NNLO hard functions in massless QCD

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ABSTRACT: We derive the hard functions for all $2 \to 2$ processes in massless QCD up to next-to-next-to-leading order (NNLO) in the strong coupling constant. By employing the known one- and two-loop helicity amplitudes for these processes, we obtain analytic expressions for the ultraviolet and infrared finite, minimally subtracted hard functions, which are matrices in color space. These hard functions will be useful in carrying out higher-order resummations in processes such as dijet and highly energetic top-quark pair production by means of soft-collinear effective theory methods.

Contents

1 Introduction

Some of the most fundamental processes at hadron colliders such as the LHC are mediated at leading order (LO) in perturbation theory by $2 \to 2$ scattering processes of colored particles — two prime examples within the Standard Model are dijet and top-quark pair production. Fixed-order perturbation theory provides an obvious and conceptually straightforward framework in which to calculate higher-order QCD corrections to the total and differential cross sections for such processes, but it is often interesting or even necessary to supplement the fixed-order calculations with certain classes of logarithmic corrections to all orders in perturbation theory.

The factorization formulas underlying such resummations depend on the way in which the observable is sensitive to soft and collinear emissions, and are thus in general different for each particular differential cross section. Concrete examples are threshold resummation for inclusive jet production [1–3], highly boosted top-quark pair production [4, 5], and inclusive hadroproduction [6], as well as the resummations used for dijet event shapes in [7]. Relatively recently, the factorization of processes with two (or more) jets was also studied by means of Soft Collinear Effective Theory (SCET) methods [8, 9]. However, a common ingredient to all resummations for $2 \to 2$ processes are so-called "hard functions", which

account for virtual corrections to the underlying Born amplitudes. There is thus one such hard function for each possible $2 \to 2$ partonic process involving massless quarks and gluons, although all can be derived from those for $q\bar{q} \to Q\bar{Q}$, $qg \to qg$, and $gg \to gg$ scattering, where q and Q are distinct quarks. The hard functions are related to the interference of vector components of color-decomposed helicity amplitudes, and are matrices of varying dimension for the different partonic scattering processes — 2×2 for four-quark processes, 3×3 for $qg \to qg$ processes, and 9×9 for the four-gluon process. The next-to-leading order (NLO) hard functions for such QCD processes were extracted in [10]; these are a necessary ingredient for the resummation of any dijet hadronic process up to next-to-next-to-leading logarithmic (NNLL) accuracy.

The goal of the current work is to build on previous results by presenting the complete set of next-to-next-to-leading order (NNLO) hard functions. The main building blocks needed in this task are the NNLO UV-renormalized, color-decomposed helicity amplitudes for $2 \to 2$ massless QCD processes calculated in [11–15]. We turn these computations into results for the hard functions by performing an IR renormalization procedure on the color decomposed amplitudes, and then constructing the spin-averaged hard matrices by interfering all possible combinations of the fully renormalized color-decomposed amplitudes. Needless to say, the final results are quite lengthy, and are therefore given in electronic form with the arXiv submission of this work. To facilitate use by other groups, we also provide a `Mathematica` interface to the results.

The hard functions we calculate in the present work are a necessary ingredient for pushing any resummed calculation of a dijet observable at hadron colliders to next-to-next-to-next-to-leading logarithmic (NNNLL) order, as they provide the boundary terms for the renormalization-group evolution equations to that order. However, especially in cases where NNLO results are known, it is a frequent practice to include these boundary terms on top of NNLL resummations to achieve "NNLL$'$+NNLO" accuracy,[1] even in the absence of the three-loop non-cusp and four-loop cusp anomalous dimensions needed for a full NNNLL resummation. We thus anticipate that the results collected here will be useful for practitioners of higher-order resummation in the near and distant future.

The organization of the paper is as follows: we describe our calculational procedure in section 2, give results in section 3, and conclude in section 4.

2 Hard functions to NNLO: calculational procedure

The goal of this paper is to obtain the NNLO hard functions for all scattering processes with two incoming and two outgoing partons in massless QCD. These processes can be classified into groups containing four quarks, two quarks and two gluons, and four gluons.

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[1]Here we use the nomenclature of, e.g., [16].

Including all possible crossings, the four-quark processes are

$$q(p_1) + \bar{q}(p_2) \longrightarrow Q(p_3) + \bar{Q}(p_4) \,, \tag{2.1}$$

$$q(p_1) + \bar{Q}(p_2) \longrightarrow q(p_3) + \bar{Q}(p_4) \,, \tag{2.2}$$

$$q(p_1) + Q(p_2) \longrightarrow q(p_3) + Q(p_4) \,, \tag{2.3}$$

$$q(p_1) + Q(p_2) \longrightarrow Q(p_3) + q(p_4) \,, \tag{2.4}$$

$$q(p_1) + \bar{q}(p_2) \longrightarrow q(p_3) + \bar{q}(p_4) \,, \tag{2.5}$$

$$q(p_1) + q(p_2) \longrightarrow q(p_3) + q(p_4) \,, \tag{2.6}$$

where q and Q indicate quarks of different flavors. For scattering processes involving two quarks and two gluons, we focus on the following three possibilities:

$$g(p_1) + g(p_2) \longrightarrow q(p_3) + \bar{q}(p_4) \,, \tag{2.7}$$

$$q(p_1) + g(p_2) \longrightarrow q(p_3) + g(p_4) \,, \tag{2.8}$$

$$q(p_1) + g(p_2) \longrightarrow g(p_3) + q(p_4) \,. \tag{2.9}$$

There is also a $q\bar{q} \to gg$ process, but with our definitions its hard function is the same as (2.7) up to an overall factor $(N^2 - 1)^2/N^2$ (where $N = 3$ is the number of colors) which accounts for the color average over the incoming quarks rather than the incoming gluons. Furthermore, one needs to consider processes analogous to (2.8), (2.9) containing antiquarks, for which the hard functions are also identical to the corresponding processes involving quarks. We therefore omit these from the discussion. Finally, we consider the four-gluon scattering process

$$g(p_1) + g(p_2) \longrightarrow g(p_3) + g(p_4) \,. \tag{2.10}$$

Here and in the following, we associate to the particle carrying momentum p_i a helicity index $\lambda_i \in \{+, -\}$ and a color index a_i, which is understood to be in the fundamental representation of SU(3) if the particle is a quark, and in the adjoint representation if the particle is a gluon. For use later on, we introduce the invariants

$$s = (p_1 + p_2)^2 \,, \qquad t = (p_1 - p_3)^2 \,, \quad u = (p_1 - p_4)^2 \,, \quad \text{and} \quad r = -t/s \,. \tag{2.11}$$

Momentum conservation implies that the Mandelstam variables satisfy $s + t + u = 0$, which we will use to write our end results as functions of s, the 't Hooft scale μ, and the dimensionless ratio r.

A unique hard function is associated to each of the processes listed above. These can all be extracted using the two-loop helicity amplitudes calculated in [11–15]. To describe the calculational procedure that goes into doing this, we first introduce some aspects of the color-space formalism of [17], which allows us to treat the different cases with a uniform notation. In this formalism the UV-renormalized helicity amplitudes are considered vectors in color space, whose perturbative expansions we define as

$$|\mathcal{M}_h(\epsilon, r, s)\rangle = 4\pi\alpha_s \left[|\mathcal{M}_h^{(0)}\rangle + \frac{\alpha_s}{2\pi} |\mathcal{M}_h^{(1)}\rangle + \left(\frac{\alpha_s}{2\pi}\right)^2 |\mathcal{M}_h^{(2)}\rangle + \mathcal{O}(\alpha_s^3) \right] \tag{2.12}$$

$$= 4\pi\alpha_s \left[|\hat{\mathcal{M}}_h^{(0)}\rangle + \frac{\alpha_s}{4\pi} |\hat{\mathcal{M}}_h^{(1)}\rangle + \left(\frac{\alpha_s}{4\pi}\right)^2 |\hat{\mathcal{M}}_h^{(2)}\rangle + \mathcal{O}(\alpha_s^3) \right] \,. \tag{2.13}$$

Here $\epsilon = (4-d)/2$ is the dimensional regulator and the subscript $h = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ labels the helicity amplitudes. Moreover, we have suppressed the arguments of the expansion coefficients on the right-hand side.[2] Finally, in order to follow SCET conventions for the perturbative expansions of the hard functions in powers of $\alpha_s/4\pi$ below, we have defined a set of coefficients $|\hat{\mathcal{M}}_h^{(L)}\rangle \equiv 2^L |\mathcal{M}_h^{(L)}\rangle$.

The amplitudes can be further decomposed in a particular color-space basis as

$$|\mathcal{M}_h^{(L)}\rangle = \sum_{I=1}^n \mathcal{M}_{hI}^{(L)} |\mathcal{C}_I\rangle, \tag{2.14}$$

where $|\mathcal{C}_I\rangle$ are basis vectors. The basis includes two vectors in processes involving four quarks and three vectors in processes involving two quarks and two gluons. For the four-gluon scattering process we will use the redundant basis involving nine vectors employed in [15].

The helicity amplitudes contain IR poles in the dimensional regulator ϵ. We can subtract these poles in the $\overline{\text{MS}}$ scheme using the renormalization procedure described in [18, 19]. We thus define renormalized amplitudes according to

$$|\mathcal{M}_h^{\text{ren}}(r, s, \mu)\rangle = \lim_{\epsilon \to 0} \boldsymbol{Z}^{-1}(\epsilon, r, s, \mu) |\mathcal{M}_h(\epsilon, r, s)\rangle. \tag{2.15}$$

The exact form of the renormalization factor $\boldsymbol{Z}$ was determined up to two-loops by means of SCET methods in [18, 19]; for now we just note that it is the same for all helicity amplitudes. For reasons that will become apparent later on, we define the perturbative expansion of the renormalization factor as (with a slight abuse of notation inherited from [20])

$$\boldsymbol{Z}^{-1}(\epsilon, r, s, \mu) = \mathbf{1} + \frac{\alpha_s}{2\pi} \boldsymbol{Z}^{(1)}(\epsilon) + \left(\frac{\alpha_s}{2\pi}\right)^2 \boldsymbol{Z}^{(2)}(\epsilon) + \mathcal{O}(\alpha_s^3), \tag{2.16}$$

where $\boldsymbol{Z}^{(L)}(\epsilon) \equiv \boldsymbol{Z}^{(L)}(\epsilon, r, s, \mu)$. We can then evaluate (2.15) order-by-order in perturbation theory, defining renormalized amplitudes and expansion coefficients analogous to (2.12) and (2.14).

With this notation it is now a simple matter to write expressions for the hard functions to NNLO. We first define expansion coefficients through

$$\boldsymbol{H}(r, s, \mu) = \frac{16\pi^2 \alpha_s^2}{\mathcal{N}_R} \left[\boldsymbol{H}^{(0)} + \frac{\alpha_s}{4\pi} \boldsymbol{H}^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \boldsymbol{H}^{(2)} + \mathcal{O}(\alpha_s^3)\right], \tag{2.17}$$

where $\boldsymbol{H}^{(L)}(r, s, \mu) \equiv \boldsymbol{H}^{(L)}$. The factor $\mathcal{N}_R$ takes into account the channel-dependent factors related to averaging over initial state colors; it is defined as $\mathcal{N}_R = N^2$ for initial states with two quarks, $\mathcal{N}_R = N(N^2 - 1)$ for quark-gluon initial states, and $\mathcal{N}_R = (N^2 - 1)^2$ for initial states with two gluons. In terms of the color-decomposed, IR and UV renormalized helicity amplitudes perturbatively expanded as in (2.13), the hard function

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[2]These coefficients depend on on r, s, and the renormalization scale μ_r, although the all-order amplitude on the left-hand side is independent of μ_r.

matrix elements read

$$H_{IJ}^{(0)} = \frac{1}{4} \sum_h \left(\hat{\mathcal{M}}_{hI}^{(0)} \right)^* \hat{\mathcal{M}}_{hJ}^{(0)} ,$$

$$H_{IJ}^{(1)} = \frac{1}{4} \sum_h \left[\left(\hat{\mathcal{M}}_{hI}^{(0)} \right)^* \hat{\mathcal{M}}_{hJ}^{(1)} + \left(\hat{\mathcal{M}}_{hI}^{(1)} \right)^* \hat{\mathcal{M}}_{hJ}^{(0)} \right] ,$$

$$H_{IJ}^{(2)} = \frac{1}{4} \sum_h \left[\left(\hat{\mathcal{M}}_{hI}^{(1)} \right)^* \hat{\mathcal{M}}_{hJ}^{(1)} + \left(\hat{\mathcal{M}}_{hI}^{(0)} \right)^* \hat{\mathcal{M}}_{hJ}^{(2)} + \left(\hat{\mathcal{M}}_{hI}^{(2)} \right)^* \hat{\mathcal{M}}_{hJ}^{(0)} \right] . \qquad (2.18)$$

The factor $1/4$ in (2.18) is related to the average over the spin of the two incoming partons. The normalization of the expansion coefficients above (but not that of the hard function itself) then coincides with the $m_t \to 0$ limit of the corresponding results for the production of massive top pairs [21]. The hard functions are 2×2 matrices for scattering processes involving four quarks and 3×3 matrices for processes involving two quarks and two gluon. The hard functions for the process involving four gluons are 9×9 matrices with our choice of color basis. All of the matrices are Hermitian.

With our definitions, the hard function is related to the square of the renormalized amplitude as

$$\frac{1}{4\mathcal{N}_R} \sum_h \langle \mathcal{M}_h^{\mathrm{ren}}(r,s,\mu) | \mathcal{M}_h^{\mathrm{ren}}(r,s,\mu) \rangle = \mathrm{Tr} \left[\boldsymbol{H}(r,s,\mu) \, \tilde{\boldsymbol{s}}^{(0)} \right] . \qquad (2.19)$$

The matrix $\tilde{\boldsymbol{s}}^{(0)}$ is a "tree-level soft function", whose elements are defined as

$$\tilde{s}_{IJ}^{(0)} = \langle \mathcal{C}_I | \mathcal{C}_J \rangle . \qquad (2.20)$$

The color bases for the various processes we consider are specified in the next section, along with the basis-dependent results for the soft functions (2.20). Furthermore, the hard function is related to the L-loop corrections to the double differential partonic cross section in s and r by

$$\frac{d^2 \hat{\sigma}^{(L)}}{ds \, dr} = \frac{\alpha_s^2}{\mathcal{N}_R} \left(\frac{\alpha_s}{4\pi} \right)^L \frac{\pi}{s^2} \mathrm{Tr} \left[\boldsymbol{H}^{(L)}(r,s,\mu) \, \tilde{\boldsymbol{s}}^{(0)} \right] . \qquad (2.21)$$

With this conceptual framework in place, we now address the more practical issue of how to extract the hard functions from the NNLO helicity amplitudes calculated in [11–15]. In all cases, we have used the helicity amplitudes evaluated in the 't Hooft-Veltman (HV) scheme. The most straightforward way to use the information in those papers would be to construct the UV-renormalized, color decomposed helicity amplitudes, perform the IR renormalization procedure (2.15), and then evaluate the matrix elements (2.18). We have indeed used this straightforward (and tedious) method in obtaining our results.

A slightly more streamlined method, detailed recently in [20], uses that [11–15] do in fact quote results for IR finite amplitudes, but in different renormalization schemes based on the structure of IR poles written down in [22]. Therefore, constructing the $\overline{\mathrm{MS}}$ subtracted amplitudes (2.15) from those works is just a matter of switching between renormalization schemes. To understand how to perform this switch, we first consider the typical split of

UV-renormalized helicity amplitudes into pole and finite remainder terms used in [11–15]. As a concrete example, the one-loop helicity amplitudes calculated in [12] are written as (see (4.10) of that work)

$$|\mathcal{M}_h^{(1)}\rangle = \boldsymbol{I}^{(1)}(\epsilon)|\mathcal{M}_h^{(0)}\rangle + |\mathcal{M}_h^{(1),\text{fin}}\rangle\,, \tag{2.22}$$

while the two-loop helicity amplitudes are written as (see (4.11) of that work)

$$|\mathcal{M}_h^{(2)}\rangle = \boldsymbol{I}^{(2)}(\epsilon)|\mathcal{M}_h^{(0)}\rangle + \boldsymbol{I}^{(1)}(\epsilon)|\mathcal{M}_h^{(1)}\rangle + |\mathcal{M}_h^{(2),\text{fin}}\rangle\,. \tag{2.23}$$

The IR poles are contained in the color-space operator $\boldsymbol{I}$ [22], whose perturbative expansion is defined as

$$\boldsymbol{I}(\epsilon, r, s, \mu) = 1 + \frac{\alpha_s}{2\pi}\boldsymbol{I}^{(1)}(\epsilon) + \left(\frac{\alpha_s}{2\pi}\right)^2 \boldsymbol{I}^{(2)}(\epsilon) + \mathcal{O}\left(\alpha_s^3\right) \tag{2.24}$$

where $\boldsymbol{I}^{(L)}(\epsilon) \equiv \boldsymbol{I}^{(L)}(\epsilon, r, s, \mu)$. This object is analogous, but not identical, to the renormalization factor $\boldsymbol{Z}$ in (2.15). The difference is that $\boldsymbol{Z}$ contains only pole terms while $\boldsymbol{I}$ contains both pole terms and some finite terms. Moreover, the $1/\epsilon$ pole term and finite parts of the two-loop coefficient $\boldsymbol{I}^{(2)}$ were not fully specified in [22], but are instead parameterized in a function $\boldsymbol{H}_{R.S.}^{(2)}$ defined in equation (19) of that work. The authors of [11–15] provide explicit expressions for this function in their calculations, but in such a way that the finite parts of $\boldsymbol{I}^{(2)}$ are not the same in each paper. For these reasons, the finite remainders quoted in [11–15] differ from the $\overline{\text{MS}}$ renormalized amplitudes (2.15). Instead, they can be viewed as renormalized amplitudes in a scheme defined by equations (2.22) and (2.23) above, which differs from calculation to calculation according to the exact choice of $\boldsymbol{H}_{R.S.}^{(2)}$. To convert these finite-remainders to the $\overline{\text{MS}}$ scheme, one can insert (2.22), (2.23) into (2.15) to find

$$
\begin{aligned}
|\mathcal{M}_h^{(1),\text{ren}}\rangle =& |\mathcal{M}_h^{(1),\text{fin}}\rangle + \left(\boldsymbol{I}^{(1)}(\epsilon) + \boldsymbol{Z}^{(1)}(\epsilon)\right)|\mathcal{M}_h^{(0)}\rangle\,, \\
|\mathcal{M}_h^{(2),\text{ren}}\rangle =& |\mathcal{M}_h^{(2),\text{fin}}\rangle + \left(\boldsymbol{I}^{(1)}(\epsilon) + \boldsymbol{Z}^{(1)}(\epsilon)\right)|\mathcal{M}_h^{(1),\text{fin}}\rangle \\
&+ \left[\boldsymbol{I}^{(2)}(\epsilon) + \left(\boldsymbol{I}^{(1)}(\epsilon) + \boldsymbol{Z}^{(1)}(\epsilon)\right)\boldsymbol{I}^{(1)}(\epsilon) + \boldsymbol{Z}^{(2)}(\epsilon)\right]|\mathcal{M}_h^{(0)}\rangle\,. \tag{2.25}
\end{aligned}
$$

One can then recast this equation into an explicitly IR finite form. As explained above, the result depends on the choice of the single pole term in $\boldsymbol{I}^{(2)}$. In the case where this term is identical to that in $\boldsymbol{Z}$, i.e. adds no extra finite parts to $\boldsymbol{I}^{(2)}$, one can write the result in terms of ϵ independent operators $\boldsymbol{\mathcal{C}}_i$ ($i = 0, 1$) and various known anomalous dimensions as in [20]

$$
\begin{aligned}
\boldsymbol{I}^{(1)}(\epsilon) + \boldsymbol{Z}^{(1)}(\epsilon) =& \boldsymbol{\mathcal{C}}_0\,, \\
\boldsymbol{I}^{(2)}(\epsilon) + \left(\boldsymbol{I}^{(1)}(\epsilon) + \boldsymbol{Z}^{(1)}(\epsilon)\right)\boldsymbol{I}^{(1)}(\epsilon) + \boldsymbol{Z}^{(2)}(\epsilon) =& \frac{1}{2}\boldsymbol{\mathcal{C}}_0^2 + \frac{\gamma_1^{\text{cusp}}}{8}\left(\boldsymbol{\mathcal{C}}_0 + \frac{\pi^2}{128}\Gamma_0'\right) \\
&+ \frac{\beta_0}{2}\left(\boldsymbol{\mathcal{C}}_1 + \frac{\pi^2}{32}\boldsymbol{\Gamma}_0 + \frac{7\zeta_3}{96}\Gamma_0'\right) - \frac{1}{8}[\boldsymbol{\Gamma}_0, \boldsymbol{\mathcal{C}}_1]\,, \tag{2.26}
\end{aligned}
$$

where

$$\boldsymbol{\Gamma}_0 = \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2} \gamma_0^{\text{cusp}} \ln \frac{\mu^2}{-s_{ij}} + \sum_i \gamma_0^i \,,$$

$$\boldsymbol{\mathcal{C}}_0 = \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{16} \left[\gamma_0^{\text{cusp}} \ln^2 \frac{\mu^2}{-s_{ij}} - 4 \frac{\gamma_0^i}{C_i} \ln \frac{\mu^2}{-s_{ij}} \right] - \frac{\pi^2}{96} \Gamma'_0 \,,$$

$$\boldsymbol{\mathcal{C}}_1 = \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{48} \left[\gamma_0^{\text{cusp}} \ln^3 \frac{\mu^2}{-s_{ij}} - 6 \frac{\gamma_0^i}{C_i} \ln^2 \frac{\mu^2}{-s_{ij}} \right] - \frac{\pi^2}{48} \boldsymbol{\Gamma}_0 - \frac{\zeta_3}{24} \Gamma'_0 \,. \tag{2.27}$$

The sums run over the unordered tuples (i,j) of distinct partons, $\boldsymbol{T}_i$ is the color generator associated with the i-th parton in the scattering amplitude, and $s_{ij} \equiv 2\sigma_{ij} p_i \cdot p_j + i0$, where the sign factor $\sigma_{ij} = +1$ if the two parton momenta are both incoming or outgoing, and $\sigma_{ij} = -1$ otherwise. The anomalous dimensions appearing in (2.26), (2.27) are collected in many places, for example in appendix A of [20]; an explicit expression for the commutator $[\boldsymbol{\Gamma}_0, \boldsymbol{C}_1]$ in terms of $\boldsymbol{T}_i$, logarithms and anomalous dimensions can be found in the same appendix. We emphasize that in [11–15] the single pole term in $\boldsymbol{H}^{(2)}_{R.S.}$ is multiplied by factors of the form $(-\mu^2/s_{ij})^{2\epsilon}$, which yield additional finite contributions to $\boldsymbol{I}^{(2)}$ and thus to the right-hand side of the second line of (2.26) upon expansion in ϵ. We do not list these explicitly, as they differ between [11–15], but we do take them into account when extracting $\overline{\text{MS}}$-renormalized helicity amplitudes from those references using (2.25) and (2.26).

We have calculated the $\overline{\text{MS}}$-renormalized helicity amplitudes using both methods described above, and checked that they agree. We then used these amplitudes to construct the hard functions through (2.18). We end this section by describing several cross-checks we have performed on our channel and basis-dependent results, which we give in the next section. First, for the $q\bar{q} \to Q\bar{Q}$ and $gg \to q\bar{q}$ channels, we verified that the trace of functions given in (2.19) is consistent with the NNLO results derived in [23]. In turn, the results in the latter reference were tested against the squared NLO and NNLO matrix elements for the processes in (2.1), (2.7), which can be found in [24–26]. For the $gg \to gg$ channel, on the other hand, we have checked (2.19) against the UV-renormalized squared matrix elements given in [27, 28]. In order to carry out this last comparison, it was necessary to renormalized away the IR poles from the squared amplitudes in [27, 28], this was done by employing once more the IR renormalization method of [18, 19].

Second, the hard functions for the channels in (2.2), (2.3), (2.4) were assembled not only by starting from the appropriate amplitudes obtained from [12], but also by applying crossing symmetries to the amplitudes for the process in (2.1). Similarly, the hard functions for the processes in (2.8), (2.9) were also obtained a second time from the amplitudes for the process in (2.7) by applying crossing symmetries.

Third, we have checked that the hard functions satisfy the renormalization-group equations implied by (2.15). These take the form

$$\frac{d}{d\ln\mu} \boldsymbol{H}(r,s,\mu) = \boldsymbol{\Gamma}(r,s,\mu) \boldsymbol{H}(r,s,\mu) + \boldsymbol{H}(r,s,\mu) \boldsymbol{\Gamma}^\dagger(r,s,\mu) \,. \tag{2.28}$$

The general form of the anomalous dimension operator $\boldsymbol{\Gamma}$ in the case of massless scattering amplitudes is exactly known up to two loops [18, 19]. In those papers it is also conjectured

that $\boldsymbol{\Gamma}$ involves only color dipoles at all orders. The n-th order term in the expansion of $\boldsymbol{\Gamma}$ can be obtained by replacing $\gamma_0^a \to \gamma_n^a$ ($a \in \{\text{cusp}, i\}$) in (2.27). We give explicit, channel and basis dependent results for the anomalous dimension in the next section.

As a byproduct of our calculation we also evaluated the NLO hard functions, which were previously calculated in [10]. We find agreement with the results in that work, after we account for differences in notation.[3]

3 Hard functions to NNLO: results

We now present our results for the hard functions. We split the discussion into three subsections for the four-quark, two quark plus two gluon, and four-gluon scattering, which in turn are subdivided according to momentum crossings. In each case we define the channel-dependent color basis in which the hard function is calculated, and give analytic results for the tree-level hard and soft functions as well as the anomalous dimension $\boldsymbol{\Gamma}$ in that basis. The color bases are defined by projections of the basis vectors onto the arbitrary vector $|\{a\}\rangle \equiv |a_1, a_2, a_3, a_4\rangle$, where a_i represents the color index of the parton i (which can be either in the fundamental or adjoint representation, depending on the process). These vectors satisfy the relation

$$\langle\{a\}|\{b\}\rangle = \delta_{a_1 b_1} \delta_{a_2 b_2} \delta_{a_3 b_3} \delta_{a_4 b_4}. \tag{3.1}$$

The action of the color operators $\boldsymbol{T}_i$ on the vectors $|\{a\}\rangle$, which is needed to construct the basis-dependent expressions, is discussed in many references, see for example section 3.2 in [21].

The main results of this work are the NNLO hard functions obtained through the last line of (2.18). The analytic results for these functions would fill about 100 pages, were they printed out explicitly. As by now customary in such situations, we instead include the results in electronic format with the arXiv submission of this work. All of the hard functions are stored in `Mathematica` input files which can be loaded in the accompanying `Mathematica` notebook. In the latter file, a simple function allows the user to obtain numerical values for the hard functions for the processes listed in (2.1)–(2.10) once the desired perturbative order (LO, NLO, or NNLO) and the values of r, s, μ and N_l (the number of fermions) are specified. As a reference for other groups which might desire to carry out this calculation, we give explicit numerical results for the NLO and NNLO hard functions at a specific benchmark point in the subsections that follow. In all cases we use

$$N = 3, \quad N_l = 5, \quad r = \frac{\sqrt{5}-1}{2}, \quad \sqrt{s} = 2\mu. \tag{3.2}$$

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[3]Equation (55) in [10] defines a real hard function in the channels with two quarks and two gluons, while we define Hermitian hard functions with complex off diagonal terms. Furthermore, one needs to be careful when applying the crossing relations listed in table 2 of [10] in cases where fermions are switched between the initial and final state, as this necessitates extra minus signs. Finally, there is a minor typo in table 5 of [10], which is related to the four-gluon channel: the color factors listed in the next to the last column of that table apply to the helicities labeled 7 and 8, while the ones listed in the last column of the table apply to the helicities labeled by the numbers 9–16.

3.1 Four-quark scattering

Here we summarize results for the four-quark scattering processes in (2.1)–(2.6). In all cases we use singlet-octet type color bases defined below, for which the tree-level soft function is[4]

$$\tilde{\boldsymbol{s}}^{(0)} = \begin{pmatrix} N^2 & 0 \\ 0 & \frac{C_F N}{2} \end{pmatrix}, \tag{3.3}$$

where $C_F = (N^2 - 1)/(2N)$. Channel dependent results for the hard functions and anomalous dimensions are gathered in the subsections below.

3.1.1 $q(p_1) + \bar{q}(p_2) \to Q(p_3) + \bar{Q}(p_4)$

The color basis which we employ to describe the four-quark process in (2.1) is

$$\mathcal{C}_1 \equiv \langle \{a\} | \mathcal{C}_1 \rangle = \delta_{a_1 a_2} \delta_{a_3 a_4}, \qquad \mathcal{C}_2 \equiv \langle \{a\} | \mathcal{C}_2 \rangle = t^c_{a_2 a_1} t^c_{a_3 a_4}. \tag{3.4}$$

with this choice, the tree-level hard function is

$$\boldsymbol{H}^{(0)} = \left(1 - 2r + 2r^2\right) \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}. \tag{3.5}$$

We also provide numerical values of the NLO and NNLO hard functions at the benchmark point (3.2):

$$\boldsymbol{H}^{(1)} = \begin{pmatrix} 0 & -0.139210 - i\,0.192224 \\ -0.139210 + i\,0.192224 & 2.51146 \end{pmatrix}, \tag{3.6}$$

$$\boldsymbol{H}^{(2)} = \begin{pmatrix} 7.16744 & -22.1589 - i\,70.2433 \\ -22.1589 + i\,70.2433 & 380.359 \end{pmatrix}. \tag{3.7}$$

The anomalous dimension $\boldsymbol{\Gamma}$ in this basis is

$$\boldsymbol{\Gamma} = \left[2C_F \gamma_{\text{cusp}}\left(\alpha_s\right) \left(\ln \frac{s}{\mu^2} - i\pi \right) + 4\gamma^q\left(\alpha_s\right) \right] \mathbf{1}$$
$$+ N\gamma_{\text{cusp}}\left(\alpha_s\right) \left(\ln r + i\pi\right) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + 2\gamma_{\text{cusp}}\left(\alpha_s\right) \ln\left(\frac{r}{1-r}\right) \begin{pmatrix} 0 & \frac{C_F}{2N} \\ 1 & -\frac{1}{N} \end{pmatrix}. \tag{3.8}$$

The anomalous dimensions γ_{cusp} and γ^q have an expansion in powers of $a \equiv \alpha_s/(4\pi)$ of the form $\gamma = \sum_i a^i \gamma_i$. The coefficients of the expansions up to NNLO are collected in many sources, for example appendix A in [20].

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[4]The soft function in (3.3) differs by an overall factor N from the one some of us employed in previous work involving four quark partonic processes (see for example [21]). This is due to the definition of $\tilde{s}^{(0)}$ in (2.20), which differs slightly from the one employed in previous papers. Analogous considerations apply to the soft function for the channels involving two gluons, which can be found in (3.20).

3.1.2 $q(p_1) + \bar{Q}(p_2) \to q(p_3) + \bar{Q}(p_4)$

The color basis which we employ to describe the four-quark process in (2.2) is the same one introduced in (3.4). The tree-level hard matrix is

$$\boldsymbol{H}^{(0)} = \frac{2 - 2r + r^2}{N^2 r^2} \begin{pmatrix} \frac{(N^2-1)^2}{2N^2} & -\frac{N^2-1}{N} \\ -\frac{N^2-1}{N} & 2 \end{pmatrix}. \tag{3.9}$$

The NLO and NNLO matrices at the benchmark point (3.2) are

$$\boldsymbol{H}^{(1)} = \begin{pmatrix} 71.4641 & -47.5473 - i\,13.8859 \\ -47.5473 + i\,13.8859 & 31.1224 \end{pmatrix},$$

$$\boldsymbol{H}^{(2)} = \begin{pmatrix} 2231.12 & -1575.43 - i\,608.692 \\ -1575.43 + i\,608.692 & 1325.99 \end{pmatrix}, \tag{3.10}$$

and the anomalous dimension $\boldsymbol{\Gamma}$ in this basis is the same as (3.8).

3.1.3 $q(p_1) + Q(p_2) \to q(p_3) + Q(p_4)$

The color basis which we employ to describe the four-quark process in (2.3) is

$$\mathcal{C}_1 \equiv \langle \{a\} | \mathcal{C}_1 \rangle = \delta_{a_3 a_2} \delta_{a_4 a_1}, \qquad \mathcal{C}_2 \equiv \langle \{a\} | \mathcal{C}_2 \rangle = t^c_{a_3 a_2} t^c_{a_4 a_1}. \tag{3.11}$$

In this channel, the tree-level hard matrix is identical to the one in (3.9). The NLO and NNLO matrices at the benchmark point (3.2) are

$$\boldsymbol{H}^{(1)} = \begin{pmatrix} 58.9143 & -50.2365 + i\,13.8859 \\ -50.2365 - i\,13.8859 & 42.2155 \end{pmatrix},$$

$$\boldsymbol{H}^{(2)} = \begin{pmatrix} 2083.37 & -1350.52 + i\,209.622 \\ -1350.52 - i\,209.622 & 1071.72 \end{pmatrix}, \tag{3.12}$$

and the anomalous dimension $\boldsymbol{\Gamma}$ in this basis is

$$\boldsymbol{\Gamma} = \left[2 C_F \gamma_{\text{cusp}} \left(\alpha_s \right) \left(\ln \frac{s}{\mu^2} - i\pi \right) + 4 \gamma^q \left(\alpha_s \right) \right] \boldsymbol{1}$$
$$+ \gamma_{\text{cusp}} \left(\alpha_s \right) \left(\ln r + i\pi \right) \begin{pmatrix} 2 C_F & \frac{C_F}{N} \\ 2 & \frac{N^2-3}{N} \end{pmatrix} + \gamma_{\text{cusp}} \left(\alpha_s \right) \ln \left(\frac{r}{1-r} \right) \begin{pmatrix} -2 C_F & 0 \\ 0 & \frac{1}{N} \end{pmatrix}. \tag{3.13}$$

3.1.4 $q(p_1) + Q(p_2) \to Q(p_3) + q(p_4)$

The color basis employed for the process in (2.4) is the one we wrote in (3.11). The LO hard function in this channel is

$$\boldsymbol{H}^{(0)} = \left(1 - \frac{2}{1-r} + \frac{2}{(1-r)^2} \right) \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}. \tag{3.14}$$

The NLO and NNLO matrices at the benchmark point (3.2) are

$$\boldsymbol{H}^{(1)} = \begin{pmatrix} 0 & -14.8225 - i\,15.5573 \\ -14.8225 + i\,15.5573 & 1086.44 \end{pmatrix},$$

$$\boldsymbol{H}^{(2)} = \begin{pmatrix} 36.6860 & 538.071 + i\,2019.16 \\ 538.071 - i\,2019.16 & 43819.6 \end{pmatrix}. \tag{3.15}$$

The anomalous dimension $\boldsymbol{\Gamma}$ in this basis is the same as (3.13).

3.1.5 $\quad q(p_1) + \bar{q}(p_2) \to q(p_3) + \bar{q}(p_4)$

We consider here the scattering process in (2.5), where we employ the color basis in (3.4). The tree-level hard matrix is in this case given by

$$\boldsymbol{H}^{(0)} = \frac{(N^2 - 1)}{N^3 r^2} \begin{pmatrix} \frac{(N^2-1)}{2N}\left[2 - r(2-r)\right] & -r\left(N(r-1)^2 + r - 2\right) - 2 \\ -r\left(N(r-1)^2 + r - 2\right) - 2 & \frac{N\left\{2r\left[N^2 r(2(r-1)r+1)+2N(r-1)^2+r-2\right]+4\right\}}{N^2-1} \end{pmatrix}. \tag{3.16}$$

The NLO and NNLO matrices at the benchmark point (3.2) are

$$\boldsymbol{H}^{(1)} = \begin{pmatrix} 71.9130 & -54.4697 - i\,14.8627 \\ -54.4697 + i\,14.8627 & 45.2964 \end{pmatrix},$$

$$\boldsymbol{H}^{(2)} = \begin{pmatrix} 2214.83 & -1745.69 - i\,590.364 \\ -1745.69 + i\,590.364 & 2303.46 \end{pmatrix}, \tag{3.17}$$

and the anomalous dimension $\boldsymbol{\Gamma}$ in this basis is the same as (3.8).

3.1.6 $\quad q(p_1) + q(p_2) \to q(p_3) + q(p_4)$

The color basis which we employ in the process in (2.6) is the one in (3.11). The tree-level hard matrix is

$$\boldsymbol{H}^{(0)} = \frac{(N^2 - 1)}{N^3 r^2} \begin{pmatrix} \frac{(N^2-1)}{2N}\left[2 - r(2-r)\right] & \frac{2-r[N+(r-3)r+4]}{r-1} \\ \frac{2-r[N+(r-3)r+4]}{r-1} & \frac{N\left\{2r\left[N^2\left(r^3+r\right)+2N(r-1)+(r-2)((r-2)r+3)\right]+4\right\}}{(N^2-1)(1-r)^2} \end{pmatrix}. \tag{3.18}$$

The NLO and NNLO matrices at the benchmark point (3.2) are

$$\boldsymbol{H}^{(1)} = \begin{pmatrix} 50.4474 & 126.555 + i\,2.12346 \\ 126.555 - i\,2.12346 & 797.387 \end{pmatrix},$$

$$\boldsymbol{H}^{(2)} = \begin{pmatrix} 2106.09 & 5787.25 + i\,2270.79 \\ 5787.25 - i\,2270.79 & 35362.4 \end{pmatrix}, \tag{3.19}$$

and the anomalous dimension $\boldsymbol{\Gamma}$ in this basis is the same as (3.13).

3.2 Two-quark two-gluon scattering

We now turn to the two-quark two-gluon processes in (2.7)–(2.9). As with the four-quark processes, we choose color bases for the three processes such that the tree-level soft function is the same for each, and reads

$$\tilde{\boldsymbol{s}}^{(0)} = V \begin{pmatrix} N & 0 & 0 \\ 0 & \frac{N}{2} & 0 \\ 0 & 0 & \frac{N^2-4}{2N} \end{pmatrix}, \tag{3.20}$$

where we introduced the quantity $V \equiv N^2 - 1$. Channel dependent results for the hard functions and anomalous dimensions are gathered in the subsections below.

3.2.1 $g(p_1) + g(p_2) \to q(p_3) + \bar{q}(p_4)$

The quark-antiquark pair production in the gluon fusion channel, (2.7), is studied by employing the color basis

$$\mathcal{C}_1 \equiv \langle\{a\}|\mathcal{C}_1\rangle = \delta^{a_1 a_2}\delta_{a_3 a_4}, \qquad\qquad\qquad \mathcal{C}_2 \equiv \langle\{a\}|\mathcal{C}_2\rangle = if^{a_1 a_2 c}t^c_{a_3 a_4},$$

$$\mathcal{C}_3 \equiv \langle\{a\}|\mathcal{C}_3\rangle = d^{a_1 a_2 c}t^c_{a_3 a_4} \tag{3.21}$$

With this basis, the tree-level hard matrix becomes

$$\boldsymbol{H}^{(0)} = \begin{pmatrix} \frac{1}{N^2}\left(\frac{1}{2r} + \frac{1}{2(1-r)} - 1\right) & \frac{1}{N}\left(\frac{1}{2r} - \frac{1}{2(1-r)} + 2r - 1\right) & \frac{1}{N}\left(\frac{1}{2r} + \frac{1}{2(1-r)} - 1\right) \\ \frac{1}{N}\left(\frac{1}{2r} - \frac{1}{2(1-r)} + 2r - 1\right) & \frac{1}{2r} + \frac{1}{2(1-r)} + 4r - 4r^2 - 3 & \frac{1}{2r} - \frac{1}{2(1-r)} + 2r - 1 \\ \frac{1}{N}\left(\frac{1}{2r} + \frac{1}{2(1-r)} - 1\right) & \frac{1}{2r} - \frac{1}{2(1-r)} + 2r - 1 & \frac{1}{2r} + \frac{1}{2(1-r)} - 1 \end{pmatrix}. \tag{3.22}$$

The NLO and NNLO matrices at the benchmark point (3.2) are

$$\boldsymbol{H}^{(1)} = \begin{pmatrix} 2.73994 & -1.90813 + i\,0.304853 & 8.37732 - i\,2.53415 \\ -1.90813 - i\,0.304853 & 1.32846 & -5.83592 + i\,0.880136 \\ 8.37732 + i\,2.53415 & -5.83592 - i\,0.880136 & 25.6045 \end{pmatrix},$$

$$\boldsymbol{H}^{(2)} = \begin{pmatrix} 108.917 & -53.1488 - i\,144.053 & 253.200 + i\,363.288 \\ -53.1488 + i\,144.053 & 52.2319 & -106.493 + i\,180.767 \\ 253.200 - i\,363.288 & -106.493 - i\,180.767 & 597.058 \end{pmatrix}, \quad (3.23)$$

and the anomalous dimension $\boldsymbol{\Gamma}$ in this basis is

$$\boldsymbol{\Gamma} = \left[(N + C_F)\gamma_{\text{cusp}}(\alpha_s)\left(\ln\frac{s}{\mu^2} - i\pi\right) + 2\gamma^g(\alpha_s) + 2\gamma^q(\alpha_s) \right]\mathbf{1}$$

$$+ N\gamma_{\text{cusp}}(\alpha_s)(\ln r + i\pi)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \gamma_{\text{cusp}}(\alpha_s)\ln\left(\frac{r}{1-r}\right)\begin{pmatrix} 0 & 1 & 0 \\ 2 & -\frac{N}{2} & \frac{N^2-4}{2N} \\ 0 & \frac{N}{2} & -\frac{N}{2} \end{pmatrix}. \tag{3.24}$$

The anomalous dimension γ^g can be expanded in powers of $a = \alpha_s/(4\pi)$ as $\gamma^g = \sum_i a^i \gamma_i^g$. The coefficients of the expansion up to NNLO can be found for example in appendix A in [20].

3.2.2 $q(p_1) + g(p_2) \longrightarrow q(p_3) + g(p_4)$

The color basis that we adopt in order to describe the process in (2.8) is

$$\mathcal{C}_1 \equiv \langle\{a\}|\mathcal{C}_1\rangle = \delta^{a_4 a_2}\delta_{a_3 a_1}\,, \qquad\qquad \mathcal{C}_2 \equiv \langle\{a\}|\mathcal{C}_2\rangle = if^{a_4 a_2 c}t^c_{a_3 a_1}\,,$$

$$\mathcal{C}_3 \equiv \langle\{a\}|\mathcal{C}_3\rangle = d^{a_4 a_2 c}t^c_{a_3 a_1} \tag{3.25}$$

The tree-level hard function is

$$\boldsymbol{H}^{(0)} = \begin{pmatrix} \frac{1}{2N^2}\left(1 - r + \frac{1}{1-r}\right) & \frac{1}{N}\left(\frac{3}{2} - \frac{2}{r} - \frac{1}{2(1-r)} - \frac{r}{2}\right) & \frac{1}{2N}\left(1 - r + \frac{1}{1-r}\right) \\ \frac{1}{N}\left(\frac{3}{2} - \frac{2}{r} - \frac{1}{2(1-r)} - \frac{r}{2}\right) & \frac{5}{2} - \frac{r}{2} + \frac{4}{r^2} - \frac{4}{r} + \frac{1}{2(1-r)} & \frac{3}{2} - \frac{2}{r} - \frac{1}{2(1-r)} - \frac{r}{2} \\ \frac{1}{2N}\left(1 - r + \frac{1}{1-r}\right) & \frac{3}{2} - \frac{2}{r} - \frac{1}{2(1-r)} - \frac{r}{2} & \frac{1}{2}\left(1 - r + \frac{1}{1-r}\right) \end{pmatrix}, \tag{3.26}$$

while the NLO and NNLO matrices at the benchmark point (3.2) are

$$\boldsymbol{H}^{(1)} = \begin{pmatrix} -0.960743 & -17.4992 + i\,23.7762 & 2.64035 - i\,6.99918 \\ -17.4992 - i\,23.7762 & 278.010 & -89.5442 - i\,24.3766 \\ 2.64035 + i\,6.99918 & -89.5442 + i\,24.3766 & 24.4888 \end{pmatrix},$$

$$\boldsymbol{H}^{(2)} = \begin{pmatrix} 466.834 & -1614.79 + i\,37.4939 & 770.946 + i\,328.609 \\ -1614.79 - i\,37.4939 & 9379.39 & -3080.95 - i\,2025.95 \\ 770.946 - i\,328.609 & -3080.95 + i\,2025.95 & 1148.26 \end{pmatrix}, \tag{3.27}$$

The anomalous dimension $\boldsymbol{\Gamma}$ in this basis is

$$\boldsymbol{\Gamma} = \left[(N + C_F)\,\gamma_{\rm cusp}\,(\alpha_s)\left(\ln\frac{s}{\mu^2} - i\pi\right) + 2\gamma^g\,(\alpha_s) + 2\gamma^q\,(\alpha_s)\right]\boldsymbol{1}$$

$$+\,\gamma_{\rm cusp}\,(\alpha_s)\,(\ln r + i\pi)\begin{pmatrix} C_F + N & -1 & 0 \\ -2 & N - \frac{1}{2N} & \frac{4 - N^2}{2N} \\ 0 & -\frac{N}{2} & N - \frac{1}{2N} \end{pmatrix}$$

$$+\,\gamma_{\rm cusp}\,(\alpha_s)\ln\left(\frac{r}{1-r}\right)\begin{pmatrix} 0 & 1 & 0 \\ 2 & -\frac{N}{2} & \frac{N^2 - 4}{2N} \\ 0 & \frac{N}{2} & -\frac{N}{2} \end{pmatrix}. \tag{3.28}$$

3.2.3 $q(p_1) + g(p_2) \to g(p_3) + q(p_4)$

The color basis employed in order to describe the scattering process in (2.9) is

$$\mathcal{C}_1 \equiv \langle\{a\}|\mathcal{C}_1\rangle = \delta^{a_3 a_2}\delta_{a_4 a_1}\,, \qquad\qquad \mathcal{C}_2 \equiv \langle\{a\}|\mathcal{C}_2\rangle = if^{a_3 a_2 c}t^c_{a_4 a_1}\,,$$

$$\mathcal{C}_3 \equiv \langle\{a\}|\mathcal{C}_3\rangle = d^{a_2 a_3 c}t^c_{a_4 a_1} \tag{3.29}$$

The tree-level hard function is

$$\boldsymbol{H}^{(0)} = \begin{pmatrix} \frac{1}{2N^2}\left(\frac{1}{r} + r\right) & \frac{1}{N}\left(1 - \frac{1}{2r} - \frac{2}{1-r} + \frac{r}{2}\right) & \frac{1}{2N}\left(\frac{1}{r} + r\right) \\ \frac{1}{N}\left(1 - \frac{1}{2r} - \frac{2}{1-r} + \frac{r}{2}\right) & 2 - \frac{4}{1-r} + \frac{1}{2r} + \frac{4}{(1-r)^2} + \frac{r}{2} & 1 - \frac{1}{2r} - \frac{2}{1-r} + \frac{r}{2} \\ \frac{1}{2N}\left(\frac{1}{r} + r\right) & 1 - \frac{1}{2r} - \frac{2}{1-r} + \frac{r}{2} & \frac{1}{2}\left(\frac{1}{r} + r\right) \end{pmatrix}. \tag{3.30}$$

As expected, the matrix above can be obtained from (3.26) by replacing r by $1 - r$. The NLO and NNLO matrices at the benchmark point (3.2) are

$$\boldsymbol{H}^{(1)} = \begin{pmatrix} -2.78334 & -17.7615 + i\,29.2185 & -2.76521 - i\,6.64060 \\ -17.7615 - i\,29.2185 & 900.938 & -124.257 - i\,3.26543 \\ -2.76521 + i\,6.64060 & -124.257 + i\,3.26543 & 8.45881 \end{pmatrix},$$

$$\boldsymbol{H}^{(2)} = \begin{pmatrix} 568.421 & -3153.14 - i\,6008.72 & 786.213 + i\,1640.19 \\ -3153.14 + i\,6008.72 & 34356.2 & -4658.03 - i\,1019.52 \\ 786.213 - i\,1640.19 & -4658.03 + i\,1019.52 & 742.940 \end{pmatrix}. \quad (3.31)$$

and the anomalous dimension $\boldsymbol{\Gamma}$ in this basis can be obtained by replacing $r \to 1 - r$ in (3.28).

3.3 Four-gluon scattering

For the four-gluon scattering case (2.10) we adopt the color basis used in [15], namely

$$\begin{aligned} \mathcal{C}_1 &\equiv \langle \{a\} | \mathcal{C}_1 \rangle = 4\mathrm{Tr}\,[t^{a_1} t^{a_2} t^{a_3} t^{a_4}]\,, \\ \mathcal{C}_2 &\equiv \langle \{a\} | \mathcal{C}_2 \rangle = 4\mathrm{Tr}\,[t^{a_1} t^{a_2} t^{a_4} t^{a_3}]\,, \\ \mathcal{C}_3 &\equiv \langle \{a\} | \mathcal{C}_3 \rangle = 4\mathrm{Tr}\,[t^{a_1} t^{a_4} t^{a_2} t^{a_3}]\,, \\ \mathcal{C}_4 &\equiv \langle \{a\} | \mathcal{C}_4 \rangle = 4\mathrm{Tr}\,[t^{a_1} t^{a_3} t^{a_2} t^{a_4}]\,, \\ \mathcal{C}_5 &\equiv \langle \{a\} | \mathcal{C}_5 \rangle = 4\mathrm{Tr}\,[t^{a_1} t^{a_3} t^{a_4} t^{a_2}]\,, \\ \mathcal{C}_6 &\equiv \langle \{a\} | \mathcal{C}_6 \rangle = 4\mathrm{Tr}\,[t^{a_1} t^{a_4} t^{a_3} t^{a_2}]\,, \\ \mathcal{C}_7 &\equiv \langle \{a\} | \mathcal{C}_7 \rangle = 4\mathrm{Tr}\,[t^{a_1} t^{a_2}]\,\mathrm{Tr}\,[t^{a_3} t^{a_4}]\,, \\ \mathcal{C}_8 &\equiv \langle \{a\} | \mathcal{C}_8 \rangle = 4\mathrm{Tr}\,[t^{a_1} t^{a_3}]\,\mathrm{Tr}\,[t^{a_2} t^{a_4}]\,, \\ \mathcal{C}_9 &\equiv \langle \{a\} | \mathcal{C}_9 \rangle = 4\mathrm{Tr}\,[t^{a_1} t^{a_4}]\,\mathrm{Tr}\,[t^{a_2} t^{a_3}]\,. \end{aligned} \quad (3.32)$$

The color basis in (3.32) is over-complete. The factor of 4 in the r.h.s. of (3.32) arises from the fact that the authors of [15] define their color basis by employing color matrices normalized as $\mathrm{Tr}[T^a T^b] = \delta_{ab}$, while we re-express their basis in terms of color matrices with the standard normalization $\mathrm{Tr}[t^a t^b] = \delta_{ab}/2$.

The tree-level hard function for the process in (2.10) is

$$\boldsymbol{H}^{(0)} = \left(\begin{array}{ccccc|c} a & b & c & c & b & a \\ b & d & e & e & d & b \\ c & e & f & f & e & c \\ c & e & f & f & e & c & \boldsymbol{0}_{3\times 6} \\ b & d & e & e & d & b \\ a & b & c & c & b & a \\ \hline & & \boldsymbol{0}_{6\times 3} & & & \boldsymbol{0}_{3\times 3} \end{array} \right), \quad (3.33)$$

where the elements $a, \cdots, f$ are

$$a = \frac{1}{r^2} - \frac{2}{r} - 2r + r^2 + 3 \,,$$
$$b = \frac{1}{r} + \frac{1}{1-r} + r - r^2 - 2 \,,$$
$$c = \frac{1}{r} - \frac{1}{r^2} - \frac{1}{1-r} + r - 1 \,,$$
$$d = \frac{1}{(1-r)^2} - \frac{2}{1-r} + r^2 + 2 \,,$$
$$e = -\frac{1}{r} - \frac{1}{(1-r)^2} + \frac{1}{1-r} - r \,,$$
$$f = 1 + \frac{1}{r^2} + \frac{1}{(1-r)^2} \,. \tag{3.34}$$

The NLO hard function for the four-gluon scattering process of (2.10) depends on nine independent functions and has the following structure:

$$\boldsymbol{H}^{(1)} = \begin{pmatrix} a_1 & b_1 & c_1 & c_1 & b_1 & a_1 & g_1 & g_1 & g_1 \\ b_1^* & d_1 & e_1 & e_1 & d_1 & b_1^* & h_1 & h_1 & h_1 \\ c_1^* & e_1^* & f_1 & f_1 & e_1^* & c_1^* & i_1 & i_1 & i_1 \\ c_1^* & e_1^* & f_1 & f_1 & e_1^* & c_1^* & i_1 & i_1 & i_1 \\ b_1^* & d_1 & e_1 & e_1 & d_1 & b_1^* & h_1 & h_1 & h_1 \\ a_1 & b_1 & c_1 & c_1 & b_1 & a_1 & g_1 & g_1 & g_1 \\ g_1^* & h_1^* & i_1^* & i_1^* & h_1^* & g_1^* & 0 & 0 & 0 \\ g_1^* & h_1^* & i_1^* & i_1^* & h_1^* & g_1^* & 0 & 0 & 0 \\ g_1^* & h_1^* & i_1^* & i_1^* & h_1^* & g_1^* & 0 & 0 & 0 \end{pmatrix} \,, \tag{3.35}$$

where the non-zero elements at the benchmark point (3.2) are

$$
\begin{array}{lll}
a_1 = 68.8613 \,, & b_1 = 111.212 + i\,18.1565 \,, & c_1 = -158.807 - i\,55.4626 \,, \\
d_1 = 179.607 \,, & e_1 = -256.410 - i\,42.2061 \,, & f_1 = 359.541 \,, \\
g_1 = 24.0246 - i\,22.3654 \,, & h_1 = 38.8726 - i\,36.1879 \,, & i_1 = -62.8973 + i\,58.5533 \,.
\end{array}
\tag{3.36}
$$

The NNLO matrix has the structure

$$\boldsymbol{H}^{(2)} = \begin{pmatrix} a_2 & b_2 & c_2 & c_2 & b_2 & a_2 & g_2 & j_2 & m_2 \\ b_2^* & d_2 & e_2 & e_2 & d_2 & b_2^* & h_2 & k_2 & n_2 \\ c_2^* & e_2^* & f_2 & f_2 & e_2^* & c_2^* & i_2 & l_2 & o_2 \\ c_2^* & e_2^* & f_2 & f_2 & e_2^* & c_2^* & i_2 & l_2 & o_2 \\ b_2^* & d_2 & e_2 & e_2 & d_2 & b_2^* & h_2 & k_2 & n_2 \\ a_2 & b_2 & c_2 & c_2 & b_2 & a_2 & g_2 & j_2 & m_2 \\ g_2^* & h_2^* & i_2^* & i_2^* & h_2^* & g_2^* & p_2 & p_2 & p_2 \\ j_2^* & k_2^* & l_2^* & l_2^* & k_2^* & j_2^* & p_2 & p_2 & p_2 \\ m_2^* & n_2^* & o_2^* & o_2^* & n_2^* & m_2^* & p_2 & p_2 & p_2 \end{pmatrix} \,, \tag{3.37}$$

and the value of the 16 independent elements at the benchmark point (3.2) is

$$
\begin{aligned}
a_2 &= 2106.67\,, & b_2 &= 3196.18 + i\,3422.95\,, & c_2 &= -4797.70 - i\,4902.02\,, \\
d_2 &= 5188.15\,, & e_2 &= -8129.75 + i\,692.435\,, & f_2 &= 13732.3\,, \\
g_2 &= 1930.67 - i\,5041.68\,, & h_2 &= 2747.40 - i\,8529.46\,, & i_2 &= -3470.79 + i\,13852.3\,, \\
j_2 &= -9.86728 + i\,3871.76\,, & k_2 &= -392.448 + i\,5892.80\,, & l_2 &= 1609.60 - i\,9483.43\,, \\
m_2 &= 148.769 - i\,202.044\,, & n_2 &= -135.769 - i\,698.760\,, & o_2 &= 1194.28 + i\,1181.94\,, \\
& & p_2 &= 1041.49\,.
\end{aligned}
\tag{3.38}
$$

The anomalous dimension $\mathbf{\Gamma}$ in this basis is

$$
\mathbf{\Gamma} = \left[2N\gamma_{\mathrm{cusp}}\left(\alpha_s\right)\left(\ln\frac{s}{\mu^2} - i\pi\right) + 4\gamma^g\left(\alpha_s\right) \right] \mathbf{1}
$$
$$
+ \gamma_{\mathrm{cusp}}\left(\alpha_s\right)\left(\ln r + i\pi\right)\mathbf{M}_1 + \gamma_{\mathrm{cusp}}\left(\alpha_s\right)\ln\left(\frac{r}{1-r}\right)\mathbf{M}_2\,,
\tag{3.39}
$$

where the matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ are

$$
\mathbf{M}_1 = \begin{pmatrix}
N & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & N & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 2N & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 2N & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & N & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & N & 0 & 0 & -1 \\
-1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 2N & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 2N
\end{pmatrix}\,,
$$
$$
\mathbf{M}_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & -N & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -N & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -N & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -N & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & -1 & -1 & -1 & 0 & 0 & -2N & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}\,.
\tag{3.40}
$$

Finally, the tree-level soft function is

$$\tilde{\boldsymbol{s}}^{(0)} = \frac{V}{N^2} \begin{pmatrix} C_1 & C_2 & C_2 & C_2 & C_2 & C_3 & NV & -N & NV \\ C_2 & C_1 & C_2 & C_2 & C_3 & C_2 & NV & NV & -N \\ C_2 & C_2 & C_1 & C_3 & C_2 & C_2 & -N & NV & NV \\ C_2 & C_2 & C_3 & C_1 & C_2 & C_2 & -N & NV & NV \\ C_2 & C_3 & C_2 & C_2 & C_1 & C_2 & NV & NV & -N \\ C_3 & C_2 & C_2 & C_2 & C_2 & C_1 & NV & -N & NV \\ NV & NV & -N & -N & NV & NV & N^2V & N^2 & N^2 \\ -N & NV & NV & NV & NV & -N & N^2 & N^2V & N^2 \\ NV & -N & NV & NV & -N & NV & N^2 & N^2 & N^2V \end{pmatrix}, \qquad (3.41)$$

with $C_1 = N^4 - 3N^2 + 3$, $C_2 = 3 - N^2$, and $C_3 = 3 + N^2$.

4   Conclusions

We have given results for the spin-averaged hard functions for all $2 \to 2$ scattering processes in massless QCD up to NNLO in the strong coupling constant. These hard functions are a necessary ingredient for resummations in processes mediated by $2 \to 2$ scatterings at Born level, typical examples being dijet and boosted top production.

We extracted our results from NNLO calculations of UV-renormalized helicity amplitudes presented in [11–15], using a calculational procedure explained in section 2. The main idea is to interpret the IR poles in the color-decomposed helicity amplitudes as the UV poles of effective-theory operators, and to subtract them in the $\overline{\text{MS}}$ scheme. The hard functions defined through this procedure depend on the basis used in the color decomposition, which we specified in section 3. In all cases we performed several non-trivial cross-checks on our (lengthy) results for the matrix valued, basis-dependent hard functions, which are 2×2 matrices for four-quark processes, 3×3 matrices for two-quark two-gluon processes, and 9×9 matrices for the four-gluon process. We have listed their explicit numerical values at a benchmark point in section 3, which will facilitate future cross-checks. Moreover, we have provided analytic results in `Mathematica` form with the electronic submission of this paper to ensure their easy accessibility. Our results will thus be useful for practitioners of higher-order resummations in the near and distant future.

Acknowledgments

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