Wilson loops on three-manifolds and their M2-brane duals

Daniel Farquet and James Sparks

Mathematical Institute, University of Oxford,
Radcliffe Observatory Quarter, Woodstock Road, Oxford OX2 6GG, U.K.

E-mail: farquet@maths.ox.ac.uk, sparks@maths.ox.ac.uk

ABSTRACT: We compute the large N limit of Wilson loop expectation values for a broad class of $\mathcal{N} = 2$ supersymmetric gauge theories defined on a general class of background three-manifolds M_3, diffeomorphic to S^3. We find a simple closed formula which depends on the background geometry only through a certain supersymmetric Killing vector field. The supergravity dual of such a Wilson loop is an M2-brane wrapping the M-theory circle, together with a complex curve Σ_2 in a self-dual Einstein manifold M_4, whose conformal boundary is M_3. We show that the regularized action of this M2-brane also depends only on the supersymmetric Killing vector, precisely reproducing the large N field theory computation.

Contents

1   Introduction and summary

There has recently been considerable interest in defining and studying supersymmetric gauge theories on compact manifolds. This stems from the fact that certain observables may be computed exactly in such quantum field theories using localization. The first examples of such computations in the literature typically studied round sphere backgrounds, but more generally the observables also depend on the choice of background geometry, leading to a richer structure. Such exact computations may be used to test and explore non-perturbative dualities, and the focus of this paper will be the gauge/gravity duality.

In [1] the partition function Z of three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories on a general class of background three-manifold geometries M_3 was computed exactly. In particular Z was shown to depend on the background geometry only through a certain supersymmetric Killing vector field K.[1] There are rich classes of $\mathcal{N} = 2$ superconformal gauge theories which have a large N gravity dual in M-theory. For these theories one can compute the large N limit of the partition function using the matrix model saddle point technique of [3]. When M_3 is diffeomorphic to S^3 with the standard action of $\mathrm{U}(1) \times \mathrm{U}(1)$ on $S^3 \subset \mathbb{R}^2 \oplus \mathbb{R}^2$, and writing $K = b_1 \partial_{\varphi_1} + b_2 \partial_{\varphi_2}$ in terms of the generators ∂_{φ_i} of $\mathrm{U}(1) \times \mathrm{U}(1)$, one finds [1] the large N free energy $\mathcal{F} = -\log Z$ satisfies

$$\lim_{N \to \infty} \frac{\mathcal{F}}{\mathcal{F}_{\mathrm{round}}} = \frac{(|b_1| + |b_2|)^2}{4|b_1 b_2|}, \tag{1.1}$$

---

[1]This was also argued independently in [2].

where $\mathcal{F}_{\text{round}}$ is the large N limit of the free energy on the *round* three-sphere, which scales as $N^{3/2}$ [4].

In [5] the field theory result (1.1) was reproduced in a dual computation in four-dimensional gauged supergravity. Here $M_3 \cong S^3$ arises as the conformal boundary of a self-dual Einstein four-manifold M_4, where the supersymmetric Killing vector K also extends over M_4. The asymptotically locally Euclidean AdS metric on M_4 is conformally Kähler, and supersymmetry requires one to turn on a graviphoton field A proportional to the Ricci one-form of this Kähler metric. A remarkable feature of the computation of the holographic free energy in [5] is that one does not need to know the form of the Einstein metric on M_4 explicitly — rather (1.1) is proven for an *arbitrary* such metric.

In [6] the vacuum expectation values of BPS Wilson loops on the round sphere were computed for a variety of gauge theories, and matched to regularized M2-brane actions in $\text{AdS}_4 \times Y_7$. Here the choice of internal space Y_7 determines the gauge theory on M_3. The purpose of this paper is to extend these computations to general supersymmetric backgrounds $M_3 = \partial M_4$. A Wilson loop is BPS if it wraps an orbit of K, and we will find that the large N Wilson loop VEV satisfies

$$\lim_{N \to \infty} \frac{\log \langle W \rangle}{\log \langle W_{\text{round}} \rangle} = \mathcal{S}_{b_1, b_2} \,, \tag{1.2}$$

where

$$\mathcal{S}_{b_1, b_2} \equiv \frac{|b_1| + |b_2|}{2} \ell \,. \tag{1.3}$$

Here $\langle W_{\text{round}} \rangle$ denotes the large N limit of the Wilson loop on the round sphere, whose logarithm scales as $N^{1/2}$, and $2\pi\ell$ denotes the length of the orbit of K. Such orbits always close over the poles of S^3, i.e. at the origins of each copy of $\mathbb{R}^2$ in $S^3 \subset \mathbb{R}^2 \oplus \mathbb{R}^2$, where the lengths are then $\ell = 1/|b_1|$ and $\ell = 1/|b_2|$, respectively. For these Wilson loops (1.2) becomes a function of b_1/b_2, exactly as in (1.1). The supergravity dual configurations are given by M2-branes wrapping a supersymmetric copy of the M-theory circle in Y_7 [6] and a complex curve $\Sigma_2 \subset M_4$, with boundary $\partial\Sigma_2 \subset M_3$ being the Wilson line. Identifying the logarithm of the VEV with minus the holographically renormalized M2-brane action, we also prove that (1.2) holds in general, thus verifying the matching of this observable in AdS/CFT in a very broad (infinite-dimensional) class of backgrounds.

The outline of the rest of this paper is as follows. In section 2 we review the geometry of M_3, the definition of the BPS Wilson loop and how it may be computed using localization techniques in the large N limit to find (1.2). Section 3 analyses supersymmetric M2-branes in $M_4 \times Y_7$ backgrounds in M-theory and we also derive the formula (1.2) in supergravity. Since our arguments are for general backgrounds they are somewhat implicit; in section 4 we therefore look at some explicit toric self-dual Einstein spaces, to exemplify our general formulae. We conclude in section 5 with a brief discussion.

2 Wilson loops in $\mathcal{N} = 2$ gauge theories on M_3

The field theories of interest have UV descriptions as $\mathcal{N} = 2$ Chern-Simons gauge theories coupled to matter on M_3, where M_3 is a supersymmetric three-manifold. We begin this

section by reviewing the geometry of M_3, and then define the BPS Wilson loops of interest. These have been studied on particular squashed sphere backgrounds in [7, 8] (see also [9, 10]), and the extension to the general backgrounds of [1, 11] is straightforward. After explaining how the Wilson loop VEVs localize in the matrix model, we then take the large N limit to derive (1.2).

2.1 Three-dimensional background geometry

The manifold M_3 belongs to a general class of "real" supersymmetric backgrounds, with two supercharges related to one another by charge conjugation [11]. If χ denotes the Killing spinor on M_3 then there is an associated Killing vector field

$$K \equiv \chi^\dagger \gamma^\mu \chi \partial_\mu \ = \ \partial_\psi \, . \tag{2.1}$$

This Killing vector is nowhere zero and therefore defines a foliation of the three-manifold. This foliation is transversely holomorphic with local complex coordinate z. In terms of these coordinates the background metric may be written as[2]

$$\mathrm{d}s^2_{M_3} = (\mathrm{d}\psi + \phi_{(0)})^2 + 4\mathrm{e}^{w_{(0)}}\mathrm{d}z\mathrm{d}\bar{z} \, , \tag{2.2}$$

where $\phi_{(0)} = \phi_{(0)}(z, \bar{z})\mathrm{d}z + \overline{\phi_{(0)}(z, \bar{z})}\mathrm{d}\bar{z}$ is a local one-form and $w_{(0)}(z, \bar{z})$ is a function. We introduce an orthonormal frame for the three-metric $\mathrm{d}s^2_{M_3}$:

$$e^1_{(3)} \ = \ \mathrm{d}\psi + \phi_{(0)} \, , \qquad e^2_{(3)} + \mathrm{i}e^3_{(3)} \ = \ 2\mathrm{e}^{w_{(0)}/2}\mathrm{d}z \, , \tag{2.3}$$

and will use indices $i, j, k = 1, 2, 3$ for this frame.

It is important to stress here that *arbitrary* choices for $\phi_{(0)}$ and $w_{(0)}$ (subject to M_3 being smooth) lead to supersymmetric backgrounds. The corresponding Killing spinor equation for χ may be found in [1, 11]. Choosing the three-dimensional gamma matrices, in the frame (2.3), to be simply the Pauli matrices, one finds that the Killing spinor solution is

$$\chi = \mathrm{e}^{\mathrm{i}\alpha(\psi, z, \bar{z})} \begin{pmatrix} \chi_0 \\ \chi_0 \end{pmatrix} \, , \tag{2.4}$$

where χ_0 is a constant and $\alpha(\psi, z, \bar{z})$ is a phase. The latter will play an important role later.

In much of what follows, and as in [1], we will assume that $M_3 \cong S^3$ with a toric structure, so that we have a $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry. If we realize $M_3 \cong S^3 \subset \mathbb{R}^2 \oplus \mathbb{R}^2$ then we may write

$$K = b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2} \, , \tag{2.5}$$

where φ_1, φ_2 are standard 2π-period coordinates on $\mathrm{U}(1) \times \mathrm{U}(1)$.

---

[2]More generally there is a conformal factor for this metric [11]. However, as in [5] we are interested in conformal field theories with gravity duals, and we may hence set this conformal factor to 1.

2.2 The Wilson loop

In $\mathcal{N} = 2$ supersymmetric gauge theories the gauge field $\mathcal{A}_i$ is part of a vector multiplet that also contains two real scalars σ and D and a two-component spinor λ, all of which are in the adjoint representation of the gauge group G. The BPS Wilson loop in a representation $\mathcal{R}$ of G is given by

$$W = \frac{1}{\dim \mathcal{R}} \mathrm{Tr}_{\mathcal{R}} \left[\mathcal{P} \exp \left(\oint_\gamma \mathrm{d}s(\mathrm{i}\mathcal{A}_i \dot{x}^i + \sigma |\dot{x}|) \right) \right] , \qquad (2.6)$$

where $x^i(s)$ parametrizes the worldline $\gamma \subset M_3$ of the Wilson loop and the path ordering operator has been denoted by $\mathcal{P}$. For a Chern-Simons theory the gauge multiplet has a kinetic term described by the supersymmetric Chern-Simons action

$$S_{\mathrm{CS}} = \frac{\mathrm{i}k}{4\pi} \int \mathrm{Tr} \left[\mathcal{A} \wedge \mathrm{d}\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} + (2D\sigma - \lambda^\dagger \lambda)\mathrm{vol}_3 \right] , \qquad (2.7)$$

where k denotes the Chern-Simons coupling and vol_3 is the Riemannian volume form on M_3.

The full set of supersymmetry transformations for a vector multiplet and matter multiplet may be found in [1]. For our purposes we need note only that localization of the path integral, discussed in the next section, requires one to choose a Killing spinor, namely χ in (2.4). We then need the following two supersymmetry transformations

$$\delta \mathcal{A}_i = -\frac{\mathrm{i}}{2}\lambda^\dagger \tau_i \chi , \qquad \delta \sigma = -\frac{1}{2}\lambda^\dagger \chi ,$$

where τ_i are the Pauli matrices. If one varies the Wilson loop (2.6) under the latter supersymmetry transformation one obtains

$$\delta W \propto \frac{1}{2}\lambda^\dagger (\tau_i \dot{x}^i - |\dot{x}|)\chi . \qquad (2.8)$$

The Wilson loop is then invariant under supersymmetry provided

$$(\tau_i \dot{x}^i - |\dot{x}|)\chi = 0 . \qquad (2.9)$$

Choosing s to parametrize arclength, so that $|\dot{x}| = 1$ along the loop, it is straightforward to show that (2.9) is satisfied if and only if the Wilson loop lies along the $e^1_{(3)}$ direction. From (2.3) we see that $e^1_{(3)}$ is the one-form dual to the supersymmetric Killing vector $K = \partial_\psi$. Thus the Wilson loop (2.6) is indeed a BPS operator provided one takes γ to be an orbit of K. Notice that the topology of M_3 has not been used in this subsection, and thus any Wilson loop wrapped along an orbit of K is BPS, regardless of the topology of M_3.

2.3 Localization in the matrix model

The VEV of the BPS Wilson loop (2.6) is, by definition, obtained by inserting W into the path integral for the theory on M_3. The computation of this is greatly simplified by the fact that this path integral *localizes* onto supersymmetric configurations of fields. This is by now a fairly standard computation, and we shall simply summarize the main steps,

referring the reader to [1, 3, 12–14] for further details. In particular the localization of the Wilson loop was explained in detail in [6] for the round S^3 case. This section generalizes that discussion to a generic supersymmetric manifold $M_3 \cong S^3$.

The central idea is that the path integral, with W inserted, is invariant under the supersymmetry variation δ corresponding to the Killing spinor χ. We have written two of the supersymmetry variations in (2.8), and the variations of other fields (including fields in the chiral matter multiplets) may be found on the curved background M_3 in [1]. Crucially, $\delta^2 = 0$ is nilpotent. There is then a form of fixed point theorem that implies that the only net contributions to this path integral come from field configurations that are invariant under δ [15].

For the $\mathcal{N} = 2$ supersymmetric Chern-Simons-matter theories of interest, one finds that the δ-invariant configurations on $M_3 \cong S^3$ are particularly simple:

$$\mathcal{A}_i = 0, \quad \sigma = \text{constant}, \quad D = -\sigma h, \tag{2.10}$$

where the function $h = \frac{1}{2} * (e^1_{(3)} \wedge \mathrm{d}e^1_{(3)})$, and with all fields in the matter multiplet set identically to zero [1]. Here we may diagonalize σ by a gauge transformation. The exact localized partition function then takes the saddle point form [1]

$$Z = \int \mathrm{d}\sigma\, \mathrm{e}^{-\frac{\mathrm{i}\pi k}{|b_1 b_2|} \mathrm{Tr}\,\sigma^2} \prod_{\alpha \in \Delta_+} 4 \sinh \frac{\pi \sigma \alpha}{|b_1|} \sinh \frac{\pi \sigma \alpha}{|b_2|} \prod_\rho s_\beta \left[\frac{\mathrm{i}Q}{2}(1-r) - \frac{\rho(\sigma)}{\sqrt{|b_1 b_2|}} \right]. \tag{2.11}$$

Here the integral is over the Cartan of the gauge group, k denotes the Chern-Simons level, the first product is over positive roots $\alpha \in \Delta_+$ of the gauge group, and the second product is over weights ρ in the weight space decomposition for a chiral matter field in an arbitrary representation $\mathcal{R}_{\text{matter}}$ of the gauge group. We have also defined

$$\beta \equiv \sqrt{\left| \frac{b_1}{b_2} \right|}, \qquad Q \equiv \beta + \frac{1}{\beta}, \tag{2.12}$$

the R-charge of the matter field is denoted r, and $s_\beta(z)$ denotes the double sine function.

In this set-up, the VEV of the BPS Wilson loop (2.6) reduces to

$$\langle W \rangle = \frac{1}{Z \dim \mathcal{R}} \int \mathrm{d}\sigma\, \mathrm{e}^{-\frac{\mathrm{i}\pi k}{|b_1 b_2|} \mathrm{Tr}\,\sigma^2} \prod_{\alpha \in \Delta_+} 4 \sinh \frac{\pi \sigma \alpha}{|b_1|} \sinh \frac{\pi \sigma \alpha}{|b_2|}$$

$$\times \prod_\rho s_\beta \left[\frac{\mathrm{i}Q}{2}(1-r) - \frac{\rho(\sigma)}{\sqrt{|b_1 b_2|}} \right] \mathrm{Tr}_\mathcal{R} \left(\mathrm{e}^{2\pi \ell \sigma} \right). \tag{2.13}$$

Notice the integrand is the same as that for the partition function (2.11), with an additional insertion of $\mathrm{Tr}_\mathcal{R}(\mathrm{e}^{2\pi \ell \sigma})$ arising from the Wilson loop operator. Note also that we have normalized the VEV relative to the partition function Z, so that $\langle 1 \rangle = 1$, as is usual in quantum field theory. We have also defined

$$\oint_\gamma \mathrm{d}s = 2\pi \ell \tag{2.14}$$

so that ℓ parametrizes the length of the Wilson line. More precisely, the integral (2.14) is well-defined only for a closed orbit of the Killing vector $K = \partial_\psi = b_1 \partial_{\varphi_1} + b_2 \partial_{\varphi_2}$. A generic orbit is closed only when $b_1/b_2 \in \mathbb{Q}$ is rational, so that K generates a circle subgroup of $U(1) \times U(1)$. Writing $b_1/b_2 = m/n$ with $m, n \in \mathbb{Z}$ relatively prime integers, these define torus knots via $\gamma \subset T^2 \subset S^3$, where the homology class $[\gamma] = (m, n) \in H_1(T^2, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. These have been studied in the present context in [8]. If on the other hand b_1/b_2 is irrational, then the *only* closed orbits are at the two "poles" of $M_3 \cong S^3$, where $\partial_{\varphi_1} = 0$ and $\partial_{\varphi_2} = 0$, respectively. Over these poles $\oint_\gamma \mathrm{d}s = 2\pi/|b_2|$, $2\pi/|b_1|$, respectively. Wherever the loop is located, we denote its length $\oint_\gamma \mathrm{d}s$ by $2\pi\ell$ as above.

For a U(N) gauge group we may write $\sigma = \mathrm{diag}(\frac{\lambda_1}{2\pi}, \dots \frac{\lambda_N}{2\pi})$, thus parametrizing $2\pi\sigma$ by its eigenvalues λ_i. Localization has then reduced the partition function Z and the Wilson loop VEV to finite-dimensional integrals (2.11), (2.13) over these eigenvalues, but in practice the formulae are difficult to evaluate explicitly. For comparison to the dual supergravity results we must take the $N \to \infty$ limit, where the number of eigenvalues, and hence integrals, tends to infinity. One can then attempt to compute this limit using a saddle point approximation of the integral. In [3] a simple *ansatz* for the large N limit of the saddle point eigenvalue distribution was introduced. One seeks saddle points with eigenvalues of the form

$$\lambda_i = x_i N^{1/2} + \mathrm{i}y_i \,, \tag{2.15}$$

with x_i and y_i real and assumed to be $\mathcal{O}(1)$ in a large N expansion. In the large N limit the real part is assumed to become dense. Ordering the eigenvalues so that the x_i are strictly increasing, the real part becomes a continuous variable x, with density $\rho(x)$, while y_i becomes a continuous function of x, $y(x)$.

Writing $Z = \mathrm{e}^{-\mathcal{F}}$ one then obtains a functional $\mathcal{F}[\rho(x), y(x)]$, with x supported on some interval $[x_{\min}, x_{\max}]$, and to apply the saddle point method one then extremizes $\mathcal{F}$ with respect to $\rho(x), y(x)$, subject to the constraint that $\rho(x)$ is a density

$$\int_{x_{\min}}^{x_{\max}} \rho(x)\mathrm{d}x = 1 \,. \tag{2.16}$$

One then finally also extremizes over the choice of interval, by varying with respect to $x_{\min}$, $x_{\max}$, to obtain the saddle point eigenvalue distribution $\rho(x), y(x)$.

As it turns out, if one caries out the large N limit with the ansatz (2.15), one finds a very simple relation between the round sphere results $\mathcal{F}_{\mathrm{round}}$ and $\log \langle W_{\mathrm{round}} \rangle$ and their squashed counterparts (with arbitrary b_1 and b_2) $\mathcal{F}$ and $\log \langle W \rangle$. To obtain this result for $\mathcal{F}$, one may first relabel σ as $|b_2|\sigma$ in (2.11). The partition function then takes the same form as that in [16], where the large N limit was computed in detail. In particular in the latter reference it was shown that in the large N limit $\mathcal{F}[\rho(x), y(x)]$ is simply a rescaling of the round sphere result by a factor $(\beta Q)^3/2^3\beta^2$, *provided* one also rescales the Chern-Simons coupling k as $k \to (2/\beta Q)^2 \cdot k$. This then leads to the large N result (1.1).

The same logic may be applied to the calculation of the Wilson loop. For the class of $\mathcal{N} = 2$ supersymmetric Chern-Simons theories coupled to matter on the round three-sphere

studied in [6], $x_{\max}$ is always proportional to $1/\sqrt{k}$. According to the above prescription, the result for $x_{\max}$ on a general background M_3 is given by rescaling the round sphere result by $|b_2| \cdot (\beta Q/2) = (|b_1|+|b_2|)/2$. Here the factor of $|b_2|$ comes from the relabelling $\sigma \to |b_2|\sigma$, while the factor of $\beta Q/2$ comes from the rescaling of the Chern-Simons coupling. Thus

$$x_{\max} = \frac{|b_1| + |b_2|}{2} \, x_{\max}^{\text{round}} , \qquad (2.17)$$

where $x_{\max}^{\text{round}}$ determines the supremum of the support of $\rho(x)$ for the field theory on the round three-sphere. For the theories studied in [6], the eigenvalue density is always a continuous piecewise linear function supported on $[x_{\min}, x_{\max}]$. Using this fact, the large N limit of the Wilson loop (2.13) in the fundamental representation may be easily computed with a saddle point approximation, and is

$$\log \langle W \rangle_{\text{QFT}} \;=\; \ell \cdot x_{\max} \, N^{1/2} + o(N^{1/2}) . \qquad (2.18)$$

Here recall that the length $\oint_\gamma \mathrm{d}s$ is in general $2\pi\ell$. The round three-sphere Wilson loop in particular is obtained by setting $b_1 = b_2 = 1$ and $\ell = 1$ and is, as shown in [6],

$$\log \langle W_{\text{round}} \rangle_{\text{QFT}} \;=\; x_{\max}^{\text{round}} \, N^{1/2} + o(N^{1/2}) . \qquad (2.19)$$

We thus obtain

$$\lim_{N \to \infty} \frac{\log \langle W \rangle_{\text{QFT}}}{\log \langle W_{\text{round}} \rangle_{\text{QFT}}} \;=\; \frac{|b_1| + |b_2|}{2}\ell . \qquad (2.20)$$

This is the field theory result for the VEV of a supersymmetric Wilson loop on a general supersymmetric manifold $M_3 \cong S^3$. In the next section we will look at the M2-brane dual to this Wilson loop, and show quite generally that the holographic dual computaton of the VEV agrees with (2.20).

3 Dual M2-branes

In this section we analyse the supersymmetric M2-brane probes that are relevant for computing the holographic dual of the Wilson loop VEV (2.20). The dual solution is constructed in four-dimensional gauged supergravity [5], and we begin by summarizing the geometry of these solutions. We then look at the eleven-dimensional uplift, and finally we compute the regularized action of the M2-brane.

3.1 Four-dimensional supergravity dual

In [5] it was shown that supersymmetric three-manifolds M_3 of precisely the form described in section 2.1 arise as the conformal boundaries of Euclidean self-dual solutions to four-dimensional gauged supergravity. For $M_3 \cong S^3$ the four-dimensional supergravity solution is defined on a four-ball $M_4 \cong B^4$, and is asymptotically locally Euclidean AdS with conformal boundary M_3. The Killing vector K defined by (2.1) extends as a Killing vector bilinear over M_4, and the four-metric is then Einstein, has anti-self-dual Weyl tensor, and

is conformal to a Kähler metric. Supersymmetry also requires one to turn on a specific graviphoton field A. After summarizing these solutions, and deriving some relevant formulae, we then use them to study the BPS M2-branes dual to the Wilson loops of the previous section.

The four-dimensional metric on the manifold M_4 takes the form

$$\mathrm{d}s^2_{M_4} = \frac{1}{y^2}\left[V^{-1}(\mathrm{d}\psi + \phi)^2 + V(\mathrm{d}y^2 + 4\mathrm{e}^w\mathrm{d}z\mathrm{d}\bar{z})\right], \tag{3.1}$$

where

$$\begin{aligned} V &= 1 - \frac{1}{2}y\partial_y w, \\ \mathrm{d}\phi &= \mathrm{i}\partial_z V \mathrm{d}y \wedge \mathrm{d}z - \mathrm{i}\partial_{\bar{z}} V \mathrm{d}y \wedge \mathrm{d}\bar{z} + 2\mathrm{i}\partial_y(V\mathrm{e}^w)\mathrm{d}z \wedge \mathrm{d}\bar{z}, \end{aligned} \tag{3.2}$$

and $w = w(y, z, \bar{z})$ satisfies the Toda equation

$$\partial_z\partial_{\bar{z}}w + \partial_y^2\mathrm{e}^w = 0. \tag{3.3}$$

The metric (3.1) is equipped with the Killing vector $K = \partial_\psi$, which extends the vector (2.1) on the conformal boundary, which is at $y = 0$. The coordinate y may be regarded as a radial coordinate, $y \in (0, y_0]$, with the conformal boundary at $y = 0$ and the origin of $M_4 \cong B^4$ being at $y = y_0 > 0$. The local complex coordinate z similarly extends that on the conformal boundary M_3. The metric (3.1) is then entirely determined by the solution $w = w(y, z, \bar{z})$ to the Toda equation (3.3).

Supersymmetry requires that the graviphoton gauge field A takes the *local* form

$$A = -\frac{1}{4}V^{-1}\partial_y w(\mathrm{d}\psi + \phi) + \frac{\mathrm{i}}{4}\partial_z w\mathrm{d}z - \frac{\mathrm{i}}{4}\partial_{\bar{z}}w\mathrm{d}\bar{z}, \tag{3.4}$$

where the field strength F is defined by $F = \mathrm{d}A$. Indeed, the metric (3.1) is conformal to the Kähler metric $\mathrm{d}s^2_{\text{Kahler}} = y^2\mathrm{d}s^2_{M_4}$, which is asymptotic to a cylinder $\mathbb{R} \times M_3$ near to the conformal boundary $y = 0$. The gauge field (3.4) is then $\frac{1}{2}$ of the Ricci one-form for this Kähler metric. These solutions were referred to as *self-dual* solutions in [5], since the Weyl tensor is anti-self-dual[3] and F is anti-self-dual, *i.e.* $*_4F = -F$. Moreover, the metric (3.1) is Einstein with negative cosmological constant. We shall use the following orthonormal frame for (3.1)

$$e^0 = \frac{1}{y}V^{1/2}\mathrm{d}y, \quad e^1 = \frac{1}{y}V^{-1/2}(\mathrm{d}\psi + \phi), \quad e^2 + \mathrm{i}e^3 = \frac{2}{y}(V\mathrm{e}^w)^{1/2}\mathrm{d}z. \tag{3.5}$$

As already mentioned, the solutions of interest are asymptotically locally Euclidean AdS (asymptotically hyperbolic), with the conformal boundary at $y = 0$. In particular imposing boundary conditions such that $w(y, z, \bar{z})$ is analytic around $y = 0$, i.e.

$$w(y, z, \bar{z}) = w_{(0)}(z, \bar{z}) + yw_{(1)}(z, \bar{z}) + \frac{1}{2}y^2 w_{(2)}(z, \bar{z}) + \mathcal{O}(y^3), \tag{3.6}$$

---

[3]With respect to the canonical orientation of the conformal Kähler metric.

then setting $r = 1/y$ the metric (3.1) expands to leading order as

$$\mathrm{d}s_{M_4}^2 \simeq \frac{\mathrm{d}r^2}{r^2} + r^2 \left[(\mathrm{d}\psi + \phi_{(0)})^2 + 4\mathrm{e}^{w_{(0)}} \mathrm{d}z\mathrm{d}\bar{z} \right] , \qquad (3.7)$$

when $r \to \infty$. Here we have also expanded the one-form ϕ tangent to M_3

$$\phi(y, z, \bar{z}) \,|_{TM_3} = \phi_{(0)}(z, \bar{z}) + y\phi_{(1)}(z, \bar{z}) + \mathcal{O}(y^2). \qquad (3.8)$$

In fact by expanding (3.2) one can show that $\phi_{(1)} = 0$. Equation (3.7) shows explicitly that the metric is asymptotically locally Euclidean AdS around $y = 0$, and moreover there is a natural choice of conformal class for the metric on the boundary M_3 given precisely by (2.2).

The four-dimensional geometry that we have just described, together with the gauge field A, form a supersymmetric solution to Euclidean gauged supergravity. There is correspondingly a Dirac spinor ϵ satisfying the Killing spinor equation of this theory. In the orthonormal frame (3.5) and using the gamma matrices

$$\Gamma_i = \begin{pmatrix} 0 & \tau_i \\ \tau_i & 0 \end{pmatrix} , \qquad \Gamma_0 = \begin{pmatrix} 0 & \mathrm{i}\mathbb{I}_2 \\ -\mathrm{i}\mathbb{I}_2 & 0 \end{pmatrix} , \qquad (3.9)$$

with τ_i the Pauli matrices, the Killing spinor ϵ is given by

$$\epsilon = \frac{1}{\sqrt{2y}} \left(1 + V^{-1/2}\Gamma_0 \right) \zeta . \qquad (3.10)$$

with

$$\zeta = \begin{pmatrix} \chi \\ 0 \end{pmatrix} \quad \text{where} \quad \chi = \begin{pmatrix} \chi_0 \\ \chi_0 \end{pmatrix} . \qquad (3.11)$$

In particular the bulk spinor (3.10) precisely matches onto the boundary two-component spinor χ given by (2.4). The phase in (2.4) may be shifted locally by making gauge transformations of A, since the Killing spinor is charged under the latter. However, for these solutions one may write A as a *global* one-form on M_4. This requires making an appropriate gauge transformation on the local expression (3.4), as we shall see in the next subsection.

So far we have not imposed the $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry we imposed on the boundary M_3 at the end of section 2.1. Doing so will simplify the subsequent discussion. Thus as in [5] we assume that the four-manifold M_4 is $M_4 \cong B^4 \cong \mathbb{R}^2 \oplus \mathbb{R}^2$ and that the torus $\mathrm{U}(1) \times \mathrm{U}(1)$ acts in the standard way on $\mathbb{R}^2 \oplus \mathbb{R}^2$. The Killing vector $K = \partial_\psi$ is then parametrized as

$$K = b_1 \partial_{\varphi_1} + b_2 \partial_{\varphi_2} , \qquad (3.12)$$

again precisely as in (2.5) on the conformal boundary. It will be important to fix carefully the orientations here. Since the metrics are defined on a ball, diffeomorphic to $\mathbb{R}^4 \cong \mathbb{R}^2 \oplus \mathbb{R}^2$

with $U(1) \times U(1)$ acting in the obvious way, we choose ∂_{φ_i} so that the orientations on $\mathbb{R}^2$ induce the given orientation on $\mathbb{R}^4$ (with respect to which the metric has anti-self-dual Weyl tensor). This fixes the relative signs of b_1 and b_2. Given that K has no fixed points near the conformal boundary, we must also have b_1 and b_2 non-zero. Thus $b_1/b_2 \in \mathbb{R} \setminus \{0\}$, and its sign will be important in what follows.

In order to construct such backgrounds one can start with a toric ($U(1) \times U(1)$-invariant) self-dual Einstein metric on a ball, which is asymptotically locally Euclidean AdS. There are many examples of such metrics — we discuss the two simplest in section 4, but as explained in [5] the moduli space is in fact infinite-dimensional (each metric inducing a different conformal structure on the boundary $M_3 \cong S^3$). One can then choose a Killing vector (3.12), and then writing $K = \partial_\psi$ the metric will necessarily take the form (3.1). Thus in particular the choice of K determines the conformal Kähler metric, which in turn determines the instanton gauge field A and Killing spinor ϵ. However, not all choices of K in (3.12) give non-singular gauge fields. While the metric (3.1) is smooth by assumption, the instanton $F = \mathrm{d}A$ and Killing spinor ϵ are singular where the conformal Kähler metric is singular. Regularity is in fact equivalent to having either $b_1/b_2 > 0$ or $b_1/b_2 = -1$. Moreover, the origin $y = y_0$ of $M_4 \cong \mathbb{R}^2 \oplus \mathbb{R}^2$ is then at

$$y_0 = \frac{1}{|b_1 + b_2|}, \tag{3.13}$$

which notice is $y_0 = \infty$ when $b_1/b_2 = -1$.

3.2 Global gauge for A

As remarked after equation (3.11), we will want to choose a gauge for A in which it is a global, smooth one-form on M_4. The reason for this is that we will evaluate the Wess-Zumino term in the M2-brane action in section 3.4 by using Stokes' theorem for $F = \mathrm{d}A$, which requires us to write A as a global one-form. This was also discussed to some extent in [5], but for the computation of the Wilson loop we need a little more information.

The key point is to recall that A is proportional to the Ricci one-form for the conformal Kähler metric $\mathrm{d}s_{\mathrm{Kahler}}^2 = y^2 \mathrm{d}s_{M_4}^2$. When $b_1/b_2 > 0$ the associated complex structure identifies $M_4 \cong \mathbb{R}^2 \oplus \mathbb{R}^2 \cong \mathbb{C}^2$. The orientation in which the Weyl tensor is anti-self-dual is the same as the canonical orientation on $\mathbb{C}^2$. One can then introduce standard complex coordinates $z_i = \rho_i \mathrm{e}^{\mathrm{i}\psi_i}$, $i = 1, 2$, on $\mathbb{C}^2$. The spinor ζ in (3.11), which is used to construct the Killing spinor (3.10), is the canonical spinor that exists on any Kähler manifold [5]. As such we have

$$\mathcal{L}_{\partial_{\psi_i}} \epsilon = \frac{\mathrm{i}}{2} \epsilon, \qquad i = 1, 2. \tag{3.14}$$

Denoting the complex structure tensor by J we also have that $J(V^{-1}\partial_y) = \partial_\psi = K$. Since y is *decreasing* as we move away from the origin of $\mathbb{C}^2$, where recall that the origin is at $y = y_0 > 0$, this means that for $b_1 > 0$ and $b_2 > 0$ we must then identify $\varphi_i = -\psi_i$, where φ_i are the coordinates on $U(1) \times U(1)$ in (3.12). This is because for r any radial coordinate on $\mathbb{C}^2$ we have $J(r\partial_r) = a_1 \partial_{\psi_1} + a_2 \partial_{\psi_2}$ where necessarily $a_1, a_2 > 0$ (that is, the Reeb cone

is the positive quadrant in $\mathbb{R}^2$ — see, for example, [17]). On the other hand for $b_1 < 0$ and $b_2 < 0$ we instead have $\varphi_i = +\psi_i$, $i = 1, 2$.

The other non-singular case is $b_1/b_2 = -1$. This is qualitatively different from the case $b_1/b_2 > 0$ in the last paragraph, as here $y_0 = \infty$ (3.13). Moreover, the origin $y = y_0$ of $M_4 \cong \mathbb{R}^2 \oplus \mathbb{R}^2$ is now identified with the point at infinity in $\mathbb{C}^2$, rather than the origin. One can see this from the conformal Kähler metric $\mathrm{d}s_{\text{Kahler}}^2 = y^2 \mathrm{d}s_{M_4}^2$, which is asymptotically Euclidean around $y = y_0$. Thus now $V^{-1}\partial_y$ has the correct orientation for a radial vector on $\mathbb{C}^2$, and we deduce that for $b_1 < 0$ and $b_2 > 0$ we have $\varphi_1 = -\psi_1$, $\varphi_2 = +\psi_2$, while for $b_1 > 0$ and $b_2 < 0$ we instead have $\varphi_1 = +\psi_1$, $\varphi_2 = -\psi_2$.

Putting all of the above together, we may compute the charge of the Killing spinor ϵ under the supersymmetric Killing vector $K = \partial_\psi$:

$$\mathcal{L}_K \epsilon = \mathrm{i}\gamma\epsilon\,, \tag{3.15}$$

where[4]

$$\gamma \equiv -\text{sign}\left(\frac{b_1}{b_2}\right) \cdot \frac{|b_1| + |b_2|}{2}\,. \tag{3.16}$$

Since in all cases the Kähler structure is defined on $\mathbb{C}^2$, the canonical bundle is trivial and one may indeed take A to be a global one-form on M_4. We denote the restriction of this global A to the conformal boundary $M_3 = \partial M_4$ at $y = 0$ by $A_{(0)}$. Then the formula (3.15) for the charge of ϵ under $K = \partial_\psi$ means that

$$A_{(0)} = \gamma\mathrm{d}\psi - \frac{1}{4}w_{(1)}(\mathrm{d}\psi + \phi_{(0)}) + \frac{\mathrm{i}}{4}\partial_z w_{(0)}\mathrm{d}z - \frac{\mathrm{i}}{4}\partial_{\bar{z}} w_{(0)}\mathrm{d}\bar{z}\,. \tag{3.17}$$

This is the restriction of (3.4) to $y = 0$, together with a gauge transformation $A \to A + \gamma\mathrm{d}\psi$ which accounts for the charge (3.15). One can show independently that (3.17) then defines a global one-form on $M_3 \cong S^3$, which leads to another formula for γ that was derived in section 3.3 of [5], although we will not need this in the present paper.

3.3 Uplifting to $D = 11$ supergravity

In order to study the M2-branes dual to Wilson loops, we need to uplift the four-dimensional geometry to an eleven-dimensional supergravity solution. More precisely, we are interested in a class of $\mathcal{N} = 2$ supersymmetric $M_4 \times Y_7$ backgrounds of M-theory in *Euclidean* signature. In Euclidean signature there are certain factors of i that appear relative to the uplifting formula in Lorentzian signature of [18]. Again, this will be important for correctly evaluating the M2-brane action.

The action of $D = 11$ supergravity in Euclidean signature is

$$S_{11} = \frac{1}{(2\pi)^8\ell_p^9}\left(\int \mathrm{d}^{11}x\sqrt{g_{11}}\left[-\mathcal{R} + \frac{1}{2}\mathrm{d}C \wedge *_{11}\mathrm{d}C\right] + \frac{\mathrm{i}}{6}\int C \wedge \mathrm{d}C \wedge \mathrm{d}C\right)\,. \tag{3.18}$$

---

[4]We also denoted the Wilson line curve by $\gamma : S^1 \to M_3$ in section 2, but have chosen to use the same symbol for the charge of ϵ under K as this was also used in [1, 5].

Here we have denoted by g_{11} the eleven-dimensional metric, with associated Ricci scalar $\mathcal{R}$, C is the three-form potential and ℓ_p denotes the eleven-dimensional Planck length. The equations of motion for the metric and C-field follow immediately:

$$\mathcal{R}_{AB} - \frac{1}{12}\left(G_{AC_1C_2C_3}G_B{}^{C_1C_2C_3} - \frac{1}{12}g_{AB}G^2\right) = 0\,,$$
$$\mathrm{d}\ast_{11}G + \frac{\mathrm{i}}{2}G\wedge G = 0\,, \tag{3.19}$$

where we have defined $G \equiv \mathrm{d}C$ and $A, B, C = 1, \ldots, 11$. It is also useful to define $G_7 = \mathrm{i}(\ast_{11}G + \frac{\mathrm{i}}{2}C\wedge G)$ so that the equation of motion for G is simply $\mathrm{d}G_7 = 0$.

An ansatz that leads to a consistent truncation to four-dimensional gauged supergravity in Lorentz signature was given in [18]. Here the internal space Y_7 is taken to be any Sasaki-Einstein seven-manifold Y_7 with contact one-form η, transverse Kähler-Einstein six-metric $\mathrm{d}s_T^2$ with Kähler form $\omega_T = \mathrm{d}\eta/2$, and with the seven-dimensional metric normalized so that $\mathrm{Ric} = 6g_{Y_7}$. The consistent truncation ansatz in Euclidean signature then becomes

$$\mathrm{d}s_{11}^2 = R^2\left[\frac{1}{4}\mathrm{d}s_{M_4}^2 + \left(\eta + \frac{1}{2}A\right)^2 + \mathrm{d}s_T^2\right]\,,$$
$$G = -\mathrm{i}R^3\left(\frac{3}{8}\mathrm{vol}_4 - \frac{1}{4}\ast_4 F\wedge \mathrm{d}\eta\right)\,. \tag{3.20}$$

As before, $\mathrm{d}s_{M_4}^2$ is the four-dimensional gauged supergravity metric on M_4 with gauge field A, field-strength $F = \mathrm{d}A$ and volume form vol_4. The radius R is

$$R^6 = \frac{(2\pi\ell_p)^6 N}{6\mathrm{Vol}(Y_7)}\,, \tag{3.21}$$

where N is the number of units of flux

$$N = \frac{1}{(2\pi\ell_p)^6}\int_{Y_7} G_7\,. \tag{3.22}$$

Substituting the ansatz (3.20) into the equations of motion (3.19), we find the latter are equivalent to the metric $g_{\mu\nu}$ corresponding to $\mathrm{d}s_{M_4}^2$ and F satisfying

$$R_{\mu\nu} + 3g_{\mu\nu} = 2\left(F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4}F^2 g_{\mu\nu}\right)\,,$$
$$\mathrm{d}\ast_4 F = 0\,. \tag{3.23}$$

The ansatz (3.20) then solves the eleven-dimensional Euclidean equations of motion if and only if the four-dimensional metric $g_{\mu\nu}$ and gauge field A are a solution of four-dimensional Euclidean gauged supergravity.

3.4 BPS M2-branes

We are interested in calculating the action of M2-branes that are dual to Wilson loops of the dual gauge theory on M_3. These M2-branes wrap $\Sigma_2 \times S_M^1$, where the surface $\Sigma_2 \subset M_4$

has boundary given by the Wilson line $\partial \Sigma_2 = S^1 \subset M_3 = \partial M_4$, and $S^1_M \subset Y_7$ is a copy of the M-theory circle. In particular we will show that submanifolds $\Sigma_2 \subset M_4$ parametrized by the radial direction y in M_4 and an orbit of the Killing vector K are complex with respect to the complex structure J of the conformal Kähler metric to $\mathrm{d}s^2_{M_4}$. The wrapped M2-brane is then supersymmetric.[5] Over the poles $S^1 \subset M_3 \cong S^3$ the topology of Σ_2 is a disc, where $y \in (0, y_0]$ serves as a radial coordinate with the origin of the disc at $y = y_0 > 0$.

The action of the M2-brane is

$$S_{\mathrm{M2}} = \frac{1}{(2\pi)^2 \ell_p^3} \left[\mathrm{Vol}(\Sigma_2 \times S^1_M) + \mathrm{i} \int_{\Sigma_2 \times S^1_M} C \right] . \tag{3.24}$$

A supersymmetric M2-brane satisfies an appropriate κ-symmetry condition, which may be written as [19]

$$\mathbb{P}\epsilon_{11} = 0, \quad \text{where} \quad \mathbb{P} \equiv \frac{1}{2} \left(1 - \frac{\mathrm{i}}{3!} \varepsilon^{ijk} \partial_i X^M \partial_j X^N \partial_k X^P \hat{\Gamma}_{MNP} \right), \tag{3.25}$$

with i, j, k indices on the worldvolume. Here ϵ_{11} is the eleven-dimensional Killing spinor for the background (3.20), which is constructed as a tensor product of the four-dimensional spinor ϵ and the Killing spinor on the internal space Y_7. The $\hat{\Gamma}_M$ are eleven-dimensional gamma matrices, with X^M describing the M2-brane embedding. One can analyse (3.25) precisely as the authors did in [6]. The upshot is that $S^1_M \subset Y_7$ must be a calibrated circle in Y_7 [6], while taking $\Sigma_2 \subset M_4$ to be a surface at constant z, parametrized by y and ψ, one finds (3.25) is equivalent to the projection condition

$$(1 - \mathrm{i}\Gamma_5 \Gamma_{01})\epsilon = 0 . \tag{3.26}$$

Here we have used the orthonormal frame (3.5), and $\Gamma_5 \equiv \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3$ with Γ_μ defined by (3.9) (in the orthonormal frame). Using the explicit form for ϵ in (3.10) it is trivial to see that (3.26) indeed holds. Moreover, Σ_2 is calibrated with respect to the Kähler form for the conformal Kähler metric, making it a complex curve.

Let us now calculate the action (3.24) for our M2-brane. Using the self-dual four-dimensional supergravity solution of section 3.1 and the uplift (3.20) the C-field is computed to be

$$C = -\mathrm{i}R^3 \left(-\frac{1}{8}\Gamma + \frac{1}{4}F \wedge \eta \right), \tag{3.27}$$

where

$$\Gamma \equiv \frac{1}{2y^2}(\mathrm{d}\psi + \phi) \wedge \mathrm{d}\phi + \frac{1}{y^3}(\mathrm{d}\psi + \phi) \wedge 2\mathrm{i}V\mathrm{e}^w \mathrm{d}z \wedge \mathrm{d}\bar{z}, \tag{3.28}$$

and $\mathrm{d}\Gamma = -3\mathrm{vol}_4$. The area of the surface Σ_2 in M_4 is divergent, but can be regularized by subtracting the length of its boundary, i.e. the length of the S^1 in M_3^δ at $y = \delta \to 0$.

---

[5]More precisely the copy of $S^1_M \subset Y_7$ must also be calibrated by the contact one-form η on Y_7. Since this internal space geometry is identical to the AdS$_4 \times Y_7$ backgrounds studied in [6], in this paper we focus on the geometry of M_4.

Notice this is then a local boundary counterterm. If we denote by M_4^δ the manifold M_4 with boundary $M_3^\delta = \{y = \delta\}$ (with $0 < \delta < y_0$), and similarly for Σ_2^δ *etc*, the action of the M2-brane is

$$S_{\mathrm{M2}} \;=\; \frac{1}{(2\pi)^2 \ell_p^3} \int_{S_M^1} \frac{R^3}{4}\, \mathrm{vol}_{S_M^1} \cdot \lim_{\delta \to 0} \left[\int_{\Sigma_2^\delta} \mathrm{vol}_{\Sigma_2} - \int_{\partial \Sigma_2^\delta} e_\mu^1 \mathrm{d}x^\mu + \int_{\Sigma_2^\delta} F \right] \;. \tag{3.29}$$

Here we have written $\mathrm{vol}_{S_M^1}$ for the volume form on S_M^1 induced from the metric g_7, and similarly for vol_{Σ_2} and the metric g_{M_4}. Applying Stokes' theorem for the gauge field term $F = \mathrm{d}A$ we then compute[6]

$$\begin{aligned}
S_{\mathrm{M2}} &= \frac{1}{(2\pi)^2 \ell_p^3} \int_{S_M^1} \mathrm{vol}_{S_M^1} \cdot \frac{\pi \ell R^3}{2} \lim_{\delta \to 0} \left[\left(\int_\delta^{y_0} \frac{\mathrm{d}y}{y^2} - \frac{1}{\delta \sqrt{V(\delta, z, \bar{z})}} \right) - \frac{1}{2\pi\ell} \int_{\partial \Sigma_2^\delta} A \right] \\
&= \frac{1}{(2\pi)^2 \ell_p^3} \int_{S_M^1} \mathrm{vol}_{S_M^1} \cdot \frac{\pi \ell R^3}{2} \left[-\left(\frac{1}{y_0} + \frac{1}{4} w_{(1)} \right) - \frac{1}{2\pi\ell} \int_{\partial \Sigma_2} A \right] \;.
\end{aligned} \tag{3.30}$$

Recall here that $2\pi\ell$ denotes the length of the orbit of K, as in (2.14). The contribution of the M-theory circle S_M^1 is exactly the same as for the $\mathrm{AdS}_4 \times Y_7$ backgrounds studied in [6], and is expressed in terms of the contact form η on Y_7 and the Dirac quantized number N of (3.22). The gauge field integral is easily computed, thanks to (3.17)

$$\int_{\partial \Sigma_2} A \;=\; \int_{\partial \Sigma_2} A_{(0)} \;=\; 2\pi\ell \left(-\frac{1}{4} w_{(1)} + \gamma \right) \;. \tag{3.31}$$

Putting everything together, and using the formula (3.13) for y_0, we have

$$\log \langle W \rangle_{\mathrm{gravity}} \;=\; -S_{\mathrm{M2}} \;=\; \ell \left(|b_1 + b_2| + \gamma \right) \cdot \frac{(2\pi)^2 \int_{S_M^1} \eta}{\sqrt{2 \int_{Y_7} \eta \wedge (\mathrm{d}\eta)^3}} \, N^{1/2} \;. \tag{3.32}$$

Using the round sphere result of [6]

$$\log \langle W_{\mathrm{round}} \rangle_{\mathrm{gravity}} \;=\; \frac{(2\pi)^2 \int_{S_M^1} \eta}{\sqrt{2 \int_{Y_7} \eta \wedge (\mathrm{d}\eta)^3}} \, N^{1/2} \;, \tag{3.33}$$

and the formula (3.16) for γ, in both cases $b_1/b_2 > 0$ and $b_1/b_2 = -1$ we obtain

$$\log \langle W \rangle_{\mathrm{gravity}} \;=\; \frac{|b_1| + |b_2|}{2} \ell \cdot \log \langle W_{\mathrm{round}} \rangle_{\mathrm{gravity}} \;. \tag{3.34}$$

In [6] it was shown in numerous families of examples that the large N limit of the Wilson loop on the round three-sphere and the M2-brane in AdS_4 have the same VEV, i.e. $\log \langle W_{\mathrm{round}} \rangle_{\mathrm{QFT}} = \log \langle W_{\mathrm{round}} \rangle_{\mathrm{gravity}}$ holds to leading order at large N. Assuming this

---

[6]The sign in front of the gauge field term arises because y is decreasing towards the boundary of M_4, and hence $\mathrm{d}y$ points inwards from M_3. Thus the natural orientation of the boundary we take is opposite to that in Stokes' theorem.

to be the case, equations (2.20) and (3.34) mean that we have shown very generally that in the large N limit

$$\log \langle W \rangle_{\text{QFT}} = \log \langle W \rangle_{\text{gravity}} \qquad (3.35)$$

where now the field theory is defined on a general class of background three-manifolds M_3, with fillings M_4 in four-dimensional gauged supergravity.

We conclude this section with two further comments. Firstly, it is interesting to note that when the orbit of K is one of the poles of S^3, where correspondingly $\ell = 1/|b_1|$ or $\ell = 1/|b_2|$ respectively, the Wilson loops are then functions only of $|b_1/b_2|$, just as for the free energy (1.1). Secondly, in the case that $b_1/b_2 = m/n$ is rational and the Wilson line wraps a generic orbit $\gamma \subset T^2 \subset S^3$ (i.e. not at either pole), then the curve $\Sigma_2 \subset M_4 \cong \mathbb{C}^2$ wrapped by the dual M2-brane is the Brieskorn-Pham curve $\{z_1^n = z_2^m\} \subset \mathbb{C}^2$. This follows since supersymmetry pairs the orbit of K with its complexification in $M_4 \cong \mathbb{C}^2$, meaning that Σ_2 is swept out as a generic $\mathbb{C}^*$ orbit of $(z_1, z_2) \to (\lambda^m z_1, \lambda^n z_2)$, with $\lambda \in \mathbb{C}^*$. The curve $\{z_1^n = z_2^m\}$ adds the origin in $\mathbb{C}^2$ at $y = y_0$, which is a singular point when $m, n > 1$, although notice this does not affect our computation of the M2-brane action, which is finite. It is well-known that (m, n) torus knots in S^3 may be realized as links of the above Brieskorn-Pham curves, and it is interesting to see that this construction is realized as the holographic dual of the knot.

4 Examples

Our derivation of the formula (3.34) was necessarily somewhat indirect, as we have shown that it holds for a very general (infinite-dimensional) class of solutions. In particular we didn't need to use the explicit form of the solution to the Toda equation (3.3). In this section we illustrate our general results by discussing two explicit families of solutions, where all quantities in the previous section may be written down in closed form. We will focus on the four-dimensional part of the M2-brane calculation, in particular showing how the factor $\ell(|b_1| + |b_2|)/2$ in (3.34) arises explicitly in these cases. In order to do so we will use the results of the previous section that allow us to write

$$\log \langle W \rangle_{\text{gravity}} = \mathcal{S}_{b_1, b_2} \cdot \log \langle W_{\text{round}} \rangle_{\text{gravity}}, \qquad (4.1)$$

where

$$\mathcal{S}_{b_1, b_2} \equiv \frac{1}{2\pi} \left(-\int_{\Sigma_2} \text{vol}_{\Sigma_2} + \int_{\partial \Sigma_2} \text{vol}_{\partial \Sigma_2} + \int_{\partial \Sigma_2} A \right) . \qquad (4.2)$$

Here we cut off Σ_2 at $y = \delta$, and (4.2) is then understood to be the limit $\delta \to 0$. We compute (4.2) directly in the examples, confirming that (3.34) indeed holds in these cases.

AdS$_4$. We begin with the metric on Euclidean AdS$_4$, which can be written

$$ds^2_{\text{EAdS}_4} = \frac{dq^2}{q^2 + 1} + q^2 \left(d\vartheta^2 + \cos^2 \vartheta d\varphi_1^2 + \sin^2 \vartheta d\varphi_2^2 \right) . \qquad (4.3)$$

Here q is a radial variable with $q \in [0, \infty)$, so that the origin of $M_4 \cong \mathbb{R}^4$ is at $q = 0$ while the conformal boundary is at $q = \infty$. The coordinate $\vartheta \in [0, \frac{\pi}{2}]$, with the endpoints being the two axes of $\mathbb{R}^2 \oplus \mathbb{R}^2 \cong \mathbb{R}^4$.

Of course the metric (4.3) is conformally flat, which leads to a trivial graviphoton $A = 0$. However, we may instead pick a general supersymmetric Killing vector $K = b_1 \partial_{\varphi_1} + b_2 \partial_{\varphi_2}$. This leads to a family of conformal Kähler structures on $\mathbb{C}^2$, where the explicit formulae for the conformal factor y and the metric function $w(y, z, \bar{z})$ may be found in [5]. In particular one calculates the *local* gauge field given by (3.4) to be

$$A_{\text{local}} = \frac{\left(b_1 + b_2\sqrt{q^2+1}\right)\mathrm{d}\varphi_1 + \left(b_2 + b_1\sqrt{q^2+1}\right)\mathrm{d}\varphi_2}{2\sqrt{(b_2 + b_1\sqrt{q^2+1})^2 \cos^2\vartheta + (b_1 + b_2\sqrt{q^2+1})^2 \sin^2\vartheta}}\,, \qquad (4.4)$$

which is a non-trivial instanton on Euclidean AdS$_4$. In fact this solution was first found in [16] using very different methods. One can check that the field strength $F = \mathrm{d}A$ for (4.4) indeed defines a smooth, non-singular instanton on EAdS$_4$ precisely when $b_1/b_2 > 0$ or $b_1/b_2 = -1$, with $b_1/b_2 = \pm 1$ both giving trivial instantons. When $b_1/b_2 < 0$ and $b_1/b_2 \neq -1$ the instanton is singular along one axis or the other.

Writing A as a global one-form and restricting to the conformal boundary at $q = \infty$ we obtain

$$A_{(0)} = \frac{b_2\mathrm{d}\varphi_1 + b_1\mathrm{d}\varphi_2}{2\sqrt{b_1^2\cos^2\vartheta + b_2^2\sin^2\vartheta}} - \frac{1}{2}(\text{sign}(b_2)\mathrm{d}\varphi_1 + \text{sign}(b_1)\mathrm{d}\varphi_2)\,. \qquad (4.5)$$

In particular notice this is well-defined at both poles $\vartheta = 0$ and $\vartheta = \pi/2$. The submanifold Σ_2 is parametrized by the radial direction q in AdS$_4$ and the S^1 wrapping φ_1 or φ_2 when $\vartheta = 0$ or $\vartheta = \pi/2$, respectively.

We now turn to computing (4.2). Notice that the dependence on b_1 and b_2 arises only via the gauge field A, and not from the metric. Indeed, we compute

$$\left[-\int_{\Sigma_2} \text{vol}_{\Sigma_2} + \int_{\partial\Sigma_2} \text{vol}_{\partial\Sigma_2} \right] = 2\pi\,, \qquad (4.6)$$

and

$$\int_{\partial\Sigma_2} A_{(0)} = \begin{cases} \pi(\frac{b_2}{|b_1|} - \text{sign}(b_2)) \cdot \text{sign}(b_1) & \text{if } \vartheta = 0\,, \\ \pi(\frac{b_1}{|b_2|} - \text{sign}(b_1)) \cdot \text{sign}(b_2) & \text{if } \vartheta = \pi/2\,. \end{cases} \qquad (4.7)$$

The overall factors of $\text{sign}(b_1)$, $\text{sign}(b_2)$ for $\vartheta = 0, \pi/2$ arise because the orientation of $\partial\Sigma_2$ is determined by K. Equation (4.2) immediately gives for all regular cases that

$$\mathcal{S}_{b_1, b_2} = \begin{cases} \dfrac{|b_1| + |b_2|}{2|b_1|} & \text{if } \vartheta = 0\,, \\[2mm] \dfrac{|b_1| + |b_2|}{2|b_2|} & \text{if } \vartheta = \pi/2\,. \end{cases} \qquad (4.8)$$

In particular using the variable ℓ introduced previously, which is given by $\ell = 1/|b_1|$ and $1/|b_2|$ for the $\vartheta = 0$ pole and $\vartheta = \pi/2$ pole, respectively, we obtain for both poles and all regular cases that

$$\mathcal{S}_{b_1, b_2} = \frac{|b_1| + |b_2|}{2} \ell \,, \tag{4.9}$$

as expected.

Taub-NUT-AdS$_4$. The Taub-NUT-AdS$_4$ metrics are a one-parameter family of self-dual Einstein metrics on the four-ball, and have been studied in detail in [20, 21]. The metric may be written

$$\mathrm{d}s_4^2 = \frac{r^2 - s^2}{\Omega(r)} \mathrm{d}r^2 + (r^2 - s^2)(\sigma_1^2 + \sigma_2^2) + \frac{4s^2 \Omega(r)}{r^2 - s^2} \sigma_3^2 \,, \tag{4.10}$$

where

$$\Omega(r) \equiv (r \mp s)^2 [1 + (r \mp s)(r \pm 3s)] \,, \tag{4.11}$$

and $\sigma_1, \sigma_2, \sigma_3$ are left-invariant one-forms on $\mathrm{SU}(2) \simeq \mathrm{S}^3$. The latter may be written in terms of Euler angle variables as

$$\sigma_1 + \mathrm{i}\sigma_2 = \mathrm{e}^{-\mathrm{i}\varsigma}(\mathrm{d}\theta + \mathrm{i}\sin\theta\mathrm{d}\varphi)\,, \qquad \sigma_3 = \mathrm{d}\varsigma + \cos\theta\mathrm{d}\varphi \,. \tag{4.12}$$

Here ς has period 4π, while $\theta \in [0, \pi]$ with φ having period 2π. The radial coordinate r lies in the range $r \in [s, \infty)$, with the origin of the ball $\cong \mathbb{R}^4$ being at $r = s$. The parameter $s > 0$ is referred to as the *squashing parameter*, with $s = \frac{1}{2}$ being the Euclidean AdS$_4$ metric studied in the previous section. The metric is asymptotically locally Euclidean AdS as $r \to \infty$, with

$$\mathrm{d}s_4^2 \simeq \frac{\mathrm{d}r^2}{r^2} + r^2(\sigma_1^2 + \sigma_2^2 + 4s^2\sigma_3^2) \,, \tag{4.13}$$

so that the conformal boundary at $r = \infty$ is a biaxially squashed S^3.

While the Taub-NUT-AdS metric (4.10) has $\mathrm{SU}(2) \times \mathrm{U}(1)$ isometry, a generic choice of the Killing vector $K = b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2} = (b_1 + b_2)\partial_\varphi + (b_1 - b_2)\partial_\varsigma$ breaks the symmetry of the full solution to $\mathrm{U}(1) \times \mathrm{U}(1)$. In particular, this symmetry is broken by the corresponding instanton A. On the other hand, in [20, 21] the $\mathrm{SU}(2) \times \mathrm{U}(1)$ symmetry of the metric was *also* imposed on the gauge field, which results in two subfamilies of the above solutions, which are 1/4 BPS and 1/2 BPS, respectively. In each case this effectively fixes the Killing vector K (or rather the parameter b_1/b_2) as a function of the squashing parameter s.

1/4 BPS solution: The supersymmetric Killing vector for this solution is $K = -\frac{1}{2s}\partial_\varsigma$ and we have

$$b_1 = -b_2 = -\frac{1}{4s} \,. \tag{4.14}$$

Here $\varsigma = \varphi_1 - \varphi_2$, $\varphi = \varphi_1 + \varphi_2$ is the change of angular coordinates. The boundary gauge field $A_{(0)}$ is [21]

$$A_{(0)} = \frac{1}{2}(4s^2 - 1)\sigma_3,\qquad(4.15)$$

which is a global one-form on $M_3 \cong S^3$. We may now take the surface Σ_2 wrapped by the M2-brane to be *any* S^1 orbit of the Hopf Killing vector ∂_ς (at any point on the base $S^2 = S^3/\mathrm{U}(1)_\varsigma$), together with the radial direction r. This is supersymmetric, and the regularized volume of Σ_2 is

$$\left[-\int_{\Sigma_2} \mathrm{vol}_{\Sigma_2} + \int_{\partial\Sigma_2} \mathrm{vol}_{\partial\Sigma_2} \right] = 8\pi s^2,\qquad(4.16)$$

while the gauge field integral is

$$\int_{\partial\Sigma_2} A_{(0)} = -2\pi(4s^2 - 1).\qquad(4.17)$$

This leads to

$$\mathcal{S}_{b_1,b_2} = 1 = \frac{|b_1| + |b_2|}{2}\ell,\qquad(4.18)$$

where $\ell = 4s$ is the length of K divided by 2π.

1/2 BPS solution: The Taub-NUT-AdS metric (4.10) also admits a 1/2 BPS solution [20, 21]. There are thus two linearly independent Killing spinors, and an appropriate linear combination preserves $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry, leading to the Killing vector

$$K = \left(2s + \sqrt{4s^2 - 1} \right)\partial_\varphi + \left(\tfrac{1}{2s} - 2s - \sqrt{4s^2 - 1} \right)\partial_\varsigma,\qquad(4.19)$$

so that

$$b_1 = \frac{1}{4s},\qquad b_2 = -\frac{1}{4s} + 2s + \sqrt{4s^2 - 1}.\qquad(4.20)$$

The boundary gauge field is

$$A_{(0)} = s\sqrt{4s^2 - 1}\,\sigma_3.\qquad(4.21)$$

This time we take the Wilson loop to wrap one of the two poles $\theta = 0$, $\theta = \pi$. These are both copies of S^1, and Σ_2 is again formed by adding the radial direction r. The boundary gauge field is

$$A_{(0)}\,|_{\mathrm{pole}} = \begin{cases} 2s\sqrt{4s^2 - 1}\,\mathrm{d}\varphi_1 & \text{if } \theta = 0, \\ -2s\sqrt{4s^2 - 1}\,\mathrm{d}\varphi_2 & \text{if } \theta = \pi. \end{cases}\qquad(4.22)$$

The regularized volume is again $8\pi s^2$, which then gives

$$\mathcal{S}_{b_1,b_2} = \begin{cases} 2s(2s + \sqrt{4s^2 - 1}) & \text{if } \theta = 0, \\ 2s(2s - \sqrt{4s^2 - 1}) & \text{if } \theta = \pi. \end{cases}\qquad(4.23)$$

In both cases we indeed have

$$\mathcal{S}_{b_1,b_2} = \frac{|b_1| + |b_2|}{2}\ell,\qquad(4.24)$$

where $\ell = 1/|b_1|$, $\ell = 1/|b_2|$ for the two poles.

5 Discussion

In this paper we have derived the formula (1.2), (1.3) for the expectation values of large N BPS Wilson loops, in both gauge theory and in supergravity. In particular the gauge theories are defined on a general class of supersymmetric backgrounds $M_3 \cong S^3$, which in the supergravity dual arise as the conformal boundaries of self-dual solutions to gauged supergravity. A key feature of the gravity calculation is that we are able to evaluate the regularized M2-brane action, that is identified with the Wilson loop VEV, without using the explicit form of the metric and graviphoton field. This seems to be a general feature of such computations of BPS quantities in AdS/CFT, and allows us to verify the correspondence for these observables in a very broad class of solutions.

   The results described in this paper lead to a number of questions, and possible future directions to pursue. First, in supergravity we have restricted to self-dual solutions, while more generally there are also non-self-dual solutions to gauged supergravity. A local study of these solutions appears in [22], while global asymptotically locally Euclidean AdS solutions were constructed in [21]. Presumably the methods we have used extend to this general class of solutions. In particular the Wilson loop was computed for a charged topological black hole background in [23], and successfully compared to a field theory calculation. The non-self-dual solutions in [21] all have the feature that the bulk M_4 has non-trivial topology, which in turn leads to issues in interpreting them holographically (and in particular uplifting to eleven dimensions restricts the choice of Y_7, implying the solutions are only relevant to specific gauge theories on M_3). It would be interesting to try to calculate Wilson loops in such examples, and compare to a dual field theory computation. Indeed, in [21] it was argued that in appropriate circumstances $\langle W \rangle = 0$ in supergravity, via a similar mechanism to that in [24]. Typically here the boundary M_3 in such examples has a non-trivial fundamental group, as in the large N gauge theory computation in [25], and there is indeed evidence that if the R-symmetry gauge field on M_3 has non-trivial topology then the large N Wilson loop VEV vanishes also in the gauge theory.[7] Finally, it is now clear that similar results should also hold in higher dimensions. A very similar formula to (1.2), (1.3) was found to hold for certain supersymmetric squashed five-sphere conformal boundaries and their gravity duals in [27, 28], and was conjectured to hold for general backgrounds in those references. It would also be interesting to compute Wilson loops in the general class of $S^1 \times S^3$ Hopf surface geometries in [29].

Acknowledgments

---

[7]The matrix model behaviour is then much more complicated, and seems difficult to analyse analytically, but very roughly speaking the Wilson loop VEV averages to zero due to the sum $\sum_{k=0}^{p-1} \omega_p^k = 0$, where ω_p is a primitive pth root of unity. This arises from the fact that the dominant contribution to the Wilson loop at large N comes from a non-trivial flat connection, with ω_p^k related to its holonomies [26].

References

[1] L.F. Alday, D. Martelli, P. Richmond and J. Sparks, *Localization on Three-Manifolds*, *JHEP* **10** (2013) 095 [arXiv:1307.6848] [inSPIRE].

[2] C. Closset, T.T. Dumitrescu, G. Festuccia and Z. Komargodski, *The Geometry of Supersymmetric Partition Functions*, *JHEP* **01** (2014) 124 [arXiv:1309.5876] [inSPIRE].

[3] C.P. Herzog, I.R. Klebanov, S.S. Pufu and T. Tesileanu, *Multi-Matrix Models and Tri-Sasaki Einstein Spaces*, *Phys. Rev.* **D 83** (2011) 046001 [arXiv:1011.5487] [inSPIRE].

[4] N. Drukker, M. Mariño and P. Putrov, *From weak to strong coupling in ABJM theory*, *Commun. Math. Phys.* **306** (2011) 511 [arXiv:1007.3837] [inSPIRE].

[5] D. Farquet, J. Lorenzen, D. Martelli and J. Sparks, *Gravity duals of supersymmetric gauge theories on three-manifolds*, arXiv:1404.0268 [inSPIRE].

[6] D. Farquet and J. Sparks, *Wilson loops and the geometry of matrix models in AdS_4/CFT_3*, *JHEP* **01** (2014) 083 [arXiv:1304.0784] [inSPIRE].

[7] A. Kapustin and B. Willett, *Wilson loops in supersymmetric Chern-Simons-matter theories and duality*, arXiv:1302.2164 [inSPIRE].

[8] A. Tanaka, *Comments on knotted 1/2 BPS Wilson loops*, *JHEP* **07** (2012) 097 [arXiv:1204.5975] [inSPIRE].

[9] M. Fujitsuka, M. Honda and Y. Yoshida, *Higgs branch localization of 3d $N = 2$ theories*, arXiv:1312.3627 [inSPIRE].

[10] A. Lewkowycz and J. Maldacena, *Exact results for the entanglement entropy and the energy radiated by a quark*, *JHEP* **05** (2014) 025 [arXiv:1312.5682] [inSPIRE].

[11] C. Closset, T.T. Dumitrescu, G. Festuccia and Z. Komargodski, *Supersymmetric Field Theories on Three-Manifolds*, *JHEP* **05** (2013) 017 [arXiv:1212.3388] [inSPIRE].

[12] A. Kapustin, B. Willett and I. Yaakov, *Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter*, *JHEP* **03** (2010) 089 [arXiv:0909.4559] [inSPIRE].

[13] D.L. Jafferis, *The Exact Superconformal R-Symmetry Extremizes Z*, *JHEP* **05** (2012) 159 [arXiv:1012.3210] [inSPIRE].

[14] D. Martelli and J. Sparks, *The large-N limit of quiver matrix models and Sasaki-Einstein manifolds*, *Phys. Rev.* **D 84** (2011) 046008 [arXiv:1102.5289] [inSPIRE].

[15] E. Witten, *Mirror manifolds and topological field theory*, hep-th/9112056 [inSPIRE].

[16] D. Martelli, A. Passias and J. Sparks, *The gravity dual of supersymmetric gauge theories on a squashed three-sphere*, *Nucl. Phys.* **B 864** (2012) 840 [arXiv:1110.6400] [inSPIRE].

[17] J. Sparks, *Sasaki-Einstein Manifolds*, *Surveys Diff. Geom.* **16** (2011) 265 [arXiv:1004.2461] [inSPIRE].

[18] J.P. Gauntlett and O. Varela, *Consistent Kaluza-Klein reductions for general supersymmetric AdS solutions*, *Phys. Rev.* **D 76** (2007) 126007 [arXiv:0707.2315] [inSPIRE].

[19] K. Becker, M. Becker and A. Strominger, *Five-branes, membranes and nonperturbative string theory*, *Nucl. Phys.* **B 456** (1995) 130 [hep-th/9507158] [INSPIRE].

[20] D. Martelli and J. Sparks, *The gravity dual of supersymmetric gauge theories on a biaxially squashed three-sphere*, *Nucl. Phys.* **B 866** (2013) 72 [arXiv:1111.6930] [INSPIRE].

[21] D. Martelli, A. Passias and J. Sparks, *The supersymmetric NUTs and bolts of holography*, *Nucl. Phys.* **B 876** (2013) 810 [arXiv:1212.4618] [INSPIRE].

[22] M. Dunajski, J.B. Gutowski, W.A. Sabra and P. Tod, *Cosmological Einstein-Maxwell Instantons and Euclidean Supersymmetry: Beyond Self-Duality*, *JHEP* **03** (2011) 131 [arXiv:1012.1326] [INSPIRE].

[23] T. Nishioka, *The Gravity Dual of Supersymmetric Rényi Entropy*, *JHEP* **07** (2014) 061 [arXiv:1401.6764] [INSPIRE].

[24] E. Witten, *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [hep-th/9803131] [INSPIRE].

[25] L.F. Alday, M. Fluder and J. Sparks, *The Large-N limit of M2-branes on Lens spaces*, *JHEP* **10** (2012) 057 [arXiv:1204.1280] [INSPIRE].

[26] L. F. Alday, D. Martelli and J. Sparks, unpublished.

[27] L.F. Alday, M. Fluder, P. Richmond and J. Sparks, *Gravity Dual of Supersymmetric Gauge Theories on a Squashed Five-Sphere*, *Phys. Rev. Lett.* **113** (2014) 141601 [arXiv:1404.1925] [INSPIRE].

[28] L.F. Alday, M. Fluder, C.M. Gregory, P. Richmond and J. Sparks, *Supersymmetric gauge theories on squashed five-spheres and their gravity duals*, *JHEP* **09** (2014) 067 [arXiv:1405.7194] [INSPIRE].

[29] B. Assel, D. Cassani and D. Martelli, *Localization on Hopf surfaces*, *JHEP* **08** (2014) 123 [arXiv:1405.5144] [INSPIRE].