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Direct derivation of "mirror" ABJ partition function

Masazumi Honda

High Energy Accelerator Research Organization (KEK),
Tsukuba, Ibaraki 305-0801, Japan
Yukawa Institute for Theoretical Physics, Kyoto University,
Kitashirakawa Oiwakecho, Sakyo-ku, Kyoto 606-8502, Japan

E-mail: mhonda@post.kek.jp

Abstract: We study the partition function of the three-dimensional $\mathcal{N} = 6$ $\mathrm{U}(N)_k \times \mathrm{U}(N+M)_{-k}$ superconformal Chern-Simons matter theory known as the ABJ theory. We prove that the ABJ partition function on S^3 is exactly the same as a formula recently proposed by Awata, Hirano and Shigemori. While this formula was previously obtained by an analytic continuation from the $L(2,1)$ lens space matrix model, we directly derive this by using a generalization of the Cauchy determinant identity. We also give an interpretation for the formula from brane picture.

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Contents

1 Introduction

Recently there has been significant progress in understanding low-energy effective theories of multiple M2-branes. The simplest case of such theories is the so-called ABJM theory [1], which is the 3d $\mathcal{N} = 6$ supersymmetric Chern-Simons matter theory (CSM) with the gauge group $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$. The authors in [1] discussed that this theory describes N M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$ in the low-energy limit. Furthermore it has been shown by the localization method [2–7] (see also [8–10]) that a class of supersymmetric observables in $\mathcal{N} = 2$ theory on S^3 have representations in terms of certain matrix integrals. Thanks to the localization technique, several works have extensively studied the partition function and BPS Wilson loops in the ABJM theory on S^3 [11–35]. Especially a breakthrough was caused by a seminal paper [20], which rewrites the ABJM partition function as an ideal Fermi gas system (see also [17, 19, 36, 37]). Based on this formalism, recent studies have revealed structures of the partition function [27–31] and half-BPS Wilson loop [33] including worldsheet and membrane (D2-brane) instanton effects [18, 38, 39].

In this paper we study the $\mathcal{N} = 6$ CSM with more general gauge group $\mathrm{U}(N)_k \times \mathrm{U}(N+M)_{-k}$ known as the ABJ theory [40]. This theory has been expected to arise when we have N M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$, together with M fractional M2-branes sitting at the singularity. The authors in [40] also argued that the ABJ theory has good approximations by the 11d SUGRA on $AdS_4 \times S^7/\mathbb{Z}_k$ with discrete torsion for $N^{1/5} \gg k$, and the type IIA SUGRA on $AdS_4 \times \mathbb{CP}^3$ with nontrivial B-field holonomy for $N^{1/5} \ll k \ll N$, respectively. Furthermore the recent works [41, 42] have also conjectured that the ABJ theory is dual to $\mathcal{N} = 6$ parity-violating Vasiliev theory on AdS_4 with a $\mathrm{U}(N)$ gauge symmetry when $M, k \gg 1$ with M/k and N kept fixed. Thus it is worth studying the ABJ theory in detail.

Here we study the partition function of the $\mathrm{U}(N)_k \times \mathrm{U}(N+M)_{-k}$ ABJ theory on S^3. By using the localization method, the partition function is given by [3–7]

$$Z_{\mathrm{ABJ}}^{(N,N+M)}(k) = \frac{i^{-\frac{1}{2}(N^2-(N+M)^2)\mathrm{sign}(k)}}{(N+M)!N!} \int_{-\infty}^{\infty} \frac{d^{N+M}\mu}{(2\pi)^{N+M}} \frac{d^N\nu}{(2\pi)^N} e^{-\frac{ik}{4\pi}\left(\sum_{j=1}^{N+M}\mu_j^2 - \sum_{a=1}^{N}\nu_a^2\right)}$$
$$\times \left[\frac{\prod_{1\leq j<l\leq N+M} 2\sinh\frac{\mu_j-\mu_l}{2} \prod_{1\leq a<b\leq N} 2\sinh\frac{\nu_a-\nu_b}{2}}{\prod_{j=1}^{N+M}\prod_{b=1}^{N} 2\cosh\frac{\mu_j-\nu_b}{2}} \right]^2 . \quad (1.1)$$

For $M \neq 0$, this representation is not suitable for the Fermi gas approach since the Cauchy identity is not helpful in contrast to the ABJM case [20]. Nevertheless, Awata, Hirano and Shigemori (AHS) recently proposed [43] that the ABJ partition function is equivalent to

$$Z^{(N,N+M)}_{\text{AHS}}(k) = \frac{i^{-\frac{1}{2}(N^2+(N+M)^2)\text{sign}(k)+N+\frac{M}{2}}(-1)^{\frac{N}{2}(N-1)}}{2^N k^{N+M/2} N!}(1-q)^{\frac{M(M-1)}{2}} G_2(M+1;q)$$
$$\int_{-i\infty-2\pi\eta}^{i\infty-2\pi\eta} \frac{d^N s}{(-2\pi i)^N} \prod_{a=1}^{N} \frac{1}{2\sin\frac{s_a}{2}} \frac{\left(q^{\frac{s_a}{2\pi}+1}\right)_M}{\left(-q^{\frac{s_a}{2\pi}+1}\right)_M} \prod_{1\le a<b\le N} \frac{\left(1-q^{\frac{s_b-s_a}{2\pi}}\right)^2}{\left(1+q^{\frac{s_b-s_a}{2\pi}}\right)^2}, \quad (1.2)$$

where

$$q = e^{-\frac{2\pi i}{k}}, \quad (a)_n = \prod_{m=0}^{n-1}(1-aq^m), \quad G_2(z+1;q) = (1-q)^{-\frac{z}{2}(z-1)} \prod_{m=1}^{\infty}\left[\left(\frac{1-q^{z+m}}{1-q^m}\right)^m (1-q^m)^z\right].$$

Here η specifies the integral contour. The authors in [43] have determined η for $N = 1$ as

$$\eta = \begin{cases} 0_+ & \text{for} \quad \frac{|k|}{2} - M \geq 0 \\ -\frac{|k|}{2} + M + 0_+ & \text{for} \quad \frac{|k|}{2} - M \leq 0 \end{cases}, \quad (1.3)$$

to be consistent with the Seiberg-like duality [40] but not for general N.

One expects that the AHS formula (1.2) gives a generalization of the "mirror" description of the ABJM partition function. One of the strongest evidence is that $Z_{\text{AHS}}|_{M=0,k=1}$ is the same as the partition function of the $\mathcal{N} = 4$ super QCD with one adjoint and fundamental hypermultiplets [36] related through 3d mirror symmetry [1, 44, 45]. There is also an interpretation for general k from the S-dual brane construction [46–48].

The AHS representation also has several advantages. First, this is suitable for the Fermi gas approach and Tracy-Widom theorem [49], which reduces the grand canonical analysis to Thermodynamic Bethe Ansatz-like equation. Second, it is easier to perform Monte Carlo simulation as in the ABJM case [21, 22] than the original formula (1.1). Finally, the AHS formula highly simplifies analysis in the Vasiliev limit. Despite of the advantages, the AHS proposal is still conjecture in the following senses:

- The derivation of Z_{AHS} started with an analytic continuation [11–15] from the partition function of the $L(2,1)$ lens space matrix model [50, 51]. The analytic continuation has not been rigorously justified in spite of much strong evidence [11–15, 18, 21, 22, 27–31, 52–54].

- While we can represent the partition function of the $L(2,1)$ matrix model in terms of a convergent series, its analytic continuation to the ABJ theory yields a non-convergent series. The AHS formula corresponds to its well-defined integral representation and reproduces the series order by order in the perturbative expansion by $2\pi i/k$ although there would be non-perturbative ambiguity generically.

In this paper we prove the AHS conjecture $Z_{\text{ABJ}} = Z_{\text{AHS}}$ and determine the integral contour for arbitrary parameters as discussed in section 2. It will turn out that the choice (1.3) of η is still correct even for general N. Section 3 is devoted to discussion.

2 Proof

In this section we prove $Z_{\mathrm{ABJ}} = Z_{\mathrm{AHS}}$. Let us start with the localization formula (1.1). For $M = 0$, the Cauchy determinant identity is quite useful to derive its "mirror" description [19, 20, 36], but not for $M \neq 0$. Here instead we use a generalization[1] of the Cauchy identity (corresponding to Lemma. 2 of [55]):

$$\frac{\prod_{j<l}(x_j - x_l)\prod_{a<b}(y_a - y_b)}{\prod_{j,b}(x_j - y_b)} = (-1)^{-\frac{N}{2}(N-1)} \begin{vmatrix} x_1^{M-1} & x_1^{M-2} & \cdots & 1 & \frac{1}{x_1-y_1} & \frac{1}{x_1-y_2} & \cdots & \frac{1}{x_1-y_N} \\ x_2^{M-1} & x_2^{M-2} & \cdots & 1 & \frac{1}{x_2-y_1} & \frac{1}{x_2-y_2} & \cdots & \frac{1}{x_2-y_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{N+M}^{M-1} & x_{N+M}^{M-2} & \cdots & 1 & \frac{1}{x_{N+M}-y_1} & \frac{1}{x_{N+M}-y_2} & \cdots & \frac{1}{x_{N+M}-y_N} \end{vmatrix}, \tag{2.1}$$

where j, l, a and b run $1 \leq j, l \leq N + M$, $1 \leq a, b \leq N$. One of easiest way to prove this identity is to use Boson-Fermion correspondence in 2d CFT. More concretely, the left-hand side corresponds to a correlation function on $\mathbb{P}^1$ of a bc system in "Boson" representation:

$$\langle M|b(x_1)\cdots b(x_{N+M})c(y_N)\cdots c(y_1)|0\rangle, \tag{2.2}$$

where we have identified as $b \leftrightarrow : e^{\varphi} :$ and $c \leftrightarrow : e^{-\varphi} :$ by using a free boson satisfying

$$\varphi(z) = \tilde{q} + a_0 \log z - \sum_{n\neq 0} \frac{a_n}{n} z^{-n}, \qquad [a_m, \tilde{q}] = \delta_{m,0}, \qquad [a_m, a_n] = m\delta_{m+n,0},$$
$$a_{n\geq 0}|0\rangle = 0, \qquad \langle 0|\tilde{q} = \langle 0|a_{n<0} = 0, \qquad \langle 0|0\rangle = 1, \qquad |M\rangle = e^{M\tilde{q}}|0\rangle, \tag{2.3}$$

while the right-hand side is the one in "Fermion" representation with identifications: $b \leftrightarrow \bar{\psi}$ and $c \leftrightarrow \psi$ in terms of charged fermions satisfying

$$\bar{\psi}(z) = \sum_{n\in\mathbb{Z}+1/2} \bar{\psi}_n z^{-n-\frac{1}{2}}, \qquad \psi(z) = \sum_{n\in\mathbb{Z}+1/2} \psi_n z^{-n-\frac{1}{2}}, \tag{2.4}$$
$$\{\bar{\psi}_m, \bar{\psi}_n\} = \{\psi_m, \psi_n\} = 0, \qquad \{\bar{\psi}_m, \psi_n\} = \delta_{m+n,0},$$
$$\bar{\psi}_{n>0}|0\rangle = \psi_{n>0}|0\rangle = 0, \qquad \langle 0|\bar{\psi}_{n<0} = \langle 0|\psi_{n<0} = 0, \qquad |M\rangle = \bar{\psi}_{-M+\frac{1}{2}} \cdots \bar{\psi}_{-\frac{1}{2}}|0\rangle.$$

If we take $x_j = e^{\mu_j}$ and $y_a = -e^{\nu_a}$ in the determinant identity (2.1), then we find

$$\frac{\prod_{j<l} 2\sinh\frac{\mu_j-\mu_l}{2} \prod_{a<b} 2\sinh\frac{\nu_a-\nu_b}{2}}{\prod_{j,b} 2\cosh\frac{\mu_l-\nu_b}{2}} = \prod_{j=1}^{N+M} e^{-M\frac{\mu_j}{2}} \prod_{a=1}^{N} e^{M\frac{\nu_a}{2}} \det\left(\frac{\theta_{N,l}}{2\cosh\frac{\mu_j-\nu_l}{2}} + e^{\left(N+M+\frac{1}{2}-j\right)\mu_j}\theta_{l,N+1}\right), \tag{2.5}$$

[1] We thank Sanefumi Moriyama to tell us about this identity and suggest that the identity would be useful for analyzing the ABJ matrix model (1.1). He showed us a different proof for the identity in April 2012. In the meanwhile, we remembered the identity when we read a (Japanese) textbook on conformal field theory written by Yasuhiko Yamada. Therefore we are also grateful to Yasuhiko Yamada.

where

$$\theta_{j,l} = \begin{cases} 1 & \text{for} \quad j \geq l \\ 0 & \text{for} \quad j < l \end{cases}. \tag{2.6}$$

Plugging this into (1.1), we find

$$Z_{\rm ABJ}^{(N,N+M)}(k) = \frac{\mathcal{N}_{\rm ABJ}}{N!}\sum_{\sigma}(-1)^{\sigma}\int_{-\infty}^{\infty}\frac{d^{N+M}\mu}{(2\pi)^{N+M}}\frac{d^N\nu}{(2\pi)^N}\prod_{j=1}^{N+M}e^{-\frac{ik}{4\pi}\mu_j^2-M\mu_j}\prod_{a=1}^{N}\frac{e^{\frac{ik}{4\pi}\nu_a^2+M\nu_a}}{2\cosh\frac{\mu_a-\nu_a}{2}}$$
$$\prod_{l=N+1}^{N+M}e^{\left(N+M+\frac{1}{2}-l\right)\mu_l}\prod_{j=1}^{N+M}\left(\frac{\theta_{N,j}}{2\cosh\frac{\mu_{\sigma(j)}-\nu_j}{2}}+e^{\left(N+M+\frac{1}{2}-j\right)\mu_{\sigma(j)}}\theta_{j,N+1}\right),$$

where

$$\mathcal{N}_{\rm ABJ} = i^{-\frac{1}{2}\left(N^2-(N+M)^2\right)\mathrm{sign}(k)}. \tag{2.7}$$

Making a Fourier transformation

$$\frac{1}{2\cosh\frac{p}{2}} = \frac{1}{\pi}\int_{-\infty}^{\infty}dx\frac{e^{\frac{i}{\pi}px}}{2\cosh x}, \tag{2.8}$$

and introducing auxiliary variables $y_{N+1},\cdots,y_{N+M}$ constrained by

$$y_l = \frac{\pi}{i}\left(N+M+\frac{1}{2}-l\right) \quad \text{with } l = N+1,\cdots,N+M, \tag{2.9}$$

the partition function becomes

$$Z_{\rm ABJ} = \frac{\mathcal{N}_{\rm ABJ}}{N!}\sum_{\sigma}(-1)^{\sigma}\int_{-\infty}^{\infty}\frac{d^Nx}{\pi^N}\frac{d^{N+M}y}{\pi^{N+M}}\frac{d^{N+M}\mu}{(2\pi)^{N+M}}\frac{d^N\nu}{(2\pi)^N}\prod_{j=1}^{N+M}e^{-\frac{ik}{4\pi}\mu_j^2-M\mu_j}$$
$$\times\prod_{a=1}^{N}\frac{e^{\frac{ik}{4\pi}\nu_a^2+M\nu_a+\frac{i}{\pi}x_a(\mu_a-\nu_a)+\frac{i}{\pi}(y_{\sigma(a)}\mu_a-y_a\nu_a)}}{2\cosh x_a\cdot 2\cosh y_a}$$
$$\times\prod_{l=N+1}^{N+M}\left[\pi\delta\left(y_l-\frac{\pi}{i}(N+M+1/2-l)\right)e^{\left(N+M+\frac{1}{2}-l\right)\mu_l+\frac{i}{\pi}y_{\sigma(l)}\mu_l}\right]. \tag{2.10}$$

Here we used $\sum_{j=1}^{N+M}y_j\mu_{\sigma(j)} = \sum_{j=1}^{N+M}y_{\sigma^{-1}(j)}\mu_j$ and redefined the permutation symbol as $\sigma^{-1}\to\sigma$. Performing the Fresnel integrals over μ_i and ν_a allows us to find

$$Z_{\rm ABJ} = \frac{e^{-\frac{iM\pi}{4}\mathrm{sign}(k)}\mathcal{N}_{\rm ABJ}}{N!|k|^{N+\frac{M}{2}}}\sum_{\sigma}(-1)^{\sigma}\int_{-\infty}^{\infty}\frac{d^Nx}{\pi^N}\frac{d^{N+M}y}{\pi^{N+M}}\prod_{a=1}^{N}\frac{e^{-\frac{2i}{k\pi}x_a\left(y_a-y_{\sigma(a)}\right)+\frac{2}{k}M\left(y_a-y_{\sigma(a)}\right)}}{2\cosh x_a\cdot 2\cosh y_a}$$
$$\prod_{l=N+1}^{N+M}\left[\pi e^{-\frac{i\pi}{k}\left(N+\frac{1}{2}-l\right)^2}\delta\left(y_l-\frac{\pi}{i}(N+M+1/2-l)\right)e^{\frac{i}{k\pi}y_l^2+\frac{2}{k}\left(N+\frac{1}{2}-l\right)y_{\sigma(l)}}\right]. \tag{2.11}$$

Note that the integration over x_a is convergent only for[2] $\left|\frac{2}{k\pi}\mathrm{Im}\left(y_a-y_{\sigma(a)}\right)\right| \leq 1$. Since $y_{\sigma(a)}$ would be $-i\pi(M-1/2)$ depending on the permutation, the integration is always

[2]The saturated case $\left|\frac{2}{k\pi}\mathrm{Im}\left(y_a-y_{\sigma(a)}\right)\right| = 1$ is understood as a limit from $\left|\frac{2}{k\pi}\mathrm{Im}\left(y_a-y_{\sigma(a)}\right)\right| < 1$.

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safe for $2M \leq |k|+1$. On the other hand, a part of the integrations is divergent for $2M > |k|+1$. However, these divergences must be apparent and cancel out after summing over the permutation since this case is equivalent to the safe case through the Seiberg-like duality [40] between the ABJ theories with gauge groups

$$\mathrm{U}(N)_k \times \mathrm{U}(N+M)_{-k} \quad \text{and} \quad \mathrm{U}(N+|k|-M)_k \times \mathrm{U}(N)_{-k}, \tag{2.12}$$

which has been proven for the S^3 partition functions [56, 57]. Although we could regularize the divergences, instead we will adopt another way as discussed in section 2.2.

2.1 For $2M \leq |k|+1$

We can continue our computation straightforwardly for this case. Integrating over x_a leads us to

$$Z_{\mathrm{ABJ}}^{(N,N+M)}(k) = \frac{e^{-\frac{iM\pi}{4}\mathrm{sign}(k)}\mathcal{N}_{\mathrm{ABJ}}}{N!|k|^{N+\frac{M}{2}}} \sum_\sigma (-1)^\sigma \int_{-\infty}^{\infty} \frac{d^{N+M}y}{\pi^{N+M}} \prod_{a=1}^N \frac{e^{\frac{2}{k}M\left(y_a - y_{\sigma(a)}\right)}}{2\cosh\frac{y_a - y_{\sigma(a)}}{k}\cdot 2\cosh y_a}$$
$$\prod_{l=N+1}^{N+M}\left[\pi e^{-\frac{i\pi}{k}\left(N+\frac12-l\right)^2}\delta\left(y_l - \frac{\pi}{i}(N+M+1/2-l)\right)e^{\frac{i}{k\pi}y_l^2+\frac{2}{k}\left(N+\frac12-l\right)y_{\sigma(l)}}\right].$$

Noting $\sum_a \left(y_a - y_{\sigma(a)}\right) = -\sum_{l=N+1}^{N+M}\left(y_l - y_{\sigma(l)}\right)$ and rescaling y_a as $y_a \to y_a/2$, one finds

$$Z_{\mathrm{ABJ}} = \frac{e^{-\frac{iM\pi}{4}\mathrm{sign}(k)}\mathcal{N}_{\mathrm{ABJ}}}{N!|k|^{N+\frac{M}{2}}} \int_{-\infty}^{\infty} \frac{d^{N+M}y}{(2\pi)^{N+M}} \prod_{a=1}^N \frac{1}{2\cosh\frac{y_a}{2}}$$
$$\times \prod_{l=N+1}^{N+M}\left[\pi e^{-\frac{i\pi}{k}\left(N+\frac12-l\right)^2}\delta\left(\frac{y_l}{2} - \frac{\pi}{i}(N+M+1/2-l)\right)e^{\frac{i}{4k\pi}y_l^2 - \frac{M}{k}y_l}\right]$$
$$\times \det\left(\frac{\theta_{N,l}}{2\cosh\frac{y_j-y_l}{2k}} + e^{\frac{1}{k}(N+M+1/2-l)y_j}\theta_{l,N+1}\right). \tag{2.13}$$

If we use the identity (2.5) again and integrate over $y_{N+1}, \cdots, y_{N+M}$, then we obtain

$$Z_{\mathrm{ABJ}}^{(N,N+M)}(k)$$
$$= \frac{i^{-\frac{\mathrm{sign}(k)}{2}(N^2+(N+M)^2)}(-1)^{\frac{N}{2}(N-1)+\frac{M}{2}(M-1)+NM} i^{N+\frac{M}{2}}}{N!2^N k^{N+\frac{M}{2}}} q^{\frac{M}{12}(M^2-1)} \prod_{1\leq l<m\leq M}\left[2i\sin\frac{\pi(l-m)}{k}\right]$$
$$\int_{-\infty}^{\infty}\frac{d^N y}{(2\pi)^N}\prod_{a<b}\tanh^2\frac{y_a-y_b}{2k}\prod_{a=1}^N\left[\frac{1}{2\cosh\frac{y_a}{2}}\prod_{l=0}^{M-1}\tanh\frac{y_a+2\pi i(l+1/2)}{2k}\right]. \tag{2.14}$$

Taking account of

$$\prod_{1\leq l<m\leq M}\left[2i\sin\frac{\pi(l-m)}{k}\right] = (-1)^{\frac{M}{2}(M-1)} q^{-\frac{M}{12}(M^2-1)} (1-q)^{\frac{M}{2}(M-1)} G_2(M+1;q),$$

$$\prod_{l=0}^{M-1}\tanh\frac{x_j+2\pi i(l+1/2)}{2k} = (-1)^M \frac{\left(q^{\frac{ix_a}{2\pi}+\frac12}\right)_M}{\left(-q^{\frac{ix_a}{2\pi}+\frac12}\right)_M},$$

and making a transformation $s_a = iy_a - \pi$, the partition function takes the form of

$$Z_{\rm ABJ} = \frac{i^{-\frac{1}{2}(N^2+(N+M)^2)\mathrm{sign}(k)+N+\frac{M}{2}}(-1)^{\frac{1}{2}N(N-1)}}{2^N k^{N+M/2} N!}(1-q)^{\frac{M(M-1)}{2}} G_2(M+1;q)$$
$$\int_{-i\infty-2\pi\eta}^{i\infty-2\pi\eta} \frac{d^N s}{(-2\pi i)^N} \prod_{a=1}^{N} \frac{1}{2\sin\frac{s_a}{2}} \frac{\left(q^{\frac{s_a}{2\pi}+1}\right)_M}{\left(-q^{\frac{s_a}{2\pi}+1}\right)_M} \prod_{1\le a<b\le N} \frac{\left(1-q^{\frac{s_b-s_a}{2\pi}}\right)^2}{\left(1+q^{\frac{s_b-s_a}{2\pi}}\right)^2}. \qquad (2.15)$$

Although η is naively $\eta = 1/2$, we can change η in a range $0 < \eta < 1$ by the Cauchy integration theorem. If we take $\eta = 0_+$ for $2M \le |k|$ and $\eta = \frac{1}{2} + 0_+$ for $2M = |k|+1$, then this is nothing but the AHS formula for $2M \le |k|$ and $2M = |k|+1$ (as a special case of $2M \ge |k|$), respectively. Note that the choice (1.3) of η is still correct for general N.

For a later convenience, we introduce the $\mathrm{U}(M)_k$ pure Chern-Simons partition function (without level shift) on S^3 as [50, 51, 58, 59]

$$\begin{aligned} Z_{\rm CS}^{(M)}(k) &= i^{-\mathrm{sign}(k)\frac{M^2}{2}} i^{\frac{M}{2}} k^{-\frac{M}{2}} q^{-\frac{M}{12}(M^2-1)}(1-q)^{\frac{M}{2}(M-1)} G_2(M+1;q) \\ &= |k|^{-\frac{M}{2}} \prod_{l=1}^{M-1}\left(2\sin\frac{\pi l}{|k|}\right)^{M-l}. \end{aligned} \qquad (2.16)$$

In terms of this, the ABJ partition function can be rewritten as

$$Z_{\rm ABJ} = \frac{i^{-\mathrm{sign}(k)N(N-1)}(-1)^{\frac{N}{2}(N-1)}}{N! 2^N |k|^N} q^{\frac{M}{12}(M^2-1)} Z_{\rm CS}^{(M)}(k) \qquad (2.17)$$
$$\int_{-i\infty-2\pi\eta}^{i\infty-2\pi\eta} \frac{d^N s}{(-2\pi i)^N} \prod_{a=1}^{N} \frac{1}{2\sin\frac{s_a}{2}} \prod_{l=1}^{M} \tan\frac{s+2l\pi}{2|k|} \prod_{1\le a<b\le N} \frac{\left(1-q^{\frac{s_b-s_a}{2\pi}}\right)^2}{\left(1+q^{\frac{s_b-s_a}{2\pi}}\right)^2}.$$

Remark

We can also express the ABJ partition function as

$$Z_{\rm ABJ}^{(N,N+M)}(k) = \frac{e^{-\frac{iM\pi}{4}\mathrm{sign}(k)} q^{\frac{M}{12}(M^2-1)} \mathcal{N}_{\rm ABJ}}{N! |k|^{N+\frac{M}{2}}} \int_{-\infty}^{\infty} \frac{d^{N+M} y}{(2\pi)^{N+M}} \frac{\prod_{j<l} 2\sinh\frac{y_j-y_l}{2k} \prod_{a<b} 2\sinh\frac{y_a-y_b}{2k}}{\prod_{j,b} 2\cosh\frac{y_j-y_b}{2k}}$$
$$\prod_{a=1}^{N} \frac{1}{2\cosh\frac{y_a}{2}} \prod_{l=N+1}^{N+M}\left[2\pi\delta\left(y_l - \frac{2\pi}{i}(N+M+1/2-l)\right)\right]. \qquad (2.18)$$

Each factor in this integrand has an interpretation from the brane picture. Recall that the type IIB brane construction for the $\mathrm{U}(N)_k \times \mathrm{U}(N+M)_{-k}$ ABJ theory consists of N circular D3-branes, $(1,-k)$5-brane, NS5-brane and M D3-branes suspended between the two 5-branes [40]. Taking S-transformation, the $(1,-k)$5-brane and NS5-brane become $(-k,1)$5-brane and D5-brane, respectively. First, note that the second factor

$$\prod_{a=1}^{N} \frac{1}{2\cosh\frac{y_a}{2}}$$

agrees with the contribution from a bi-fundamental hypermultiplet [3–7]. This multiplet comes from strings ending on the D5-brane and D3-branes. Next, the first factor

$$\frac{1}{N!|k|^{N+\frac{M}{2}}}\frac{\prod_{1\le j<l\le N+M}2\sinh\frac{y_j-y_l}{2k}\prod_{1\le a<b\le N}2\sinh\frac{y_a-y_b}{2k}}{\prod_{j=1}^{N+M}\prod_{b=1}^{N}2\cosh\frac{y_j-y_b}{2k}} \tag{2.19}$$

is a bit nontrivial. For $M=0$, the authors in [46] argued that this factor comes from a system of the $(1,-k)$5-brane and D3-branes (see also [47, 48]). Hence we can interpret (2.19) as natural generalization of this contribution. Finally, the last factor

$$\prod_{l=N+1}^{N+M}\delta\left(y_l-\frac{2\pi}{i}(N+M+1/2-l)\right)$$

reflects a fact that the M suspended D3-branes are locked into position by the two 5-branes.

2.2 For $2M\ge|k|-1$

The integration (2.11) is apparently divergent for this case. Instead of imposing some regularizations, we use the Seiberg-like duality (2.12) for the ABJ theory [40]. Note that this duality has been already proven for the S^3 partition functions because the duality comes [57] from the Giveon-Kutasov duality [60] proven in [56]. Since the dual ABJ partition function is given by (2.17) for $2M\ge|k|-1$, we can express $Z_{\rm ABJ}^{(N,N+M)}(k)$ in terms of $Z_{\rm ABJ}^{(N+|k|-M,N)}(-k)$ through the duality.

Let us show $Z_{\rm ABJ}=Z_{\rm AHS}$ for[3] $2M\ge|k|-1$. Via the Seiberg-like duality (2.12) as mathematical identity, the ABJ partition function is given by

$$\begin{aligned}Z_{\rm ABJ}^{(N,N+M)}(k)=&(-1)^{\frac{N+M}{2}(N+M-1)+\frac{N+|k|-M}{2}(N+|k|-M-1)}i^{-\frac{1}{2}\left(2N^2-(N+M)^2-(N+|k|-M)^2\right)\mathrm{sign}(k)}\\&e^{\frac{\pi i}{12}\left(k^2+6N|k|-6|k|+2\right)\mathrm{sign}(k)}Z_{\rm ABJ}^{(N,N+|k|-M)}(-k)\,.\end{aligned} \tag{2.20}$$

Here the phase factor has been determined[4] in [57]. Plugging (2.17) into this leads us to

$$\begin{aligned}Z_{\rm ABJ}^{(N,N+M)}(k)=&\frac{i^{-\mathrm{sign}(k)N(N-1)}(-1)^{\frac{N}{2}(N-1)}}{N!2^N|k|^N}q^{\frac{M}{12}(M^2-1)}Z_{\rm CS}^{(|k|-M)}(-k)\\&\int_{-i\infty-0_+}^{i\infty-0_+}\frac{d^Ns}{(-2\pi i)^N}\prod_{a=1}^{N}\frac{1}{2\sin\frac{s_a}{2}}\prod_{l=1}^{M}\tan\frac{s+2l\pi}{2|k|}\prod_{1\le a<b\le N}\frac{\left(1-q^{\frac{s_b-s_a}{2\pi}}\right)^2}{\left(1+q^{\frac{s_b-s_a}{2\pi}}\right)^2}\,.\end{aligned}$$

By using the level-rank duality[5] for the pure CS theory: $Z_{\rm CS}^{(|k|-M)}(-k)=Z_{\rm CS}^{(M)}(k)$ and eq. (E.2) in [43]:

$$\frac{1}{\sin\frac{s}{2}}\prod_{l=1}^{M}\tan\frac{s+2\pi l}{2|k|}=\frac{1}{\sin\frac{s}{2}}\prod_{l=1}^{M}\tan\frac{s+2\pi l}{2|k|}\Bigg|_{s\to s+2\pi\left(-\frac{|k|}{2}+M\right),M\to|k|-M}\,, \tag{2.21}$$

[3] For $2M=|k|$ and $2M=|k|+1$, we can also apply the argument in section 2.1. We set the condition $2M\ge|k|-1$ such that the dual ABJ partition function is given by (2.17).

[4] Note that our normalization for the partition function differs from the original paper [57] by $Z_{\rm ours}^{(N,N+M)}(k)=(-1)^{\frac{N}{2}(N-1)+\frac{N+M}{2}(N+M-1)}\mathcal{N}_{\rm ABJ}Z_{\rm KWY}^{(N,N+M)}(k)$.

[5] See e.g. appendix. B of [57] for a proof.

we obtain

$$Z^{(N,N+M)}_{\rm ABJ}(k) = \frac{i^{-\frac{1}{2}(N^2+(N+M)^2){\rm sign}(k)+N+\frac{M}{2}}(-1)^{\frac{1}{2}N(N-1)}}{2^N k^{N+M/2} N!}(1-q)^{\frac{M(M-1)}{2}} G_2(M+1;q)$$
$$\int_{-i\infty-2\pi\eta}^{i\infty-2\pi\eta} \frac{d^N s}{(-2\pi i)^N} \prod_{a=1}^{N} \frac{1}{2\sin\frac{s_a}{2}} \frac{\left(q^{\frac{s_a}{2\pi}+1}\right)_M}{\left(-q^{\frac{s_a}{2\pi}+1}\right)_M} \prod_{1\le a<b\le N} \frac{\left(1-q^{\frac{s_b-s_a}{2\pi}}\right)^2}{\left(1+q^{\frac{s_b-s_a}{2\pi}}\right)^2}, \quad (2.22)$$

where

$$\eta = -\frac{|k|}{2} + M + 0_+ . \quad (2.23)$$

For $2M \geq |k|$, this exactly agrees with $Z_{\rm AHS}$. For $2M = |k|-1$, we can show that making a transformation $s_a \to -s_a$ with a choice $\eta = 0_-$ and using the periodicity of the integrand: $s_a \sim s_a + 4k\pi i$ give the AHS formula. Thus we find again that the choice (1.3) of the integral contour is valid for general N.

Remark

As already discussed in [43], the ABJ partition function vanishes for $M > k$ since the pure CS partition function in the prefactors vanishes for this case. This manifests an expectation that the supersymmetries are spontaneously broken in this case [40, 61–63] (see also [64, 65]).

3 Discussion

In this paper we have proven that the ABJ partition function on S^3 is exactly the same as the formula (1.2) recently proposed by Awata, Hirano and Shigemori [43], which can be interpreted as the "mirror" description of the ABJ partition function. It has also turned out that the choice (1.3) of the integral contour, previously determined only for $N = 1$, is still correct for general N. Our proof heavily relied on the determinant identity (2.1) and the following illuminating structure:

$$Z_{\rm ABJ} \sim \int d^{N+M}\mu d^N\nu \left[\langle M|b(e^{\mu_1})\cdots b(e^{\mu_{N+M}})c(-e^{\nu_N})\cdots c(-e^{\nu_1})|0\rangle\right]^2 e^{-\frac{ik}{4\pi}\left(\sum_i \mu_i^2 - \sum_a \nu_a^2\right)}, \quad (3.1)$$

which might be reminiscent of the AGT relation [66]. This would imply that somehow the Boson-Fermion correspondence knows how to simplify the ABJ partition function on S^3. One can see similar structure also in general $\mathcal{N} = 3$ quiver CSM. It is interesting if we find any physical origin of this structure. A recent work [67] in three dimensions and topological string perspective [68] might provide valuable insights along this direction.

For $2M > |k| + 1$, we have to change the integral contour. We can also express the partition function for this case as

$$Z_{\rm ABJ}|_{2M>|k|+1} \sim \int_{-\infty}^{\infty} d^{N+M}y \frac{\prod_{j<l} 2\sinh\frac{y_j-y_l}{2k} \prod_{a<b} 2\sinh\frac{y_a-y_b}{2k}}{\prod_{j,b} 2\cosh\frac{y_j-y_b}{2k}} \prod_{a=1}^{N} \frac{1}{2\cosh\frac{y_a}{2}}$$
$$\prod_{l=N+1}^{N+M} \left[2\pi\delta\left(y_l - \frac{2\pi}{i}(N+M+1/2-l)\right)\right]\Bigg|_{y_a\to y_a+2\pi i\left(-\frac{|k|+1}{2}+M\right)} .$$

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The similar representation (2.18) was useful to find the brane interpretation for $2M \leq |k|+1$. Formally this shift is similar to anomalous R-charge, imaginary mass and voertex loop [3–7]. It would be intriguing to interpret this shift in the integrand from the brane picture.

As mentioned in section 1, the AHS representation (1.2) is suitable for the Fermi gas approach and Monte Carlo simulation, and easier to study the higher spin limit than the original representation (1.1). Therefore it is illuminating to apply these approaches to the ABJ partition function and investigate wide parameter region.

Finally we comment on a relation between the ABJ theory and $L(2,1)$ matrix model. The authors in [43] obtained (1.2) by an analytic continuation from the partition function of the $L(2,1)$ matrix model represented by a convergent series. Because its analytic continuation to the ABJ theory yields an ill-defined series, the AHS formula corresponds to its well-defined integral representation and correctly reproduces the formal series order by order in the perturbative expansion. Since here we have proven $Z_{\rm ABJ} = Z_{\rm AHS}$, our argument combined with the one in [43] might give key idea for rigorous proof of the analytic continuation.

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