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Color kinematic symmetric (BCJ) numerators in a light-like gauge

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Abstract: Color-ordered tree level scattering amplitudes in Yang-Mills theories can be written as a sum over terms which display the various propagator poles of Feynman diagrams. The numerators in these expressions which are obtained by straightforward application of Feynman rules are not satisfying any particular relations, typically. However, by reshuffling terms, it is known that one can arrive at a set of numerators which satisfy the same Jacobi identity as the corresponding color factors. By extending previous work by us we show how this can be systematically accomplished within a Lagrangian framework. We construct an effective Lagrangian which yields tree-level color-kinematic symmetric numerators in Yang-Mills theories in a light-like gauge at five-points. The five-point effective Lagrangian is non-local and it is zero by Jacobi identity. The numerators obtained from it respect the original pole structure of the color-ordered amplitude. We discuss how this procedure can be systematically extended to higher order.

Keywords: Scattering Amplitudes, Effective field theories, Gauge Symmetry

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1 Introduction

In field theory, a very intriguing result is the Kawai-Lewellen-Tye (KLT) relation [1], which connects gauge theories with gravity. A particular version of the KLT relations was given by Bern, Dennen, Huang and Kiermaier in [2].[1] Following [2], let us give a brief summary about it in the language of Lagrangian field theory. We symbolically write the tree color-dressed n gluon amplitude as

$$A^{(n)} = \sum_i \frac{c_i n_i}{(\Pi s_j)_i}, \tag{1.1}$$

where c_i are color factors, n_i are numerators made of momenta and polarization vectors, and (Πs_j) are appropriate products of inverse propagators, all constructed according to a well defined set of Feynman rules once a gauge choice is made. Bern, Carrasco and Johansson [4] stated that for channels which satisfy the Jacobi identity

$$c_i + c_j + c_k = 0, \tag{1.2}$$

[1] More KLT-type relations can be found in [3].

one can reshuffle terms and obtain a new set of numerators so that they seem to have been constructed completely through some effective three point vertices and thus also satisfy

$$\bar{n}_i + \bar{n}_j + \bar{n}_k = 0. \tag{1.3}$$

The relations (1.2) and (1.3) have been called color-kinematics duality and the numerators which have this property have been called BCJ or color-kinematic symmetric.[2]

Together with the antisymmetry of the effective three vertices, (1.2) leads to a reduction in the number of independent color coefficients to $(n-2)!$, and via (1.3) to the same number of independent numerators. Thus, naively one may also conclude that there are $(n-2)!$ independent color-ordered amplitudes. Any of such a set is called a Kleiss-Kuijf basis [11]. We can form a column vector for a set of the independent numerators $|\bar{N}\rangle$ and another column vector for the set of color-ordered amplitudes in the chosen Kleiss-Kuijf basis $|A\rangle$. They are related by

$$|A\rangle = M|\bar{N}\rangle, \tag{1.4}$$

where the elements of the propagator matrix M are made of sums of products of propagators. An appropriate set of independent color coefficients will form a row vector $\langle C|$, which will yield the color-dressed n particle amplitude

$$A^{(n)} = \langle C|M|\bar{N}\rangle. \tag{1.5}$$

In [12] we pointed out that in fact there are $(n-3)(n-3)!$ degrees of arbitrariness in changing the elements in $|\bar{N}\rangle$, which will yield the same $|A\rangle$. This freedom in writing up the BCJ numerators was called the generalized gauge transformations in [12].[3] The underlying reason for this is that there are $(n-3)(n-3)!$ eigenvectors with zero eigenvalue for M,[4] and therefore one can add to $|\bar{N}\rangle$ this number of arbitrary functions, each multiplied to one of the zero eigenvectors. Clearly, it has no effect on $|A\rangle$. Seen through this, the true number of independent elements in $|A\rangle$ is in fact only $(n-3)!$.

It is important that we should be in our possession a set of dual symmetric numerators, because the KLT relation, as expressed by Bern et al. in [2], in the present context is a statement that up to coupling constants, the tree level n graviton amplitude is given by

$$A^{(n)}_{gr} = \langle\tilde{N}|M|\bar{N}\rangle, \tag{1.6}$$

in which $\tilde{n}_i$ can be numerators due to a different gauge theory or not, which satisfy the color-kinematic duality relations

$$\tilde{c}_i + \tilde{c}_j + \tilde{c}_k = 0, \tag{1.7}$$

[2]The color-kinematic duality has sparked a great interest: connections with string theory were probed [5, 6], string inspired monodromy techniques were used to write relations between color-ordered amplitudes in [7], a kinematic algebra (for the self-dual sector only) was proposed in [8] and extensions to loop amplitudes in pure gauge theories were found [9, 10], to list a few.

[3]In [2] the authors exploited this freedom in their proof of the relationship between gravity and gauge theory amplitudes.

[4]Recent work by Cachazo et al. expressed the entries of what we called the propagator matrix M in [12] as partial amplitudes of a double-copy scalar theory with cubic interactions and wrote them in terms solutions to the so-called scattering equations [13].

and

$$\tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0. \tag{1.8}$$

There are various proposals to construct concretely these color-kinematic symmetric numerators.[5] However, in our view they are not straightforwardly implementable via a set of conventional Feynman rules.[6] We will demonstrate that in fact a general approach can be so prescribed. We have chosen in what follows to work in a light-like gauge of which space-cone gauge [19] is an example. The reason is our hidden desire to ultimately understand the connection of the gauge Lagrangian to the gravity Lagrangian in some way. The light-like gauges seem to be the most promising, because explicitly there are only two independent fields in each theory in four spacetime dimensions. Indeed, in [25], by working in a light-like gauge, the authors were able to expose the squaring relation between the gravity and gauge theory four-point tree-level amplitudes, at the level of the Lagrangian. However, as we will see, our procedure does not rely on any particular gauge choice, in the sense of a specific choice of the light-like vector which can dramatically reduce the number of Feynman diagrams as in [19],[7] if all we care is to obtain color-kinematic symmetric numerators.

We have described how to relate the Kleiss-Kuijf set of amplitudes to numerators which are color-kinematic symmetric. When the numerators are not initially color-kinematic symmetric, as it is generally the case if we just apply Feynman rules as we normally would to calculate amplitudes, then we must give a recipe how to modify them to make them so. The important criterion to observe is that the color-ordered amplitudes should be the same under such modifications. To be more specific, if we start out with

$$n_i + n_j + n_k = \Delta_{ijk} \neq 0, \tag{1.9}$$

we shall make changes

$$n_l \to \bar{n}_l = n_l + \delta n_l \tag{1.10}$$

such that

$$\bar{n}_i + \bar{n}_j + \bar{n}_k = 0, \tag{1.11}$$

[5]See for example [14–17].

[6]The exception is [18] who set out to derive BCJ numerators using a covariant (Feynman) gauge. In doing so they extended the particular effective five-point Lagrangian obtained by Bern et al. in [2]. However their approach is somewhat less transparent than the steps we undertake in this paper and we were unable to see a direct translation of their algorithm into ours.

[7]Another benefit of the space-cone gauge is that it allows a straightforward proof of the BCFW on-shell recursion relations [20], at the level of Feynman diagrams [21]. By choosing the null space-cone gauge fixing vector such that it is expressed in terms of the two external gluon momenta which are analytically continued in the BCFW recursion, the only z-dependence in the analytically continued Feynman diagrams comes from the propagators. Then BCFW factorization is simply a statement about partial fractioning of the propagators in the Feynman diagrams followed by a regrouping into products of lower n-point amplitudes. In another application, the MHV Lagrangian was shown by Mansfield [22] to be derived from a unitary transformation acting on the fields of the light-cone gauge fixed Lagrangian. Light-like gauges are useful beyond tree-level as well. We recall that Mandelstam used light-cone gauge for his proof of the UV finiteness of maximally supersymmetric Yang-Mills theories in four space-time dimensions [23]. On-shell recursion at one loop is also somewhat subtle, but space-cone gauge makes it for an easier approach [24].

which is equivalent to having the changes to absorb the violation

$$\delta n_i + \delta n_j + \delta n_k = -\Delta_{ijk}. \tag{1.12}$$

Now, we demand that from (1.1)

$$\sum_i \frac{c_i \delta n_i}{(\Pi s_j)_i} = 0. \tag{1.13}$$

It is easy to see that there are only $(n-2)!$ independent δn_l and upon expressing the others in terms of them and Δ's, we find that we end up with an equation

$$|D\rangle = M|\delta N\rangle, \tag{1.14}$$

in which $|\delta N\rangle$ is a column vector with the independent δn's as entries, $|D\rangle$ is made of the Δ's, and M is the same matrix as in (1.4).

Just as before, because of the existence of eigenvectors with null eigenvalue in M, we cannot invert the equation for δn_l uniquely; there are only $(n-3)!$ linear combinations of them which are active. We must make some ansatz for the functional forms of these δn's and solve for them, which also points to the fact that there is in principle a whole host of choices one can make to render the numerators dual symmetric. What we would like to reiterate is that the Δ's are constructed through Feynman rules. They are uniquely given, once a gauge is picked. On the other hand, there are $(n-3)(n-3)!$ degrees of freedom in choosing $\delta n's$ (and hence dual symmetric $n's$), which agree with the number of generalized gauge transformations one can make. By the same token, there are $(n-2)!$ entries in D, and we can use any $(n-3)!$ of them for the 'inversion' of (1.14).

We will fix, in part, this freedom by requiring that the numerator shifts δn do not introduce spurious poles. In other words, we require that the original pole structure expressed in writing the color-ordered amplitudes as in (1.1), where n_i are obtained via Feynman rules, is preserved. This will result in a tighter set of constraints imposed on the numerator shifts. For the five-points, the freedom in the numerator shifts reduces then to two arbitrary constants. As a consequence, we obtain color-symmetric numerators $\bar{n}_i$ in (1.4) which will also preserve the original pole structure.

In the next few sections we will explicitly follow the program just outlined for $n = 4,\ 5$, to obtain a set of shifts which render the numerators color-kinematic symmetric. We will write an effective Lagrangian for them. The parametrization of the other dual symmetric shifts will be given. It will become obvious that the same procedure should work for any number of particles and in any light-like gauge.

The plan of this article is as follows. In section 2 we give a quick overview of Yang-Mills theories in non-covariant, light-like gauges. We also introduce here our notation. In section 3 we discuss four-point amplitudes as derived from the gauge-fixed Lagrangian. We notice that similar to results derived in covariant gauges, the numerators are already BCJ symmetric. The next two sections, 4 and 5, are dedicated to the five-point amplitudes and the corresponding numerators. The numerators obtained via Feynman rules are not BCJ symmetric. However, we show that there is an effective null five-point Lagrangian which induces shifts of the numerators such that the end result is BCJ symmetric. We relegate technical details and intermediate results to four of the appendices. We make some final remarks in section 6 and comment on extending our procedure to six-point functions in appendix E.

2 Notation, conventions, and a quick overview of light-like gauges

Throughout this paper we work in four space-time dimensions. The Lorentz metric we use is defined via the scalar product

$$P_\mu Q^\mu = -P^0 Q^0 + \vec{P}\cdot\vec{Q} = p\bar{q} + \bar{p}q - p^+ q^- - p^- q^+, \tag{2.1}$$

where we have introduced the notation

$$p^\pm \equiv \frac{1}{\sqrt{2}}(P^0 \pm P^3), \qquad p \equiv \frac{1}{\sqrt{2}}(P^1 + iP^2), \qquad \bar{p} \equiv \frac{1}{\sqrt{2}}(P^1 - iP^2). \tag{2.2}$$

We reserve capital letter notation for vectors carrying Greek indices: $P_\mu = (P_0, \vec{P})$.

Following [19], we introduce the reference (commuting) spinors $|\pm\rangle$ and $|\pm]$, normalized to

$$\langle +-\rangle = [-+] = 1 \tag{2.3}$$

but otherwise arbitrary. Then the set of null vectors $\{|+\rangle[+|, |-\rangle[+|, |+\rangle[-|, |-\rangle[-|\}$ forms a basis and the four-vector components introduced earlier in (2.2) are obtained from the decomposition

$$P = p^+|+\rangle[+| \, + \, p^-|-\rangle[-| \, + \, p|+\rangle[-| \, + \, \bar{p}|-\rangle[+|. \tag{2.4}$$

If P^μ is a null four-vector, i.e. there exist spinors such that $P = |p\rangle[p|$, then

$$p = \langle p+\rangle[p-], \; \bar{p} = \langle p-\rangle[p+], \; p^+ = \langle p-\rangle[-p], \; p^- = \langle p+\rangle[+p]. \tag{2.5}$$

Starting with the Yang-Mills Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu\, a} F^{\mu\nu}{}_a \tag{2.6}$$

where a is an adjoint color index and the field strength $F_{\mu\nu\, a}$ is given as

$$F_{\mu\nu\, a} = \partial_\mu A_{\nu\, a} - \partial_\nu A_{\mu\, a} + g f_{abc} A_{\mu\, b} A_{\nu\, c}, \tag{2.7}$$

one can reach the non-covariant gauge [19]

$$a_b = 0. \tag{2.8}$$

This is analogous to the more familiar light-cone gauge fixing condition $a_b^+ = 0$. Both gauges are light-like, in the sense that one sets to zero a component of the gauge field along a given null vector.

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In components, the Yang-Mills Lagrangian is

$$\begin{aligned}\mathcal{L} = -\frac{1}{4}\Big[&2(\partial^+ a_a^- - \partial^- a_a^+ + g f_{abc} a_b^+ a_c^-)(\partial^- a_a^+ - \partial^+ a_a^- + g f_{ade} a_d^- a_e^+) \\ &-4(\partial^+ a_a - \partial a_a^+ + g f_{abc} a_b^+ a_c)(\partial^- \bar{a}_a - \bar{\partial} a_a^- + g f_{ade} a_d^- \bar{a}_e) \\ &-4(\partial^+ \bar{a}_a - \bar{\partial} a_a^+ + g f_{abc} a_b^+ \bar{a}_c)(\partial^- a_a - \partial a_a^- + g f_{ade} a_d^- a_e) \\ &+2(\partial \bar{a}_a - \bar{\partial} a_a + g f_{abc} a_b \bar{a}_c)(\bar{\partial} a_a - \partial \bar{a}_a + g f_{ade} \bar{a}_d a_e)\Big]\end{aligned} \tag{2.9}$$

where all derivatives are understood to be ∂^μ. For example, $\partial^+ = -\frac{\partial}{\partial x^-}$, $\partial = \frac{\partial}{\partial \bar{x}}$ etc. In momentum space these derivatives convert simply to factors of the corresponding momentum components: ∂^+ becomes ip^+, ∂ becomes ip etc.

After using the gauge fixing condition $a_b = 0$, $\bar{a}_b$ is independent of the "time"-derivative $\bar{\partial}$ and so it can be eliminated from its equation of motion,

$$\bar{a}_b = \frac{1}{\partial}\left[\partial^+ a_b^- + \partial^- a_b^+ - g\frac{f_{bcd}}{\partial}(\partial a_c^- a_d^+ + \partial a_c^+ a_d^-)\right] \tag{2.10}$$

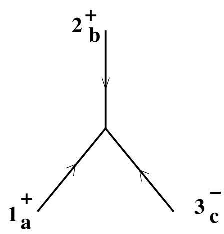
The gauge fixed Lagrangian becomes

$$\begin{aligned}\mathcal{L} = &-a_a^- \partial_\mu \partial^\mu a_a^+ + 2g f_{abc}\left(\frac{\partial^+}{\partial} a_a^-\right) a_b^- \partial a_c^+ + 2g f_{abc}\left(\frac{\partial^-}{\partial} a_a^+\right) a_b^+ \partial a_c^- \\ &+2g^2\left(f_{abc} a_b^- \partial a_c^+\right)\frac{1}{\partial^2}\left(f_{ade} a_d^+ \partial a_e^-\right).\end{aligned} \tag{2.11}$$

This Lagrangian contains now only the two physical degrees of freedom of a gauge field in four space-time dimensions: positive and negative helicities corresponding respectively to the a^+ and a^- components.[8]

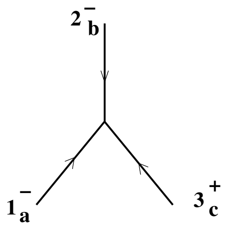
This yields the following Feynman rules:

Propagator: $= \frac{-i\delta_{ab}}{P^2}$

Three-point vertices: 

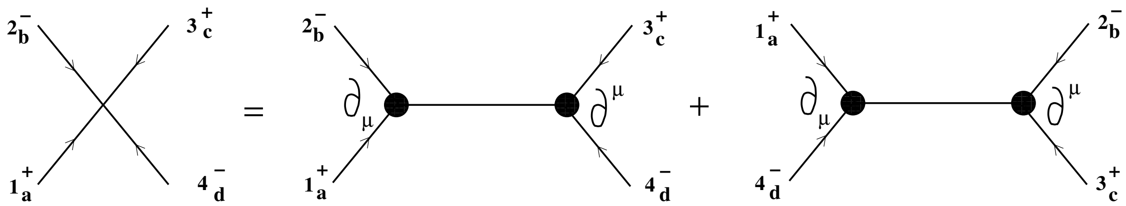
$= -2g f_{abc}\left(\frac{p_1^-}{p_1} - \frac{p_2^-}{p_2}\right) p_3$

[8]Note that in [26], the gauge-fixed Lagrangian given in (14) has a sign typo in the kinetic term. For another comparison, [19] have their Lagrangian being normalized as $\mathcal{L} = \frac{1}{8g^2} Tr(F_{\mu\nu}F^{\mu\nu})$ with $F_{\mu\nu} \equiv F_{\mu\nu\,a}T_a = (\partial_\mu A_{\nu\,a} - \partial_\nu A_{\mu\,a})T_a + A_{\mu\,b}A_{\nu\,c}[T_b, T_c]$. After rescaling the gauge fields $A_{\mu\,a} \to g A_{\mu\,a}$, and using that in the adjoint representation the gauge group generators equal $(T_a)_{bc} = i f_{abc}$ where f_{abc} are the structure constants ($[T_a, T_b] = -i f_{abc} T_c$), their Lagrangian becomes our (2.11) up to an overall factor $-\frac{1}{2}$ times a normalization factor N obtained from the evaluation of the traces $Tr(T_a T_b) = N\delta_{ab}$. Taking $N = 2$ results in agreement between the Lagrangian in [19] and (2.11) up to an overall sign.



$$= -2g f_{abc} \left(\frac{p_1^+}{p_1} - \frac{p_2^+}{p_2} \right) p_3$$

Four-point vertices:[9]



$$= -2g^2 i f_{abe} f_{cde} \frac{p_1 p_4 + p_2 p_3}{(p_1+p_2)^2} \frac{s_{12}}{s_{12}} + (-2g^2 i) f_{dae} f_{bce} \frac{p_1 p_2 + p_3 p_4}{(p_1+p_4)^2} \frac{s_{14}}{s_{14}}.$$

These Feynman rules need to be supplemented by the insertion of external line factors. These originate in the polarization vectors and their components corresponding to the positive or negative helicity of a given external line. More concretely,

$$\epsilon(P)^+ = \frac{[-p]}{\langle +p \rangle}, \ \epsilon(P)^- = \frac{\langle +p \rangle}{[-p]}, \tag{2.12}$$

need to be inserted for each external line with momentum P and positive or negative helicity respectively.

Chalmers and Siegel noted the advantage which comes from choosing the reference spinors in such a way that $|+\rangle[+|$ is the momentum of an external negative helicity gluon and $|-\rangle[-|$ is the momentum of an external positive helicity gluon (up to a normalization factor). This choice leads to the least number of Feynman diagrams for a given process (far less then in other more covariant gauges), and they referred to it as space-cone gauge [19]. However, here we work in full generality, and keep the reference spinors arbitrary. To emphasize this distinction we will refer to the gauge condition in (2.8) as a light-like gauge condition.

It should be pointed out that in a light-like gauge, where we separate out particles of two different helicities, the symmetries among like helicity particles are explicit, while those between unlikes must be imposed by hand.

In the usual fashion, we will convert the structure constant factors into traces over the group generators, and compute color ordered amplitudes.The tree level n-gluon scattering

[9]We choose to interpret the quartic term in the Lagrangian $2g^2(f_{abc} a_b^- \partial a_c^+) \frac{1}{\partial^2} (f_{ade} a_d^+ \partial a_e^-)$ as $-2g^2 \partial_\mu (f_{abc} a_b^- \partial a_c^+) \frac{1}{\Box \partial^2} \partial^\mu (f_{ade} a_d^+ \partial a_e^-)$. The manifest propagator in the denominator makes it clear how we choose to assign the contribution of the four-point vertex to the numerators. We note that a similar choice was made by [18].

amplitude is then equal to the sum over the color-ordered partial amplitudes

$$A^{(n)} = -ig^{n-2} \sum_{\sigma} Tr[T_{a(\sigma(1))} T_{a(\sigma(2))} \dots T_{a(\sigma(n-1))} T_{a(\sigma(n))}] A(\sigma(1), \sigma(2), \dots \sigma(n)), \quad (2.13)$$

where σ is a non-cyclic permutation of the external gluons.

To identify a given numerator by the labelling of the indices, we follow the convention we developed in [12]. Thus, for n gluons, we have a set of color ordered numerators. As a consequence of clockwise vs. counterclockwise tracing, the numerators satisfy

$$n(i_n \cdots i_1) = (-1)^n n(i_1 \cdots i_n), \quad (2.14)$$

The indices can be further refined: if two adjacent indices, say j and $j+1$, share the same structure constant $f_{a_j a_{j+1} a_k}$, we shall separate them from the other indices by two sets of semi-colons $n(i_1 \cdots i_{j-1}; i_j i_{j+1}; i_{j+2} \cdots i_n)$. Clearly they are antisymmetric

$$n(i_1 \cdots i_{j-1}; i_j i_{j+1}; i_{j+2} \cdots i_n) = -n(i_1 \cdots i_{j-1}; i_{j+1} i_j; i_{j+2} \cdots i_n), \quad (2.15)$$

because of the form taken by the cubic and quartic vertices.

It is also clear that for each pair of such indices, a propagator $\frac{i}{2s_{j\ j+1}}$ will go with them in an amplitude, where $s_{j\ j+1} = -\frac{1}{2}(P_j + P_{j+1})^2$. For example in writing $n(i_1 i_2; i_3; \dots; i_{n-1} i_n)$, the pole structure associated with it is $s_{i_1 i_2} = -\frac{1}{2}(P_{i_1} + P_{i_2})^2$, $s_{i_1 i_2 i_3} = -\frac{1}{2}(P_{i_1} + P_{i_2} + P_{i_3})^2, \dots\ s_{i_1 i_2 i_3 \dots i_{n-2}} = -\frac{1}{2}(P_{i_n} + P_{i_{n-1}})^2$.

3 Duality for four particles

In this section we carry out the program outlined in the Introduction for the simplest case when $n = 4$. As it is well-known, the configurations $\pm\pm\pm\pm$ and $\pm\pm\pm\mp$ are trivial, because the amplitudes vanish. The Jacobi permutation of the numerators $n_i + n_j + n_k$ vanishes even off-shell. For the maximal helicity violation case $1^+ 2^- 3^+ 4^-$ and other helicity assignments, we show that when the particles are all on shell the numerators are automatically dual symmetric, if we just apply the Feynman rules to obtain them in any light-like gauge, and particularly in the space-cone gauge. We then extend this to obtain a result for the relevant Jacobi cyclic permutation when the particles are off-shell, which will be used in the next section as an insertion. We find here that each term is proportional to the invariant mass of one of the four particles, which is an affirmation that the numerators are BCJ symmetric on-shell, as said already mentioned.

Upon using (2.14) and (2.15), we see that it is sufficient to deal with the numerators $n(12;34)$, $n(13;24)$, and $n(23;14)$. Let us focus on cyclically permuting the first three indices,

$$n(12;34) + n(23;14) + n(31;24) \equiv \Delta(123|4), \quad (3.1)$$

or

$$n(23;41) = n(12;34) - n(13;24) - \Delta(123|4). \quad (3.2)$$

The last equation means that $n(12;34)$ and $n(13;24)$ can be taken as the independent numerators, while $n(23;41)$ being given by them and the amount of duality violation $\Delta(123|4)$. Similarly, if we Jacobi permute the last three indices

$$n(12;34) + n(13;42) + n(14;23) \equiv \Delta(1|234), \tag{3.3}$$

we see that using (2.14) and (2.15), we are yielded

$$\Delta(123|4) = \Delta(1|234), \tag{3.4}$$

For the color-ordered amplitudes in the Kleiss-Kuijf basis, chosen for concreteness to be composed of $A(1234)$ and $A(1324)$, we have

$$\begin{aligned} A(1234) &= \frac{n(12;34)}{s_{12}} + \frac{n(23;41)}{s_{14}} \\ &= n(12;34)\left(\frac{1}{s_{12}} + \frac{1}{s_{14}}\right) + n(13;24)\left(-\frac{1}{s_{14}}\right) + \Delta(123|4)\left(-\frac{1}{s_{14}}\right), \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} A(1324) &= \frac{n(13;24)}{s_{13}} - \frac{n(23;41)}{s_{14}} \\ &= n(12;34)\left(-\frac{1}{s_{14}}\right) + n(13;24)\left(\frac{1}{s_{13}} + \frac{1}{s_{14}}\right) + \Delta(123|4)\left(\frac{1}{s_{14}}\right). \end{aligned} \tag{3.6}$$

The next step is to modify the numerators derived from the use of Feynman rules by adding δn terms such that the resulting $\bar{n} = n + \delta n$ numerators obey Jacobi identity, and such that the amplitudes are unchanged. More concretely,

$$\begin{aligned} &\bar{n}(12;34) = n(12;34) + \delta n(12;34),\ \ \bar{n}(13;42) = n(13;42) + \delta n(13;42), \\ &\bar{n}(14;23) = n(14;23) + \delta n(14;23), \end{aligned} \tag{3.7}$$

are defined so that

$$\bar{n}(12;34) + \bar{n}(13;42) + \bar{n}(14;23) = 0, \tag{3.8}$$

or

$$\delta n(12;34) + \delta n(13;42) + \delta n(14;23) = -\Delta(123/4), \tag{3.9}$$

such that the values of the color-ordered amplitudes are not changed. Please note that by definition we are referring to on-shell quantities here. When we extend the amplitudes to amputated Green's functions, we cannot make such a demand. From the requirement that the change made to the numerators does not change the amplitudes, which in terms of the color-kinematic symmetric numerators $\bar{n}$ are written as

$$\begin{pmatrix} A(1234) \\ A(1324) \end{pmatrix} = M^{(4)} \begin{pmatrix} \bar{n}(12;34) \\ \bar{n}(13;24) \end{pmatrix}, \tag{3.10}$$

we are led to the following constraint on the numerator shifts in the chosen Kleiss-Kuijf basis:

$$\begin{pmatrix} \frac{-\Delta(123|4)}{s_{14}} \\ \frac{\Delta(123|4)}{s_{14}} \end{pmatrix} = M^{(4)} \begin{pmatrix} \delta n(12;34) \\ \delta n(13;24) \end{pmatrix}, \tag{3.11}$$

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with $M^{(4)}$ the four-point propagator matrix introduced in [12]

$$M^{(4)} = \begin{pmatrix} \frac{1}{s_{12}} + \frac{1}{s_{14}} & -\frac{1}{s_{14}} \\ -\frac{1}{s_{14}} & \frac{1}{s_{13}} + \frac{1}{s_{14}} \end{pmatrix}. \tag{3.12}$$

An important observation made in [12] is that $M^{(4)}$ has an eigenvector with zero eigenvalue

$$\langle \lambda^0 | = \langle -s_{12},\ s_{13}|. \tag{3.13}$$

Then one has the freedom to change the numerators by adding these zero eigenvectors. In doing so, the defining equation (3.12) remains the same. This freedom was called generalized gauge transformation in [12].[10] For the four point amplitudes, the implication is that there is only one effective $\bar{n}$ and one effective δn. For the latter, we make the following generalized gauge transformation

$$\begin{pmatrix} \delta n(12;34) \\ \delta n(13;24) \end{pmatrix} \to \begin{pmatrix} \delta n(12;34) \\ \delta n(13;24) \end{pmatrix} - \frac{\delta n(13;24)}{s_{13}} |\lambda^0\rangle = \begin{pmatrix} \delta n \\ 0 \end{pmatrix} \tag{3.14}$$

where

$$\delta n \equiv \delta n(12;34) + \frac{s_{12}}{s_{13}} \delta n(13;24). \tag{3.15}$$

This results in a reduced equation

$$\begin{pmatrix} \frac{-\Delta(123|4)}{s_{14}} \\ \frac{\Delta(123|4)}{s_{14}} \end{pmatrix} = M \begin{pmatrix} \delta n \\ 0 \end{pmatrix}, \tag{3.16}$$

which demands

$$\Delta(123|4) = 0. \tag{3.17}$$

We will verify this explicitly in a direct calculation below. In the mean time, it tells us that the numerators calculated through Feynman rules for the on shell $n = 4$ amplitudes are dual symmetric without any need for modification. We should point out that up to this point, we need not refer to any specific choice of gauge, light-like or covariant, to come to this conclusion.

Let us turn to the off-shell situation, by which we mean of course that the invariant mass of each individual particle is non-zero. Also, we do not let the numerator matrix elements act on the polarization tensors. Since we have

$$n(12;34) + n(31;24) + n(23;14) = n(12;34) + n(14;23) + n(13;42) \tag{3.18}$$

there is only one Jacobi permutation. Also, for a given number of + and −, the specific assignment to each individual particle can be arbitrary, because we cycle them through the permutations above, which will cover all the cases if we relabel the particle number. For

[10] We would like to emphasize that the shifts δn cannot be obtained in general by making generalized gauge transformations. The four-point case is somewhat special since we will argue that the numerators satisfy the color-kinematic duality without any need to make these shifts. However, this does not extend to the higher n-point numerators.

$\pm\pm\pm\pm$, the case is trivial, because the vertices cannot be matched to make the scattering go. For $\pm\pm\pm\mp$, we need only three-point vertices of the same type $(++-)$. In the case we are considering the numerators are

$$\begin{aligned} n(1^+2^+;3^+4^-) &= \left(\frac{p_1^-}{p_1}-\frac{p_2^-}{p_2}\right)(-(p_1+p_2))\left(\frac{p_1^-+p_2^-}{p_1+p_2}-\frac{p_3^-}{p_3}\right)p_4, \\ n(1^+4^-;2^+3^+) &= \left(\frac{p_2^-}{p_2}-\frac{p_3^-}{p_3}\right)(-(p_2+p_3))\left(\frac{p_2^-+p_3^-}{p_2+p_3}-\frac{p_1^-}{p_1}\right)p_4, \\ n(1^+3^+;4^-2^+) &= \left(\frac{p_3^-}{p_3}-\frac{p_1^-}{p_1}\right)(-(p_3+p_1))\left(\frac{p_3^-+p_1^-}{p_3+p_1}-\frac{p_2^-}{p_2}\right)p_4. \end{aligned} \tag{3.19}$$

When we add them, we find that all terms in the sum cancel completely.

The $\pm\pm\mp\mp$ case is the non-trivial one. For one thing, four-vertices make their appearance. In view of the somewhat tedious algebra to bring the expressions to the final form, we are relegating the details to appendix A. The results are

$$\begin{aligned} n(1^-2^+;3^+4^-) &= s_{23}+\left(\frac{p_2^-}{p_2}-\frac{p_3^-}{p_3}\right)\left(\frac{p_4^+}{p_4}-\frac{p_1^+}{p_1}\right)(p_1p_3+p_2p_4) \\ &\quad+\frac{p_1p_3}{p_1+p_2}\left(-\frac{P_4^2}{2p_4}+\frac{P_2^2}{2p_2}\right)+\frac{p_2p_4}{p_1+p_2}\left(-\frac{P_3^2}{2p_3}+\frac{P_1^2}{2p_1}\right), \end{aligned} \tag{3.20}$$

$$\begin{aligned} n(3^+1^-;2^+4^-) &= -s_{23}+\left(\frac{p_2^-}{p_2}-\frac{p_3^-}{p_3}\right)\left(\frac{p_4^+}{p_4}-\frac{p_1^+}{p_1}\right)(p_1p_2+p_3p_4) \\ &\quad+\frac{p_3p_4}{p_2+p_4}\left(-\frac{P_2^2}{2p_2}+\frac{P_1^2}{2p_1}\right)+\frac{p_1p_2}{p_2+p_4}\left(-\frac{P_4^2}{2p_4}+\frac{P_3^2}{2p_3}\right), \end{aligned} \tag{3.21}$$

and

$$n(2^+3^+;1^-4^-) = -\left(\frac{p_2^-}{p_2}-\frac{p_3^-}{p_3}\right)\left(\frac{p_4^+}{p_4}-\frac{p_1^+}{p_1}\right)(p_1+p_4)(p_2+p_3), \tag{3.22}$$

where we have omitted a product of the four polarization vectors because we are extending the result to off-shell $P_i^2=\vec{P}_i^2-(P_i^0)^2\neq 0$. We now add them and find

$$\begin{aligned} \Delta(2^+|1^-3^+4^-) &\equiv n(2^+1^-;3^+4^-)+n(2^+3^+;4^-1^-)+n(2^+4^-;1^-3^+) \\ &= \tfrac{1}{2}P_1^2p_4\left(\frac{1}{p_1+p_2}-\frac{1}{p_1+p_3}\right)+\tfrac{1}{2}P_2^2p_3\left(\frac{1}{p_1+p_2}-\frac{1}{p_2+p_4}\right) \\ &\quad+\tfrac{1}{2}P_3^2p_2\left(\frac{1}{p_3+p_4}-\frac{1}{p_1+p_3}\right)+\tfrac{1}{2}P_4^2p_1\left(\frac{1}{p_3+p_4}-\frac{1}{p_2+p_4}\right). \end{aligned} \tag{3.23}$$

When we go on-shell, by setting $P_i^2\to 0$, we have $\Delta\to 0$, which, as advertised, means that duality holds by the on-shell numerators as calculated through regular Feynman rules, without any need for additional adjustment. We will find the off-shell $\Delta(2^+|1^-3^+4^-)$ useful as an insertion in the next section when we look into the five particle case. The fact that it is non-vanishing is an indication that it is non-trivial in enforcing dual symmetry for higher point numerators. We can associate the off-shell parts of (3.20)–(3.21)$\approx P_i^2$ with an operator insertion

$$\begin{aligned}&f_{bac}f_{b'a'c}\left[\left(\frac{1}{\Box}\frac{1}{\partial}\left(\partial a_b^-\frac{\Box}{\partial}a_a^+\right)\right)(\partial a_{b'}^+a_{a'}^-)-\left(\frac{1}{\Box}\frac{1}{\partial}(\partial a_b^-a_a^+)\right)\left(\partial a_{b'}^+\frac{\Box}{\partial}a_{a'}^-\right)\right]\\&\propto Tr\left(\frac{1}{\partial}\frac{1}{\Box}([a^-,\partial a^+])\left[\partial a^-,\frac{\Box}{\partial}a^+\right]\right),\end{aligned}\tag{3.24}$$

which generates the off-shell $\Delta(2^+|1^-3^+4^-)$ and others with $\Box=\partial^\mu\partial_\mu$.[11] We note that by adding this operator to the Lagrangian we can ensure that the four-points obey the BCJ duality even off-shell. However, we will refrain in the following from doing so, and instead use the four-point Δ's as building blocks for computing the five-point and higher violations of the BCJ duality discussed in detail in appendix B and appendix E.

For completeness, let us use these numerators to calculate (and check) one of the color ordered amplitudes. It helps to note that when on-shell

$$s_{ij}=-p_ip_j\left(\frac{p_i^+}{p_i}-\frac{p_j^+}{p_j}\right)\left(\frac{p_i^-}{p_i}-\frac{p_j^-}{p_j}\right),\tag{3.25}$$

and

$$\frac{p_a^+}{p_a}-\frac{p_b^+}{p_b}=\frac{\langle ab\rangle\langle +-\rangle}{\langle +a\rangle\langle +b\rangle},\quad \frac{p_a^-}{p_a}-\frac{p_b^-}{p_b}=\frac{[ab][-+]}{[-a][-b]},\tag{3.26}$$

Then some simple algebra gives

$$\begin{aligned}A(1^-2^+3^+4^-)&=\frac{n(1^-2^+;3^+4^-)}{s_{12}}-\frac{n(2^+3^+,1^-4^-)}{s_{14}}\\&=-\frac{p_1p_2p_3p_4}{s_{12}s_{14}}\left(\frac{p_2^-}{p_2}-\frac{p_3^-}{p_3}\right)^2\left(\frac{p_4^+}{p_4}-\frac{p_1^+}{p_1}\right)^2\epsilon_1^-\epsilon_2^+\epsilon_3^+\epsilon_4^-\\&=\frac{\langle 14\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\end{aligned}\tag{3.27}$$

a well-known result.

We did not make any choice of the reference vectors $|+\rangle[+|$ or $|-\rangle[-|$ up to this point in order to show generality. However, if the intention is to shorten a calculation, then some particular choices can be expeditious. For example, if we take $|1\rangle[1|\propto|+\rangle[+|$ and $|2\rangle[2|\propto|-\rangle[-|$, we have $\epsilon_1^-,\ \epsilon_2^+,\ p_1,\ p_2\to 0$. However

$$\epsilon_1^-\frac{p_1^+}{p_1}\to 1,\ \epsilon_2^+\frac{p_2^-}{p_2}\to 1,\tag{3.28}$$

and many terms can be dropped to give immediately

$$A(1^-2^+3^+4^-)=-\epsilon_3^+\epsilon_4^-\frac{p_3p_4}{s_{14}}=\frac{\langle 14\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}.\tag{3.29}$$

[11] We note in passing that the operator insertion (3.24) which insures that the off-shell four-point enjoys the color-kinematic duality can be obtained via the following field redefinition $a_b^-\to a_b^--\frac{1}{\partial}\left(f_{bcd}f_{deg}\partial a_c^-\frac{1}{\partial\Box}(a_e^-\partial a_g^+)\right)$, and its parity conjugate counterpart.

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4 Duality for five particles

For the five particle amplitudes following [12] we choose the Kleiss-Kuijf basis to be composed of $A(12345), A(14325), A(13425), A(12435), A(14235), A(13245)$. Each of these amplitudes has simple poles in the various kinematic invariants. There are fifteen numerators associated with these poles, owing to symmetries such as (2.14) and (2.15). Keeping the same notation as in [12] we denote six of them as follows:[12]

$$\begin{aligned} n_1 &= n(12;3;45), & n_{12} &= n(12;4;35), & n_{15} &= n(13;2;45), \\ n_9 &= n(13;4;25), & n_{14} &= n(14;2;35), & n_6 &= n(14;3;25). \end{aligned} \tag{4.1}$$

Then we incorporate Jacobi permutations of the last three indices to express

$$\begin{aligned} n(12;5;34) &= -n_1 + n_{12} + \Delta(12|345), \\ n(13;5;24) &= -n_{15} + n_9 + \Delta(13|245), \\ n(14;5;23) &= -n_{14} + n_6 + \Delta(14|352), \\ n(15;2;34) &= -n_1 + n_{12} + n_9 - n_6 + \Delta(12|345) + \Delta(25|134) + \Delta(34|125), \\ n(15;3;42) &= -n_{12} + n_{15} - n_9 + n_{14} - \Delta(35|124) - \Delta(24|135) - \Delta(13|245), \\ n(15;4;23) &= n_1 - n_{15} - n_{14} + n_6 + \Delta(14|352) + \Delta(45|123) + \Delta(23|145), \\ n(23;1;45) &= -n_1 + n_{15} - \Delta(45|123), \\ n(24;1;35) &= -n_{12} + n_{14} - \Delta(35|124), \\ n(25;1;34) &= n_9 - n_6 + \Delta(25|134). \end{aligned} \tag{4.2}$$

We will later give concrete expressions for the Δ's for the configuration $1^+2^-3^+4^-5^+$. Actually there is one extra equation which over-determines the quantities in (4.2). Thus, for consistency, one has to have

$$\begin{aligned} \Delta(13|245) = \; & \Delta(45|123) + \Delta(23|145) + \Delta(34|125) \\ & +\Delta(12|345) + \Delta(25|134) + \Delta(14|352) \\ & -\Delta(35|124) - \Delta(24|135) - \Delta(15|234), \end{aligned} \tag{4.3}$$

which will be checked.

Then color-kinematic duality statement is that there is a set of numerators $\bar{n}$'s, obeying Jacobi identity under cyclic permutation of three indices. The algebraic relation between

[12] If we use color-kinematics duality, these six numerators would be the independent set in terms of which all others are expressed. However, here we are concerned with a Lagrangian-based approach, and as we will see the numerators obtained via Feynman diagrams in a generic light-like gauge do not obey color-kinematics duality. We denote the violation of color-kinematic duality by Δ and we compute the specific Δ's. Only after modifying the numerators by δn shifts will the resulting numerators obey color-kinematic duality. Of course the shifts are required to leave the amplitudes unchanged, as we did in the previous section.

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the amplitudes and the BCJ numerators is

$$\begin{pmatrix} A(12345) \\ A(14325) \\ A(13425) \\ A(12435) \\ A(14235) \\ A(13245) \end{pmatrix} = M^{(5)} \begin{pmatrix} \bar{n}(12;3;45) \\ \bar{n}(14;3;25) \\ \bar{n}(13;4;25) \\ \bar{n}(12;4;35) \\ \bar{n}(14;2;35) \\ \bar{n}(13;2;45) \end{pmatrix} = M^{(5)} \begin{pmatrix} \bar{n}_1 \\ \bar{n}_6 \\ \bar{n}_9 \\ \bar{n}_{12} \\ \bar{n}_{14} \\ \bar{n}_{15} \end{pmatrix}, \tag{4.4}$$

where the propagator matrix $M^{(5)}$ is given by the following:

$$\begin{pmatrix}
\frac{1}{s_{12}s_{45}}+\frac{1}{s_{15}s_{34}}+\frac{1}{s_{23}s_{15}}+\frac{1}{s_{12}s_{34}}+\frac{1}{s_{23}s_{45}} & \frac{1}{s_{15}s_{34}}+\frac{1}{s_{23}s_{15}} & \frac{-1}{s_{15}s_{34}} & \frac{-1}{s_{15}s_{34}}+\frac{-1}{s_{12}s_{34}} & \frac{-1}{s_{23}s_{15}} & \frac{-1}{s_{23}s_{45}}+\frac{-1}{s_{23}s_{15}} \\
\frac{1}{s_{15}s_{34}}+\frac{1}{s_{15}s_{23}} & \frac{1}{s_{14}s_{25}}+\frac{1}{s_{14}s_{23}}+\frac{1}{s_{15}s_{23}}+\frac{1}{s_{15}s_{34}}+\frac{1}{s_{34}s_{25}} & \frac{-1}{s_{15}s_{34}}+\frac{-1}{s_{34}s_{25}} & \frac{-1}{s_{15}s_{34}} & \frac{-1}{s_{14}s_{23}}+\frac{-1}{s_{15}s_{23}} & \frac{-1}{s_{15}s_{23}} \\
\frac{-1}{s_{15}s_{34}} & \frac{-1}{s_{15}s_{34}}+\frac{-1}{s_{34}s_{25}} & \frac{1}{s_{13}s_{25}}+\frac{1}{s_{13}s_{24}}+\frac{1}{s_{15}s_{24}}+\frac{1}{s_{15}s_{34}}+\frac{1}{s_{34}s_{25}} & \frac{1}{s_{15}s_{24}}+\frac{1}{s_{15}s_{34}} & \frac{-1}{s_{15}s_{24}} & \frac{-1}{s_{13}s_{24}}+\frac{-1}{s_{15}s_{24}} \\
\frac{-1}{s_{12}s_{34}}+\frac{-1}{s_{15}s_{34}} & \frac{-1}{s_{15}s_{34}} & \frac{1}{s_{15}s_{34}}+\frac{1}{s_{15}s_{24}} & \frac{1}{s_{12}s_{35}}+\frac{1}{s_{12}s_{34}}+\frac{1}{s_{15}s_{34}}+\frac{1}{s_{15}s_{24}}+\frac{1}{s_{24}s_{35}} & \frac{-1}{s_{15}s_{24}}+\frac{-1}{s_{24}s_{35}} & \frac{-1}{s_{15}s_{24}} \\
\frac{-1}{s_{15}s_{23}} & \frac{-1}{s_{14}s_{23}}+\frac{-1}{s_{15}s_{23}} & \frac{-1}{s_{15}s_{24}} & \frac{-1}{s_{15}s_{24}}+\frac{-1}{s_{24}s_{35}} & \frac{1}{s_{14}s_{35}}+\frac{1}{s_{14}s_{23}}+\frac{1}{s_{15}s_{23}}+\frac{1}{s_{15}s_{24}}+\frac{1}{s_{24}s_{35}} & \frac{1}{s_{15}s_{23}}+\frac{1}{s_{15}s_{24}} \\
\frac{-1}{s_{23}s_{45}}+\frac{-1}{s_{23}s_{15}} & \frac{-1}{s_{15}s_{23}} & \frac{-1}{s_{13}s_{24}}+\frac{-1}{s_{15}s_{24}} & \frac{-1}{s_{15}s_{24}} & \frac{1}{s_{15}s_{23}}+\frac{1}{s_{15}s_{24}} & \frac{1}{s_{13}s_{45}}+\frac{1}{s_{13}s_{24}}+\frac{1}{s_{15}s_{24}}+\frac{1}{s_{15}s_{23}}+\frac{1}{s_{23}s_{45}}
\end{pmatrix}. \tag{4.5}$$

As in the four particle case, we achieve color-kinematic symmetry by adding δn to the Feynman rule determined numerators n

$$\bar{n}_i = n_i + \delta n_i, \tag{4.6}$$

such that $\bar{n}$'s have the required symmetry. The net result is that we have the same set of equation as in (4.2) with n_i's replaced by δn_i and with each term with a Δ gaining a minus sign. We now impose the requirement that the Feynman numerator shifts by δn_i

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must leave the color ordered amplitudes untouched. For example, we have

$$\begin{aligned} A(12345) = &\frac{n(12;3;45)}{s_{12}s_{45}} - \frac{n(12;5;34)}{s_{12}s_{34}} + \frac{n(23;4;51)}{s_{23}s_{51}} \\ &- \frac{n(23;1;45)}{s_{23}s_{45}} - \frac{n(34;2;51)}{s_{34}s_{51}}, \end{aligned} \tag{4.7}$$

which yields

$$\begin{aligned} &\frac{\delta n_1}{s_{12}s_{45}} + \frac{\delta n_1 - \delta n_{12} + \Delta(12|345)}{s_{12}s_{34}} - \frac{-\delta n_1 + \delta n_{15} + \Delta(45|123)}{s_{23}s_{45}} \\ &+ \frac{\delta n_1 - \delta n_{15} - \delta n_{14} + \delta n_6 - \Delta(14|352) - \Delta(45|123) - \Delta(23|145)}{s_{23}s_{15}} \\ &- \frac{-\delta n_1 + \delta n_{12} + \delta n_9 - \delta n_6 - \Delta(12|345) - \Delta(25|134) - \Delta(34|125)}{s_{34}s_{15}} = 0, \end{aligned} \tag{4.8}$$

or, collecting all Jacobi-violating Δ's into a single quantity

$$\begin{aligned} &\delta n_1 \left(\frac{1}{s_{12}s_{45}} + \frac{1}{s_{12}s_{34}} + \frac{1}{s_{23}s_{15}} + \frac{1}{s_{23}s_{45}} + \frac{1}{s_{34}s_{15}} \right) \\ &-\delta n_{12} \left(\frac{1}{s_{12}s_{34}} + \frac{1}{s_{34}s_{15}} \right) + \delta n_{15} \left(-\frac{1}{s_{23}s_{15}} - \frac{1}{s_{23}s_{45}} \right) \\ &-\delta n_9 \left(\frac{1}{s_{34}s_{15}} \right) + \delta n_{14} \left(-\frac{1}{s_{23}s_{15}} \right) - \delta n_6 \left(-\frac{1}{s_{23}s_{15}} - \frac{1}{s_{34}s_{15}} \right) = D\,(12345)\,, \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} D\,(12345) \equiv\; &\Delta\,(12|345) \left(-\frac{1}{s_{12}s_{34}} - \frac{1}{s_{34}s_{15}} \right) + \Delta\,(45|123) \left(\frac{1}{s_{23}s_{15}} + \frac{1}{s_{23}s_{45}} \right) \\ &+\Delta\,(14|352) \left(\frac{1}{s_{23}s_{15}} \right) + \Delta\,(23|145) \left(\frac{1}{s_{23}s_{15}} \right) \\ &+\Delta\,(25|134) \left(-\frac{1}{s_{34}s_{15}} \right) + \Delta\,(34|125) \left(-\frac{1}{s_{34}s_{15}} \right). \end{aligned} \tag{4.10}$$

In a similar fashion we obtain all the other D's corresponding to the amplitudes in our chosen Kleiss-Kuijf basis, and we list them in appendix C.

Succinctly, starting from the defining relation (4.4),

$$|A\rangle = M^{(5)}|\bar{N}\rangle, \tag{4.11}$$

where $|A\rangle$ denotes the set of Kleiss-Kuijf amplitudes and $|\bar{N}\rangle$ the set of BCJ numerators, we replace $\bar{n}$'s by $n + \delta n$'s. On the other hand, the Feynman numerators $\langle N| = (n_1, n_6, n_9, n_{12}, n_{14}, n_{15})$ satisfy

$$|A\rangle - |D\rangle = M^{(5)}|N\rangle, \tag{4.12}$$

where we collected the Jacobi-violating terms into a six-component vector $|D\rangle$. Then, the requirement for the shifts δn is that they should satisfy

$$|D\rangle = M^{(5)}|\delta N\rangle, \tag{4.13}$$

or, more explicitly,

$$\begin{pmatrix} D(12345) \\ D(14325) \\ D(13425) \\ D(12435) \\ D(14235) \\ D(13245) \end{pmatrix} = M^{(5)} \begin{pmatrix} \delta n_1 \\ \delta n_6 \\ \delta n_9 \\ \delta n_{12} \\ \delta n_{14} \\ \delta n_{15} \end{pmatrix}. \tag{4.14}$$

The solution for δn_i is not unique, because $M^{(5)}$ has four eigenvectors with zero eigenvalue. We gave a rather thorough discussion on this in [12] with regard to the origin of generalized gauge transformations. The effects are that we can determine only two linear combinations of δn_i, which are

$$\begin{aligned} \delta n' &= \delta n_1 - \delta n_9 \frac{s_{12}s_{45}}{s_{13}s_{24}} + \delta n_{12} \frac{s_{45}(s_{12}+s_{24})}{s_{24}s_{35}} \\ &\quad - \delta n_{14} \frac{s_{12}s_{45}}{s_{24}s_{35}} + \delta n_{15} \frac{s_{12}(s_{24}+s_{45})}{s_{13}s_{24}} \\ &= s_{12}(s_{25}D(13425) - (s_{15}+s_{25})D(12435)), \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} \delta n'' &= \delta n_6 + \delta n_9 \frac{s_{14}(s_{24}+s_{25})}{s_{13}s_{24}} - \delta n_{12} \frac{s_{14}s_{25}}{s_{24}s_{35}} \\ &\quad + \delta n_{14} \frac{s_{25}(s_{14}+s_{24})}{s_{24}s_{35}} - \delta n_{15} \frac{s_{14}s_{25}}{s_{13}s_{24}} \\ &= s_{25}(-(s_{12}+s_{15})D(13425) + s_{12}D(12435)). \end{aligned} \tag{4.16}$$

Another noteworthy remark is that they imply that there should be only two independent $D_i's$, which requires checking for consistency.

Using D's and Δ's in appendix C and appendix B, respectively, we find that

$$\delta n' = s_{12} \frac{s_{45}}{s_{24}} X, \quad \delta n'' = -s_{25} \frac{s_{14}}{s_{24}} X, \tag{4.17}$$

where

$$X = \frac{\bar{p_1}}{p_1}(p_{52} - p_{54}) + \frac{\bar{p_5}}{p_5}(p_{12} - p_{14}) - \frac{\bar{p_3}}{p_3}(p_{12} - p_{14} + p_{52} - p_{54}), \tag{4.18}$$

and

$$\begin{aligned} p_{12} &= \frac{p_1 p_2}{p_1 + p_4}, \quad p_{14} = \frac{p_1 p_4}{p_1 + p_2}, \\ p_{32} &= \frac{p_3 p_2}{p_3 + p_4}, \quad p_{34} = \frac{p_3 p_4}{p_3 + p_2}, \\ p_{52} &= \frac{p_5 p_2}{p_5 + p_4}, \quad p_{54} = \frac{p_5 p_4}{p_5 + p_2}. \end{aligned} \tag{4.19}$$

(We should attach a product of the five polarization tensors to X, which will be understood, because we are dealing with on-shell amplitudes at this point.)

At this point it seems that the numerator shifts are bound to contain a large degree of ambiguity, since we are only placing a constraint on $\delta n'$ (4.15) and on $\delta n''$ (4.16). However,

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this is not the case if we impose the additional condition that the numerator shifts should not introduce spurious poles. For example, this would require that

$$\begin{aligned}&\delta n_1 = s_{12}s_{45}a_1,\ \delta n_6 = s_{14}s_{25}a_6,\ \delta n_9 = s_{13}s_{25}a_9,\\&\delta n_{12} = s_{12}s_{35}a_{12},\ \delta n_{14} = s_{14}s_{35}a_{14},\ \delta n_{15} = s_{13}s_{45}a_{15},\end{aligned} \tag{4.20}$$

where a_1 should have at most simple poles in s_{12} and s_{45}, a_6 should have at most simple poles in s_{14} or s_{25} etc.

We take note that X is symmetric under $1 \leftrightarrow 5$, but antisymmetric under $2 \leftrightarrow 4$. Then (4.17) can be written as

$$s_{24}a_1 - s_{25}a_9 + (s_{12}+s_{24})a_{12} - s_{14}a_{14} + (s_{24}+s_{45})a_{15} = X, \tag{4.21}$$

and

$$s_{24}a_6 + (s_{24}+s_{25})a_9 - s_{12}a_{12} + (s_{14}+s_{24})a_{14} - s_{45}a_{15} = -X. \tag{4.22}$$

Actually (4.21) and (4.22) follow from each other, because under

$$1 \leftrightarrow 5:\quad a_1 \leftrightarrow -a_6,\ a_{12} \leftrightarrow -a_9,\ a_{15} \leftrightarrow -a_{14}, \tag{4.23}$$

and under

$$2 \leftrightarrow 4:\quad a_1 \leftrightarrow a_6,\ a_{15} \leftrightarrow a_9,\ a_{12} \leftrightarrow a_{14}. \tag{4.24}$$

When we add (4.21)–(4.22), we further obtain

$$a_1 + a_6 + a_9 + a_{12} + a_{14} + a_{15} = 0. \tag{4.25}$$

To solve for δn_i, or equivalently for the a_i separately, instead of just the combinations $\delta n'$ and $\delta n''$, we are guided by symmetry and by the requirement that a's must have at most simple poles in the allowed channels: e.g. a_1 can have at most simple poles in s_{12} and in s_{45} etc.

$$\begin{aligned}a_1 &= \frac{p_1^-}{p_1}\left(x_1^{32}p_{32} + x_1^{34}p_{34} + x_1^{52}p_{52} + x_1^{54}p_{54}\right)\\&\quad+\frac{p_3^-}{p_3}\left(y_1^{12}p_{12} + y_1^{14}p_{14} + y_1^{52}p_{52} + y_1^{54}p_{54}\right)\\&\quad+\frac{p_5^-}{p_5}\left(z_1^{12}p_{12} + z_1^{14}p_{14} + z_1^{32}p_{32} + z_1^{34}p_{34}\right),\end{aligned} \tag{4.26}$$

$$\begin{aligned}a_{15} &= \frac{p_1^-}{p_1}\left(x_{15}^{32}p_{32} + x_{15}^{34}p_{34} + x_{15}^{52}p_{52} + x_{15}^{54}p_{54}\right)\\&\quad+\frac{p_3^-}{p_3}\left(y_{15}^{12}p_{12} + y_{15}^{14}p_{14} + y_{15}^{52}p_{52} + y_{15}^{54}p_{54}\right)\\&\quad+\frac{p_5^-}{p_5}\left(z_{15}^{12}p_{12} + z_{15}^{14}p_{14} + z_{15}^{32}p_{32} + z_{15}^{34}p_{34}\right),\end{aligned} \tag{4.27}$$

where the x's, y's, and z's are functions of s_{ij}. By inspection, from (4.17) and (4.18) we infer that they are of the order $1/s$. We obtain the other a's through (4.23)–(4.24). After some straightforward but tedious algebra, recorded in appendix D, we obtain

$$\begin{aligned}
x_1^{32} &= \frac{1}{s_{12}} + \frac{\alpha}{s_{45}}, & x_1^{34} &= \frac{\beta}{s_{45}}, & x_1^{52} &= -\frac{1}{s_{12}} + \frac{\beta}{s_{45}}, & x_1^{54} &= \frac{\alpha}{s_{45}}, \\
y_1^{12} &= \alpha\left(\frac{1}{s_{12}} - \frac{1}{s_{45}}\right), & y_1^{14} &= \beta\left(\frac{1}{s_{12}} - \frac{1}{s_{45}}\right), & y_1^{52} &= \beta\left(\frac{1}{s_{12}} - \frac{1}{s_{45}}\right), & y_1^{54} &= \alpha\left(\frac{1}{s_{12}} - \frac{1}{s_{45}}\right), \\
z_1^{12} &= -\frac{\alpha}{s_{12}}, & z_1^{14} &= \frac{1}{s_{45}} - \frac{\beta}{s_{45}}, & z_1^{32} &= -\frac{\beta}{s_{12}}, & z_1^{34} &= -\frac{1}{s_{45}} - \frac{\alpha}{s_{12}},
\end{aligned} \tag{4.28}$$

$$\begin{aligned}
x_{15}^{32} &= \frac{1}{s_{13}} - \frac{\alpha}{s_{45}}, & x_{15}^{34} &= -\frac{1}{s_{13}} - \frac{\beta}{s_{45}}, & x_{15}^{52} &= -\frac{\beta}{s_{45}}, & x_{15}^{54} &= -\frac{\alpha}{s_{45}}, \\
y_{15}^{12} &= -\frac{1}{s_{13}} + \frac{\alpha}{s_{45}}, & y_{15}^{14} &= \frac{1}{s_{13}} + \frac{\beta}{s_{45}}, & y_{15}^{52} &= \frac{\beta}{s_{45}}, & y_{15}^{54} &= \frac{\alpha}{s_{45}}, \\
z_{15}^{12} &= \frac{\beta - \alpha}{s_{13}}, & z_{15}^{14} &= -\frac{1}{s_{45}} + \frac{\alpha - \beta}{s_{13}}, & z_{15}^{32} &= \frac{\alpha - \beta}{s_{13}}, & z_{15}^{34} &= \frac{\beta - \alpha}{s_{13}} + \frac{1}{s_{45}}.
\end{aligned} \tag{4.29}$$

Please be reminded that $\delta n_1 = s_{12}s_{45}a_1$ and $\delta n_{15} = s_{13}s_{45}a_{15}$. Thus, there is no spurious singularity in the forms of $\frac{1}{s_{12}}$, $\frac{1}{s_{13}}$ or $\frac{1}{s_{45}}$ in δn_1 or δn_{15}, nor is there any in other δn's. The numerator shifts are not uniquely determined, there is still some arbitrariness as parametrized by the constants α and β. This is due to the fact that we have the freedom of shifting the numerators using the zero-modes of the propagator matrix. This freedom was further restricted here by requiring that the shifts preserve the original pole structure of the Feynman-rules amplitude decomposition (1.1), leaving only the undetermined α and β.

Also, we would like to point out that if we choose $|+\rangle[+| \propto P_2$ or $|+\rangle[+| \propto P_4$, which makes $\epsilon_2^- = 0$ or $\epsilon_4^- = 0$, respectively, then the shifts $\delta n_i = 0$, or the numerators are already BCJ symmetric to begin with.[13]

[13] For MHV amplitudes, space-cone gauge with $|+\rangle[+| \propto P_i$, where i denotes an on-shell negative helicity gluon, yields BCJ numerators. The other external legs can be kept off-shell. The choice made such that one of negative helicity gluons is reference (and the space-cone gauge is defined relative to it) means that the vertices used to generate the MHV diagrams will be of type $(++-)$ and only one $(+--)$, where one of the negative helicity gluons participating in the only vertex $(+--)$ is our reference gluon. The quartic vertex $(++--)$ is zero provided that we make this choice. With this structure one can easily check that the numerators generated by Feynman rules are color-kinematic symmetric. See also [8]. There is a class of one-loop amplitudes, that of rational amplitudes, where this features extends, in the sense that the amplitude integrand obtained from Feynman rulles is manifestly color-kinematic symmetric. These are one-loop amplitudes which vanish at the tree level: the amplitudes with all external gluons of the same helicity, and those where all external gluons but one have the same helicity. They are special in the sense that they are all constructed from cubic vertices as follows. It is easy to see that in a light-like gauge, the all plus helicity one-loop amplitude is constructed only from $(++-)$ vertices. The one-loop all plus one negative helicity amplitude is built mostly from $(++-)$ vertices, with a single cubic $(--+)$ vertex attached to the negative helicity external line. The quartic vertex can be removed in a light-like gauge where the momentum of the negative helicity gluon is used in defining the gauge. This fact was noted and used earlier in [24]. More recently in [9] it was shown that the cubic vertex structure of the one-loop rational amplitudes yields BCJ dual expressions for the integrands.

5 Effective Lagrangian

Using the results of (4.28)–(4.29), we can derive all the

$$a(ij;k;lm) \equiv \frac{\delta n(ij;k;lm)}{s_{ij}s_{lm}}, \tag{5.1}$$

which appear naturally in the scattering amplitudes, with an effective Lagrangian

$$\begin{aligned}\mathcal{L}_5 = \Bigg[&-\frac{f_{c_1c_2d}}{s_{c_1c_2}}(f_{dc_3e}f_{ec_4c_5}+f_{dc_5e}f_{ec_3c_4}+f_{dc_4e}f_{ec_5c_3}) \\ &-\frac{f_{c_1c_5d}}{s_{c_1c_5}}(f_{dc_2e}f_{c_3c_4e}+f_{dc_3e}f_{c_4c_2e}+f_{dc_4e}f_{c_2c_3e}) \\ &-(1-\alpha+\beta)\frac{f_{c_2c_4d}}{s_{c_2c_4}}(f_{dc_1e}f_{c_3c_5e}+f_{dc_3e}f_{c_5c_1e}+f_{dc_5e}f_{c_1c_3e}) \\ &+\alpha\left(\frac{f_{c_5c_2d}}{s_{c_2c_5}}(f_{dc_1e}f_{ec_4c_3}+f_{dc_4e}f_{ec_3c_1}+f_{dc_3e}f_{ec_1c_4})+\frac{f_{c_3c_4d}}{s_{c_3c_4}}(f_{dc_1e}f_{ec_2c_5}+f_{dc_2e}f_{ec_5c_1}+f_{dc_5e}f_{ec_1c_2})\right) \\ &+\beta\left(\frac{f_{c_5c_4d}}{s_{c_4c_5}}(f_{dc_1e}f_{ec_2c_3}+f_{dc_2e}f_{ec_3c_1}+f_{dc_3e}f_{ec_1c_2})+\frac{f_{c_3c_2d}}{s_{c_3c_2}}(f_{dc_1e}f_{ec_4c_5}+f_{dc_4e}f_{ec_5c_1}+f_{dc_5e}f_{ec_1c_4})\right) \\ &+(\beta-\alpha)\frac{f_{c_5c_3d}}{s_{c_3c_5}}(f_{dc_2e}f_{ec_4c_1}+f_{dc_4e}f_{ec_1c_2}+f_{dc_1e}f_{ec_2c_4})\Bigg]\frac{\partial^-}{\partial}a^+_{c_1}\partial a^-_{c_2}\frac{1}{\partial}(\partial a^+_{c_5}a^-_{c_4})a^+_{c_3}+h.c.\end{aligned} \tag{5.2}$$

where α and β are arbitrary constants. We should note that in view of the Jacobi identity obeyed by the structure constants f_{abc} this effective Lagranigan is null, which is of course a succinct statement that the shifts we performed in the numerators have no effects on the physical amplitudes.

It is easy to check that this effective Lagrangian implements the desired shifts:

$$\begin{aligned}a_1 = a(1^+2^-;3^+;4^-5^+) = \frac{p_1^-}{p_1}&\left[\frac{1}{s_{12}}(p_{32}-p_{52})+\frac{1}{s_{45}}\Big(\alpha(p_{32}+p_{54})+\beta(p_{34}+p_{52})\Big)\right] \\ +\frac{p_3^-}{p_3}&\left[\frac{1}{s_{12}}\Big(\alpha(p_{12}+p_{54})+\beta(p_{14}+p_{52})\Big)-\frac{1}{s_{45}}\Big(\alpha(p_{12}+p_{54})+\beta(p_{14}+p_{52})\Big)\right] \\ +\frac{p_5^-}{p_5}&\left[-\frac{1}{s_{12}}\Big(\alpha(p_{12}+p_{34})+\beta(p_{14}+p_{32})\Big)+\frac{1}{s_{45}}(p_{14}-p_{34})\right]\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}a_{15} = a(1^+3^+;2^-;4^-5^+) = \frac{p_1^-}{p_1}&\left[\frac{1}{s_{13}}(p_{32}-p_{34})-\frac{1}{s_{45}}\Big(\alpha(p_{32}+p_{54})+\beta(p_{34}+p_{52})\Big)\right] \\ +\frac{p_3^-}{p_3}&\left[\frac{1}{s_{13}}(p_{14}-p_{12})+\frac{1}{s_{45}}\Big(\alpha(p_{12}+p_{54})+\beta(p_{14}+p_{52})\Big)\right] \\ +\frac{p_5^-}{p_5}&\left[\frac{\beta-\alpha}{s_{13}}(p_{12}-p_{14}-p_{32}+p_{34})+\frac{1}{s_{45}}(p_{34}-p_{14})\right]\end{aligned} \tag{5.4}$$

We will not write out all the $a(ij;k;lm)$ explicitly, as their particular form is not especially illuminating. Suffices it to say that they fall into three groups, according to their helicity arrangements:

$$(A) \qquad (\pm\mp;+;\pm\mp), \tag{5.5}$$

which consists of

$$a(12;5;34),\ a(12;3;45),\ a(14;5;23),\ a(14;3;52),\ a(23;1;45),\ a(25;1;34). \tag{5.6}$$

The $\alpha = \beta = 0$ contribution to the set of numerators (A) comes from the piece of $\mathcal{L}_5$ which is proportional to

$$(f_{dec_3}f_{c_4c_5e} + f_{dec_5}f_{c_3c_4e})f_{c_2c_1d}\frac{1}{s_{c_1c_2}} \tag{5.7}$$

$$(B) \qquad (\pm\mp;-;++),\quad (++;-;\pm\mp), \tag{5.8}$$

which consists of

$$a(12;4;53),\ a(13;2;45),\ a(13;4;52),\ a(14;2;35),\ a(15;2;34),\ a(15;4;23). \tag{5.9}$$

The $\alpha = \beta = 0$ contribution to this set of numerators comes from terms proportional to

$$f_{dec_4}f_{c_5c_3e}f_{c_2c_1d}\frac{1}{s_{c_1c_2}} + (f_{dec_2}f_{c_3c_4e} + f_{dec_4}f_{c_2c_3e})f_{c_5c_1d}\frac{1}{s_{c_1c_5}} \tag{5.10}$$

$$(C) \qquad (++;+;--),\quad (--;+;++), \tag{5.11}$$

which consists of

$$a(13;5;24),\ a(15;3;42),\ a(24;1;35). \tag{5.12}$$

Lastly, the $\alpha = \beta = 0$ contribution to the numerators of type (C) comes from terms proportional to

$$f_{dec_3}f_{c_4c_2e}f_{c_5c_1d}\frac{1}{s_{c_1c_5}} + (f_{dec_1}f_{c_3c_5e} + f_{dec_3}f_{c_5c_1e} + f_{dec_5}f_{c_1c_3e})f_{c_4c_2d}\frac{1}{s_{c_2c_4}}. \tag{5.13}$$

6 Concluding remarks

We would like to digress at this point and to explain how BCFW on-shell recursion [20] is performed in the space-cone gauge. We note that the Lagrangian in (2.11) has no $\bar{\partial}$ dependence in its interaction terms. Thus, analytical continuation is done by making shifts in some $\bar{p}$ direction with a complex number z (and if necessary by also choosing some appropriate reference vector $\pm$ so that $A(z) \to 0$ as $z \to \infty$.) Because the numerators have no $\bar{p}$ dependence, the continuation does not affect them and the poles of the amplitude are due to the vanishing of some inverse propagators. This polology makes it very transparent the meaning of cuts of the amplitudes in evaluating the integral $\int \frac{dz}{z}A(z)$. In other words, the cutting of the amplitude into two halves gives an easy organization to yield BCFW recurrence [21].

A question which can be asked is whether one can circumvent the Lagrangian approach and write down BCJ numerators from amplitudes. In particular, as noticed in [12], there is a set of BCJ numerators which can be obtained from knowledge of the amplitudes provided that we use the zero modes of the propagator matrix M fully to set to zero $(n-3)(n-3)!$ components of the BCJ numerators $\bar{N}$. Then the relation $|A\rangle = M|\bar{N}\rangle$ can be inverted. However, the numerators obtained in such fashion will generally contain spurious

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poles. In [17], with the same starting point, it was noticed that one can obtain 'virtuous" numerators by applying a certain symmetrization procedure. While these expression carry certain 'virtues' [15], there are also issues which demand attention. Consider the virtuous four-point numerator given by [15, 17]

$$\hat{n}(1^-2^+;3^+4^-) = \frac{1}{3}(s_{12}A(1^-2^+3^+4^-) - s_{14}A(1^-4^-2^+3^+)). \tag{6.1}$$

When we use

$$A(1^-4^-2^+3^+) = A(4^-2^+3^+1^-) = A(1^-2^+3^+4^-)|_{1\leftrightarrow 4} \tag{6.2}$$

and (3.27), we express it as

$$\hat{n}(1^-2^+;3^+4^-) = \frac{1}{3}p_1p_2p_3p_4\left(\frac{p_2^-}{p_2}-\frac{p_3^-}{p_3}\right)^2\left(\frac{p_4^+}{p_4}-\frac{p_1^+}{p_1}\right)^2\left(-\frac{1}{s_{14}}+\frac{1}{s_{24}}\right)\epsilon(1)^-\epsilon(2)^+\epsilon(3)^+\epsilon(4)^-. \tag{6.3}$$

We see that the numerators obtained in this way contain spurious poles at s_{14} and s_{24}. This defeats to some extent the purpose of decomposing the amplitude in the form (1.1), with the propagator poles manifestly written.

Color-kinematics duality allows for a particular version of the KLT relations, expressing the gravity amplitudes in terms of gauge theory amplitudes, with the key ingredient being the color-kinematic symmetric (BCJ) gauge theory numerators. However, the BCJ symmetry is not automatic, if one is to compute the numerators from a gauge Lagrangian. To summarize our results, what we have shown is that the violation of this symmetry can be systematically computed and absorbed into shifts of the Feynman numerators. These shifts do not change the color-ordered amplitudes. We specifically work in a light-like gauge, because it is physical and therefore makes the on-shell limit transparent. We have set up a set of equations for four and five particle cases, which are used to solve for the shifts in terms of the violations. In the four particle case, there is no need to make any shift on-shell. For the five particle case, we have obtained the general solution for shifts which are consistent with the acceptable pole structure. We have also constructed the null five-point Lagrangian which augments the light-like gauge fixed Lagrangian and which yields color-kinematic symmetric numerators. It is clear that this program should work for any number of particles and with an arbitrary choice of the light-like gauge fixing vector.

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A Four-point off-shell numerators

In this appendix, we calculate the Jacobi-permutation of three indices of four particle numerators.

It is useful to notice that for a tree-level amplitude, the net factor of 2 from the propagators $i/(2s_{ij})$ and from the vertices given in section 2 will cancel against the normalization factor of the group generators. [See footnote 6.] In what follows we decompose the amplitude as in (1.1) and (2.13). The factors of g^{n-2} and $(-i)$ which accompany a tree-level n-point amplitude are implicit. We will omit them in writing out the numerators of a color-ordered amplitude. With this observation, we have the following ingredients for the color-ordered amplitudes: the propagator is $1/P^2$, the color-ordered three-point vertex is

$$\begin{aligned}
&3\,\text{pt vertex}(1^-2^+k^-) \equiv (1^-2^+k^-) = p_2\left(\frac{k^+}{k} - \frac{p_1^+}{p_1}\right), \\
&3\,\text{pt vertex}(1^-2^+k^+) \equiv (1^-2^+k^+) = p_1\left(\frac{p_2^-}{p_2} - \frac{k^-}{k}\right),
\end{aligned} \tag{A.1}$$

and color-ordered four-point vertex is

$$\begin{aligned}
4\,\text{pt vertex}(1^-2^+;3^+4^-) &= -\frac{p_2p_4 + p_1p_3}{(p_1+p_2)^2} \qquad\qquad (A.2)\\
4\,\text{pt vertex}(4^-1^-;2^+3^+) &= 0 \\
4\,\text{pt vertex}(2^+1^-;3^+4^-) &= 4\,\text{pt vertex}(1^-2^+;4^-3^+) \\
&= -4\,\text{pt vertex}(1^-2^+;3^+4^-) = -4\,\text{pt vertex}(2^+1^-;4^-3^+).
\end{aligned}$$

Of course, the color-ordered 4 pt vertex(1234) is the sum of 4 pt vertex(12; 34) and 4 pt vertex(41; 23). The split we make is relevant only in assigning each contribution to a certain numerator: 4 pt vertex(12; 34) times the inverse propagator $-s_{12}$ contributes to $n(12;34)$ and 4 pt vertex(41; 23) times the inverse propagator $-s_{14}$ contributes to $n(41;23)$.

Schematically, we write

$$n(12;34) = (12k)(k34) + (12;34), \tag{A.3}$$

where

$$(12;34) \equiv 4\,\text{pt vertex}(12;34) \times (-s_{12}) \tag{A.4}$$

now includes the inverse propagator.[14] Thus, with the understanding that a product of the four polarization tensors is omitted and that the particles can then be off-shell, we have

$$\begin{aligned}
n\left(1^-2^+;3^+4^-\right) = &\left(\frac{p_2^-}{p_2} - \frac{p_1^- + p_2^-}{p_1+p_2}\right) p_1 \left(\frac{p_4^+}{p_4} - \frac{p_1^+ + p_2^+}{p_1+p_2}\right) p_3 \\
&+ \left(\frac{p_1^+ + p_2^+}{p_1+p_2} - \frac{p_1^+}{p_1}\right) p_2 \left(\frac{p_1^- + p_2^-}{p_1+p_2} - \frac{p_3^-}{p_3}\right) p_4 \\
&+ \frac{p_2p_4 + p_1p_3}{(p_1+p_2)^2}\frac{1}{2}\left(P_1+P_2\right)^2,
\end{aligned} \tag{A.5}$$

[14] To avoid cluttering the notation further we write $(12k)(k34)$ for the cubic vertex contribution even though we mean that in each vertex all momenta are incoming, and so this should really be written as $(12k)(-k34)$, with $P_k = -P_1 - P_2 = P_3 + P_4$. We hope that this is an obvious omission and will refrain from writing the sign of the momentum in the other cubic vertex.

where the last term is $(12;34)$, and as explained before it includes the $\frac{1}{2}(P_1+P_2)^2$ factor. We pick out one term each from the three lines above to form

$$\frac{p_2p_4+p_1p_3}{(p_1+p_2)^2}\left[(p_1^++p_2^+)(p_1^-+p_2^-)+\tfrac{1}{2}(P_1+P_2)^2\right]=\frac{p_2p_4+p_1p_3}{(p_1+p_2)^2}(p_1+p_2)(\bar{p}_1+\bar{p}_2). \quad \text{(A.6)}$$

Now we write

$$\bar{p}_1+\bar{p}_2=-\bar{p}_3-\bar{p}_4=-\left(\frac{p_3^+p_3^-}{p_3}+\frac{\frac{1}{2}P_3^2}{p_3}\right)-\left(\frac{p_4^+p_4^-}{p_4}+\frac{\frac{1}{2}P_4^2}{p_4}\right). \quad \text{(A.7)}$$

Putting (A.5), (A.6) into (A.7), we have

$$\begin{aligned}n(1^-2^+;3^+4^-)=p_1p_3\Bigg(&\frac{p_3^-}{p_1+p_2}\frac{p_4^+}{p_4}+\frac{p_2^-}{p_2}\frac{p_4^+}{p_4}+\frac{p_2^-}{p_2}\frac{p_3^++p_4^+}{p_1+p_2}\\&-\frac{1}{p_1+p_2}\frac{p_3^+p_3^-}{p_3}-\frac{1}{p_1+p_2}\frac{\frac{1}{2}P_3^2}{p_3}-\frac{1}{p_1+p_2}\frac{\frac{1}{2}P_4^2}{p_4}\Bigg)\\+p_2p_4\Bigg(&\frac{p_4^+}{p_1+p_2}\frac{p_3^-}{p_3}+\frac{p_1^+}{p_1}\frac{p_3^-}{p_3}+\frac{p_1^+}{p_1}\frac{p_3^-+p_4^-}{p_1+p_2}\\&-\frac{1}{p_1+p_2}\frac{p_4^+p_4^-}{p_4}-\frac{1}{p_1+p_2}\frac{\frac{1}{2}P_3^2}{p_3}-\frac{1}{p_1+p_2}\frac{\frac{1}{2}P_4^2}{p_4}\Bigg). \end{aligned}\quad \text{(A.8)}$$

It is useful to add and subtract $-(p_1p_3+p_2p_4)\left(\frac{p_2^-}{p_2}\frac{p_4^+}{p_4}-\frac{p_2^-}{p_2}\frac{p_1^+}{p_1}-\frac{p_3^-}{p_3}\frac{p_4^+}{p_4}+\frac{p_3^-}{p_3}\frac{p_1^+}{p_1}\right)$ to the expression above. Then we use

$$p_1p_3\left(\frac{p_3^-}{p_1+p_2}\frac{p_4^+}{p_4}+\frac{p_3^-}{p_3}\frac{p_4^+}{p_4}\right)+p_2p_4\left(\frac{p_4^+}{p_1+p_2}\frac{p_3^-}{p_3}+\frac{p_3^-}{p_3}\frac{p_4^+}{p_4}\right)=-p_3^-p_4^+, \quad \text{(A.9)}$$

and

$$\begin{aligned}&p_1p_3\left(\frac{p_2^-}{p_2}\frac{p_3^++p_4^+}{p_1+p_2}+\frac{p_2^-}{p_2}\frac{p_1^+}{p_1}\right)+p_2p_4\left(\frac{p_1^-}{p_1}\frac{p_3^-+p_4^-}{p_1+p_2}+\frac{p_2^-}{p_2}\frac{p_1^+}{p_1}\right)\\&=-\frac{p_1p_3}{p_1+p_2}\frac{p_2^+p_2^-}{p_2}-\frac{p_2p_4}{p_1+p_2}\frac{p_3^+p_3^-}{p_3}-p_1^+p_2^-,\end{aligned} \quad \text{(A.10)}$$

to obtain

$$\begin{aligned}n\left(1^-2^+;3^+4^-\right)&=-\frac{1}{p_1+p_2}\Bigg((p_1p_3+p_2p_4)\left(\frac{P_3^2}{2p_3}+\frac{P_4^2}{2p_4}\right)\\&\quad+p_1p_3\left(\frac{p_2^+p_2^-}{p_2}+\frac{p_3^+p_3^-}{p_3}\right)+p_2p_4\left(\frac{p_1^+p_1^-}{p_1}+\frac{p_4^+p_4^-}{p_4}\right)\Bigg)\\&\quad+(p_1p_3+p_2p_4)\left(\frac{p_2^-}{p_2}\frac{p_4^+}{p_4}-\frac{p_2^-}{p_2}\frac{p_1^+}{p_1}-\frac{p_3^-}{p_3}\frac{p_4^+}{p_4}-\frac{p_3^-}{p_3}\frac{p_1^+}{p_1}\right)\\&=s_{23}+\left(\frac{p_2^-}{p_2}-\frac{p_3^-}{p_3}\right)\left(\frac{p_4^+}{p_4}-\frac{p_1^+}{p_1}\right)(p_1p_3+p_2p_4)\\&\quad+\frac{p_1p_3}{2\,(p_1+p_2)}\left(-\frac{P_4^2}{p_4}+\frac{P_2^2}{p_2}\right)+\frac{p_2p_4}{2\,(p_1+p_2)}\left(-\frac{P_3^2}{p_3}+\frac{P_1^2}{p_1}\right).\end{aligned} \quad \text{(A.11)}$$

In a similar way, we obtain $n(3^+1^-;2^+4^-)$ and $n(2^+3^+;1^-4^-)$ given in (3.21)–(3.22).

B Five-point Δ's

In this appendix we calculate the Δ's for $1^+2^-3^+4^-5^+$. We put all the external particles on-shell. To shorten the expression of various terms we continue to omit the common factor of the product of the polarizations (i.e. the external line factors). Let us take one specific case and the others will be treated similarly. For $\Delta(1^+2^-|3^+4^-5^+)$, there are two sets of contributions. The first set is due to a four-vertex multiplied by a three vertex for each graph. The second set is due to a three vertex $(1^+, 2^-, -(1+2)^+)$ multiplied by the off-shell $\Delta((1+2)^-|3^+4^-5^+)$. For the first set, we have

$$\begin{aligned}
\Delta\left(1^+2^-|3^+4^-5^+\right)_1 &= \left(1^+2^-;3^+\left(4+5\right)^-\right)\left(-\left(4+5\right)^+4^-5^+\right)\\
&\quad+\left(1^+2^-;5^+\left(3+4\right)^-\right)\left(-\left(3+4\right)^+3^+4^-\right)\\
&\quad+\left(1^+2^-;4^+\left(5+3\right)^+\right)\left(-\left(5+3\right)^-5^+3^+\right)\\
&=\frac{s_{12}}{\left(p_1+p_2\right)^2}\left[\left(p_3p_2+p_1\left(p_4+p_5\right)\right)\left(\frac{p_5^-}{p_5}-\frac{p_4^-+p_5^-}{p_4+p_5}\right)p_4\right.\\
&\quad+\left(p_5p_2+p_1\left(p_3+p_4\right)\right)\left(\frac{p_3^-+p_4^-}{p_3+p_4}-\frac{p_3^-}{p_3}\right)p_4\\
&\quad\left.+\left(p_2\left(p_3+p_5\right)\right)+p_1p_4\right)\left(\frac{p_5^-}{p_5}-\frac{p_3^-}{p_3}\right)\left(p_3+p_5\right)\right],
\end{aligned}\tag{B.1}$$

which after some algebra is simplified to

$$\begin{aligned}
\Delta\left(1^+2^-|3^+4^-5^+\right)_1 &= \frac{s_{12}}{\left(p_1+p_2\right)}\left[-\frac{p_5^-}{p_5}\left(p_2p_3+p_1p_4\right)+\frac{p_3^-}{p_3}\left(p_1p_4+p_2p_5\right)\right.\\
&\quad\left.+\frac{p_3^-+p_4^-}{p_3+p_4}p_2p_3-\frac{p_4^-+p_5^-}{p_4+p_5}p_2p_5\right].
\end{aligned}\tag{B.2}$$

The contribution of the other set is

$$\begin{aligned}
\Delta\left(1^+2^-|3^+4^-5^+\right)_2 &= \left(1^+,2^-,-\left(1+2\right)^+\right)\Delta\left(\left(1+2\right)^-/3^+4^-5^+\right)\\
&= s_{12}\left(\frac{p_1^-+p_2^-}{p_1+p_2}-\frac{p_1^-}{p_1}\right)p_2p_4\left(\frac{1}{p_4+p_5}-\frac{1}{p_3+p_4}\right).
\end{aligned}\tag{B.3}$$

The sum of these two contributions gives

$$\begin{aligned}
\Delta\left(1^+2^-|3^+4^-5^+\right) &= s_{12}\left[\frac{p_1^-}{p_1}p_2p_4\left(-\frac{1}{p_4+p_5}+\frac{1}{p_3+p_4}\right)\right.\\
&\quad+\frac{p_3^-}{p_3}\left(\frac{p_1p_4}{p_1+p_2}-\frac{p_2p_5}{p_4+p_5}\right)\\
&\quad\left.+\frac{p_5^-}{p_5}\left(\frac{p_2p_3}{p_3+p_4}-\frac{p_1p_4}{p_1+p_2}\right)\right].
\end{aligned}\tag{B.4}$$

Please note that

$$\Delta\left(ij|klm\right)=-\Delta\left(klm|ij\right)=-\Delta\left(ji|klm\right)=-\Delta\left(ij|lkm\right)=-\Delta\left(mlk|ji\right),\tag{B.5}$$

and therefore, we have

$$\begin{aligned}\Delta\left(1^+2^-3^+|4^-5^+\right) &= -\Delta\left(5^+4^-|3^+2^-1^+\right) = -\Delta\left(1^+2^-|3^+4^-5^+\right)|_{1\leftrightarrow5,2\leftrightarrow4}\\ &= -s_{45}\left[\frac{p_1^-}{p_1}\left(\frac{p_3p_4}{p_2+p_3}-\frac{p_2p_5}{p_4+p_5}\right)\right.\\ &\quad+\frac{p_3^-}{p_3}\left(\frac{p_2p_5}{p_4+p_5}-\frac{p_1p_4}{p_1+p_2}\right)\\ &\quad\left.+\frac{p_5^-}{p_5}p_2p_4\left(-\frac{1}{p_1+p_2}+\frac{1}{p_2+p_3}\right)\right]. \end{aligned} \tag{B.6}$$

In a similar fashion, we obtain

$$\begin{aligned}\Delta\left(1^+4^-5^+|2^-3^+\right) &= \Delta\left(1^+2^-3^+|4^-5^+\right)|_{2\leftrightarrow4,1\leftrightarrow5}\\ &= -s_{23}\left[\frac{p_1^-}{p_1}\left(\frac{p_2p_5}{p_4+p_5}-\frac{p_3p_4}{p_2+p_3}\right)\right.\\ &\quad+\frac{p_5^-}{p_5}\left(\frac{p_3p_4}{p_2+p_3}-\frac{p_1p_2}{p_1+p_4}\right)\\ &\quad\left.+\frac{p_3^-}{p_3}p_2p_4\left(-\frac{1}{p_1+p_4}+\frac{1}{p_4+p_5}\right)\right], \end{aligned} \tag{B.7}$$

$$\begin{aligned}\Delta\left(1^+2^-5^+|3^+4^-\right) &= -\Delta\left(1^+2^-3^+|4^-5^+\right)|_{3\leftrightarrow5}\\ &= s_{34}\left[\frac{p_1^-}{p_1}\left(\frac{p_4p_5}{p_2+p_5}-\frac{p_2p_3}{p_3+p_4}\right)\right.\\ &\quad+\frac{p_5^-}{p_5}\left(\frac{p_2p_3}{p_3+p_4}-\frac{p_1p_4}{p_1+p_2}\right)\\ &\quad\left.+\frac{p_3^-}{p_3}p_2p_4\left(-\frac{1}{p_1+p_2}+\frac{1}{p_2+p_5}\right)\right], \end{aligned} \tag{B.8}$$

$$\begin{aligned}\Delta\left(3^+4^-5^+|1^+2^-\right) &= \Delta\left(1^+2^-5^+|3^+4^-\right)|_{2\leftrightarrow4,1\leftrightarrow3}\\ &= s_{12}\left[\frac{p_3^-}{p_3}\left(\frac{p_2p_5}{p_4+p_5}-\frac{p_1p_4}{p_1+p_2}\right)\right.\\ &\quad+\frac{p_5^-}{p_5}\left(\frac{p_1p_4}{p_1+p_2}-\frac{p_2p_3}{p_3+p_4}\right)\\ &\quad\left.+\frac{p_1^-}{p_1}p_2p_4\left(-\frac{1}{p_3+p_4}+\frac{1}{p_4+p_5}\right)\right], \end{aligned} \tag{B.9}$$

$$\begin{aligned}\Delta\left(1^+3^+4^-|2^-5^+\right) &= -\Delta\left(3^+4^-5^+|1^+2^-\right)|_{1\leftrightarrow5}\\ &= -s_{25}\left[\frac{p_3^-}{p_3}\left(\frac{p_1p_2}{p_1+p_4}-\frac{p_4p_5}{p_2+p_5}\right)\right.\\ &\quad+\frac{p_1^-}{p_1}\left(\frac{p_4p_5}{p_2+p_5}-\frac{p_2p_3}{p_3+p_4}\right)\\ &\quad\left.+\frac{p_5^-}{p_5}p_2p_4\left(-\frac{1}{p_3+p_4}+\frac{1}{p_1+p_4}\right)\right], \end{aligned} \tag{B.10}$$

$$\begin{aligned}
\Delta\left(2^-3^+5^+|1^+4^-\right) &= -\Delta\left(1^+2^-3^+|4^-5^+\right)|_{1\leftrightarrow 5} \\
&= -s_{14}\left[\frac{p_3^-}{p_3}\left(\frac{p_4p_5}{p_2+p_5}-\frac{p_1p_2}{p_1+p_4}\right)\right. \\
&\quad +\frac{p_5^-}{p_5}\left(\frac{p_1p_2}{p_1+p_4}-\frac{p_3p_4}{p_2+p_3}\right) \\
&\quad \left.+\frac{p_1^-}{p_1}p_2p_4\left(-\frac{1}{p_2+p_3}+\frac{1}{p_2+p_5}\right)\right], \qquad \text{(B.11)}
\end{aligned}$$

$$\Delta\left(1^+2^-4^-|3^+5^+\right) = -s_{35}\left[p_1\left(p_3+p_5\right)\left(\frac{p_3^-}{p_3}-\frac{p_5^-}{p_5}\right)\left(\frac{1}{p_1+p_4}-\frac{1}{p_1+p_2}\right)\right], \qquad \text{(B.12)}$$

$$\begin{aligned}
\Delta\left(2^-3^+4^-|1^+5^+\right) &= -\Delta\left(1^+2^-4^-|3^+5^+\right)|_{1\leftrightarrow 3} \\
&= s_{15}\left[p_3\left(p_1+p_5\right)\left(\frac{p_5^-}{p_5}-\frac{p_1^-}{p_1}\right)\left(\frac{1}{p_2+p_3}-\frac{1}{p_3+p_4}\right)\right], \qquad \text{(B.13)}
\end{aligned}$$

$$\begin{aligned}
\Delta\left(2^-4^-5^+|1^+3^+\right) &= -\Delta\left(1^+2^-4^-/3^+5^+\right)|_{1\leftrightarrow 5} \\
&= s_{13}\left[p_5\left(p_1+p_3\right)\left(\frac{p_3^-}{p_3}-\frac{p_1^-}{p_1}\right)\left(\frac{1}{p_4+p_5}-\frac{1}{p_2+p_5}\right)\right], \qquad \text{(B.14)}
\end{aligned}$$

and

$$\Delta\left(1^+3^+5^+|2^-4^-\right) = 0. \qquad \text{(B.15)}$$

When we add all the equations from (B.6) to (B.15), we find that (4.6) holds. This serves as a check on the algebra.

C Five-point D's

Following the procedure from (4.6) to (4.10), we arrive at the other D's:

$$\begin{aligned}
D\left(14325\right) &= \Delta\left(14|352\right)\left(\frac{1}{s_{14}s_{23}}+\frac{1}{s_{23}s_{15}}\right)+\Delta\left(12|345\right)\left(-\frac{1}{s_{34}s_{15}}\right) \\
&\quad +\Delta\left(25|134\right)\left(-\frac{1}{s_{34}s_{15}}-\frac{1}{s_{25}s_{34}}\right)+\Delta\left(34|125\right)\left(-\frac{1}{s_{34}s_{15}}\right) \\
&\quad +\Delta\left(45|123\right)\left(\frac{1}{s_{23}s_{15}}\right)+\Delta\left(23|145\right)\left(\frac{1}{s_{23}s_{15}}\right), \qquad \text{(C.1)}
\end{aligned}$$

$$\begin{aligned}
D\left(13425\right) &= \Delta\left(13|245\right)\left(\frac{1}{s_{13}s_{24}}+\frac{1}{s_{24}s_{15}}\right)+\Delta\left(12|345\right)\left(\frac{1}{s_{34}s_{15}}\right) \\
&\quad +\Delta\left(25|134\right)\left(\frac{1}{s_{34}s_{15}}+\frac{1}{s_{25}s_{34}}\right)+\Delta\left(34|125\right)\left(\frac{1}{s_{34}s_{15}}\right) \\
&\quad +\Delta\left(35|124\right)\left(\frac{1}{s_{24}s_{15}}\right)+\Delta\left(24/135\right)\left(\frac{1}{s_{24}s_{15}}\right), \qquad \text{(C.2)}
\end{aligned}$$

$$\begin{aligned}
D\left(12435\right) &= \Delta\left(12|345\right)\left(\frac{1}{s_{12}s_{34}}+\frac{1}{s_{34}s_{15}}\right)+\Delta\left(24|135\right)\left(\frac{1}{s_{24}s_{15}}\right) \\
&\quad +\Delta\left(35|124\right)\left(\frac{1}{s_{24}s_{15}}+\frac{1}{s_{24}s_{35}}\right)+\Delta\left(13|245\right)\left(\frac{1}{s_{24}s_{15}}\right) \\
&\quad +\Delta\left(34|125\right)\left(\frac{1}{s_{34}s_{15}}\right)+\Delta\left(25|134\right)\left(\frac{1}{s_{34}s_{15}}\right), \qquad \text{(C.3)}
\end{aligned}$$

$$
\begin{aligned}
D\,(14235) =&\ \Delta\,(14|352)\left(-\frac{1}{s_{14}s_{23}}-\frac{1}{s_{23}s_{15}}\right)+\Delta\,(24|135)\left(-\frac{1}{s_{24}s_{15}}\right)\\
&+\Delta\,(35|124)\left(-\frac{1}{s_{24}s_{15}}-\frac{1}{s_{24}s_{35}}\right)+\Delta\,(13|245)\left(-\frac{1}{s_{24}s_{15}}\right)\\
&+\Delta\,(45|123)\left(-\frac{1}{s_{23}s_{15}}\right)+\Delta\,(23|145)\left(-\frac{1}{s_{23}s_{15}}\right), \qquad \text{(C.4)}\\
D\,(13245) =&\ \Delta\,(13|245)\left(-\frac{1}{s_{13}s_{24}}-\frac{1}{s_{24}s_{15}}\right)+\Delta\,(14|352)\left(-\frac{1}{s_{23}s_{15}}\right)\\
&+\Delta\,(45|123)\left(-\frac{1}{s_{23}s_{15}}-\frac{1}{s_{23}s_{45}}\right)+\Delta\,(23|145)\left(-\frac{1}{s_{23}s_{15}}\right)\\
&+\Delta\,(35|124)\left(-\frac{1}{s_{24}s_{15}}\right)+\Delta\,(24|135)\left(-\frac{1}{s_{24}s_{15}}\right). \qquad \text{(C.5)}
\end{aligned}
$$

D Solving for the five-point numerator shifts

In this appendix we give the details of the steps taken to arrive at the solution given in the main text for the numerator shifts. First we notice that because of

$$
a_6 = a_1(2 \leftrightarrow 4) = -a_1(1 \leftrightarrow 5), \tag{D.1}
$$

we have

$$
\begin{aligned}
z_1^{12}(1\leftrightarrow 5) &= -x_1^{54}(2\leftrightarrow 4), z_1^{14}(1\leftrightarrow 5) = -x_1^{52}(2\leftrightarrow 4),\\
z_1^{32}(1\leftrightarrow 5) &= -x_1^{34}(2\leftrightarrow 4), z_1^{34}(1\leftrightarrow 5) = -x_1^{32}(2\leftrightarrow 4),\\
y_1^{12}(1\leftrightarrow 5) &= -y_1^{54}(2\leftrightarrow 4), y_1^{14}(1\leftrightarrow 5) = -y_1^{52}(2\leftrightarrow 4).
\end{aligned} \tag{D.2}
$$

Instead of using (4.21) or (4.22) to normalize the x, y, z's we use instead equivalently[15]

$$
\delta n_1 - \delta n_3 - \delta n_{12} = -\Delta(12|345), \tag{D.3}
$$

where

$$
\delta n_3 = \delta n(12;5;43) = \delta n_1(3 \leftrightarrow 5). \tag{D.4}
$$

We now use (4.25) to obtain four independent equations:

$$
x_1^{32} + x_1^{34}(2\leftrightarrow 4) + x_{15}^{32} + x_{15}^{34}(2\leftrightarrow 4) - z_{15}^{34}(2\leftrightarrow 4; 1\leftrightarrow 5) - z_{15}^{32}(1\leftrightarrow 5) = 0, \tag{D.5}
$$

$$
x_1^{52} + x_1^{54}(2\leftrightarrow 4) + x_{15}^{52} + x_{15}^{54}(2\leftrightarrow 4) - z_{15}^{14}(2\leftrightarrow 4; 1\leftrightarrow 5) - z_{15}^{12}(1\leftrightarrow 5) = 0, \tag{D.6}
$$

$$
y_1^{12} + y_1^{14}(2\leftrightarrow 4) + y_{15}^{12} + y_{15}^{14}(2\leftrightarrow 4) - y_{15}^{54}(2\leftrightarrow 4; 1\leftrightarrow 5) - y_{15}^{52}(1\leftrightarrow 5) = 0, \tag{D.7}
$$

$$
y_1^{52} + y_1^{54}(2\leftrightarrow 4) + y_{15}^{52} + y_{15}^{54}(2\leftrightarrow 4) - y_{15}^{14}(2\leftrightarrow 4; 1\leftrightarrow 5) - y_{15}^{12}(1\leftrightarrow 5) = 0. \tag{D.8}
$$

[15]We are using here notation introduced earlier in eq. (3.1) in [12].

By equating coefficients multiplied to different $\frac{p_i^-}{p_i} p_{jk}$ from (D.3), we obtain a set of twelve equations:

$$-s_{34}x_1^{52}(3\leftrightarrow 5) + s_{45}x_1^{32} + s_{35}z_{15}^{34}(2\leftrightarrow 4;1\leftrightarrow 5) = 1,, \tag{D.9}$$
$$-s_{34}x_1^{54}(3\leftrightarrow 5) + s_{45}x_1^{34} + s_{35}z_{15}^{32}(2\leftrightarrow 4;1\leftrightarrow 5) = 0, \tag{D.10}$$
$$-s_{34}x_1^{32}(3\leftrightarrow 5) + s_{45}x_1^{52} + s_{35}z_{15}^{14}(2\leftrightarrow 4;1\leftrightarrow 5) = -1, \tag{D.11}$$
$$-s_{34}x_1^{34}(3\leftrightarrow 5) + s_{45}x_1^{54} + s_{35}z_{15}^{12}(2\leftrightarrow 4;1\leftrightarrow 5) = 0; \tag{D.12}$$
$$s_{34}x_1^{54}(2\leftrightarrow 4;1\to 3\to 5\to 1) + s_{45}y_1^{12} + s_{35}y_{15}^{54}(2\leftrightarrow 4;1\leftrightarrow 5) = 0, \tag{D.13}$$
$$s_{34}x_1^{52}(2\leftrightarrow 4;1\to 3\to 5\to 1) + s_{45}y_1^{14} + s_{35}y_{15}^{52}(2\leftrightarrow 4;1\leftrightarrow 5) = -1, \tag{D.14}$$
$$s_{34}x_1^{34}(2\leftrightarrow 4;1\to 3\to 5\to 1) + s_{45}y_1^{52} + s_{35}y_{15}^{14}(2\leftrightarrow 4;1\leftrightarrow 5) = 1, \tag{D.15}$$
$$s_{34}x_1^{32}(2\leftrightarrow 4;1\to 3\to 5\to 1) + s_{45}y_1^{54} + s_{35}y_{15}^{12}(2\leftrightarrow 4;1\leftrightarrow 5) = 0; \tag{D.16}$$
$$-s_{34}y_1^{12}(3\leftrightarrow 5) + s_{45}z_1^{12} + s_{35}x_{15}^{54}(2\leftrightarrow 4;1\leftrightarrow 5) = 0, \tag{D.17}$$
$$-s_{34}y_1^{14}(3\leftrightarrow 5) + s_{45}z_1^{14} + s_{35}x_{15}^{52}(2\leftrightarrow 4;1\leftrightarrow 5) = 1, \tag{D.18}$$
$$-s_{34}y_1^{52}(3\leftrightarrow 5) + s_{45}z_1^{32} + s_{35}x_{15}^{34}(2\leftrightarrow 4;1\leftrightarrow 5) = -1, \tag{D.19}$$
$$-s_{34}y_1^{54}(3\leftrightarrow 5) + s_{45}z_1^{34} + s_{35}x_{15}^{32}(2\leftrightarrow 4;1\leftrightarrow 5) = 0. \tag{D.20}$$

When we make $3\leftrightarrow 5$ to (D.11), we obtain

$$-s_{45}x_1^{32} + s_{34}x_1^{52} + s_{35}z_{15}^{14}(2\leftrightarrow 4;1\to 3\to 5\to 1) = -1, \tag{D.21}$$

which is added to (D.9) to give

$$z_{15}^{32}(1\leftrightarrow 5) + z_{15}^{12}(1\to 3\to 5\to 1) = 0.. \tag{D.22}$$

In a similar fashion, we obtain from (D.10) and (D.12)

$$z_{15}^{34}(1\leftrightarrow 5) + z_{15}^{14}(1\to 3\to 5\to 1) = 0.. \tag{D.23}$$

If we use the results above, then we should of course keep only one of (D.9) and (D.11) and one of (D.10) and (D.12).

Using $z_1^{12} = -x_1^{54}(2\leftrightarrow 4;1\leftrightarrow 5)$ and making $3\leftrightarrow 5$, we write (D.17) as

$$-s_{45}y_1^{12} - s_{34}x_1^{54}(2\leftrightarrow 4;1\to 3\to 5\to 1) + s_{35}x_{15}^{54}(2\leftrightarrow 4;1\to 3\to 5\to 1) = 0. \tag{D.24}$$

When we combine this with (D.13). we have

$$x_{15}^{54}(1\leftrightarrow 3) + y_{15}^{54} = 0. \tag{D.25}$$

The same operations will lead to

$$x_{15}^{52}(1\leftrightarrow 3) + y_{15}^{52} = 0. \tag{D.26}$$
$$x_{15}^{34}(1\leftrightarrow 3) + y_{15}^{14} = 0. \tag{D.27}$$
$$x_{15}^{32}(1\leftrightarrow 3) + y_{15}^{32} = 0. \tag{D.28}$$

We should then keep either the set (D.13) to (D.16) or the set (D.17)–(D.20). Therefore we have only six of equations (D.9) to (D.20) and the four of equations (D.5) to (D.8), which add up to ten. Taking into account (D.2) and (D.22) to (D.28), we have twelve independent equations for x's, y's and z's. We noted earlier that these coefficients have dimension $1/s$. We will be solving for them with the requirement that they are of the form of a sum of terms each being a simple pole in the allowed kinematic invariant (such that the numerator shifts do not introduce spurious poles). This leads to the solution given in (4.28) and (4.29).

E Beyond five-point

In this appendix we discuss how one can extend recursively the current results beyond five-points. Consider the six-point case. We begin by choosing a Kleiss-Kuijf basis as in [12]: $A(1i_2i_3i_4i_56)$ with (i_2, i_3, i_4, i_5) equal to a permutation of indices (2,3,4,5). We use the shorthand notation[16]

$$\begin{array}{llll}
n(12;3;4;56) = n_1, & n(13;2;4;56) = n_2, & n(13;4;2;56) = n_3, & n(13;4;5;26) = n_4 \\
n(12;4;3;56) = n_5, & n(14;2;3;56) = n_6, & n(14;3;2;56) = n_7, & n(14;3;5;26) = n_8 \\
n(12;5;4;36) = n_9, & n(15;2;4;36) = n_{10}, & n(15;4;2;36) = n_{11}, & n(15;4;3;26) = n_{12} \\
n(12;3;5;46) = n_{13}, & n(13;2;5;46) = n_{14}, & n(13;5;2;46) = n_{15}, & n(13;5;4;26) = n_{16} \\
n(12;4;5;36) = n_{17}, & n(14;2;5;36) = n_{18}, & n(14;5;2;36) = n_{19}, & n(14;5;3;26) = n_{20} \\
n(12;5;3;46) = n_{21}, & n(15;2;3;46) = n_{22}, & n(15;3;2;46) = n_{23}, & n(15;3;4;26) = n_{24}.
\end{array} \tag{E.1}$$

For each color-ordered amplitude we decompose into terms which display the propagator pole structure as in eq. (A.2) in [12]. For example,

$$\begin{aligned}
A(123456) = & \frac{n_1}{s_{12}s_{123}s_{1234}} - \frac{n(12;3;6;45)}{s_{12}s_{123}s_{1236}} - \frac{n(12;6;3;45)}{s_{12}s_{126}s_{1236}} \\
& + \frac{n(61;2;3;45)}{s_{16}s_{126}s_{1236}} + \frac{n(12;6;5;34)}{s_{12}s_{126}s_{1256}} + \frac{n(23;4;5;61)}{s_{23}s_{234}s_{2345}} \\
& - \frac{n(23;4;1;56)}{s_{23}s_{234}s_{1234}} - \frac{n(23;1;4;56)}{s_{23}s_{123}s_{1234}} + \frac{n(34;2;1;56)}{s_{34}s_{234}s_{1234}} \\
& - \frac{n(34;5;2;61)}{s_{34}s_{345}s_{2345}} - \frac{n(34;2;5;61)}{s_{34}s_{234}s_{2345}} + \frac{n(23;1;6;45)}{s_{23}s_{123}s_{1236}} \\
& + \frac{n(12;34;56)}{s_{12}s_{34}s_{56}} + \frac{n(61;23;45)}{s_{16}s_{23}s_{45}}.
\end{aligned} \tag{E.2}$$

However, the Feynman-rules numerators will not satisfy the BCJ relations (as opposed to the numerators in eq. (A.5) of [12]). Instead, there will be violations which we parametrized

[16] This type of numerators has been later called half-ladder in [15]. The reason is that the external legs are all arranged along an internal line with two external legs joined together only at the two ends of that internal line.

as in appendix B by Δ's. These can be constructed as follows. For concreteness let us focus on

$$n(12;3;4;56)+n(12;3;6;45)+n(12;3;5;64)=\Delta(12;3|456), \tag{E.3}$$

where

$$\Delta(12;3|456) = (12k)\Delta(k3|456)+(12;3k)\Delta(k|456). \tag{E.4}$$

and as before $(12k)$ denotes a three-point vertex and $(12;3k)$ denotes a four-point vertex.[17] The off-shell five-point Δ's are given by the corresponding version of (B.4) plus off-shell terms. For example $\Delta(1^+2^-|3^+4^-5^+)$, where all legs are taken to be off-shell, has the following off-shell pieces (representing the contributions of the off-shell four-point Δ to[18] $(1^+2^-k^+)\Delta(k^-|3^+4^-5^+)$) in addition to (B.4):

$$\begin{aligned}&\left[\frac{p_3(p_1+p_2)}{p_3+p_4}\left(-\frac{P_5^2}{2p_5}+\frac{P_4^2}{2p_4}\right)+\frac{p_4p_5}{p_4+p_3}\left(-\frac{(P_1+P_2)^2}{2(p_1+p_2)}+\frac{P_3^2}{2p_3}\right)\right.\\&\left.+\frac{p_5(p_1+p_2)}{p_4+p_5}\left(-\frac{P_3^2}{2p_3}+\frac{P_4^2}{2p_4}\right)-\frac{p_4p_3}{p_4+p_5}\left(-\frac{(P_1+P_2)^2}{2(p_1+p_2)}+\frac{P_5^2}{2p_5}\right)\right]\\&\times\left(\frac{p_1^-+p_2^-}{p_1+p_2}-\frac{p_1^-}{p_1}\right)p_2.\end{aligned} \tag{E.5}$$

Consider $n(12;6;3;45)$ as obtained by Feynman rules. We can write this as $(12k)n(k6;3;45)+(12;6k)n(k3;45)$. The second term is necessary since it is a contribution from the 4-point vertex $(12;6k)$ which is not included in the first term where a cubic vertex is affixed to the off-shell 5-point numerator.

Next we use that

$$n(k6;3;45)+n(3k;6;45)+n(63;k;45)=\Delta(k63|45) \tag{E.6}$$

where this is the off-shell 5-point Δ described earlier in this section.

Then

$$n(3k;6;45)=n(k3;4;56)+n(k3;5;64)+\Delta(3k|645), \tag{E.7}$$

while

$$n(63;k;45)=n(36;4;5k)+n(36;5;k4)+\Delta(63|k45). \tag{E.8}$$

[17] Recall that according to the Feynman rules the four-point vertex contribution $(12;3k)$ is non-zero only when the gluons in each pair (12) and $(3k)$ have opposite helicities, and that $(12;3k)$ is proportional to s_{12}.

[18] Recall that $\Delta(k^+|3^+4^-5^+)=0$ and that $\Delta(k|345)=\Delta(543|k)=\Delta(435|k)$.

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Putting everything together,

$$\begin{aligned}
n(12;6;3;45) &= (12k)n(k6;3;45)+(12;6k)n(k3;45)\\
&= (12k)[\Delta(k63|45)+\Delta(k3|645)-\Delta(63|k45)]+(12;6k)n(k3;45)\\
&\quad -(12k)n(k3;4;56)-[(12;3k)n(k456)-(12;3k)n(k4;56)]\\
&\quad +(12k)n(k3;5;46)+[(12;3k)n(k5;46)-(12;3k)n(k5;46)]\\
&\quad +(12k)n(k5;4;63)+[(12;5k)n(k4;63)-(12;5k)n(k4;63)]\\
&\quad +(12k)n(k4;5;36)+[(12;4k)n(k5;36)-(12;4k)n(k5;36)]\\
&= (12k)[\Delta(k63|45)+\Delta(k3|645)-\Delta(63|k45)]+(12;6k)n(k3;45)\\
&\quad -n(12;3;4;56)+n(12;3;5;46)-n(12;5;4;36)+n(12;4;5;36)\\
&\quad +(12;3k)n(k4;56)-(12;3k)n(k5;46)+(12;5k)n(k4;36)-(12;4k)n(k5;36)\\
&= n(12;3;4;56)+n(12;3;5;46)-n(12;5;4;36)+n(12;4;5;36)\\
&\quad +\Delta(12;3|645)\\
&\quad +(12k)\Delta(36|k45)+(12;6k)n(k3;45)-(12;3k)n(k6;45)\\
&\quad +(12k)\Delta(45|36k)+(12;5k)n(k4;36)-(12;4k)n(k5;36). \qquad \text{(E.9)}
\end{aligned}$$

The following numerators can be expressed in this way and obtained by relabelling of external legs:

$$n(12;6;5;34) \text{ from } n(12;6;3;45) \text{ with } (3\to 5, 4\to 3, 5\to 4) \qquad \text{(E.10)}$$

$$n(23;1;4;56)=n(65;4;1;32) \text{ from } n(12;6;3;45) \text{ with } (1\leftrightarrow 6, 5\leftrightarrow 2, 4\leftrightarrow 3) \qquad \text{(E.11)}$$

$$n(34;2;1;56)=n(65;1;2;43) \text{ from } n(12;6;3;45) \text{ with } (6\leftrightarrow 1, 5\to 2, 2\to 3, 3\to 5). \qquad \text{(E.12)}$$

Yet another type of terms is $n(23;1;6;45)$. We write it as $n(23;1k)n(k6;45)+(23k)(k1;6l)(l45)$ and manipulate it such that we express it in terms of the chosen basis of numerators plus violating terms.

$$\begin{aligned}
n(23;1;6;45) &= (-n(12;3k)-n(31;2k)+\Delta(123|k))(-n(k4;56)-n(k5;64)+\Delta(k|645))\\
&\quad +(23k)(k1;6l)(l45)\\
&= n(12;3;4;56)-n(12;3;5;46)-n(13;2;4;56)+n(13;2;5;46)\\
&\quad -\Delta(231|k)[n(k4;56)+n(k5;64)]-\Delta(k|645)[n(12;3k)+n(31;2k)]\\
&\quad -(12k)[(k3;4l)(l56)+(k3;5l)(l64)]+(13k)[(k2;4l)(l56)+(k2;5l)(l64)]\\
&\quad +(23k)(k1;6l)(l45). \qquad \text{(E.13)}
\end{aligned}$$

Then we have the snowflake $n(12;34;56)$. This can be expressed as $-(12k)n(34;k;56)+(34;k)(kl;12)(l56)+(12k)(34;lk)(l56)+(12k)(kl;56)(l34)$, so

$$\begin{aligned}
n(12;34;56) &= -(12k)\Delta(34k|56)+(12k)n(4k;3;56)+(12k)n(k3;4;56)\\
&\quad +(34;k)(kl;12)(l56)+(12k)(34;lk)(l56)+(12k)(kl;56)(l34)\\
&= -(12k)\Delta(34k|56)-n(12;4;3;56)+n(12;3;4;56)\\
&\quad +(12k)(k4;3l)(l56)-(12k)(k4;3l)(l56)\\
&\quad +(34;k)(kl;12)(l56)+(12k)(34;lk)(l56)+(12k)(kl;56)(l34). \qquad \text{(E.14)}
\end{aligned}$$

The more complicated numerators have an s_{61} associated pole. Let's consider $n(61;2;3;45)$. We can write it as $n(61;2k)n(k3;45)+(61k)(k2;3l)(l45)$, which gives

$$\begin{aligned}
n(61;2;3;45) &= [-n(12;6k)-n(26;1k)+\Delta(162|k)]n(k3;45)+(61k)(k2;3l)(l45)\\
&= -n(12;6;3;45)+n(62;1;3;45)+\Delta(162|k)n(k3;45)\\
&\quad +(61k)(k2;3l)(l45)+(12k)(k6;3l)(l45)+(26k)(k1;3l)](l45), \qquad \text{(E.15)}
\end{aligned}$$

then each of the numerators $n(12;6;3;45)$ and $n(26;1;3;45)$ receives the same treatment as before. For the final expression,

$$\begin{aligned}
n(61;2;3;45) &= n(12;3;4;56)-n(12;3;5;46)+n(12;5;4;36)-n(12;4;5;36)\\
&\quad -n(62;3;4;51)+n(62;3;5;41)-n(62;5;4;31)+n(62;4;5;31)\\
&\quad -\Delta(12;3|645)\\
&\quad -(12k)\Delta(36|k45)-(12;6k)n(k3;45)-(12;k3)n(k6;45)\\
&\quad -(12k)\Delta(45|36k)-(12;5k)n(k4;36)-(12;k4)n(k5;36)\\
&\quad +\Delta(15;4|632)\\
&\quad +(15k)\Delta(46|k32)+(15;6k)n(k4;32)-(15;k4)n(k6;32)\\
&\quad +(15k)\Delta(32|46k)+(15;2k)n(k3;46)-(15;k3)n(k2;46)\\
&\quad +\Delta(162|k)n(k3;45)\\
&\quad +(61k)(k2;3l)(l45)+(12k)(k6;3l)(l45)+(26k)(k1;3l)(l45). \qquad \text{(E.16)}
\end{aligned}$$

The other numerators in the same family are obtained as follows

$$\begin{aligned}
n(23;4;5;61) &= n(16;5;4;32) = -n(61;5;4;32) \text{ from } -n(61;2;3;45) \text{ with } (2\leftrightarrow 5, 3\leftrightarrow 4)\\
n(34;5;2;61) &= -n(61;2;5;43) \text{ from } -n(61;2;3;45) \text{ with } (3\leftrightarrow 5)\\
n(34;2;5;61) &= -n(61;5;2;43) \text{ from } -n(61;2;3;45) \text{ with } (2\to 5, 3\to 2, 5\to 3). \qquad \text{(E.17)}
\end{aligned}$$

Lastly, we have the snowflake with a s_{61} inverse propagator. We write it as

$$n(61;23;45) = n(61kl)(k23)(l45)+(16k)(23;kl)(l45)+(16k)(45;kl)(l23), \qquad \text{(E.18)}$$

to obtain

$$\begin{aligned}
n(61;23;45) &= -(23k)n(k1;6l)(l45)+(23k)n(k6;1l)(l45)\\
&\quad +\Delta(61k|l)(k23)(l45)+(16k)(23;kl)(l45)+(16k)(45;kl)(l23)\\
&= -n(23;1;6;45)+n(23;6;1;45)\\
&\quad +(23;1k)n(k6;45)+n(23;1k)(k6;45)-(23;6k)n(k1;45)-n(23;6k)(k1;45)\\
&\quad +\Delta(61k|l)(k23)(l45)+(16k)(23;kl)(l45)+(16k)(45;kl)(l23). \qquad \text{(E.19)}
\end{aligned}$$

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The numerators $n(23;1;6;45)$ and $n(23;6;1;45)$ have been discussed before, leading to the following expression for the 61-snowflake:

$$
\begin{aligned}
n(61;23;45) = -\Big(& n(12;3;4;56) - n(12;3;5;46) - n(13;2;4;56) + n(13;2;5;46) \\
& -\Delta(231|k)[n(k4;56) + n(k5;64)] - \Delta(k|645)[n(12;3k) + n(31;2k)] \\
& -(12k)[(k3;4l)(l56) + (k3;5l)(l64)] + (13k)[(k2;4l)(l56) + (k2;5l)(l64)] \\
& +(23k)(k1;6l)(l45)\Big) \\
& +\Big(n(62;3;4;51) - n(62;3;5;41) - n(63;2;4;51) + n(63;2;5;41) \\
& -\Delta(236|k)[n(k4;51) + n(k5;64)] - \Delta(k|145)[n(62;3k) + n(36;2k)] \\
& -(62k)[(k3;4l)(l51) + (k3;5l)(l14)] + (63k)[(k2;4l)(l51) + (k2;5l)(l14)] \\
& +(23k)(k6;1l)(l45)\Big) \\
& +(23;1k)n(k6;45) + n(23;1k)(k6;45) - (23;6k)n(k1;45) - n(23;6k)(k1;45) \\
& +\Delta(61k|l)(k23)(l45) + (16k)(23;kl)(l45) + (16k)(45;kl)(l23). \qquad \text{(E.20)}
\end{aligned}
$$

Armed with this we can proceed to computing the D's. For example, by collecting together all the δn-independent terms in the expression below gives $D(123456)$, in a natural extension of the five-point relations (4.6)–(4.10):

$$
\begin{aligned}
& \frac{\delta n_1}{s_{12}s_{123}s_{1234}} - \frac{-\delta n_1 + \delta n_{13} + \Delta(12;3|456)}{s_{12}s_{123}s_{1236}} + \frac{-\delta n_1 + \delta n_2 + \Delta(231|4;56)}{s_{23}s_{123}s_{1234}} \\
& - \frac{1}{s_{12}s_{126}s_{1236}}\Big(- \delta n_1 - \delta n_9 + \delta n_{13} + \delta n_{17} + \Delta(12;3|645) \\
& + (12k)\Delta(36|k45) + (12;6k)n(k3;45) + (12;3k)n(6k;45) \\
& + (12k)\Delta(45|36k) + (12;5k)n(k4;36) + (12;k4)n(k5;36)\Big) \\
& + \frac{1}{s_{12}s_{126}s_{1256}}\Big(\delta n_1 - \delta n_5 + \delta n_9 - \delta n_{21} + \Delta(12;5|634) \\
& + (12k)\Delta(56|k34) + (12;6k)n(k5;34) + (12;5k)n(6k;34) \\
& + (12k)\Delta(34|56k) + (12;4k)n(k3;56) + (12;k3)n(k4;56)\Big) \\
& - \frac{1}{s_{23}s_{243}s_{1234}}\Big(- \delta n_1 + \delta n_2 + \delta n_6 - \delta n_7 + +\Delta(65;4|132) \\
& + (65k)\Delta(41|k32) + (65;1k)n(k4;32) + (65;4k)n(1k;32) \\
& + (65k)\Delta(32|41k) + (65;2k)n(k3;41) + (65;k3)n(k2;41)\Big) \\
& + \frac{1}{s_{34}s_{234}s_{1234}}\Big(\delta n_1 - \delta n_3 - \delta n_5 + \delta n_7 + \Delta(62;4|135) \\
& + (64k)\Delta(31|k52) + (64;1k)n(k3;52) + (64;3k)n(1k;52) \\
& + (64k)\Delta(52|31k) + (64;2k)n(k5;31) + (64;k5)n(k2;41)\Big) \\
& + \frac{1}{s_{23}s_{123}s_{1236}}\Big(\delta n_1 - \delta n_2 - \delta n_{13} + \delta n_{14} \\
& - \Delta(231|k)[n(k4;56) + n(k5;64)] - \Delta(k|645)[n(12;3k) + n(31;2k)]
\end{aligned}
$$

$$
\begin{aligned}
&-(12k)[(k3;4l)(l56)+(k3;5l)(l64)]+(13k)[(k2;4l)(l56)+(k2;5l)(l64)]\\
&+(23k)(k1;6l)(l45)\Big)\\
&+\frac{1}{s_{12}s_{34}s_{56}}\Big(\delta n_1-\delta n_5-(12k)\Delta(34k|56)\\
&+(12k)(k4;3l)(l56)-(12k)(k4;3l)(l56)\\
&+(34;k)(kl;12)(l56)+(12k)(34;lk)(l56)+(12k)(kl;56)(l34)\Big)\\
&+\frac{1}{s_{16}s_{126}s_{1236}}\Big(\delta n_1-\delta n_4+\delta n_9-\delta n_{12}-\delta n_{13}+\delta n_{16}-\delta n_{17}+\delta n_{20}\\
&-\Delta(12;3|645)+\Delta(15;4|632)-(12k)\Delta(36|k45)-(12;6k)n(k3;45)-(12;k3)n(k6;45)\\
&-(12k)\Delta(45|36k)-(12;5k)n(k4;36)-(12;k4)n(k5;36)+(15k)\Delta(46|k32)+(15;6k)n(k4;32)\\
&-(15;k4)n(k6;32)+(15k)\Delta(32|46k)+(15;2k)n(k3;46)-(15;k3)n(k2;46)\\
&+\Delta(162|k)n(k3;45)+(61k)(k2;3l)(l45)+(12k)(k6;3l)(l45)+(26k)(k1;3l)(l45)\Big)\\
&-\frac{1}{s_{16}s_{156}s_{1456}}\Big(\delta n_1-\delta n_2-\delta n_6+\delta n_7+\delta n_{11}-\delta n_{12}-\delta n_{22}+\delta n_{23}\\
&-\Delta(15;4|632)+\Delta(12;3|645)-(15k)\Delta(46|k32)-(15;6k)n(k4;32)-(15;k4)n(k6;32)\\
&-(15k)\Delta(32|46k)-(15;2k)n(k3;46)-(15;k3)n(k2;46)+(12k)\Delta(36|k45)+(12;6k)n(k3;45)\\
&-(12;k3)n(k6;45)+(12k)\Delta(45|36k)+(12;5k)n(k4;36)-(12;k4)n(k5;36)\\
&+\Delta(165|k)n(k4;32)+(61k)(k5;4l)(l32)+(15k)(k6;4l)(l32)+(56k)(k1;4l)(l32)\Big)\\
&-\frac{1}{s_{16}s_{156}s_{1256}}\Big(-\delta n_1+\delta n_3+\delta n_5-\delta n_7-\delta n_{10}+\delta n_{12}+\delta n_{22}-\delta n_{24}\\
&-\Delta(15;2|643)+\Delta(13;4|625)-(15k)\Delta(26|k43)-(15;6k)n(k2;43)-(15;k2)n(k6;43)\\
&-(15k)\Delta(43|26k)-(15;3k)n(k4;26)-(15;k4)n(k3;26)+(13k)\Delta(46|k25)+(13;6k)n(k4;25)\\
&-(13;k4)n(k6;25)+(13k)\Delta(25|46k)+(13;5k)n(k2;46)-(13;k2)n(k5;46)\\
&+\Delta(165|k)n(k2;43)+(61k)(k5;2l)(l43)+(15k)(k6;2l)(l43)+(56k)(k1;2l)(l43)\Big)\\
&-\frac{1}{s_{16}s_{126}s_{1256}}\Big(-\delta n_1+\delta n_4+\delta n_5-\delta n_8-\delta n_9+\delta n_{12}+\delta n_{21}-\delta n_{24}\\
&-\Delta(12;3|645)+\Delta(15;4|632)-(12k)\Delta(56|k43)-(12;6k)n(k5;43)-(12;k5)n(k6;43)\\
&-(12k)\Delta(43|56k)-(12;3k)n(k4;56)-(12;k4)n(k3;56)+(13k)\Delta(46|k52)+(13;6k)n(k4;52)\\
&-(13;k4)n(k6;52)+(13k)\Delta(52|46k)+(13;2k)n(k5;46)-(13;k5)n(k2;46)\\
&+\Delta(162|k)n(k5;43)+(61k)(k2;5l)(l43)+(12k)(k6;5l)(l43)+(26k)(k1;5l)(l43)\Big)\\
&+\frac{1}{s_{16}s_{23}s_{45}}\Big(\delta n_1-\delta n_2+\delta n_{11}-\delta n_{12}-\delta n_{13}+\delta n_{14}-\delta n_{19}+\delta n_{20}\\
&+\Delta(231|k)[n(k4;56)+n(k5;64)]+\Delta(k|645)[n(12;3k)+n(31;2k)]\\
&+(12k)[(k3;4l)(l56)+(k3;5l)(l64)]-(13k)[(k2;4l)(l56)+(k2;5l)(l64)]-(23k)(k1;6l)(l45)\\
&-\Delta(236|k)[n(k4;51)+n(k5;64)]-\Delta(k|145)[n(62;3k)+n(36;2k)]\\
&-(62k)[(k3;4l)(l51)+(k3;5l)(l14)]+(63k)[(k2;4l)(l51)+(k2;5l)(l14)]+(23k)(k6;1l)(l45)\\
&+(23;1k)n(k6;45)+n(23;1k)(k6;45)-(23;6k)n(k1;45)-n(23;6k)(k1;45)\\
&+\Delta(61k|l)(k23)(l45)+(16k)(23;kl)(l45)+(16k)(45;kl)(l23)\Big)\\
&=0
\end{aligned}
\tag{E.21}
$$

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