

$N = 2$ Born-Infeld attractors

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ABSTRACT: We derive new types of $U(1)^n$ Born-Infeld actions based on $N = 2$ special geometry in four dimensions. As in the single vector multiplet ($n = 1$) case, the non-linear actions originate, in a particular limit, from quadratic expressions in the Maxwell fields. The dynamics is encoded in a set of coefficients d_{ABC} related to the third derivative of the holomorphic prepotential and in an $SU(2)$ triplet of $N = 2$ Fayet-Iliopoulos charges, which must be suitably chosen to preserve a residual $N = 1$ supersymmetry.

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1 Introduction

The supersymmetric Born-Infeld (BI) Lagrangian [1], obtained in [2] and in a closed superspace form in [3], was shown in [4] to encode the dominant low-energy couplings of the goldstino sector in the presence of a $2 \rightarrow 1$ partial breaking of supersymmetry. The original Volkov-Akulov [5, 6] action plays a similar role in the $1 \rightarrow 0$ case, so that this result can be also summarized by saying that in the $2 \rightarrow 1$ case the goldstino is accompanied by an $N = 1$ partner, the Abelian vector field strength

$$W_\alpha = \bar{D}^2 D_\alpha V . \tag{1.1}$$

The supersymmetric BI action possesses a number of special features. Clearly, setting to zero the gaugino it reduces to the standard BI action for the vector field, while setting to zero the vector field it reduces to the standard Volkov-Akulov action. Moreover, it is invariant under a second non-linearly realized supersymmetry, whose transformations can be conveniently expressed in terms of W_α and of the chiral superfield X , related to W_α by the non-linear constraint [4]

$$W^2 + X \left(m - \frac{1}{4} \bar{D}^2 \bar{X} \right) = 0 . \tag{1.2}$$

Here m is a parameter with dimension of $[\text{mass}]^2$ and the additional supersymmetry transformations read

$$\delta W_\alpha = \left(m - \frac{1}{4} \bar{D}^2 \bar{X} \right) \eta_\alpha - i \partial_{\alpha\bar{\alpha}} X \bar{\eta}^{\bar{\alpha}} , \tag{1.3}$$

$$\delta X = -2W^\alpha \eta_\alpha . \tag{1.4}$$

Eq. (1.2) can be regarded as a non-linear nilpotency constraint for an $N = 2$ chiral superfield \mathcal{X} [11], which can be built by combining a pair of $N = 1$ chiral superfields X and W_α according to

$$\mathcal{X}(\theta_1, \theta_2) = X(\theta_1) - 2\theta_2^\alpha W_\alpha(\theta_1) - \theta_2^\alpha \theta_{2\alpha} \left(m - \frac{1}{4} \bar{D}^2 \bar{X} \right) . \tag{1.5}$$

The $N = 2$ superfield \mathcal{X} obeys the generalized superfield constraints [7–11]

$$D_{ab}^2 \mathcal{X} = \epsilon_{ac} \epsilon_{bd} \bar{D}^{2cd} \bar{\mathcal{X}} + i m_{ab}, \quad (a, b = 1, 2) \quad (1.6)$$

where $m_{ab} = \sigma_{ab}^x m_x$, and $m_x = (m, 0, 0)$ is a magnetic charge triplet. \mathcal{X} also obeys the nilpotency constraint [11]

$$\mathcal{X}^2 = 0. \quad (1.7)$$

The solution of eq. (1.7) is provided by eq. (1.2), which also implies the additional constraints¹

$$X^2 = 0, \quad X W_\alpha = 0. \quad (1.8)$$

These are the $N = 1$ nilpotency constraints proposed in [12–15]. Finally, the BI Lagrangian is simply

$$\mathcal{L} = \Im e \int d^2\theta X. \quad (1.9)$$

Here e is a complex parameter, X is subject to the constraint (1.2), while \Re and \Im will always denote real and imaginary parts.

Alternatively, in terms of \mathcal{X} , the Lagrangian becomes the half-integral of an $N = 2$ Fayet-Iliopoulos term, projected on the $SU(2)$ triplet half-chiral measure (see [9–11]). The authors of [11] also showed that the non-linear action (1.9) can be obtained starting from the quadratic $N = 2$ action considered by Antoniadis, Partouche and Taylor in [9]. In that paper, the superpotential and the $N = 1$ Fayet-Iliopoulos terms were chosen to give an $N = 1$ vacuum with broken $N = 2$ supersymmetry. A convenient way to obtain this result is via an electric charge $(e_1, e_2, 0)$, aligned with the first two components of a triplet and a magnetic charge $(m, 0, 0)$, aligned with the first component. In this fashion, the first supersymmetry is unbroken and the $N = 1$ Fayet-Iliopoulos terms vanish [11]. On the other hand, the partial breaking $N = 2 \rightarrow N = 1$ is only possible if the $N = 2$ Fayet-Iliopoulos magnetic charge m does not vanish [9, 10, 16, 17].

Our goal here is to extend the construction to an arbitrary $N = 2$ special geometry with n vector multiplets, thus identifying the $U(1)^n$ generalization of eqs. (1.2) and (1.9).

The model is defined by magnetic and electric charges, m^A and e_A , which will be defined in the next sections, and by the superpotential²

$$U(X) = \frac{i}{2} C_{AB} X^A X^B + \frac{1}{3!M} d_{ABC} X^A X^B X^C, \quad (1.10)$$

where C_{AB} and d_{ABC} are totally symmetric and real and M sets the scale of the problem. For brevity, in the following we shall set $M = 1$, keeping in mind that the dimensionless charge triplets $Q_x = (m_x^A, e_{xA})$ are meant to be accompanied by a factor M^2 in the final result.

We shall find it convenient to introduce shifted superfields Y^A , ($A = 1, \dots, n$), defined by

$$X^A = x^A + Y^A. \quad (1.11)$$

¹This corresponds to dropping W_α in eq. (1.2) while keeping the nilpotency constraint in eq. (1.8).

²The cubic truncation leaves out higher-order non-renormalizable terms that are expected to be subdominant at low energies. Interestingly, this choice results in a shift symmetry of the axion fields $\Re X^A$.

The vacuum expectation values (VEV)s $x^A = \langle X^A \rangle$ are determined by the $N = 1$ vacuum condition

$$U_{AB}(x) m^B = e_A, \tag{1.12}$$

with

$$U_{AB}(X) = \frac{\partial^2 U(X)}{\partial X^A \partial X^B}. \tag{1.13}$$

As we shall see, the Y^A satisfy the generalized BI constraints

$$d_{ABC} [W^B W^C + Y^B (m^C - \bar{D}^2 \bar{Y}^C)] = 0, \tag{1.14}$$

which involve the totally symmetric sets of coefficients d_{ABC} and reduce to eq. (1.2) for $n = 1$, up to a slight change of conventions. As a result, the $U(1)^n$ generalized BI actions will depend on the choice of such symmetric tensors. We shall also examine in detail the available choices for the d_{ABC} in the $n = 2$ case. Moreover, we shall see that the n -extended Lagrangians can be cast in the form

$$\mathcal{L} = \Im \int d^2\theta e_A Y^A - \Re \int d^2\theta C_{AB} [W^A W^B + Y^A (m^B - \bar{D}^2 \bar{Y}^B)], \tag{1.15}$$

or alternatively, making use of the vacuum condition (1.12) and of the non-linear constraint (1.14), in the form

$$\mathcal{L} = - \Im \left[U_{AB}(x) \int d^2\theta (W^A W^B - Y^A \bar{D}^2 \bar{Y}^B) \right]. \tag{1.16}$$

Note that in the $n = 1$ case the second term in eq. (1.15) vanishes identically on account of the constraint (1.14). This reflects the fact the single C_{AB} that is present in that case can be eliminated by a field redefinition. However, for $n > 1$ the C_{AB} are needed, in general, to guarantee positivity, as is manifest from the alternative form of the Lagrangian in eq. (1.16).

2 Special geometry, Fayet-Iliopoulos terms and $N = 1$ attractors

In this section we generalize the models of refs. [9] and [11] to the multi-field case. To this end, let us first observe that the data of the problem are the $N = 2$ Fayet-Iliopoulos terms, which build up an $Sp(2n)$ symplectic triplet of electric and magnetic charges $Q_x = (m_x^A, e_{xA})$, with $x = 1, 2, 3$, $A = 1, \dots, n$, and the prepotential of eq. (1.10).

Eq. (1.10) clearly identifies the d_{ABC} as third derivatives of the prepotential U . Moreover, the $N = 2$ Lagrangian with an $N = 2$ Fayet-Iliopoulos term, written in $N = 1$ language, acquires a symplectic structure due to the underlying special geometry, which is encoded in the symplectic vector [18–21]

$$\mathcal{V} = \left(X^A, U_A \equiv \frac{\partial U}{\partial X^A} \right). \tag{2.1}$$

The scalar-field dependent $n \times n$ symmetric matrices g_{AB} and θ_{AB} determine the quadratic terms in the vector fields as

$$\mathcal{L} = - \frac{1}{4} g_{AB} G_{\mu\nu}^A G^{B\mu\nu} + \frac{1}{8} \theta_{AB} G_{\mu\nu}^A G_{\rho\sigma}^B \epsilon^{\mu\nu\rho\sigma}. \tag{2.2}$$

Moreover, in $N = 2$ special geometry

$$g_{AB} = \Im U_{AB}, \quad \theta_{AB} = \Re U_{AB}, \quad U_{AB} = \frac{\partial^2 U}{\partial X^A \partial X^B}, \quad (2.3)$$

and it is convenient to define the symplectic metric

$$\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (2.4)$$

The $2n \times 2n$ matrix \mathcal{M} , with entries

$$\mathcal{M} = \begin{pmatrix} g + \theta g^{-1} \theta & -\theta g^{-1} \\ -g^{-1} \theta & g^{-1} \end{pmatrix}, \quad (2.5)$$

then satisfies the two conditions of being symplectic and positive definite:

$$\mathcal{M} = \mathcal{M}^T, \quad \mathcal{M} \Omega \mathcal{M} = \Omega, \quad (2.6)$$

for a positive definite g , as required by the Lagrangian terms in eq. (2.2).

The contributions to the potential involve the triplets $Q_x = (m_x^A, e_{xA})$ of electric and magnetic charges. The first two combine into the complex sets

$$Q \equiv (m^A, e^A) = (m_1^A + i m_2^A, e_{1A} + i e_{2A}) \quad (2.7)$$

and determine the superpotential

$$\mathcal{W} = \mathcal{V}^T \Omega Q = (U_A m^A - X^A e_A). \quad (2.8)$$

The last,

$$Q_3 = (m_3^A, e_{3A}), \quad (2.9)$$

is real and determines, in $N = 1$ language, magnetic and electric Fayet-Iliopoulos D -terms.

The potential of the theory can thus be expressed, in $N = 1$ language, as

$$V = V_F + V_D, \quad (2.10)$$

where

$$V_F = (\Im U^{-1})^{AB} \frac{\partial \mathcal{W}}{\partial X^A} \frac{\partial \mathcal{W}}{\partial X^B} = \bar{Q}^T (\mathcal{M} - i \Omega) Q, \quad (2.11)$$

$$V_D = Q_3^T \mathcal{M} Q_3. \quad (2.12)$$

Vacua preserving an $N = 1$ supersymmetry aligned with the $N = 1$ superspace [11] are determined by critical points of the potential³, and thus by the attractor equations

$$\frac{\partial \mathcal{W}}{\partial X^A} = 0, \quad (2.13)$$

³Notice that this is not the case in the model of [9] where the D -term has a non-vanishing VEV, so that the unbroken supersymmetry is a mixture of the two original $N = 2$ superspace supersymmetries [22]. The $N = 2$ $SU(2)$ R-symmetry allows in fact to rotate $N = 1$ Fayet-Iliopoulos terms into superpotential terms.

which are in this case

$$(\mathcal{M} - i\Omega)Q = 0 . \tag{2.14}$$

This equation can admit a solution for nonzero Q only if

$$i\bar{Q}^T \Omega Q = i(m^B \bar{e}_B - \bar{m}^B e_B) > 0 , \tag{2.15}$$

while the condition $V_D = 0$ implies $Q_3 = 0$, since at the critical point \mathcal{M} is positive definite. In solving the attractor equations we shall take m^A real and e_A complex, so that eq. (2.15) will translate into the condition

$$m^A e_{2A} > 0 . \tag{2.16}$$

These are indeed attractor equations for the $N = 2$ theory quadratic in vector field strengths. It is interesting to stress the analogy with the attractor equations for $N = 2$ extremal black holes with symplectic vector $Q = (m^A, e_A)$. In terms of the \mathcal{M} matrix the black hole potential [23–28],

$$V_{\text{BH}} = \frac{1}{2} Q^T \mathcal{M} Q , \tag{2.17}$$

is also determined by the last expression in eq. (2.11), but for a real Q , so that the Ω term vanishes identically. However, in this case the value attained by V_{BH} at the attractor point is positive and gives the Bekenstein-Hawking entropy

$$V_{\text{BH}}(X_{\text{attr}}) = \frac{A}{4\pi} = \frac{S(Q)}{\pi} . \tag{2.18}$$

On the other hand, when expressed in terms of the central charge Z , which is the counterpart of \mathcal{W} , the black-hole potential contains an additional term [23–28], and reads

$$V_{\text{BH}} = |\mathcal{D}_i Z|^2 + |Z|^2 . \tag{2.19}$$

Hence, at the $\frac{1}{2}$ — BPS critical point, where $\mathcal{D}_i Z = 0$,

$$V_{\text{crit}} \equiv V_{\text{BH}}(X_{\text{attr}}) = |Z|_{\text{attr}}^2 . \tag{2.20}$$

Instead, in our case $V_{\text{crit}} = 0$, which implies $\frac{\partial \mathcal{W}}{\partial X^A} = 0$ in order to leave $N = 1$ supersymmetry unbroken.

3 Born-Infeld attractors

We can now exhibit a limit where the original theory quadratic in the field strengths gives rise to a generalized supersymmetric BI system, characterized by eqs. (1.14) and (1.15). In $N = 1$ language, the initial action reads

$$\mathcal{L} = - \Im \int d^2\theta \left[U_{AB} W^A W^B + \mathcal{W}(X) + \frac{b}{2} \bar{D}^2 (X^A \bar{U}_A - \bar{X}^A U_A) \right] , \tag{3.1}$$

where

$$\mathcal{W}(X) = U_A m^A - X^A e_A . \tag{3.2}$$

Therefore, the Euler-Lagrange equations for X^A are

$$U_{ABC} W^B W^C + U_{AB} (m^B - b \bar{D}^2 \bar{X}^B) - e_A + b \bar{D}^2 \bar{U}_A = 0 . \quad (3.3)$$

The Lagrangian (3.1) is manifestly $N = 1$ supersymmetry, while $N = 2$ supersymmetry fixes the relative coefficients of the second and third terms. However, the coefficient of the superpotential can be changed by a rescaling of the complex charge vector Q , while the normalization b of the scalar kinetic term reflects itself in the normalization of the $N = 2$ supersymmetry anticommutator.

In the Introduction we have anticipated that $U(1)^n$ generalized BI Lagrangians can be defined via a set of n constrained $N = 2$ vector multiplets satisfying eq. (1.14). As we have stressed already, the parameters that enter the action combine into a constant, totally symmetric tensor d_{ABC} and into a matrix $U_{AB}(x)$, which depends via eq. (1.12) on the d_{ABC} and on the charge vector. The generalization of eqs. (1.3) and (1.4) is

$$\delta W_\alpha^A = m^A \eta_\alpha - b \bar{D}^2 \bar{X}^A \eta_\alpha - i c \partial_{\alpha\bar{\alpha}} X^A \bar{\eta}^{\bar{\alpha}} , \quad (3.4)$$

$$\delta X^A = -2W^{A\alpha} \eta_\alpha . \quad (3.5)$$

The closure of the supersymmetry algebra fixes the parameter $c = 4b$, and in the following we shall choose $b = 1$.⁴ Note that only the magnetic charges, and not the electric ones, enter the supersymmetry transformations. The reason is that the contribution to the superpotential \mathcal{W} containing the electric charge is linear in X^A , and therefore is also invariant under the second supersymmetry [4]. Note also that the action (3.1) contains no other parameters.

The explicit form of the vacuum equations (2.13) and (2.14) is given in (3.10) and implies that the goldstino is

$$\lambda_g = \left(\frac{i}{2} \bar{Q}^T \Omega Q \right)^{-\frac{1}{2}} m^A \Im U_{AB}(x) \lambda^B = \left(\frac{i}{2} \bar{Q}^T \Omega Q \right)^{-\frac{1}{2}} e_{2A} \lambda^A ; \quad (3.6)$$

therefore, the corresponding superfield takes the form

$$W_{g\alpha} = \left(\frac{i}{2} \bar{Q}^T \Omega Q \right)^{-\frac{1}{2}} e_{2A} W_\alpha^A . \quad (3.7)$$

Its non-linear variation under the second supersymmetry, making use of eqs. (2.15) and (3.4), reads

$$\delta W_{g\alpha} = \left(\frac{i}{2} \bar{Q}^T \Omega Q \right)^{-\frac{1}{2}} e_{2A} m^A \eta_\alpha + \dots = \left(\frac{i}{2} \bar{Q}^T \Omega Q \right)^{\frac{1}{2}} \eta_\alpha + \dots , \quad (3.8)$$

so that, in units of M , the supersymmetry breaking scale is

$$E = \left(\frac{i}{2} \bar{Q}^T \Omega Q \right)^{\frac{1}{4}} , \quad (3.9)$$

which is a symplectic invariant, as expected.

⁴If one demands, as in [4], that the $N = 2$ anticommutators have the same normalization, then $c = 4b = 1$.

Because of the nilpotency constraints on X , some care will be needed to obtain the non-linear actions of eqs. (1.15) and (1.16) from the spontaneously broken theory of n linear vector multiplets of section 2. In particular, in order to satisfy the vacuum conditions (2.13) it is necessary to introduce VEVs $\langle X^A \rangle = x^A \neq 0$. In fact, eq. (2.14) is

$$U_{AB}(x) m^B \equiv (i C_{AB} + d_{ABC} x^C) m^B = e_A, \quad m^B \text{ real}, \quad e_A = e_{1A} + i e_{2A}, \quad (3.10)$$

and implies the two real equations

$$(C_{AB} + d_{ABC} \Im x^C) m^B = e_{2A}, \quad d_{ABC} \Re x^C m^B = e_{1A}. \quad (3.11)$$

A non-vanishing C_{AB} is needed to restore positivity of the kinetic term when the matrix $d_{ABC} m^C$ is not positive definite.

If we now define chiral superfields Y^A with vanishing VEV, letting $X^A = x^A + Y^A$, with the x^A c-numbers, the equations of motion (3.3) become

$$d_{ABC} \left[W^B W^C + Y^B (m^C - \bar{D}^2 \bar{Y}^C) + \frac{1}{2} \bar{D}^2 (\bar{Y}^B \bar{Y}^C) \right] + [\bar{U}_{AB}(x) - U_{AB}(x)] \bar{D}^2 \bar{Y}^B = 0. \quad (3.12)$$

Only the last term depends on x^A (and also on e_A via the vacuum equations (3.10)).

The BI Lagrangians emerge in the limit in which $U_{AB}(x)$ is negligible with respect to the d_{ABC} , where the equations of motion reduce to

$$d_{ABC} \left[W^B W^C + Y^B (m^C - \bar{D}^2 \bar{Y}^C) + \frac{1}{2} \bar{D}^2 (\bar{Y}^B \bar{Y}^C) \right] = 0. \quad (3.13)$$

The last contribution contains only overall derivatives, and therefore can be neglected in the IR limit where our effective actions will be well defined. One can then insert the *ansatz*

$$U_{ABC} Y^B Y^C = 0 \quad (3.14)$$

in (3.13), solve the resulting equation and check the self-consistency of the solution. This leads to the multi-field generalization of the BI constraint of [4, 11],

$$d_{ABC} [W^B W^C + Y^B (m^C - \bar{D}^2 \bar{Y}^C)] = 0, \quad (3.15)$$

which was already presented in eq. (1.14) in the Introduction. Taking into account that the equations of motion are solved by $D^A = 0$, the θ^2 component of (3.15) reads

$$d_{ABC} [G_+^B \cdot G_+^C + F^B (m^C - \bar{F}^C)] = 0, \quad (3.16)$$

where G_+ is the self-dual vector field strength and, here and in the next section, “dots” indicate full Lorentz contractions⁵

These complex algebraic equations determine the auxiliary fields F^A as non-linear functions of $G_+ \cdot G_+$ and $G_- \cdot G_-$, and are the seed of the generalized BI non-linear Lagrangians. For $n = 1$ the Lagrangian corresponding to eq. (1.15) reduce to the form

$$\mathcal{L} = \left(e_1 \Im F + e_2 \Re F \right) = -\frac{e_1}{m} G \cdot \tilde{G} + \frac{e_2 m}{2} \left[1 - \sqrt{1 + \frac{4}{m^2} G \cdot G - \frac{4}{m^4} (G \cdot \tilde{G})^2} \right]. \quad (3.17)$$

⁵The superfield expansion corresponding to our definition of F^A is $X^A(\theta_1) = \dots - \frac{1}{4} \theta_1^\alpha \theta_{1\alpha} F^A$.

A simple way to verify that a field solving eq. (3.15) does indeed satisfy the ansatz (3.14) is to notice that the lowest component of (3.15) is

$$d_{ABC} [\lambda^B \lambda^C + y^B (m^C - \bar{F}^C)] = 0, \quad (3.18)$$

where $\lambda^A = W^A|_{\theta=0}$ and F^A is the auxiliary θ^2 — component of Y^A . Multiplying eq. (3.18) by λ^A and using the Fierz identity $\lambda^{(A} \lambda^B \lambda^{C)} = 0$ implies that

$$d_{ABC} \lambda^A y^B (m^C - 2\bar{F}^C) = 0, \quad (3.19)$$

with $y^A = Y^A|_{\theta=0}$. Since the factor within parentheses is arbitrary, this condition requires that

$$d_{ABC} \lambda^A y^B = 0, \quad (3.20)$$

and multiplying eq. (3.18) by y^A and using (3.20) one then finds

$$d_{ABC} y^A y^B = 0. \quad (3.21)$$

Therefore, eq. (3.14) holds at $\theta = 0$, and $N = 1$ supersymmetry then implies that the entire multiplets vanishes. We have thus shown that the Y^A obey the nilpotency equations

$$d_{ABC} Y^B Y^C = 0, \quad d_{ABC} Y^B W_\alpha^C = 0. \quad (3.22)$$

Eq. (3.22) and eq. (3.15) combine in the $N = 2$ superspace constraint

$$d_{ABC} \mathcal{X}^B \mathcal{X}^C = 0, \quad (3.23)$$

which is the multi-field generalization of eq. (1.7) of the Introduction.

The Lagrangian corresponding to eqs. (3.12) is

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2i} \int d^2\theta \left[U_{AB}(x) W^A W^B + d_{ABC} \left(W^A W^B + \frac{1}{2} m^A Y^B \right) Y^C \right] + \text{h.c.} \\ & -\frac{1}{2i} \int d^2\theta d^2\bar{\theta} \left[\bar{U}_{AB}(\bar{x}) Y^A \bar{Y}^B - U_{AB}(x) Y^A \bar{Y}^B \right. \\ & \left. + \frac{1}{2} d_{ABC} (Y^A \bar{Y}^B \bar{Y}^C - \bar{Y}^A Y^B Y^C) \right]. \quad (3.24) \end{aligned}$$

Let us notice that the solution of eq. (3.12) can be expressed as the solution of (3.13) with an additional term linear in $U_{AB}(x)$: $Y = Y|_{U_{AB}=0} + \delta Y$. As a result $\delta Y = \mathcal{O}(1/\xi)$, where ξ is an overall rescaling of the d_{ABC} , and after using the constraints (3.15) and (3.22) and some integrations by parts, one is then led in the $\xi \rightarrow \infty$ limit to the two equivalent Lagrangians presented in eqs. (1.15) and (1.16) of the Introduction.

Before concluding this section, we would like to comment on two aspects of multi-field BI actions. First of all, let us emphasize some analogies and some differences with the multi-field case considered in [11, 29–31]. In those papers, the chiral superfield X is a matrix, while in our case it is a vector. Moreover, their constraints are stronger. In fact, a $U(n)$ e.m. duality is imposed, while in our case we generally expect only a $U(1)^n$ duality even if the vectors are coupled.

Finally, we notice that our $U(1)^n$ construction is not a mere complexification of the construction in [4], since for one matter it also applies for odd values of n . Moreover the terms containing C_{AB} are crucial, in general, to grant positivity. This will be manifest in the simple examples that we are about to discuss, one of which could be related to the complexified nilpotency constraints $(X \pm iY)^2 = 0$ in superspace. However, the corresponding action would contain ghosts unless a quadratic term involving C_{AB} were added to the prepotential, and this term necessarily breaks the complex structure. Therefore, even in that particular case the model is different from the $U(1)^{2n}$ generalizations proposed in [29].

4 Explicit examples: the $n = 2$ case

The generalized BI Lagrangians are determined by the superfield constraints

$$d_{ABC} [W^A W^B + Y^B (m^C - \bar{D}^2 \bar{Y})] = 0 . \quad (4.1)$$

To find them explicitly one needs only the F-term equations (3.16), since the D^A -terms vanish. Since eq. (4.1) is clearly solved by $F^A = 0$ when $G_+^A = 0$, it is useful to perform the change of variables

$$\Re F^A = \frac{1}{2} m^A - H^A , \quad (4.2)$$

thus turning imaginary and real parts of eq. (4.1) into

$$d_{ABC} \Im F^B m^C = - d_{ABC} G^B \cdot \tilde{G}^C , \quad (4.3)$$

$$d_{ABC} \left(\frac{1}{2} m^B + H^B \right) \left(\frac{1}{2} m^C - H^C \right) = d_{ABC} (- G^B \cdot G^C + \Im F^B \Im F^C) . \quad (4.4)$$

Notice that eqs. (4.3) are linear, while eqs. (4.4) are quadratic.

Any specific class of models solving these constraints is defined by the U polynomial modulo field redefinitions by $Sl(n, R)$ transformations.⁶ Inequivalent theories are thus classified by the $Sl(n, R)$ orbits of the cubic polynomials

$$U = \frac{1}{3!} d_{ABC} X^A X^B X^C . \quad (4.5)$$

As a first nontrivial example, let us consider the $n = 2$ case, where the d_{ABC} , with $A, B, C = 1, 2$, take values in the $\text{spin}-\frac{3}{2}$ representation of $Sl(2, R)$. This possesses a unique quartic invariant, which also corresponds to the discriminant of the cubic. The quartic invariant is

$$I_4 = - 27 d_{222}^2 d_{111}^2 + d_{221}^2 d_{112}^2 + 18 d_{222} d_{111} d_{112} d_{221} - 4 d_{111} d_{122}^3 - 4 d_{222} d_{211}^3 , \quad (4.6)$$

and is a truncation of Cayley's hyperdeterminant, an object that also emerges from studies of black-hole entropies [32, 33] and of q-bit entanglement in Quantum Information Theory [33–35]. Different types of roots are associated to different properties of its four orbits: O_t, O_s, O_l, O_c .

⁶The more general symplectic duality $\text{Sp}(2n, R)$ is broken by the presence of the $N = 2$ Fayet-Iliopoulos electric and magnetic charges (m_x^A, e_{xA}) .

For $I_4 > 0$ the cubic has three real simple roots and O_t is a *time-like* orbit. When the roots are simple but two are complex conjugates, $I_4 < 0$ and the orbit O_s is *space-like*. A double root $I_4 = 0$, $\partial I_4 \neq 0$ corresponds to a *light-like* orbit O_l , and finally a triple root corresponds to $I_4 = \partial I_4 = 0$ and to the critical orbit O_c made of a single point.

The four inequivalent theories can be associated to the four representative polynomials determined by the conditions

$$\begin{aligned}
 I_4 > 0 & \quad d_{222} = d_{211} \neq 0 & \quad O_t, \\
 I_4 < 0 & \quad d_{222} = d_{111} \neq 0 & \quad O_s, \\
 I_4 = 0 & \quad d_{222} = d_{221} \neq 0 & \quad O_l, \\
 \partial I_4 = 0 & \quad d_{222} \neq 0 & \quad O_c,
 \end{aligned}
 \tag{4.7}$$

which read

$$O_t = \frac{1}{3!} X^3 - \frac{1}{2} X Y^2, \tag{4.8}$$

$$O_s = \frac{1}{3!} (X^3 + Y^3), \tag{4.9}$$

$$O_l = \frac{1}{3!} X^3 - \frac{1}{2} X^2 Y, \tag{4.10}$$

$$O_c = \frac{1}{3!} X^3. \tag{4.11}$$

The imaginary parts of the Hessian matrices of these polynomials contribute to the kinetic terms. It is simple to see that only in the O_s case the Hessian is positive definite. On the other hand, the Hessians of the O_t and O_l cases have negative determinant, so that their eigenvalues have opposite signs. Finally, in the O_c case there is a vanishing eigenvalue. Hence, aside from the O_s case a C_{AB} term is needed in the generalized BI Lagrangians.

We can now consider the solutions of the constraints given in eqs. (4.3) and (4.4). The O_c and O_s cases are trivial, since there is no coupling between the two vectors in the non-linear constraints. The other two cases are nontrivial and are determined by the nilpotency constraints

$$\begin{aligned}
 O_t : \quad X^2 - Y^2 = 0, & \quad X Y = 0, \\
 O_l : \quad X^2 = 0, & \quad X Y = 0.
 \end{aligned}
 \tag{4.12}$$

Still, eqs. (4.4) can be solved by elementary techniques, since they only involve quadratic radicals.

For example, the explicit solution of eqs. (4.3) for the O_t case is

$$\Im F^X = \frac{m^X R^X + m^Y R^Y}{(m^X)^2 + (m^Y)^2}, \quad \Im F^Y = \frac{-m^Y R^X + m^X R^Y}{(m^X)^2 + (m^Y)^2}, \tag{4.13}$$

where

$$R^X = -G^X \cdot \tilde{G}^X + G^Y \cdot \tilde{G}^Y, \quad R^Y = -2G^X \cdot \tilde{G}^Y. \tag{4.14}$$

On the other hand, eqs. (4.4) become

$$-(H^X)^2 + (H^Y)^2 = S^X, \quad 2H^X H^Y = S^Y, \tag{4.15}$$

where

$$S^X = T^X - \frac{(m^X)^2}{4} + \frac{(m^Y)^2}{4}, \quad S^Y = T^Y + \frac{m^X m^Y}{2}, \quad (4.16)$$

and

$$T^X = -G^X \cdot G^X + G^Y \cdot G^Y + (\Im F^X)^2 - (\Im F^Y)^2, \quad T^Y = 2(G^X \cdot G^Y + \Im F^X \Im F^Y). \quad (4.17)$$

In terms of these quantities, the explicit solutions for H^X and H^Y read

$$H^X = \frac{1}{\sqrt{2}} \left(\sqrt{(S^X)^2 + (S^Y)^2} - S^X \right)^{\frac{1}{2}}, \quad H^Y = \frac{1}{\sqrt{2}} \left(\sqrt{(S^X)^2 + (S^Y)^2} + S^X \right)^{\frac{1}{2}}. \quad (4.18)$$

The solutions of eqs. (4.4) become apparently more and more complicated with increasing n , when the number of inequivalent cases and their degeneracies also increase. Their classification rests on the theory of invariant polynomials, which was only completed for the $n = 3$ and $n = 4$ cases so far [36–39].

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