_i^{L,R} |0\rangle_{L,R} = B_i^{L,R} |0\rangle_{L,R} = 0,$$
$$|0\rangle = |0\rangle_L \otimes |0\rangle_R. \tag{4.3}$$

The late-time algebra which quasi-particles obey is given by

$$\left[E_i^{L,R}, E_j^{L,R\dagger} \right] = C \delta_{ij},$$
$$\left[B_i^{L,R}, B_j^{L,R\dagger} \right] = C \delta_{ij}, \tag{4.4}$$

which is obtained so that the results by the replica trick are reproduced. Here C is a real number.[8] In the gauge theory in addition to (4.2), we need the following commutation relation for different particles:

$$\left[E_3^{L,R}, B_2^{L,R\dagger} \right] = X_{R,L}, \quad \left[E_2^{L,R}, B_3^{L,R\dagger} \right] = Y_{R,L}, \tag{4.5}$$

where $X_{R,L}$ and $Y_{R,L}$ are given by

$$X_R = -X_L = Y_L = -Y_R,$$
$$X_{R,L}^2 = Y_{R,L}^2 = \frac{9}{16} C^2. \tag{4.6}$$

---

[8]The redefinition of quasi-particles can absorb the constant C.

Here $X_{R,L}$ and $Y_{R,L}$ are real numbers.[9] The commutation relation between electric (magnetic) quasi-particles is determined so that the effective density matrices computed by (4.4) are consistent with (3.11) respectively. The relation for the quasi-particles of E_1 should be the same as that for B_1 so that the effective density matrix in (4.1) reproduces $\Delta S_A^{(n)}$ for the matrix in (3.14). That relation between quasi-particles generated by E_2 (E_3) and those by B_3 (B_2) reproduces the matrix in (3.16). We also check that $\Delta S_A^{(n)}$ for $\mathcal{O} = \mathbf{B} \cdot \mathbf{E}$ ($F^{\mu\nu} F_{\mu\nu}$), $B_3 E_2 - B_2 E_3$ are reproduced by using the commutation relation in (4.4) and (4.5).

The relation in (4.5) shows that the effect of fields along the direction vertical to ∂A is significantly different from that along the direction parallel to ∂A on the late-time structure. It makes the effects of electromagnetic fields different from that of free scalar fields on the late-time structure of quantum entanglement.

5 Conclusion and future problems

We have studied how gauge invariant operators such as E_i, B_i and the composite operators constructed of them changes the structure of quantum entanglement by studying $\Delta S_A^{(n)}$. We studied whether $\Delta S_A^{(n)}$ for locally excited states created by gauge invariant local operators reflects the electric-magnetic duality. $\Delta S_A^{(n)}$, which we studied in this paper, is invariant under the duality transformation. If only E_i or B_i acts on the ground state, without taking the late time limit, the time evolution of $\Delta S_A^{(n)}$ depends on them. Due to the duality, $\Delta S_A^{(n)}$ for E_i is equal to that for B_i. Around $t = l$, $\Delta S_A^{(n)}$ for $E_{2,3}$ ($B_{2,3}$) increases slower than that for E_1 (B_1). However they can not show the difference between the effects of electromagnetic fields and that of scalar fields on the late-time structure because the late-time values of $\Delta S_A^{(n)}$ for them can be interpreted in terms of quasi-particle created by scalar fields.

On the other hand, the late-time values of $\Delta S_A^{(n)}$ for the specific operators constructed of both electric and magnetic fields can not be interpreted in terms of quasi-particles by scalar fields. They show that there are differences between the effects of electromagnetic fields and that of scalar fields on the late-time structure of quantum entanglement. If their late-time values are interpreted in terms of electromagnetic quasi-particles in (4.2), there are commutation relations between E_2 (E_3) and B_3 (B_2), which make the effect of electromagnetic field significantly different from that of scalar fields on the late-time structure. The effect of E_1 and B_1 on the late-time structure is different from that of $E_{2,3}$ and $B_{2,3}$.

Future problems. We finish with comments on a few future problems:

- In this paper we only consider $4d$ Maxwell gauge theory which has conformal symmetry. $D(\neq 4)$ dimensional Maxwell gauge theory is not a CFT. Therefore, it is interesting to generalize the analysis in $4d$ Maxwell theory to that in the theories with general dimensions.

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[9]We find the correspondence between propagators and commutation relations. The commutations can be defined by the late time limit of propagators. We will discuss the detail of the correspondence in [62]. When we use this correspondence, $X_L = -\frac{3}{4} C$.

- We expect that the structure of the late-time algebra depends on the spacetime dimension D. Then, it is also interesting to study it in general dimensions.

Acknowledgments

A Green's functions

The relation between E_i, B_i and field strengths which are defined in Euclidean space is given by

$$E_i = -iF_{\tau i}, \quad B_1 = -F_{23}, \quad B_2 = F_{13}, \quad B_3 = -F_{12}. \tag{A.1}$$

The analytic continued Green's functions are defined by

$$
\begin{aligned}
\left\langle E_1(\theta)E_1(\theta') \right\rangle &= F_{E1E1}(\theta - \theta'), \\
\left\langle E_2(\theta)E_2(\theta') \right\rangle = \left\langle E_3(\theta)E_3(\theta') \right\rangle &= F_{E2E2}(\theta - \theta'), \\
\left\langle B_1(\theta)B_1(\theta') \right\rangle &= F_{B1B1}(\theta - \theta'), \\
\left\langle B_2(\theta)B_2(\theta') \right\rangle = \left\langle B_3(\theta)B_3(\theta') \right\rangle &= F_{B2B2}(\theta - \theta'), \\
\left\langle E_2(\theta)B_3(\theta') \right\rangle &= F_{E2B3}(\theta - \theta'), \\
\left\langle B_3(\theta)E_2(\theta') \right\rangle &= F_{B3E2}(\theta - \theta'), \\
\left\langle E_3(\theta)B_2(\theta') \right\rangle &= F_{E3B2}(\theta - \theta'), \\
\left\langle B_2(\theta)E_3(\theta') \right\rangle &= F_{B2E3}(\theta - \theta').
\end{aligned}
\tag{A.2}
$$

If the limit $\epsilon \to 0$ is taken, their leading terms for $n = 1$ are given by

$$
\begin{aligned}
F_{E1E1}(\theta_1 - \theta_2) &\sim \frac{1}{16\pi^2\epsilon^4}, \\
F_{E2E2}(\theta_1 - \theta_2) &\sim \frac{1}{16\pi^2\epsilon^4}, \\
F_{B1B1}(\theta_1 - \theta_2) &\sim \frac{1}{16\pi^2\epsilon^4}, \\
F_{B2B2}(\theta_1 - \theta_2) &\sim \frac{1}{16\pi^2\epsilon^4}.
\end{aligned}
\tag{A.3}
$$

That for the arbitrary n in $0 < t < l$ are the same as given in (A.3).

That for the arbitrary n in $0 < l \leq t$ are given by

$$F_{E1E1}(\theta_1-\theta_2)=F_{E1E1}(\theta_2-\theta_1) \sim -\frac{(l-2t)(l+t)^2}{64\pi^2 t^3 \epsilon^4},$$

$$F_{E2E2}(\theta_1-\theta_2)=F_{E2E2}(\theta_2-\theta_1) \sim \frac{l^3+3lt^2+4t^3}{128\pi^2 t^3 \epsilon^4},$$

$$F_{B1B1}(\theta_1-\theta_2)=F_{B1B1}(\theta_2-\theta_1) \sim -\frac{(l-2t)(l+t)^2}{64\pi^2 t^3 \epsilon^4},$$

$$F_{B2B2}(\theta_1-\theta_2)=F_{B2B2}(\theta_2-\theta_1) \sim \frac{l^3+3lt^2+4t^3}{128\pi^2 t^3 \epsilon^4},$$

$$F_{E2B3}(\theta_1-\theta_2)=F_{E2B3}(\theta_2-\theta_1) \sim \frac{3(t-l)(l+t)}{128\pi^2 t^2 \epsilon^4},$$

$$F_{B3E2}(\theta_1-\theta_2)=F_{B3E2}(\theta_2-\theta_1) \sim \frac{3(t-l)(l+t)}{128\pi^2 t^2 \epsilon^4},$$

$$F_{E3B2}(\theta_1-\theta_2)=F_{E3B2}(\theta_2-\theta_1) \sim \frac{3(l-t)(l+t)}{128\pi^2 t^2 \epsilon^4},$$

$$F_{B2E3}(\theta_1-\theta_2)=F_{B2E3}(\theta_2-\theta_1) \sim \frac{3(l-t)(l+t)}{128\pi^2 t^2 \epsilon^4},$$

$$F_{E1E1}(\theta_1-\theta_2+2\pi)=F_{E1E1}(\theta_2-\theta_1-2\pi)$$
$$=F_{E1E1}(\theta_1-\theta_2-2(n-1)\pi)=F_{E1E1}(\theta_2-\theta_1+2(n-1)\pi) \sim \frac{(l-t)^2(l+2t)}{64\pi^2 t^3 \epsilon^4},$$

$$F_{E2E2}(\theta_1-\theta_2+2\pi)=F_{E2E2}(\theta_2-\theta_1-2\pi)$$
$$=F_{E2E2}(\theta_1-\theta_2-2(n-1)\pi)=F_{E2E2}(\theta_2-\theta_1+2(n-1)\pi) \sim -\frac{l^3+3lt^2-4t^3}{128\pi^2 t^3 \epsilon^4}, \qquad \text{(A.4)}$$

$$F_{B1B1}(\theta_1-\theta_2+2\pi)=F_{B1B1}(\theta_2-\theta_1-2\pi)$$
$$=F_{B1B1}(\theta_1-\theta_2-2(n-1)\pi)=F_{B1B1}(\theta_2-\theta_1+2(n-1)\pi) \sim \frac{(l-t)^2(l+2t)}{64\pi^2 t^3 \epsilon^4},$$

$$F_{B2B2}(\theta_1-\theta_2+2\pi)=F_{B2B2}^{(n,l)}(\theta_2-\theta_1-2\pi)$$
$$=F_{B2B2}(\theta_1-\theta_2-2(n-1)\pi)=F_{B2B2}^{(n,l)}(\theta_2-\theta_1+2(n-1)\pi) \sim -\frac{l^3+3lt^2-4t^3}{128\pi^2 t^3 \epsilon^4},$$

$$F_{E2B3}(\theta_1-\theta_2+2\pi)=F_{E2B3}(\theta_2-\theta_1-2\pi)$$
$$=F_{E2B3}(\theta_1-\theta_2-2(n-1)\pi)=F_{E2B3}(\theta_2-\theta_1+2(n-1)\pi) \sim \frac{3(l-t)(l+t)}{128\pi^2 t^2 \epsilon^4},$$

$$F_{B3E2}(\theta_1-\theta_2+2\pi)=F_{B3E2}(\theta_2-\theta_1-2\pi)$$
$$=F_{B3E2}(\theta_1-\theta_2-2(n-1)\pi)=F_{B3E2}(\theta_2-\theta_1+2(n-1)\pi) \sim \frac{3(l-t)(l+t)}{128\pi^2 t^2 \epsilon^4},$$

$$F_{E3B2}(\theta_1-\theta_2+2\pi)=F_{E3B2}(\theta_2-\theta_1-2\pi)$$
$$=F_{E3B2}(\theta_1-\theta_2-2(n-1)\pi)=F_{E3B2}(\theta_2-\theta_1+2(n-1)\pi) \sim \frac{3(t-l)(l+t)}{128\pi^2 t^2 \epsilon^4},$$

$$F_{B2E3}(\theta_1-\theta_2+2\pi)=F_{B2E3}(\theta_2-\theta_1-2\pi)$$
$$=F_{B2E3}(\theta_1-\theta_2-2(n-1)\pi)=F_{B2E3}(\theta_2-\theta_1+2(n-1)\pi) \sim \frac{3(t-l)(l+t)}{128\pi^2 t^2 \epsilon^4}.$$

The contribution of the other propagators is much smaller than those in (A.4).

B   Other bases

We introduce new bases $\mathcal{O}_{1,2} = \frac{E_2 \pm B_3}{2}$, $\mathcal{O}_{3,4} = \frac{E_3 \pm B_2}{2}$. Their Green's functions in $(0 < t < l)$ are given by

$$
\begin{aligned}
\langle \mathcal{O}_1 \mathcal{O}_1 \rangle (\theta_1 - \theta_2) &\sim \frac{1}{4 \cdot 8\pi^2 \epsilon^4}, \\
\langle \mathcal{O}_2 \mathcal{O}_2 \rangle (\theta_1 - \theta_2) &\sim \frac{1}{4 \cdot 8\pi^2 \epsilon^4}, \\
\langle \mathcal{O}_3 \mathcal{O}_3 \rangle (\theta_1 - \theta_2) &\sim \frac{1}{4 \cdot 8\pi^2 \epsilon^4}, \\
\langle \mathcal{O}_4 \mathcal{O}_4 \rangle (\theta_1 - \theta_2) &\sim \frac{1}{4 \cdot 8\pi^2 \epsilon^4},
\end{aligned}
\tag{B.1}
$$

and those in $(0 < l \leq t)$ are given by

$$
\begin{aligned}
\langle \mathcal{O}_1 \mathcal{O}_1 \rangle (\theta_1 - \theta_2) = \langle \mathcal{O}_1 \mathcal{O}_1 \rangle (\theta_2 - \theta_1) &\sim \frac{(l+t)\left(l^2 - 4lt + 7t^2\right)}{4 \cdot 64\pi^2 t^3 \epsilon^4}, \\
\langle \mathcal{O}_2 \mathcal{O}_2 \rangle (\theta_1 - \theta_2) = \langle \mathcal{O}_2 \mathcal{O}_2 \rangle (\theta_2 - \theta_1) &\sim \frac{(l+t)^3}{4 \cdot 64\pi^2 t^3 \epsilon^4}, \\
\langle \mathcal{O}_3 \mathcal{O}_3 \rangle (\theta_1 - \theta_2) = \langle \mathcal{O}_3 \mathcal{O}_3 \rangle (\theta_2 - \theta_1) &\sim \frac{(l+t)^3}{4 \cdot 64\pi^2 t^3 \epsilon^4}, \\
\langle \mathcal{O}_4 \mathcal{O}_4 \rangle (\theta_1 - \theta_2) = \langle \mathcal{O}_4 \mathcal{O}_4 \rangle (\theta_2 - \theta_1) &\sim \frac{(l+t)\left(l^2 - 4lt + 7t^2\right)}{4 \cdot 64\pi^2 t^3 \epsilon^4}, \\
\langle \mathcal{O}_1 \mathcal{O}_1 \rangle (\theta_1 - \theta_2 + 2\pi) &= \langle \mathcal{O}_1 \mathcal{O}_1 \rangle (\theta_2 - \theta_1 - 2\pi) \\
= \langle \mathcal{O}_1 \mathcal{O}_1 \rangle (\theta_1 - \theta_2 - 2(n-1)\pi) = \langle \mathcal{O}_1 \mathcal{O}_1 \rangle (\theta_2 - \theta_1 + 2(n-1)\pi) &\sim -\frac{(l-t)^3}{4 \cdot 64\pi^2 t^3 \epsilon^4}, \\
\langle \mathcal{O}_2 \mathcal{O}_2 \rangle (\theta_1 - \theta_2 + 2\pi) &= \langle \mathcal{O}_2 \mathcal{O}_2 \rangle (\theta_2 - \theta_1 - 2\pi) \\
= \langle \mathcal{O}_2 \mathcal{O}_2 \rangle (\theta_1 - \theta_2 - 2(n-1)\pi) = \langle \mathcal{O}_2 \mathcal{O}_2 \rangle (\theta_2 - \theta_1 + 2(n-1)\pi) &\sim -\frac{(l-t)\left(l^2 + 4lt + 7t^2\right)}{4 \cdot 64\pi^2 t^3 \epsilon^4}, \\
\langle \mathcal{O}_3 \mathcal{O}_3 \rangle (\theta_1 - \theta_2 + 2\pi) &= \langle \mathcal{O}_3 \mathcal{O}_3 \rangle (\theta_2 - \theta_1 - 2\pi) \\
= \langle \mathcal{O}_3 \mathcal{O}_3 \rangle (\theta_1 - \theta_2 - 2(n-1)\pi) = \langle \mathcal{O}_3 \mathcal{O}_3 \rangle (\theta_2 - \theta_1 + 2(n-1)\pi) &\sim -\frac{(l-t)\left(l^2 + 4lt + 7t^2\right)}{4 \cdot 64\pi^2 t^3 \epsilon^4}, \\
\langle \mathcal{O}_4 \mathcal{O}_4 \rangle (\theta_1 - \theta_2 + 2\pi) &= \langle \mathcal{O}_4 \mathcal{O}_4 \rangle (\theta_2 - \theta_1 - 2\pi) \\
= \langle \mathcal{O}_4 \mathcal{O}_4 \rangle (\theta_1 - \theta_2 - 2(n-1)\pi) = \langle \mathcal{O}_4 \mathcal{O}_4 \rangle (\theta_2 - \theta_1 + 2(n-1)\pi) &\sim -\frac{(l-t)^3}{4 \cdot 64\pi^2 t^3 \epsilon^4}.
\end{aligned}
\tag{B.2}
$$

The Green's functions $\langle \mathcal{O}_i \mathcal{O}_{j \neq i} \rangle$ vanish.

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