

Classical irregular block, $\mathcal{N} = 2$ pure gauge theory and Mathieu equation

Marcin Piątek^{a,c} and Artur R. Pietrykowski^{b,c}

^a*Institute of Physics, University of Szczecin,
ul. Wielkopolska 15, 70-451 Szczecin, Poland*

^b*Institute of Theoretical Physics, University of Wrocław,
pl. M. Borna 9, 50-204 Wrocław, Poland*

^c*Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research,
141980 Dubna, Russia*

E-mail: piatek@fermi.fiz.univ.szczecin.pl, pietrie@theor.jinr.ru

ABSTRACT: Combining the semiclassical/Nekrasov-Shatashvili limit of the AGT conjecture and the Bethe/gauge correspondence results in a triple correspondence which identifies classical conformal blocks with twisted superpotentials and then with Yang-Yang functions. In this paper the triple correspondence is studied in the simplest, yet not completely understood case of pure SU(2) super-Yang-Mills gauge theory. A missing element of that correspondence is identified with the classical irregular block. Explicit tests provide a convincing evidence that such a function exists. In particular, it has been shown that the classical irregular block can be recovered from classical blocks on the torus and sphere in suitably defined decoupling limits of classical external conformal weights. These limits are “classical analogues” of known decoupling limits for corresponding quantum blocks. An exact correspondence between the classical irregular block and the SU(2) gauge theory twisted superpotential has been obtained as a result of another consistency check. The latter determines the spectrum of the 2-particle periodic Toda (sin-Gordon) Hamiltonian in accord with the Bethe/gauge correspondence. An analogue of this statement is found entirely within $2d$ CFT. Namely, considering the classical limit of the null vector decoupling equation for the degenerate irregular block a celebrated Mathieu’s equation is obtained with an eigenvalue determined by the classical irregular block. As it has been checked this result reproduces a well known weak coupling expansion of Mathieu’s eigenvalue. Finally, yet another new formulae for Mathieu’s eigenvalue relating the latter to a solution of certain Bethe-like equation are found.

KEYWORDS: Supersymmetric gauge theory, Field Theories in Lower Dimensions, Solitons Monopoles and Instantons, Conformal and W Symmetry

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1 Introduction

Studying the problem of the oscillations of an elliptical membrane E. Mathieu [1] obtained the following two ordinary differential equations with real coefficients [2, 3]:¹

$$\frac{d^2\psi}{dx^2} + (\lambda - 2h^2 \cos 2x) \psi = 0, \quad x \in \mathbb{R} \tag{1.1}$$

¹In the present paper we adopt the notation from [4].

and

$$\frac{d^2\psi}{dx^2} - (\lambda - 2h^2 \cosh 2x) \psi = 0 \quad \text{i.e.} \quad \frac{d^2\psi}{d(ix)^2} + (\lambda - 2h^2 \cos 2ix) \psi = 0. \quad (1.2)$$

In honor of their originator the eqs. (1.1), (1.2) are now called Mathieu and modified Mathieu equations respectively. As has been explicitly written down the modified Mathieu eq. (1.2) may be derived from eq. (1.1) by writing ix for x and vice versa.

For certain λ and h^2 there exists a general solution $\psi(x)$ of the eq. (1.1) and a *Floquet exponent* ν such that

$$\psi(x + \pi) = e^{i\pi\nu} \psi(x).$$

If $\psi_+(x)$ is the solution of the Mathieu equation satisfying the initial conditions $\psi_+(0) = \text{const.} = a$ and $\psi'_+(0) = 0$, the parameter ν can be determined from the relation [4]:

$$\cos \pi\nu = \frac{\psi_+(\pi)}{a}. \quad (1.3)$$

Thus the Floquet exponent is determined by the value at $x = \pi$ of the solution $\psi_+(x)$ which is even around $x = 0$. To the lowest order ($h^2 = 0$) the even solution around $x = 0$ is $\psi_+^{(0)}(x) = a \cos \sqrt{\lambda}x$. Hence, from (1.3) for $h^2 = 0$ we have $\nu = \sqrt{\lambda}$, and more in general

$$\nu^2 = \lambda + \mathcal{O}(h^2).$$

One can derive various terms of this expansion perturbatively. For instance, in the weak coupling regime, for small h^2 , the eigenvalue λ as a function of ν and h^2 explicitly reads as follows [4]:

$$\begin{aligned} \lambda = \nu^2 + \frac{h^4}{2(\nu^2 - 1)} + \frac{(5\nu^2 + 7)h^8}{32(\nu^2 - 4)(\nu^2 - 1)^3} \\ + \frac{(9\nu^4 + 58\nu^2 + 29)h^{12}}{64(\nu^2 - 9)(\nu^2 - 4)(\nu^2 - 1)^5} + \mathcal{O}(h^{16}). \end{aligned} \quad (1.4)$$

One sees that this expansion cannot hold for integer values of ν . These cases have to be dealt with separately, cf. [4].

The solutions of the (modified) Mathieu equation govern a vast number of problems in physics: (i) the propagation of electromagnetic waves along elliptical cylinder, (ii) the vibrations of a membrane in the shape of an ellipse with a rigid boundary (Mathieu's problem) [2], (iii) the motion of a rod fixed on one end and being under periodic tension at the other end [3], (iv) the motion of particles in electromagnetic traps [5], (v) the inverted pendulum and the quantum pendulum, (vi) the wave scattering by D-brane [6], (vii) fluctuations of scalar field about a D3-brane [7, 8], (viii) reheating process in inflationary models [9], (ix) determination of the mass spectrum of a scalar field in a world with latticized and circular continuum space [10], are just some of them.

Eq. (1.1) can be looked at as a one-dimensional Schrödinger equation $\mathbf{H}\psi = E\psi$ with the Hamiltonian (the Mathieu operator) $\mathbf{H} = -d^2/dx^2 + 2h^2 \cos 2x$ and the energy eigenvalue

$E = \lambda$. The Mathieu operator H belongs to the class of Schrödinger operators with periodic potentials [11] which are of special importance in solid state physics.

Mathieu’s equation has recently emerged in a fascinating context, namely in the studies of the interrelationships between quantum integrable systems (QIS), $\mathcal{N} = 2$ super-Yang-Mills (SYM) theories and two-dimensional conformal field theory (2d CFT). In order to spell out aims of the present work let us discuss certain aspects of these investigations in detail. First, recall that eq. (1.1) with $\lambda = 8u\hbar^{-2}$, $h^2 = 4\hat{\Lambda}^2\hbar^{-2}$, $2x = \varphi$, i.e.:

$$\left[-\frac{\hbar^2}{2} \frac{\partial^2}{\partial \varphi^2} + \hat{\Lambda}^2 \cos \varphi \right] \psi(\varphi) = u \psi(\varphi) \tag{1.5}$$

is a Schrödinger equation for the quantum one-dimensional sine-Gordon system defined by the Lagrangian $\mathcal{L} = \frac{1}{2}\dot{\varphi}^2 - \hat{\Lambda}^2 \cos \varphi$. In this context quantities $\hat{\Lambda}$, φ , u can be complex valued.²

Secondly, it is a well known fact that the (classical) sine-Gordon model encodes an information about the $\mathcal{N} = 2$ SU(2) pure gauge Seiberg-Witten theory [12].³ More precisely, the so-called Bohr-Sommerfeld periods

$$\Pi(\Gamma) \equiv \oint_{\Gamma} P_0(\varphi) d\varphi = \oint_{\Gamma} \sqrt{2(u - \hat{\Lambda}^2 \cos \varphi)} d\varphi$$

for two complementary contours $\Gamma = A, B$ encircling two turning points $\varphi_0 = \pm \cos^{-1}(u/\hat{\Lambda}^2)$ define the Seiberg-Witten system [15]: $a = \Pi(A)$, $\partial\mathcal{F}(a)/\partial a = \Pi(B)$.⁴ Here, a is a modulus and $\mathcal{F}(a)$ denotes the Seiberg-Witten prepotential determining the low energy effective dynamics of the $\mathcal{N} = 2$ supersymmetric SU(2) pure gauge theory.

As has been observed in [16] (see also [17, 18]) the above statement has its “quantum analogue” or “quantum generalization” which can be formulated as follows. Namely, the monodromies⁵

$$\tilde{\Pi}(\Gamma) \equiv \oint_{\Gamma} P(\varphi, \hbar) d\varphi$$

of the exact WKB solution

$$\psi(\varphi) = \exp \left\{ \frac{i}{\hbar} \int^{\varphi} P(\rho, \hbar) d\rho \right\}$$

to the eq. (1.5) define the Nekrasov-Shatashvili system [19]: $a = \tilde{\Pi}(A)$, $\partial\mathcal{W}(\hat{\Lambda}, a, \hbar)/\partial a = \tilde{\Pi}(B)$.⁶ Here, $\mathcal{W}(\hat{\Lambda}, a, \hbar) = \mathcal{W}_{\text{pert}}(\hat{\Lambda}, a, \hbar) + \mathcal{W}_{\text{inst}}(\hat{\Lambda}, a, \hbar)$ is the *effective twisted superpotential* of the 2d SU(2) pure gauge (Ω -deformed) SYM theory defined in [19] as the following (Nekrasov-Shatashvili) limit

$$\mathcal{W}(\hat{\Lambda}, a, \hbar) \equiv \lim_{\epsilon_2 \rightarrow 0} \epsilon_2 \log \mathcal{Z}(\hat{\Lambda}, a, \epsilon_1 = \hbar, \epsilon_2)$$

of the Nekrasov partition function $\mathcal{Z}(\hat{\Lambda}, a, \epsilon_1, \epsilon_2) = \mathcal{Z}_{\text{pert}}(\hat{\Lambda}, a, \epsilon_1, \epsilon_2) \mathcal{Z}_{\text{inst}}(\hat{\Lambda}, a, \epsilon_1, \epsilon_2)$ [21, 22].

²In other words we are dealing here with quantized integrable model defined on the complex plane which is not hermitian quantum-mechanical system.

³See also [13, 14].

⁴Cf. appendix C.

⁵Here, $P(\varphi, \hbar) = P_0(\varphi) + P_1(\varphi)\hbar + P_2(\varphi)\hbar^2 + \dots$

⁶For SU(N) generalization of this result, see [20].

classical block on	twisted superpotential	Yang-Yang function
4-punctured sphere	$SU(2) N_f = 4$	SL(2)-type Gaudin model
1-punctured torus	$SU(2) \mathcal{N} = 2^*$	2-particle eCM model
?	$SU(2)$ pure gauge	2-particle pToda chain

Table 1. Examples of 2dCFT/gauge/Bethe correspondence. Here eCM and pToda stand for elliptic Calogero-Moser model and periodic Toda chain, respectively.

Twisted superpotentials determine the low energy effective dynamics of $2d$ SYM theories restricted to the Ω -background. These quantities play also a pivotal role in the so-called Bethe/gauge correspondence [19, 23–25] that maps supersymmetric vacua of the $\mathcal{N} = 2$ theories to Bethe states of quantum integrable systems. A result of that duality is that the twisted superpotentials are identified with the *Yang-Yang functions* [19] which describe the spectrum of the corresponding quantum integrable systems.⁷

Twisted superpotentials occur also in the context related to the AGT correspondence [26]. The AGT conjecture states that the Liouville field theory (LFT) correlators on the Riemann surface $C_{g,n}$ with genus g and n punctures can be identified with the partition functions of a class $T_{g,n}$ of four-dimensional $\mathcal{N} = 2$ supersymmetric $SU(2)$ quiver gauge theories. A significant part of the AGT conjecture is an exact correspondence between the Virasoro blocks on $C_{g,n}$ and the instanton sectors of the Nekrasov partition functions of the gauge theories $T_{g,n}$. Soon after its discovery, the AGT hypothesis has been extended to the $SU(N)$ -gauge theories/conformal Toda correspondence [27]. The AGT duality works at the level of the quantum Liouville field theory. At this point arises a question, what happens if we proceed to the classical limit of the Liouville theory. This is the limit in which the central charge and the external and intermediate conformal weights of LFT correlators tend to infinity in such a way that their ratios are fixed. It is commonly believed that such limit exists and the Liouville correlation functions, and in particular, conformal blocks behave in this limit exponentially. It turns out that the semiclassical limit of the LFT correlation functions corresponds to the Nekrasov-Shatashvili limit ($\epsilon_2 \rightarrow 0$, $\epsilon_1 = \text{const.}$) of the Nekrasov partition functions. A consequence of that correspondence is that the instanton parts of the effective twisted superpotentials can be identified with *classical conformal blocks*.

Therefore, combining both the classical/Nekrasov-Shatashvili limit of the AGT duality and the Bethe/gauge correspondence one thus gets the triple correspondence which connects the classical blocks with the twisted superpotentials and then with the Yang-Yang functions. The simplest, although not yet completely understood examples of that correspondence are listed in the table 1.

Our goals in this paper are twofold. First, we study the triple correspondence in case where the SYM theory is the $SU(2)$ pure gauge theory (the third example in the table 1).⁸ We identify the question mark in the table 1 as the *classical irregular block* f_{irr} . The latter is the classical limit of the quantum irregular block \mathcal{F}_{irr} [29].

⁷The Yang-Yang functions are potentials for Bethe equations.

⁸For a discussion of the first example in the table 1, see [28].

Motivations to study classical blocks were, for a long time, mainly confined to applications in pure mathematics, in particular, to the celebrated uniformization problem of Riemann surfaces [30, 31] which is closely related to the monodromy problem for certain ordinary differential equations [32–40].⁹ The importance of the classical blocks is not only limited to the uniformization theorem, but gives also information about the solution of the Liouville equation on surfaces with punctures. Recently, a mathematical application of classical blocks emerged in the context of Painlevé VI equation [42]. Due to the recent discoveries the classical blocks are also relevant for physics. Indeed, in addition to the correspondence discussed in the present paper lately the classical blocks have been of use to studies of the entanglement entropy within the AdS₃/2d CFT holography [43] and the topological string theory [44].

The second purpose of the present work is to discuss implications of the aforementioned triple correspondence for the eigenvalue problem of Mathieu’s operator. More concretely, as has been observed in [16] the eigenvalue u in eq. (1.5) (or equivalently λ in eq. (1.1)) can be found from eq. $a = \tilde{\Pi}(A)$ as a series in \hbar by applying the exact WKB method. The result can be re-expressed as a logarithmic derivative of the twisted superpotential $\mathcal{W}(\hat{\Lambda}, a, \epsilon_1)$ w.r.t. $\hat{\Lambda}$.¹⁰ The same “should be visible” at the conformal field theory side. Indeed, Mathieu’s equation occurs entirely within formalism of 2d CFT as a classical limit of the null vector decoupling equation satisfied by the 3-point degenerate irregular block (= matrix element of certain primary degenerate chiral vertex operator between Gaiotto states [29]). As expected, the eigenvalue in this equation is given by the logarithmic derivative of the classical irregular block f_{irr} w.r.t. $\hat{\Lambda}$. Concluding, these observations pave the way for working out new methods for calculating Mathieu’s eigenvalues and eigenfunctions. The second goal of this paper is to check this possibility.

Our studies of Mathieu’s equation and the classical irregular block are in particular motivated by recent results obtained by one of the authors in [39]. There have been derived novel expressions for the so-called accessory parameter \mathbf{B} of the Lamé equation:¹¹

$$\frac{d^2\eta}{dz^2} - [\kappa \wp(z) + \mathbf{B}] \eta = 0.$$

In particular, it has been found that

$$\frac{\mathbf{B}(\tau)}{4\pi^2} = q \frac{d}{dq} f_{\text{torus}}(-\kappa, \delta_*; q) + \frac{\kappa}{12} E_2(\tau), \tag{1.6}$$

where $q = \exp(2\pi i\tau)$ and τ is a torus modular parameter; $f_{\text{torus}}(\cdot, \cdot; q)$ denotes the classical torus block.¹² It is a well known fact that in a certain limit the Lamé equation

⁹In a somewhat different context, see also [41].

¹⁰See appendix C.

¹¹Here, $\wp(z)$ is the Weierstrass elliptic function and $E_2(\tau)$ denotes the second Eisenstein series.

¹²In eq. (1.6) the classical torus block is evaluated on the so-called saddle point intermediate classical weight $\delta_* = \frac{1}{4} + p_*^2$, where p_* is a solution of the following equation ($p \in \mathbb{R}$):

$$\frac{\partial}{\partial p} S_L^{(3)} \left(2p, -2i\sqrt{\kappa + \frac{1}{4}}, 2p \right) = 2\Re f_{\text{torus}} \left(-\kappa, \frac{1}{4} + p^2; q \right) \Big|_{p=p_*}.$$

$S_L^{(3)}$ is known classical Liouville action on the Riemann sphere with three hyperbolic singularities (holes), cf. [39].

becomes the Mathieu equation. Hence, one may expect that the Lamé equation with the eigenvalue expressed in terms of the classical torus block f_{torus} consistently reduces to the Mathieu equation with the eigenvalue given by the classical irregular block f_{irr} if in such a limit $f_{\text{torus}} \rightarrow f_{\text{irr}}$. If this statement is true it will give a strong evidence that conjectured formula (1.6) is correct.

The organization of the paper is as follows. In section 2 we define the quantum irregular block \mathcal{F}_{irr} related to the $SU(2)$ pure gauge Nekrasov instanton partition function, in accordance with the so-called *non-conformal AGT relation* [29]. Inspired by the latter and the results of [19] we then conjecture that \mathcal{F}_{irr} exponentiates to the classical irregular block f_{irr} in the classical limit. Indeed, for the low orders of expansion of the quantum irregular block one can see that the classical limit of \mathcal{F}_{irr} exists yielding consistent definition of the classical irregular block. The latter corresponds to the twisted superpotential of the $2d \mathcal{N} = 2$ $SU(2)$ pure gauge theory. In addition, we perform another consistency checks suggesting that the function f_{irr} really “lives its own life”. In particular, we verify that classical blocks on the 1-punctured torus $C_{1,1}$ and on the 4-punctured sphere $C_{0,4}$ reduce to f_{irr} in certain properly defined *decoupling limit* of external classical weights. This limit is a classical analogue of known decoupling limits for quantum blocks on $C_{1,1}$ and $C_{0,4}$.

In section 3 (see also appendix B) we consider the classical limit of the null vector decoupling equation satisfied by the 3-point degenerate irregular block [18] and find an expression for the Mathieu eigenvalue. As has been already mentioned the latter is determined by the classical irregular block. This formula yields the well known weak coupling expansion (1.4) of the eigenvalue of the Mathieu operator.

Section 4 is devoted to the derivation of the Mathieu eigenvalue from the non-conformal AGT counterpart of the classical irregular block, namely the twisted superpotential. The latter is obtained from the Nekrasov instanton partition function for pure $SU(2)$ gauge theory as a zero limit in one of the two deformation parameters ϵ_1, ϵ_2 . Since the Nekrasov partition function can be represented by the sum over profiles of the Young diagrams [21] (see appendix D for details) the two deformation parameters are associated with two edges of elementary box of anisotropic Young diagrams. As a result the superpotential is obtained from the critical Young diagram which is determined by a dominating contribution to the partition function. The Mathieu eigenvalue can be thus found by means of the Bethe/gauge correspondence postulated in ref. [19].

In section 5 we present our conclusions. The problems that are still open and the possible extensions of the present work are discussed.

In the appendix A are collected formulae for expansion coefficients of $2d$ CFT and gauge theory functions used in the main text. In the appendix C the Mathieu eigenvalue is obtained from the exact WKB method. Appendices D and E contain supplementary material to the section 4. Specifically, the Nekrasov partition function is given in terms of the profiles of the Young diagrams as was originally done by Nekrasov and Okounkov [21].

2 Quantum and classical irregular blocks

2.1 Quantum irregular block

In order to define the quantum irregular block first we need to introduce the notion of the Gaiotto state. This is the vector $|\Delta, \Lambda^2\rangle$ defined by the following conditions [18, 29]:

$$L_0|\Delta, \Lambda^2\rangle = \left(\Delta + \frac{\Lambda}{2} \frac{\partial}{\partial \Lambda}\right) |\Delta, \Lambda^2\rangle, \quad (2.1)$$

$$L_1|\Delta, \Lambda^2\rangle = \Lambda^2|\Delta, \Lambda^2\rangle, \quad (2.2)$$

$$L_n|\Delta, \Lambda^2\rangle = 0 \quad \forall n \geq 2. \quad (2.3)$$

In [45] it was shown that the state $|\Delta, \Lambda^2\rangle$ which solves the Gaiotto constraint equations has the following form

$$|\Delta, \Lambda^2\rangle = \sum_{n \geq 0} \Lambda^{2n} |\Delta, n\rangle = \sum_{n \geq 0} \Lambda^{2n} \sum_{|J|=n} [G_{c,\Delta}^n]^{(1^n)J} L_{-J}|\Delta\rangle. \quad (2.4)$$

The quantity $[G_{c,\Delta}^n]^{IJ}$ denotes the inverse of the Gram matrix $[G_{c,\Delta}^n]_{IJ} = \langle \Delta | L_I L_{-J} | \Delta \rangle$ in the standard basis

$$|\Delta, J\rangle := L_{-J}|\Delta\rangle = L_{-k_1} \dots L_{-k_n}|\Delta\rangle, \quad J = (k_1 \geq \dots \geq k_n \geq 1), \quad (2.5)$$

$$|J| = k_n + \dots + k_1 = n$$

of the Verma module with the central charge c and the highest weight Δ .

The quantum irregular block is defined as the scalar product $\langle \Delta, \Lambda^2 | \Delta, \Lambda^2 \rangle$ of the Gaiotto state. Hence, taking into account (2.4) one gets¹³

$$\langle \Delta, \Lambda^2 | \Delta, \Lambda^2 \rangle = \sum_{n \geq 0} \Lambda^{4n} \langle \Delta, n | \Delta, n \rangle = \sum_{n \geq 0} \Lambda^{4n} [G_{c,\Delta}^n]^{(1^n)(1^n)}. \quad (2.6)$$

Due to the AGT relation the irregular block can be expressed through the SU(2) pure gauge Nekrasov instanton partition function $\mathcal{Z}_{\text{inst}}^{\text{Nf}=0, \text{SU}(2)}$. Indeed, the following relation

$$\begin{aligned} \mathcal{F}_{c,\Delta}(\Lambda) &:= \langle \Delta, \Lambda^2 | \Delta, \Lambda^2 \rangle = \sum_{n \geq 0} \Lambda^{4n} [G_{c,\Delta}^n]^{(1^n)(1^n)} \\ &= \sum_{n \geq 0} \hat{\Lambda}^{4n} \mathcal{Z}_n(a, \epsilon_1, \epsilon_2) = \mathcal{Z}_{\text{inst}}^{\text{Nf}=0, \text{SU}(2)}(\hat{\Lambda}, a, \epsilon_1, \epsilon_2) \end{aligned} \quad (2.7)$$

holds for

$$\Lambda = \frac{\hat{\Lambda}}{\sqrt{\epsilon_1 \epsilon_2}}, \quad \Delta = \frac{(\epsilon_1 + \epsilon_2)^2 - 4a^2}{4\epsilon_1 \epsilon_2}, \quad c = 1 + 6 \left(\frac{\epsilon_1 + \epsilon_2}{\sqrt{\epsilon_1 \epsilon_2}} \right)^2 \equiv 1 + 6Q^2, \quad (2.8)$$

where

$$Q = b + \frac{1}{b} \equiv \sqrt{\frac{\epsilon_2}{\epsilon_1}} + \sqrt{\frac{\epsilon_1}{\epsilon_2}} \Leftrightarrow b = \sqrt{\frac{\epsilon_2}{\epsilon_1}}. \quad (2.9)$$

For a nonrigorous derivation of eq. (2.7), see [46, 47]. Rigorous proof can be found in ref. [48].

¹³For an explicit computation of the first few coefficients in (2.6), see appendix A.

2.2 Classical irregular block

In [19] it was observed that in the limit $\epsilon_2 \rightarrow 0$ the Nekrasov partition functions behave exponentially. In particular, for the instantonic sector we have

$$\mathcal{Z}_{\text{inst}}^{\text{Nf}=0, \text{SU}(2)}(\hat{\Lambda}, a, \epsilon_1, \epsilon_2) \stackrel{\epsilon_2 \rightarrow 0}{\sim} \exp \left\{ \frac{1}{\epsilon_2} \mathcal{W}_{\text{inst}}^{\text{Nf}=0, \text{SU}(2)}(\hat{\Lambda}, a, \epsilon_1) \right\}. \quad (2.10)$$

In other words, there exists the limit

$$\mathcal{W}_{\text{inst}}^{\text{Nf}=0, \text{SU}(2)}(\hat{\Lambda}, a, \epsilon_1) \equiv \lim_{\epsilon_2 \rightarrow 0} \epsilon_2 \log \mathcal{Z}_{\text{inst}}^{\text{Nf}=0, \text{SU}(2)} = \sum_{k \geq 1} \hat{\Lambda}^{4k} \mathcal{W}_k(a, \epsilon_1) \quad (2.11)$$

called the effective twisted superpotential.

Taking into account (2.7), (2.8), (2.9) and (2.10) one can expect the exponential behaviour of the irregular block in the limit $b \rightarrow 0$.¹⁴ Indeed, let $b = \sqrt{\epsilon_2/\epsilon_1}$ and $\Lambda = \hat{\Lambda}/(\epsilon_1 b)$, $c = 1 + 6Q^2$, $\Delta = \frac{1}{b^2} \delta$, $\delta = \mathcal{O}(b^0)$ then, we conjecture that¹⁵

$$\mathcal{F}_{c, \Delta}(\Lambda) = \langle \Delta, \Lambda^2 | \Delta, \Lambda^2 \rangle \stackrel{b \rightarrow 0}{\sim} \exp \left\{ \frac{1}{b^2} f_\delta \left(\frac{\hat{\Lambda}}{\epsilon_1} \right) \right\}. \quad (2.12)$$

Equivalently, there exists the limit

$$\begin{aligned} f_\delta \left(\frac{\hat{\Lambda}}{\epsilon_1} \right) &= \lim_{b \rightarrow 0} b^2 \log \mathcal{F}_{c, \Delta} \left(\frac{\hat{\Lambda}}{\epsilon_1 b} \right) = \lim_{b \rightarrow 0} b^2 \log \left[1 + \sum_{n \geq 1} \left(\frac{\hat{\Lambda}}{\epsilon_1 b} \right)^{4n} [G_{c, \Delta}^n]^{(1^n)(1^n)} \right] \\ &= \sum_{n \geq 1} \left(\frac{\hat{\Lambda}}{\epsilon_1} \right)^{4n} f_\delta^n \end{aligned} \quad (2.13)$$

called the classical irregular block. It should be stressed that the asymptotical behaviour (2.12) is a nontrivial statement concerning the quantum irregular block. Although there is no proof of this property the existence of the classical irregular block can be checked, first, by direct calculation. For instance, up to $n = 3$ one finds

$$f_\delta^1 = \frac{1}{2\delta}, \quad f_\delta^2 = \frac{5\delta - 3}{16\delta^3(4\delta + 3)}, \quad f_\delta^3 = \frac{9\delta^2 - 19\delta + 6}{48\delta^5(4\delta^2 + 11\delta + 6)}. \quad (2.14)$$

Secondly, there exist two other equivalent ways to get the function $f_\delta(\hat{\Lambda}/\epsilon_1)$. As has been shown in ref. [45] the quantum irregular block $\mathcal{F}_{c, \Delta}(\Lambda)$ can be recovered from the 4-point

¹⁴It was first observed in [16] (however not spelled out explicitly) that the Nekrasov-Shatashvili limit of the pure gauge SU(2) Nekrasov partition function might have something to do with certain (in fact classical) limit of the quantum irregular block.

¹⁵Motivated by [16, 19] and the AGT correspondence it is not difficult to realize that eq. (2.12) should hold. Nevertheless, up to the best of our knowledge no one in literature has identified this as the classical irregular block and discussed it entirely in terms of 2d CFT, at least we have not noticed it anywhere. The present paper is though to partially fill this gap (at the level of direct check, though without the proof which is yet to be done). This proposal is in the line with the research initiated by A. and Al. Zamolodchikovs in the seminal paper [32] (see also [49]), where they conjectured exponential behavior of conformal blocks in the classical limit. In our paper we present novel arguments supporting this hypothesis.

block on the sphere $\mathcal{F}_{c,\Delta} \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix} \right] (x)$ in a properly defined decoupling limit of the external conformal weights Δ_i ,¹⁶

$$\tilde{\mathcal{F}}_{c,\Delta} \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix} \right] (x) \xrightarrow[x\mu_1\mu_2\mu_3\mu_4=\hat{\Lambda}^4]{\mu_1,\mu_2,\mu_3,\mu_4 \rightarrow \infty} \mathcal{F}_{c,\Delta}(\Lambda), \quad (2.15)$$

where $(\epsilon = \epsilon_1 + \epsilon_2)$:

$$\begin{aligned} \Delta_4 &= \frac{(\epsilon + \mu_3 - \mu_4)(\epsilon - \mu_3 + \mu_4)}{4\epsilon_1\epsilon_2}, & \Delta_3 &= \frac{(\mu_3 + \mu_4)(2\epsilon - \mu_3 - \mu_4)}{4\epsilon_1\epsilon_2}, \\ \Delta_2 &= \frac{(\mu_1 + \mu_2)(2\epsilon - \mu_1 - \mu_2)}{4\epsilon_1\epsilon_2}, & \Delta_1 &= \frac{(\epsilon + \mu_1 - \mu_2)(\epsilon - \mu_1 + \mu_2)}{4\epsilon_1\epsilon_2}. \end{aligned} \quad (2.16)$$

The same phenomenon occurs in the case of the 1-point block on the torus $\mathcal{F}_{c,\Delta}^{\tilde{\Delta}}(q)$ with

$$\tilde{\Delta} = [m(\epsilon_1 + \epsilon_2 - m)]/(\epsilon_1\epsilon_2). \quad (2.17)$$

The torus 1-point block yields $\mathcal{F}_{c,\Delta}(\Lambda)$ after decoupling of the external weight $\tilde{\Delta}$ [50],

$$\tilde{\mathcal{F}}_{c,\Delta}^{\tilde{\Delta}}(q) \xrightarrow[qm^4=\hat{\Lambda}^4]{m \rightarrow \infty} \mathcal{F}_{c,\Delta}(\Lambda). \quad (2.18)$$

Our claim is that the decoupling limits (2.15), (2.18) work also at the ‘‘classical level’’, i.e. after taking the classical limit of the quantum conformal blocks $\mathcal{F}_{c,\Delta} \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix} \right] (x)$ and $\mathcal{F}_{c,\Delta}^{\tilde{\Delta}}(q)$. We describe this observation in detail in the next subsection.

2.3 Decoupling limits

To begin with let us recall definitions of the 4-point block on the sphere and the 1-point block on the torus. Let x be the modular parameter of the 4-punctured Riemann sphere then the s -channel conformal block on $C_{0,4}$ is defined as the following formal x -expansion:

$$\begin{aligned} \mathcal{F}_{c,\Delta} \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix} \right] (x) &= x^{\Delta - \Delta_2 - \Delta_1} \left(1 + \sum_{n=1}^{\infty} \mathcal{F}_{c,\Delta}^n \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix} \right] x^n \right) \\ &=: x^{\Delta - \Delta_2 - \Delta_1} \tilde{\mathcal{F}}_{c,\Delta} \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix} \right] (x), \\ \mathcal{F}_{c,\Delta}^n \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix} \right] &= \lim_{z \rightarrow 1} \sum_{n=|I|=|J|} \langle \Delta_4 | V_{\Delta_3}(z) | \Delta, I \rangle \left[G_{c,\Delta} \right]^{IJ} \langle \Delta, J | V_{\Delta_2}(z) | \Delta_1 \rangle. \end{aligned} \quad (2.19)$$

Let $q = e^{2\pi i\tau}$ be the elliptic variable on the torus with modular parameter τ then the conformal block on $C_{1,1}$ is given by the following formal q -series:

$$\begin{aligned} \mathcal{F}_{c,\Delta}^{\tilde{\Delta}}(q) &= q^{\Delta - \frac{c}{24}} \left(1 + \sum_{n=1}^{\infty} \mathcal{F}_{c,\Delta}^{\tilde{\Delta},n} q^n \right) \\ &=: q^{\Delta - \frac{c}{24}} \tilde{\mathcal{F}}_{c,\Delta}^{\tilde{\Delta}}(q), \\ \mathcal{F}_{c,\Delta}^{\tilde{\Delta},n} &= \lim_{z \rightarrow 1} \sum_{n=|I|=|J|} \langle \Delta, I | V_{\tilde{\Delta}}(z) | \Delta, J \rangle \left[G_{c,\Delta} \right]^{IJ}. \end{aligned} \quad (2.20)$$

¹⁶Here and below, $\Lambda = \hat{\Lambda}/(\epsilon_1 b)$.

The above formulae in eqs. (2.19) and (2.20) contain the matrix elements of the primary chiral vertex operator between the basis states (2.5). In order to calculate them it is sufficient to know:

- (i) the covariance properties of the primary chiral vertex operator w.r.t. the Virasoro algebra,

$$[L_n, V_\Delta(z)] = z^n \left(z \frac{d}{dz} + (n+1)\Delta \right) V_\Delta(z), \quad n \in \mathbb{Z};$$

- (ii) the form of the normalized matrix element of the primary chiral vertex operator,¹⁷

$$\langle \Delta_i | V_{\Delta_j}(z) | \Delta_k \rangle = z^{\Delta_i - \Delta_j - \Delta_k}.$$

Let us consider the classical limit of conformal blocks, i.e. the limit in which the central charge $c = 1 + 6(b + \frac{1}{b})^2$, external and intermediate conformal weights tend to infinity in such a way that their ratios are fixed. It is known (however not yet proved) that in such limit quantum conformal blocks exponentiate. In particular, for conformal blocks on $C_{0,4}$ and $C_{1,1}$ we have

$$\tilde{\mathcal{F}}_{c,\Delta} \left[\begin{matrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{matrix} \right] (x) \stackrel{b \rightarrow 0}{\sim} \exp \left\{ \frac{1}{b^2} f_\delta \left[\begin{matrix} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{matrix} \right] (x) \right\}, \quad (2.21)$$

$$\tilde{\mathcal{F}}_{c,\Delta}^{\tilde{\Delta}}(q) \stackrel{b \rightarrow 0}{\sim} \exp \left\{ \frac{1}{b^2} f_{\tilde{\delta}}^{\tilde{\Delta}}(q) \right\}. \quad (2.22)$$

In the above formulae it is assumed that the quantum conformal weights are *heavy*:

$$\Delta_i = \frac{1}{b^2} \delta_i, \quad \tilde{\Delta} = \frac{1}{b^2} \tilde{\delta}, \quad \Delta = \frac{1}{b^2} \delta, \quad \delta_i, \tilde{\delta}, \delta = \mathcal{O}(b^0).$$

The functions:

$$f_\delta \left[\begin{matrix} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{matrix} \right] (x) = \sum_{n=1}^{\infty} f_\delta^n \left[\begin{matrix} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{matrix} \right] x^n, \quad f_{\tilde{\delta}}^{\tilde{\Delta}}(q) = \sum_{n=1}^{\infty} f_{\tilde{\delta}}^{\tilde{\Delta},n} q^n$$

are known as the classical conformal blocks on the sphere [32, 49, 51] and on the torus [39] respectively. The coefficients $f_\delta^n \left[\begin{matrix} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{matrix} \right]$ and $f_{\tilde{\delta}}^{\tilde{\Delta},n}$ can be found directly from the semiclassical asymptotics (2.21), (2.22) and the power expansions of quantum blocks:

$$\sum_{n=1}^{\infty} f_\delta^n \left[\begin{matrix} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{matrix} \right] x^n = \lim_{b \rightarrow 0} b^2 \log \left(1 + \sum_{n=1}^{\infty} \mathcal{F}_{c,\Delta}^n \left[\begin{matrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{matrix} \right] x^n \right),$$

$$\sum_{n=1}^{\infty} f_{\tilde{\delta}}^{\tilde{\Delta},n} q^n = \lim_{b \rightarrow 0} b^2 \log \left(1 + \sum_{n=1}^{\infty} \mathcal{F}_{c,\Delta}^{\tilde{\Delta},n} q^n \right),$$

see appendix A.

¹⁷The normalization condition takes the form $\langle \Delta_i | V_{\Delta_j}(1) | \Delta_k \rangle = 1$.

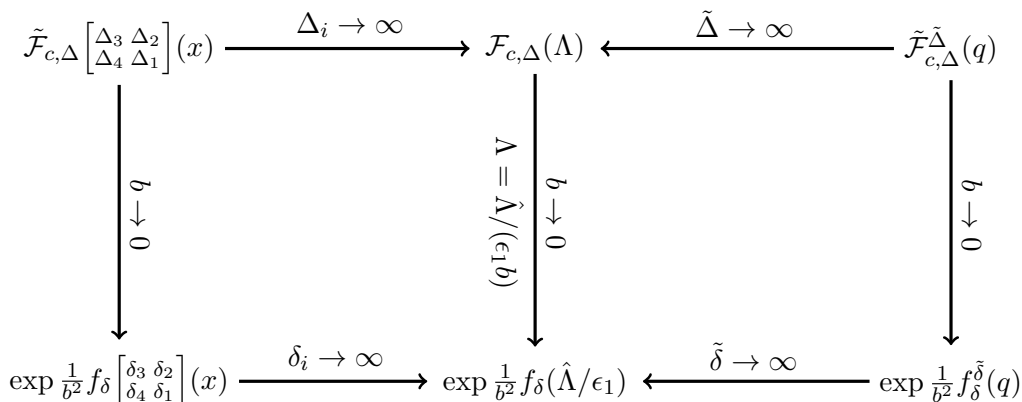


Figure 1. Equivalent ways to get the classical irregular block $f_\delta(\hat{\Lambda}/\epsilon_1)$ (above $c = 1 + 6(b + \frac{1}{b})^2$ and $(\Delta, \Delta_i, \tilde{\Delta}) = \frac{1}{b^2}(\delta, \delta_i, \tilde{\delta})$).

Let us observe that the quantum external weights (2.16), (2.17) are heavy in the terminology of [32], i.e. the following limits exist:

$$\delta_i = \lim_{b \rightarrow 0} b^2 \Delta_i = \lim_{\epsilon_2 \rightarrow 0} \frac{\epsilon_2}{\epsilon_1} \Delta_i, \quad \tilde{\delta} = \lim_{b \rightarrow 0} b^2 \tilde{\Delta} = \lim_{\epsilon_2 \rightarrow 0} \frac{\epsilon_2}{\epsilon_1} \tilde{\Delta}.$$

The classical external weights explicitly read as follows

$$\begin{aligned} \delta_4 &= [\epsilon_1^2 - (\mu_3 - \mu_4)^2] / (4\epsilon_1^2), & \delta_3 &= [(\mu_3 + \mu_4)(2\epsilon_1 - \mu_3 - \mu_4)] / (4\epsilon_1^2), \\ \delta_2 &= [(\mu_1 + \mu_2)(2\epsilon_1 - \mu_1 - \mu_2)] / (4\epsilon_1^2), & \delta_1 &= [\epsilon_1^2 - (\mu_1 - \mu_2)^2] / (4\epsilon_1^2) \end{aligned} \quad (2.23)$$

and

$$\tilde{\delta} = [m(\epsilon_1 - m)] / \epsilon_1^2.$$

Hence, one can consider the $b \rightarrow 0$ limit of both sides of the decoupling limits (2.15), (2.18). For the classical weights given by (2.23) one can verify order by order that

$$\begin{aligned} f_\delta \left[\begin{matrix} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{matrix} \right] (x) &\xrightarrow[x_{\mu_1 \mu_2 \mu_3 \mu_4 = \hat{\Lambda}^4}]{\mu_1, \mu_2, \mu_3, \mu_4 \rightarrow \infty} f_\delta(\hat{\Lambda}/\epsilon_1), \\ f_\delta^\tilde{\delta}(q) &\xrightarrow[qm^4 = \hat{\Lambda}^4]{m \rightarrow \infty} f_\delta(\hat{\Lambda}/\epsilon_1). \end{aligned}$$

Calculations presented above can be visualized on the diagram (see figure 1). Let us stress that the commutativity of this diagram lend a strong support to the exponentiation hypothesis (2.12), (2.21), (2.22) of conformal blocks in the limit $b \rightarrow 0$.

As a final remark in this section let us observe that joining together (2.7)–(2.9), (2.10) and (2.12) one gets

$$f_\delta(\hat{\Lambda}/\epsilon_1) = \frac{1}{\epsilon_1} \mathcal{W}_{\text{inst}}^{\text{Nf}=0, \text{SU}(2)}(\hat{\Lambda}, a, \epsilon_1), \quad (2.24)$$

where

$$\delta = \lim_{b \rightarrow 0} b^2 \Delta = \lim_{\epsilon_2 \rightarrow 0} \frac{\epsilon_2}{\epsilon_1} \Delta = \frac{1}{4} - \frac{a^2}{\epsilon_1^2}. \quad (2.25)$$

Indeed, to show the consistency of our approach one can use (2.14) and formulae from the appendix A, and check that¹⁸

$$\frac{1}{\epsilon_1} \mathcal{W}_n^{\text{Nf}=0, \text{SU}(2)} = \frac{1}{\epsilon_1^{4n}} f_{\frac{1}{4} - \frac{\alpha^2}{\epsilon_1^2}}^n, \quad n = 1, 2, 3, \dots \quad (2.26)$$

3 Mathieu equation in 2d CFT

3.1 Null vector decoupling equation for degenerate irregular block

Let

$$V_+(z) \equiv V_{\Delta_+}(z), \quad \Delta_+ = -\frac{1}{2} - \frac{3}{4}b^2$$

denotes the degenerate primary chiral vertex operator [52]

$$V_+(z) \equiv V_{\infty}^{\Delta'} \underset{z}{V}_{\tilde{\Delta}}^{\tilde{\Delta}} (|\Delta_+\rangle \otimes \cdot) : \mathcal{V}_{\tilde{\Delta}} \rightarrow \mathcal{V}_{\Delta'}$$

acting between the Verma modules $\mathcal{V}_{\tilde{\Delta}}$ and $\mathcal{V}_{\Delta'}$. We will assume that the conformal weights $\tilde{\Delta}$ and Δ' are related by the fusion rule, i.e.:

$$\tilde{\Delta} = \Delta \left(\sigma - \frac{b}{4} \right), \quad \Delta' = \Delta \left(\sigma + \frac{b}{4} \right), \quad \text{where } \Delta(\sigma) = \frac{Q^2}{4} - \sigma^2. \quad (3.1)$$

Let us consider the descendant¹⁹

$$\chi_+(z) = \left[\hat{L}_{-2}(z) - \frac{3}{2(2\Delta_+ + 1)} \hat{L}_{-1}^2(z) \right] V_+(z)$$

of $V_+(z)$ which corresponds to the null vector at the second level of the Verma module \mathcal{V}_{Δ_+} . According to the null vector decoupling theorem [54] (see also [55]) the matrix element of $\chi_+(z)$, between the states with the highest weights obeying (3.1), must vanish. In particular, vanishes the matrix element

$$\begin{aligned} \langle \Delta', \Lambda^2 | \chi_+(z) | \tilde{\Delta}, \Lambda^2 \rangle &= \langle \Delta', \Lambda^2 | \hat{L}_{-2}(z) V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle \\ &+ \frac{1}{b^2} \langle \Delta', \Lambda^2 | \hat{L}_{-1}^2(z) V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle = 0, \end{aligned}$$

where $\langle \Delta', \Lambda^2 |$ and $| \tilde{\Delta}, \Lambda^2 \rangle$ are the Gaiotto states introduced in (2.1)–(2.3). The above null vector decoupling condition can be converted to the following partial differential equation for the degenerate irregular 3-point block $\Psi(\Lambda, z) = \langle \Delta', \Lambda^2 | V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle$ [18],

$$\left[\frac{1}{b^2} z^2 \frac{\partial^2}{\partial z^2} - \frac{3z}{2} \frac{\partial}{\partial z} + \Lambda^2 \left(z + \frac{1}{z} \right) + \frac{\Lambda}{4} \frac{\partial}{\partial \Lambda} + \frac{\tilde{\Delta} + \Delta' - \Delta_+}{2} \right] \Psi(\Lambda, z) = 0. \quad (3.3)$$

¹⁸Cf. [16].

¹⁹Recall that [53]

$$\hat{L}_{-k}(z) \equiv \frac{1}{2\pi i} \oint_{C_z} dw (w-z)^{1-k} T(w). \quad (3.2)$$

Indeed, using the Ward identity

$$\begin{aligned} \langle \Delta', \Lambda^2 | T(w) V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle &= \left[\frac{z}{w(w-z)} \frac{\partial}{\partial z} + \frac{\Delta_+}{(w-z)^2} + \left(\frac{\Lambda^2}{w} + \frac{\Lambda^2}{w^3} \right) \right. \\ &\quad \left. + \frac{1}{2w^2} \left(\frac{\Lambda}{2} \frac{\partial}{\partial \Lambda} + \tilde{\Delta} + \Delta' - \Delta_+ - z \frac{\partial}{\partial z} \right) \right] \Psi(\Lambda, z), \end{aligned} \quad (3.4)$$

where $T(w) = \sum_{n \in \mathbb{Z}} w^{-n-2} L_n$ is the holomorphic component of the energy-momentum tensor, one can find

$$\begin{aligned} \langle \Delta', \Lambda^2 | \widehat{L}_{-2}(z) V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle &= \left[-\frac{1}{z} \frac{\partial}{\partial z} + \left(\frac{\Lambda^2}{z} + \frac{\Lambda^2}{z^3} \right) \right. \\ &\quad \left. + \frac{1}{2z^2} \left(\frac{\Lambda}{2} \frac{\partial}{\partial \Lambda} + \tilde{\Delta} + \Delta' - \Delta_+ - z \frac{\partial}{\partial z} \right) \right] \Psi(\Lambda, z), \end{aligned} \quad (3.5)$$

$$\langle \Delta', \Lambda^2 | \widehat{L}_{-1}^2(z) V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle = \frac{\partial^2}{\partial z^2} \Psi(\Lambda, z). \quad (3.6)$$

For a derivation of eqs. (3.4), (3.5), see appendix B.

At this point, a few comments concerning $\Psi(\Lambda, z)$ are necessary. First, let us note that from (2.4) we have

$$\begin{aligned} \Psi(\Lambda, z) &= \langle \Delta', \Lambda^2 | V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle = z^{\Delta' - \Delta_+ - \tilde{\Delta}} \sum_{m, n \geq 0} \Lambda^{2(m+n)} z^{m-n} \\ &\quad \times \sum_{|I|=m} \sum_{|J|=n} [G_{c, \Delta'}^m]^{(1^m)I} \langle \Delta', I | V_+(1) | \tilde{\Delta}, J \rangle [G_{c, \tilde{\Delta}}^n]^{J(1^n)} \end{aligned} \quad (3.7)$$

$$\equiv z^\kappa \Phi(\Lambda, z), \quad (3.8)$$

where $\kappa \equiv \Delta' - \Delta_+ - \tilde{\Delta}$.

Let us observe that $\Phi(z, \Lambda)$ can be split into two parts, i.e. when $m = n$ and $m \neq n$:

$$\Phi(\Lambda, z) = \Phi^{(m=n)}(\Lambda) + \Phi^{(m \neq n)}(\Lambda, z), \quad (3.9)$$

$$\Phi^{(m=n)}(\Lambda) = \sum_{n \geq 0} \Lambda^{4n} \sum_{|I|=n} [G_{c, \Delta'}^n]^{(1^n)I} \langle \Delta', I | V_+(1) | \tilde{\Delta}, I \rangle [G_{c, \tilde{\Delta}}^n]^{I(1^n)},$$

$$\Phi^{(m \neq n)}(\Lambda, z) = \sum_{\substack{m \neq n \\ m, n \geq 0}} \Lambda^{2(m+n)} z^{m-n} \sum_{\substack{|I|=m \\ |J|=n}} [G_{c, \Delta'}^m]^{(1^m)I} \langle \Delta', I | V_+(1) | \tilde{\Delta}, J \rangle [G_{c, \tilde{\Delta}}^n]^{J(1^n)}.$$

Then, one can write

$$\begin{aligned} \Psi(\Lambda, z) &= z^\kappa \exp \left\{ \log \left(\Phi^{(m=n)}(\Lambda) + \Phi^{(m \neq n)}(\Lambda, z) \right) \right\} \\ &= z^\kappa \exp \left\{ \log \Phi^{(m=n)}(\Lambda) + \log \left(1 + \frac{\Phi^{(m \neq n)}(\Lambda, z)}{\Phi^{(m=n)}(\Lambda)} \right) \right\} \equiv z^\kappa e^{\phi_1(\Lambda)} e^{\phi_2(\Lambda, z)}. \end{aligned} \quad (3.10)$$

Inserting (3.8) into the eq. (3.3) we get

$$\begin{aligned} \left[\frac{1}{b^2} z^2 \partial_z^2 + \left(\frac{2\kappa}{b^2} - \frac{3}{2} \right) z \partial_z + \frac{\Lambda}{4} \partial_\Lambda + \frac{\kappa(\kappa-1)}{b^2} - \frac{3\kappa}{2} \right. \\ \left. + \Lambda^2 \left(z + \frac{1}{z} \right) + \frac{\tilde{\Delta} + \Delta' - \Delta_+}{2} \right] \Phi(\Lambda, z) = 0. \end{aligned} \quad (3.11)$$

3.2 Classical limit

Now, we want to find the limit $b \rightarrow 0$ of the eq. (3.11). This firstly requires to rescale the parameter σ in $\tilde{\Delta}$ and Δ' , i.e. $\sigma = \xi/b$ and to express Λ as $\Lambda = \hat{\Lambda}/(\epsilon_1 b)$. After rescaling we have

$$\Delta', \tilde{\Delta} \xrightarrow{b \rightarrow 0} \frac{1}{b^2} \delta, \quad \text{where} \quad \delta = \lim_{b \rightarrow 0} b^2 \Delta' = \lim_{b \rightarrow 0} b^2 \tilde{\Delta} = \frac{1}{4} - \xi^2, \quad (3.12)$$

$$\tilde{\Delta} + \Delta' - \Delta_+ \xrightarrow{b \rightarrow 0} \frac{1}{b^2} 2 \left(\frac{1}{4} - \xi^2 \right) = \frac{1}{b^2} 2\delta, \quad (3.13)$$

$$\kappa \xrightarrow{b \rightarrow 0} \frac{1}{2} - \xi, \quad \kappa(\kappa - 1) \xrightarrow{b \rightarrow 0} - \left(\frac{1}{4} - \xi^2 \right) = -\delta. \quad (3.14)$$

Note, that $\Delta_+ \xrightarrow{b \rightarrow 0} \mathcal{O}(b^0)$.

Secondly, one has to determine the behavior of the normalized degenerate irregular block $\Phi = z^{-\kappa} \Psi$ when $b \rightarrow 0$. For $\Lambda = \hat{\Lambda}/(\epsilon_1 b)$ and $\Delta', \tilde{\Delta} \xrightarrow{b \rightarrow 0} \frac{1}{b^2} \delta$ it is reasonable to expect, that²⁰

$$\Phi(\Lambda, z) = z^{-\kappa} \langle \Delta', \Lambda^2 | V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle \xrightarrow{b \rightarrow 0} v(\hat{\Lambda}/\epsilon_1, z) \exp \left\{ \frac{1}{b^2} f_\delta(\hat{\Lambda}/\epsilon_1) \right\} \quad (3.15)$$

where (cf. (3.10))

$$v(\hat{\Lambda}/\epsilon_1, z) = \lim_{b \rightarrow 0} e^{\phi_2(\Lambda, z)}, \quad (3.16)$$

$$f_\delta(\hat{\Lambda}/\epsilon_1) = \lim_{b \rightarrow 0} b^2 \phi_1(\Lambda) = \lim_{b \rightarrow 0} b^2 \log \Phi^{(m=n)}(\Lambda). \quad (3.17)$$

Let us stress that the asymptotic (3.15) is a nontrivial statement concerning the 3-point irregular block $\Phi(\Lambda, z)$. We have no rigorous proof of this conjecture. However, the eq. (3.15) can be well confirmed by direct calculations. Indeed, one can check order by order that the limits (3.16) and (3.17) exist. Moreover, the latter yields the classical irregular block.

Then, after substituting (3.15) into the eq. (3.11), multiplying by b^2 , and taking the limit $b \rightarrow 0$ one gets²¹

$$\left[z^2 \partial_z^2 + 2 \left(\frac{1}{2} - \xi \right) z \partial_z + \frac{\hat{\Lambda}^2}{\epsilon_1^2} \left(z + \frac{1}{z} \right) + \frac{\hat{\Lambda}}{4} \partial_{\hat{\Lambda}} f_\delta(\hat{\Lambda}/\epsilon_1) \right] v(z) = 0. \quad (3.18)$$

²⁰The meaning of eq. (3.15) is that the light degenerate chiral vertex operator do not contribute to the classical limit. In other words its presence in the matrix element does not affect the ‘‘classical dynamics’’ (i.e. the ‘‘classical action’’). Let us note that eq. (3.15) is a ‘‘chiral version’’ of Zamolodchikovs’ conjecture [32] (see also [56]) concerning semiclassical behavior of Liouville correlators with heavy and light vertices on the sphere. Let us stress that there are only a few explicitly known tests verifying this hypothesis. For instance, the derivation of a large intermediate conformal weight limit $\Delta \rightarrow \infty$ of the 4-point block on the sphere as well as its expansion in power of the so-called elliptic variable is based on the assumption about such semiclassical behavior of the 5-point function with the light degenerate vertex operator [49, 51]. The calculation performed in this subsection can be seen as yet another new test of semiclassical behavior of the form (3.15) (see also [42]).

²¹The key point here is the fact that $\lim_{b \rightarrow 0} b^2 \frac{\hat{\Lambda}}{4} \partial_{\hat{\Lambda}} v = 0$.

In order to get the Mathieu equation we define a new function $\psi(z)$ related to the old one by

$$v(z) = z^\xi \psi(z). \tag{3.19}$$

Now, what we obtain equals

$$\left[z^2 \partial_z^2 + z \partial_z + \frac{\hat{\Lambda}^2}{\epsilon_1^2} \left(z + \frac{1}{z} \right) + \frac{\hat{\Lambda}}{4} \partial_{\hat{\Lambda}} f_\delta(\hat{\Lambda}/\epsilon_1) - \xi^2 \right] \psi(z) = 0. \tag{3.20}$$

Since for $z = e^w$ the derivatives transform as $(z^2 \partial_z^2 + z \partial_z) \psi(z) = \partial_w^2 \psi(e^w)$ then, the eq. (3.20) goes over to the form

$$\left[\frac{d^2}{dw^2} + 2 \frac{\hat{\Lambda}^2}{\epsilon_1^2} \cosh(w) + \frac{\hat{\Lambda}}{4} \partial_{\hat{\Lambda}} f_\delta(\hat{\Lambda}/\epsilon_1) - \xi^2 \right] \psi(e^w) = 0. \tag{3.21}$$

Finally, the substitution $w = 2ix$, $x \in \mathbb{R}$ in (3.21) yields

$$\left[-\frac{d^2}{dx^2} + 8 \frac{\hat{\Lambda}^2}{\epsilon_1^2} \cos 2x + \hat{\Lambda} \partial_{\hat{\Lambda}} f_\delta(\hat{\Lambda}/\epsilon_1) - 4\xi^2 \right] \psi(e^{2ix}) = 0. \tag{3.22}$$

Now, one can identify the parameters λ and h appearing in the Mathieu equation (1.1) as follows

$$\lambda = -\hat{\Lambda} \partial_{\hat{\Lambda}} f_\delta(\hat{\Lambda}/\epsilon_1) + 4\xi^2, \quad h = \pm \frac{2\hat{\Lambda}}{\epsilon_1}. \tag{3.23}$$

Let us observe that using (2.14) and postulating the following relation

$$\xi = \frac{\nu}{2} \tag{3.24}$$

between the parameter ξ and the Floquet exponent ν one finds

$$\begin{aligned} \lambda &= -\hat{\Lambda} \partial_{\hat{\Lambda}} \left[\sum_{n \geq 1} \left(\hat{\Lambda}/\epsilon_1 \right)^{4n} f_\delta^n \right] + 4\xi^2 \\ &= -\frac{4h^4}{16} f_{\frac{1}{4}-\frac{\nu^2}{4}}^1 - \frac{8h^8}{256} f_{\frac{1}{4}-\frac{\nu^2}{4}}^2 - \frac{12h^{12}}{4096} f_{\frac{1}{4}-\frac{\nu^2}{4}}^3 - \dots + 4 \left(\frac{\nu^2}{4} \right) \\ &= \nu^2 + \frac{h^4}{2(\nu^2 - 1)} + \frac{(5\nu^2 + 7) h^8}{32(\nu^2 - 4)(\nu^2 - 1)^3} + \frac{(9\nu^4 + 58\nu^2 + 29) h^{12}}{64(\nu^2 - 9)(\nu^2 - 4)(\nu^2 - 1)^5} + \dots \end{aligned}$$

Hence, the formula (3.23) reproduces the expansion (1.4).

On the other hand one can check that the formula (3.23) exactly matches that obtained by means of the WKB method. Indeed, taking into account that (cf. (2.25))

$$\delta = \frac{1}{4} - \frac{a^2}{\epsilon_1^2} = \frac{1}{4} - \xi^2 \iff \xi = \frac{a}{\epsilon_1}$$

one can rewrite the eq. (3.22) to the following Schrödinger-like form:

$$\left[-\epsilon_1^2 \frac{d^2}{dx^2} + 8\hat{\Lambda}^2 \cos 2x \right] \psi = \mathbf{E} \psi, \tag{3.25}$$

where

$$E = \epsilon_1^2 \lambda = 4a^2 - \epsilon_1^2 \hat{\Lambda} \partial_{\hat{\Lambda}} f_{\frac{1}{4} - \frac{a^2}{\epsilon_1^2}}(\hat{\Lambda}/\epsilon_1). \quad (3.26)$$

The “energy eigenvalue” E in eq. (3.25) can be computed by making use of the WKB method. These calculations are performed in the appendix C. The result of the WKB calculations coincides with E given by eq. (3.26). This check is yet another example of an interesting link between the semiclassical limit of conformal blocks and the WKB approximation.

4 Mathieu eigenvalue from $\mathcal{N} = 2$ gauge theory

We saw in previous sections that the eigenvalue of the Mathieu operator is related to the classical irregular conformal block. The latter appeared to be a limit of the quantum irregular conformal block when $b^2 \sim \hbar \rightarrow 0$. The irregular quantum conformal block is found to be related [45] to the Nekrasov’s instanton partition function of the pure ($N_f = 0$) $\mathcal{N} = 2$ SYM. Nekrasov’s partition function in the zero limit of one of its parameters (in what follows we take it to be ϵ_2) yields an effective twisted superpotential. Therefore it is natural to expect that thus established correspondence between the two theories at the quantum level also extends to the classical level. In the preceding section we found a relation between the expansion parameters on both sides of the correspondence $\mathcal{F}_{c,\Delta}(\Lambda) = \mathcal{Z}_{\text{inst}}^{N_f=0, \text{SU}(2)}(\hat{\Lambda}/\epsilon_1, a, b)$ and verified the agreement between expansion coefficients of classical irregular block and twisted superpotential up to the third order. In this section we address the derivation of the twisted superpotential \mathcal{W} from the representation of the instanton partition function for $\mathcal{N} = 2$ pure gauge SYM in terms of profile functions of the Young diagrams employed in this context first by Nekrasov and Okounkov [21].

4.1 Nekrasov-Shatashvili limit

The Nekrasov’s partition function for $\mathcal{N} = 2$ pure SYM with $\text{SU}(N)$ symmetry on Ω -background relates instanton configurations on a moduli space with partitions a graphical representation of whose are Young diagrams. This relationship makes the mentioned gauge theory tantamount to the theory of random partitions. There are five equivalent forms of the instanton partition function (see appendix D for notation and more information about Nekrasov partition function). For our purpose we use the following one

$$\mathcal{Z}^{N_f=0, \text{SU}(N)}(\hat{\Lambda}, \mathbf{a}, \epsilon_1, \epsilon_2) = \exp \left\{ - \sum_{\alpha, \beta=1} \gamma_{\epsilon_1, \epsilon_2}(a_\alpha - a_\beta; \hat{\Lambda}) \right\} \mathcal{Z}_{\text{inst}}^{N_f=0, \text{SU}(N)}(\hat{\Lambda}, \mathbf{a}, \epsilon_1, \epsilon_2), \quad (4.1a)$$

where the first factor on the right hand side is the perturbative part and the second one reads

$$\mathcal{Z}_{\text{inst}}^{N_f=0, \text{SU}(N)}(\hat{\Lambda}, \mathbf{a}, \epsilon_1, \epsilon_2) \equiv \sum_{k \geq 0} \left(\frac{\hat{\Lambda}}{\epsilon_1} \right)^{2Nk} Z_k(\mathbf{a}, \epsilon_1, \epsilon_2), \quad (4.1b)$$

$$Z_k(\mathbf{a}, \epsilon_1, \epsilon_2) \equiv \sum_{\mathbf{k}: |\mathbf{k}|=k} b^{-4Nk} \prod_{(\alpha, i) \neq (\beta, j)} \frac{\Gamma(\epsilon_2^{-1}(x_{\alpha, i} - x_{\beta, j} - \epsilon_1)) \Gamma(\epsilon_2^{-1}(x_{\alpha, i}^0 - x_{\beta, j}^0))}{\Gamma(\epsilon_2^{-1}(x_{\alpha, i} - x_{\beta, j})) \Gamma(\epsilon_2^{-1}(x_{\alpha, i}^0 - x_{\beta, j}^0 - \epsilon_1))}. \quad (4.1c)$$

The arguments of the Euler gamma functions are defined as follows²²

$$x_{\alpha,i} \equiv a_\alpha + \epsilon_1(i-1) + \epsilon_2 k_{\alpha i}, \quad x_{\alpha,i}^0 \equiv a_\alpha + \epsilon_1(i-1). \quad (4.2)$$

Thus, the instanton partition function is expressed in terms of partitions of an integer instanton number k i.e., $k = |\mathbf{k}| = \sum |\mathbf{k}_\alpha| = \sum k_{\alpha,i}$, $\alpha = 1, \dots, N$ and $i \in \mathbb{N}$, such that for any $i < j$ and fixed α , $k_{\alpha,i} \geq k_{\alpha,j} \geq 0$. This particular form of it is related to the one, defined in terms of the profiles of the deformed Young diagrams. Namely,

$$\mathcal{Z}^{\text{Nf}=0, \text{SU}(N)}(\hat{\Lambda}, \mathbf{a}, \epsilon_1, \epsilon_2) = \sum_{f_{\mathbf{a},\mathbf{k}} \in \mathcal{P}(\mathbb{Y}^N)} \exp \left\{ -\frac{1}{4} \iint_{\mathbb{R}^2} dx dy f''_{\mathbf{a},\mathbf{k}}(x|\epsilon_1, \epsilon_2) \gamma_{\epsilon_1, \epsilon_2}(x-y; \hat{\Lambda}) f''_{\mathbf{a},\mathbf{k}}(y|\epsilon_1, \epsilon_2) \right\}, \quad (4.3)$$

where $\mathcal{P}(\mathbb{Y}^N)$ denotes the space of all profiles of N -tuple of Young diagrams. The profile function is defined as follows²³

$$f_{\mathbf{a},\mathbf{k}}(x|\epsilon_1, \epsilon_2) \equiv \sum_{\alpha=1}^N |x - a_\alpha| + \sum_{\alpha=1}^N \sum_{i \geq 1} \left(|x - x_{\alpha,i}^0 - \epsilon_1| - |x - x_{\alpha,i} - \epsilon_1| - |x - x_{\alpha,i}^0| + |x - x_{\alpha,i}| \right). \quad (4.4)$$

In what follows we work with the instanton density rather than with the profile functions. The linear density function of instantons at position \mathbf{a} and configuration \mathbf{k} is defined as

$$\rho_{\mathbf{a},\mathbf{k}}(x|\epsilon_1, \epsilon_2) \equiv f_{\mathbf{a},\mathbf{k}}(x|\epsilon_1, \epsilon_2) - f_{\mathbf{a},\emptyset}(x|\epsilon_1, \epsilon_2), \quad (4.5)$$

where the second term is the empty profile given by the first term on the right hand side of eq. (4.4). The instanton density satisfies the following normalization condition

$$k = \frac{1}{-2\epsilon_1\epsilon_2} \int_{\mathbb{R}} dx \rho_{\mathbf{a},\mathbf{k}}(x|\epsilon_1, \epsilon_2). \quad (4.6)$$

The above equation shows that the density function stores the information about both, the number of instantons and their configuration. The partition function expressed in terms of profiles in eq. (4.3) can now be written in terms of instanton densities. Discarding the perturbative part which takes the form of the multiplier in eq. (4.1a) the instanton part reads

$$\mathcal{Z}_{\text{inst}}^{\text{Nf}=0, \text{SU}(N)}(\hat{\Lambda}, \mathbf{a}, \epsilon_1, \epsilon_2) = \sum_{\rho_{\mathbf{a},\mathbf{k}} \in \mathcal{R}(\mathbb{Y}^N)} \exp \left\{ -H_{\text{inst}}[\rho_{\mathbf{a},\mathbf{k}}](\epsilon_1, \epsilon_2, \hat{\Lambda}) \right\}, \quad (4.7)$$

where $\mathcal{R}(\mathbb{Y}^N) \subset \mathcal{P}(\mathbb{Y}^N)$.²⁴ The Hamiltonian for instanton configurations structurally reads

$$H_{\text{inst}}[\rho_{\mathbf{a},\mathbf{k}}](\epsilon_1, \epsilon_2, \hat{\Lambda}) \equiv \frac{1}{4} (f''_{\emptyset})^i \gamma_{\epsilon_1, \epsilon_2}(\hat{\Lambda})_{ij} (\rho_{\mathbf{k}}'')^j + \frac{1}{4} (\rho_{\mathbf{k}}'')^i \gamma_{\epsilon_1, \epsilon_2}(\hat{\Lambda})_{ij} (f''_{\emptyset})^j + \frac{1}{4} (\rho_{\mathbf{k}}'')^i \gamma_{\epsilon_1, \epsilon_2}(\hat{\Lambda})_{ij} (\rho_{\mathbf{k}}'')^j. \quad (4.8)$$

²²For the graphical meaning of $x_{\alpha,i}$ see appendix D and figure 3.

²³Note, that this definition of the profile function, when defined for real argument, is meaningful if and only if both parameters ϵ_1 or ϵ_2 have opposite signs. For definiteness we take $\epsilon_2 < 0 < \epsilon_1$.

²⁴ $\mathcal{R}(\mathbb{Y}^N)$ is a space of all functions over N -tuple of Young diagrams with a compact support.

In the above equation we introduced the following notation

$$(\rho''_{\mathbf{k}})^i \gamma_{\epsilon_1, \epsilon_2}(\hat{\Lambda})_{ij} (\rho''_{\mathbf{k}})^j \equiv \iint_{\mathbb{R}^2} dx dy \rho''_{\mathbf{a}, \mathbf{k}}(x|\epsilon_1, \epsilon_2) \gamma_{\epsilon_1, \epsilon_2}(x-y; \hat{\Lambda}) \rho''_{\mathbf{a}, \mathbf{k}}(y|\epsilon_1, \epsilon_2). \quad (4.9)$$

In what follows we are concerned with the form of the instanton partition function within the Nekrasov-Shatashvili limit i.e., when one of the deformation parameters tends to zero. This limit of the Nekrasov partition function defines the effective twisted superpotential. Explicitly

$$\mathcal{W}^{\text{Nf}=0, \text{SU}(N)}(\hat{\Lambda}, \mathbf{a}, \epsilon_1) \equiv - \lim_{\epsilon_2 \rightarrow 0} \epsilon_2 \log \mathcal{Z}^{\text{Nf}=0, \text{SU}(N)}(\hat{\Lambda}, \mathbf{a}, \epsilon_1, \epsilon_2), \quad (4.10)$$

where $\mathcal{W} = \mathcal{W}_{\text{pert}} + \mathcal{W}_{\text{inst}}$. In the case under study this limit can be approached by taking the thermodynamic limit for the number of instantons and simultaneously squeezing the boxes in ϵ_2 direction. Let us consider the relation between the instanton number and the density of instantons given in eq. (4.6). By definition this formula is satisfied by any partition of k . Next, let us choose the one that corresponds to the colored Young diagram with the highest column for $\alpha = 1$ color index, i.e., $\mathbf{k}_0 \equiv \{k_{\alpha, i}\} = \{\delta_{\alpha, 1} \delta_{i, 1} k\}$ and take the limit of $\epsilon_2 k$ while keeping the area under ρ constant, i.e.,

$$\lim_{\substack{\epsilon_2 \rightarrow 0 \\ k \rightarrow \infty}} \epsilon_2 k = - \frac{1}{2\epsilon_1} \lim_{\substack{\epsilon_2 \rightarrow 0 \\ k \rightarrow \infty}} \int_{\mathbb{R}} dx \rho_{\mathbf{a}, \mathbf{k}_0}(x|\epsilon_1, \epsilon_2) = - \frac{1}{2\epsilon_1} \int_{\mathbb{R}} dx \rho_{\mathbf{a}, \boldsymbol{\omega}_0}(x|\epsilon_1) = \omega. \quad (4.11)$$

Within this limit columns of the diagram $\boldsymbol{\omega}$ become nonpositive real numbers and $\rho_{\mathbf{a}, \boldsymbol{\omega}}$ becomes a function of infinite many variables $\omega_{\alpha, i} \in \mathbb{R}_{\leq 0}$.

With this picture in mind we would like to determine a squeezed colored Young diagram upon which the weight in the partition function, given in the deformed version of eq. (4.7), yields a dominant contribution over the other summands. In order to find it, let us expand the instanton Hamiltonian in ϵ_2 about zero, namely

$$H_{\text{inst}}[\rho_{\mathbf{a}, \mathbf{k}}](\hat{\Lambda}, \epsilon_1, \epsilon_2) = \sum_{g \geq 0} \epsilon_2^{g-1} c_g H_{\text{inst}}^{(g)}[\rho_{\mathbf{a}, \boldsymbol{\omega}}](\hat{\Lambda}, \epsilon_1), \quad (4.12a)$$

$$c_0 = 1, \quad c_1 = -\frac{1}{2}, \quad c_2 = \frac{1}{12}, \quad \dots,$$

where $H_{\text{inst}}^{(g)}$ is defined in eq. (4.8) with the kernel $\gamma_{\epsilon_1, \epsilon_2}$ traded for the following one²⁵

$$\gamma_{\epsilon_1}^{(g)}(x; \hat{\Lambda}) = \epsilon_1^{1-g} \frac{d}{ds} \left[\left(\frac{\hat{\Lambda}}{\epsilon_1} \right)^s (s)_{g-1} \zeta \left(s + g - 1, \frac{x}{\epsilon_1} + 1 \right) \right]_{s=0}. \quad (4.12b)$$

From explicit form of the expansion in eq. (4.12a) it is seen that the dominant term within the limit $\epsilon_2 \rightarrow 0$ equals

$$H_{\text{inst}}[\rho_{\mathbf{a}, \mathbf{k}}](\hat{\Lambda}, \epsilon_1, \epsilon_2) \approx \frac{1}{\epsilon_2} H_{\text{inst}}^{(0)}[\rho_{\mathbf{a}, \boldsymbol{\omega}}](\hat{\Lambda}, \epsilon_1) \equiv \frac{1}{\epsilon_2} \mathcal{H}[\rho_{\mathbf{a}, \boldsymbol{\omega}}](\hat{\Lambda}, \epsilon_1). \quad (4.13)$$

²⁵ $(s)_n \equiv \Gamma(n+s)/\Gamma(n)$.

The squeezed Hamiltonian $\mathcal{H}[\rho_{\mathbf{a},\omega}]$ takes the analogous form to the one in eq. (4.8) with the following substitutions: $\rho_{\mathbf{a},\mathbf{k}}(x|\epsilon_1, \epsilon_2) \rightarrow \rho_{\mathbf{a},\omega}(x|\epsilon_1)$, $\gamma_{\epsilon_1, \epsilon_2}(x, \hat{\Lambda}) \rightarrow \gamma_{\epsilon_1}^{(0)}(x, \hat{\Lambda}) \equiv \gamma_{\epsilon_1}(x, \hat{\Lambda})$. It can also be cast into the form that proves useful in what follows, namely²⁶

$$\begin{aligned} \mathcal{H}[\rho_{\mathbf{a},\omega}](\hat{\Lambda}, \epsilon_1) &= \frac{N}{\epsilon_1} \log \left(\frac{\hat{\Lambda}}{\epsilon_1} \right) \sum (\rho_{\omega})^i + \frac{1}{4} (f_{\theta}'')^i \gamma_{\epsilon_1}(\epsilon_1)_{ij} (\rho_{\omega}'')^j + \frac{1}{4} (\rho_{\omega}'')^i \gamma_{\epsilon_1}(\epsilon_1)_{ij} (f_{\theta}'')^j \\ &+ \frac{1}{4} (\rho_{\omega}'')^i \gamma_{\epsilon_1}(\epsilon_1)_{ij} (\rho_{\omega}'')^j, \end{aligned} \quad (4.14)$$

where

$$\sum (\rho_{\omega})^i \equiv \int_{\mathbb{R}} dx \rho_{\mathbf{a},\omega}(x|\epsilon_1).$$

It is expected that the sum over all profiles and hence over all instanton densities for a large k approaches the path integral over the latter. This can be understood through the analysis of number of configurations accessible for k instantons, i.e., the dimension of the configuration space Ω for a given k . Since the number of partitions runs with k like $p(k) \sim \exp(\pi\sqrt{2k/3} - \log k)$, which stems from the Hardy-Ramanujan theorem, the sum over colored partitions runs with k like $(k = \sum \mathbb{k}_{\alpha})$ ²⁷

$$\dim \Omega_k = \sum_{r=1}^{c(k,N)} \prod_{\alpha=1}^N p(|\mathbb{k}_{\alpha}^r|) \stackrel{k \gg 1}{\approx} \sum_{r=1}^{k^N/(N-1)!} \exp \left\{ \pi \sum_{\alpha=1}^N \sqrt{\frac{2}{3} |\mathbb{k}_{\alpha}^r|} \right\}.$$

This estimation shows that the dimension grows very fast with k such that the discrete distribution of configurations approaches the continuous one. Therefore, the sum over densities may be approximated by the functional integral over $\{\omega_i\}_{i \in \mathbb{N}} \in \mathbf{c}_0(\mathbb{R}_{\leq 0})^N$ i.e., the space of sequences monotonically convergent to zero. Hence,

$$\mathcal{Z}_{\text{inst}}^{N_f=0, \text{SU}(N)}(\Lambda, \mathbf{a}, \epsilon_1, \epsilon_2) \stackrel{\epsilon_2 \rightarrow 0}{\underset{k \rightarrow \infty}{\approx}} \int_{\mathbf{c}_0(\mathbb{R}_{\leq 0})^N} \prod_{\alpha, i} d\omega_{\alpha, i} e^{-\frac{1}{\epsilon_2} \mathcal{H}[\rho_{\mathbf{a},\omega}]}, \quad (4.15)$$

where $\omega_{\alpha, i} = x_{\alpha, i} - x_{\alpha, i}^0$.

4.2 Saddle point equation

We would like to determine the extremal density function for which the weight in the path integral in eq. (4.15) becomes the dominant contribution. In order to do this let us consider two Young diagrams that differ in small fluctuations within each column, i.e.,

$$\tilde{\omega}_{\alpha, i} = \omega_{\alpha, i} + \epsilon_2 \delta \omega_{\alpha, i} \quad \Rightarrow \quad \rho_{\mathbf{a}, \tilde{\omega}}(x) = \rho_{\mathbf{a}, \omega}(x) + \epsilon_2 \delta \rho_{\mathbf{a}, \omega}(x),$$

where²⁸

$$\delta \rho_{\mathbf{a}, \omega}(x) = \sum_{\alpha=1}^N \sum_{i \geq 1} [\text{sgn}(x - x_{\alpha, i}) - \text{sgn}(x - x_{\alpha, i} - \epsilon_1)] \delta \omega_{\alpha, i}. \quad (4.16)$$

²⁶To find this form the identity has been used: $\gamma_{\epsilon_1}(x; \hat{\Lambda}) = -\epsilon_1 \log(\hat{\Lambda}/\epsilon_1) \zeta(-1, x/\epsilon_1 + 1) + \gamma_{\epsilon_1}(x; \epsilon_1)$.

²⁷ $c(k, N)$ is defined as the number of possibilities for k to be partitioned into N nonnegative, integer parts. This number equals $c(k, N) = (k + N - 1)!/k!(N - 1)!$.

²⁸The signum function is defined as $\text{sgn}(x) \equiv |x|/x = \sum_{s \in \mathbb{Z}_2} s \theta(sx)$.

In this case the Hamiltonian (4.14) will suffer the change

$$\mathcal{H}[\rho_{\mathbf{a},\tilde{\omega}}] = \mathcal{H}[\rho_{\mathbf{a},\omega}] + \epsilon_2 \frac{\delta \mathcal{H}[\rho_{\mathbf{a},\omega}]}{\delta \rho_{\mathbf{a},\omega}^i} \delta \rho_{\mathbf{a},\omega}^i + \mathcal{O}(\epsilon_2^2).$$

For some stationary point $\rho_{*\mathbf{a},\omega}$

$$\frac{\delta \mathcal{H}[\rho_{*\mathbf{a},\omega}]}{\delta \rho_{\mathbf{a},\omega}^i} \delta \rho_{\mathbf{a},\omega}^i \equiv \int_{\mathbb{R}} dx \frac{\delta \mathcal{H}[\rho_{*\mathbf{a},\omega}]}{\delta \rho_{\mathbf{a},\omega}(x)} \delta \rho_{\mathbf{a},\omega}(x) = 0, \quad \rho_{*\mathbf{a},\omega}(x) = \rho_{\mathbf{a},\omega_*}(x), \quad (4.17)$$

where ω_* is the extremal of the colored Young diagram. Hence, within the limit $\epsilon_2 \rightarrow 0$ the only term that is left from eq. (4.15) reads

$$\mathcal{Z}_{\text{inst}}^{\text{Nf}=0, \text{SU}(N)}(\Lambda, \mathbf{a}, \epsilon_1, \epsilon_2) \sim \exp \left\{ -\frac{1}{\epsilon_2} \mathcal{H}[\rho_{*\mathbf{a},\omega}](\hat{\Lambda}, \epsilon_1) \right\},$$

and by virtue of eq. (4.10) we obtain

$$\mathcal{W}_{\text{inst}}^{\text{Nf}=0, \text{SU}(N)}(\hat{\Lambda}, \mathbf{a}, \epsilon_1) = \mathcal{H}[\rho_{*\mathbf{a},\omega}](\hat{\Lambda}, \epsilon_1). \quad (4.18)$$

Thus, in order to obtain the form of the twisted superpotential we must find the explicit form of the Hamiltonian at the extremal solving the saddle point equation (4.17). Its functional form reads

$$\begin{aligned} & \frac{N}{\epsilon_1} \log \left(\frac{\hat{\Lambda}}{\epsilon_1} \right) \int_{\mathbb{R}} dx x \delta \rho'_{\mathbf{a},\omega}(x) \\ & + \frac{1}{4} \iint_{\mathbb{R}^2} dx dy \left[f''_{\mathbf{a},\emptyset}(x) \partial_y \gamma_{\epsilon_1}(x-y) \delta \rho'_{\mathbf{a},\omega}(y) + \delta \rho'_{\mathbf{a},\omega}(y) \partial_y \gamma_{\epsilon_1}(y-x) f''_{\mathbf{a},\emptyset}(x) \right. \\ & \left. + \rho''_{\mathbf{a},\omega}(x) \partial_y \gamma_{\epsilon_1}(x-y) \delta \rho'_{\mathbf{a},\omega}(y) + \delta \rho'_{\mathbf{a},\omega}(y) \partial_y \gamma_{\epsilon_1}(y-x) \rho''_{\mathbf{a},\omega}(x) \right] = 0. \end{aligned} \quad (4.19)$$

The above equation for critical density is directly related to the equation on critical squeezed Young diagram. In order to find the latter one must express eq. (4.19) in terms of $x_{\alpha,i}$. After some algebra we find

$$\begin{aligned} 0 = & \sum_{\alpha,i} \left[2N \log \left(\frac{\hat{\Lambda}}{\epsilon_1} \right) - \sum_{\beta=1}^N \log \left(\frac{(a_\beta - x_{\alpha,i} - \epsilon_1)(x_{\alpha,i} - a_\beta)}{\epsilon_1^2} \right) \right] \delta \omega_{\alpha,i} \\ & + \sum_{(\alpha,i) \neq (\beta,j)} \log \left(\frac{x_{\alpha,i} - x_{\beta,j} - \epsilon_1}{x_{\alpha,i} - x_{\beta,j} + \epsilon_1} \right) \delta \omega_{\alpha,i} + \sum_{\alpha,i;\beta,j} \log \left(\frac{x_{\alpha,i} - x_{\beta,j}^0 + \epsilon_1}{x_{\alpha,i} - x_{\beta,j}^0 - \epsilon_1} \right) \delta \omega_{\alpha,i}. \end{aligned} \quad (4.20)$$

The ‘‘off-diagonal’’ sum comes from necessity of taking the principal values of integrals in the last line of eq. (4.19) which is due to the singularity of $\gamma'_{\epsilon_1}(x) \sim \log \Gamma(x)$ at zero. In order to put the sums on the same footing we add zero to the above equation in the form that completes the ‘‘off-diagonal’’ sum with the missing diagonal term, i.e.,

$$\sum_{\alpha,i} \left[\log \left(\frac{x_{\alpha,i} - x_{\alpha,i} - \epsilon_1}{x_{\alpha,i} - x_{\alpha,i} + \epsilon_1} \right) - \sqrt{-1} \pi \right] \delta \omega_{\alpha,i}.$$

The last step and the arbitrariness of the fluctuations $\delta\omega_{\alpha,i}$ enable us to cast the saddle point equation into the form first found in ref. [57]. Namely,

$$(-1)^N (\hat{\Lambda}/\epsilon_1)^{2N} \prod_{\beta=1}^N \prod_{j \geq 1} \frac{(x_{\alpha,i} - x_{\beta,j} - \epsilon_1)(x_{\alpha,i} - x_{\beta,j}^0 + \epsilon_1)}{(x_{\alpha,i} - x_{\beta,j} + \epsilon_1)(x_{\alpha,i} - x_{\beta,j}^0 - \epsilon_1)} \prod_{j=1}^2 \frac{\epsilon_1}{x_{\alpha,i} - x_{\beta,j}^0 + \epsilon_1} = -1. \quad (4.21)$$

The equation (4.21) in fact represents an infinite set of equations for the critical colored squeezed Young diagram. As one may infer from the above formula, the infinite sequence of columns of the critical diagram for a fixed color must be strictly increasing (or strictly decreasing if one considers $|\omega_{\alpha,i}|$).²⁹ In order to see this, consider the factor in the denominator that takes the form $x_{\alpha,i} - x_{\beta,j} + \epsilon_1$ and for $\beta = \alpha$ choose the neighboring columns with $i = j - 1$. For this specific choice just mentioned formula amounts to $\omega_{\alpha,j-1} - \omega_{\alpha,j}$ which, by definition of partitions, must be nonpositive. In fact, it must be strictly negative, because it is a factor of the product of terms in the denominator. Therefore, the critical diagram consists of columns each of which has the multiplicity at most one and as such form a *strictly monotone null sequence of negative real numbers*, i.e., $\{\omega_{*i}\}_{i \in \mathbb{N}} \in \mathbf{c}_0(\mathbb{R}_{<0})^N$.

4.3 Solution to saddle point equation

The saddle point equation (4.21) can be solved iteratively by means of the method first proposed by Poghossian [57]. Noticing, that it depends on the only free parameter $(\hat{\Lambda}/\epsilon_1)^{2N}$, the columns, that are sought quantities, can be expressed in terms of powers of $(\hat{\Lambda}/\epsilon_1)^{2N}$. The advantage of this particular form of the saddle point equation is that it does not change if we introduce an explicit cutoff length for each of N constituents of the colored Young diagram regardless of whether this cutoff is colored or common for every α . Therefore, the first step is to introduce a common cutoff length L that starts from 1 and increases along with each step of iteration, such that $|\omega_{\alpha,i}| > 0$ for $i \in \{1, \dots, L\}$ and assumes zero value elsewhere. Furthermore, the ansatz one starts with takes the form

$$x_{\alpha,i}^{(L)} = x_{\alpha,i}^0 + \delta x_{\alpha,i}^{(L)}, \quad \delta x_{\alpha,i}^{(L)} \equiv x_{\alpha,i}^{(L)} - x_{\alpha,i}^0 = \omega_{* \alpha, i}^{(L)}. \quad (4.22)$$

The form of the above ansatz can be justified with the aid of the following argument. Let us first note that by construction

$$\omega_* = \sum_{\alpha=1}^N \sum_{i \geq 1} \omega_{* \alpha, i} = \sum_{i \geq 1} \left(\sum_{\alpha=1}^N \delta x_{\alpha, i} \right) = \lim_{L \rightarrow \infty} \sum_{i=1}^L \sum_{\alpha=1}^N \delta x_{\alpha, i}^{(L)}. \quad (4.23)$$

On the other hand from eq. (4.14) we get

$$-\omega_* = (\hat{\Lambda}/\epsilon_1)^{2N} \frac{d\mathcal{H}[\rho_* \mathbf{a}, \omega]}{d(\hat{\Lambda}/\epsilon_1)^{2N}} = (\hat{\Lambda}/\epsilon_1)^{2N} \frac{\partial \mathcal{H}[\rho_* \mathbf{a}, \omega]}{\partial (\hat{\Lambda}/\epsilon_1)^{2N}},$$

where we used the saddle point equation in the form given in eq. (4.17). Employing the definition of the Nekrasov-Shatashvili limit (4.10) we can formally expand the right hand

²⁹Recall, that $\omega_{\alpha,i} = \lim_{\substack{\epsilon_2 \rightarrow 0 \\ k \rightarrow \infty}} \epsilon_2 k_{\alpha,i}$ and we have chosen $\epsilon_2 < 0$. Hence, $\omega_{\alpha,i} \leq 0$ for any $i \in \mathbb{N}$.

side of eq. (4.10), and due to the established relationship between the superpotential $\mathcal{W}_{\text{inst}}$ and \mathcal{H} in eq. (4.18) we find³⁰

$$-\omega_* = (\hat{\Lambda}/\epsilon_1)^{2N} \frac{\partial \mathcal{W}_{\text{inst}}(\hat{\Lambda}, \mathbf{a}, \epsilon_1)}{\partial (\hat{\Lambda}/\epsilon_1)^{2N}} = \sum_{i \geq 1} (\hat{\Lambda}/\epsilon_1)^{2Ni} W_i(\mathbf{a}, \epsilon_1). \quad (4.24)$$

The above equation and eq. (4.23) imply that $\sum_{\alpha} \delta x_{\alpha,i}^{(L)} \sim \mathcal{O}((\hat{\Lambda}/\epsilon_1)^{2Ni})$. Hence,

$$\delta x_{\alpha,i}^{(L)} = \omega_{*\alpha,i}^{(L)}(\hat{\Lambda}/\epsilon_1, \mathbf{a}, \epsilon_1) = \sum_{j=i}^L (\hat{\Lambda}/\epsilon_1)^{2Nj} \omega_{\alpha,i,j}(\mathbf{a}, \epsilon_1). \quad (4.25)$$

Thus, the iterative solution of eq. (4.21) amounts to finding the coefficients $\omega_{\alpha,i,j}$. Their forms for the gauge symmetry group $\text{SU}(N)$ and $N = 2$ with the above ansatz up to $j = 3$ are gathered in appendix F.

The generic case when $L \rightarrow \infty$ can be examined through the saddle point equation given in a different form. Making use of identities given in appendix E the equation (4.21) assumes yet another form

$$\left(\frac{\hat{\Lambda}}{\epsilon_1}\right)^{2N} \frac{Y(x_{\alpha,i} - \epsilon_1)}{Y(x_{\alpha,i} + \epsilon_1)} = (-1)^{N-1}, \quad (4.26)$$

where the function $Y(z)$ is defined in eq. (E.4). Since $x_{\alpha,i}$ are zeros of this function the saddle point equation in the form of eq. (4.21) is traded for the equations on zeros themselves.

4.4 Mathieu eigenvalue — contour integral representation

The iterative solution of the saddle point equation (4.21) enables us to express the effective twisted superpotential in terms of the expansion parameter $(\hat{\Lambda}/\epsilon_1)^{2N}$ as a sum over columns of the critical Young diagram. This can be formally accomplished by means of the formula (4.24) and integrating it term by term with respect to the expansion parameter. Explicitly

$$\mathcal{W}_{\text{inst}}^{\text{N}_f=0, \text{SU}(N)}(\hat{\Lambda}, \mathbf{a}, \epsilon_1) = - \int_0^{(\hat{\Lambda}/\epsilon_1)^{2N}} \frac{dq}{q} \omega_*(q) = - \sum_{i \geq 1} \int_0^{(\hat{\Lambda}/\epsilon_1)^{2N}} dq q^{i-1} \left(\sum_{\alpha=1}^N \sum_{j=1}^i \omega_{\alpha,j,i} \right). \quad (4.27)$$

In the above expression we employed eq. (4.25) where the coefficients $\omega_{\alpha,j,i}$ are introduced. For the gauge group with $N = 2$ which is the case in question, the superpotential was computed up to the third order in the expansion parameter. Namely, making use of the following formula ($\sum a_{\alpha} = 0$)

$$\mathcal{W}_{\text{inst}}^{\text{N}_f=0, \text{SU}(2)}(\hat{\Lambda}, a, \epsilon_1) = \sum_{i \geq 1} (\hat{\Lambda}/\epsilon_1)^{4i} W_i(a, \epsilon_1)/i, \quad W_i = - \sum_{\alpha=1}^2 \sum_{j=1}^i \omega_{\alpha,j,i}.$$

³⁰The relationship of the coefficients W_i in the last formula with those introduced earlier in eq. (2.11) is $W_i/(i \epsilon_1^{2Ni}) = \mathcal{W}_i$ for $N = 2$.

and the results in appendix F, the coefficients of the expansion written in the form that is suitable for comparison read

$$\begin{aligned}
 \frac{1}{\epsilon_1^4} W_1^{N_f=0, \text{SU}(2)} &= -\frac{2}{\epsilon_1^3 - 4a^2\epsilon_1}, \\
 \frac{1}{2\epsilon_1^8} W_2^{N_f=0, \text{SU}(2)} &= \frac{20a^2 + 7\epsilon_1^2}{4(\epsilon_1^2 - 4a^2)^3(\epsilon_1^3 - a^2\epsilon_1)}, \\
 \frac{1}{3\epsilon_1^{12}} W_3^{N_f=0, \text{SU}(2)} &= -\frac{4(144a^4 + 29(8a^2\epsilon_1^2 + \epsilon_1^4))}{3(\epsilon_1^2 - 4a^2)^5(4a^4\epsilon_1 - 13a^2\epsilon_1^3 + 9\epsilon_1^5)}.
 \end{aligned}
 \tag{4.28}$$

As it may be observed these coefficients coincide with those in eq. (A.1) ($W_i/(i\epsilon_1^{4i}) = \mathcal{W}_i$). The above results can be compared with coefficients of the irregular classical block. The relationship between coefficients of the latter with those of twisted superpotential is given in eq. (2.26). In terms of W_i this relationship reads

$$\frac{1}{i\epsilon_1^{4i}} W_i^{N_f=0, \text{SU}(2)}(a, \epsilon_1) = -\frac{1}{\epsilon_1^{4i}} f_{\frac{1}{4} - \frac{a^2}{\epsilon_1^2}}^i.
 \tag{4.29}$$

Coefficients of the classical irregular block given in eq. (2.14) expressed in terms of instanton parameters a, ϵ_1 read

$$\begin{aligned}
 f_{\frac{1}{4} - \frac{a^2}{\epsilon_1^2}}^1 &= \frac{2\epsilon_1^2}{\epsilon_1^2 - 4a^2}, \\
 f_{\frac{1}{4} - \frac{a^2}{\epsilon_1^2}}^2 &= -\frac{\epsilon_1^6(20a^2 + 7\epsilon_1^2)}{4(\epsilon_1^2 - 4a^2)^3(\epsilon_1^2 - a^2)}, \\
 f_{\frac{1}{4} - \frac{a^2}{\epsilon_1^2}}^3 &= \frac{4\epsilon_1^{10}(144a^4 + 29(8a^2\epsilon_1^2 + \epsilon_1^4))}{3(\epsilon_1^2 - 4a^2)^5(4a^4 - 13a^2\epsilon_1^2 + 9\epsilon_1^4)}.
 \end{aligned}
 \tag{4.30}$$

Heaving these coefficients in the explicit form it is now straightforward to check the relationship with coefficients of the superpotential stated in eq. (4.29) where the latter are expressed in terms of the critical Young diagram in eq. (4.28). Although just verified mathematical correspondence between the two contextually different theories was performed up to the third order in expansion parameter $\hat{\Lambda} \sim \mu e^{\log(a)/(4)}$, it is by induction reasonable to assume that the range of validity extends to all orders which was phrased earlier in eq. (2.26). The last statement should be yet proved. Nevertheless, the observed relationship, with aforementioned caveat, may be regarded as an extension of the non-conformal limit of AGT correspondence found in ref. [45] to the classical level.

The classical non-conformal AGT relation enables one to associate the eigenvalue of the Mathieu operator expressed in terms of the classical block in eq. (3.23) with instanton parameters through the twisted superpotential. Let us recall at this point that according to the Nekrasov-Shatashvili's conjecture put forward in [19] (the Bethe/gauge correspondence) for any $\mathcal{N} = 2$ gauge theory there exists a quantum integrable system whose Hamiltonians are given in terms of gauge independent functions of operators $\mathcal{O}_k(x)$ that form the so-called twisted chiral ring, and enumerate vacua of the relevant gauge theory. These operators can

be chosen as the traces of the lowest components of vector multiplet. Their expectation values evaluated at the minimum of the (shifted) superpotential correspond to the spectrum of Hamiltonians of the integrable system. Namely [21]

$$\mathcal{E}_k = \langle \mathcal{O}_k(x) \rangle = \langle \text{tr} \phi^k \rangle_{\mathbf{a}}|_{\epsilon_2=0} = \frac{1}{2} \int_{\mathbb{R}} dx x^k f''_{\mathbf{a},\emptyset}(x) + \frac{1}{2} \int_{\mathbb{R}} dx x^k \rho''_{\mathbf{a},\omega_*}(x|\epsilon_1),$$

where the first term on the right hand side is the classical part. In the case under consideration the gauge group is $SU(2)$ and the relevant quantum integrable system is the two-particle periodic Toda chain. Its two Hamiltonians have the following spectrum: $\mathcal{E}_1 = 0$ and

$$\begin{aligned} E = 2\mathcal{E}_2 &= 4a^2 + 2 \int_{\mathbb{R}} dx \rho_{\mathbf{a},\omega_*}(x|\epsilon_1) = 4a^2 - 4\epsilon_1 \sum_{\alpha} \sum_{i \geq 1} \omega_{*\alpha,i}(\hat{\Lambda}/\epsilon_1) \\ &= \epsilon_1 \hat{\Lambda} \partial_{\hat{\Lambda}} \mathcal{W}^{\text{Nf}=0, \text{SU}(2)}(\hat{\Lambda}, \mathbf{a}, \epsilon_1) = \epsilon_1^2 \lambda, \end{aligned} \tag{4.31}$$

where in the second line we accounted for the result (4.24) as well as the perturbative part given in eq. (4.1a) with the gamma function traded for the one for $g = 0$ order of the expansion in the limit $\epsilon_2 \rightarrow 0$ given in eq. (4.12b) that we signified as $\gamma_{\epsilon_1}(x; \hat{\Lambda})$. Explicitly, the perturbative part reads

$$\mathcal{W}_{\text{pert}}^{\text{Nf}=0, \text{SU}(2)}(\hat{\Lambda}, \mathbf{a}, \epsilon_1) = \sum_{\alpha, \beta=1}^N \gamma_{\epsilon_1}(a_{\alpha} - a_{\beta}; \hat{\Lambda}) = \frac{4}{\epsilon_1} \log \left(\frac{\hat{\Lambda}}{\epsilon_1} \right) \left(a^2 + \frac{\epsilon_1^2}{12} \right).$$

It is worth noting that the relationship between the spectrum of the second Hamiltonian E , twisted superpotential \mathcal{W} and Mathieu eigenvalue λ stated in eq. (4.31) coincides with the one deduced from the null vector decoupling in eq. (3.26), which is the CFT side of classical AGT correspondence.

The formula (4.31) can also be given yet another, concise form in terms of the $Y_{\mathbf{a},\omega}(z)$ function which does not refer to the iterative expansion of \mathcal{W} . Namely, noting that

$$\frac{1}{2} \int_{\mathbb{R}} dx x^n \rho''_{\mathbf{a},\omega_*}(x|\epsilon_1) = - \oint_C \frac{dz}{2\pi i} [(z + \epsilon_1)^n - z^n] \partial_z \log \frac{Y(z)}{Y_0(z)},$$

and

$$\frac{1}{2} \int_{\mathbb{R}} dx x^n f''_{\mathbf{a},\emptyset}(x) = \oint_C \frac{dz}{2\pi i} z^n \partial_z \log \frac{Y_0(z - \epsilon_1)}{Y_0(z)},$$

we obtain

$$E = -2 \oint_C \frac{dz}{2\pi i} z^2 \partial_z \log \frac{Y(z - \epsilon_1)}{Y(z)}, \tag{4.32}$$

where the contour C encloses all points $x_{\alpha,i}^*$ and $x_{\alpha,i}^0$, which are zeros of $Y(z)$ and $Y_0(z)$, respectively. The latter functions solve eq. (4.26).

5 Conclusions

In this paper we have postulated the existence of the classical irregular block $f_\delta(\hat{\Lambda}/\epsilon_1)$. Calculations presented in this work provide a convincing evidence that the classical limit of the quantum irregular block (2.12) exists yielding a consistent definition of the function $f_\delta(\hat{\Lambda}/\epsilon_1)$. In particular, this hypothesis is strongly supported by the tests described in section 2 (see figure 1).

As an implication of the conjectured semiclassical asymptotic (3.15) the classical irregular block enters the expression (3.23) for the Mathieu eigenvalue λ . Indeed, we have checked that the formula (3.23) reproduces well known weak coupling (small h^2) expansion of λ .³¹ Hence, the existence of the function $f_\delta(\hat{\Lambda}/\epsilon_1)$ seems to be well confirmed *a posteriori*, i.e. by consequences of its existence. Let us stress that although the classical limit of the null vector decoupling equation for the degenerate 3-point irregular block was discussed before in [18] the expression (3.23) has not appeared in the literature so far.

The existence of the classical irregular block is also apparent from the semiclassical limit of the “non-conformal” AGT relation. Two checks have been performed in the present work in order to show that the function $f_\delta(\hat{\Lambda}/\epsilon_1)$ corresponds to the instanton twisted superpotential $\mathcal{W}_{\text{inst}}^{N_f=0, \text{SU}(2)}$ of the $\mathcal{N} = 2$ SU(2) pure gauge SYM theory. One of these checks (see section 4) is highly non-trivial. It employs a representation of $\mathcal{W}_{\text{inst}}^{N_f=0, \text{SU}(2)}$ as a critical value of the gauge theory “free energy”. The criticality condition or equivalently the saddle point equation takes the form of the Bethe-like equation. This equation can be solved by a power expansion in $\hat{\Lambda}/\epsilon_1$ (see appendix F). Solution to this equation (or in fact a system of equations) describes a shape of the two critical Young diagrams extremizing the “free energy”. Finally, the latter evaluated on the “critical configuration” and expanded in $\hat{\Lambda}/\epsilon_1$ yields $f_\delta(\hat{\Lambda}/\epsilon_1)$ provided that certain relations between parameters are assumed. As a by-product, the identity (2.24) and the formula (3.26) imply another new expression for the Mathieu eigenvalue λ , namely, as a sum of columns’ lengths in critical Young diagrams, cf. (4.31).³² Moreover, as was noticed in [57],³² once critical columns’ lengths are known in a closed form such sum can be rewritten in terms of the contour integral with integrand built out with certain special functions. We have rederived this statement at the gauge theory side within the formalism of profile functions of the Young diagrams, cf. (4.32). The latter and (4.31) imply contour integral representation of λ .

Let us interpret results of this paper within the context of the triple correspondence

$$2d \text{ CFT} \Big/ 2d \mathcal{N} = 2 \text{ SU}(N) \text{ SYM} \Big/ N - \text{particle QIS} \tag{5.1}$$

mentioned in the introduction. In our case we have $N = 2$. On the conformal field theory side we have found the formula for the Mathieu eigenvalue λ expressed in terms of the classical irregular block. Its analogue on the gauge theory side is given by $\mathcal{W}_{\text{inst}}^{N_f=0, \text{SU}(2)}$ and realized here as v.e.v. of certain gauge theory operator (see section 4). Then, according

³¹Work is in progress in order to verify whether calculations performed in section 3 pave the way for working out really new methods for calculating corresponding eigenfunctions.

³²See also appendix C, where has been shown an exact matching between this result and that obtained by means of the WKB analysis.

to the Bethe/gauge correspondence λ is nothing but an eigenvalue of the corresponding integrable system 2-particle Hamiltonian (see appendix C). An interesting further line of research is to extend this observation going beyond $N = 2$. In such a case the AGT duality is extended to the correspondence between the $2d$ conformal Toda theory and the $4d \mathcal{N} = 2$ $SU(N)$ gauge theories [27]. Then, in order to get examples of (5.1) one has to study classical limit of the W_N symmetry conformal blocks.³³

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A Expansion coefficients of $2d$ CFT and gauge theory functions

Gram matrix.

$n = 1$: $\{L_{-1}|\nu_\Delta\rangle\}$,

$$G_{c,\Delta}^{n=1} = \langle L_{-1}\nu_\Delta | L_{-1}\nu_\Delta \rangle = \langle \nu_\Delta | L_1 L_{-1}\nu_\Delta \rangle = 2\Delta,$$

$n = 2$: $\{L_{-2}|\nu_\Delta\rangle, L_{-1}L_{-1}|\nu_\Delta\rangle\}$,

$$G_{c,\Delta}^{n=2} = \begin{pmatrix} \langle L_{-2}\nu_\Delta | L_{-2}\nu_\Delta \rangle & \langle L_{-1}^2\nu_\Delta | L_{-2}\nu_\Delta \rangle \\ \langle L_{-2}\nu_\Delta | L_{-1}^2\nu_\Delta \rangle & \langle L_{-1}^2\nu_\Delta | L_{-1}^2\nu_\Delta \rangle \end{pmatrix} = \begin{pmatrix} \frac{c}{2} + 4\Delta & 6\Delta \\ 6\Delta & 4\Delta(2\Delta + 1) \end{pmatrix},$$

$n = 3$: $\{L_{-3}|\nu_\Delta\rangle, L_{-2}L_{-1}|\nu_\Delta\rangle, L_{-1}L_{-1}L_{-1}|\nu_\Delta\rangle\}$,

$$G_{c,\Delta}^{n=3} = \begin{pmatrix} 2c + 6\Delta & 10\Delta & 24\Delta \\ 10\Delta & \Delta(c + 8\Delta + 8) & 12\Delta(3\Delta + 1) \\ 24\Delta & 12\Delta(3\Delta + 1) & 24\Delta(\Delta + 1)(2\Delta + 1) \end{pmatrix}.$$

Coefficients of the quantum irregular block.

$$[G_{c,\Delta}^1]^{(1)(1)} = \frac{1}{2\Delta},$$

$$[G_{c,\Delta}^2]^{(1^2)(1^2)} = \frac{c + 8\Delta}{4\Delta(2c\Delta + c + 2\Delta(8\Delta - 5))},$$

$$[G_{c,\Delta}^3]^{(1^3)(1^3)} = \frac{(11c - 26)\Delta + c(c + 8) + 24\Delta^2}{24\Delta((c - 7)\Delta + c + 3\Delta^2 + 2)(2(c - 5)\Delta + c + 16\Delta^2)},$$

\vdots

³³Interestingly, there is also a need for the classical W_N blocks in the study of the entanglement entropy within the $AdS_3/2d$ CFT holography [58].

Coefficients of the quantum 4-point block on the sphere.

$$\begin{aligned} \mathcal{F}_{c,\Delta}^1 \begin{bmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix} &= \frac{(\Delta + \Delta_3 - \Delta_4)(\Delta + \Delta_2 - \Delta_1)}{2\Delta}, \\ \mathcal{F}_{c,\Delta}^2 \begin{bmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix} &= \left[(4\Delta(1 + 2\Delta))^{-1} (\Delta - \Delta_1 + \Delta_2)(1 + \Delta - \Delta_1 + \Delta_2) \right. \\ &\quad \times (\Delta + \Delta_3 - \Delta_4)(1 + \Delta + \Delta_3 - \Delta_4) \\ &\quad + (\Delta - \Delta^2 - \Delta_1 - \Delta_2 + 3(\Delta_2 - \Delta_1)^2 - 2\Delta(\Delta_1 + \Delta_2)) \\ &\quad \times (\Delta - \Delta^2 - \Delta_3 - \Delta_4 + 3(\Delta_3 - \Delta_4)^2 - 2\Delta(\Delta_3 + \Delta_4)) \left. \right] \\ &\quad \times \left[2(1 + 2\Delta)^2 \left(c - \frac{4\Delta(5 - 8\Delta)}{2 + 4\Delta} \right) \right]^{-1}, \\ &\quad \vdots . \end{aligned}$$

Coefficients of the classical 4-point block on the sphere.

$$\begin{aligned} f_\delta^1 \begin{bmatrix} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{bmatrix} &= \frac{(\delta + \delta_3 - \delta_4)(\delta + \delta_2 - \delta_1)}{2\delta}, \\ f_\delta^2 \begin{bmatrix} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{bmatrix} &= \left[16\delta^3(4\delta + 3) \right]^{-1} \left[13\delta^5 + \delta^4(18\delta_2 - 14\delta_1 + 18\delta_3 - 14\delta_4 + 9) \right. \\ &\quad + \delta^3(\delta_1^2 + \delta_2^2 - 2\delta_1(\delta_2 + 6\delta_3 - 10\delta_4 + 6) \\ &\quad + 4\delta_2(5\delta_3 - 3\delta_4 + 3) + (\delta_3 - \delta_4)(\delta_3 - \delta_4 + 12)) \\ &\quad - 3\delta^2(\delta_1^2(2\delta_3 + 2\delta_4 - 1) + 2\delta_1(\delta_3^2 + \delta_4^2 + 2\delta_3 + \delta_2 - 2\delta_2\delta_3 - 2\delta_4(\delta_2 + \delta_3 + 1)) \\ &\quad + \delta_2^2(2\delta_3 + 2\delta_4 - 1) + 2\delta_2(\delta_3 - \delta_4 - 2)(\delta_3 - \delta_4) - (\delta_3 - \delta_4)^2) \\ &\quad \left. + 5\delta(\delta_1 - \delta_2)^2(\delta_3 - \delta_4)^2 - 3(\delta_1 - \delta_2)^2(\delta_3 - \delta_4)^2 \right], \\ &\quad \vdots . \end{aligned}$$

Coefficients of the quantum 1-point block on the torus.

$$\begin{aligned} \mathcal{F}_{c,\Delta}^{\tilde{\Delta},1} &= \frac{(\tilde{\Delta} - 1)\tilde{\Delta}}{2\Delta} + 1, \\ \mathcal{F}_{c,\Delta}^{\tilde{\Delta},2} &= [4\Delta(2c\Delta + c + 16\Delta^2 - 10\Delta)]^{-1} \\ &\quad \left[(8c\Delta + 3c + 128\Delta^2 + 56\Delta)\tilde{\Delta}^2 + (-8c\Delta - 2c - 128\Delta^2)\tilde{\Delta} \right. \\ &\quad \left. + (c + 8\Delta)\tilde{\Delta}^4 + (-2c - 64\Delta)\tilde{\Delta}^3 + 16c\Delta^2 + 8c\Delta + 128\Delta^3 - 80\Delta^2 \right]. \\ &\quad \vdots . \end{aligned}$$

Coefficients of the classical 1-point block on the torus.

$$f_\delta^{\tilde{\delta},1} = \frac{\tilde{\delta}^2}{2\delta}, \quad f_\delta^{\tilde{\delta},2} = \frac{\tilde{\delta}^2[24\delta^2(4\delta + 1) + \tilde{\delta}^2(5\delta - 3) - 48\tilde{\delta}\delta^2]}{16\delta^3(4\delta + 3)}, \quad \dots$$

Coefficients of the SU(2) pure gauge Nekrasov's instanton function.

$$\begin{aligned}
 \mathcal{Z}_1^{N_f=0, \text{SU}(2)} &= \frac{2}{\epsilon_1 \epsilon_2 ((\epsilon_1 + \epsilon_2)^2 - 4a^2)}, \\
 \mathcal{Z}_2^{N_f=0, \text{SU}(2)} &= \frac{-8a^2 + 8\epsilon_1^2 + 8\epsilon_2^2 + 17\epsilon_1 \epsilon_2}{\epsilon_1^2 \epsilon_2^2 ((\epsilon_1 + \epsilon_2)^2 - 4a^2) ((2\epsilon_1 + \epsilon_2)^2 - 4a^2) ((\epsilon_1 + 2\epsilon_2)^2 - 4a^2)}, \\
 \mathcal{Z}_3^{N_f=0, \text{SU}(2)} &= \left[2 \left(\epsilon_1^2 (594\epsilon_2^2 - 104a^2) + \epsilon_1 (363\epsilon_2^3 - 188a^2\epsilon_2) \right. \right. \\
 &\quad \left. \left. + 8(4a^4 - 13a^2\epsilon_2^2 + 9\epsilon_2^4) + 72\epsilon_1^4 + 363\epsilon_2\epsilon_1^3 \right) \right] \\
 &\quad \left[3\epsilon_1^3 \epsilon_2^3 \left((\epsilon_1 + \epsilon_2)^2 - 4a^2 \right) \left((2\epsilon_1 + \epsilon_2)^2 - 4a^2 \right) \left((3\epsilon_1 + \epsilon_2)^2 - 4a^2 \right) \right. \\
 &\quad \left. \times \left((\epsilon_1 + 2\epsilon_2)^2 - 4a^2 \right) \left((\epsilon_1 + 3\epsilon_2)^2 - 4a^2 \right) \right]^{-1}, \\
 &\quad \vdots .
 \end{aligned}$$

Coefficients of the SU(2) pure gauge twisted superpotential.

$$\begin{aligned}
 \mathcal{W}_1^{N_f=0, \text{SU}(2)} &= \frac{2}{\epsilon_1^3 - 4a^2\epsilon_1}, \\
 \mathcal{W}_2^{N_f=0, \text{SU}(2)} &= -\frac{20a^2 + 7\epsilon_1^2}{4(\epsilon_1^2 - 4a^2)^3(\epsilon_1^3 - a^2\epsilon_1)}, \\
 \mathcal{W}_3^{N_f=0, \text{SU}(2)} &= \frac{4(144a^4 + 232a^2\epsilon_1^2 + 29\epsilon_1^4)}{3(\epsilon_1^2 - 4a^2)^5(4a^4\epsilon_1 - 13a^2\epsilon_1^3 + 9\epsilon_1^5)}, \\
 &\quad \vdots .
 \end{aligned} \tag{A.1}$$

B Ward identities

In order to derive the Ward identity (3.4) first we will need to prove the following relation:

$$\langle \Delta', \Lambda^2 | V_+(z) L_0 | \tilde{\Delta}, \Lambda^2 \rangle = \frac{1}{2} \left(\frac{\Lambda}{2} \frac{\partial}{\partial \Lambda} + \Delta' + \tilde{\Delta} - \Delta_+ - z \frac{\partial}{\partial z} \right) \Psi(\Lambda, z). \tag{B.1}$$

Indeed, using $L_0 | \Delta, \Lambda^2 \rangle = (\Delta + \frac{\Lambda}{2} \partial_\Lambda) | \Delta, \Lambda^2 \rangle$ one gets

$$\begin{aligned}
 \langle \Delta', \Lambda^2 | V_+(z) (L_0 - \tilde{\Delta}) | \tilde{\Delta}, \Lambda^2 \rangle &= \langle \Delta', \Lambda^2 | V_+(z) \frac{\Lambda}{2} \frac{\partial}{\partial \Lambda} | \tilde{\Delta}, \Lambda^2 \rangle \\
 &= \frac{\Lambda}{2} \frac{\partial}{\partial \Lambda} \Psi(\Lambda, z) - \langle \Delta', \Lambda^2 | (L_0 - \Delta') V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle.
 \end{aligned}$$

Hence, taking into account that $[L_0, V_+(z)] = (z\partial_z + \Delta_+) V_+(z)$ one finds

$$\begin{aligned}
 \frac{\Lambda}{2} \frac{\partial}{\partial \Lambda} \Psi(\Lambda, z) &= \langle \Delta', \Lambda^2 | V_+(z) (L_0 - \tilde{\Delta}) | \tilde{\Delta}, \Lambda^2 \rangle + \langle \Delta', \Lambda^2 | (L_0 - \Delta') V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle \\
 &= - \left(\Delta' + \tilde{\Delta} - \Delta_+ - z \frac{\partial}{\partial z} \right) \Psi(\Lambda, z) + 2 \langle \Delta', \Lambda^2 | V_+(z) L_0 | \tilde{\Delta}, \Lambda^2 \rangle.
 \end{aligned}$$

Next, let us consider the matrix element $\langle \Delta', \Lambda^2 | T(w) V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle$ with

$$T(w) = \sum_{n \in \mathbb{Z}} w^{-n-2} L_n,$$

i.e.:

$$\begin{aligned} \langle \Delta', \Lambda^2 | T(w)V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle &= \sum_{n=-\infty}^{n=+\infty} \frac{1}{w^{n+2}} \langle \Delta', \Lambda^2 | L_n V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle \\ &= \sum_{n=-\infty}^{n=-2} \frac{1}{w^{n+2}} \langle \Delta', \Lambda^2 | L_n V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle + \frac{1}{w} \langle \Delta', \Lambda^2 | L_{-1} V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle \\ &\quad + \sum_{n=0}^{\infty} \frac{1}{w^{n+2}} \langle \Delta', \Lambda^2 | L_n V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle. \end{aligned}$$

From (2.3) and $L_n^\dagger = L_{-n}$ we have $\sum_{n=-\infty}^{n=-2} w^{-n-2} \langle \Delta', \Lambda^2 | L_n V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle = 0$. Then,

$$\begin{aligned} \langle \Delta', \Lambda^2 | T(w)V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle &= \frac{1}{w} \langle \Delta', \Lambda^2 | L_{-1} V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle \\ &\quad + \sum_{n=0}^{\infty} \frac{1}{w^{n+2}} \langle \Delta', \Lambda^2 | L_n V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle \\ &= \frac{1}{w} \Lambda^2 \Psi(\Lambda, z) + \sum_{n=0}^{\infty} \frac{1}{w^{n+2}} \langle \Delta', \Lambda^2 | [L_n, V_+(z)] | \tilde{\Delta}, \Lambda^2 \rangle \\ &\quad + \frac{1}{w^2} \langle \Delta', \Lambda^2 | V_+(z) L_0 | \tilde{\Delta}, \Lambda^2 \rangle + \frac{1}{w^3} \langle \Delta', \Lambda^2 | V_+(z) L_1 | \tilde{\Delta}, \Lambda^2 \rangle, \end{aligned}$$

where again the conditions (2.1)–(2.3) defining the Gaiotto state have been used. Now, using $[L_n, V_+(z)] = z^n (z\partial_z + (n+1)\Delta_+) V_+(z)$ one gets

$$\begin{aligned} \langle \Delta', \Lambda^2 | T(w)V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle &= \left(\frac{\Lambda^2}{w} + \frac{\Lambda^2}{w^3} \right) \Psi(\Lambda, z) + \frac{1}{w^2} \langle \Delta', \Lambda^2 | V_+(z) L_0 | \tilde{\Delta}, \Lambda^2 \rangle \\ &\quad + \sum_{n=0}^{\infty} \frac{1}{w^{n+2}} z^n \left(z \frac{\partial}{\partial z} + (n+1)\Delta_+ \right) \Psi(\Lambda, z). \end{aligned}$$

Let us note that for $|\frac{z}{w}| < 1$ we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{z^{n+1}}{w^{n+2}} &= \frac{z}{w^2} \sum_{n=0}^{\infty} \left(\frac{z}{w} \right)^n = \frac{z}{w^2} \frac{1}{1 - \frac{z}{w}} = \frac{z}{w(w-z)}, \\ \sum_{n=0}^{\infty} \frac{z^n}{w^{n+2}} (n+1) &= \frac{\partial}{\partial z} \left(\frac{z}{w(w-z)} \right) = \frac{1}{(w-z)^2}. \end{aligned}$$

Then,

$$\begin{aligned} \langle \Delta', \Lambda^2 | T(w)V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle &= \left[\left(\frac{\Lambda^2}{w} + \frac{\Lambda^2}{w^3} \right) + \frac{\Delta_+}{(w-z)^2} + \frac{z}{w(w-z)} \frac{\partial}{\partial z} \right] \Psi(\Lambda, z) \\ &\quad + \frac{1}{w^2} \langle \Delta', \Lambda^2 | V_+(z) L_0 | \tilde{\Delta}, \Lambda^2 \rangle. \end{aligned}$$

Finally, an application of (B.1) yields (3.4).

Having (3.4) one can prove (3.5). Indeed, using (3.2), (3.4) and Cauchy's contour integral formula one gets

$$\begin{aligned} \langle \Delta', \Lambda^2 | \widehat{L}_{-2}(z)V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle &= \frac{1}{2\pi i} \oint_{C_z} dw(w-z)^{-1} \langle \Delta', \Lambda^2 | T(w)V_+(z) | \tilde{\Delta}, \Lambda^2 \rangle \\ &= \frac{1}{2\pi i} \oint_{C_z} dw(w-z)^{-1} \times \text{r.h.s. of (3.4)} \\ &= \text{r.h.s. of (3.5)}. \end{aligned}$$

C Mathieu eigenvalue from WKB

In this appendix we perform computations for the spectrum of the Mathieu operator by means of the WKB method along the line of refs. [16, 18, 19, 59].

The Seiberg-Witten (SW) theory [15] with SU(2) gauge group is defined by the prepotential $\mathcal{F}_{SW}(a, \hat{\Lambda})$ which is determined by the set of quantities $(a_\alpha, \partial\mathcal{F}_{SW}/\partial a_\alpha)$ i.e., moduli and their duals and the elliptic curve

$$y^2 = (z^2 - \hat{\Lambda}^4)(z - u), \quad u \equiv \hat{\Lambda}^2 \left(1 - 2 \frac{\vartheta_2^4(0|\tau)}{\vartheta_3^4(0|\tau)} \right), \quad (\text{C.1})$$

where $\vartheta_i^4(0|\tau)$, $i = 1, 2$ are Jacobi functions and τ is a complexified coupling constant that parametrizes the modular half-plane for the one-dimensional complex tori. u is the parameter on the moduli space of the theory. The latter has three singular points: two branch points $u = \hat{\Lambda}^2$, $(\tau = 0)$, $u = -\hat{\Lambda}^2$, $(\tau = \pm 1)$ and a singular point at $u = \infty$, $(\tau = i\infty)$. The latter corresponds to the perturbative region of the moduli space (asymptotic freedom). The modulus and its dual with appropriate monodromies about the singular points are found by means of the curve (C.1) and SW differential which takes the form

$$\lambda = \frac{\sqrt{2}}{2\pi} \frac{z - u}{y(z)} dz = \frac{\sqrt{2}}{2\pi} \sqrt{\frac{z - u}{z^2 - \hat{\Lambda}^4}} dz.$$

The mentioned functions are determined by integrals of λ with appropriate cycles, i.e., $(\sum a_\alpha = 0)$

$$a = \oint_A \lambda, \quad a_D = \oint_B \lambda, \quad a_D \equiv \frac{\partial\mathcal{F}_{SW}(a, \hat{\Lambda})}{\partial a}.$$

Integration along the contour A is the one that circumvents a branch cut between the points $[-\hat{\Lambda}^2, \hat{\Lambda}^2]$, whereas the integration over contour B starts from the point on one sheet, passes on the second sheet through the branch cut between u and ∞ and returns to the starting point on the first sheet through the branch cut $[-\hat{\Lambda}^2, \hat{\Lambda}^2]$. Hence, the contour integrals can

be given the form

$$\begin{aligned}
 a_{\hat{\Lambda}^2}(u) &= \frac{\sqrt{2}}{\pi} \int_{-\hat{\Lambda}^2}^{\hat{\Lambda}^2} \sqrt{\frac{z-u}{z^2-\hat{\Lambda}^4}} dz = \frac{\sqrt{2u}}{\pi} \int_{-\hat{\Lambda}^2/u}^{\hat{\Lambda}^2/u} \sqrt{\frac{z-1}{z^2-(\hat{\Lambda}^2/u)^2}} dz \equiv a(\hat{\Lambda}^2/u), \\
 (a_{\hat{\Lambda}^2})_D(u) &= \frac{\sqrt{2}}{\pi} \int_{\hat{\Lambda}^2}^u \sqrt{\frac{z-u}{z^2-\hat{\Lambda}^4}} dz = \frac{\sqrt{2u}}{\pi} \int_{\hat{\Lambda}^2/u}^1 \sqrt{\frac{z-1}{z^2-(\hat{\Lambda}^2/u)^2}} dz \equiv a_D(\hat{\Lambda}^2/u).
 \end{aligned} \tag{C.2}$$

In a parametrization $z = \hat{\Lambda}^2 \cos(\theta)$, $\pi\lambda = \sqrt{2u - 2\hat{\Lambda}^2 \cos(\theta)} d\theta = p(\theta; u, \hat{\Lambda}) d\theta$, and

$$\begin{aligned}
 a(\hat{\Lambda}^2/u) &= \frac{1}{2\pi} \oint_A p(\theta; u, \hat{\Lambda}) d\theta = \frac{1}{\pi} \int_0^\pi p(\theta; u, \hat{\Lambda}) d\theta, \\
 a_D(\hat{\Lambda}^2/u) &= \frac{1}{2\pi} \oint_B p(\theta; u, \hat{\Lambda}) d\theta = \frac{1}{\pi} \int_\pi^{\theta_u} p(\theta; u, \hat{\Lambda}) d\theta, \quad \theta_u \equiv \arccos(u/\hat{\Lambda}^2).
 \end{aligned} \tag{C.3}$$

Equations (C.2) are integral representations of the hypergeometric functions, namely

$$a(u, v) = \frac{\sqrt{2u}}{\pi} \int_{-v}^v \sqrt{\frac{z-1}{z^2-v^2}} dz = \sqrt{2u} \sqrt{1+v} F_{2,1} \left(-\frac{1}{2}, \frac{1}{2}, 1; \frac{2v}{1+v} \right), \tag{C.4a}$$

$$a_D(u, v) = \frac{\sqrt{2u}}{\pi} \int_v^1 \sqrt{\frac{z-1}{z^2-v^2}} dz = \frac{i}{2^{3/2}} \sqrt{2u} (1-v) F_{2,1} \left(\frac{1}{2}, \frac{1}{2}, 2; -\frac{1-v}{2v} \right), \tag{C.4b}$$

where $v \equiv \hat{\Lambda}^2/u$. The SW prepotential $\mathcal{F}_{SW}(a, \hat{\Lambda})$ corresponds to the $\epsilon_1 \epsilon_2 \rightarrow 0$ limit of the exponentiated Nekrasov partition function. It is related to the classical two-particle Toda periodic chain or sine-Gordon model action of which takes the form

$$S[\theta(t_2), \theta(t_1), t_2, t_1] = \int_{t_1}^{t_2} dt \left(\frac{1}{2} \dot{\theta}^2 - \hat{\Lambda}^2 \cos \theta \right) = \int_{\theta(t_1)}^{\theta(t_2)} p(\theta; E, \hat{\Lambda}) d\theta - (t_2 - t_1) E, \tag{C.5}$$

where

$$p(\theta; E, \hat{\Lambda}) \equiv \sqrt{2E - 2\hat{\Lambda}^2 \cos \theta}. \tag{C.6}$$

Canonical quantization of the system in eq. (C.5) results in the following form of the time independent Schroedinger equation ($\Psi(t, \theta) = e^{-iEt/\epsilon_1} \psi(\theta)$)

$$\left[-\frac{\epsilon_1^2}{2} \frac{d^2}{d\theta^2} + \hat{\Lambda}^2 \cos \theta - E \right] \psi(\theta) = 0, \tag{C.7}$$

with the energy given in eq. (4.31). This equation can be solved with the aid of the *exact WKB method*. Namely, the WKB ansatz takes the form

$$\psi_{\text{WKB}}(\theta) = \exp \left\{ \frac{i}{\epsilon_1} \int_{\theta_0}^{\theta} P(\vartheta; E, \hat{\Lambda}) d\vartheta \right\}. \tag{C.8}$$

The turning points are at $\theta_{1,2}^{(\text{tp})} = \pm \arccos(E/\hat{\Lambda}^2)$. Inserting the above wave function into the eq. (C.7) we obtain equation for $P(\vartheta; E, \hat{\Lambda})$. In general it can be solved for

$$P(\theta; u, \hat{\Lambda}) d\theta = \sum_{m \geq 0} \epsilon_1^m p_m(\theta; u, \hat{\Lambda}), \quad (\text{C.9})$$

order by order. The first term $p_0(\theta; u, \hat{\Lambda}) \equiv p(\theta; u, \hat{\Lambda})$ corresponds to quasi-classical solution. The modulus a related to this solution is related to the ‘‘classical’’ momentum through

$$a(\hat{\Lambda}^2/u) = \frac{1}{2\pi} \oint_A p_0(\theta; u, \hat{\Lambda}) d\theta.$$

Its quantum counterpart corresponds to modulus on the ‘‘quantum’’ moduli space and reads

$$\mathfrak{a}[a(\hat{\Lambda}^2/u)] = \frac{1}{2\pi} \oint_A P(\theta; u, \hat{\Lambda}) d\theta = \frac{1}{2\pi} \sum_{m \geq 0} \epsilon_1^m \oint_A p_m(\theta; u, \hat{\Lambda}) d\theta. \quad (\text{C.10})$$

The above corrections to momentum can be obtained by means of the following relation

$$p_m(\theta; u, \hat{\Lambda}) = \mathcal{O}_m(\gamma, \mathcal{E}, \partial_\gamma, \partial_\mathcal{E}) p_0(\theta; u, \hat{\Lambda}), \quad (\text{C.11})$$

such that

$$\mathfrak{a}[a(\hat{\Lambda}^2/u)] = \sum_{m \geq 0} \epsilon_1^m \mathcal{O}_m(\gamma, \mathcal{E}, \partial_\gamma, \partial_\mathcal{E}) \mathfrak{a}(\gamma/\mathcal{E}), \quad (\text{C.12})$$

where $\mathcal{E} \equiv 2u$, $\gamma \equiv 2\hat{\Lambda}^2$. We have found these operators up to third order in ϵ_1 . The result reads

$$\mathcal{O}_1(\gamma, \mathcal{E}, \partial_\gamma, \partial_\mathcal{E}) \equiv \frac{\gamma}{2^2 \cdot 3} \partial_\gamma \partial_\mathcal{E}, \quad (\text{C.13a})$$

$$\mathcal{O}_2(\gamma, \mathcal{E}, \partial_\gamma, \partial_\mathcal{E}) \equiv \frac{\gamma}{2^5 \cdot 3^2 \cdot 5} \left[5\gamma \partial_\mathcal{E}^2 \partial_\gamma^2 - 2\mathcal{E} \partial_\mathcal{E}^3 \partial_\gamma \right], \quad (\text{C.13b})$$

$$\begin{aligned} \mathcal{O}_3(\gamma, \mathcal{E}, \partial_\gamma, \partial_\mathcal{E}) \equiv & \frac{\gamma}{2^7 \cdot 3^6 \cdot 5 \cdot 7} \left[463 \gamma^2 \partial_\mathcal{E}^3 \partial_\gamma^3 - 118 \mathcal{E} \gamma \partial_\mathcal{E}^4 \partial_\gamma^2 \right. \\ & \left. + (20\mathcal{E}^2 + 16\gamma^2) \partial_\mathcal{E}^5 \partial_\gamma + 16 \mathcal{E} \gamma \partial_\mathcal{E}^6 \right]. \end{aligned} \quad (\text{C.13c})$$

In order to find the energy for the full quantum system we have to invert (C.10) so that $E = E(\mathfrak{a}(a))$. The result for the quantum modulus takes the form

$$\begin{aligned} \mathfrak{a}(a(\gamma/\mathcal{E})) = & \sqrt{\mathcal{E}} - \gamma^2 \left(\frac{1}{16 \mathcal{E}^{3/2}} + \frac{\epsilon_1^2}{64 \mathcal{E}^{5/2}} + \frac{\epsilon_1^4}{256 \mathcal{E}^{7/2}} + \frac{\epsilon_1^6}{1024 \mathcal{E}^{9/2}} \right) \\ & - \gamma^4 \left(\frac{15}{1024 \mathcal{E}^{7/2}} + \frac{35 \epsilon_1^2}{2048 \mathcal{E}^{9/2}} + \frac{273 \epsilon_1^4}{16384 \mathcal{E}^{11/2}} + \frac{33 \epsilon_1^6}{2048 \mathcal{E}^{13/2}} \right) \\ & - \gamma^6 \left(\frac{105}{16384 \mathcal{E}^{11/2}} + \frac{1155 \epsilon_1^2}{65536 \mathcal{E}^{13/2}} + \frac{5005 \epsilon_1^4}{131072 \mathcal{E}^{15/2}} + \frac{42185 \epsilon_1^6}{524288 \mathcal{E}^{17/2}} \right) + \mathcal{O}(\gamma^8). \end{aligned} \quad (\text{C.14})$$

Inverting the above series we obtain the energy up to the third order in $\hat{\Lambda}^4$,

$$\begin{aligned} 2E(a) = & a^2 + \hat{\Lambda}^4 \left(\frac{1}{2a^2} + \frac{\epsilon_1^2}{8a^4} + \frac{\epsilon_1^4}{32a^6} + \frac{\epsilon_1^6}{128a^8} \right) \\ & + \hat{\Lambda}^8 \left(\frac{5}{32a^6} + \frac{21 \epsilon_1^2}{64a^8} + \frac{219 \epsilon_1^4}{512a^{10}} + \frac{121 \epsilon_1^6}{256a^{12}} \right) \\ & + \hat{\Lambda}^{12} \left(\frac{9}{64a^{10}} + \frac{55 \epsilon_1^2}{64a^{12}} + \frac{1495 \epsilon_1^4}{512a^{14}} + \frac{4035 \epsilon_1^6}{512a^{16}} \right) + \mathcal{O}(\hat{\Lambda}^{16}). \end{aligned} \quad (\text{C.15})$$

As one may check, all the coefficients of $\hat{\Lambda}^{4n}$ in the above formula for energy are Maclaurin expansions of coefficients of $-\epsilon_1 \omega_*(a, \Lambda^4/\epsilon_1^4)$ in ϵ_1 . Explicitly,

$$\begin{aligned}
 2E(a) &= a^2 - \epsilon_1 \hat{\Lambda}^4 \sum_{\alpha=1}^2 \omega_{*\alpha,1,1} - \epsilon_1 \hat{\Lambda}^8 \sum_{\alpha=1}^2 \sum_{j=1}^2 \omega_{*\alpha,j,2} - \epsilon_1 \hat{\Lambda}^{12} \sum_{\alpha=1}^2 \sum_{j=1}^3 \omega_{*\alpha,j,3} \\
 &= a^2 - \epsilon_1 \omega_*,
 \end{aligned}
 \tag{C.16}$$

where the coefficients $\omega_{\alpha,i,j}$ are given in eqs. (F.2a)–(F.2k). This result should be related to the two particle periodic Toda chain (pToda). The system is defined either by means of the canonical Poisson-commuting Hamiltonians with potential $U(x_1, x_2) = 2\hat{\Lambda}^2 \cosh(x_1 - x_2)$ or by the Hamiltonians obtained as coefficients of the characteristic polynomial of Lax operator (for details see [19]) with potential $U(x_1, x_2) = 2\hat{\Lambda}^2 \cosh(ix_1 - ix_2)$. In both cases canonical quantization leads to the Schroedinger equation which can be cast in the form of Mathieu equation, where the relationship between parameters E_{Toda} , $\hat{\Lambda}^2$ and Mathieu parameters λ and h^2 reads

$$\lambda = \frac{2E_{\text{Toda}}}{\epsilon_1^2}, \quad h^2 = \frac{\hat{\Lambda}^2}{\epsilon_1^2}.$$

The relation between the spectrum of the sin-Gordon operator and Toda operator is $4E_{\text{SG}} = E_{\text{Toda}}$. Hence, multiplying eq. (C.16) by two we get the formula given in eq. (4.31).

D Young diagrams, their profiles and Nekrasov partition function

In this appendix we introduce basic notions concerning partitions and their use in the context of instanton partition function for pure $\mathcal{N} = 2$ gauge theory with $SU(N)$ symmetry group.

Let $\mathbb{k}^s(n)$ denote a partition of a number $n \in \mathbb{N}$ into ℓ positive integers $k_i^s(n)$, such that for any $i, j \in \{1, \dots, \ell\}$ and $\ell \geq i > j$, $k_i^s \leq k_j^s$. The upper index s indicates that the partition of n may be done in $p(n)$ distinct ways. Thus s^{th} partition of n for some $s \in \{1, \dots, p(n)\}$ is a finite ordered sequence of nonincreasing positive integers, which for any $n \in \mathbb{N}$ and $s \in \{1, \dots, p(n)\}$ may be represented by an ordered ℓ -tuple of numbers

$$\mathbb{k}^s(n) = (k_1^s(n), k_2^s(n), \dots, k_\ell^s(n)), \quad n \equiv |\mathbb{k}^s(n)| = \sum_{i=1}^{\ell} k_i^s(n).
 \tag{D.1}$$

The number $\ell = \ell(\mathbb{k}^s(n))$ is called a length of partition $\mathbb{k}^s(n)$ and it may vary from 1 to $\ell_{\text{max}} = n$. Let \mathcal{P}_n denote a set of all partitions of $n \in \mathbb{N}$, i.e., $\mathcal{P}_n \equiv \{\mathbb{k}^s(n)\}_{s=1}^{p(n)}$. Note, that $\ell(\mathcal{P}_n) = p(n)$. If we also include the null partition, i.e., $\mathcal{P}_0 \equiv \{k^1(0)\}$ ($p(0) = 1$), where $\mathbb{k}^1(0) = (0)$, then the space of all partitions of each element of $\mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$ termed *Young graph* and denoted by \mathbb{Y} is a disjoint union of sets \mathcal{P}_n , that is

$$\mathbb{Y} = \bigsqcup_{n \in \mathbb{N}_0} \mathcal{P}_n.$$

The partition $\mathbb{k}^s(n)$ can be represented graphically by means of the Young diagram, whose columns contain $k_i^s(n)$ unenumerated boxes for each $1 \leq i \leq \ell$ and $n > 0$.

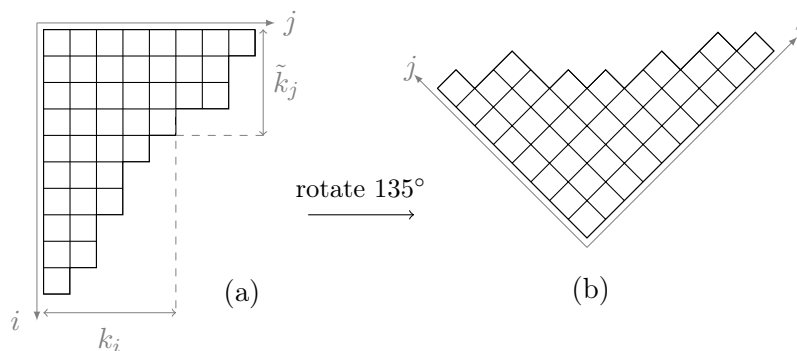


Figure 2. The Young diagram for the partition $\mathbb{k} = (8, 7, 7, 5, 4, 3, 3, 2, 2, 1)$ of $|\mathbb{k}| = 42$ with quantities defined in the text and the relationship between the two different conventions: English (panel (a)) and Russian (panel (b)).

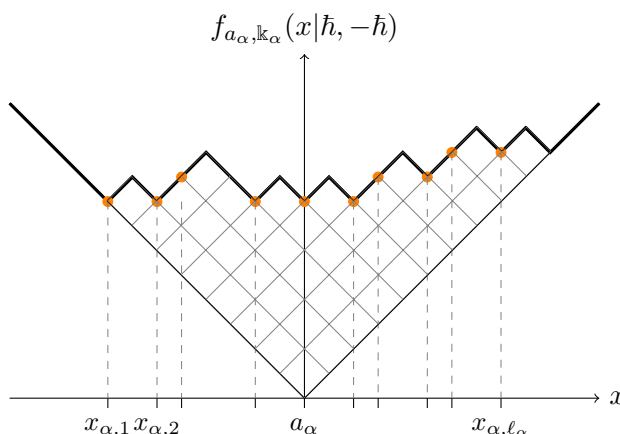


Figure 3. The thick line represents the profile function for the colored Young diagram depicted on panel (b) of figure 2. The point $x_{\alpha,i}$ is a coordinate of i^{th} column within a α^{th} Young diagram in the Russian convention and projected onto the real axis. The decline of columns coincides with the linear order of \mathbb{R} . Since $\epsilon_1 \neq \epsilon_2$ the boxes of a Young diagram are deformed to rectangles with edges $l_i = \sqrt{2}|\epsilon_i|$, $i = 1, 2$.

Let $k \in \mathbb{N}_0$ be the instanton number (charge). In order to extend the notion of partitions to the colored ones we assume that this number can be partitioned into a sum of N nonnegative integers n_α for $\alpha = 1, \dots, N$. Therefore, in opposite to the earlier partitioning the maximal length of the latter is constrained to N . However, each of n_α can be partitioned as before i.e., in an unconstrained way ($\ell_{\alpha \max} = n_\alpha$). Let $\mathbf{k}^s(k)$ be a s^{th} colored partition of k . The space of all colored partitions termed *colored Young graph* and its elements are defined as follows³⁴

$$\mathbb{Y}^N \equiv \prod_{i=1}^N \mathbb{Y}, \quad \mathbf{k}^s(k) \in \mathbb{Y}^N, \quad \mathbf{k}^s(k) \equiv (\mathbb{k}^{s1}(n_1), \mathbb{k}^{s2}(n_2), \dots, \mathbb{k}^{sN}(n_N)), \quad (\text{D.2})$$

³⁴In what follows we ascribe Greek letters to color indexes $\alpha = 1, \dots, N$ and Latin letters to coordinates of the box within a Young diagram, $(i, j) \in \mathbb{k}^s(n)$.

where for a given $k \in \mathbb{N}_0$

$$|\mathbf{k}^s(k)| \equiv k = \sum_{\alpha=1}^N n_\alpha = \sum_{\alpha=1}^N |\mathbb{k}^{s_\alpha}(n_\alpha)| = \sum_{\alpha=1}^N \sum_{i_\alpha=1}^{\ell_\alpha} k_{i_\alpha}^{s_\alpha}(n_\alpha). \quad (\text{D.3})$$

for any $s_\alpha \in \{1, \dots, p(n_\alpha)\}$ with $\alpha = 1, \dots, N$. Note that any $\mathbf{k}^s(k)$ is uniquely specified by the set of numbers s_α and n_α , such that for a fixed n_α for all α we have $\prod p(n_\alpha)$ number of different $\mathbf{k}^s(k)$'s. The above equation represents a specific decomposition of k into a sum of parts n_α . However, this decomposition is not unique. In fact, there is a $c(n, N) = (n + N - 1)! / (N - 1)! n!$ possible ways to do this, where $c(n, N)$ is a number of constrained partitions (so called *compositions*) of $n \in \mathbb{N}_0$ into a sum of exactly N parts. Let $\mathcal{S}(k, N)$ denote a *shell* in \mathbb{Y}^N , that is a set of points defined as follows $\mathcal{S}(k, N) \equiv \{\mathbf{k} \in \mathbb{Y}^N \mid |\mathbf{k}| = k\}$, then the number of points in the shell amounts to

$$\#\mathcal{S}(k, N) = \sum_{r=1}^{c(k, N)} \prod_{\alpha=1}^N p(n_\alpha^r).$$

This enables us to decompose the sum in the instanton part of eq. (4.1b) into a sum over distinct shells and the sum over points in a given shell, namely

$$\sum_{\mathbf{k} \in \mathbb{Y}^N} \hat{\Lambda}^{2N|\mathbf{k}|} Z_{\mathbf{k}}(\mathbf{a}, \epsilon_1, \epsilon_2) = \sum_{k \geq 0} \hat{\Lambda}^{2Nk} \sum_{\mathbf{k} \in \mathcal{S}(k, N)} Z_{\mathbf{k}}(\mathbf{a}, \epsilon_1, \epsilon_2). \quad (\text{D.4})$$

Sum over shells amounts to

$$\sum_{\mathbf{k} \in \mathcal{S}(k, N)} Z_{\mathbf{k}}(\mathbf{a}, \epsilon_1, \epsilon_2) = \sum_{r=1}^{c(k, N)} \prod_{\alpha=1}^N \sum_{s_\alpha=1}^{p(n_\alpha^r)} Z_{\text{inst}}(\mathbf{a}, \{\mathbb{k}^{s_\alpha}(n_\alpha^r)\}, \epsilon_1, \epsilon_2). \quad (\text{D.5})$$

Instead of regarding s^{th} partition $\mathbb{k}^s(n)$ as a ℓ -tuple of non-increasing, positive integers one can extend this notion to an infinite sequence of non-increasing, *nonnegative* integers k_i whose length is $\ell(\mathbb{k}) = \ell$, and for $i > \ell$, $k_i = 0$. Moreover, we follow the practice to keep implicit both, the label s that positions a partition in \mathcal{P}_n and the number n the relevant partition corresponds to. Within this extended notion, the partition is the following sequence

$$\mathbb{k} = \{k_i\}_{i \in \mathbb{N}}, \quad \text{where } k_i \in \mathbb{N}_0, \quad \bigvee_{i, j \in \mathbb{N}} i > j \Rightarrow k_i \leq k_j, \quad |\mathbb{k}| = \sum_{i \geq 1} k_i = n.$$

Note that this implies that for $i \rightarrow \infty$, $k_i \rightarrow 0$. Form this point of view, \mathbb{Y} is a subset of nonnegative, integer null sequences $\mathbf{c}_0(\mathbb{Z}_{\geq 0})$. For any partition (Young diagram) $\mathbb{k} \in \mathbb{Y}$ there is a dual partition $\tilde{\mathbb{k}} \in \mathbb{Y}^D$. It is related to the original one through the transposition, i.e., $\tilde{\mathbb{k}} = \mathbb{k}^T$ ($\tilde{\mathbb{k}}^T = \mathbb{k}$) and $|\tilde{\mathbb{k}}| = |\mathbb{k}|$. Hence, $\mathbb{Y} \cong \mathbb{Y}^D = \text{Reverse}(\mathbb{Y})$, where the last map is a specific permutation that reverses the ordering of partitions within each \mathcal{P}_n . Extension to a colored partition is straightforward and for any $\mathbf{k} \in \mathbb{Y}^N$ it takes the following form

$$\mathbf{k} = (\mathbb{k}_1, \dots, \mathbb{k}_N) = \{k_{\alpha, i}\}_{\alpha \in \{1, \dots, N\}, i \in \mathbb{N}} \quad \text{and} \quad k = |\mathbf{k}| = \sum_{\alpha=1}^N |\mathbb{k}_\alpha| = \sum_{\alpha=1}^N \sum_{i \geq 1} k_{\alpha, i}.$$

Notions of partition theory play the crucial role in counting of instanton configurations. As it has been shown by Nekrasov [22] and independently by Nakajima and Yoshioka [60] the instanton part of the partition function for the $\mathcal{N} = 2$ nonabelian gauge theory with $SU(N)$ symmetry, when computed on the so called Ω -background [21], appears to be a sum over all colored Young diagrams. The instanton charge k corresponds to the total number of boxes that are distributed into N parts (compositions) each of which can be further partitioned, thus forming a colored Young diagram as described above. The form of Nekrasov's instanton partition function for a pure $SU(N)$ gauge theory reads [21]

$$Z_{\text{inst}}(\hat{\Lambda}, \mathbf{a}, \epsilon_1, \epsilon_2) = \sum_{\mathbf{k} \in \mathbb{Y}^N} \hat{\Lambda}^{2Nk} Z_{\mathbf{k}}(\mathbf{a}, \epsilon_1, \epsilon_2), \quad (\text{D.6})$$

where

$$Z_{\mathbf{k}}(\mathbf{a}, \epsilon_1, \epsilon_2) = \prod_{\alpha, \beta=1}^N \prod_{i, j \geq 1} \frac{a_{\alpha} - a_{\beta} + \epsilon_1(i-1) + \epsilon_2(-j)}{a_{\alpha} - a_{\beta} + \epsilon_1(i - \tilde{k}_{\beta, j} - 1) + \epsilon_2(k_{\alpha, i} - j)}, \quad (\text{D.7a})$$

where $\hat{\Lambda}$ is an infrared effective scale which is related to the complexified $SU(N)$ gauge coupling $\tau = 4\pi i/g^2 + \theta/2\pi$ through $\hat{\Lambda}^{2N} \sim \mu_{UV}^{2N} \exp\{2\pi i\tau\}$. μ_{UV} is the scale where the classical (microscopic) theory is defined. The deformation parameters are in general $\epsilon_{1,2} \in \mathbb{C}$ and either $\Re(\epsilon_1) > 0, \Re(\epsilon_2) < 0$ or $\Re(\epsilon_1) < 0, \Re(\epsilon_2) > 0$. For the sake of definiteness in what follows we assume $\Re(\epsilon_1) > 0, \Re(\epsilon_2) < 0$. The contribution (D.7a) to eq. (D.6) may be cast into three equivalent forms, which we quote following refs. [21, 60, 61], namely

$$\begin{aligned} Z_{\mathbf{k}}(\mathbf{a}, \epsilon_1, \epsilon_2) &= \prod_{\alpha, \beta=1}^N \prod_{i, j \geq 1} \frac{a_{\alpha\beta} + \epsilon_1(-i) + \epsilon_2(j-1)}{a_{\alpha\beta} + \epsilon_1(\tilde{k}_{\alpha, j} - i) + \epsilon_2(j - k_{\beta, i} - 1)} \end{aligned} \quad (\text{D.7b})$$

$$= \frac{1}{\epsilon_2^{2Nk}} \prod_{(\alpha, i) \neq (\beta, j)} \frac{\Gamma(b_{\alpha\beta} + \nu(j-i+1) + k_{\alpha i} - k_{\beta j}) \Gamma(b_{\alpha\beta} + \nu(j-i))}{\Gamma(b_{\alpha\beta} + \nu(j-i) + k_{\alpha i} - k_{\beta j}) \Gamma(b_{\alpha\beta} + \nu(j-i+1))} \quad (\text{D.7c})$$

$$= \prod_{\alpha, \beta=1}^N \prod_{s \in \mathbb{k}_{\alpha}} \frac{1}{a_{\alpha\beta} - \epsilon_1 L_{\beta}(s) + \epsilon_2 (A_{\alpha}(s) + 1)} \prod_{s \in \mathbb{k}_{\beta}} \frac{1}{a_{\alpha\beta} + \epsilon_1 (L_{\alpha}(s) + 1) - \epsilon_2 A_{\beta}(s)}, \quad (\text{D.7d})$$

where

$$a_{\alpha\beta} \equiv a_{\alpha} - a_{\beta}, \quad b_{\alpha\beta} \equiv a_{\alpha\beta}/\epsilon_2, \quad \nu \equiv -\epsilon_1/\epsilon_2,$$

and $A_{\alpha}(s) \equiv k_{\alpha, i} - j$, $L_{\alpha}(s) \equiv \tilde{k}_{\alpha, j} - i$ are arm-length and leg-length, respectively. $s \equiv (i, j)$ represents a box in a Young diagram \mathbb{k}_{α} , where $1 \leq i \leq \ell_{\alpha} = \tilde{k}_{\alpha, 1}$ and $1 \leq j \leq k_{\alpha, i}$. The forms (D.7a)–(D.7c) although defined in terms of infinite product are finite and well defined due to a finiteness of the relevant Young diagrams. In what follows we are concerned with yet another form, namely the contribution expressed in terms of the profiles of Young diagrams. This form can be obtained by means of the ζ -function regularization techniques that enable to deal with divergent expressions like the numerator or the denominator of eq. (D.7a) when considered separately. The contribution (D.7a) to the instanton partition

function (D.6) along with the perturbative part introduced in eq. (4.1a) can be cast into the following form

$$\begin{aligned} & \hat{\Lambda}^{2Nk} \exp \left\{ - \sum_{\alpha, \beta=1}^N \gamma_{\epsilon_1, \epsilon_2}(a_{\alpha\beta}; \hat{\Lambda}) \right\} Z_{\mathbf{k}}(\mathbf{a}, \epsilon_1, \epsilon_2) \\ &= \exp \left\{ - \frac{d}{ds} \Big|_{s=0} \frac{\hat{\Lambda}^s}{\Gamma(s)} \int_0^\infty dt t^{s-1} \frac{1}{(e^{\epsilon_1 t} - 1)(e^{\epsilon_2 t} - 1)} \sum_{\alpha, \beta=1}^N \left[e^{a_{\alpha\beta} t} \right. \right. \\ & \quad \left. \left. + (e^{-\epsilon_1 t} - 1)(e^{\epsilon_2 t} - 1) \sum_{i, j \geq 1} \left(e^{t[a_{\alpha\beta} + \epsilon_1(i - \tilde{k}_{\beta j}) + \epsilon_2(k_{\alpha i} - j)]} - e^{t[a_{\alpha\beta} + \epsilon_1 i - \epsilon_2 j]} \right) \right] \right\}, \quad (\text{D.8}) \end{aligned}$$

where the perturbative part is defined in eq. (4.1a) with the $\epsilon_1 \epsilon_2$ -parametric gamma function assumes a form (recall that we take $\Re \epsilon_1 > 0$, $\Re \epsilon_2 < 0$)

$$\gamma_{\epsilon_1, \epsilon_2}(x; \hat{\Lambda}) \equiv \frac{d}{ds} \Big|_{s=0} \frac{\hat{\Lambda}^s}{\Gamma(s)} \int_0^\infty dt t^{s-1} \frac{e^{-xt}}{(e^{\epsilon_1 t} - 1)(e^{\epsilon_2 t} - 1)}, \quad \Re x > 0, \Re s > 2. \quad (\text{D.9})$$

The above gamma function satisfies the following second order functional partial difference equation

$$\Delta_{\epsilon_1} \Delta_{\epsilon_2} \gamma_{\epsilon_1, \epsilon_2}(x; \hat{\Lambda}) = \log \left(\frac{\hat{\Lambda}}{x + \epsilon_1 + \epsilon_2} \right),$$

where $\Delta_{\epsilon_i} f(x) \equiv f(x + \epsilon_i) - f(x)$ for $i = 1, 2$. For convenience we introduce the following notation

$$e_\alpha(t) \equiv e^{a_\alpha t}, \quad p(t) \equiv e^{-\epsilon_1 t}, \quad q(t) \equiv e^{\epsilon_2 t}, \quad \Re(\epsilon_1) > 0, \Re(\epsilon_2) < 0. \quad (\text{D.10})$$

With this notation the formula in the last line of eq. (D.8) reads

$$\begin{aligned} & - \frac{d}{ds} \Big|_{s=0} \frac{\hat{\Lambda}^s}{\Gamma(s)} \int_0^\infty dt t^{s-1} \frac{1}{(p^{-1} - 1)(q - 1)} \\ & \quad \times \sum_{\alpha, \beta=1}^N e_\alpha e_\beta^{-1} \left\{ 1 + (p - 1)(q - 1) \sum_{i, j \geq 1} \left(p^{\tilde{k}_{\beta j} - i} q^{k_{\alpha i} - j} - p^{-i} q^{-j} \right) \right\}, \quad (\text{D.11}) \end{aligned}$$

The above series can be transformed into the finite form, namely ($\tilde{\ell}_\alpha = \ell(\mathbb{k}_\alpha^T) = k_{\alpha,1}$)

$$\begin{aligned}
& 1 + (p-1)(q-1) \sum_{i,j \geq 1} \left(p^{\tilde{k}_{\beta j} - i} q^{k_{\alpha i} - j} - p^{-i} q^{-j} \right) \\
&= 1 + (p-1)(q-1) \sum_{i=1}^{\ell_\alpha} \sum_{j=1}^{\tilde{\ell}_\beta} p^{-i} q^{-j} \left(p^{\tilde{k}_{\beta j}} - 1 \right) \left(q^{k_{\alpha i}} - 1 \right) \\
&\quad + (p-1) \sum_{i=1}^{\ell_\alpha} p^{-i} \left(q^{k_{\alpha i}} - 1 \right) + (q-1) \sum_{j=1}^{\tilde{\ell}_\beta} q^{-j} \left(p^{\tilde{k}_{\alpha j}} - 1 \right) \\
&= \left[1 + (p-1) \sum_{i=1}^{\ell_\alpha} p^{-i} \left(q^{k_{\alpha i}} - 1 \right) \right] \left[1 + (q-1) \sum_{j=1}^{\tilde{\ell}_\beta} q^{-j} \left(p^{\tilde{k}_{\beta j}} - 1 \right) \right]. \tag{D.12}
\end{aligned}$$

In what follows it proves useful to define the following quantity

$$\varphi_{\mathbb{k}_\alpha}(p, q) \equiv 1 + (p-1) \sum_{i=1}^{\ell_\alpha} p^{-i} \left(q^{k_{\alpha i}} - 1 \right). \tag{D.13}$$

Note, that in the above formula we can extend the upper sum limit ℓ_α up to infinity which leaves its form intact. Multiplying eq. (D.13) by e_α , summing over color index and rewriting it back in the original form as in eq. (D.10) we obtain

$$\begin{aligned}
& \sum_{\alpha=1}^N e_\alpha \varphi_{\mathbb{k}_\alpha}(p, q) \\
&= \sum_{\alpha=1}^N e^{a_\alpha t} + \sum_{\alpha=1}^N \sum_{i \geq 1} \left(e^{[a_\alpha + \epsilon_1(i-1) + \epsilon_2 k_{\alpha i}]t} - e^{[a_\alpha + \epsilon_1 i + \epsilon_2 k_{\alpha i}]t} - e^{[a_\alpha + \epsilon_1(i-1)]t} + e^{[a_\alpha + \epsilon_1 i]t} \right).
\end{aligned}$$

which can be written as

$$\frac{1}{2} \int_{\mathbb{R}} dx f''_{\mathbf{a}, \mathbf{k}}(x | \epsilon_1, \epsilon_2) e^{xt} = \sum_{\alpha=1}^N e_\alpha(t) \varphi_{\mathbb{k}_\alpha}(p(t), q(t)), \tag{D.14a}$$

where

$$\begin{aligned}
f_{\mathbf{a}, \mathbf{k}}(x | \epsilon_1, \epsilon_2) &= \sum_{\alpha=1}^N |x - a_\alpha| + \sum_{\alpha=1}^N \sum_{i \geq 1} \left(|x - x_{\alpha, i}| - |x - x_{\alpha, i} - \epsilon_1| \right. \\
&\quad \left. - |x - x_{\alpha, i}^0| + |x - x_{\alpha, i}^0 - \epsilon_1| \right), \tag{D.14b}
\end{aligned}$$

is the *profile function* of the colored Young diagram \mathbf{k} . In analogy to the above, one defines for the second factor of the last line of eq. (D.12) the following form of the profile

$$\frac{1}{2} \int_{\mathbb{R}} dx f''_{\mathbf{a}, \mathbf{k}^T}(x | \epsilon_2, \epsilon_1) e^{-xt} = \sum_{\alpha=1}^N e_\alpha^{-1}(t) \varphi_{\mathbb{k}_\alpha^T}(q(t), p(t)), \tag{D.14c}$$

and

$$\begin{aligned}
 f_{\mathbf{a},\mathbf{k}^T}(x|\epsilon_2, \epsilon_1) &= \sum_{\alpha=1}^N |x + a_\alpha| + \sum_{\alpha=1}^N \sum_{i \geq 1} \left(|x - \tilde{x}_{\alpha,i}| - |x - \tilde{x}_{\alpha,i} - \epsilon_2| \right. \\
 &\quad \left. - |x - \tilde{x}_{\alpha,i}^0| + |x - \tilde{x}_{\alpha,i}^0 - \epsilon_2| \right). \tag{D.14d}
 \end{aligned}$$

This form of the profile function equals to the one in eq. (D.14b) as it takes its form from subjecting the former to the transposition of the Young diagram and subsequent reversion of the order of pair of parameters $(\epsilon_1, \epsilon_2) \rightarrow (\epsilon_2, \epsilon_1)$. It stems from the fact that in case of the arbitrary Young diagram the following identity holds true

$$\sum_{(i,j) \in \mathbf{k}} p^i q^j = \sum_{i=1}^{\ell} \sum_{j=1}^{k_i} p^i q^j = \sum_{j=1}^{\bar{\ell}} \sum_{i=1}^{\bar{k}_j} p^i q^j = \sum_{(j,i) \in \mathbf{k}^T} p^i q^j. \tag{D.15}$$

It entails that $\varphi_{\mathbf{k}_\alpha}(p(t), q(t)) = \varphi_{\mathbf{k}_\alpha^T}(q(t), p(t))$ and from eq. (D.10) it results that $p^{-1}(t) = p(-t)$ which holds true also for q and e_α . Hence, we obtain the equality $f_{\mathbf{a},\mathbf{k}}(x|\epsilon_1, \epsilon_2) = f_{\mathbf{a},\mathbf{k}^T}(x|\epsilon_2, \epsilon_1)$. Using the two expressions for the profile functions we obtain the exponent of right hand side of eq. (D.8) in the sought form, namely

$$\begin{aligned}
 & - \frac{d}{ds} \Big|_{s=0} \frac{\hat{\Lambda}^s}{\Gamma(s)} \int_0^\infty dt t^{s-1} \frac{1}{(p^{-1}-1)(q-1)} \sum_{\alpha,\beta=1}^N e_\alpha e_\beta^{-1} \varphi_{\mathbf{k}_\alpha}(p, q) \varphi_{\mathbf{k}_\beta^T}(q, p) \\
 &= -\frac{1}{4} \iint_{\mathbb{R}^2} dx dy f''_{\mathbf{a},\mathbf{k}}(x|\epsilon_1, \epsilon_2) \gamma_{\epsilon_1, \epsilon_2}(x-y; \hat{\Lambda}) f''_{\mathbf{a},\mathbf{k}^T}(y|\epsilon_2, \epsilon_1). \tag{D.16}
 \end{aligned}$$

The partition function for the $\mathcal{N} = 2$ pure gauge theory with $SU(N)$ symmetry takes now the form

$$\mathcal{Z}(\Lambda, \mathbf{a}, \epsilon_1, \epsilon_2) = \sum_{f_{\mathbf{a},\mathbf{k}} \in \mathcal{F}} \exp \left\{ -\frac{1}{4} \iint_{\mathbb{R}^2} dx dy f''_{\mathbf{a},\mathbf{k}}(x|\epsilon_1, \epsilon_2) \gamma_{\epsilon_1, \epsilon_2}(x-y; \Lambda) f''_{\mathbf{a},\mathbf{k}^T}(y|\epsilon_2, \epsilon_1) \right\}, \tag{D.17}$$

Due to the fact that the profile function equals its (parameter) reverse (Young diagram) transpose, the exponent in the above formula can be cast into yet two equivalent forms with exponents that have schematic form³⁵

$$(f''_{\mathbf{k}^T})^i(\epsilon_2, \epsilon_1) \gamma_{ij}(\hat{\Lambda}) (f''_{\mathbf{k}})^j(\epsilon_1, \epsilon_2) \quad \text{and} \quad (f''_{\mathbf{k}})^i(\epsilon_1, \epsilon_2) \gamma_{ij}(\hat{\Lambda}) (f''_{\mathbf{k}^T})^j(\epsilon_1, \epsilon_2).$$

The last two correspond to eqs. (D.7b) and (D.7c), respectively multiplied by the perturbative part. This statement can be phrased yet in another form. Let us define the following quantity

$$N_{\alpha\beta}(p, q) \equiv \frac{\varphi_{\mathbf{k}_\alpha}(p, q) \varphi_{\mathbf{k}_\beta^T}(q, p) - 1}{(p^{-1}-1)(q-1)}. \tag{D.18}$$

³⁵For notation see eq. (4.9).

Then the left hand side of eq. (D.16) can be put in the form

$$\begin{aligned}
 & -\frac{1}{4} \iint_{\mathbb{R}^2} dx dy f''_{\mathbf{a},\mathbf{k}}(x|\epsilon_1, \epsilon_2) \gamma_{\epsilon_1, \epsilon_2}(x-y; \hat{\Lambda}) f''_{\mathbf{a},\mathbf{k}^T}(y|\epsilon_2, \epsilon_1) \\
 & = -\sum_{\alpha, \beta=1}^N \gamma_{\epsilon_1, \epsilon_2}(a_{\alpha\beta}; \hat{\Lambda}) - \frac{d}{ds} \Big|_{s=0} \frac{\hat{\Lambda}^s}{\Gamma(s)} \int_0^\infty dt t^{s-1} \sum_{\alpha, \beta=1}^N e_\alpha e_\beta^{-1} N_{\alpha\beta}(p, q).
 \end{aligned}$$

The quantity under the second integral with the use of eq. (D.15) can be brought to the following equivalent forms

$$\sum_{\alpha, \beta=1}^N e_\alpha e_\beta^{-1} N_{\alpha\beta}(p, q) = -\sum_{\alpha, \beta=1}^N e_\alpha e_\beta^{-1} \sum_{i, j \geq 1} \left(p^{\tilde{k}_{\beta j} - i + 1} q^{k_{\alpha i} - j} - p^{-i+1} q^{-j} \right) \quad (\text{D.19a})$$

$$= -\sum_{\alpha, \beta=1}^N e_\alpha e_\beta^{-1} \sum_{i, j \geq 1} \left(p^{-\tilde{k}_{\alpha j} + i} q^{-k_{\beta i} + j - 1} - p^i q^{j-1} \right) \quad (\text{D.19b})$$

$$= \sum_{(\alpha, i) \neq (\beta, j)} e_\alpha e_\beta^{-1} \frac{p-1}{q-1} \left(q^{k_{\alpha i} - k_{\beta j}} p^{j-i} - p^{j-i} \right) \quad (\text{D.19c})$$

$$= -\sum_{\alpha, \beta=1}^N e_\alpha e_\beta^{-1} \left(\sum_{s \in \mathbb{k}_\alpha} p^{L_\beta(s)+1} q^{A_\alpha(s)} + \sum_{s \in \mathbb{k}_\beta} p^{-L_\alpha(s)} q^{-A_\beta(s)-1} \right). \quad (\text{D.19d})$$

The ordering of the above equations corresponds to the one in eqs. (D.7a)–(D.7d). The definition of A_α and L_α can be found below the latter equation. The “off-diagonal” sum in eq. (D.19c) is understood as follows

$$\sum_{(\alpha, i) \neq (\beta, j)} A_{\alpha, i; \beta, j} = \sum_{\alpha=1}^N \sum_{i \neq j} A_{\alpha, i; \alpha, j} + \sum_{\alpha \neq \beta} \sum_{i, j} A_{\alpha, i; \beta, j}.$$

E Product identities and Y functions

The saddle point equation can be given yet another form. First, let us note that the following identities hold

$$\prod_{\alpha=1}^N \prod_{j=1}^L \frac{x - x_{\alpha, j}^0 + \epsilon_1}{x - x_{\alpha, j}^0} = \prod_{\alpha=1}^N \frac{x - x_{\alpha, 1}^0 + \epsilon_1}{x - x_{\alpha, 1}^0 - \epsilon_1(L-1)}. \quad (\text{E.1a})$$

Shifting the above equation in $x \rightarrow x - \epsilon_1$ and multiplying thus obtained result by the eq. (E.1a) leads to the following identity

$$\prod_{\alpha=1}^N \prod_{j=1}^L \frac{x - x_{\alpha, j}^0 + \epsilon_1}{x - x_{\alpha, j}^0 - \epsilon_1} \prod_{j=1}^2 \frac{\epsilon_1}{x - x_{\alpha, j}^0 + \epsilon_1} = \prod_{\alpha=1}^N \prod_{j=1}^2 \frac{\epsilon_1}{x - x_{\alpha, j}^0 - \epsilon_1(L-1)}. \quad (\text{E.1b})$$

As the first step to the iterative solution of eq. (4.21) one assumes that there exists some $L \in \mathbb{N}$ at which a sequence of columns terminates, i.e., $\omega_{\alpha, i} = 0$ for $i > L$. In this case the

only change in the saddle point equation (4.21) is

$$\begin{aligned} & \prod_{\beta=1}^N \prod_{j \geq 1} (x_{\alpha,i} - x_{\beta,j} - \epsilon_1)(x_{\alpha,i} - x_{\beta,j}^0 + \epsilon_1) \prod_{j=1}^2 \frac{\epsilon_1}{x_{\alpha,i} - x_{\beta,j}^0 + \epsilon_1} \\ &= \prod_{\beta=1}^N \prod_{j=1}^L \frac{(x_{\alpha,i} - x_{\beta,j} - \epsilon_1)(x_{\alpha,i} - x_{\beta,j}^0 + \epsilon_1)}{(x_{\alpha,i} - x_{\beta,j} + \epsilon_1)(x_{\alpha,i} - x_{\beta,j}^0 - \epsilon_1)} \prod_{j=1}^2 \frac{\epsilon_1}{x_{\alpha,i} - x_{\beta,j}^0 + \epsilon_1}. \end{aligned}$$

Due to identity in eq. (E.1b) it is possible to rewrite the right hand side of the above equation in the form

$$\prod_{\beta=1}^N \prod_{j=1}^L \frac{x_{\alpha,i} - x_{\beta,j} - \epsilon_1}{x_{\alpha,i} - x_{\beta,j} + \epsilon_1} \prod_{j=1}^2 \frac{\epsilon_1}{x_{\alpha,i} - x_{\beta,j}^0 - \epsilon_1(L-1)} = \epsilon_1^{2N} \prod_{\beta=1}^N \prod_{j \geq 1} \frac{x_{\alpha,i} - x_{\beta,j} - \epsilon_1}{x_{\alpha,i} - x_{\beta,j} + \epsilon_1}.$$

Hence, the saddle point equation takes the infinite product form ($\xi_{\alpha,i} \equiv x_{\alpha,i}/\epsilon_1$)

$$\hat{\Lambda}^{2N} \prod_{\beta=1}^N \prod_{j \geq 1} \frac{x_{\alpha,i} - x_{\beta,j} - \epsilon_1}{x_{\alpha,i} - x_{\beta,j} + \epsilon_1} = (\hat{\Lambda}/\epsilon_1)^{2N} \prod_{\beta=1}^N \prod_{j \geq 1} \frac{\xi_{\alpha,i} - \xi_{\beta,j} - 1}{\xi_{\alpha,i} - \xi_{\beta,j} + 1} = (-1)^{N-1}, \quad (\text{E.2})$$

It is now possible to release the condition for the finiteness of the number of columns such that $\{\omega_i\}_{i=1}^L \rightarrow \{\omega_i\}_{i \in \mathbb{N}}$, which leaves the form of the equation intact. As one may notice, this form resembles TBA equation.

Before we proceed, let us rewrite the saddle point equation using the following function

$$U(z) \equiv \prod_{\beta=1}^N \prod_{j \geq 1} \frac{z/\epsilon_1 - 1 - \xi_{\beta,j}}{z/\epsilon_1 + 1 - \xi_{\beta,j}}, \quad (\hat{\Lambda}/\epsilon_1)^{2N} U(x_{\alpha,i}) = (-1)^{N-1}.$$

Note, that numerator and denominator if taken separately are divergent. For the study of non-iterative solution of eq. (E.2) it is convenient to work with well defined functions. Therefore, following Poghossian [57] we bring the saddle point equation into yet another form. Namely, let us cast $U(z)$ function back into the form of eq. (4.21) with the cutoff L for safety reasons, i.e.,

$$\begin{aligned} U_L(z) &= \frac{\epsilon_1^{2N}}{P(z)P(z+\epsilon_1)} \prod_{\beta=1}^N \prod_{j=1}^L \frac{(z - x_{\beta,j} - \epsilon_1)(z - x_{\beta,j}^0 + \epsilon_1)}{(z - x_{\beta,j} + \epsilon_1)(z - x_{\beta,j}^0 - \epsilon_1)} \\ &= \frac{\epsilon_1^{2N}}{P(z)P(z+\epsilon_1)} \prod_{\beta=1}^N \prod_{j=1}^L \frac{\left(1 - \frac{z-\epsilon_1}{x_{\beta,j}}\right) \left(1 - \frac{z+\epsilon_1}{x_{\beta,j}^0}\right)}{\left(1 - \frac{z+\epsilon_1}{x_{\beta,j}}\right) \left(1 - \frac{z-\epsilon_1}{x_{\beta,j}^0}\right)}, \end{aligned} \quad (\text{E.3})$$

where $P(z) \equiv \prod (z - a_\alpha)$. Each factor of the above product may be transformed as follows

$$\begin{aligned} & \prod_{\beta=1}^N \prod_{j=1}^L \left(1 - \frac{z}{x_{\beta,j}}\right) e^{z/x_{\beta,j}^0} e^{-z/x_{\beta,j}} = \prod_{\beta=1}^N e^{-(z/\epsilon_1) \sum_{j=1}^L (\epsilon_1/x_{\beta,j}^0)} \prod_{j=1}^L \left(1 - \frac{z}{x_{\beta,j}}\right) e^{z/x_{\beta,j}^0} \\ &= e^{-\frac{z}{\epsilon_1} N \left(\gamma + \sum_{j=1}^L \frac{1}{j}\right)} \prod_{\beta=1}^N e^{(z/\epsilon_1) \psi_L(a_\beta/\epsilon_1)} \prod_{j=1}^L \left(1 - \frac{z}{x_{\beta,j}}\right) e^{z/x_{\beta,j}^0}, \end{aligned}$$

where

$$\psi_L(z+1) \equiv -\gamma + \sum_{i=1}^L \left(\frac{1}{i} - \frac{1}{z+i} \right) \xrightarrow{L \rightarrow \infty} \psi(z+1) = \frac{d}{dz} \log \Gamma(z+1).$$

Inserting this result in the appropriate form into the eq. (E.3) we get

$$U_L(z) = \frac{\epsilon_1^{2N}}{P(z)P(z+\epsilon_1)} \prod_{\beta=1}^N \frac{e^{2(z/\epsilon_1)\psi_L(a_\beta/\epsilon_1)}}{e^{2(z/\epsilon_1)\psi_L(a_\beta/\epsilon_1)}} \\ \times \prod_{j=1}^L \frac{\left(1 - \frac{z-\epsilon_1}{x_{\beta,j}}\right) e^{(z-\epsilon_1)/x_{\beta,j}^0} \left(1 - \frac{z+\epsilon_1}{x_{\beta,j}^0}\right) e^{(z+\epsilon_1)/x_{\beta,j}^0}}{\left(1 - \frac{z+\epsilon_1}{x_{\beta,j}}\right) e^{(z+\epsilon_1)/x_{\beta,j}^0} \left(1 - \frac{z-\epsilon_1}{x_{\beta,j}^0}\right) e^{(z-\epsilon_1)/x_{\beta,j}^0}}.$$

Each factor in the numerator and the denominator are now separately well defined. Within the limit $L \rightarrow \infty$ these functions resemble inverse of the Gamma function which is known to be entire function. Define the following quantities

$$Y(z) \equiv \prod_{\beta=1}^N e^{-(z/\epsilon_1)\psi(a_\beta/\epsilon_1)} \prod_{j \geq 1} \left(1 - \frac{z}{x_{\beta,j}}\right) e^{z/x_{\beta,j}^0}, \\ Y_0(z) \equiv \prod_{\beta=1}^N e^{-(z/\epsilon_1)\psi(a_\beta/\epsilon_1)} \prod_{j \geq 1} \left(1 - \frac{z}{x_{\beta,j}^0}\right) e^{z/x_{\beta,j}^0}. \quad (\text{E.4})$$

These are either entire functions with zeros at points $x_{\beta,j}, x_{\beta,j}^0$ respectively. We can now safely send $L \rightarrow \infty$ so that the result reads

$$U(z) = \frac{\epsilon_1^{2N}}{P(z)P(z+\epsilon_1)} \frac{Y(z-\epsilon_1)Y_0(z+\epsilon_1)}{Y(z+\epsilon_1)Y_0(z-\epsilon_1)} = \frac{Y(z-\epsilon_1)}{Y(z+\epsilon_1)}. \quad (\text{E.5})$$

With the aid of these functions the saddle point equation (E.2) can now be expressed in yet another enlightening form given in eq. (4.26).

F Result of iterative solution of saddle point equation for SU(2)

In this appendix we present results of iterative solution of the saddle point equation (4.21). This is done up to $L = 3$, where L is a cutoff for number of columns within each colored Young diagram. Quantities obtained this way are coefficients of expansion in $(\hat{\Lambda}/\epsilon_1)^{2N}$ of colored column in an extreme Young diagram, namely

$$\omega_{*\alpha,i} = \sum_{j \geq i} \omega_{\alpha,i,j} \mathfrak{q}^j, \quad \mathfrak{q} \equiv (\hat{\Lambda}/\epsilon_1)^{2N}.$$

The sum over colors within a given order of \mathfrak{q} expansion yields coefficients of the twisted superpotential. The saddle point equation given in eq. (4.21) which is to be solved itera-

tively in this appendix may be rewritten in a more useful form, namely

$$\begin{aligned} & \mathfrak{q}\epsilon_1^{2N} \prod_{\alpha=1}^N \prod_{j=1}^L \left(x_{\alpha,i}^{(L)} - x_{\beta,j}^0 + \epsilon_1 \right) \left(x_{\alpha,i}^{(L)} - x_{\beta,j}^{(L)} - \epsilon_1 \right) \\ & + \prod_{\alpha=1}^N \prod_{j=1}^2 \left(x_{\alpha,i}^{(L)} - x_{\beta,j}^0 + \epsilon_1 \right) \prod_{j=1}^L \left(x_{\alpha,i}^{(L)} - x_{\beta,j}^0 - \epsilon_1 \right) \left(x_{\alpha,i}^{(L)} - x_{\beta,j}^{(L)} + \epsilon_1 \right) = 0. \end{aligned} \quad (\text{F.1})$$

- $N = 2, L = 1$

$$\omega_{1,1,1} = -\frac{\epsilon_1^3}{2a(2a + \epsilon_1)}, \quad \omega_{2,1,1} = -\frac{\epsilon_1^3}{2a(2a - \epsilon_1)}. \quad (\text{F.2a})$$

- $N = 2, L = 2$

$$\omega_{1,1,2} = -\frac{\epsilon_1^5 (-8a^5 + 4a^4\epsilon_1 + 22a^3\epsilon_1^2 + 3a^2\epsilon_1^3 + 3a\epsilon_1^4 + \epsilon_1^5)}{8a^3(2a - \epsilon_1)^2(a + \epsilon_1)(2a + \epsilon_1)^3}, \quad (\text{F.2b})$$

$$\omega_{1,2,2} = -\frac{\epsilon_1^5}{8a(a + \epsilon_1)(2a + \epsilon_1)^2}, \quad (\text{F.2c})$$

$$\omega_{2,1,2} = \frac{\epsilon_1^5 (8a^5 + 4a^4\epsilon_1 - 22a^3\epsilon_1^2 + 3a^2\epsilon_1^3 - 3a\epsilon_1^4 + \epsilon_1^5)}{8a^3(a - \epsilon_1)(2a - \epsilon_1)^3(2a + \epsilon_1)^2} \quad (\text{F.2d})$$

$$\omega_{2,2,2} = -\frac{\epsilon_1^5}{8a(a - \epsilon_1)(2a - \epsilon_1)^2}. \quad (\text{F.2e})$$

- $N = 2, L = 3$

In this case the solutions are for $\alpha = 1$

$$\begin{aligned} \omega_{1,1,3} = & -\frac{\epsilon_1^7 (896a^{12} - 1600a^{11}\epsilon_1 - 2656a^{10}\epsilon_1^2)}{96a^5(a - \epsilon_1)^2(\epsilon_1 - 2a)^4(a + \epsilon_1)(2a + \epsilon_1)^5(2a + 3\epsilon_1)} \\ & - \frac{\epsilon_1^7 (7344a^9\epsilon_1^3 + 3000a^8\epsilon_1^4 - 7788a^7\epsilon_1^5 + 7102a^6\epsilon_1^6)}{96a^5(a - \epsilon_1)^2(\epsilon_1 - 2a)^4(a + \epsilon_1)(2a + \epsilon_1)^5(2a + 3\epsilon_1)} \\ & - \frac{\epsilon_1^7 (-2999a^5\epsilon_1^7 - 677a^4\epsilon_1^8 - 144a^3\epsilon_1^9 - 96a^2\epsilon_1^{10} + 30a\epsilon_1^{11} + 18\epsilon_1^{12})}{96a^5(a - \epsilon_1)^2(\epsilon_1 - 2a)^4(a + \epsilon_1)(2a + \epsilon_1)^5(2a + 3\epsilon_1)}, \end{aligned} \quad (\text{F.2f})$$

$$\omega_{1,2,3} = -\frac{2\epsilon_1^7 (-2a^3 - a^2\epsilon_1 + 5a\epsilon_1^2 + \epsilon_1^3)}{3a(\epsilon_1 - 2a)^2(a + \epsilon_1)(2a + \epsilon_1)^4(2a + 3\epsilon_1)}, \quad (\text{F.2g})$$

$$\omega_{1,3,3} = -\frac{\epsilon_1^7}{96a(2a + 3\epsilon_1)(2a^2 + 3a\epsilon_1 + \epsilon_1^2)^2}, \quad (\text{F.2h})$$

and for $\alpha = 2$

$$\omega_{2,1,3} = - \frac{\epsilon_1^7 (896a^{12} + 1600a^{11}\epsilon_1 - 2656a^{10}\epsilon_1^2)}{96a^5 (2a - \epsilon_1)^5 (a + \epsilon_1)^2 (2a + \epsilon_1)^4 (2a^2 - 5a\epsilon_1 + 3\epsilon_1^2)} - \frac{\epsilon_1^7 (-7344a^9\epsilon_1^3 + 3000a^8\epsilon_1^4 + 7788a^7\epsilon_1^5 + 7102a^6\epsilon_1^6)}{96a^5 (2a - \epsilon_1)^5 (a + \epsilon_1)^2 (2a + \epsilon_1)^4 (2a^2 - 5a\epsilon_1 + 3\epsilon_1^2)} - \frac{\epsilon_1^7 (2999a^5\epsilon_1^7 - 677a^4\epsilon_1^8 + 144a^3\epsilon_1^9 - 96a^2\epsilon_1^{10} - 30a\epsilon_1^{11} + 18\epsilon_1^{12})}{96a^5 (2a - \epsilon_1)^5 (a + \epsilon_1)^2 (2a + \epsilon_1)^4 (2a^2 - 5a\epsilon_1 + 3\epsilon_1^2)}, \quad (\text{F.2i})$$

$$\omega_{2,2,3} = \frac{2\epsilon_1^7 (2a^3 - a^2\epsilon_1 - 5a\epsilon_1^2 + \epsilon_1^3)}{3a (\epsilon_1 - 2a)^4 (2a + \epsilon_1)^2 (2a^2 - 5a\epsilon_1 + 3\epsilon_1^2)}, \quad (\text{F.2j})$$

$$\omega_{2,3,3} = - \frac{\epsilon_1^7}{96a (2a - 3\epsilon_1) (2a^2 - 3a\epsilon_1 + \epsilon_1^2)^2}. \quad (\text{F.2k})$$

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