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On some 3-point functions in the W_4 CFT and related braiding matrix

P. Furlan[a] **and V.B. Petkova**[b]

[a] *Dipartimento di Fisica dell'Università di Trieste, Trieste, Italy*
[b] *Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Sofia, Bulgaria*

E-mail: furlan@ts.infn.it, petkova@inrne.bas.bg

Abstract: We construct a class of 3-point constants in the $sl(4)$ Toda conformal theory W_4, extending the examples in Fateev and Litvinov [1]. Their knowledge allows to determine the braiding/fusing matrix transforming 4-point conformal blocks of one fundamental, labelled by the 6-dimensional $sl(4)$ representation, and three partially degenerate vertex operators. It is a 3×3 submatrix of the generic 6×6 fusing matrix consistent with the fusion rules for the particular class of representations. We check a braiding relation which has wider applications to conformal models with $sl(4)$ symmetry. The 3-point constants in dual regions of central charge are compared in preparation for a BPS like relation in the $\hat{sl}(4)$ WZW model.

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Contents

1 Introduction

The 2d conformal field theories (CFT) related to the $sl(2)$ algebra, like the Virasoro, the WZW models with the affine sl(2) KM algebra and their supersymmetric extensions, are by now well established. This includes explicit expressions for basic data as the operator product expansion (OPE) coefficients (3-point functions) and the braiding/fusing matrices transforming conformal blocks. Much less is known about these structures in the CFT with higher rank symmetries, although a considerable progress in Toda CFT [2] was made by Fateev and Litvinov (FL) [1, 3]. Further advances in the field are important for the development of the higher rank 2d CFT as well as for potential applications in the string theory side of the AdS/CFT correspondence.

In the free field (Coulomb gas) approach the OPE constants are represented by complicated integrals which have to be computed explicitly before analytic continuation. The alternative derivation of functional relations arising from locality (crossing symmetry) of particular 4-point functions involving degenerate vertex operators requires the knowledge of fundamental braiding/fusing matrix elements, which in general are also part of the problem.

In [1, 3] Fateev and Litvinov developed a general method of recursively computing certain class of conformal integrals and gave explicit examples of 3-point constants.[1] In the case of Toda W_3 theory they have as well computed the fundamental fusing matrix directly from the integral representations of the 4-point blocks; some partial results in the general W_n case were also obtained.

In this paper we are dealing with the sl(4) Toda conformal theory W_4. The 3-point functions known so far involve one vertex operator V_β with a degenerate charge β proportional

[1] Apart from these traditional 2d methods a novel approach to the computation of the 3-point constants is provided by the (5d version of the) AGT-W relation [4, 5], see [6, 7], where the main example of [1] has been recently reproduced, as well as references therein.

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to the fundamental weight ω_1, or $\omega_3 = \omega_1^*$, i.e., the highest weight of the 4-dimensional $sl(4)$ representation. Our focus instead is on the symmetric representations $\beta = \beta^*$ and, in particular, $\beta = -k\omega_2 b$, where ω_2 is the highest weight of the 6-dimensional fundamental $sl(4)$ representation and k is arbitrary.[2] The real parameter b parametrises Toda central charge. In section 2 we present a 3-point OPE constant for two partially degenerate ("4d scalars") and one symmetric representations

$$(\beta_1, \alpha_i) = 0 = (\beta_2, \alpha_i)\,, \text{ for } i = 1, 3\,, \; (\beta_3, \alpha_1) = (\beta_3, \alpha_3)\,. \tag{1.1}$$

The 3-point constant is obtained by deriving and solving a recurrence relation for the corresponding Coulomb gas integrals along the method of [1, 8], which is then analytically continued.

In section 3 we use this data to derive the fusing matrix F transforming the corresponding 4-point conformal blocks with one fundamental vertex $V_{-\omega_2 b}$. Here we follow a path somewhat opposite to the standard consideration in which - given the fusing matrix, one solves for the 3-point constants the system of equations implied by locality of the 4-point function. We shall not need the explicit integral realisation of this particular Toda 4-point function with three more partially degenerate representations of the type $\beta_a = -k_a \omega_2 b\,, a = 1, 2, 3\,.$ In the intermediate channels appear also vertex operators with symmetric weights so that in the equations the more general constants of the type (1.1) derived in section 2 are needed. The restriction to chiral vertex operators V_{β_a} of such particular highest weights effectively restricts the braiding/fusing matrix to a 3×3 submatrix; its matrix elements are explicitly described.

Finally in this section we check a braiding identity, which is equivalent to a standard identity for the modular group on the sphere with 4 holes. This relation imposes restrictions solely on certain products of F matrix elements and allows in principle for more general solutions for the individual F matrix elements than the ones computed in the W_4 CFT. The semi-classical "heavy charges" limit of the identity is a particular $sl(4)$ analog of the one exploited in the strong coupling $sl(2)$ sigma model constructions in [9, 10]. This suggests that the explicit expressions for the products of the fusing matrix elements extracted from Toda CFT (or their closely related WZW model counterparts) may eventually be used as a first step in higher rank generalisations of that work.

In the last section 4 we compare the 3-point constants in two regions of the central charge, W_4 analogs of the two Virasoro theory ingredients of Liouville gravity (non-critical string theory) with $c > 25$ (Liouville) and $c < 1$ ("matter"). We discuss a BPS-like relation for the two sets of weights which is intrinsic for the vertex operators of the $\hat{sl}(4)$ WZW models, related to Toda theory by the quantum Hamiltonian Drinfeld Sokolov (DS) reduction. Although it seems that there are no direct W_4 analogs of the physical fields of Liouville gravity, we show that the product of the two 3-point Toda constants with weights subject to the BPS constraint trivialises in the semi-classical "light" charges" limit. Furthermore we speculate on the possible implications for the related WZW 3-point

[2] These highest weights correspond to scalars in the context of 4d conformal group representations, where in general the (nonnegative, integer) components $2j_i = -(\beta, \alpha_i)/b, i = 1, 3$ label the $SL(2, \mathbb{C})$ spins, while $\triangle = (\beta, \omega_2)/b$ corresponds to the 4d conformal dimension.

correlators (the determination of which is still an open problem) and make a comparison with computations of related 3-point correlators in the supergravity approximations of the $AdS_5 \times S^5$ strings [11, 12].

The appendix contains some details of the computation of the 3-point functions, including one slightly more general constant not presented in section 2, as well as an alternative Coulomb like representation of the 4-point functions discussed in section 3. It reveals a connection to certain Liouville correlators.

2 3-point W_4 constants

We consider the W_4 CFT with central charge

$$c_T = 3(1+20Q^2) = 3\left(41+20\left(b^2+\frac{1}{b^2}\right)\right) > 243\,, \quad Q = \frac{1}{b}+b\,, \tag{2.1}$$

for real values of the parameter b. We shall skip the detailed presentation of the basics of Toda conformal theory and the free field (Coulomb gas) representation of the correlation functions: the reader is referred to [1], as well as to the original paper of Fateev and Lukyanov [2], formulated in the dual region of central charge with $b \to ib$ in (2.1).

The OPE constant of 2d scalar vertex operators is

$$c(\beta_1, \beta_2\,, 2\rho Q - \beta_3) = \lim_{x_3 \to \infty} (x_3^2)^{2\triangle(\beta_3)} \langle V_{2\rho Q - \beta_3}(x_3) V_{\beta_2}(1) V_{\beta_1}(0)\rangle_{\text{Coulomb}}$$

where the conformal dimension is given by the $sl(4)$ inner product

$$\triangle(\beta) = \frac{1}{2}(\beta, 2\rho Q - \beta) \tag{2.2}$$

and $\rho = \sum_{i=1}^{3} \omega_i$ is the Weyl vector. The dimension (2.2), as well as the two other W_4 quantum numbers, are invariant with respect to an action of the Weyl reflection group

$$w \star \beta = Q\rho + w(\beta - Q\rho) \tag{2.3}$$

so that any of the vertex operators $V_{w\star\beta}$ represents the same field. The Coulomb gas representation of the OPE constant is defined for the charge conservation condition

$$\beta_{12}^3 + b\sum_i s_i\alpha_i := \beta_1 + \beta_2 - \beta_3 + b\sum_i s_i \alpha_i = 0 \;\Rightarrow\; bs_i = -(\beta_{12}^3, \omega_i)\,. \tag{2.4}$$

The integers s_i in front of the simple roots α_i in (2.4) count the number of screening charge vertex operators $V_{\alpha_i b}(z,\bar z)$, $i = 1,2,3$, from the interaction term of Toda action. These operators are spinless fields of dimension $\triangle(\alpha_i b) = 1$. Formula (2.4) describes a generic $sl(4)$ type fusion rule in which β_3 is obtained by a shift of, say, β_1 with the weight diagram $\Gamma_{-\beta_2/b} = \{-\beta_2/b - s_i\alpha_i\}$ of the representation of highest weight $-\beta_2/b$, times $(-b)$.

The OPE constant is given by a $\sum_i s_i-$ multiple 2d integral $I_{s_1,s_2,s_3}(\beta_1,\beta_2)$ (formula (1.33) of [1] recalled in (A.1) below). We compute this integral in the particular case when the three highest weights are chosen as in (1.1). The components (β_a, α_2) of the weights in (1.1) take arbitrary values, subject of the condition (2.4); the latter implies that $s_1 = s_3$ and we shall assume that the integer $l := -(\beta_3, \alpha_1)/b = s_2 - 2s_1$ is nonnegative.

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We shall skip the detailed computation of the OPE constant since it follows straightforwardly the steps of the method explained in [1], which is based on the use of a $sl(2)$ type duality formula [8] in order to derive recursion relations for $sl(n)$ Toda multiple 2d integrals; see the appendix for a short summary of the procedure. In our case after s steps one gets an integral of type

$$I_{s_1-s, s_2-2s, s_3-s}(\beta_1 + s\omega_2 b, \beta_2 + s\omega_2 b)$$

so that setting $s = s_3 = s_1$ the integral is reduced to known Liouville Coulomb integral. In particular for $\beta_2 = -\omega_2 b$ the resulting formula reproduces the structure constants $c_h := c(\beta_1, -\omega_2 b, 2\rho Q - (\beta_1 - hb))$ of the fusion of V_{β_1} with the fundamental field $V_{-\omega_2 b}$ corresponding to three of the six points of the weight diagram Γ_{ω_2}, i.e.,

$$\begin{aligned} &\beta_3 = \beta_1 - hb \quad \text{with} \\ &h = \omega_2 \, ; \; h = \omega_2 - \alpha_2 = w_2(\omega_2) \, ; \; h = -\omega_2 = \omega_2 - 2\alpha_2 - \alpha_1 - \alpha_3 = w_{2312}(\omega_2) \, . \end{aligned} \tag{2.5}$$

These are the three weights of Γ_{ω_2} preserving the symmetric type $\beta_3 = \beta_3^*$ with $l \geq 0$. The expressions for these OPE constants reproduce special cases of the general formula (1.51) of [1] valid for arbitrary β_1. For the partially degenerate weights of type $(\beta_1, \alpha_i) = 0\,, i = 1, 3$ the remaining three OPE constants in [1] vanish, in agreement with the vanishing of the corresponding $sl(4)$ tensor product decomposition multiplicities: the $sl(4)$ (or $\hat{sl}(4)$) Verma modules of highest weights $\lambda = -\beta/b$ with non-negative integer components $l^{(i)} = (\lambda, \alpha_i), i = 1, 3$ have two singular vectors, whose factorisation imposes additional restrictions on the fundamental fusion rule. In particular in the case $(\lambda_1, \alpha_1) = 0 = (\lambda_1, \alpha_3)$ for each of the three weights $h = \pm(-\omega_1 + \omega_3)\,, \omega_2 - (\omega_1 + \omega_3) \in \Gamma_{\omega_2}$ there is an odd Weyl group element w_1, or w_3, or both, the shifted action of which keeps $\lambda_3 = \lambda_1 + h$ invariant, a property which does not depend on the value of (λ_1, α_2), and which implies the vanishing of the corresponding fusion multiplicities.[3]

Our next step is the standard analytic continuation of the OPE constant, to be denoted $C(\beta_1, \beta_2, 2\rho Q - \beta_3)$ for weights of the type (1.1) not restricted by (2.4), so that the Coulomb gas OPE constant is reproduced as a double residue

$$c(\beta_1, \beta_2, 2\rho Q - \beta_3) = \mathrm{res}_{(\beta_{12}^3, \omega_2 - 2\omega_1) = -(s_2 - 2s_1)b} \, \mathrm{res}_{(\beta_{12}^3, \omega_1) = -s_1 b} C(\beta_1, \beta_2, 2\rho Q - \beta_3) \tag{2.6}$$

where s_1 and $s_2 - 2s_1$ are nonnegative integers.

[3] The sl(4) pattern of the W_4 fusion rules multiplicities is independently proved in the particular case of integer dominant weights $-\beta/b$ extrapolating the rational b^2 result in [13], derived by reduction of the $\hat{sl}(n)$ WZW Verlinde formula. The fusion rule of $f = -\omega_2 b$ with representations of generic highest weight can be derived algebraically accounting for the factorisation of the null states in the corresponding completely degenerate W_4 Verma module. The three independent singular vectors - two at level 1 and one at level 2, inherited via the quantum DS reduction from the singular vectors of $\hat{sl}(4)$ module, are not sufficient. Together with the two projective Ward identities corresponding to the zero modes $W_0^{(j)}$ of the spin $j = 3$ and $j = 4$ currents, they provide five relations which eliminate the 3-point matrix elements containing the negative modes $W_{-k}^{(j)}, 1 \leq k \leq j - 1$: the latter determine the action on the fields of all higher negative modes $W_{-n}^{(j)}, n \geq j$. To derive the fusion rule itself one needs to explore the factorisation of three more descendent null states, presumably at levels up to 5, as suggested by the classical KZ equation for this representation; see the general discussion in [1, 14, 15] applied to W_3 examples.

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We shall write down the related formula for $\beta_3 = \beta_3^*$ replaced by $2\rho Q - \beta_3$, equivalently obtained by multiplication with the reflection amplitude $R(\beta_3)$ corresponding to the longest Weyl group element w_{121321} [1]

$$R(\beta) = (b^{2(1-b^2)}\lambda_T)^{\frac{(2\rho Q-2\beta,\rho)}{b}} \prod_{\alpha>0} \frac{\Upsilon_b((\rho Q-\beta,\alpha))}{\Upsilon_b((\beta-\rho Q,\alpha))}\,, \tag{2.7}$$

namely,

$$\begin{aligned} C(\beta_1,\beta_2,\beta_3) &= R(\beta_3)\, C(\beta_1,\beta_2,2\rho Q-\beta_3) \qquad\qquad (2.8)\\ &= \left(b^{2(1-b^2)}\lambda_T\right)^{\frac{(2\rho Q-\beta_{123},\rho)}{b}} \frac{\prod_{\alpha=\alpha_1,\alpha_{14}} \Upsilon_b((\beta_3-\rho Q,\alpha)+Q)}{\prod_{\alpha=\alpha_1,\alpha_{13}} \Upsilon_b((\beta_3-\rho Q,\alpha))}\times \\ &\prod_{a=1,2} \frac{\Upsilon_b((\beta_a,\alpha_2))}{\Upsilon_b((\beta_{123}-2\beta_a,\omega_2-\omega_1))} \frac{\Upsilon_b(b)\,\Upsilon_b((\beta_3,\alpha_2))}{\Upsilon_b((\beta_{12}^3,\omega_1))\Upsilon_b((\beta_{123},\omega_2-\omega_1)-2Q)}\times \\ &\prod_{a=1,2} \frac{\Upsilon_b((\beta_a-\rho Q,\alpha_{24})+Q)}{\Upsilon_b((\beta_{123}-2\beta_a,\omega_1)-Q)} \frac{\Upsilon_b(b)\,\Upsilon_b((\beta_3-\rho Q,\alpha_{24})+Q)}{\Upsilon_b((\beta_{12}^3,\omega_2-\omega_1)-Q)\Upsilon_b((\beta_{123},\omega_1)-3Q)}\,. \end{aligned}$$

Here λ_T is proportional to the Toda cosmological constant, $\lambda_T = \pi\mu_T\gamma(b^2)$. Recall that $\Upsilon_b(x)$ is an entire function with zeros at $x = -nb-m/b$ and $x = Q+nb+m/b$, $n, m \in \mathcal{Z}_{\geq 0}$, satisfying the functional relations

$$\Upsilon_b(x+b^\epsilon) = \gamma(xb^\epsilon)\, b^{\epsilon(1-2xb^\epsilon)}\, \Upsilon_b(x)\,, \epsilon = \pm 1\,, \tag{2.9}$$

$\gamma(x) = \Gamma(x)/\Gamma(1-x)$. In the products over positive roots in (2.8) the root α_1 can be replaced with α_3 (and $\alpha_{13} = \alpha_1+\alpha_2$ - with $\alpha_{24} = \alpha_2+\alpha_3$) since $\beta_3 = \beta_3^*\,, \rho = \rho^*$. The ratio in the second line of (2.8) produces a finite constant for $(\beta_3,\alpha_1) \to -lb\,, l-$nonnegative integer so that (2.8) has sense for such values of β_3 whenever the components (β_a,α_2) of the three weights are generic. We have also used that the two weights β_1,β_2 have zero components $(\beta_a,\alpha_i) = 0$ for $i = 1,3$ in order to write (2.8) in a form which makes it explicitly symmetric when the third weight β_3 is also chosen of this type, i.e., $l = 0$, as we shall need it below: in that particular case $(\beta,\omega_2-\omega_1) = (\beta,\omega_1)$ in all products of this type in the denominators in the last two lines.

Using (2.3) the terms in (2.8) depending on the three vertices, i.e., the eight Υ_b-factors in the denominators of the last two lines, can be also written as points on an orbit of the Weyl group acting on the three weights (as discussed, e.g., in [6, 7] for the FL example)

$$\frac{\left(\Upsilon_b((\beta_{123}-2\rho Q,\omega_1))\Upsilon_b((w_{13}^{(3)}\star\beta_{123}-2\rho Q,\omega_1))\,\Upsilon_b((w_{121321}^{(3)}\star\beta_{123}-2\rho Q,\omega_1))\right)^{-1}}{\prod_{j=1}^3 \Upsilon_b((w_{2132}^{(j)}\star\beta_{123}-2\rho Q,\omega_1))\,\prod_{i=1,2}\Upsilon_b((w_{2132}^{(i)}w_{13}^{(3)}\star\beta_{123}-2\rho Q,\omega_1))} \tag{2.10}$$

where $w'^{(1)}w^{(3)}\star\beta_{123} = w'\star\beta_1+\beta_2+w\star\beta_3$, etc..

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3 Locality, fusing matrix, braiding identity

Consider the local 4-point function $\langle V_f V_{\beta_1} V_{\beta_2} V_{\beta_3}\rangle$ of primary spinless operators $V_\beta(z,\bar z)$ one of which is labelled by a fundamental highest weight, in our case $f=-b\omega_2$. The most interesting for the applications are the cases in which the remaining three primary fields have highest weights β_a with nonnegative integer components $l_a^{(i)}=-(\beta_a,\alpha_i)/b\,, i=1,3$ and generic (β_a,α_2) (or, any of their Weyl group related values providing equivalent vertex representations). These representations arise from doubly reducible Verma modules with two singular vectors. The projective Ward identities and the factorisation of all singular vectors - as well as the descendent null states in the fundamental representation f give restrictions on the conformal blocks reducing the space of descendent states described in terms of powers of the modes $W^{(j)}_{-k}, 1\le k\le j-1$, to a finite dimensional subspace, see [1, 16, 17] for different approaches to this problem. Instead of the detailed analysis of this space one can give, as in [3], an alternative argument showing that at least a subclass of these 4-point functions admit an integral Coulomb gas like representation, see the appendix. This indicates that all descendent states of the above space are eliminated so that the fusion channels of these highly degenerated 4-point functions follow the $sl(4)$ pattern dictated by the completely degenerate field $V_{-\omega_2 b}$; in what follows we restrict to this subclass of 4-point functions. In the case when all $l_a^{(i)}=0\,, i=1,3\,, a=1,2,3$ - which is the main case under consideration below, the alternative Coulomb representation allows to identify the linear differential equation satisfied by the 4-point function.

The 4-point function admits different equivalent diagonal decompositions in conformal blocks. They are related by braiding transformations, i.e., matrix realisation of the braiding group with generators $e_i\,, i=1,2,3$ on the plane (Riemann sphere) with 4 holes; e_i is exchanging the chiral vertex operators at the i-th and $i+1$-th points and the notation refers to the fixed ordered points, not to the labels of the concrete interchanged operators. In particular the generators e_2 (for the above order of the corresponding chiral vertex operators) is represented by non-trivial braiding matrix B proportional to the fusing matrix F

$$\begin{aligned}
&B_{\beta_1-h_sb,\beta_2-h_tb}\begin{bmatrix}\beta_1 & \beta_2\\ f & \beta_3\end{bmatrix}(\epsilon)=e^{i\pi\epsilon(\triangle(\beta_3)+\triangle(f)-\triangle(\beta_1-h_sb)-\triangle(\beta_2-h_tb))}F_{\beta_1-h_sb,\beta_2-h_tb}\begin{bmatrix}\beta_1 & \beta_3\\ f & \beta_2\end{bmatrix},\\
&B_{\gamma,\delta}\begin{bmatrix}\beta_1 & \beta_2\\ \beta_4 & \beta_3\end{bmatrix}(\epsilon)B_{\delta,\gamma'}\begin{bmatrix}\beta_2 & \beta_1\\ \beta_4 & \beta_3\end{bmatrix}(-\epsilon)=\delta_{\gamma\gamma'}
\end{aligned}\tag{3.1}$$

while e_1 and e_3, which exchange the operators in the first two, respectively last two, fixed points, reduce due to triviality of F, to diagonal matrices. Locality (symmetry under exchange of two 2d fields) requires that the function is invariant under such transformations relating different diagonal chiral decompositions. This results in equations involving fusing matrix elements and products of 3-point constants. In the case under consideration the equations take the form of a finite sum. E.g., for the exchange of V_{β_1} and V_{β_2} they read

$$\begin{aligned}
&\sum_{h_s\in\Gamma_{\omega_2}}\frac{c_{h_s}(\beta_1)C(\beta_1-h_sb,\beta_2,\beta_3)}{c_{h_t}(\beta_2)\,C(\beta_1,\beta_2-h_tb,\beta_3)}F_{\beta_1-h_sb,\beta_2-h_tb}\,F_{\beta_1-h_sb,\beta_2-h_ub}=\delta_{h_t,h_u}\\
&\qquad=\sum_{h_s\in\Gamma_{\omega_2}}(F^{-1})_{h_t\,h_s}F_{h_s\,h_u}\,.
\end{aligned}\tag{3.2}$$

Here $c_{h_s}(\beta_1)$ is a shorthand notation for the OPE constant $c(-\omega_2 b, \beta_1, 2\rho Q - (\beta_1 - h_s b))$, see the general formula (1.51) in [1]. In particular $c_{h=\omega_2} = 1$. In general h_s stands for the weights of the weight diagram Γ_{ω_2} of the 6 dimensional representation, but for our restricted set of highest weights $\beta_a = -k_a \omega_2 b$, $a = 1, 2, 3$, three of the OPE coefficients c_{h_s} given in [1] vanish, as discussed above, so we are left with summation over 3 of the weights, as given in (2.5). A shorthand notation for the matrix $F_{h_s,h_t} = F_{\beta_1 - h_s b, \beta_2 - h_t b}$ in the last equality in (3.2) is used.

As indicated in the r.h.s. of (3.2) the matrix formed by the ratio of constants times F can be identified with the inverse matrix F^{-1}

$$\frac{c_{h_s}(\beta_1) C(\beta_1 - h_s b, \beta_2, \beta_3)}{c_{h_t}(\beta_2)\, C(\beta_2 - h_t b, \beta_1, \beta_3)} F_{\beta_1 - h_s b, \beta_2 - h_t b} = (F^{-1})_{h_t, h_s}\,. \tag{3.3}$$

It is furthermore required that

$$(F^{-1})_{h_t,h_s} = F_{h_t,h_s}(\beta_2, \beta_1), \tag{3.4}$$

a consequence of the pentagon relation for F (or of the normalization relation in (3.1)). In a shorthand notation we shall denote $F_{\beta_1 - s\omega_2 b, \beta_2 - t\omega_2 b}(\beta_1, \beta_2) = F_{s,t}(\beta_1, \beta_2) = F_{s,t}\,,\ s, t = \pm 1\,,\ F_{\beta_1 - \bar{h} b, \beta_2 - t\omega_2 b} = F_{\bar{h},t}\,$, for $\bar{h} := \omega_2 - \alpha_2\,$, etc., suppressing the dependence on the third argument β_3.

The ratios in (3.3) will be denoted

$$U_{h,h'}(\beta_1, \beta_2) := \frac{c_h(\beta_1) C(\beta_1 - hb, \beta_2, \beta_3)}{c_{h'}(\beta_2)\, C(\beta_2 - h'b, \beta_1, \beta_3)} = \frac{U_{h,+}(\beta_1, \beta_2)}{U_{+,+}(\beta_1, \beta_2) U_{h',+}(\beta_2, \beta_1)} \tag{3.5}$$

and thus one needs to compute all $U_{h,+}$. We give the explicit expression of the first of these ratios, computed from (2.8)

$$U_{+,+}(\beta_1, \beta_2) := \frac{C(\beta_1 - b\omega_2, \beta_2, \beta_3)}{C(\beta_1, \beta_2 - b\omega_2, \beta_3)} = \frac{\gamma(1 + b(\beta_2 - \rho Q, \alpha_2))\gamma(1 + b(\beta_2 - 2\rho Q, \alpha_2))}{\gamma(1 + b(\beta_1 - \rho Q, \alpha_2))\gamma(1 + b(\beta_1 - 2\rho Q, \alpha_2))} \times \\ \frac{\gamma\big(b((\beta_{13}^2, \omega_1) - \frac{b}{2})\big)\, \gamma\big(b((\beta_{13}^2, \omega_1) - Q - \frac{b}{2})\big)}{\gamma\big(b((\beta_{23}^1, \omega_1) - \frac{b}{2})\big)\, \gamma\big(b((\beta_{23}^1, \omega_1) - Q - \frac{b}{2})\big)}\,. \tag{3.6}$$

By analogy with the Liouville case this suggests the following ansatz for F:

$$F_{\beta_1 - \omega_2 b, \beta_2 - \omega_2 b} \begin{bmatrix} \beta_1 & \beta_3 \\ -\omega_2 b & \beta_2 \end{bmatrix} = F_{+,+}(\beta_1, \beta_2) = \frac{\Gamma(b(Q\rho - \beta_2, \alpha_2))\, \Gamma(1 - b(\rho Q - \beta_1, \alpha_2))}{\Gamma(b((\beta_{31}^2, \omega_1) - \frac{b}{2}))\, \Gamma(1 - b((\beta_{23}^1, \omega_1) - \frac{b}{2}))} \times \\ \frac{\Gamma(b(2\rho Q - \beta_2, \alpha_2))\, \Gamma(1 + b(\beta_1 - 2\rho Q, \alpha_2))}{\Gamma(b((\beta_{31}^2, \omega_1) - \frac{b}{2} - Q))\, \Gamma(1 - b((\beta_{23}^1, \omega_1) - \frac{b}{2} - Q))}\,. \tag{3.7}$$

From (3.7) one computes $F_{+,+}(\beta_2, \beta_1)$ and confirms, using (3.3), that it indeed satisfies (3.4)

$$F_{+,+}(\beta_2, \beta_1) = \frac{C(\beta_1 - \omega_2 b, \beta_2, \beta_3)}{C(\beta_1, \beta_2 - \omega_2 b, \beta_3)}\, F_{+,+}(\beta_1, \beta_2) = (F^{-1})_{+,+}\,.$$

Altogether we have

$$\begin{aligned}U_{+,+}\,F^2_{+,+} &= F_{+,+}(\beta_2,\beta_1)F_{+,+}(\beta_1,\beta_2)\\ &= \frac{\sin\pi b((\beta^2_{31},\omega_1)-\frac{b}{2})\sin\pi b((\beta^1_{23},\omega_1)-\frac{b}{2})}{\sin\pi b(\rho Q-\beta_1,\alpha_2)\,\sin\pi b(Q\rho-\beta_2,\alpha_2)}\times\\ &\qquad\frac{\sin\pi b((\beta^2_{31},\omega_1)-\frac{b}{2}-Q))\,\sin\pi b((\beta^1_{23},\omega_1)-\frac{b}{2}-Q))}{\sin\pi b(2\rho Q-\beta_1,\alpha_2)\,\sin\pi b(2\rho Q-\beta_2,\alpha_2)}\\ &=: \frac{ABA'B'}{P1[\beta_1]P1[\beta_2]P2[\beta_1]P2[\beta_2]}\,. \end{aligned} \tag{3.8}$$

Here $Pk[\beta]:=\sin\pi b((\beta,\alpha_2)-kQ)$ and A,B,A',B' denote the four sin's in the numerator correspondingly.

We proceed in this way to obtain $F_{h,+}$ for the other two shifts of $\beta_1\to\beta_1-hb$. Then with the help of simple trigonometric relations one checks and proves the first of the diagonal equations in (3.2), for $h_t=h_u=\omega_2$. Similarly one finds eight of the nine F matrix elements checking the related equations. The expression for $U_{\bar h,\bar h}$, however, has a different structure, not suggesting straightforwardly an expression for $F_{\bar h,\bar h}$

$$U_{\bar h,\bar h}(\beta_1,\beta_2)=\tfrac{\gamma(3\rho Q-b(\beta_1,\alpha_2))}{\gamma(1+b(\rho Q-\beta_1,\alpha_2))}\tfrac{\gamma(1+b(\beta_2-3\rho Q,\alpha_2))}{\gamma(b(\beta_2-\rho Q,\alpha_2))}\,. \tag{3.9}$$

On the other hand writing the general expression of an inverse of a 3×3 matrix, $F^{-1}_{ij}=\frac{\varepsilon_{ikl}\varepsilon_{jmn}}{2\det F}F_{mk}F_{nl}$ with $i,j,k,l,m,n=+\,,-\,,\bar h$ we have, e.g.,

$$F_{\bar h,+}(\beta_2,\beta_1)=\frac{1}{\det F}(F_{-,+}F_{\bar h,-}-F_{-,-}F_{\bar h,+})$$

etc. From this we can determine $\det F$:

$$\det F(\beta_1,\beta_2) == -\prod_{\alpha=\alpha_2,\alpha_{24},\alpha_{14}}\frac{(\beta_1-\rho Q,\alpha)}{(\beta_2-\rho Q,\alpha)}\,. \tag{3.10}$$

Then, e.g., from

$$F_{\bar h,\bar h}(\beta_2,\beta_1)=\frac{1}{\det F}(F_{+,+}F_{-,-}-F_{+,-}F_{-,+})$$

we can determine $F_{\bar h,\bar h}$ and check the remaining identities in (3.2).

Summarising we get for the matrix elements of F starting with (3.7)

$$\begin{aligned}&F_{-,+}(\beta_1,\beta_2)=F_{+,+}(w_{2132}\star\beta_1\,,\beta_2)\\ &F_{\bar h,+}(\beta_1,\beta_2)=F_{\beta_1-\bar h b,\beta_2-\omega_2 b}\begin{bmatrix}\beta_1 & \beta_3\\ -\omega_2 b & \beta_2\end{bmatrix}=\frac{\Gamma(1-Qb)}{\Gamma(1-2Qb)}\times\\ &\quad\frac{\Gamma(1+b(\beta_1-3\rho Q,\alpha_2))\,\Gamma(1-b(\beta_1-\rho Q,\alpha_2))\,\Gamma(b((Q\rho-\beta_2,\alpha_2))\,\Gamma(b((2\rho Q-\beta_2,\alpha_2))}{\Gamma(b((\beta^2_{13},\omega_1)-\frac{b}{2}-Q))\Gamma(1-b((\beta^1_{23},\omega_1)-\frac{b}{2}))\Gamma(1-b((\beta^3_{12},\omega_1)-\frac{b}{2}))\,\Gamma(1-b((\beta_{123},\omega_1)-\frac{b}{2}-2Q))}\\ &F_{+,-}(\beta_1,\beta_2)=F_{+,+}(\beta_1,w_{2132}\star\beta_2)\\ &F_{-,-}(\beta_1,\beta_2)=F_{+,+}(w_{2132}\star\beta_1\,,w_{2132}\star\beta_2)\\ &F_{\bar h,-}(\beta_1,\beta_2)=F_{\bar h,+}(\beta_1,w_{2132}\star\beta_2)\end{aligned} \tag{3.11}$$

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$$F_{+,\bar h}(\beta_1,\beta_2) = F_{\beta_1-\omega_2 b,\beta_2-\bar h b}\begin{bmatrix}\beta_1 & \beta_3\\ -\omega_2 b & \beta_2\end{bmatrix} = \frac{\Gamma(Qb)}{\Gamma(2Qb)}\times$$
$$\frac{\Gamma(1-b(Q\rho-\beta_1,\alpha_2))\,\Gamma(1-b(2\rho Q-\beta_1,\alpha_2))\,\Gamma(b(3\rho Q-\beta_2,\alpha_2))\,\Gamma(b(\beta_2-\rho Q,\alpha_2))}{\Gamma(b((\beta_{12}^3,\omega_1)-\frac{b}{2}))\,\Gamma(b((\beta_{123},\omega_1)-\frac{b}{2}-2Q))\Gamma(b((\beta_{13}^2,\omega_1)-\frac{b}{2}))\,\Gamma(1-b((\beta_{23}^1,\omega_1)-\frac{b}{2}-Q))}$$
$$F_{-,\bar h}(\beta_1,\beta_2) = F_{+,\bar h}(w_{2132}\star\beta_1,\beta_2)$$
$$F_{\bar h,\bar h}(\beta_1,\beta_2) = \frac{\Gamma((\beta_2-\rho Q,\alpha_2)b)\Gamma(1+(\rho Q-\beta_1,\alpha_2)b)}{\Gamma(1+(\beta_2-3\rho Q,\alpha_2)b)\Gamma((3\rho Q-\beta_1,\alpha_2)b)}\times$$
$$\left(1+\frac{2\cos\pi Qb\,\sin\pi b((\beta_{12}^3,\omega_1)-\frac{b}{2}-Q)\,\sin\pi b((\beta_{123},\omega_1)-\frac{b}{2}-3Q)}{\sin\pi b((\beta_1,\alpha_2)-3Q)\ \sin\pi b((\beta_2,\alpha_2)-3Q)}\right).$$

The last matrix element can be written in various different ways.

Let us introduce some additional notation

$$\begin{aligned}
D = D[\beta_1,\beta_2,\beta_3] &:= \sin\pi b\left((\beta_{123}-2\rho Q,\omega_1)-\frac{b}{2}\right),\\
D' = \sin\pi b&\left((\beta_{123}-2\rho Q,\omega_1)-\frac{b}{2}+Q\right) = D[\beta_1,\beta_2,w_{13}\star\beta_3],\\
C = \sin\pi b&\left((\beta_{12}^3,\omega_1)-\frac{b}{2}\right) = D'[\beta_1,\beta_2,w_{2132}\star\beta_3],\\
C' = \sin\pi b&\left((\beta_{12}^3,\omega_1)-\frac{b}{2}-Q\right) = D[\beta_1,\beta_2,w_{2132}\star\beta_3],
\end{aligned} \tag{3.12}$$

and A, A', B, B', explicitly described above, are also written in terms of Weyl group action

$$\begin{aligned}
A = D'[\beta_1,w_{2132}\star\beta_2,\beta_3], &\quad A' = D[\beta_1,w_{2132}\star\beta_2,\beta_3],\\
B = D'[w_{2132}\star\beta_1,\beta_2,\beta_3], &\quad B' = D[w_{2132}\star\beta_1,\beta_2,\beta_3].
\end{aligned} \tag{3.13}$$

Denoting $\tilde F = F(\beta_2,\beta_1)$, we have for the products $\tilde F_{h_s,h_t}F_{h_t,h_s}$ of matrix elements in (3.2)

$$\tilde F_{+,+}F_{+,+} = \frac{AA'BB'}{P1[\beta_1]P2[\beta_1]P1[\beta_2]P2[\beta_2]},\qquad \tilde F_{+,-}F_{-,+} = \frac{CC'DD'}{P3[\beta_1]P2[\beta_1]P1[\beta_2]P2[\beta_2]},$$
$$\tilde F_{+,\bar h}F_{\bar h,+} = \frac{2\cos\pi b^2\,A'BCD'}{P1[\beta_1]P3[\beta_1]P1[\beta_2]P2[\beta_2]},$$
$$\tilde F_{-,+}F_{+,-} = \frac{CC'DD'}{P1[\beta_1]P2[\beta_1]P3[\beta_2]P2[\beta_2]},\qquad \tilde F_{-,-}F_{-,-} = \frac{AA'BB'}{P3[\beta_1]P2[\beta_1]P3[\beta_2]P2[\beta_2]},$$
$$\tilde F_{-,\bar h}F_{\bar h,-} = \frac{2\cos\pi b^2\,AB'C'D}{P1[\beta_1]P3[\beta_1]P3[\beta_2]P2[\beta_2]},$$
$$\tilde F_{\bar h,+}F_{+,\bar h} = \frac{2\cos\pi b^2\,AB'CD'}{P1[\beta_1]P2[\beta_1]P1[\beta_2]P3[\beta_2]},\qquad \tilde F_{\bar h,-}F_{-,\bar h} = \frac{2\cos\pi b^2\,A'BC'D}{P3[\beta_1]P2[\beta_1]P1[\beta_2]P3[\beta_2]},$$

$$\begin{aligned}
\tilde F_{\bar h,\bar h}F_{\bar h,\bar h} &= \frac{(AA'BB'-CC'DD')^2}{P2[\beta_1]P2[\beta_2]\prod_{k=1}^3 Pk[\beta_1]Pk[\beta_2]}\\
&= \frac{(\cos\pi b^2\ \cos\pi b((\beta_3,\alpha_2)-2b)-\cos\pi b((\beta_1,\alpha_2)-2b)\cos\pi b((\beta_2,\alpha_2)-2b))^2}{P1[\beta_1]P1[\beta_2]P3[\beta_1]P3[\beta_2]}.
\end{aligned} \tag{3.14}$$

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Compare with the Liouville case where $(\beta,\alpha)=2\beta^L\,,(\beta,\omega)=\beta^L$ and

$$F^L_{s,t}=F^L_{\beta_1-s\omega b,\beta_2-t\omega b}\begin{bmatrix}\beta_1 & \beta_3\\ -\omega b & \beta_2\end{bmatrix}\,,\quad s,t=\pm1$$

satisfying[4]

$$\begin{aligned}F^L_{+,+}\tilde F^L_{+,+}&=F^L_{+,+}\frac{1}{\det F}F^L_{-,-}=\tilde F^L_{-,-}F^L_{-,-}=\frac{AB}{P1[\beta_1]P1[\beta_2]}\,,\\ F^L_{+,-}\tilde F^L_{-,+}&=-F^L_{+,-}\frac{1}{\det F}F^L_{-,+}=\tilde F^L_{+,-}F^L_{-,+}=\frac{CD}{P1[\beta_1]P1[\beta_2]}\,.\end{aligned}\tag{3.15}$$

One can analogously compute the fusing matrix elements corresponding to the $sl(4)$ fundamental weights $\omega_1=\omega_3^*$ using the 3-point constant computed by Fateev and Litvinov [1] in which two of the weights are arbitrary and the third is proportional to one of these fundamental weights; this will be a special case of the 4×4 F matrix. Partial data on the braiding matrices in that case is also provided (though in a different gauge) by the Boltzmann weights defining integrable $A_3^{(1)}$ lattice models [18] taking a proper limit of the spectral parameter.

Finally we check a braiding relation relevant for the 4-point chiral blocks under consideration, namely (choosing the sign $\epsilon=1$ in (3.1))

$$\Omega_1\Omega_2\Omega_3:=(e_1^2)(e_2e_1^2e_2^{-1})(e_3e_2e_1^2e_2^{-1}e_3^{-1})=e^{-4\pi i\triangle(f)}\,.\tag{3.16}$$

In the limit $b\to0$ the r.h.s. of (3.16) becomes an identity for any of the fundamental weights $f=-\omega_ib$. In our case $\triangle(f)=\triangle(-\omega_2b)=-\frac{5}{2}b^2-2$.

The meaning of the l.h.s. of (3.16) is a composition of monodromies around the three vertex coordinates. On the sphere with 4 (ordered) points e_1 and e_3 are represented by diagonal braiding matrices; e_2 in Ω_2 is represented by (3.1), while in Ω_3 it is represented by the same braiding matrix with β_3 and β_1 exchanged. Using the defining relations of the braiding group

$$e_ie_{i+1}e_i=e_{i+1}e_ie_{i+1}\,,\ e_ie_j=e_je_i\ \text{for}\ j\neq i\pm1\tag{3.17}$$

(3.16) is reduced to the first of the two additional relations on the generators $\{e_i\}$ which characterise the modular group on the sphere with 4 holes [19]

$$e_1e_2e_3^2e_2e_1=e^{-4\pi i\triangle(f)}\,,\ (e_1e_2e_3)^4=e^{-2\pi i(\triangle(f)+\sum_{a=1}^3\triangle(\beta_a))}\,.\tag{3.18}$$

Take the trace of (3.16)

$$\mathrm{Tr}(\Omega_1\Omega_2)=e^{-4\pi i\triangle(-\omega_2b)}\mathrm{Tr}(\Omega_3^{-1})\,.\tag{3.19}$$

The eigenvalues of the monodromy Ω are computed from the difference of Toda dimensions

$$p(\beta,h)=\triangle(\beta-hb)-\triangle(\beta)-\triangle(-b\omega_2)=2bQ+b(\beta-\rho Q,h)\tag{3.20}$$

[4] Analogous relations hold in the $sl(2)$ WZW case. The braiding matrices differ by β - independent phases (Q is replaced by b) and by normalisation so that effectively $\det F=1=\det B$, i.e., $F\in\mathrm{SL}(2)$.

where in general $h \in \Gamma_{\omega_2}$ is a weight in the 6-dim weight diagram Γ_{ω_2} of the fundamental representation $\omega_2 = (0,1,0)$, i.e., $h = \pm\omega_2, \pm(\omega_2 - \omega_1 - \omega_3), \pm(\omega_1 - \omega_3)$. Thus in general the trace of the diagonal monodromy

$$\mathrm{Tr}(\Omega) = \mathrm{Tr}(e_1^2) = \sum_{h \in \Gamma_{\omega_2}} e^{2\pi i p(\beta,h)} = e^{i\pi 4bQ} \chi_{\omega_2}(2\pi i b(\beta - \rho Q)) \tag{3.21}$$

is proportional to the character $\chi_{\omega_2}(\mu)$ of the fundamental representation $\omega_2 = (0,1,0)$ evaluated at the "angle" $\mu = 2\pi b(\beta - \rho Q)$. (For $f = \omega_i$ in the formula corresponding to (3.20) $h \in \Gamma_{\omega_i}$ and the constant term in the r.h.s. is given by $(\omega_i, \rho)bQ$.) Denote by $q(\beta)$ the normalized diagonal matrix

$$q_{h_s,h_t}(\beta) = \delta_{h_s,h_t}\, e^{2\pi i b(\beta - \rho Q, h_s)} = e^{-4\pi i Q}\Omega\,.$$

In terms of F and its inverse $\tilde{F}$ the relation (3.19) reads (collecting the three overall terms $e^{4i\pi b^2}$ in the r.h.s.)

$$\sum_{h_s,h_t} q_{h_s}(\beta_1) F_{h_s,h_t} \tilde{F}_{h_t,h_s} q_{h_t}(\beta_2) = e^{-4i\pi(3b^2 + \triangle(-\omega_2 b))} \mathrm{Tr}\, q^{-1}(\beta_3)\,. \tag{3.22}$$

In our case of "scalar" (in 4d sense) weights β_a the relation involves a 3×3 submatrix and accordingly $q(\beta)$ reduces to the diagonal submatrix with matrix elements $e^{2\pi i(\beta - \rho Q, h_s)}$, $h_s = \pm\omega_2\,, \omega_2 - \alpha_2$. Thus the sums in the l.h.s. of (3.22) run over these three weights, while the r.h.s. reduces to

$$\text{r.h.s.}' = e^{-2i\pi b^2}(2\cos 2\pi b(\beta - 2\rho Q, \alpha_2) + e^{2\pi i b^2})\,. \tag{3.23}$$

Each of the products in (3.14) which appear in the l.h.s. of (3.22) in this case is a second order polynomial in $2\cos \pi b(\beta - 2\rho Q, \alpha_2)$, as is the expression in (3.23), so the reduced relation is checked order by order.

In (3.22) F enters only through the products $F_{h_s,h_t}\tilde{F}_{h_t,h_s}$, hence it is a restriction on these products. In principle the identity (3.16) with diagonal braiding determined from (3.20) may admit more general solutions for the individual F matrix elements than the present Toda CFT solution (3.7), (3.11). Indeed the $sl(2)$ analog of the identity (3.16) with trivial r.h.s. has been exploited in the recent papers [9, 10] on $\mathrm{AdS}_3 \times S^3$ sigma model 3-point correlators in the semi-classical strong 't Hooft coupling λ limit with large quantum numbers. Identifying $b^2 = 1/\sqrt{\lambda}$ this corresponds to the semiclassical limit $b \to 0$ with three heavy charges $\beta_a/b = \eta_a/b^2$, η_a - finite. In the sigma model case the eigenvalues of the monodromy matrix $e^{2\pi i(\eta(x),h)}$ depend on the spectral parameter x and the solution for the individual $F = F(x)$ matrix elements depends nontrivially on the specific spectral curve. On the other hand the expression for the products $F_{h_s,h_t}\tilde{F}_{h_t,h_s}$ as functions of $\eta(x)$ coincides with those in the WZW model, or, up to normalization, with those of Virasoro theory (cf. (3.15) and footnote 4). One may expect that the Toda theory data and their WZW extensions for the fundamental representations $f = -\omega_i b$ can similarly be used as a starting point, although in this case the equation (3.19) is less restrictive by itself, compared with the sl(2) case where it uniquely determines the fusing matrix products.

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4 The 3-point functions in the compact ("matter") region and BPS-like relation

By analogy with the Liouville gravity described by two dual Virasoro CFT with $c > 25$ (Liouville) and $c < 1$ ("matter") we shall extend here the results of section 2 to another region of central charge of the W_4 CFT, parametrised by the same real parameter b as (2.1),

$$c_m = 3(1 - 20e_0^2) < 3\,, \qquad e_0 = \frac{1}{b} - b\,. \tag{4.1}$$

The sum of central charges (2.1) and (4.1) is compensated by the contribution of the ghosts (pairs (b_k, c_k) of dimensions $(k, 1-k)\,, k = 2, 3, 4$), i.e., $c + c_m + c_{\rm ghost} = 0$.

The conformal dimension of vertex operator $V_e^{(\rm m)}$ is given by

$$\triangle_m(e) = \frac{1}{2}(e, e - 2\rho e_0)\,, \tag{4.2}$$

invariant under the action of the Weyl group

$$\hat{w}(e) := \rho e_0 + w(e - \rho e_0) = b\left(w \cdot \frac{e}{b} - \frac{1}{b^2} w \cdot 0\right) \tag{4.3}$$

(i.e., the horizontal projection of the shifted action of the affine Weyl group elements $t_{-w.0}w$ on $(e/b + k\omega_0)$, times b, where $k + 4 = 1/b^2$).

The minimal W_4 theory in the region (4.1) for rational b^2 has been discussed in [2]. Here the real parameter b is generic and we shall consider vertex operators $V_e^{(\rm m)}$ with symmetric charges $e = (r\omega_2 + s(\omega_1 + \omega_3))b = e^*$. Such W_4 representations are degenerate for nonnegative integers r, s. Once again we consider a 3-point function of vertex operators two of which have highest weights of type $e_a = r_a\omega_2 b\,, a = 1, 2$, and one - a general symmetric weight $e_3 = e_3^*$. The Coulomb gas computation is performed as before, with interaction term defined by vertex operators $V_{-\alpha_i b}^{(\rm m)}$, or, one can directly continue the Toda Coulomb gas OPE constants (being given by finite products of ratios of γ functions) to $b^2 \to -b^2\,, Qb \to e_0 b\,, \beta b \to eb$. This OPE constant can be expressed directly in terms of Υ_b-functions with the result

$$\begin{aligned}
&C_m(e_1, e_2, e_3) = R_m(e_3)\, C_m(e_1, e_2, 2\rho e_0 - e_3) \\
&= \left(b^{2Qb}\lambda_m\right)^{\frac{(e_{123} - 2\rho e_0, \rho)}{b}} \frac{\prod_{\alpha=\alpha_1,\alpha_{13}} \Upsilon_b((e_3 - \rho e_0, \alpha) + b)}{\prod_{\alpha=\alpha_1,\alpha_{14}} \Upsilon_b((e_3 - \rho e_0, \alpha) + e_0 + b)} \times \\
&\quad \prod_{a=1,2} \frac{\Upsilon_b((e_{123} - 2e_a, \omega_2 - \omega_1) + b)}{\Upsilon_b((e_a, \alpha_2) + b)} \frac{\Upsilon_b((e_{12}^3, \omega_1) + b)\Upsilon_b((e_{123}, \omega_1) - 2e_0 + b)}{\Upsilon_b(b)\Upsilon_b((e_3, \alpha_2) + b)} \times \\
&\quad \prod_{a=1,2} \frac{\Upsilon_b((e_{123} - 2e_a, \omega_1) - e_0 + b)}{\Upsilon_b((e_a, \alpha_2) - e_0 + b)} \frac{\Upsilon_b((e_{12}^3, \omega_2 - \omega_1) - e_0 + b)\Upsilon_b((e_{123}, \omega_2 - \omega_1) - 3e_0 + b)}{\Upsilon_b(b)\Upsilon_b((e_3, \alpha_{24}) - e_0 + b)}
\end{aligned} \tag{4.4}$$

where $\lambda_m = \pi\mu_m\gamma(-b^2)$ with μ_m - the analog of the cosmological constant, multiplying the interaction term in the action. The reflection amplitude corresponding to the longest

Weyl group element w_{121321} is the analytic continuation of (2.7) (written first as a finite ratio of γ-functions and then rewritten in terms of Υ_b-functions)

$$R_m(e_3) = \left(b^{2Qb}\lambda_m\right)^{\frac{(2e_3-2\rho e_0,\rho)}{b}} \prod_{\alpha>0} \frac{\Upsilon_b((e_3-\rho e_0,\alpha)+b)}{\Upsilon_b((e_3-\rho e_0,\alpha)+e_0+b)}\,. \tag{4.5}$$

Analogously to (2.10) the eight three charge factors in (4.4) can be written as points on an orbit with respect to the shifted Weyl action (4.3). The F- matrix elements are obtained by the same analytic continuation of the Toda ones in (3.7), e.g.,

$$\begin{aligned} F^m_{e_1+\omega_2 b, e_2+\omega_2 b}\begin{bmatrix} e_1 & e_3 \\ \omega_2 b & e_2 \end{bmatrix} &= \frac{\Gamma(b(e_0\rho - e_2,\alpha_2))\,\Gamma(1-b(\rho e_0 - e_1,\alpha_2))}{\Gamma(b((e^2_{31},\omega_1)+\frac{b}{2}))\,\Gamma(1-b((e^1_{23},\omega_1)+\frac{b}{2}))}\times \\ &\quad \frac{\Gamma(b(2\rho e_0 - e_2,\alpha_2))\,\Gamma(1+b(e_1-2\rho e_0,\alpha_2))}{\Gamma(b((e^2_{31},\omega_1)+\frac{b}{2}-e_0))\,\Gamma(1-b((e^1_{23},\omega_1)+\frac{b}{2}-e_0))} \end{aligned} \tag{4.6}$$

etc..

The W_4 CFT is described alternatively as the (principal) quantum DS reduction of a $\hat{sl}(4)$ WZW model (or its dual). With the parametrisation in (2.1) and (4.1) in the non-compact and compact WZW analogs the corresponding Sugawara dimensions are given by

$$\triangle^{\mathrm{Su}}(\beta) = \frac{1}{2}(\beta, 2\rho b - \beta)\,, \quad \triangle^{\mathrm{Su}}_m(e) = \frac{1}{2}(e, e+2\rho b)\,, \tag{4.7}$$

invariant (along with the higher Casimir eigenvalues) under the standard shifted action of the Weyl group on the $sl(4)$ weights $-\beta/b$ and e/b. The dimensions of the vertex operators in the WZW theory and their reduced Toda counterparts are related as[5]

$$\triangle(\beta) = \triangle^{\mathrm{Su}}(\beta) + \frac{1}{b}(\beta,\rho)\,, \quad \triangle_m(e) = \triangle^{\mathrm{Su}}_m(e) - \frac{1}{b}(e,\rho)\,. \tag{4.8}$$

For any pair of weights β and e related by an element w of the Weyl group one has a BPS-like relation

$$\begin{aligned} \beta = -b\,w\cdot\frac{e}{b} &= b\rho - w(e+b\rho) \\ \Rightarrow \quad \triangle^{\mathrm{Su}}(\beta) &+ \triangle^{\mathrm{Su}}_m(e) = 0\,. \end{aligned} \tag{4.9}$$

In particular there is only one nontrivial element of Weyl group, w_{2132}, s.t. its shifted action preserves the $sl(4)$ representations of type $(\lambda,\alpha_1) = 0 = (\lambda,\alpha_3)$, namely, $w_{2132}\cdot\lambda = -\lambda - 4\omega_2$, so that the first line of (4.9) reads

$$\beta = -bw_{2132}\cdot\frac{e}{b} = e + 4\omega_2 b \;\;\Rightarrow (\beta,\alpha_2)/b = (e,\alpha_2)/b + 4\,. \tag{4.10}$$

[5]On the level of 2-and 3-point functions the reduction amounts (up to a constant) to a "$x\to z$" limit of the $sl(4)$ isospin variables, see [20] and references therein. In particular, for the vertex highest weights of type $(\lambda,\alpha_1) = 0 = (\lambda,\alpha_3)$, to which we shall restrict in what follows, they are described by a 4d vector x_μ and the "$x\to z$" limit reads $x^2_{ij} \to |z_{ij}|^4$. E.g., applied to the WZW 2-point functions $G^{(m)}_e(x_{12}; z_{12}, \bar{z}_{12}) = (x^2_{12})^{(e,\rho)/2b}\,|z_{12}|^{-2\triangle^{\mathrm{Su}}_m(e)}$ and $G_\beta(y_{12}; z_{12}, \bar{z}_{12})) = (y^2_{12})^{-(\beta,\rho)/2b}\,|z_{12}|^{-2\triangle^{\mathrm{Su}}(\beta)}$ this reproduces, in agreement with (4.8), the corresponding 2-point functions of the W_4 fields up to constants.

While the sum of Sugawara dimensions vanishes according to (4.9), for the sum of the related by (4.8) W_4 dimensions one has $\triangle(e+4b\omega_2)+\triangle_m(e)=8$.

Recall that in the $sl(2)$ case the relation in the first line of (4.9) and its dual yield for the Virasoro dimension $\triangle(\epsilon e+\alpha b^{\epsilon})+\triangle_m(e)=1\,,\epsilon=\pm 1$. Accordingly the products $c\bar{c}\,V_\beta V_e^{(\mathrm{m})}$ (where $c,\bar{c}$ are the chiral components of the ghost of dimension -1), or $\int d^2x V_\beta V_e^{(\mathrm{m})}$, describe BRST invariant operators - the tachyons of the Liouville gravity. They have trivial, up to leg factors, 3-point function [21, 22]. Apparently unlike the Virasoro case one cannot realise W_4 analogs of such operators through products of vertex operators from the two regions of the theory.

Nevertheless in view of the relation between the $\hat{sl}(4)$ WZW and the W_4 conformal theories we may expect that the 3-point constants in the two W_4 regions are closely related. Indeed, take all $e_a=(0,r_a,0)b$ and impose (4.10), i.e., $C(\beta_1,\beta_2,\beta_3)=C(e_1+4\omega_2 b,e_2+4\omega_2 b,e_3+4\omega_2 b)$. One then has for the product of the two related constants

$$\begin{aligned}C_m(e_1,e_2,e_3)\,C(\beta_1,\beta_2,\beta_3)&=\prod_{a=1}^{3}\phi((\beta_a,\alpha_2))\,A(\beta_1,\beta_2,\beta_3)\,\bar{C}_m(e_1,e_2,e_3)\,\bar{C}(\beta_1,\beta_2,\beta_3)\\&=\lambda_T^{\frac{(2\rho Q-\beta_{123},\rho)}{b}}\lambda_m^{\frac{(e_{123}-2\rho e_0,\rho)}{b}}\prod_{a=1}^{3}\frac{(b^2)^3\prod_{\alpha=\alpha_2,\alpha_{24},\alpha_{14}}\gamma((\beta_a-\rho b,\alpha)b)}{\gamma((\rho Q-\beta_a,\alpha_{24})b)}\,A(\beta_1,\beta_2,\beta_3)\,,\end{aligned}\tag{4.11}$$

where

$$A(\beta_1,\beta_2,\beta_3)=\left((1-b^4)^2((\beta_{123}-2\rho Q,\omega_1)b)^2\prod_{a=1}^{3}((\beta_{123}-2\beta_a,\omega_1)-Q)b)^2\right)^{-1}.$$

The γ-factors in the second line of (4.11) (analogs of the leg factors in Liouville gravity) can always be removed by proper field normalisation. The intermediate notation $\bar{C}$ and $\bar{C}_m$ in the r.h.s. of the first equality refers to the constants obtained from the corresponding C in (2.8) and C_m in (4.4) by replacing $Q\to b$ and $e_o\to -b$, respectively, in the Υ_b-functions. This is achieved by the use of one of the functional relations (2.9) and produces finite products of γ-functions for each of the two constants, that are furthermore compensated in the product CC_m up to the factor $A(\beta_1,\beta_2,\beta_3)$ in (4.11) and $\prod_a\phi((\beta_a,\alpha_2))$, the explicit expression of which we skip. As clear from (4.11) the product $\bar{C}\bar{C}_m$ itself is trivial up to field renormalisation: the modified denominator from the third (fourth) line in (2.8) cancels the modified numerator from the fourth (third) line in (4.4) respectively.

One may expect that the two constants $\bar{C}_m(e_1,e_2,e_3)$ and $\bar{C}(\beta_1,\beta_2,\beta_3)$ will describe the corresponding 3-point constants of the compact and noncompact WZW model. This conjecture remains to be checked. In any case the triviality of the product $\bar{C}\,\bar{C}_m$ whenever the relation (4.10) is imposed is a property expected for the correlators of BRST invariant states in the non-critical string model described by a G/G topological CFT, see, e.g., [23].

In the semi-classical limit $b\to 0$ with "light" charges, i.e., $(\beta_a,\alpha_2)/b=\sigma_a$ are assumed finite, the factor in (4.11) which depends nontrivially on the three charges goes to a numerical constant, $A(\sigma_1 b,\sigma_2 b,\sigma_3 b)\to 1/9$. In other words in this limit the cancellation expected for the WZW counterparts of the W_4 constants holds true for the Toda constants themselves.

We conclude with a remark about the "light-charge" limit of each of the constants $\bar{C}(\beta_1,\beta_2,\beta_3)$ and $\bar{C}_m(e_1,e_2,e_3)$ computed using the asymptotics of $\Upsilon_b(x)$

$$\lim_{b\to 0}\Upsilon_b(b)/\Upsilon_b(\sigma b) = \Gamma(\sigma) b^{\sigma-1}\,.$$

As explained above in these constants compared to the initial Toda ones one replaces $Q\to b$ and $e_0\to -b$. All weights are taken to be proportional to the second fundamental weight ω_2, $\beta_a = \sigma_a\omega_2 b\,, e_a = r_a\omega_2 b$. We have in the limit $b\to 0$ with finite $\sigma_a\,, r_a$

$$\bar{C}(\beta_1,\beta_2,\beta_3)\sim \Gamma\left(\frac{\sigma_{123}}{2}-2\right)\prod_a \frac{\Gamma(\frac{\sigma_{123}}{2}-\sigma_a)}{\Gamma(\sigma_a)}\times\Gamma\left(\frac{\sigma_{123}}{2}-3\right)\prod_a \frac{\Gamma\left(\frac{\sigma_{123}}{2}-\sigma_a-1\right)}{\Gamma(\sigma_a-1)} \tag{4.12}$$

and

$$\bar{C}_m(e_1,e_2,e_3)\sim \frac{1}{\Gamma(\frac{r_{123}}{2}+3)}\prod_a \frac{\Gamma(r_a+1)}{\Gamma(\frac{r_{123}}{2}-r_a+1)}\times \frac{1}{\Gamma(\frac{r_{123}}{2}+4)}\prod_a \frac{\Gamma(r_a+2)}{\Gamma(\frac{r_{123}}{2}-r_a+2)}\,. \tag{4.13}$$

One recognizes in the Γ-function ratios of the first lines in (4.12) and (4.13) precisely the expressions of the 3-points constants of scalar 4d fields computed by integrating the bulk-bounday kernels (classical vertex operators) over the cosets AdS_5 and S^5, respectively [11, 12]. In this comparison we identify the charges $(\beta_a,\alpha_2)/b$ - with the 4d scalar field conformal dimensions $\triangle_a$ and the weights e_a/b (taking nonzero integer values) with the 4d isospins given by the SU(4) representation $(0,J_a,0)$.[6] The condition (4.10) for which the product of (4.12) and (4.13) trivialises implies with such identification $J_a=\triangle_a-4$.

On the other hand we can identify $(\beta_a,\alpha_2)/b = (\beta_a,\omega_2)/b$ with $\triangle_a+4$ instead. Then neither of the two factors in $\bar{C}$ (4.12) reproduces the AdS_5 result, but the trivialisation of the full $\bar{C}\,\bar{C}_m$ (and, in this limit, of the Toda constants product CC_m itself) due to (4.10) holds true for $J_a=\triangle_a$, which is the actual 4d supersymmetric BPS condition for the given class of representations; the second line in (4.9) is equivalent to the vanishing of the second Casimir of the superconformal algebra $sl(2,2|4)$.[7] Note that Toda light charge classical correlators can be computed alternatively by integrals of the exponential fields over the "bulk" SL(4) group, as shown on examples in [1], generalising the computation [25] in the Liouville case.

5 Concluding remarks

We have constructed 3-point functions in the W_4 Toda theory and have used them to derive novel data on a fundamental braiding/fusing matrix extending the rank 1 results. The solution described by a 3×3 matrix applies to a particular class of representations

[6]These are the AdS_5 and S^5 free field ingredients of the 3-point function of "chiral primary operators" with $\triangle_a = J_a$ at strong coupling λ [12]. The full correlator involves an additional factor coming from the coupling constant of the supergravity cubic interaction term, which compensates the product of the expressions in the r.h.s. of the first lines of (4.12) and (4.13) taken with $\sigma_a=\triangle_a\,, r_a = J_a$ (formula (3.40) of [12]).

[7]Different identifications for the three weights are also possible (reminiscent of the mixed correlators discussed in [24]).

arising from partially degenerate Verma modules with highest weights proportional to the sl(4) fundamental weight ω_2. The examples of OPE structure constants computed here and in [1] are still quite simple and need to be extended to positive integer "4d spin" components $l_a^{(i)} = -(\beta_a, \alpha_i)/b\,, i = 1, 3$. For that purpose the AGT-W approach [6, 7] might be more constructive. On the other hand one can try to exploit the pentagon equation for the 6×6 F matrix as a recursive relation given the initial data computed here and in [1].

We have analysed a higher rank analog of the braiding relation which played a basic role in the construction of the semi-classical limit of a class of 3-point functions on $AdS_3 \times S^3$ [9, 10] and have identified it with a standard identity in the modular group on the plane with four holes. The explicit data for the solutions of the braiding identity provided by Toda CFT, in particular their "heavy charge" limit, may thus find application to the quasiclassics of conformal sigma models described by compact and noncompact forms of SL$(4, \mathcal{C})$, generalising the SL$(2, \mathcal{C})$ results. Here again for a realistic application one needs first to extend the result beyond the particular class of representations.

More precisely, for this application one needs the extension of the Toda modular data to that of its WZW model counterpart; we hope to return to this problem. The computation of the corresponding $\hat{sl}(4)$ WZW 3-point functions is important also in view of the possible application to the G/G models. As we have observed, the affine sl(4) WZW theories can alternatively describe the simplest BPS states in the "light charge" classical limit by a different mechanism than the one provided by the supergravity approximation. The 2d CFT expected to describe the worldsheet realisation of the $\mathcal{N} = 4$ YM theory lacks the affine symmetry of the (super)conformal WZW models. Nevertheless further development of the latter may provide some inside on the structure of the former.

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A Details on the calculation of the Coulomb integrals

We start with briefly recalling the technique [1] of computation of some multiple integrals generalising Selberg integrals. The Toda 3-point Coulomb integral (with one type of screening charges) is given by

$$I_{s_1,s_2,s_3}(\beta_1, \beta_2) = \int \prod_{k=1}^{3} d\mu_{s_k}(t^{(k)})\, D_{s_k}^{-2b^2}(t^{(k)}) \times \tag{A.1}$$

$$\prod_{i,j}^{s_1,s_2} \left(|t_i^{(1)} - t_j^{(2)}|^{2b^2}\right) \prod_{i,j}^{s_3,s_2} \left(|t_i^{(3)} - t_j^{(2)}|^{2b^2}\right) \prod_{k=1}^{3} \prod_{i=1}^{s_k} |t_i^{(k)}|^{-2(\beta_1,\alpha_k)b} |t_i^{(k)} - 1|^{-2(\beta_2,\alpha_k)b}$$

where

$$D_s(t) = \prod_{i<j}^{s} |t_i - t_j|^2\,, \quad d\mu_s(t) = \frac{1}{\pi^s s!} \prod_{j=1}^{s} d^2 t_j\,.$$

The integral can be computed recursively for particular sets of weights β_1, β_2, exploiting a duality formula [8] originating from the Virasoro theory of central charge $c = -2$

$$\int d\mu_n(y) D_n(y) \prod_{i=1}^{n} \prod_{j=1}^{n+m+1} |y_i - t_j|^{2p_j} = \frac{\prod_{j=1}^{n+m+1} \gamma(1+p_j)}{\gamma(1+n+\sum_j p_j)} \prod_{i<j}^{n+m+1} |t_i - t_j|^{2+2p_i+2p_j} \times \int d\mu_m(u) D_m(u) \prod_{i=1}^{m} \prod_{j=1}^{n+m+1} |u_i - t_j|^{-2-2p_j}\,. \tag{A.2}$$

This formula results from two alternative Coulomb gas representations of the $n+m+2$-point function, obtained by replacing each vertex with its dual of the same conformal dimension; the compatibilty of the two charge conservation conditions, involving different numbers of screening charges, fixes the parameter b parametrizing the central charge. The two integral representations coincide up to a constant $C_n(\{p_j\}) = C_m^{-1}(\{-1-p_j\})$, indicated in the r.h.s. of (A.2), which is given by a product of reflection amplitudes.

For the particular integral discussed in section 2 the dependence on the two charges in (A.1) simplifies since $(\beta_1, \alpha_i) = 0 = (\beta_2, \alpha_i)$ for $i = 1, 3$. The calculation of the integral starts applying (A.2) for $n = s_1 - 1\,, m = 0$ and $p_j = -Qb\,, j = 1, \ldots, s_1\,,$ identifying the power of coordinate differences in the r.h.s. with the factor $D_{s_1}^{1+2p_j}(t^{(1)}) = D_{s_1}^{-2b^2}(t^{(1)})/D_{s_1}(t^{(1)})$ in (A.1). The formula (A.2) is then applied to the integrals over $\{t^{(k)}\}$, sequentially for $k = 1, 2, 3$ with $m = s_2 - 2\,, m = s_3 - 1\,, m = 0\,,$ respectively. The result is an integral of the same type as (A.1), with shifted indices and arguments described in section 2. After s_1 steps one obtains

$$I_{s_1,s_2,s_3}(\beta_1, \beta_2) = K I_{0,s_2-2s_1,0}(\beta_1 + s_1 b\omega_2, \beta_2 + s_1 b\omega_2) \tag{A.3}$$

where the integral in the r.h.s. is a Coulomb Liouville integral with $s_2 - 2s_1$ screening charges

$$\sum_{a=1}^{3} \beta_a^L + (s_2 - 2s_1)b = Q\,, \; 2\beta_a^L := 2(\beta_a, \omega_1) + s_1 b\,, a = 1, 2\,. \tag{A.4}$$

It is a residue of the DOZZ formula at the values corresponding to the charge conservation condition in (A.4), or, in Toda variables - at $(\beta_{12}^3, \omega_2 - 2\omega_1) = (\beta_3, \alpha_1) = -lb$ with nonnegative integer $l = s_2 - 2s_1$.

The constant K in (A.3) is given by

$$K = \prod_{s=0}^{s_1-1} \left(b^{((s_1+1)b^2+1)}\, \gamma((s-s_1)b^2)\right) \lim_{\varepsilon\to 0} \frac{\Upsilon_b(2Q + (s_2 - 2s_1)b + \varepsilon)}{\Upsilon_b(2Q + (s_2 - s_1)b + \varepsilon)}$$
$$\left(\frac{b^{-2Qb}}{\gamma(-b^2)}\right)^{\frac{-4(\beta_{12}^3,\omega_3)}{b}} \prod_{\alpha=\alpha_2,\alpha_{24}} \prod_{a=1,2} \frac{\Upsilon_b((Q\rho - \beta_a, \alpha))}{\Upsilon_b((Q\rho - \beta_a, \alpha) - s_1 b)} \tag{A.5}$$
$$\frac{\Upsilon_b(3Q - s_2 b - (\beta_1 + \beta_2, \alpha_2))\Upsilon_b(4Q - 2s_1 b - (\beta_1 + \beta_2, \alpha_2))}{\Upsilon_b(3Q + (s_1 - s_2)b - (\beta_1 + \beta_2, \alpha_2))\Upsilon_b(4Q - s_1 b - (\beta_1 + \beta_2, \alpha_2))}\,.$$

In writing (A.5) we have used the functional relation (2.9) to replace products of γ - functions with ratio of Υ_b- functions. The ratio of (regularized) Υ_b-functions in the r.h.s. of the first line is a finite product of γ's, written in a compact form. This factor can be rewritten getting rid of the nonnegative integers s_1, s_2 using (2.4) and then can be continued for arbitrary β_a of the type in (1.1) without the restrictions implied by (2.4), thus giving $\frac{\Upsilon_b((\beta_3,\alpha_1)-Q)}{\Upsilon_b((\beta_{12}^3,\omega_2-\omega_1)-Q)}$. Analogously are rewritten and continued the factors in the second and the third line. On the other hand the product in the first line can be written as a residue of an analytically continued expression

$$\mathrm{res}_{(\beta_{12}^3,\omega_1)=-s_1 b}\frac{\Upsilon(b)}{\Upsilon((\beta_{12}^3,\omega_1))} = b^{s_1[(s_1+1)b^2+1]}\prod_{s=0}^{s_1-1}\gamma((s-s_1)b^2)\,. \tag{A.6}$$

Altogether, combining with the Liouville constant discussed above, one obtains the expression for the OPE constant $C(\beta_1,\beta_2,2\rho Q-\beta_3)$ in (2.8). It is valid for weights of the type in (1.1), while the Coulomb OPE is recovered as in (2.6).

A slightly more general case, in which the OPE constant $c(\beta_1,\beta_2,2\rho Q-\beta_3)$ can be computed along the same path, is provided by charges β_1,β_2 s.t., say, β_2 is of the same kind as before, $(\beta_2,\alpha_i)=0, i=1,3$, while β_1 has two nonvanishing components, e.g., $(\beta_1,\alpha_1)=0$. The integral is computed under the condition $s_3\geq s_1$. After s_1 steps the $sl(4)$ type integral reduces to a $sl(3)$ type I_{0,s_2-2s_1,s_3-s_1} which furthermore is reduced to Liouville type $I_{0,s_2-s_1-s_3,0}$. In particular the resulting expression for the example $\beta_2 = -\omega_2 b\,, s_1=0\,,\, s_3=1=s_2$ reproduces the OPE formula (1.51) in [1] for the shift $\beta_3 = \beta_1-(\omega_2-\alpha_{24})b=\beta_1-(\omega_1-\omega_3)b$:

$$c(\beta_1,-\omega_2 b,2\rho Q-(\beta_1-\omega_2 b+\alpha_{24}b)) = \left(\frac{\pi\mu}{\gamma(-b^2)}\right)^2\frac{\gamma((\beta_1-\rho Q,\alpha_3)b)\gamma((\beta_1-\rho Q,\alpha_{24})b)}{\gamma((\beta_1,\alpha_3)b)\gamma(Qb+(\beta_1-\rho Q,\alpha_{24})b)}\,.$$

If we set $(\beta_1,\alpha_3)=0$ - as in the case considered in section 2, the r.h.s. vanishes.

For the analytic continuation of $c(\beta_1,\beta_2,2\rho Q-\beta_3)$ one obtains

$$\begin{aligned}C(\beta_1,\beta_2,2\rho Q-\beta_3) = &\frac{(b^{2e_0b}\lambda)^{\frac{-(\beta_{12}^3,\rho)}{b}}\Upsilon_b^3(b)}{\Upsilon_b((\beta_{12}^3,\omega_1)\Upsilon_b((\beta_{12}^3,\omega_2-\omega_1)-Q)}\\
&\frac{\Upsilon_b((\rho Q-\beta_1,\alpha_2))\Upsilon_b((\rho Q-\beta_1,\alpha_{24}))\Upsilon_b((\rho Q-\beta_2,\alpha_2))\Upsilon_b((\rho Q-\beta_2,\alpha_{13}))}{\Upsilon_b((\beta_{123^*}-2\rho Q,\omega_2-\omega_1)-Q)\Upsilon_b((\beta_{123^*}-2\rho Q,\omega_1))}\\
&\frac{1}{\Upsilon_b((\beta_{23}^1,\omega_1)-Q)\Upsilon_b((\beta_{23}^1,\omega_2-\omega_1))}\frac{1}{\Upsilon_b((\beta_{13^*}^2,\omega_2-\omega_1))\Upsilon_b((\beta_{13^*}^2,\omega_1)-Q)}\\
&\frac{\prod_{\alpha>0}\Upsilon_b((\beta_3-\rho Q,\alpha))}{\Upsilon_b((\beta_{12}^3,\omega_3-\omega_1)+(\beta_3-\rho Q,\alpha_{24}))\Upsilon_b((\beta_{123^*},\omega_3-\omega_1)+(\rho Q-\beta_3^*,\alpha_{24}))}\\
&\frac{\Upsilon_b((\rho Q-\beta_1,\alpha_3))}{\Upsilon_b((\beta_{123^*}-2\rho Q,\alpha_2-\omega_2))\Upsilon_b((\beta_{12}^3,\alpha_2-\omega_2))\Upsilon_b((\rho Q-\beta_1,\alpha_3)+(\beta_{12}^3,\omega_3-\omega_1))\Upsilon_b((\beta_{12}^3,\omega_3-\omega_1))}\end{aligned} \tag{A.7}$$

valid for arbitrary β_3 and $(\beta_2,\alpha_1)=0=(\beta_2,\alpha_3)\,,(\beta_1,\alpha_1)=0$. Taking the residue at $(\beta_{12}^3,\omega_3-\omega_1)=0$ and then setting $(\beta_1,\alpha_3)=0$ one reproduces the OPE constant $C(\beta_1,\beta_2,2\rho Q-\beta_3)$ with $\beta_3=\beta_3^*$ of section 2.

Similarly one derives the analog of the constant (A.7) with $(\beta_1,\alpha_3)=0$ and nonzero components $(\beta_1,\alpha_i)\neq 0\,, i=1,2$.

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The duality formula (A.2) can be used to show that the 4-point Toda functions of the type discussed in section 3 admit an alternative integral representation. The derivation is a certain generalisation of the one in [26] for the case of Liouville correlators with one degenerate field, shown to be proportional to a Coulomb Liouville correlator with generic weights; similar consideration appears in [1, 3].

Consider the 4-point function $\langle V_{-\omega_2 b}(x)V_{\beta_1}(0)V_{\beta_2}(1)V_{2\rho Q-\beta_4}(\infty)\rangle_{CG}$ with vertex highest weights $(\beta_a,\alpha_i)=0, i=1,3, a=1,2$, $\beta_3=-\omega_2 b$ and $(\beta_4,\alpha_i)=-lb, i=1,3$ with non-negative integer l. We assume that this is a Coulomb correlator with weights satsfying the charge conservation condition $-(\beta^4_{123},\omega_1)/b=s_1=s_3$, with a positive integer s_1. It is given by the multiple integral $I_{s_1,s_2,s_1}(\beta_1,\beta_2,\beta_3)$ with $s_2-2s_1=-(\beta^4_{123},\omega_2-2\omega_1)/b=-(\beta_4,\alpha_1)/b=l$. This integral is converted recursively with the help of (A.2) similarly to what was done above for the 3-point function. Unlike that computation the recursion does not preserve the type of the integral, since at the first step formula (A.2) is applied in the last integration with respect to $\{t^{(3)}_j, j=1,2,\ldots,s_3\}$ with $m=2$, i.e., one more double integral is added and this structure after the first step is recursively repeated, yielding for $1\le s\le s_1$ the integral

$$I_{s_1-s,s_2-2s,s_3-s+1;2}(\beta^{(s)}_1,\beta^{(s)}_2,0;\beta^{(s)}_3)(x):= \tag{A.8}$$

$$\int d\mu_{s_3-s+1}(t^{(3)})\Phi^{(s)}(x;t^{(3)})\int d\mu_2(t^{(4)})D_2(t^{(4)})\prod_{j=1}^{s_3-s+1}\prod_{i=1}^{2}|t^{(3)}_j-t^{(4)}_i|^{2b^2}\times$$
$$|t^{(4)}_i|^{2(\beta^{(s)}_1-2\rho Q,\alpha_2)b}|t^{(4)}_i-1|^{2(\beta^{(s)}_2-2\rho Q,\alpha_2)b}|t^{(4)}_i-x|^{-2(\beta^{(s)}_3+2\rho Q,\alpha_2)b}$$

where

$$\beta^{(s)}_a=(\beta_a+s\omega_2 b)\,, a=1,2,3 \tag{A.9}$$

and $\Phi^{(s)}(x;t^{(3)})$ is the integrand of $I_{s_1-s,s_2-2s,s_3-s+1}(\beta^{(s)}_1,\beta^{(s)}_2,0)(x)$ integrated over the first two sets of variables $\{t^{(1)}_j|, j=1,2\ldots s_1-s\}\,,\{t^{(2)}_i|, i=1,2\ldots s_2-2s\}$ so that

$$\int d\mu_{s_3-s+1}(t^{(3)})\Phi^{(s)}(x;t^{(3)})=I_{s_1-s,s_2-2s,s_3-s+1}(\beta^{(s)}_1,\beta^{(s)}_2,0)(x)\,.$$

Setting $s=s_1=s_3$ we obtain the integral $I_{0,l,1;\,2}$ up to a constant $\Omega_{s_1,l}(\{\beta_a\})$. In this integral s_1 is a parameter appearing in the new weights $\beta^{(s_1)}_a$, so the integral can be continued to generic values of β_a. The constant $\Omega_{s_1,l}(\{\beta_a\})$ is analytically continued then for non-integer $s_1=-(\beta^4_{123},\omega_1)/b$ keeping l non-negative integer. This determines the initial correlator with the charge conservation condition dropped

$$\langle V_{-\omega_2 b}(x)V_{\beta_1}(0)V_{\beta_2}(1)V_{2\rho Q-\beta_4}(\infty)\rangle=\Omega_l(\{\beta_a\})|x|^{2b(4Q\rho-(\beta_1,\alpha_2))}|x-1|^{2b(4Q-(\beta_2,\alpha_2))}\times$$
$$I_{0,l,1;\,2}(\beta_1-(\beta^4_{123},\omega_1)\omega_2,\beta_2-(\beta^4_{123},\omega_1)\omega_2,0;\beta_3-(\beta^4_{123},\omega_1)\omega_2)(x)\,, \tag{A.10}$$

$$\Omega_l(\{\beta_a\}) = \frac{b^{4(\beta_{123}^4,\omega_1)b} b^{2Qbl} \gamma(-b^2)^{l+1}}{\gamma(-Qb+(\beta_{123}^4,\omega_1)b)\gamma((\beta_{123}^4,\omega_1)b)} \frac{\Upsilon_b(2Q)}{\Upsilon_b(Q)} \times$$
$$\frac{\left(b^{2(1-b^2)}\lambda_T\right)^{-\frac{(\beta_{123}^4,\rho)}{b}} \Upsilon_b^2(b)}{\Upsilon_b((\beta_{123}^4,\omega_1))\Upsilon_b((\beta_{123}^4,\omega_2-\omega_1)-Q)} \prod_{k=0}^{l-1} b^{-b(3Q+lb)}\ \gamma(kb^2+2Qb)\times$$
$$\prod_{a=1}^{2} \frac{\Upsilon_b((\beta_a,\alpha_2)\Upsilon_b((\beta_a-\rho Q,\alpha_2))}{\Upsilon_b(Q+(\beta_{123}^4-2\beta_a,\omega_1))\Upsilon_b(2Q+(\beta_{123}^4-2\beta_a,\omega_1))} \times$$
$$\frac{\Upsilon_b((\beta_4-\rho Q,\alpha_{24}))\Upsilon_b((\beta_4-\rho Q,\alpha_{14}))}{\Upsilon_b((\beta_{1234},\omega_2-\omega_1)-2Q)\Upsilon_b((\beta_{1234}-2\rho Q,\omega_1))}\,. \tag{A.11}$$

The constant $\Omega_{s_1,l}(\{\beta_a\})$ is recovered as the coefficient of an order two pole of $\Omega_l(\{\beta_a\})$ in (A.11) at $(\beta_{123}^4,\omega_1) = -s_1 b\,, s_1 \in \mathcal{Z}_{>0}$. The appearance of the pole of order two is due to the fact that l is a non-negative integer. Alternatively the expression in (A.11) can be further extended for generic values of l so that the second order pole splits into two first order poles - then the initial Coulomb representation of the l.h.s. is recovered by a double residue as in (2.6).

For $l=0$ the integral in the r.h.s. can be interpreted, after integrating over $t^{(3)}$, as a Liouville Coulomb integral $I_2^{(\tilde{b})}(\tilde{\beta}_1,\tilde{\beta}_2,\tilde{\beta}_3)(x)$ with a modified parameter $b^2 \to\ \tilde{b}^2 = -Qb$,

$$\begin{aligned}
&2\tilde{\beta}_a\tilde{b} = -(\beta_a^{(s_1)},\alpha_2)b + 2Qb = -(\beta_a,\alpha_2)b + (\beta_{123}^4,\omega_1) + 2Qb\,, a=1,2\,,\\
&2\tilde{\beta}_3\tilde{b} = (\beta_3^{(s_1)},\alpha_2)b + 2Qb = (\beta_3,\alpha_2)b - (\beta_{123}^4,\omega_1) + 2Qb\,,\\
&2\tilde{\beta}_4\tilde{b} = 2(\tilde{\beta}_{123}+2\tilde{b})\tilde{b} = -(\beta_4,\alpha_2)b - (\beta_{123}^4,\omega_1)b + 2\,.
\end{aligned} \tag{A.12}$$

This integral represents a 4-point function which admits three fusion channels - in agreement with the truncated to three terms Toda fusion rule. According to the result in [26] this Coulomb Liouville 4-point function is furthermore related to a Liouville 4-point function with one degenerate field $V_{-\tilde{b}}$, which satisfies a third order BPZ differential equation. The observed relation between 4-point functions in the W_4 and the W_2 theory (with modified parameter $\tilde{b} = -Qb$) suggests that there will be also a relation for the fusing matrices. Indeed, the 3×3 F matrix transforming the Virasoro block with the degenerate vertex highest weight $\beta^L = -\tilde{b}$ and three arbitrary representations has similar structure to the F matrix computed in section 3; the precise identification will be presented elsewhere.

The derivation of the above Coulomb representation can be extended to a more general set of weights, e.g., restricting only the components $(\beta_a,\alpha_1) = 0\,, a=1,2$ and with non-symmetric β_4 s.t. $-(\beta_4,\alpha_1)/b = l \in \mathcal{Z}_{\geq 0}$. To ensure that $s_3 - s_1 \in \mathcal{Z}_{\geq 0}$ one has to impose additional restrictions on the combination of components $(\beta_{12}^4/b, \alpha_1-\alpha_3)$.The set includes the doubly degenerate weights with $l_a^{(3)} = -(\beta_a,\alpha_3)/b \in \mathcal{Z}_{\geq 0}\,, a=1,2,3$.

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