Breaking conformal invariance — Large N Chern-Simons theory coupled to massive fundamental fermions

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ABSTRACT: We analyze the theory of massive fermions in the fundamental representation coupled to a U(N) Chern-Simons gauge theory at level K. It is done in the large N, large K limits where $\lambda = \frac{N}{K}$ is kept fixed. Following [1] we obtain the solution of a Schwinger-Dyson equation for the two point function, the exact expression for the fermion propagator and the partition function at finite temperature. We prove that in the large K limit there exists an infinite set of classically conserved high spin currents also when a mass is introduced, breaking the conformal invariance. In analogy to the seminal work of 't Hooft on two dimensional QCD, we write down a Bethe-Salpeter equation for the wave function of a "quark anti-quark" bound state. We show that unlike the two dimensional QCD case, the three dimensional Chern-Simons theory does not admit a confining spectrum.

Contents

1 Introduction

In recent years a major progress has been made in the understanding of large N three dimensional Chern-Simons theory coupled to matter in the fundamental representation [1]–[14]. Interesting exact results have been derived without the aid of supersymmetry. Among these achievements is the determination of the exact planar free energy of the theory at finite temperature on $\mathcal{R}^2$ as a function of the 't Hooft coupling $\lambda = \frac{N}{K}$, where K is the level of the Chern-Simons term. Another property of these theories is the fact that classically in the large N there is an infinite tower of high-spin conserved currents. It was shown in [1] that the divergence of these currents is equal to a double and a triple trace of currents that vanish in the large N limit. In [4] it was shown that in the large N limit the theory of N scalars coupled to U(N) CS theory at level K is equivalent to the Legendre transform of the theory of K fermions coupled to a U(K) CS theory at level N.

In [1] the fact that one can extract exact results is attributed to the discrete nature of the CS coupling constant, the large N limit, the light-cone gauge and the fact that for the massless case the theory is conformal invariant. The main question addressed in the this work is to what extent can one decipher the large N CS theory coupled to massive fundamental fermions. Thus our question is essentially whether two of the three ingredients of the CS coupling, large N and the light-cone gauge are enough to enable us to solve it exactly or is conformal symmetry necessary for that. Our answer is that there are interesting physical quantities that can be determined even without conformal invariance. Concretely we have addressed the following three questions: (i) The fermion propagator

and the thermal free energy. (ii) The hight spin currents and their classical conservations. (iii) The spectrum bound state mesons.

Following [1] we show that by solving a Schwinger-Dyson equation, the fermion propagator and the partition function at finite temperature can be determined exactly. We have generalized the result of [1] to the massive case while using a somewhat different technique. In [2] it was shown that the result of [1] is incomplete and that there is an additional contribution to the thermal free energy from winding modes. The full expression written down in that paper holds for fermions of any mass, with an appropriate modification of the parameters.

We prove that in the large N limit there exists an infinite set of classically conserved high spin currents. The conservation holds classically for high spin currents which are similar to the ones used in the massless case apart from the following replacement

$$(\overleftarrow{D_\sigma}\overrightarrow{D^\delta}) \to (\overleftarrow{D_\sigma}\overrightarrow{D^\delta}) - m^2 \tag{1.1}$$

The divergence of these currents is equal to double trace operators which vanish in the large N limit. This is the same structure as for the conformal invariant setup.

As for the spectrum of bound state mesons, we write down, in analogy to the seminal work of 't Hooft on two dimensional QCD, a Bethe-Salpeter equation for the wave function of a "quark anti-quark" bound state. We show that unlike the two dimensional QCD case, the three dimensional Chern-Simons theory does not admit a confining spectrum. In fact, no high mass bound states exist.

The paper is organized as follows: the next section describes the basic setup of a Chern-Simons theory in Euclidean three dimensions in the large N and large level K limits with fixed ratio, coupled to a fermion in the fundamental representation. Section 3 is devoted to the determination of the fermion propagator at zero temperature. In section 4 we determine the fermion propagator at finite temperature. This is in fact a straightforward generalization of the result found in [2] for the theory which at zero temperature is conformal invariant. In section 5 we discuss the free energy case. Section 6 is devoted to the high spin currents. In section 7 we write down a 't Hooft-like equation for the bound states of the theory at zero temperature, and transform it to a form closer to the two dimensional case. We then apply in subsection 7.1 the high mass approximation, finding a solution that is not consistent with the approximation. The conclusion is that there are no high mass bound states, where by high mass we mean much larger than the quark mass. Section 8 is devoted to analyzing higher spin currents. We show that the same structure that occurs for the conformal theory is also characterizing the massive theory. In the last section we summarize our results and present several open questions.

2 The setup

The $\mathcal{R}^3$ Euclidean action of the U(N) CS theory coupled to a massive fermion in the fundamental representation is

$$S = \frac{iK}{4\pi} \int d^3x Tr\left[AdA + \frac{2}{3}A^3\right] + \int d^3x \bar{\psi}(\gamma^\mu D_\mu + m_{\text{bare}})\psi \tag{2.1}$$

where $A = A^a T^a$, T^a is a fundamental generator normalized so that $Tr[(T^a)^2] = \frac{1}{2}$ and $D_\mu \psi = \partial_\mu \psi - i A^a_\mu T^a \psi$. Note that we set the coupling constant to one. Using light-cone coordinates x^+, x^-, x^3 and light-front gauge $A_- = 0$ the action in momentum space reads

$$S = \int \frac{d^3p}{(2\pi)^3} \left[-\frac{iK}{2\pi} Tr[A_3(-p)p_- A_+(p)] + \bar{\psi}(-p)(i\gamma^\mu p_\mu + m_{\text{bare}})\psi(p) \right]$$
$$-i \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \left[\bar{\psi}(-p)[\gamma^+ A_+(-q) + \gamma^3 A_3(-q)]\psi(p+q) \right] \qquad (2.2)$$

Here

$$x^\pm = \frac{1}{\sqrt{2}}[x^1 \pm x^2]$$
$$A^\pm = \frac{1}{\sqrt{2}}[A^1 \pm A^2] \qquad (2.3)$$

It follows from this action that the gauge field propagator takes the form

$$< A^a_\mu(p) A^b_\nu(-q) >= (2\pi)^3 \delta(p-q) \delta^{ab} G_{\mu\nu}(p) \qquad (2.4)$$

where the only non-trivial components of $G_{\mu\nu}(p)$ are

$$G_{+3}(p) = -G_{3+}(p) = \frac{4\pi i}{K} \frac{1}{p^+} \qquad (2.5)$$

This translates in configuration space to

$$< A_3(x) A_+(0) >= - < A_+(x) A_3(0) >= \frac{2}{K} \frac{\delta(x_3)}{x^+} \qquad (2.6)$$

3 The fermion propagator

The fermion propagator is given by

$$< \psi^m(q) \psi_n(-p) >= (2\pi)^3 \delta(q-p) \delta^m_n S(q) = (2\pi)^3 \delta(q-p) \delta^m_n \frac{1}{i\gamma^\mu q_\mu + m_{\text{bare}} + \Sigma(q)} \qquad (3.1)$$

and

$$\Sigma(q) = i\Sigma_\mu \gamma^\mu + \Sigma_I I - m_{\text{bare}} I \qquad (3.2)$$

The equation for Σ takes the form

$$\Sigma(p) = -i4\pi\lambda \int \frac{d^3q}{(2\pi)^3} \frac{\gamma^+ \Sigma_I + iI(q + \Sigma(q))_-}{(q_\mu + \Sigma_\mu(q))(q^\mu + \Sigma^\mu(q)) + \Sigma_I(q)^2} \frac{1}{(p-q)^+} \qquad (3.3)$$

where $\lambda = N_c/K$. This is depicted in figure 1.

Equating the coefficients of the various γ^μ matrices it is clear that Σ is independent of p_3 and

$$\Sigma_- = \Sigma_3 = 0 \qquad \Sigma_I = p_s f_0(\lambda, p_s, m_{\text{bare}}) \qquad \Sigma_+ = p_+ g_0(\lambda, p_s, m_{\text{bare}}) \qquad (3.4)$$
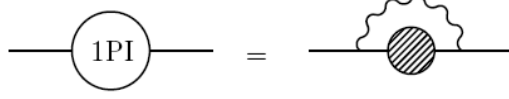
Figure 1. Fermion self energy.

with

$$p_s = \sqrt{p_1^2 + p_2^2} = \sqrt{2}|p^-| = \sqrt{2}|p^+|. \tag{3.5}$$

Substituting (3.4) into (3.3) we get the following integral equations for f_0 and g_0

$$g_0 = -\frac{4\pi\lambda}{p^-} \int \frac{d^3q}{(2\pi)^3} \frac{q_s f_0}{q_3^2 + q_s^2(1 + g_0 + f_0^2)} \frac{1}{(p^+ - q^+)}$$

$$f_0 p_s - m_{\text{bare}} = 4\pi\lambda \int \frac{d^3q}{(2\pi)^3} \frac{q^+}{q_3^2 + q_s^2(1 + g_0 + f_0^2)} \frac{1}{(p^+ - q^+)} \tag{3.6}$$

To solve for the functions $f_0(\lambda, p_s, m_{\text{bare}})$ and $f_0(\lambda, p_s, m_{\text{bare}})$ we now employ the identity

$$\frac{\partial}{\partial p^-} \frac{1}{p^+} = 2\pi\delta^2(p) \tag{3.7}$$

Applying this to eq. (3.6) we get

$$\frac{\partial}{\partial p^-}(p_s f_0) = \frac{\lambda p^+}{p_s\sqrt{1 + g_0 + f_0^2}}$$

$$\frac{\partial}{\partial p^-}(p^- g_0) = -\frac{\lambda f_0}{\sqrt{1 + g_0 + f_0^2}} \tag{3.8}$$

Multiplying the first with $f_0 p_s$, the second by p^+ and adding, we get zero for the right hand side, thus obtaining

$$\left(1 + \frac{1}{2}p_s\frac{\partial}{\partial p_s}\right)(g_0 + f_0^2) = 0 \tag{3.9}$$

Which gives the solution

$$g_0 + f_0^2 = \frac{m^2}{p_s^2} \tag{3.10}$$

The constant of integration comes out to be m^2, where m is the pole in the full propagator. Using this, the first equation in (3.8) can be integrated, to give

$$p_s f_0 = \lambda\sqrt{p_s^2 + m^2} + C \tag{3.11}$$

To determine C, we will evaluate the integral in (3.3) for $p_s = 0$. Actually, it is enough to evaluate the scalar part. So we have

$$p_s \to 0 : (p_s f_0 - m_{\text{bare}}) \to -4\pi\lambda \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2 + m^2} \tag{3.12}$$

The integral is equal to

$$-\lambda\frac{2}{\pi} \int_0^\infty dq + \lambda m \tag{3.13}$$

Setting the linearly divergent integral to zero, by dimensional regularization, we get that $C = m_{\text{bare}}$, and so

$$p_s f_0 = \lambda \sqrt{p_s^2 + m^2} + m_{\text{bare}} \qquad (3.14)$$

As for $p_+ g_0$, it follows from the integral equation (3.3) that it vanishes at $p_s = 0$. This means that $p_s f_0$ equals m for $p_s = 0$, entailing

$$m_{\text{bare}} = m(1 - \lambda) \qquad (3.15)$$

Thus the functions f_0 and g_0 and hence the non-trivial components of Σ are given by

$$p_s f_0(\lambda, p_s, m_{\text{bare}}) = m + \lambda[\sqrt{p_s^2 + m^2} - m]$$
$$p_s^2 g_0(\lambda, p_s, m_{\text{bare}}) = -\lambda \left[2m(1 - \lambda)[\sqrt{p_s^2 + m^2} - m] + \lambda p_s^2 \right] \qquad (3.16)$$

Note that we got this solution without solving for the integrals, just by their form and their values at $p = 0$.

It follows from (3.16) that Σ takes the form

$$\Sigma(p) = ip_+ \left[-\lambda^2 - 2\lambda \left(\frac{m_{\text{bare}}}{p_s} \right) \left(\sqrt{1 + \frac{m^2}{p_s^2}} - \frac{m}{p_s} \right) \right] \gamma^+ + \lambda p_s \sqrt{1 + \frac{m^2}{p_s^2}} I \qquad (3.17)$$

Thus the coefficient of the unit matrix in Σ, which was for the massless case λp_s, is still linear in λ but there is a re-scaling of $p_s \to p_s \sqrt{1 + \frac{m^2}{p_s^2}}$. The coefficient of γ^+, $ip_+ g_0$, which for the massless case was $-ip_+ \lambda^2$, is determined in the massive case from the relation (3.10).

It is easy to check that for the massless limit these results go back to

$$f_0 = \lambda \qquad g_0 = -\lambda^2 \qquad (3.18)$$

To summarize the propagator of the massive fermion takes the form

$$S(q) = -\frac{iq_- \gamma^- + iq_+(1 - g_0)\gamma^+ + iq_3 \gamma^3 - f_0 q_s}{q^2 + m^2} \qquad (3.19)$$

4   The fermion propagator at finite temperature

The physical system considered in the previous section can be brought into an equilibrium with a thermal bath of temperature $T = \frac{1}{\beta}$. To analyze such a case one is required to compactify the Euclidean time direction and thus put the system on a $\mathcal{R}^2 \times S^1$ background manifold rather than on an $\mathcal{R}^3$ manifold. The fermion propagator for such a system at finite temperature was computed for the massless case in [1]. This result was later corrected in [2] where the contribution from windings around the Euclidean time direction were incorporated and in addition a mass term was added. The result derived in [2] takes the following form

$$y f_0(y) - \beta m_{\text{bare}} = -\lambda y \sqrt{1 + \frac{\mu_T^2}{y^2}} + \frac{1}{\pi i} \left[Li_2 \left(-e^{-y \sqrt{1 + \frac{\mu_T^2}{y^2}} + \pi i \lambda} \right) - c.c \right] \qquad (4.1)$$

where $y = \beta p_s$ and $Li_2[\ldots]$ is the dilogarithm function. The "thermal mass parameter" μ_T is determined by the equation

$$\mu_T(\lambda) = \left| \beta m_{\text{bare}} + \lambda \mu_T + \frac{1}{\pi i} \left[Li_2 \left(-e^{-\mu_T - \pi i \lambda} \right) - c.c \right] \right| \tag{4.2}$$

where the positive solution for μ_T was chosen. At finite temperature there is a non-trivial thermal mass parameter both for the underlying massive as well as the massless theories. In terms of the fermion propagator the difference between the massless and massive theories is the fact that for the former case one has to substitute $m_{\text{bare}} = 0$ in the equations above. Note also that when $T \to 0$, and with $\mu_T = \beta m_T$, we get that $m_T(1 - \lambda) \to m_{\text{bare}}$, as expected (use also (3.15)).

Once the $f(y)$ function is determined we are left over only with $g(y)$ which determines the coefficient of γ^+ in Σ. This function is determined in this case also from a relation similar to (3.10), namely

$$f_0^2(y) + g_0(y) = \frac{\mu_T^2(\lambda)}{y^2} \tag{4.3}$$

5 The free energy

Once the fermion self energy is determined we can also compute the free energy of the system. In [2] it was shown that the free energy of the thermal system is given by

$$\beta F = -NV_2 \int_{-1/2}^{1/2} du \int \frac{d^2q}{(2\pi)^2}$$
$$\sum_{n=-\infty}^{\infty} Tr \left[\log \left(i\tilde{q}_\mu \gamma^\mu + \Sigma_T + m_{\text{bare}} I \right) - \frac{1}{2} \Sigma_T \frac{1}{i\tilde{q}_\mu \gamma^\mu + \Sigma_T + m_{\text{bare}}} \right] \tag{5.1}$$

where we have replaced the trace with an integral over the uniform spread of eigenvalues and where $\tilde{q}_\mu = q_\mu - \frac{2\pi |\lambda| u}{\beta} \delta_{3,\mu}$.

The final result derived in [2] takes the form

$$\beta F = \frac{NV_2}{2\pi\beta^2} \left\{ \frac{\mu_T^3}{3} \left(1 - \frac{1}{|\lambda|} \right) \right.$$
$$\left. + \frac{\beta m_{\text{bare}} \mu_T^2}{2\lambda} - \frac{(\beta m_{\text{bare}})^3}{6\lambda} + \frac{1}{\pi i \lambda} \int_{\mu_T}^{\infty} dy y [Li_2 \left(-e^{-\mu_T + \pi i \lambda} \right) - c.c] \right\} \tag{5.2}$$

6 High spin currents

In [1] it was shown that the free massless theory admits an infinite set of high spin currents which are classically conserved. The conserved currents of dimension $s+1$ and spin s have the following structure

$$J_{\mu_1,\ldots,\mu_s}^{(s)} = \bar{\psi} \gamma_{\mu_1} \left[polynomial \ of \ order^{(s-1)} \ of (\overleftarrow{\partial} \ and \ \overrightarrow{\partial}) \right] \psi + Symmetrized \tag{6.1}$$

where there is a full symmetrization of all the vector indices. The polynomial is determined using a generating function [9]

$$\mathcal{O}(x, \epsilon) = \sum J^{(s)}_{\mu_1, \ldots, \mu_s} \epsilon^{\mu_1} \ldots \epsilon^{\mu_s} = \bar{\psi} \vec{\gamma} \cdot \vec{\epsilon} f(\overleftarrow{\partial}, \overrightarrow{\partial}, \vec{\epsilon}) \psi \tag{6.2}$$

where

$$f(\vec{u}, \vec{v}, \vec{\epsilon}) = \frac{\exp{(\vec{u} \cdot \vec{\epsilon} - \vec{v} \cdot \vec{\epsilon})} \sinh \sqrt{2\vec{u} \cdot \vec{v}\vec{\epsilon} \cdot \vec{\epsilon} - 4\vec{u} \cdot \vec{\epsilon}\vec{v} \cdot \vec{\epsilon}}}{\sqrt{2\vec{u} \cdot \vec{v}\vec{\epsilon} \cdot \vec{\epsilon} - 4\vec{u} \cdot \vec{\epsilon}\vec{v} \cdot \vec{\epsilon}}} \tag{6.3}$$

By Taylor expanding this function the four first conserved currents are found to be

$$\begin{aligned}
J_\mu &= \bar{\psi} \gamma_\mu \psi \\
J_{\mu_1 \mu_2} &= \bar{\psi} \gamma_{\mu_1} \left(\overrightarrow{\partial_{\mu_2}} - \overleftarrow{\partial_{\mu_2}} \right) \psi \\
J_{\mu_1 \mu_2 \mu_3} &= \frac{1}{6} \bar{\psi} \gamma_{\mu_1} \left(3\overleftarrow{\partial_{\mu_2}}\overleftarrow{\partial_{\mu_3}} - 10\overleftarrow{\partial_{\mu_2}}\overrightarrow{\partial_{\mu_3}} + 3\overrightarrow{\partial_{\mu_2}}\overrightarrow{\partial_{\mu_3}} + 2(\overleftarrow{\partial_\sigma}\overrightarrow{\partial^\sigma})\eta_{\mu_2 \mu_3} \right) \psi \\
J_{\mu_1 \mu_2 \mu_3 \mu_4} &= \frac{1}{6} \bar{\psi} \gamma_{\mu_1} \left(\overleftarrow{\partial_{\mu_2}}\overleftarrow{\partial_{\mu_3}}\overleftarrow{\partial_{\mu_4}} - 7\overleftarrow{\partial_{\mu_2}}\overleftarrow{\partial_{\mu_3}}\overrightarrow{\partial_{\mu_4}} + 7\overleftarrow{\partial_{\mu_2}}\overrightarrow{\partial_{\mu_3}}\overrightarrow{\partial_{\mu_4}} - \overrightarrow{\partial_{\mu_2}}\overrightarrow{\partial_{\mu_3}}\overrightarrow{\partial_{\mu_4}} \right. \\
&\qquad \left. + 2(\overleftarrow{\partial_\sigma}\overrightarrow{\partial^\sigma})\overleftarrow{\partial_{\mu_2}}\eta_{\mu_3 \mu_4} - 2(\overleftarrow{\partial_\sigma}\overrightarrow{\partial^\sigma})\overrightarrow{\partial_{\mu_2}}\eta_{\mu_3 \mu_4} \right) \psi
\end{aligned} \tag{6.4}$$

It is easy to check that the first two currents are conserved also for the massive theory. The higher spin currents are conserved provided we make the following replacement

$$(\overleftarrow{\partial_\sigma}\overrightarrow{\partial^\sigma}) \to (\overleftarrow{\partial_\sigma}\overrightarrow{\partial^\sigma}) - m^2 \tag{6.5}$$

To show this, we use the generating function, after we make this replacement, applying to it $(\overrightarrow{\partial} + \overleftarrow{\partial}) \cdot \frac{\partial}{\partial \vec{\epsilon}}$. The contribution of the factor $[\exp{\left(\overrightarrow{\partial} \cdot \vec{\epsilon} - \overleftarrow{\partial} \cdot \vec{\epsilon} \right)}]$ is zero also in the massive case, since

$$(\overrightarrow{\partial} + \overleftarrow{\partial}) \cdot (\overrightarrow{\partial} - \overleftarrow{\partial}) = \overrightarrow{\Box} - \overleftarrow{\Box} \tag{6.6}$$

which vanishes when taken between $\bar{\psi}$ and ψ. As for the other factors in the generating function, it is enough to show that the contribution of the argument vanishes, namely

$$2(\overrightarrow{\partial} \cdot \overleftarrow{\partial} - m^2)\vec{\epsilon} \cdot \vec{\epsilon} - 4\overrightarrow{\partial} \cdot \vec{\epsilon}\overleftarrow{\partial} \cdot \vec{\epsilon} = 0. \tag{6.7}$$

This is indeed the case, as

$$\begin{aligned}
&4(\overrightarrow{\partial} \cdot \overleftarrow{\partial} - m^2)\vec{\epsilon} \cdot (\overrightarrow{\partial} + \overleftarrow{\partial}) - 4\overrightarrow{\partial} \cdot (\overrightarrow{\partial} + \overleftarrow{\partial})\overleftarrow{\partial} \cdot \vec{\epsilon} - 4\overrightarrow{\partial} \cdot \vec{\epsilon}\overleftarrow{\partial} \cdot (\overrightarrow{\partial} + \overleftarrow{\partial}) \\
&= -4(\overleftarrow{\partial} \cdot \vec{\epsilon})(\overrightarrow{\Box} + m^2) - 4(\overrightarrow{\partial} \cdot \vec{\epsilon})(\overleftarrow{\Box} + m^2)
\end{aligned} \tag{6.8}$$

which also vanishes when taken between $\bar{\psi}$ and ψ.

It remains to check the part where the divergence is contracted with the γ matrix, which follows from

$$\bar{\psi}(\overrightarrow{\partial}_\mu + \overleftarrow{\partial}_\mu)\gamma_\mu \psi = 0 \tag{6.9}$$

also in the massive case.

For the interacting massless theory it was proven in [1] that there is a similar set of higher spin currents, gotten by replacing ordinary derivatives with covariant derivatives,

$$\overleftarrow{\partial} \to \overleftarrow{D} \qquad \overrightarrow{\partial} \to \overrightarrow{D} \tag{6.10}$$

The similarity is in the sense that although not conserved, their divergence does not have a single trace part.

The classical divergence of the corresponding currents was derived using the equation of motion. The one associated with the gauge fields does not change in form when a mass is introduced, and it remains as

$$(F_{\mu\nu})^i_j = \frac{\pi}{K}\epsilon_{\mu\nu\rho}\bar{\psi}^i\gamma^\rho\psi_j \tag{6.11}$$

However, the equation associated with the fermion does of course change, and it now reads

$$D^\mu\gamma_\mu\psi = im\psi \tag{6.12}$$

Now, when checking the equation for the divergence of the currents, we have to change orders of covariant derivatives. For this we use

$$\begin{aligned}
\overrightarrow{D}_\mu\overrightarrow{D}_\nu - \overrightarrow{D}_\nu\overrightarrow{D}_\mu &= -iF_{\mu\nu} \\
\overleftarrow{D}_\mu\overleftarrow{D}_\nu - \overleftarrow{D}_\nu\overleftarrow{D}_\mu &= iF_{\mu\nu} \\
\overrightarrow{D}_\mu\overleftarrow{D}_\nu - \overleftarrow{D}_\nu\overrightarrow{D}_\mu &= iF_{\mu\nu}
\end{aligned} \tag{6.13}$$

Combining with (6.11), we see that for the single trace terms, we can treat the covariant derivatives as commuting, since the extra terms are higher trace, with each one multiplied by an extra $\frac{1}{K}$ factor.

We now claim that also in the massive case, the divergence of the currents will have no single trace term, provided we make the replacement

$$(\overleftarrow{D}_\sigma\overrightarrow{D^\sigma}) \rightarrow (\overleftarrow{D}_\sigma\overrightarrow{D^\sigma}) - m^2 \tag{6.14}$$

The currents are now

$$\tilde{J}^{(s)}_{\mu_1,\dots,\mu_s} = \bar{\psi}\gamma_{\mu_1}\left[polynomial\ of\ order^{(s-1)}\ of(\overleftarrow{D}\ and\ \overrightarrow{D})\right]\psi + Symmetrized \tag{6.15}$$

In the following we will omit the "Symmetrized".

Take the divergence

$$\partial_{\mu_l}\tilde{J}^{(s)}_{\mu_1,\dots,\mu_s} = \bar{\psi}(\overleftarrow{\partial}_{\mu_l} + \overrightarrow{\partial}_{\mu_l})\gamma_{\mu_1}\left[polynomial\ of\ order^{(s-1)}\ of(\overleftarrow{D}\ and\ \overrightarrow{D})\right]\psi =$$
$$\bar{\psi}(\overleftarrow{D}_{\mu_l} + \overrightarrow{D}_{\mu_l})\gamma_{\mu_1}\left[polynomial\ of\ order^{(s-1)}\ of(\overleftarrow{D}\ and\ \overrightarrow{D})\right]\psi \tag{6.16}$$

Now, when μ_l is contracted with γ, we have

$$\bar{\psi}(\overleftarrow{D}_{\mu_l} + \overrightarrow{D}_{\mu_l})\gamma_{\mu_l}\left[polynomial\ of\ order^{(s-1)}\ of(\overleftarrow{D}\ and\ \overrightarrow{D})\right]\psi =$$
$$\bar{\psi}(im + \overrightarrow{D}_{\mu_l}\gamma_{\mu_l})\left[polynomial\ of\ order^{(s-1)}\ of(\overleftarrow{D}\ and\ \overrightarrow{D})\right]\psi =$$
$$\bar{\psi}\left[polynomial\ of\ order^{(s-1)}\ of(\overleftarrow{D}\ and\ \overrightarrow{D})\right](im + \overrightarrow{D}_{\mu_l}\gamma_{\mu_l})\psi +$$
$$Terms\ with\ double\ trace\ multiplied\ by\ \frac{1}{K} =$$
$$Only\ terms\ with\ double\ trace\ multiplied\ by\ \frac{1}{K} \tag{6.17}$$

For the rest of the proof, we use the generating function, with the replacement (6.10) to covariant derivatives, and also the m^2 shift (1.1).

As for the term

$$\exp\left(\overrightarrow{D} \cdot \vec{\epsilon} - \overleftarrow{D} \cdot \vec{\epsilon}\right) \tag{6.18}$$

it is sufficient to check

$$(\overrightarrow{D} + \overleftarrow{D}) \cdot (\overrightarrow{D} - \overleftarrow{D}) \tag{6.19}$$

which, for the single trace terms, is equal to

$$\overrightarrow{D} \cdot \overrightarrow{D} - \overleftarrow{D} \cdot \overleftarrow{D} \tag{6.20}$$

This is actually zero, by the use of

$$\overrightarrow{D} \cdot \overrightarrow{D} \psi = -\left(\frac{1}{2}\epsilon^{ijk}F_{ij}\gamma_k + m^2\right)\psi$$

$$\bar{\psi}\overleftarrow{D} \cdot \overleftarrow{D} = -\bar{\psi}\left(\frac{1}{2}\epsilon^{ijk}F_{ij}\gamma_k + m^2\right) \tag{6.21}$$

which follows from (6.12).

It remains to examine the contribution of the factor

$$2(\overrightarrow{D} \cdot \overleftarrow{D} - m^2)\vec{\epsilon} \cdot \vec{\epsilon} - 4\overrightarrow{D} \cdot \vec{\epsilon}\overleftarrow{D} \cdot \vec{\epsilon} \tag{6.22}$$

Its contribution to the divergence, for the single trace terms, is

$$4(\overrightarrow{D} \cdot \overleftarrow{D} - m^2)\vec{\epsilon} \cdot (\overrightarrow{D} + \overleftarrow{D}) - 4\overrightarrow{D} \cdot (\overrightarrow{D} + \overleftarrow{D})\overleftarrow{D} \cdot \vec{\epsilon} - 4\overrightarrow{D} \cdot \vec{\epsilon}\overleftarrow{D} \cdot (\overrightarrow{D} + \overleftarrow{D})$$

$$= -4(\overleftarrow{D} \cdot \vec{\epsilon})(\overrightarrow{D} \cdot \overrightarrow{D} + m^2) - 4(\overrightarrow{D} \cdot \vec{\epsilon})(\overleftarrow{D} \cdot \overleftarrow{D} + m^2) \tag{6.23}$$

It has no single trace terms, by (6.21). This completes our proof.

7 't Hooft like equation for the spectrum of bound states

In the conformal setup when the fermions are massless a natural question to address is the spectrum of dimensions of the primaries operators and their descendants. The primaries are the operator $\bar{\psi}\psi$ and the tower of symmetric traceless currents $J^{(s)}_{\mu_1,\ldots\mu_s}$ which are constructed from a fermion anti-fermion bilinear sandwiching derivatives and a gamma matrix. The analysis of the spectrum of dimensions was carried out in [1]. The analogous question for the massive theory is the mass spectrum of bound states. The latter can be built in the same way as in the conformal theory. Here we will discuss a special class of the mesonic bound states. Note also that since the theory is invariant under local U(N) symmetry, and not only SU(N), baryons bound states are not gauge invariant.[1] We address the question of the spectrum of masses only at zero temperature.

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[1]Note, however, that at large N the U(1) part is down by $\frac{1}{N}$.
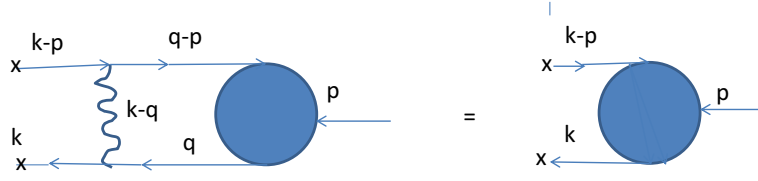
Figure 2. Schwinger-Dyson equation for the fermion anti-fermion bound state.

The spectrum of fermion anti-fermion bound states of two dimensional QCD in the planar limit was solved in the seminal work of 't Hooft [15]. Since like in that work, here we are also using (i) light-front coordinates, (ii) light-cone gauge and (iii) the planar limit, it calls for the use of a similar approach to the one used in [15] for our system. The key player is the bound state "wave-function" or the "blob" which is the Fourier transform of the matrix element of the operator $\psi(x)\bar{\psi}(0)$ between the vacuum and the meson states,

$$\phi(p,k) = \int \frac{d^3x}{(2\pi)^3} e^{ikx} < meson(p)|T\psi(x)\bar{\psi}(0)|0 > \tag{7.1}$$

To determine the "wave-function" one has to solve a Bethe-Salpeter which is depicted in figure 2.

For gauge invariance we need also a factor of $e^{\int_0^x A_\mu(y)dy^\mu}$ between ψ and $\bar{\psi}$. In the gauge $A_- = 0$, only the integrals over $A_+ dy^+$ and $A_3 dy^3$ will appear. As we will integrate over k_3 (see following (7.11)), this means $x^3 = 0$, leaving us with only an integral over $A_+ dy^+$. The discussion after eq. (7.13) indicates that actually $x^+ = 0$. Thus we discuss here directly the matrix element (7.1).

Note that the correlator in the definition of $\phi(p,k)$ includes the operators ψ and $\bar{\psi}$ at different points $\psi(x)\bar{\psi}(0)$. Expanding $\psi(x)$ around $x = 0$ we get bilinear operators of ψ and $\bar{\psi}$ with any arbitrary number of derivatives $(\partial_{\mu_1} \dots \partial_{\mu_n}\psi)\bar{\psi}|_{x=0}$. Thus the blob describes a bound state of a quark and an anti-quark with all orbital momenta. As for the internal spin, $\phi(p,k)$ is a 2x2 matrix, so it includes the spin zero and one components, and those are all the Dirac bilinear combinations in 3 dimensions.

To determine the masses of the bound states, we will have to go back to Minkowski space. But let us first continue in Euclidean space.

The integral equation reads

$$\phi(k,p) = -\frac{\lambda}{2\pi^2} S(k) \int \frac{d^3q}{(k^+ - q^+)} \left[\gamma^+\phi(q,p)\gamma^3 - \gamma^3\phi(q,p)\gamma^+\right] S(k-p) \tag{7.2}$$

Using (3.7) we can transform the integral equation into the following differential equation

$$\frac{\partial}{\partial k^-} \left[S^{-1}(k)\phi(k,p)S^{-1}(k-p)\right] = \frac{\lambda}{\pi} \left[\gamma^3\tilde{\phi}(\tilde{k},p)\gamma^+ - \gamma^+\tilde{\phi}(\tilde{k},p)\gamma^3\right] \tag{7.3}$$

where $\tilde{k}$ is the vector (k_1, k_2), and

$$\tilde{\phi}(\tilde{k},p) = \int dk^3\phi(p,k) \tag{7.4}$$

Next we expand the blob in terms of the coefficients of γ^μ and I similar to (3.2)

$$\phi = \phi_- \gamma^- + \phi_+ \gamma^+ + \phi_3 \gamma^3 + \phi_s I \tag{7.5}$$

which gives

$$\gamma^3 \phi \gamma^+ - \gamma^+ \phi \gamma^3 = 2[\phi_s \gamma^+ - \phi_- I] \tag{7.6}$$

and similarly for $\tilde{\phi}$.

Thus the right hand side of (7.2) involves ϕ_s and ϕ_- only. This results in two coupled integral equations for ϕ_s and ϕ_-, with ϕ_+ and ϕ_3 determined from ϕ_s and ϕ_-.

The integral equation now implies

$$\frac{\pi^2}{\lambda} \phi_s(k,p) = [a_-(b_s - b_3) + b_-(a_3 + a_s)] \int \frac{d^3q}{(k^+ - q^+)} \phi_s(q,p)$$

$$- [a_- b_+ + a_+ b_- + a_3 b_3 + a_s b_s] \int \frac{d^3q}{(k^+ - q^+)} \phi_-(q,p)$$

$$\frac{\pi^2}{\lambda} \phi_-(k,p) = [2a_- b_-] \int \frac{d^3q}{(k^+ - q^+)} \phi_s(q,p)$$

$$- [a_-(b_3 + b_s) + b_-(a_s - a_3)] \int \frac{d^3q}{(k^+ - q^+)} \phi_-(q,p) \tag{7.7}$$

where

$$a_s = -\frac{f_0 k_s}{k^2 + m^2} \qquad a_- = \frac{ik_-}{k^2 + m^2} \qquad a_+ = \frac{ik_+(1 - g_0)}{k^2 + m^2} \qquad a_3 = \frac{ik_3}{k^2 + m^2} \tag{7.8}$$

and (b_s, b_-, b_+, b_3) are given similarly with the same expressions but with $k - p$ replacing k.

Let us choose now the frame

$$p = (0, 0, p_3) \tag{7.9}$$

Then

$$a_- = \frac{ik_-}{[k^2 + m^2]} \qquad\qquad b_- = \frac{ik_-}{[k_s^2 + (k_3 - p_3)^2 + m^2]}$$

$$a_+ = \frac{ik_+[1 - g_0(k_s)]}{[k^2 + m^2]} \qquad\qquad b_+ = \frac{ik_+[1 - g_0(k_s)]}{[k_s^2 + (k_3 - p_3)^2 + m^2]}$$

$$a_s = -\frac{k_s f_0(k_s)}{k^2 + m^2} \qquad\qquad b_s = -\frac{k_s f_0(k_s)}{k_s^2 + (k_3 - p_3)^2 + m^2}$$

$$a_3 = \frac{ik_3}{k^2 + m^2} \qquad\qquad b_3 = \frac{i(k_3 - p_3)}{k_s^2 + (k_3 - p_3)^2 + m^2} \tag{7.10}$$

The integral equations become

$$\frac{\pi^2}{\lambda}\phi_s(k,p) = -\frac{k^+[p_3 + 2ik_s f_0(k_s)]}{[k^2 + m^2][k_s^2 + (k_3 - p_3)^2 + m^2]}\int \frac{d^3q}{(k^+ - q^+)}\phi_s(q,p)$$

$$+\frac{k_s^2 + k_3(k_3 - p_3) - m^2}{[k^2 + m^2][k_s^2 + (k_3 - p_3)^2 + m^2]}\int \frac{d^3q}{(k^+ - q^+)}\phi_-(q,p)$$

$$\frac{\pi^2}{\lambda}\phi_-(k,p) = -\frac{(k^+)^2}{[k^2 + m^2][k_s^2 + (k_3 - p_3)^2 + m^2]}\int \frac{d^3q}{(k^+ - q^+)}\phi_s(q,p)$$

$$+\frac{k^+[2ik_s f_0(k_s) - p_3]}{[k^2 + m^2][k_s^2 + (k_3 - p_3)^2 + m^2]}\int \frac{d^3q}{(k^+ - q^+)}\phi_-(q,p) \quad (7.11)$$

We can now perform an integration over k_3 on both sides. Note that the integrals on the right-hand-sides do not depend on k_3. Thus the integration can be done directly, yielding an equation for $\tilde{\phi}(\tilde{k},p) = \int dk_3 \phi(k,p)$ with $\tilde{k} = (k_+, k_-)$.

To find the bound states, we have to go to Minkowski space, by analytic continuation to $p_3 = iM_b$. The solutions of the integral equation should provide us with the masses of the bound states M_b.

To perform the integrals over k_3, we will make use of the following integrals

$$\int \frac{dk_3}{[k_s^2 + k_3^2 + m^2][k_s^2 + (k_3 - p_3)^2 + m^2]} = \frac{2\pi}{\sqrt{k_s^2 + m^2}}\frac{1}{[p_3^2 + 4(k_s^2 + m^2)]}$$

$$\int dk_3 \frac{k_3(k_3 - p_3)}{[k_s^2 + k_3^2 + m^2][k_s^2 + (k_3 - p_3)^2 + m^2]} = \frac{2\pi\sqrt{k_s^2 + m^2}}{[p_3^2 + 4(k_s^2 + m^2)]} \quad (7.12)$$

The integral equations, after the k_3 integration, become

$$\frac{\pi}{2\lambda k^+}\sqrt{k_s^2 + m^2}[p_3^2 + 4(k_s^2 + m^2)]\tilde{\phi}_s(\tilde{k},p_3) = -[2ik_s f_0(k_s) + p_3]\int \frac{d^2\tilde{q}}{(k^+ - q^+)}\tilde{\phi}_s(\tilde{q},p_3)$$

$$+4k^-\int \frac{d^2\tilde{q}}{(k^+ - q^+)}\tilde{\phi}_-(\tilde{q},p_3)$$

$$\frac{\pi}{2\lambda k^+}\sqrt{k_s^2 + m^2}[p_3^2 + 4(k_s^2 + m^2)]\tilde{\phi}_-(\tilde{k},p_3) = -k^+\int \frac{d^2\tilde{q}}{(k^+ - q^+)}\tilde{\phi}_s(\tilde{q},p_3) \quad (7.13)$$

$$+[2ik_s f_0(k_s) - p_3]\int \frac{d^2\tilde{q}}{(k^+ - q^+)}\tilde{\phi}_-(\tilde{q},p_3)$$

Note that, in the integrals on the right-hand-sides, what appears is $\int \frac{dq^+}{k^+ - q^+}\int dq^- \tilde{\phi}(\tilde{q},p)$. The integration over dq^- indicates that it is actually $x_- = 0$, and hence no need to have an integral of the form $\int_0^{x_-} A_+(y)dy_-$ in the Wilson line between ψ and $\bar{\psi}$.

7.1 Large bound state mass approximations

The analysis of the spectrum of bound sates of large N QCD in two space-time dimensions simplifies in the limit of high excitation, or large bound state mass. It was found out [15] that for high excitations the spectrum becomes a Regge-like spectrum, namely, $M_b^2 \sim n$ where n is the excitation number. Thus, we would like to first to investigate whether there

is a region of large bound sate masses and if yes what is the structure of the spectrum in that region.

Take, in the last equation, the limit of $|ip_3| \Rightarrow \infty$. Then,

$$\frac{\pi p_3}{2\lambda k_-}\sqrt{k_s^2 + m^2}\tilde{\phi}_s(\tilde{k}, p_3) = -\int \frac{d^2\tilde{q}}{(k^+ - q^+)}\tilde{\phi}_s(\tilde{q}, p_3) \tag{7.14}$$

and an identical one for $\tilde{\phi}_-(\tilde{q}, p_3)$. Define

$$\psi(\tilde{k}, p_3) = \sqrt{k_s^2 + m^2}\tilde{\phi}_s(\tilde{k}, p_3) \tag{7.15}$$

Then

$$\frac{\partial}{\partial k^-}\ln\psi(\tilde{k}, p_3) = -\frac{4\lambda}{p_3}\frac{k^+}{\sqrt{k_s^2 + m^2}} \tag{7.16}$$

from which

$$\ln\psi(\tilde{k}, p_3) = -\frac{4\lambda}{p_3}\sqrt{k_s^2 + m^2} + F \tag{7.17}$$

The additional function F, by rotational invariance in the (1,2) plane, depends on k_s only, and as it does not depend on k^-, it therefore is independent of $\tilde{k}$ altogether. It can thus depend on p_3 only. Finally, we get

$$\psi(\tilde{k}, M_b) = \exp[F(M_b)]\exp\left[i\frac{\lambda}{M_b}4\sqrt{k_s^2 + m^2}\right] \tag{7.18}$$

where the proportionality factor may depend on M_b. Note that factor $4\sqrt{k_s^2 + m^2}$ is the energy of the relative motion, in the rest frame of the bound state. Also, since the equations are homogeneous, we might as well put the proportionality factor to 1.

Substituting the solution into the integral on the right hand side of (7.14), we actually get

$$\exp[F(M_b)]\left\{\exp\left[i\frac{\lambda}{M_b}4\sqrt{k_s^2 + m^2}\right] - \exp\left[i\frac{\lambda}{M_b}4m\right]\right\}$$

which means we have no solution to the integral equation (7.14), except the trivial one $\tilde{\phi}_s(\tilde{k}, M_b) = 0$.

Turning now to $\tilde{\phi}_-(\tilde{k}, M_b)$, we have here the form k_- times a function of k_s^2. Looking for a solution for an equation like (7.14), we do find that

$$\sqrt{k_s^2 + m^2}\phi_-(\tilde{k}, M_b) = k_-G(M_b)\exp\left[i\frac{\lambda}{M_b}4\sqrt{k_s^2 + m^2}\right]$$

is a solution, provided that we take

$$lim_{k_s^2 \to \infty}\exp\left[i\frac{\lambda}{M_b}4\sqrt{k_s^2 + m^2}\right] = 0$$

in view of the oscillatory factor.

To fix the eigenvalues of M_b, we need some boundary conditions. Assuming, as an example, the requirement that the solution is real at $k_s = 0$, we get for the n-th state,

$$M_b^n = \frac{4\lambda m}{\pi n}$$

which means that the highest mass is actually the one for n=1, and it is not extending to ∞, contradicting our initial assumption.

It is thus clear that unlike spectrum of two dimensional QCD, in the large N large K limit of the CS theory coupled to fermions in the fundamental representation the spectrum does not admit a confining behavior. This should not surprise us since we are not considering for the gauge fields the YM theory but rather the CS one.

8 Summary and open questions

Two main questions have been addressed in this note:

(i) In the CS theory coupled to fermions in the fundamental representation, in the large N large level limit, is conformal invariance really necessary for:

1. Having high-spin currents which are classically conserved.

2. For the exact determination of the fermion propagator and the thermal free energy.

(ii) The structure of the spectrum of bound states of the massive theory.

As for the fermion propagator and the thermal free energy we have generalized the results of [1] to the case of massive fermions. While working on this project these properties were re-derived including a correction due to the presence of modes associated with the holonomy around the Euclidean circle in [2]. These results were determined for both the massless and massive cases. We have shown in this paper that the same structure of classically conserved high spin currents of the massless theory occurs also for the massive one. This was derived by a simple modification of the currents.

The question of the spectrum of bound states of the three dimensional CS theory is less tractable than the analogous problem in two dimensional QCD [15]. For once there are two coupled integral equations for two independent components of the "wave function" $\phi_s(k,p)$ and $\phi_-(k,p)$. The fact that the integrals are three dimensional rather than just two light-cone dimensions add further complication. We were not able to solve the equations in full generality. However, we were able to show that, unlike the two dimensional case where there is for large excitation number a " Regge spectrum" where $[M_b^{(n)}]^2 \sim n$, there are no highly excited bound states in our case. This is in accordance with the fact that the system described in this paper is not a confining one as can be seen from (6.11).

The research of the massive CS theory is still in an infant state and there are plenty of open quesitons to further investigate.

- For the conformal theories both with fermions and with scalars the high spin currents were analyzed also beyond the classical limit. In particular three point functions involving high spin currents have been computed [1, 3]. A natural question to ask is can one determine the analogous correlators for the massive theory.

- By no means the investigation of the coupled 't Hooft like equations has been exhausted. One can try solving them in other special cases similar to the $|ip_3| \to \infty$ that we have used, and one may try to use also numerical methods.

- Following the work of 't Hooft, a similar though non-homogeneous, integral equation was written down for the scattering amplitude of a quark anti-quark [16]. In a similar manner to the coupled equations that we derived for the "blob", one can write equations also for the scattering amplitude and analyze them.

- We have addressed the issue of mesonic bound-states at zero temperature. Provided that there are such bound-states at zero temperature an interesting question is to examine their fate once a temperature is turned on.

- In [3] a bosonization duality was proposed that relates a certain CS theory coupled to massive fermions in the fundamental representation to a CS theory couple to complex scalar in the fundamental representation. Thus one can investigate the issue of the spectrum of the bounds states of the massive theory also in the bosonic theory.

- An interesting limit to address is that of $m_{\text{bare}} \to \infty$ where the theory is supposed to flow to a "topological" pure CS theory.

Acknowledgments

References

[1] S. Giombi et al., *Chern-Simons theory with vector fermion matter*, *Eur. Phys. J.* **C 72** (2012) 2112 [`arXiv:1110.4386`] [INSPIRE].

[2] O. Aharony, S. Giombi, G. Gur-Ari, J. Maldacena and R. Yacoby, *The thermal free energy in large-N Chern-Simons-Matter theories*, *JHEP* **03** (2013) 121 [`arXiv:1211.4843`] [INSPIRE].

[3] O. Aharony, G. Gur-Ari and R. Yacoby, *Correlation functions of large-N Chern-Simons-Matter theories and bosonization in three dimensions*, *JHEP* **12** (2012) 028 [`arXiv:1207.4593`] [INSPIRE].

[4] O. Aharony, G. Gur-Ari and R. Yacoby, *D = 3 bosonic vector models coupled to Chern-Simons gauge theories*, *JHEP* **03** (2012) 037 [`arXiv:1110.4382`] [INSPIRE].

[5] G. Gur-Ari and R. Yacoby, *Correlators of large-N fermionic Chern-Simons vector models*, *JHEP* **02** (2013) 150 [`arXiv:1211.1866`] [INSPIRE].

[6] J. Maldacena and A. Zhiboedov, *Constraining conformal field theories with a slightly broken higher spin symmetry*, *Class. Quant. Grav.* **30** (2013) 104003 [`arXiv:1204.3882`] [INSPIRE].

[7] J. Maldacena and A. Zhiboedov, *Constraining conformal field theories with a higher spin symmetry*, *J. Phys.* **A 46** (2013) 214011 [`arXiv:1112.1016`] [INSPIRE].

[8] S. Giombi and X. Yin, *On higher spin gauge theory and the critical O(N) model*, *Phys. Rev.* **D 85** (2012) 086005 [arXiv:1105.4011] [ɪɴSPIRE].

[9] S. Giombi and X. Yin, *Higher spin gauge theory and holography: the three-point functions*, *JHEP* **09** (2010) 115 [arXiv:0912.3462] [ɪɴSPIRE].

[10] T. Takimi, *Duality and higher temperature phases of large-N Chern-Simons Matter theories on $S^2 \times S^1$*, arXiv:1304.3725 [ɪɴSPIRE].

[11] S. Jain et al., *Phases of large-N vector Chern-Simons theories on $S^2 \times S^1$*, *JHEP* **09** (2013) 009 [arXiv:1301.6169] [ɪɴSPIRE].

[12] S. Yokoyama, *Chern-Simons-fermion vector model with chemical potential*, *JHEP* **01** (2013) 052 [arXiv:1210.4109] [ɪɴSPIRE].

[13] S. Giombi and X. Yin, *The higher spin/vector model duality*, *J. Phys.* **A 46** (2013) 214003 [arXiv:1208.4036] [ɪɴSPIRE].

[14] S. Banerjee, S. Hellerman, J. Maltz and S.H. Shenker, *Light states in Chern-Simons theory coupled to fundamental matter*, *JHEP* **03** (2013) 097 [arXiv:1207.4195] [ɪɴSPIRE].

[15] G. 't Hooft, *A two-dimensional model for mesons*, *Nucl. Phys.* **B 75** (1974) 461 [ɪɴSPIRE].

[16] C.G. Callan Jr., N. Coote and D.J. Gross, *Two-dimensional Yang-Mills theory: a model of quark confinement*, *Phys. Rev.* **D 13** (1976) 1649 [ɪɴSPIRE].