An M-theoretic derivation of a 5d and 6d AGT correspondence, and relativistic and elliptized integrable systems

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ABSTRACT: We generalize our analysis in [arXiv:1301.1977], and show that a 5d and 6d AGT correspondence for $SU(N)$ — which essentially relates the relevant 5d and 6d Nekrasov instanton partition functions to the integrable representations of a q-deformed and elliptic affine $\mathcal{W}_N$-algebra — can be derived, purely physically, from the principle that the spacetime BPS spectra of *string-dual* M-theory compactifications ought to be equivalent. Via an appropriate defect, we also derive a "fully-ramified" version of the 5d and 6d AGT correspondence where integrable representations of a quantum and elliptic affine $SU(N)$-algebra at the critical level appear on the 2d side, and argue that the relevant "fully-ramified" 5d and 6d Nekrasov instanton partition functions are simultaneous eigenfunctions of commuting operators which define relativistic and elliptized integrable systems. As an offshoot, we also obtain various mathematically novel and interesting relations involving the double loop algebra of $SU(N)$, elliptic Macdonald operators, equivariant elliptic genus of instanton moduli space, and more.

Contents

1 Introduction and summary

In 2009, Alday-Gaiotto-Tachikawa verified in [1] that the Nekrasov instanton partition function of a 4d $\mathcal{N} = 2$ conformal SU(2) quiver theory is equivalent to a conformal block of a 2d CFT with $\mathcal{W}_2$-algebra symmetry that is Liouville theory. This celebrated 4d-2d correspondence, better known since as the AGT correspondence, was soon proposed to also hold for a 4d $\mathcal{N} = 2$ asymptotically-free SU(2) theory [2], and a 4d $\mathcal{N} = 2$ conformal SU(N) quiver theory where the corresponding 2d CFT is an A_{N-1} conformal Toda field theory which has $\mathcal{W}_N$-algebra symmetry [3, 4].

The basis for the AGT correspondence for SU(N) — as first pointed out in [5] — is a conjectured relation between the equivariant cohomology of the moduli space of SU(N)-instantons and the integrable representations of an affine $\mathcal{W}_N$-algebra. This conjectured relation was proved mathematically in [6, 7], and physically derived in [8] via the principle that the spacetime BPS spectra of *string-dual* M-theory compactifications ought to be equivalent.

"Ramified" generalizations of the AGT correspondence for pure SU(N) involving arbitrary surface operators were later proposed in [9, 10], where they were also physically derived in [8] via the principle that the spacetime BPS spectra of *string-dual* M-theory compactifications ought to be equivalent. For a full surface operator, the correspondence was also proved purely mathematically in [11].

The AGT correspondence also implies certain relations between the Nekrasov instanton partition function and 2d integrable systems. An example would be the conjecture in [5], which asserts that the "fully-ramified" Nekrasov instanton partition function should be related to Hitchin's integrable system. This was again physically derived using M-theory in [8].

Our main aim is to furnish in a pedagogical manner, a fundamental M-theoretic derivation of a 5d and 6d analog of the above 4d-2d relations, by generalizing our analysis in [8]. Let us now give a brief plan and summary of the paper.

A brief plan and summary of the paper. In section 2, we will first review the M-theoretic derivation of a 4d pure AGT correspondence in [8]. Then, we will generalize the analysis in loc. cit. to furnish an M-theoretic derivation of a 5d AGT correspondence for pure SU(N) in the topological string limit. In particular, we find that the corresponding 5d Nekrasov instanton partition function can be expressed as the inner product of a coherent state in an integrable module over a universal central extension of the double loop algebra of SU(N). Our result therefore sheds further light on an earlier field-theoretic result by Nekrasov-Okounkov in [12] regarding the aforementioned 5d Nekrasov instanton partition function.

In section 3, we will first review the M-theoretic derivation of a 4d AGT correspondence for a conformal linear quiver theory in [8]. Then, we will generalize the analysis in loc. cit. to furnish an M-theoretic derivation of a 5d AGT correspondence for SU(N) with $N_f = 2N$ fundamental matter. In starting with the $N = 1$ case, we will first make contact with what is known as a Ding-Iohara algebra, whence for the $N > 1$ case, we find that the corresponding 5d Nekrasov instanton partition function can be expressed as a four-point correlation function on a sphere of vertex operators of an integrable module over a q-deformed affine $\mathcal{W}_N$-algebra. Our result therefore serves as a purely physical M-theoretic proof of a mathematical conjecture by Awata-Feigin-Hoshino-Kanai-Shiraishi-Yanagida in [14], of a 5d analog of the AGT correspondence for SU(N) with $N_f = 2N$ fundamental matter. Last but not least, we will proceed in the same manner to furnish an M-theoretic derivation of a 5d AGT correspondence for pure SU(N), where we find that the corresponding 5d Nekrasov instanton partition function — which is also given by an equivariant index of the Dirac operator on the moduli space of SU(N)-instantons — can be

expressed as the inner product of a coherent state in an integrable module over a q-deformed affine $\mathcal{W}_N$-algebra — which is also given by a Whittaker function on the q-deformed affine $\mathcal{W}_N$-algebra. Our result therefore serves as an SU(N) generalization of a 5d analog of the AGT correspondence for pure SU(2) first proposed and checked by Awata-Yamada in [15]. Furthermore, as an offshoot of our physical analysis in the topological string limit, we also have the novel representation-theoretic result that a Whittaker vector in a level N module over a (certain specialization of the) Ding-Iohara algebra is also a Whittaker vector in a module over a universal central extension of the double loop algebra of SU(N).

In section 4, we will first review the M-theoretic derivation of a 4d "fully-ramified" pure AGT correspondence in [8]. Then, we will generalize the analysis in loc. cit. to furnish an M-theoretic derivation of a 5d "fully-ramified" AGT correspondence for (i) pure SU(N) and (ii) SU(N) with single adjoint matter, in the Nekrasov-Shatashvilli (NS) limit. In particular, we find that the corresponding 5d Nekrasov instanton partition function can be expressed as (i) the inner product of a coherent state in an integrable module over a quantum affine SU(N)-algebra at the critical level and (ii) a one-point correlation function on a torus of a vertex operator of an integrable module over a quantum affine SU(N)-algebra at the critical level. Along the way, we will also argue that the corresponding 5d Nekrasov instanton partition function ought to be a simultaneous eigenfunction of commuting operators which define a (i) relativistic periodic Toda integrable system and (ii) relativistic elliptic Calogero-Moser system. As an offshoot, we also have the interesting result that the simultaneous eigenfunctions of an elliptic generalization of the celebrated Macdonald operators can be understood as one-point correlation functions of a 2d QFT on a torus whose underlying symmetry is generated by a quantum affine SU(N)-algebra at the critical level.

In section 5, we will generalize our analysis in section 3 to furnish an M-theoretic derivation of a 6d AGT correspondence for the non-anomalous case of an SU(N) theory with $N_f = 2N$ fundamental matter. In starting with the $N = 1$ case, we will first make contact with what is known as an elliptic Ding-Iohara algebra, whence for the $N > 1$ case, we find that the corresponding 6d Nekrasov instanton partition function — which is also given by an equivariant elliptic genus on the moduli space of SU(N)-instantons — can be expressed as a two-point correlation function on a torus of vertex operators of an integrable module over an elliptic affine $\mathcal{W}_N$-algebra.

In section 6, we will generalize our analysis in section 4 to furnish an M-theoretic derivation of a 6d "fully-ramified" AGT correspondence for SU(N) with $N_f = 2N$ fundamental matter in the NS limit. In particular, we find that the corresponding 6d Nekrasov instanton partition function can be expressed as a two-point correlation function on a torus of vertex operators of an integrable module over an elliptic affine SU(N)-algebra at the critical level. Along the way, we will also argue that the corresponding 6d Nekrasov instanton partition function ought to be a simultaneous eigenfunction of commuting operators which define an elliptized integrable system.

2 An M-theoretic derivation of a 5d pure AGT correspondence in the topological string limit

2.1 An M-theoretic derivation of a 4d pure AGT correspondence: a review

We shall now review how an expected equivalence of the 6d spacetime BPS spectra of physically dual M-theory compactifications would allow us to derive a 4d AGT correspondence for pure $SU(N)$ [8].

To this end, recall from [8, (5.9) and (5.11)] that we have the following *physically dual* M-theory compactifications

$$\underbrace{\mathbb{R}^4|_{\epsilon_1, \epsilon_2} \times \Sigma_t \times \mathbb{R}^5|_{\epsilon_3;\, x_{6,7}}}_{N \text{ M5-branes}} \quad \Longleftrightarrow \quad \mathbb{R}^5|_{\epsilon_3;\, x_{4,5}} \times \underbrace{\mathcal{C} \times TN_N^{R \to 0}|_{\epsilon_3;\, x_{6,7}}}_{1 \text{ M5-branes}}. \tag{2.1}$$

Here, we have a common half-BPS boundary condition at the tips of the interval $\mathbb{I}_t \subset \Sigma_t = \mathbf{S}^1 \times \mathbb{I}_t$ (that is realized by a pair of M9-branes which span all directions normal to them); the radius of $\mathbf{S}^1$ is β; $\mathbb{I}_t \ll \beta$; $\mathcal{C}$ is a priori the same as Σ_t; and the ϵ_i's are parameters of the Omega-deformation along the indicated planes,[1] where the directions spanned by the M5-branes are:

	0	1	2	3	4	5	6	7	8	9	10
N M5's	–	–	–	–	–	–					
1 M5	–	–					–	–	–	–	

$$\tag{2.2}$$

with x_0 and x_1 being the coordinates on $\mathbb{I}_t$ and $\mathbf{S}^1$, respectively. (Note that if $z = x_2 + ix_3$ and $w = x_4 + ix_5$, $\mathbb{R}^4$ can be viewed as a complex surface $\mathbb{C}^2$ whose coordinates are (z, w). Likewise, if $u = x_6 + ix_7$ and $v = x_8 + ix_9$, $TN_N^{R \to 0}$ can be viewed as a complex surface whose singularity at the origin would be modeled by $\mathbb{C}^2/\mathbb{Z}_N$, where (u, v) are the coordinates on $\mathbb{C}^2$.)

The spectrum of spacetime BPS states on the l.h.s. of (2.1). Let us first discuss the spectrum of spacetime BPS states along $\mathbb{R}^5|_{\epsilon_3;\, x_{6,7}} \times \mathbb{I}_t$ on the l.h.s. of (2.1). In the absence of Omega-deformation whence $\epsilon_i = 0$, according to [8, section 5.1], the spacetime BPS states would be captured by the topological sector of the $\mathcal{N} = (4, 4)$ sigma-model on Σ_t with target the moduli space $\mathcal{M}_{SU(N)}$ of $SU(N)$-instantons on $\mathbb{R}^4$. However, in the presence of Omega-deformation, as explained in [8, section 5.1], as one traverses a closed loop in Σ_t, there would be a $\mathbf{g}$-automorphism of $\mathcal{M}_{SU(N)}$, where $\mathbf{g} \in U(1) \times U(1) \times T$, and $T \subset SU(N)$ is the Cartan subgroup. Consequently, the spacetime BPS states of interest would, in the presence of Omega-deformation, be captured by the topological sector of a non-dynamically $\mathbf{g}$-gauged version of the aforementioned sigma-model.[2] Hence, according

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[1] Thess parameters are real in our formulation, and for later convenience, we shall consider them to have the opposite sign.

[2] The relation between $\mathbf{g}$-automorphisms of the sigma-model target space and a non-dynamical $\mathbf{g}$-gauging of its worldsheet theory, is explained in [16, section 2.4 and section 5]. For self-containment, let us review the idea here. Consider a sigma-model with worldsheet Σ_t, target space $\mathcal{M}_{SU(N)}$, and bosonic scalar fields Φ. In the usual case where there is no action on $\mathcal{M}_{SU(N)}$ as we traverse a closed loop in Σ_t, one would consider in the sigma-model path-integral, the space of maps $\Phi : \Sigma_t \to \mathcal{M}_{SU(N)}$, which can

to [17] and our arguments in [8, section 3.1] which led us to [8, eq. (3.5)], we can express the Hilbert space $\mathcal{H}_{\mathrm{BPS}}^{\Omega}$ of spacetime BPS states on the l.h.s. of (2.1) as

$$\mathcal{H}_{\mathrm{BPS}}^{\Omega} = \bigoplus_m \mathcal{H}_{\mathrm{BPS},m}^{\Omega} = \bigoplus_m \mathrm{IH}_{\mathrm{U}(1)^2 \times T}^* \mathcal{U}(\mathcal{M}_{\mathrm{SU}(N),m}), \qquad (2.3)$$

where $\mathrm{IH}_{\mathrm{U}(1)^2 \times T}^* \mathcal{U}(\mathcal{M}_{\mathrm{SU}(N),m})$ is the $\mathrm{U}(1)^2 \times T$-equivariant intersection cohomology of the Uhlenbeck compactification $\mathcal{U}(\mathcal{M}_{\mathrm{SU}(N),m})$ of the (singular) moduli space $\mathcal{M}_{\mathrm{SU}(N),m}$ of $\mathrm{SU}(N)$-instantons on $\mathbb{R}^4$ with instanton number m.

Notably, as explained in [8, section 5.1], the partition function of these spacetime BPS states would be given by the following 5d (brane) worldvolume expression

$$Z_{\mathrm{BPS}}(\epsilon_1, \epsilon_2, \vec{a}, \beta) = \sum_m \mathrm{Tr}_{\mathcal{H}_m} \exp \beta(\epsilon_1 J_1 + \epsilon_2 J_2 + \vec{a} \cdot \vec{T}), \qquad (2.4)$$

where $\vec{T} = (T_1 \ldots, T_{\mathrm{rank\,SU}(N)})$ are the generators of the Cartan subgroup of $\mathrm{SU}(N)$; $\vec{a} = (a_1, \ldots, a_{\mathrm{rank\,SU}(N)})$ are the corresponding purely imaginary Coulomb moduli of the $\mathrm{SU}(N)$ gauge theory on $\mathbb{R}^4|_{\epsilon_1, \epsilon_2}$; $J_{1,2}$ are the rotation generators of the x_2-x_3 and x_4-x_5 planes, respectively, corrected with an appropriate amount of the $\mathrm{SU}(2)_R$-symmetry to commute with the two surviving worldvolume supercharges; and $\mathcal{H}_m$ is the space of holomorphic functions on the moduli space $\mathcal{M}_{\mathrm{SU}(N),m}$ of $\mathrm{SU}(N)$-instantons on $\mathbb{R}^4$ with instanton number m.

The spectrum of spacetime BPS states on the r.h.s. of (2.1). Let us next discuss the corresponding spectrum of spacetime BPS states along $\mathbb{R}^5|_{\epsilon_3;\, x_{4,5}} \times \mathbb{I}_t$ on the r.h.s. of (2.1). As explained in [8, section 5.2], the spacetime BPS states would be furnished by the I-brane theory in the following type IIA configuration:

$$\mathrm{IIA}: \qquad \underbrace{\mathbb{R}^5|_{\epsilon_3;x_{4,5}} \times \mathcal{C} \times \mathbb{R}^3|_{\epsilon_3;x_{6,7}}}_{\text{I-brane on } \mathcal{C} \,=\, N\mathrm{D}6 \cap 1\mathrm{D}4}. \qquad (2.5)$$

Here, we have a stack of N coincident D6-branes whose worldvolume is given by $\mathbb{R}^5|_{\epsilon_3;x_{4,5}} \times \mathcal{C}$, and a single D4-brane whose worldvolume is given by $\mathcal{C} \times \mathbb{R}^3|_{\epsilon_3;x_{6,7}}$.

Let us for a moment turn off Omega-deformation in (2.5), i.e., let $\epsilon_3 = \epsilon_1 + \epsilon_2 = 0$. Then, as explained in [8, section 5.2], the spacetime BPS states would be furnished by N complex chiral fermions on $\mathcal{C}$ which effectively realize a chiral WZW model at level 1 on $\mathcal{C}$, $\mathrm{WZW}_{\mathfrak{su}(N)_{\mathrm{aff}}}^{\mathrm{level\,1}}$, where $\mathfrak{su}(N)_{\mathrm{aff}}$ is the affine $\mathrm{SU}(N)$-algebra.

Now turn Omega-deformation back on. As indicated in (2.5), as one traverses around a closed loop in $\mathcal{C}$, the x_4-x_5 plane in $\mathbb{R}^4|_{\epsilon_3;\, x_{4,5}} \subset \mathbb{R}^5|_{\epsilon_3;\, x_{4,5}}$ would be rotated by an angle of ϵ_3 together with an $\mathrm{SU}(2)_R$-symmetry rotation of the supersymmetric $\mathrm{SU}(N)$

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be viewed as the space of trivial sections of a trivial bundle $X = \mathcal{M}_{\mathrm{SU}(N)} \times \Sigma_t$. If however there is a **g**-automorphism of $\mathcal{M}_{\mathrm{SU}(N)}$ as we traverse a closed loop in Σ_t, X would have to be a nontrivial bundle given by $\mathcal{M}_{\mathrm{SU}(N)} \hookrightarrow X \to \Sigma_t$; then, $\Phi(z, \bar{z})$ will not represent a map $\Sigma_t \to \mathcal{M}_{\mathrm{SU}(N)}$, but rather, it will be a nontrivial section of X. Thus, since Φ is no longer a function but a nontrivial section of a nontrivial bundle, its ordinary derivatives must be replaced by covariant derivatives. As the nontrivial structure group of X is now **g**, replacing ordinary derivatives by covariant derivatives would mean introducing on Σ_t gauge fields A^a, which, locally, can be regarded as (Lie **g**)-valued one-forms with the usual gauge transformation law $A^{a\prime} = g^{-1}A^a g + g^{-1}dg$, where $g \in \mathbf{g}$. This is equivalent to gauging the sigma-model non-dynamically by **g**.

gauge theory along $\mathbb{R}^4|_{\epsilon_3; x_{4,5}}$. As such, Omega-deformation in this instance would effect a $\mathbf{g}'$-automorphism of $\mathcal{M}_{\mathrm{SU}(N),l}$ as we traverse around a closed loop in $\mathcal{C}$, where $\mathcal{M}_{\mathrm{SU}(N),l}$ is the moduli space of SU(N)-instantons on $\mathbb{R}^4$ with instanton number l; $\mathbf{g}' = \exp \beta(\epsilon_3 J_3 + \vec{a} \cdot \vec{T}')$; J_3 is the rotation generator of the x_4-x_5 plane corrected with an appropriate amount of SU(2)$_R$-symmetry to commute with the D6-brane worldvolume supercharges; $\vec{T}' = (T_1' \ldots, T_{\mathrm{rank\,SU}(N)}')$ are the generators of the Cartan subgroup $T' \subset \mathrm{SU}(N)$; and $\vec{a} = (a_1, \ldots, a_{\mathrm{rank\,SU}(N)})$ are the corresponding purely imaginary Coulomb moduli of the SU(N) gauge theory on $\mathbb{R}^4|_{\epsilon_3; x_{4,5}}$. In fact, since $\mathcal{M}_{\mathrm{SU}(N),l}$ is also the space of self-dual connections of an SU(N)-bundle on $\mathbb{R}^4$, and since these self-dual connections correspond to differential one-forms valued in the Lie algebra $\mathfrak{su}(N)$, Omega-deformation also means that there is a $\mathbf{g}'$-automorphism of the space of elements of $\mathfrak{su}(N)$ and thus SU(N), as we traverse a closed loop in $\mathcal{C}$.

Note at this point that $\mathrm{WZW}_{\mathfrak{su}(N)_{\mathrm{aff}}}^{\mathrm{level\,1}}$ can be regarded as a (chiral half of a) SU(N) WZW model at level 1 on $\mathcal{C}$. Since a $\mathcal{G}$ WZW model on Σ is a bosonic sigma-model on Σ with target the $\mathcal{G}$-manifold, according to the last paragraph, it would mean that Omega-deformation would effect a $\mathbf{g}'$-automorphism of the target space of $\mathrm{WZW}_{\mathfrak{su}(N)_{\mathrm{aff}}}^{\mathrm{level\,1}}$ as we traverse a closed loop in $\mathcal{C}$, where $\mathbf{g}' \in \mathrm{U}(1) \times T'$. In turn, according to footnote 2, it would mean that in the presence of Omega-deformation, we would have to non-dynamically gauge $\mathrm{WZW}_{\mathfrak{su}(N)_{\mathrm{aff}}}^{\mathrm{level\,1}}$ by $\mathrm{U}(1) \times T'$.

That being said, notice also from (2.5) that as one traverses around a closed loop in $\mathcal{C}$, the x_6-x_7 plane in $\mathbb{R}^3|_{\epsilon_3; x_{6,7}}$ would be rotated by an angle of ϵ_3 together with an R-symmetry rotation of the supersymmetric U(1) gauge theory living on the single D4-brane, i.e., Omega-deformation is also being turned on along the D4-brane. As explained in [8, section 5.2], this means that we would in fact have to non-dynamically gauge $\mathrm{WZW}_{\mathfrak{su}(N)_{\mathrm{aff}}}^{\mathrm{level\,1}}$ not by $\mathrm{U}(1) \times T'$ but by $\mathcal{T} \subset T'$.

At any rate, because $\mathrm{SU}(N)/T' \simeq \mathrm{SL}(N, \mathbb{C})/B_+$, where B_+ is a Borel subgroup, it would mean that $\mathrm{SU}(N)/\mathcal{T} \simeq (\mathrm{SL}(N, \mathbb{C})/B_+) \times (T'/\mathcal{T})$. Also, $T'/\mathcal{T}$ is never bigger than the Cartan subgroup $C \subset B_+ = C \times N_+$, where N_+ is the subgroup of strictly upper triangular matrices which are nilpotent and traceless whose Lie algebra is $\mathfrak{n}_+$. Altogether, this means that our gauged WZW model which corresponds to the coset model $\mathrm{SU}(N)/\mathcal{T}$, can also be studied as an S-gauged $\mathrm{SL}(N, \mathbb{C})$ WZW model which corresponds to the coset model $\mathrm{SL}(N, \mathbb{C})/S$, where $N_+ \subseteq S \subset B_+$. As physically consistent $\mathcal{H}$-gauged $\mathcal{G}$ WZW models are such that $\mathcal{H}$ is necessarily a connected subgroup of $\mathcal{G}$, it will mean that $S = N_+$. Therefore, what we ought to ultimately consider is an N_+-gauged $\mathrm{SL}(N, \mathbb{C})$ WZW model.

Before we proceed any further, let us make a slight deviation to highlight an important point regarding the effective geometry of $\mathcal{C}$. As the simple roots of N_+ form a subset of the simple roots of $\mathrm{SL}(N, \mathbb{C})$, the level of the affine N_+-algebra ought to be the equal to the level of the affine $\mathrm{SL}(N, \mathbb{C})$-algebra [18] which is 1. However, it is clear from our discussion hitherto that the affine N_+-algebra, in particular its level, will depend nontrivially on the Omega-deformation parameters which may or may not take integral values; in other words, its level will *not* be equal to 1. A resolution to this conundrum is as follows. A deviation of the level of the affine N_+-algebra from 1 would translate into a corresponding deviation of its central charge; since a central charge arises due to an introduction of a macroscopic

scale in the 2d system which results from a curvature along $\mathcal{C}$ [19], it would mean that Omega-deformation ought to deform the a priori flat $\mathcal{C} = \Sigma_t$ into a curved Riemann surface with the same topology — that is, a Riemann sphere with two punctures — such that the anomalous deviation in the central charge and therefore level, can be consistently "absorbed" in the process. Thus, we effectively have $\mathcal{C} = \mathbf{S}^2/\{0, \infty\}$, so $\mathcal{C}$ can be viewed as an $\mathbf{S}^1$ fibration of $\mathbb{I}_t$ whose fiber has zero radius at the two end points $z = 0$ and $z = \infty$, where 'z' is a holomorphic coordinate on $\mathcal{C}$.

Coming back to our main discussion, it is clear that in the schematic notation of [8, section 3.1], our N_+-gauged $\mathrm{SL}(N, \mathbb{C})$ WZW model can be expressed as the partially gauged chiral CFT

$$\mathfrak{sl}(N)_{\mathrm{aff},1}/\mathfrak{n}_{+\,\mathrm{aff},p} \tag{2.6}$$

on $\mathcal{C}$, where the level p would, according to our discussions thus far, necessarily depend on the Omega-deformation parameters $\epsilon_1' = \beta\epsilon_1$ and $\epsilon_2' = \beta\epsilon_2$. ($p$, being a purely real number, should not depend on the purely imaginary parameter $\vec{a}' = \beta\vec{a}$).

In sum, the sought-after spacetime BPS states ought to be given by the states of the partially gauged chiral CFT in (2.6), and via [8, section B] and [20, eq. (6.67)], we find that this chiral CFT realizes an affine $\mathcal{W}_N$-algebra obtained from $\mathfrak{sl}(N)_{\mathrm{aff}}$ via a quantum Drinfeld-Sokolov reduction. In other words, the states of the chiral CFT would be furnished by a Verma module $\widehat{\mathcal{W}}_N$ over the affine $\mathcal{W}_N$-algebra, and the Hilbert space $\mathcal{H}_{\mathrm{BPS}}^{\Omega'}$ of spacetime BPS states on the r.h.s. of (2.1) can be expressed as

$$\mathcal{H}_{\mathrm{BPS}}^{\Omega'} = \widehat{\mathcal{W}}_N. \tag{2.7}$$

An AGT correspondence for pure SU(N). Clearly, the physical duality of the compactifications in (2.1) will mean that $\mathcal{H}_{\mathrm{BPS}}^{\Omega}$ in (2.3) is equivalent to $\mathcal{H}_{\mathrm{BPS}}^{\Omega'}$ in (2.7), i.e.,

$$\bigoplus_m \mathrm{IH}_{\mathrm{U}(1)^2 \times T}^* \, \mathcal{U}(\mathcal{M}_{\mathrm{SU}(N),m}) = \widehat{\mathcal{W}}_N. \tag{2.8}$$

Moreover, as explained in [8, section 5.2], the central charge and level of the affine $\mathcal{W}_N$-algebra are given by

$$c_{A,\epsilon_{1,2}} = (N-1) + (N^3 - N)\frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1\epsilon_2} \qquad \text{and} \qquad k' = -N - \epsilon_2/\epsilon_1, \tag{2.9}$$

respectively.

In the limit that $\beta \to 0$, it is well-known [21] that

$$Z_{\mathrm{BPS}}(\epsilon_1, \epsilon_2, \vec{a}, \beta) = \sum_m Z_{\mathrm{BPS},m}(\epsilon_1, \epsilon_2, \vec{a}, \beta) \tag{2.10}$$

of (2.4) behaves as $Z_{\mathrm{BPS},m}(\epsilon_1, \epsilon_2, \vec{a}, \beta \to 0) \sim \beta^{-2mN} Z_{\mathrm{BPS},m}^{\mathrm{4d}}(\epsilon_1, \epsilon_2, \vec{a})$, whence the 4d Nekrasov instanton partition function $Z_{\mathrm{inst}}(\Lambda, \epsilon_1, \epsilon_2, \vec{a}) = \sum_m \Lambda^{2mN} Z_{\mathrm{BPS},m}^{\mathrm{4d}}(\epsilon_1, \epsilon_2, \vec{a})$ can be written as

$$Z_{\mathrm{inst}}(\Lambda, \epsilon_1, \epsilon_2, \vec{a}) = \sum_m \Lambda^{2mN} Z_{\mathrm{BPS},m}'(\epsilon_1, \epsilon_2, \vec{a}, \beta \to 0), \tag{2.11}$$

where $Z'_{\text{BPS},m} = l_m \beta^{2mN} Z_{\text{BPS},m}$; l_m is some constant; and Λ can be interpreted as the inverse of the observed scale of the $\mathbb{R}^4|_{\epsilon_1,\epsilon_2}$ space on the l.h.s. of (2.1).

Note that equivariant localization [22] implies that $\text{IH}^*_{\text{U}(1)^2 \times T} \mathcal{U}(\mathcal{M}_{\text{SU}(N),m})$ must be endowed with an orthogonal basis $\{|\vec{p}_m\rangle\}$, where the $\vec{p}_m$'s denote the fixed points of a $\text{U}(1)^2 \times T$-action on $\mathcal{U}(\mathcal{M}_{\text{SU}(N),m})$. Thus, since $Z'_{\text{BPS},m}$ is a weighted count of the states in $\mathcal{H}^\Omega_{\text{BPS},m} = \text{IH}^*_{\text{U}(1)^2 \times T} \mathcal{U}(\mathcal{M}_{\text{SU}(N),m})$, it would mean that one can write

$$Z'_{\text{BPS},m}(\epsilon_1, \epsilon_2, \vec{a}, \beta \to 0) = \sum_{\vec{p}_m} l^2_{\vec{p}_m}(\epsilon_1, \epsilon_2, \vec{a}) \langle \vec{p}_m | \vec{p}_m \rangle, \tag{2.12}$$

where $l_{\vec{p}_m}(\epsilon_1, \epsilon_2, \vec{a}) \in \mathbb{R}$, and the dependence on ϵ_1, ϵ_2 and $\vec{a}$ arises because the energy level of each state — given by the eigenvalue m of the L_0 operator which generates translation along $\mathbf{S}^1 \subset \Sigma_t$ in (2.1) whence there is an Omega-deformation twist of the theory along the orthogonal spaces indicated therein — ought to depend on these Omega-deformation parameters.

Notice that (2.12) also means that

$$Z'_{\text{BPS},m}(\epsilon_1, \epsilon_2, \vec{a}, \beta \to 0) = \langle \Psi_m | \Psi_m \rangle, \tag{2.13}$$

where

$$|\Psi_m\rangle = \bigoplus_{\vec{p}_m} l_{\vec{p}_m} |\vec{p}_m\rangle. \tag{2.14}$$

Here, the state $|\Psi_m\rangle \in \text{IH}^*_{\text{U}(1)^2 \times T} \mathcal{U}(\mathcal{M}_{\text{SU}(N),m})$, and $\langle \cdot | \cdot \rangle$ is a Poincaré pairing in the sense of [11, section 2.6].

Now consider the state

$$|\Psi\rangle = \bigoplus_m \Lambda^{mN} |\Psi_m\rangle. \tag{2.15}$$

By substituting (2.13) in the r.h.s. of (2.11), and by noting that $\langle \Psi_m | \Psi_n \rangle = \delta_{mn}$, one can immediately see that

$$Z_{\text{inst}}(\Lambda, \epsilon_1, \epsilon_2, \vec{a}) = \langle \Psi | \Psi \rangle, \tag{2.16}$$

where $|\Psi\rangle \in \bigoplus_m \text{IH}^*_{\text{U}(1)^2 \times T} \mathcal{U}(\mathcal{M}_{\text{SU}(N),m})$. In turn, the duality relation (2.8) would mean that

$$|\Psi\rangle = |\text{coh}\rangle \tag{2.17}$$

whence

$$Z_{\text{inst}}(\Lambda, \epsilon_1, \epsilon_2, \vec{a}) = \langle \text{coh} | \text{coh} \rangle, \tag{2.18}$$

where $|\text{coh}\rangle \in \widehat{\mathcal{W}}_N$. Since the r.h.s. of (2.18) is defined at $\beta \to 0$ (see the r.h.s. of (2.11)), and since we have in $\mathcal{C}$ a *common* boundary condition at $z = 0$ and $z = \infty$, $|\text{coh}\rangle$ and $\langle \text{coh}|$ ought to be a state and its dual associated with the puncture at $z = 0$ and $z = \infty$, respectively (as $z = 0, \infty$ are the points in $\mathcal{C}$ where the $\mathbf{S}^1$ fiber has zero radius). Furthermore, as the r.h.s. of (2.15) is a sum over states of all possible energy levels, it would mean from (2.17) that $|\text{coh}\rangle$ is actually a *coherent state*. This is depicted in figure 1.

At any rate, since we have N D6-branes and 1 D4-brane wrapping $\mathcal{C}$ (see (2.5)), we effectively have an $N \times 1 = N$-fold cover Σ_{SW} of $\mathcal{C}$. This is also depicted in figure 1. Incidentally, Σ_{SW} is also the Seiberg-Witten curve which underlies $Z_{\text{inst}}(\Lambda, \epsilon_1, \epsilon_2, \vec{a})$! Moreover,
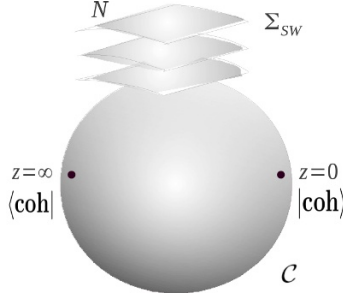
Figure 1. $\mathcal{C}$ and its N-fold cover Σ_{SW} with the states $\langle\mathrm{coh}|$ and $|\mathrm{coh}\rangle$ at $z = 0$ and ∞.

it is by now well-established (see [23] and references therein) that Σ_{SW} can be described in terms of the algebraic relation

$$\Sigma_{SW} : \lambda^N + \phi_2(z)\lambda^{N-2} + \cdots + \phi_N(z) = 0, \tag{2.19}$$

where $\lambda = y\,dz/z$ (for some complex variable y) is a section of $T^*\mathcal{C}$; the $\phi_s(z)$'s are $(s, 0)$-holomorphic differentials on $\mathcal{C}$ given by

$$\phi_j(z) = u_j \left(\frac{dz}{z}\right)^j \quad \text{and} \quad \phi_N(z) = \left(z + u_N + \frac{\Lambda^N}{z}\right)\left(\frac{dz}{z}\right)^N, \tag{2.20}$$

where $j = 2, 3, \ldots, N - 1$; while for weights $\lambda_1, \ldots, \lambda_N$ of the N-dimensional representation of $\mathrm{SU}(N)$, and for $s = 2, 3, \ldots, N$, $u_s = (-1)^{s+1}\sum_{k_1 \neq \cdots \neq k_s} e_{\lambda_{k_1}} e_{\lambda_{k_2}} \ldots e_{\lambda_{k_s}}(\vec{a})$ and $e_{\lambda_r} = \vec{a} \cdot \lambda_r$. This is consistent with our results that we have, on $\mathcal{C}$, the following $(s_i, 0)$-holomorphic differentials

$$W^{(s_i)}(z) = \left(\sum_{l \in \mathbb{Z}} \frac{W_l^{(s_i)}}{z^l}\right)\left(\frac{dz}{z}\right)^{s_i}, \quad \text{where} \quad s_i = e_i + 1 = 2, 3, \ldots, N, \tag{2.21}$$

whence we can naturally identify, up to some constant factor, $\phi_s(z)$ with $W^{(s)}(z)$. (In fact, a U(1) R-symmetry of the 4d theory along $\mathbb{R}^4|_{\epsilon_1,\epsilon_2}$ on the l.h.s. of (2.1) which underlies $Z_{\mathrm{inst}}(\Lambda, \epsilon_1, \epsilon_2, \vec{a})$ and Σ_{SW}, can be identified with the rotational symmetry of $\mathbf{S}^1$; the duality relation (2.1) then means that the corresponding U(1) R-charge of the $\phi_s(z)$ operators that define Σ_{SW}, ought to match, up to a constant, the conformal dimension of the $W^{(s)}(z)$ operators on $\mathcal{C}$, which is indeed the case.)

In arriving at the relations (2.8), (2.9), (2.17) and (2.18), and the identification between the $W^{(s_i)}$'s and the ϕ_j's, we have just derived an AGT correspondence for pure $\mathrm{SU}(N)$.

2.2 An M-theoretic derivation of a 5d pure AGT correspondence in the topological string limit

Let us now consider the topological string limit $\epsilon_1 + \epsilon_2 = \epsilon_3 = 0$ in our derivation of the 4d pure AGT correspondence in the last subsection. In this limit, Omega-deformation on the r.h.s. of (2.1) effectively vanishes. According to our discussions in the last subsection, (i) $\mathcal{C}$ in the I-brane configuration (2.5) would become a flat finite cylinder $\Sigma_t = \mathbf{S}^1 \times \mathbb{I}_t$

again; (ii) the partially gauged chiral CFT behind (2.6) would be *ungauged*. This means that instead of (2.18), we now have

$$Z_{\text{inst}}(\Lambda, \hbar, \vec{a}) = \langle u_\hbar | \Lambda^{2N\mathcal{L}_0} | u_\hbar \rangle. \tag{2.22}$$

Here, Λ is the energy scale; $\hbar = \epsilon_1 = -\epsilon_2$; $\vec{a}$ is the Coulomb moduli of the underlying 4d pure SU(N) theory; $|u_\hbar\rangle \in \widehat{\mathfrak{su}(N)}_{\text{aff},1}$, where $\widehat{\mathfrak{su}(N)}_{\text{aff},1}$ is an integrable highest weight module over an affine Lie algebra $\mathfrak{su}(N)_{\text{aff},1}$ of level 1; $|u_\hbar\rangle$ is a *coherent state* generated from the primary state $|\Delta_\hbar\rangle$ of conformal dimension $\Delta_\hbar \sim \vec{a}^2/\hbar^2$; and $\mathcal{L}_0$ is the generator of time translations along Σ_t which propagates the state $|u_\hbar\rangle$ at one end by a distance $\sim 1/g^2 \sim \ln \Lambda^{2N}$ to the other end whence it is annihilated by the state $\langle u_\hbar|$, where g is the underlying gauge coupling.[3]

The 4d Nekrasov instanton partition function and complex chiral fermions.
Note that if $\{e_l, h_l, f_l\}_{l=0,\dots,N-1}$ are the Chevalley generators of $\mathfrak{su}(N)_{\text{aff},1}$, where the e_l's and f_l's correspond to lowering and raising operators, respectively, we can write

$$|u_\hbar\rangle = \exp\left(\alpha \sum_{l=1}^N f_{l-1}\right)|v_0\rangle, \tag{2.23}$$

where α is some normalization constant, and $|v_0\rangle$ is the state corresponding to the highest weight vector v_0.

As mentioned in the last subsection, we have N complex chiral fermions

$$\psi^a(z) = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \psi_r^a \, z^{-r} \left(\frac{dz}{z}\right)^{\frac{1}{2}}, \quad \tilde{\psi}^a(z) = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \tilde{\psi}_r^a \, z^{-r} \left(\frac{dz}{z}\right)^{\frac{1}{2}}, \quad a = 1, \dots N, \tag{2.24}$$

which realize $\mathfrak{su}(N)_{\text{aff},1}$ on Σ_t. Consequently, by choosing $\alpha = N/\hbar$, we can also express (2.23) as (cf. [12])

$$|u_\hbar\rangle = e^{\frac{\mathcal{J}_{-1}}{\hbar}}|p\rangle, \tag{2.25}$$

whence from (2.22), the 4d Nekrasov instanton partition function would be given by

$$Z_{\text{inst}}(\Lambda, \hbar, \vec{a}) = \langle p | e^{\frac{\mathcal{J}_1}{\hbar}} \Lambda^{2N\mathcal{L}_0} e^{\frac{\mathcal{J}_{-1}}{\hbar}} | p \rangle, \tag{2.26}$$

where $|p\rangle$ is a vacuum state in a standard fermionic Fock space $\mathcal{H}_{[p]}$ whose energy level is $p^2/2 \sim \vec{a}^2/\hbar^2$, and $\mathcal{J}_{\mp n}$ are creation and annihilation operators in $\mathcal{H}_{[p]}$, respectively, which are constructed out of the chiral fermions. In particular, these $\mathcal{J}_{\pm 1}$ operators obey the Heisenberg algebra

$$[\mathcal{J}_1, \mathcal{J}_{-1}] = 1, \tag{2.27}$$

because they span the modes of a free chiral boson

$$\phi(z) = \phi_0 - i\mathcal{J}_0 \ln(z) + i \sum_{n \neq 0} \frac{\mathcal{J}_n}{n} z^{-n}, \tag{2.28}$$

---

[3]As the 4d gauge theory along $\mathbb{R}^4|_{\epsilon_1,\epsilon_2}$ in the original compactification in (2.1) is, in this case, asymptotically-free, the observed scale of the eleven-dimensional spacetime $\mathbb{R}^4|_{\epsilon_1,\epsilon_2} \times \Sigma_t \times \mathbb{R}^5$ ought to be inversely proportional to g^2; in particular, this means that the length of Σ_t ought to be proportional to $1/g^2$.

where

$$\Psi(z) = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \Psi_r \, z^{-r} =: e^{i\phi(z)} : \quad \text{and} \quad \tilde{\Psi}(z) = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \tilde{\Psi}_r \, z^{-r} =: e^{-i\phi(z)} : \quad (2.29)$$

is a *single* chiral fermion and its conjugate such that

$$\Psi_{N(r+\rho_a)} = \psi_r^a, \qquad \tilde{\Psi}_{N(r-\rho_a)} = \tilde{\psi}_r^a, \qquad \rho_a = \frac{2a - N - 1}{2N}. \qquad (2.30)$$

We also have

$$\mathcal{L}_0 = \sum_{r \in \mathbb{Z}+\frac{1}{2}} r : \Psi_r \tilde{\Psi}_{-r} : . \qquad (2.31)$$

Note that by using the commutator $[\mathcal{L}_0, \mathcal{J}_{\pm 1}] = \mathcal{J}_{\pm 1}$, we can, up to a constant factor of $\Lambda^{-Np^2/2}$, write

$$Z_{\text{inst}}(\Lambda', \hbar, \vec{a}) = \langle p | e^{\frac{\Lambda'}{\hbar} \mathcal{J}_1} \, e^{\frac{\Lambda'}{\hbar} \mathcal{J}_{-1}} | p \rangle = \langle \text{coh}' | \text{coh}' \rangle, \qquad (2.32)$$

where $\Lambda' = \Lambda^N$. Moreover, by comparing this with (2.18), we find that we can interpret $|\text{coh}'\rangle$ and $\langle \text{coh}'|$ as a coherent state and its dual at the $z = 0$ and $z = \infty$ point on $\mathcal{C}_{\text{eff}} = \mathbf{S}^2$.

A 5d AGT correspondence for pure SU(N) in the topological string limit.
We are now ready to derive a 5d version of (2.32). To this end, first note that the 5d version of (2.18) is just (2.18) itself but at $\beta \neq 0$. As such, if $Z_{\text{inst}}^{5d}(\Lambda', \hbar, \beta, \vec{a})$ is the 5d Nekrasov instanton partition function at $\epsilon_1 = -\epsilon_2 = \hbar$, then, according to our discussion following (2.18), we can write

$$Z_{\text{inst}}^{5d}(\Lambda', \hbar, \beta, \vec{a}) = \langle \text{cir}' | \text{cir}' \rangle, \qquad (2.33)$$

where $|\text{cir}'\rangle$ is a coherent state that has a projection onto a circle C_β of radius β in $\mathcal{C}_{\text{eff}} = \mathbf{S}^2$.

The explicit form of $|\text{cir}'\rangle$ can be determined as follows. Firstly, recall that when a quantum system is confined to a space of infinitesimal size, its higher excited states would be decoupled; nevertheless, as we gradually increase the size of the space, we would start to observe these states. Now notice that $|\text{coh}'\rangle = |p\rangle + \frac{\Lambda'}{\hbar} \mathcal{J}_{-1}|p\rangle + (\frac{\Lambda'}{\hbar})^2 \mathcal{J}_{-1}\mathcal{J}_{-1}|p\rangle + \dots$; the preceding statement and the last two paragraphs then mean that $|\text{cir}'\rangle = |p\rangle + f_1(\Lambda', \hbar, \beta)\mathcal{J}_{-1}|p\rangle + f_2(\Lambda', \hbar, \beta)\mathcal{J}_{-2}|p\rangle + f_{1,1}(\Lambda', \hbar, \beta)\mathcal{J}_{-1}\mathcal{J}_{-1}|p\rangle + f_{2,1}(\Lambda', \hbar, \beta)\mathcal{J}_{-2}\mathcal{J}_{-1}|p\rangle + f_{1,2}(\Lambda', \hbar, \beta)\mathcal{J}_{-1}\mathcal{J}_{-2}|p\rangle + f_{2,2}(\Lambda', \hbar, \beta)\mathcal{J}_{-2}\mathcal{J}_{-2}|p\rangle + \dots$, where the f's are constants depending on the indicated parameters.

Secondly, notice that (i) Z_{BPS} in (2.4) which underlies Z_{inst}^{5d} in (2.33) is invariant under the simultaneous rescalings $(\beta, \hbar, -\hbar, \vec{a}) \to (\zeta\beta, \zeta^{-1}\hbar, -\zeta^{-1}\hbar, \zeta^{-1}\vec{a})$, where ζ is some real constant; (ii) Z_{inst} in (2.32) (which is just the $\beta \to 0$ limit of Z_{inst}^{5d}) is invariant under the simultaneous rescalings $(\hbar, \Lambda') \to (\zeta^{-1}\hbar, \zeta^{-1}\Lambda')$; and (iii) since the underlying worldvolume theory of the N M5-branes on the l.h.s. of (2.1) is scale-invariant, it would mean that the physics ought to be invariant under the simultaneous rescalings $(\beta, \Lambda') \to (\zeta\beta, \zeta^{-1}\Lambda')$. Altogether, this means that the f's should be invariant under the simultaneous rescalings $(\beta, \hbar, \Lambda') \to (\zeta\beta, \zeta^{-1}\hbar, \zeta^{-1}\Lambda')$. Thus, the f's should depend on $(\Lambda', \hbar, \beta)$ through the rescaling-invariant combinations $\beta\hbar$ and $\beta\Lambda'$.

Thirdly, notice that in (2.32), we can write the exponent $(\Lambda'/\hbar)\mathcal{J}_{-1}$ as $g(\Lambda', \hbar)\frac{\mathcal{J}_{-m}}{-m}$, where $m = 1$ and g is a constant depending on the indicated parameters. The preceding two paragraphs then mean that in order to arrive at $|\mathrm{cir}'\rangle$, we ought to replace this exponent by $\sum_{n=1}^{\infty} f^n(\Lambda', \hbar, \beta)\frac{\mathcal{J}_{-n}}{-n}$, where the f^n's are distinct constants depending on $\beta\hbar$ and $\beta\Lambda'$.

Fourthly, note that when $\beta \to 0$ whence only the $\mathcal{J}_{-1}$ mode acts on the vacuum state $|p\rangle$, $|\mathrm{cir}'\rangle$ must reduce to $|\mathrm{coh}'\rangle$. On the other hand when $\beta \to \infty$, $|\mathrm{cir}'\rangle$ would be a state that has a projection onto a circle of infinite radius whence it would only have zero energy; in other words, $|\mathrm{cir}'\rangle$ must reduce to the vacuum state $|p\rangle$ when $\beta \to \infty$.

Last but not least, notice that (2.15) and (2.17) mean that the mth power of Λ' ought to accompany the state of energy level m in $|\mathrm{coh}'\rangle$ (which is indeed the case as can be seen from the expansion of $|\mathrm{coh}'\rangle$ four paragraphs earlier). In turn, this means that f^n ought to depend on $(\beta\Lambda')^n$.

The above five points and a little thought then lead one to conclude that

$$Z_{\mathrm{inst}}^{5d}(\Lambda', \hbar, \beta, \vec{a}) = \langle \mathrm{cir}'|\mathrm{cir}'\rangle = \langle p|\tilde{\Gamma}_+ \tilde{\Gamma}_-|p\rangle \qquad (2.34)$$

where

$$\tilde{\Gamma}_- = \exp\left(-\sum_{n=1}^{\infty} \frac{(\beta\Lambda')^n}{1-q^n}\frac{\mathcal{J}_{-n}}{n}\right) \qquad \text{and} \qquad \tilde{\Gamma}_+ = \exp\left(\sum_{n=1}^{\infty} \frac{(\beta\Lambda')^n}{1-q^{-n}}\frac{\mathcal{J}_n}{n}\right) \qquad (2.35)$$

Here, $q = e^{\beta\hbar}$, and

$$[\mathcal{J}_m, \mathcal{J}_n] = m\delta_{m+n,0} \qquad (2.36)$$

At any rate, note that by using the commutator $[\mathcal{L}_0, \mathcal{J}_{\pm m}] \sim \mathcal{J}_{\pm m}$, we can, up to a constant factor, also write (2.34) as

$$Z_{\mathrm{inst}}^{5d}(\Lambda', \hbar, \beta, \vec{a}) = \langle p|\Gamma_+ (\beta\Lambda')^{2\mathcal{L}_0}\Gamma_-|p\rangle, \qquad (2.37)$$

where

$$\Gamma_- = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{1-q^n}\frac{\mathcal{J}_{-n}}{n}\right) \qquad \text{and} \qquad \Gamma_+ = \exp\left(\sum_{n=1}^{\infty} \frac{1}{1-q^{-n}}\frac{\mathcal{J}_n}{n}\right). \qquad (2.38)$$

The relations (2.37) and (2.38) agrees with the results of Nekrasov-Okounkov in [12, section 7.2.3], as expected.

Comparing (2.34)–(2.35) with (2.32), it is clear that one can interpret $|\mathrm{cir}'\rangle$ as a state in an integrable module over some q-dependent affine algebra. One can ascertain this q-dependent affine algebra as follows. Firstly, recall that the (chiral) WZW model which underlies $\mathfrak{su}(N)_{\mathrm{aff},1}$ on $\mathcal{C}_{\mathrm{eff}}$, can be regarded as a bosonic sigma model with worldsheet $\mathcal{C}_{\mathrm{eff}}$ and target an $\mathrm{SU}(N)$ group manifold. Thus, $|\mathrm{coh}'\rangle$, which is defined over a point in $\mathcal{C}_{\mathrm{eff}}$, would be associated with a point in the space of all points into the target that is the $\mathrm{SU}(N)$ group itself. Similarly, $|\mathrm{cir}'\rangle$, whose projection is onto a loop in $\mathcal{C}_{\mathrm{eff}}$, would be associated with a point in the space of all loops into the target that is the loop group of $\mathrm{SU}(N)$. Via (2.34), this implies that the 5d $\mathrm{SU}(N)$ theory underlying Z_{inst}^{5d} can also

be interpreted as a 4d LSU(N) theory, where LSU(N) is the loop group of SU(N). In turn, this means that we ought to replace $\mathfrak{su}(N)$ with $\mathfrak{lsu}(N)$, the Lie algebra of LSU(N), throughout. Hence, since $|\mathrm{coh}'\rangle \in \widehat{\mathfrak{su}(N)}_{\mathrm{aff}}$, we ought to have

$$\boxed{|\mathrm{cir}'\rangle \in \widehat{\mathfrak{lsu}(N)}_{\mathrm{aff}}} \tag{2.39}$$

where $\widehat{\mathfrak{lsu}(N)}_{\mathrm{aff}}$ is an integrable module over $\mathfrak{lsu}(N)_{\mathrm{aff}}$, a universal central extension of the double loop algebra associated with $\mathfrak{su}(N)$. This double loop algebra is the Lie algebra of smooth maps from the torus to $\mathfrak{su}(N)$.[4]

Clearly, in arriving at the boxed relations (2.34), (2.35), (2.36) and (2.39), we have just derived a 5d AGT correspondence for pure SU(N) in the topological string limit!

The appearance of a q-deformed affine $\mathcal{W}_N$-algebra. Before we end this section, let us make a final comment. We have hitherto studied only the $\epsilon_1 = -\epsilon_2$ topological string limit. What about the general case of arbitrary $\epsilon_{1,2}$? Since the pure case is just the case with a single adjoint hypermultiplet matter whose mass is infinitely large, as we shall see in the next section, the corresponding 5d pure Nekrasov instanton partition function can be expressed in terms of coherent states in an integrable module over a q-deformed affine $\mathcal{W}_N$-algebra.

3 An M-theoretic derivation of a 5d AGT correspondence

3.1 An M-theoretic derivation of a 4d AGT correspondence with matter: a review

Let us now review the M-theoretic derivation of a 4d AGT correspondence with matter in [8]. For brevity, we shall restrict our discussion to the conformal linear quiver of n SU(N) gauge groups.

In figure 2(1) and figure 2(2), we have the quiver diagram and its convenient schematic representation, respectively. As explained in [8], the resulting effective correspondence between the original M-theory compactification and the I-brane $\mathcal{C}_{\mathrm{eff}} = \mathbf{S}^2$ in the dual compactification is as shown in figure 2(3). Here, $\Sigma_t = \mathbf{S}^1 \times \mathbb{I}_t$, where β is the radius of $\mathbf{S}^1$; the vertical planes represent the spatial part of M9-branes; $X^9|_{\epsilon_i} = \mathbb{R}^4|_{\epsilon_1,\epsilon_2} \times \mathbb{R}^5|_{\epsilon_3;x_{6,7}}$, where four of the spatial directions of the M5-branes are along $\mathbb{R}^4|_{\epsilon_1,\epsilon_2} \subset X^9|_{\epsilon_i}$; $\epsilon_3 = \epsilon_1 + \epsilon_2$; $l = l_{\min}$ is the minimal instanton number of the underlying 4d SU(N) theory along $\mathbb{R}^4|_{\epsilon_1,\epsilon_2}$; the m_k's are the mass parameters associated with the flavor groups; $\Phi_{\vec{\alpha},m_i,\vec{\zeta}}$ is an operator representing the shaded plane which transforms the theory with underlying SU(N) Coulomb moduli $\vec{\alpha}$ to the adjacent theory with underlying SU(N) Coulomb moduli $\vec{\zeta}$; the subscript '$\vec{j}_p$' is the highest weight that defines the primary operators $V_{\vec{j}_p}^Q$ and $V_{\vec{j}_p}$, where the superscript 'Q' means that it also depends on $Q = \epsilon_3/\sqrt{\epsilon_1\epsilon_2}$; $q_r = e^{2\pi i \tau_r}$, where $\tau_r = 4\pi i/g_r^2 + \theta_r/2\pi$ is the complexified gauge coupling associated with the rth gauge group; and the points where the $V_{\vec{j}_p}$'s are inserted are $z = 1, q_1, q_1 q_2, \ldots, q_1 q_2 \ldots q_n$.

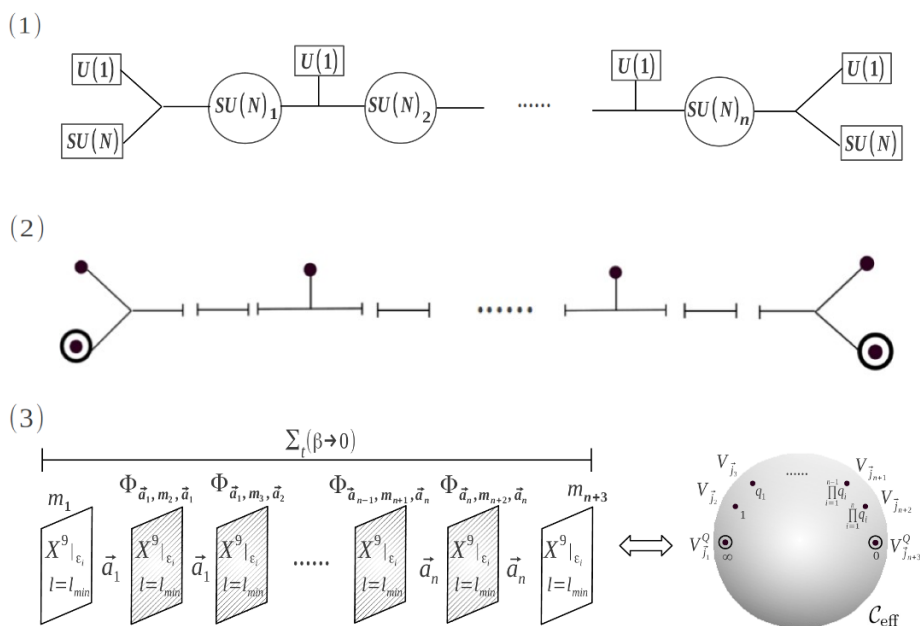[4]See [24, Part II, section 5] for more on this double loop algebra.

Figure 2. (1) The linear quiver diagram; (2) its schematic representation; (3) the resulting effective correspondence.

Clearly, the operator $\Phi_{\vec{\alpha}, m_i, \vec{\zeta}}$ also transforms the space (2.3) of BPS states of the theory with parameter $\vec{\alpha}$ to that of the adjacent theory with parameter $\vec{\zeta}$. Thus, we can also describe Φ as the following map:

$$\Phi_{\vec{\alpha}, m_i, \vec{\zeta}} : \mathcal{H}_{\vec{\alpha}} \to \mathcal{H}_{\vec{\zeta}}, \quad \text{where} \quad \mathcal{H}_{\vec{a}_p} = \bigoplus_l \text{IH}^*_{\text{U}(1)^2 \times T} \mathcal{U}(\mathcal{M}_{\text{SU}(N), l}) \otimes \mathbb{C}(\epsilon_1, \epsilon_2, \vec{a}_p), \quad (3.1)$$

and $\mathcal{H}_{\vec{a}_p}$ is the space of BPS states of the theory with parameter $\vec{a}_p$.

At any rate, in the case of a conformal linear quiver of n SU(N) gauge groups, the expression (2.11) for the 4d Nekrasov instanton partition function ought to be replaced by

$$Z^{\text{lin}}_{\text{inst}}(\mathbf{q}, \epsilon_1, \epsilon_2, \vec{\mathbf{a}}, \mathbf{m}) = \sum_{l_1, l_2 \ldots, l_n} q_1^{l_1} q_2^{l_2} \cdots q_n^{l_n} Z^{\text{lin}}_{\text{BPS}, l_1, l_2, \ldots, l_n}(\epsilon_1, \epsilon_2, \vec{\mathbf{a}}, \mathbf{m}, \beta \to 0), \quad (3.2)$$

where l_i is the instanton number associated with the SU(N)$_i$ gauge group, and $Z^{\text{lin}}_{\text{BPS}, l_1, l_2, \ldots, l_n}$ is the partition function of the aforementioned BPS states associated with the left diagram in figure 2(3). This partition function can be viewed as a sum over BPS states that propagate from the rightmost to the leftmost end of the diagram which undergo transformations of the kind described in (3.1) due to the presence of the shaded planes; in other words, one can also write

$$Z^{\text{lin}}_{\text{inst}}(\mathbf{q}, \epsilon_1, \epsilon_2, \vec{\mathbf{a}}, \mathbf{m}) = {}_{m_1}\langle \emptyset | \Phi_{\vec{a}_1, m_2, \vec{a}_1} \, q_1^{\mathbf{l}_1} \, \Phi_{\vec{a}_1, m_3, \vec{a}_2} \, q_2^{\mathbf{l}_2} \cdots \Phi_{\vec{a}_{n-1}, m_{n+1}, \vec{a}_n} \, q_n^{\mathbf{l}_n} \, \Phi_{\vec{a}_n, m_{n+2}, \vec{a}_n} | \emptyset \rangle_{m_{n+3}}, \quad (3.3)$$

where ${}_{m_1}\langle \emptyset |$ and $| \emptyset \rangle_{m_{n+3}}$ are the minimum energy BPS states at the leftmost and rightmost end of the diagram that are associated with m_1 and m_{n+3}, respectively, while $\mathbf{l}_i$ is an

instanton number operator whose eigenvalue is the instanton number l_i associated with the BPS states.

Note at this point that the duality relation in (2.8), the discussion following it, and the map (3.1), also mean that

$$\Phi_{\vec{\alpha}, m_i, \vec{\zeta}} \colon \mathcal{V}_{\mathbf{j}(\vec{\alpha})} \to \mathcal{V}_{\mathbf{j}(\vec{\zeta})}, \tag{3.4}$$

where $\mathcal{V}_{\mathbf{j}(\vec{a}_p)}$ is a Verma module over an affine $\mathcal{W}_N$-algebra of central charge and highest weight

$$c = (N-1) + (N^3 - N)\frac{\epsilon_3^2}{\epsilon_1 \epsilon_2} \quad \text{and} \quad \mathbf{j}(\vec{a}_p) = \frac{-i\vec{a}_p}{\sqrt{\epsilon_1 \epsilon_2}} + iQ\vec{\rho}, \tag{3.5}$$

with $\vec{\rho}$ being the Weyl vector of $\mathfrak{su}(N)$. Consequently, Φ can also be interpreted as a primary vertex operator V acting on $\mathcal{V}$; this underlies the correspondence between $\Phi_{\vec{\alpha}, m_i, \vec{\zeta}}$ and $V_{\vec{j}_i}$ in figure 2(3). Similarly, the duality relation in (2.8), and the discussion following it, underlie the correspondence between $_{m_1}\langle\emptyset|$ and $|\emptyset\rangle_{m_{n+3}}$ and $\langle V_{\vec{j}_1}^Q|$ and $|V_{\vec{j}_{n+3}}^Q\rangle$ in figure 2(3).

Hence, the correspondence depicted in figure 2(3), and our explanations in the last three paragraphs, mean that we can write

$$Z_{\text{inst}}^{\text{lin}}(\mathbf{q}, \epsilon_1, \epsilon_2, \vec{\mathbf{a}}, \mathbf{m}) = Z^{\text{lin}}(\mathbf{q}, \epsilon_i, \mathbf{m}) \cdot \left\langle V_{\vec{j}_1}^Q(\infty) V_{\vec{j}_2}(1) V_{\vec{j}_3}(q_1) \ldots V_{\vec{j}_{n+2}}(q_1 q_2 \ldots q_n) V_{\vec{j}_{n+3}}^Q(0) \right\rangle_{\mathbf{S}^2}. \tag{3.6}$$

The independence of the factor Z^{lin} on $\vec{\mathbf{a}}$ is because the $\vec{a}_p$'s have already been "contracted" in the correlation function — see the r.h.s. of (3.3).

According to (3.5), the fact that the a_p's and the m_k's have the same dimension, and the fact that $V_{\vec{j}_1}^Q$ and $V_{\vec{j}_{n+3}}^Q$ ought to depend on m_1 and m_{n+3}, respectively, one can conclude that

$$\vec{j}_1 = \frac{-i\vec{m}_1}{\sqrt{\epsilon_1 \epsilon_2}} + \frac{i\vec{\rho}\,\epsilon_3}{\sqrt{\epsilon_1 \epsilon_2}} \quad \text{and} \quad \vec{j}_{n+3} = \frac{-i\vec{m}_{n+3}}{\sqrt{\epsilon_1 \epsilon_2}} + \frac{i\vec{\rho}\,\epsilon_3}{\sqrt{\epsilon_1 \epsilon_2}}, \tag{3.7}$$

where the $N-1$ component vectors $\vec{m}_1$ and $\vec{m}_{n+3}$ depend on m_1 and m_{n+3}.

Similarly, one can conclude, after noting that Q vanishes where the $V_{\vec{j}_u}$ operators are inserted [8], that

$$\vec{j}_u = \frac{-i\vec{m}_u}{\sqrt{\epsilon_1 \epsilon_2}} \quad \text{for} \quad u = 2, 3, \ldots, n+2, \tag{3.8}$$

where the $N-1$ component vector $\vec{m}_i$ depends on m_i.

Last but not least, note that like in section 2.1, we effectively have N D6-branes and 1 D4-brane wrapping $\mathcal{C}_{\text{eff}}$, i.e., we effectively have an $N \times 1 = N$-fold cover Σ_{SW} of $\mathcal{C}_{\text{eff}}$. This is depicted in figure 3. Incidentally, Σ_{SW} is also the Seiberg-Witten curve which underlies $Z_{\text{inst}}^{\text{lin}}(\mathbf{q}, \epsilon_1, \epsilon_2, \vec{\mathbf{a}}, \mathbf{m})$! In fact, Σ_{SW} can be described in terms of the algebraic relation [25]

$$\Sigma_{SW} : \lambda^N + \sum_{k=2}^{N} \lambda^{N-k} \phi_k(z) = 0, \tag{3.9}$$

where $\lambda = y\,dz/z$ (for some complex variable y) is a section of $T^*\mathcal{C}_{\text{eff}}$, and the $\phi_k(z)$'s are $(k, 0)$-holomorphic differentials on $\mathcal{C}_{\text{eff}}$ with poles at the punctures $z = \infty, 1, q_1, q_1 q_2, \ldots$,
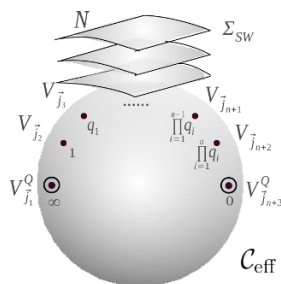
Figure 3. $\mathcal{C}_{\text{eff}}$ and its N-fold cover Σ_{SW} with primary operators inserted at the $n+3$ punctures $z = \infty, 1, q_1, q_1 q_2, \ldots, q_1 q_2 \ldots q_n, 0$.

$q_1 q_2 \ldots q_n$ that are determined by the matter content of the 4d theory. In particular, near the puncture $z = z_s$, we have

$$\phi_2(z) \sim \frac{u_s^{(2)} dz^2}{(z - z_s)^2}, \tag{3.10}$$

and from the correspondence between $\phi_2(z)$ and the holomorphic stress tensor $W^{(2)}(z)$ (established in section 2.1, which thus applies here too), we have

$$W^{(2)}(z) \mathbb{V}_{\vec{j}_s}(z_s) \sim \frac{u_s^{(2)}}{(z - z_s)^2} \mathbb{V}_{\vec{j}_s}(z_s), \tag{3.11}$$

where $\mathbb{V}_{\vec{j}_s}(z_s)$ can be $V_{\vec{j}_s}^Q(z_s)$ or $V_{\vec{j}_s}(z_s)$. In other words, the conformal dimension of the primary operator $\mathbb{V}_{\vec{j}_s}(z_s)$ is equal to $u_s^{(2)}$, i.e., we have

$$\frac{\vec{j}_s^2}{2} - \frac{i\epsilon_3}{\sqrt{\epsilon_1 \epsilon_2}} \vec{j}_s \cdot \vec{\rho} = u_s^{(2)}, \quad \text{where} \quad s = 1, 2, \ldots, n+3, \tag{3.12}$$

from which we can ascertain the explicit form of the mass vectors $\vec{m}_s$ in (3.7) and (3.8).

In arriving at the relations (3.1), (3.4), (3.5), (3.6), (3.7), (3.8) and (3.12), we have just derived the 4d AGT correspondence for a conformal linear quiver of n SU(N) gauge groups.

3.2 An M-theoretic derivation of a 5d AGT correspondence with matter

Let us now proceed to derive a 5d analog of the 4d AGT correspondence with matter. For brevity, and to make contact with relevant results in the mathematical literature, we shall restrict ourselves to an SU(N) theory with $N_f = 2N$ fundamental matter, i.e., $n = 1$ in the last subsection.

A 5d AGT correspondence for U(1) with $N_f = 2$ fundamental matter. It would be useful to first consider the $N = 1$ case, i.e., the U(1) theory with $N_f = 2$ fundamental matter. To this end, first notice that we can also write (2.32) as the three-point correlation function on $\mathbf{S}^2$:

$$Z_{\text{inst}}(\Lambda', \hbar, \vec{a}) = \langle V_0(\infty) V_\Gamma(1) V_0(0) \rangle_{\mathbf{S}^2}, \tag{3.13}$$

where the vertex operators $V_0(\infty)$ and $V_0(0)$ represent the vacuum states $\langle 0|$ and $|0\rangle$, respectively, while the operator

$$V_\Gamma(1) = \exp\left(\frac{\Lambda'}{\hbar}\mathcal{J}_1\right)\exp\left(\frac{\Lambda'}{\hbar}\mathcal{J}_{-1}\right).\tag{3.14}$$

Here, Λ' is the energy scale, and $\mathcal{J}_{\pm 1}$ are operators that obey the Heisenberg algebra (2.27). As a pure U(1) theory can also be interpreted as the $m \to \infty$, $e^{2\pi i \tau} \to 0$ limit of a U(1) theory with an adjoint hypermultiplet matter of mass m and complexified gauge coupling τ, the analysis in [8, section 6.3] would mean that we can regard the V's here as *primary* vertex operators.[5]

Similarly, (2.34) can also be expressed as the three-point correlation function on $\mathbf{S}^2$:

$$Z_{\text{inst}}^{5d}(\Lambda', \hbar, \beta, \vec{a}) = \langle V_0(\infty)\,\tilde{V}_\Gamma(1)\,V_0(0)\rangle_{\mathbf{S}^2},\tag{3.15}$$

where the vertex operator

$$\tilde{V}_\Gamma(1) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\frac{(\beta\Lambda')^n}{1-q^{-n}}\,\mathcal{J}_n\right)\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\frac{(\beta\Lambda')^n}{1-q^n}\,\mathcal{J}_{-n}\right),\tag{3.16}$$

$q = e^{\beta\hbar}$, β is the radius of the 5th dimension circle, and $\mathcal{J}_{\pm m}$ are operators that obey the Heisenberg algebra (2.36). The last paragraph then means that we can regard $\tilde{V}_\Gamma$ as a 5d analog of the 4d primary vertex operator V_Γ. In particular, all 5d analogs of 4d primary vertex operators ought to take a form similar to $\tilde{V}_\Gamma$ in (3.16), where one recalls that $\beta\Lambda'$ is a rescaling-invariant combination of parameters.

Second, note that for general values of $\epsilon_{1,2}$, according to our discussion in section 2.1 leading up to (2.6), and the fact that the smallest connected subgroup of U(1) is itself, the relevant 2d CFT along $\mathcal{C}_{\text{eff}} = \mathbf{S}^2$ is (the chiral half of) a G/G WZW model, where $G = \text{U}(1)$. As explained in section 2.2, any state in the 5d case has a projection onto a circle C_β of radius β in $\mathcal{C}_{\text{eff}}$. Notably, an arbitrary state $|\lambda\rangle$ in the 5d case ought to be represented by a corresponding gauge-invariant loop operator in the G/G WZW model. Now cut out two circles C_β at the poles of $\mathcal{C}_{\text{eff}}$ and wrap a Wilson loop $\text{Tr}_{R_\nu}\exp\oint_{C_\beta} A$ in some representation R_ν around them; this should correspond to the inner product $\langle\lambda|\lambda\rangle$. As the G/G WZW model is topological, one can also put the theory on the annulus, where the monodromies of the restrictions of the G-connection A coincide on the two circle boundaries. The inner product $\langle\lambda|\lambda\rangle$ can then be computed as the path integral over this annulus integrated over the space of all boundary monodromies weighted by $\text{Tr}_{R_\nu}\exp\oint_{C_\beta} A$. In turn, according to [26, section 2.4], $\langle\lambda|\lambda\rangle$ would be expressed in terms of Macdonald polynomials; in other

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[5]This can be understood as follows. Firstly, note that the case of a U(1) theory with an adjoint hypermultiplet matter corresponds to [8, figure 8] with $N = n = 1$. In the limit $e^{2\pi i \tau} = e^{-\frac{8\pi^2}{g^2}+i\theta} \to 0$, i.e., when $\frac{1}{g^2} \to \infty$, the I-brane $\mathcal{C}_{\text{eff}}$ in loc. cit. — after an allowed conformal transformation to bring the two infinitely-long tubes (of length $\frac{1}{g^2}$) back to the origin whilst replacing the resulting holes with vertex operators — would be an $\mathbf{S}^2$ with vertex operators inserted at $z = 0$, 1 and ∞. Then, as explained in loc. cit., the original vertex operator at $z = 1$ is primary, and since the vertex operators at $z = 0, \infty$ are supposed to represent far away states whose excitations have decayed to zero, they must also be primary.

words, there is a correspondence between $|\lambda\rangle$ and Macdonald polynomials.[6] Consequently, according to [14, section 2.3], this means that we can express

$$|\lambda\rangle = a_{-\lambda_1} a_{-\lambda_2} \ldots |\emptyset\rangle, \quad a_{\lambda_i > 0}|\emptyset\rangle = 0, \tag{3.17}$$

where the λ_i's define a partition $\lambda = (\lambda_1, \lambda_2, \cdots)$ associated with the corresponding Macdonald polynomial, $|\emptyset\rangle$ is the vacuum state, while the a_i's are operators obeying a deformed Heisenberg algebra:

$$[a_m, a_n] = m \frac{1 - \tilde{q}^{|m|}}{1 - \tilde{t}^{|m|}} \delta_{m+n,0}, \tag{3.18}$$

with $\tilde{q}$ and $\tilde{t}$ being independent parameters.

Third, from (3.1), [27, Theorem 1], and (3.4), we find that in the $N = 1$ case at hand, the 4d Nekrasov instanton partition function (3.6) would (up to some constant factor) be given by

$$Z_{\text{inst, U(1)}}^{\text{lin}}(q_1, \epsilon_1, \epsilon_2, \vec{m}) = \langle 0 | V_{\tilde{m}_1}(1) V_{\tilde{m}_2}(q_1) | 0 \rangle_{\mathbf{S}^2}. \tag{3.19}$$

Here, $q_1 = e^{2\pi i \tau}$, where τ is the complexified gauge coupling; $\vec{m} = (m_1, m_2)$ are the $N_f = 2$ fundamental matter masses; $\tilde{m}_{1,2}$ are independent linear combinations of $m_{1,2}$; the vacuum states $|0\rangle$ and $\langle 0|$ are represented by the V^Q primary vertex operators inserted at $z = 0$ and ∞, respectively; and the primary vertex operators

$$V_{\tilde{m}_k}(z) = \exp\left[-\frac{i\tilde{m}_k}{\sqrt{\epsilon_1 \epsilon_2}} \varphi_+(z) \right] \exp\left[-\frac{i\tilde{m}_k - (\epsilon_1 + \epsilon_2)}{\sqrt{\epsilon_1 \epsilon_2}} \varphi_-(z) \right], \tag{3.20}$$

where $\varphi(z)$ is a free-boson field with OPE $\varphi(z)\varphi(0) \sim -\ln z$, and $\varphi(z) = \varphi_+(z) + \varphi_-(z)$ is its decomposition into positive and negative modes, i.e.,

$$\varphi_-(z) = -\sum_{n>0} \frac{b_{-n}}{n} z^n \qquad \text{and} \qquad \varphi_+(z) = \sum_{n>0} \frac{b_n}{n} z^{-n}, \tag{3.21}$$

with

$$[b_m, b_n] = m \delta_{m+n,0}. \tag{3.22}$$

In particular, in the limit $\beta \to 0$, the 5d analog of $V_{\tilde{m}_k}$ should reduce to (3.20) with the accompanying relations (3.21) and (3.22).

In sum, the above three points mean that we can express the 5d Nekrasov instanton partition function for U(1) with $N_f = 2$ fundamental matter as

$$\boxed{Z_{\text{inst, U(1)}}^{\text{lin, 5d}}(q_1, \epsilon_1, \epsilon_2, \beta, \vec{m}) = \langle \emptyset | \Phi_v^w(z_1) \Phi_u^v(z_2) | \emptyset \rangle_{\mathbf{S}^2}} \tag{3.23}$$

---

[6]In loc. cit., the relevant theory actually has action $S = S_{G/G} + \int \text{Tr}(\ldots)$, where $S_{G/G}$ is the action of the G/G WZW model. Nevertheless, in the computation of the corresponding path integral (which gives rise to Macdonald polynomials), the contributions coming from the additional term $\int \text{Tr}(\ldots)$ just serve to correct $\langle \lambda | \lambda \rangle$; since a Macdonald polynomial can always be written as a sum of other Macdonald polynomials, $\langle \lambda | \lambda \rangle$ would also be expressed in terms of Macdonald polynomials, as claimed.

where the vacuum states $|\emptyset\rangle$ and $\langle\emptyset|$ represent the 5d analogs of the 4d primary vertex operators V^Q inserted at $z = 0$ and ∞, respectively,

$$\Phi_d^c(z) = \exp\left(\sum_{n>0} \frac{1}{n} \frac{c^{-n} - d^{-n}}{1 - q^{-n}} a_n z^{-n}\right) \exp\left(-\sum_{n>0} \frac{1}{n} \frac{c^n - (t/q)^n d^n}{1 - q^n} a_{-n} z^n\right) \quad (3.24)$$

and the rescaling-invariant parameters

$$w = e^{-\beta m_1}, \quad v = e^{-2\beta(m_1+m_2)}, \quad u = e^{-\beta m_2}, \quad q = e^{-i\beta\sqrt{\epsilon_1\epsilon_2}}, \quad t = e^{-i\beta(\epsilon_1+\epsilon_2+\sqrt{\epsilon_1\epsilon_2})} \quad (3.25)$$

while

$$[a_m, a_n] = m \frac{1 - q^{|m|}}{1 - t^{|m|}} \delta_{m+n,0}, \quad a_{m>0}|\emptyset\rangle = 0 \quad (3.26)$$

Clearly, in arriving at the boxed relations (3.23), (3.24), (3.25) and (3.26), we have just derived a 5d AGT correspondence for U(1) with $N_f = 2$ fundamental matter!

The Ding-Iohara algebra. In fact, (3.23) and (3.24) (bearing in mind (3.26)) coincide with [14, eqs. (2.7) and (2.6)]. As such, $\Phi_d^c(z)$ must be a vertex operator for a level one representation of the Ding-Iohara algebra [13]; in other words, $Z_{\text{inst, U}(1)}^{\text{lin, 5d}}$ is actually expressed in terms of a level one module over the Ding-Iohara algebra. This important observation will soon allow us to connect the $N > 1$ case with a q-deformed affine $\mathcal{W}_N$-algebra.

A 5d AGT correspondence for SU(N) with $N_f = 2N$ fundamental matter. Let us now proceed to consider an SU(N) theory with $N_f = 2N$ fundamental matter. Recall from figure 3 that for general N, we effectively have N D6-branes and 1 D4-brane wrapping $\mathcal{C}_{\text{eff}} = \mathbf{S}^2$ — that is, we effectively have an $N \times 1 = N$-fold cover of $\mathcal{C}_{\text{eff}}$. This means that we ought to make the following replacements in (3.23) to get the corresponding result for the SU(N) case.

Firstly, the vacuum states $\langle\emptyset|$ and $|\emptyset\rangle$ ought to be replaced by their N-tensor product:

$$\langle\mathbf{0}| = \langle\emptyset|^{\otimes N} \quad \text{and} \quad |\mathbf{0}\rangle = |\emptyset\rangle^{\otimes N} \quad (3.27)$$

Secondly, the vertex operators Φ_v^w and Φ_u^v ought to likewise be replaced by their N-tensor product

$$\Phi_{\mathbf{v}}^{\mathbf{w}}(z_1) = \Phi_{v_N}^{w_N} \cdots \Phi_{v_1}^{w_1}(z_1) \quad \text{and} \quad \Phi_{\mathbf{u}}^{\mathbf{v}}(z_2) = \Phi_{u_1}^{v_1} \cdots \Phi_{u_N}^{v_N}(z_2) \quad (3.28)$$

where $\mathbf{c} = (c_1, c_2, \ldots, c_N)$.

In sum, we can express the 5d Nekrasov instanton partition function for SU(N) with $N_f = 2N$ fundamental matter as

$$Z_{\text{inst, SU}(N)}^{\text{lin, 5d}}(q_1, \epsilon_1, \epsilon_2, \beta, \vec{a}, \vec{m}) = \langle\mathbf{0}|\Phi_{\mathbf{v}}^{\mathbf{w}}(z_1)\Phi_{\mathbf{u}}^{\mathbf{v}}(z_2)|\mathbf{0}\rangle_{\mathbf{S}^2} \quad (3.29)$$

and if the vertex operator Φ_d^c defines a map $\mathcal{F}_d \to \mathcal{F}_c$ from one Fock space associated with parameter d to another associated with parameter c, then

$$\Phi_{\mathbf{d}}^{\mathbf{c}} : \mathcal{F}_{d_1} \otimes \mathcal{F}_{d_2} \otimes \cdots \otimes \mathcal{F}_{d_N} \longrightarrow \mathcal{F}_{c_1} \otimes \mathcal{F}_{c_2} \otimes \cdots \otimes \mathcal{F}_{c_N}. \quad (3.30)$$

Note that in the U(1) case, the dependence of Φ_d^c on the Coulomb modulus actually drops out where it is replaced by a linear combination of the masses $m_{1,2}$ which appears in the parameter v in (3.25). In other words, in the SU(N) case, $\mathbf{v} = (v_1, \ldots, v_N)$ would be associated with the Coulomb moduli $\vec{a} = (a_1, \ldots, a_N)$ (where $\sum_i a_i = 0$), while $\mathbf{w} = (w_1, \ldots, w_N)$ and $\mathbf{u} = (u_1, \ldots, u_N)$ would continue to be associated with the $2N$ masses $\vec{m} = (m_1, m_2, \ldots, m_{2N})$. As the parameters $\mathbf{w}, \mathbf{v}, \mathbf{u}$ must reduce to w, v, u in (3.25) when $N = 1$, we conclude that

$$\boxed{w_i = e^{-\beta m_i}, \quad v_i = e^{-\beta a_i}, \quad u_i = e^{-\beta m_{N+i}}} \tag{3.31}$$

As mentioned, Φ_d^c is a vertex operator of a level one representation of the Ding-Iohara algebra — that is, $\mathcal{F}_{c,d}$ is a level one module over the Ding-Iohara algebra. As such, (3.30) means that $\Phi_{\mathbf{d}}^{\mathbf{c}}$ is a vertex operator of a level N representation of the Ding-Iohara algebra [14]. In turn, this means that

$$\boxed{\Phi_{\mathbf{d}}^{\mathbf{c}} : \widehat{\mathcal{W}_N^q} \to \widehat{\mathcal{W}_N^q}} \tag{3.32}$$

where $\widehat{\mathcal{W}_N^q}$ is a Verma module over $\mathcal{W}_N^q$, a q-deformed affine $\mathcal{W}_N$-algebra [28].[7]

Clearly, in arriving at the boxed relations (3.27), (3.28), (3.29), (3.31) and (3.32), we have just derived a 5d AGT correspondence for SU(N) with $N_f = 2N$ fundamental matter! This serves as a purely physical M-theoretic proof of [14, Conjecture 3.13], and also lends further weight to the interesting analysis in [29] (which is based on the assumption that the correspondence holds).

3.3 An M-theoretic derivation of a 5d pure AGT correspondence

Let us now proceed to derive a 5d analog of the 4d pure AGT correspondence. Again, it would be useful to start with the U(1) case.

A 5d AGT correspondence for pure U(1). As mentioned in the previous subsection, since a pure U(1) theory can also be interpreted as the $m \to \infty$, $e^{2\pi i \tau'} \to 0$ limit of a U(1) theory with an adjoint hypermultiplet matter of mass m and complexified gauge coupling τ', where $me^{2\pi i \tau'} = \Lambda$ remains fixed, the analysis in [8, section 6.3] and footnote 5 would mean that the 5d Nekrasov instanton partition function for pure U(1) can be expressed as

$$Z_{\text{inst, U(1)}}^{\text{pure, 5d}}(\epsilon_1, \epsilon_2, \beta, \Lambda) = \langle \emptyset | \Phi_{m \to \infty}(1) | \emptyset \rangle_{\mathbf{S}^2}, \tag{3.33}$$

where $\Phi_{m \to \infty}(1)$ is the 5d analog of the 4d primary vertex operator $V_m(1)$ in (3.20) in the $m \to \infty$ limit. In other words, according to our derivation of (3.24)–(3.26) from (3.20) etc., we find that

$$\Phi_{m \to \infty}(1) = \exp\left(\sum_{n>0} \frac{1}{n} \frac{c^{-n} - d^{-n}}{1 - q^{-n}} a_n\right) \exp\left(-\sum_{n>0} \frac{1}{n} \frac{c^n - (t/q)^n d^n}{1 - q^n} a_{-n}\right), \tag{3.34}$$

---

[7]In loc. cit., it was actually shown that the aforementioned level N representation of the Ding-Iohara algebra is realized by a representation of $\mathcal{W}_N^q \otimes \mathfrak{u}(1)_{\text{aff}}$. However, as explained in [8, section 5.2], because the U(1) gauge field on the D4-brane in (2.5) — unlike the SU(N) gauge field on the D6-branes which it intersects — is dynamical, one has to reduce away in the I-brane system the U(1) WZW model associated with the D4-brane. This is equivalent to reducing away the $\mathfrak{u}(1)_{\text{aff}}$ factor, whence (3.32) is indeed consistent.

so we can write

$$Z^{\text{pure},5d}_{\text{inst, U(1)}}(\epsilon_1, \epsilon_2, \beta, \Lambda) = \langle G_{\text{U(1)}} | \mathscr{G}_{\text{U(1)}} | G_{\text{U(1)}} \rangle, \tag{3.35}$$

where

$$\mathscr{G}_{\text{U(1)}} = \exp\left(\sum_{n>0} \frac{1}{n} \frac{d^n(t^n - q^n)}{q^n(1-q^n)} a_{-n} \right), \tag{3.36}$$

$$|G_{\text{U(1)}}\rangle = \exp\left(-\sum_{n>0} \frac{1}{n} \frac{c^n - d^n}{1-q^n} a_{-n} \right) |\emptyset\rangle, \quad \langle G_{\text{U(1)}}| = \langle\emptyset| \exp\left(\sum_{n>0} \frac{1}{n} \frac{c^{-n} - d^{-n}}{1-q^{-n}} a_n \right), \tag{3.37}$$

and the rescaling-invariant parameters $c = e^{-\beta(m \to \infty)}$, $d = e^{-2\beta(m \to \infty)}$, $q = e^{-i\beta\sqrt{\epsilon_1\epsilon_2}}$, $t = e^{-i\beta(\epsilon_1+\epsilon_2+\sqrt{\epsilon_1\epsilon_2})}$, while $[a_p, a_n] = p\frac{1-q^{|p|}}{1-t^{|p|}}\delta_{p+n,0}$.

Nonetheless, $d \to 0$ so $\mathscr{G}_{\text{U(1)}} \to 1$ whence we can write (3.35) as

$$\boxed{Z^{\text{pure},5d}_{\text{inst, U(1)}}(\epsilon_1, \epsilon_2, \beta, \Lambda) = \langle G_{\text{U(1)}} | G_{\text{U(1)}} \rangle} \tag{3.38}$$

Also, note that (i) for $n > 0$, we have $(c^{-n} - c^n) = (d^{-n} - d^n) \to \infty$, i.e., $(c^{-n} - d^{-n}) = (c^n - d^n)$; (ii) since c and d are rescaling-invariant, so must $(c^n - d^n)$; (iii) for $(c^n - d^n)$ to be well-defined as $m \to \infty$, it would mean that m in $(c^n - d^n)$ ought to be replaced by a physically equivalent parameter such that $(c^n - d^n)$ continues to be well-defined in this limit (although its dependence on β and independence of $\epsilon_{1,2}$ should remain) — such a parameter is the energy scale Λ, which is fixed as $m \to \infty$; (iv) (2.15) and (2.17) mean that the nth power of Λ ought to accompany the state of energy level n in $|G_{\text{U(1)}}\rangle$. Altogether, this means that we can also write (3.37) as

$$\boxed{|G_{\text{U(1)}}\rangle = \exp\left(-\sum_{n>0} \frac{1}{n} \frac{(\beta\Lambda)^n}{1-q^n} a_{-n} \right) |\emptyset\rangle, \quad \langle G_{\text{U(1)}}| = \langle\emptyset| \exp\left(\sum_{n>0} \frac{1}{n} \frac{(\beta\Lambda)^n}{1-q^{-n}} a_n \right)} \tag{3.39}$$

where

$$\boxed{[a_p, a_n] = p\frac{1-q^{|p|}}{1-t^{|p|}}\delta_{p+n,0}, \quad a_{p>0}|\emptyset\rangle = 0} \tag{3.40}$$

and

$$\boxed{q = e^{-i\beta\sqrt{\epsilon_1\epsilon_2}}, \quad t = e^{-i\beta(\epsilon_1+\epsilon_2+\sqrt{\epsilon_1\epsilon_2})}} \tag{3.41}$$

Clearly, in arriving at the boxed relations (3.38), (3.39), (3.40) and (3.41), we have just derived a 5d AGT correspondence for pure U(1)!

Moreover, notice that $|G_{\text{U(1)}}\rangle$ takes the same form as [14, eq. (5.1)] (where (α, β) in loc. cit. correspond to $(\beta\Lambda, 0)$). In other words, $|G_{\text{U(1)}}\rangle$ is a *coherent state* in a level one module over a Ding-Iohara algebra. Thus, we have a U(1) version of the SU(2) result in [15].

Rederiving the 5d AGT correspondence for pure U(1) in the topological string limit. In the topological string limit of $\epsilon_1 = -\epsilon_2 = \hbar$, we have from (3.41), the relation $q = t$. As such, we can write the 5d Nekrasov instanton partition function for pure U(1) in this limit as

$$\boxed{Z^{\text{pure},5d,\hbar}_{\text{inst, U(1)}}(\hbar, \beta, \Lambda) = \langle G^{\hbar}_{\text{U(1)}} | G^{\hbar}_{\text{U(1)}} \rangle} \tag{3.42}$$

where

$$\langle G^{\hbar}_{\mathrm{U}(1)}| = \langle 0| \exp\left(\sum_{n>0} \frac{1}{n} \frac{(\beta\Lambda)^n}{1-q^{-n}} \mathcal{J}_n\right), \quad |G^{\hbar}_{\mathrm{U}(1)}\rangle = \exp\left(-\sum_{n>0} \frac{1}{n} \frac{(\beta\Lambda)^n}{1-q^n} \mathcal{J}_{-n}\right)|0\rangle, \quad q = e^{\beta\hbar}$$

(3.43)

and[8]

$$[\mathcal{J}_p, \mathcal{J}_n] = p\,\delta_{p+n,0}, \quad \mathcal{J}_p|0\rangle = 0 \quad \text{for} \quad p \geq 0$$

(3.44)

Clearly, in arriving at the boxed relations (3.42), (3.43) and (3.44), we have just rederived a 5d AGT correspondence for pure U(1) in the topological string limit. The above three relations coincide with (2.34)–(2.36), as expected.

Level one representation of the Ding-Iohara algebra and the double loop algebra of U(1). Just like $|G_{\mathrm{U}(1)}\rangle$, $|G^{\hbar}_{\mathrm{U}(1)}\rangle$ is a coherent state or Whittaker vector in a level one module over a Ding-Iohara algebra with $c, d \to 0$, albeit at $q = t$. Comparing (3.42) with (2.33), we find a correspondence between $|G^{\hbar}_{\mathrm{U}(1)}\rangle$ and $|\mathrm{cir}'\rangle$; in turn, (2.39) means that *a Whittaker vector in a level one module over a Ding-Iohara algebra with $c, d \to 0$ and $q = t$, is also a Whittaker vector in a module over a universal central extension of the double loop algebra of* U(1)!

A 5d AGT correspondence for pure SU(N). Likewise, the pure SU(N) case with Coulomb moduli $\vec{a} = (a_1, \ldots, a_N)$ can also be obtained by making the replacements (3.27)–(3.28) in (3.33):

$$Z^{\mathrm{pure},\,5d}_{\mathrm{inst},\,\mathrm{SU}(N)}(\epsilon_1, \epsilon_2, \vec{a}, \beta, \Lambda) = \langle \mathbf{0}|\Phi^{\otimes N}_{m\to\infty}(1)|\mathbf{0}\rangle_{\mathbf{S}^2}.$$

(3.45)

Recall from the previous subsection that $\Phi_{m\to\infty}$ is also a vertex operator of a level one representation of the Ding-Iohara algebra, so $\Phi^{\otimes N}_{m\to\infty}$ is a vertex operator of a level N representation of the Ding-Iohara algebra, or rather

$$\Phi^{\otimes N}_{m\to\infty} : \widehat{\mathcal{W}^q_N} \to \widehat{\mathcal{W}^q_N}.$$

(3.46)

Explicitly,

$$\Phi^{\otimes N}_{m\to\infty}(1) = e^{\sum_{n_1>0}\left(\frac{c^{-n_1}-d^{-n_1}}{1-q^{-n_1}}\frac{a_{n_1}}{n_1} - \frac{c^{n_1}-(t/q)^{n_1}d^{n_1}}{1-q^{n_1}}\frac{a_{-n_1}}{n_1}\right)}\ldots$$
$$\times\, e^{\sum_{n_N>0}\left(\frac{c^{-n_N}-d^{-n_N}}{1-q^{-n_N}}\frac{a_{n_N}}{n_N} - \frac{c^{n_N}-(t/q)^{n_N}d^{n_N}}{1-q^{n_N}}\frac{a_{-n_N}}{n_N}\right)},$$

(3.47)

so we can write

$$Z^{\mathrm{pure},\,5d}_{\mathrm{inst},\,SU(N)}(\epsilon_1, \epsilon_2, \vec{a}, \beta, \Lambda) = \langle G_{\mathrm{SU}(N)}|\mathscr{G}_{\mathrm{SU}(N)}|G_{\mathrm{SU}(N)}\rangle,$$

(3.48)

where

$$\mathscr{G}_{\mathrm{SU}(N)} = e^{\sum_{n_1>0}\frac{1}{n_1}\frac{d^{n_1}(t^{n_1}-q^{n_1})}{q^{n_1}(1-q^{n_1})}a_{-n_1}}\ldots e^{\sum_{n_N>0}\frac{1}{n_N}\frac{d^{n_N}(t^{n_N}-q^{n_N})}{q^{n_N}(1-q^{n_N})}a_{-n_N}},$$

(3.49)

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[8]In the following, we have $\mathcal{J}_0|0\rangle = 0$ because the Heisenberg algebra is realized by a single complex chiral fermion on the I-brane $\mathcal{C}_{\mathrm{eff}} = \mathbf{S}^2$.

$$|G_{\text{SU}(N)}\rangle = (e^{-\sum_{n_1>0} \frac{1}{n_1} \frac{c^{n_1}-d^{n_1}}{1-q^{n_1}} a_{-n_1}} \cdots e^{-\sum_{n_N>0} \frac{1}{n_N} \frac{c^{n_N}-d^{n_N}}{1-q^{n_N}} a_{-n_N}}) \cdot (|\emptyset\rangle_1 \otimes \cdots \otimes |\emptyset\rangle_N), \tag{3.50}$$

$$\langle G_{\text{SU}(N)}| = (_N\langle\emptyset| \otimes \cdots \otimes {}_1\langle\emptyset|) \cdot (e^{\sum_{n_N>0} \frac{1}{n_N} \frac{c^{-n_N}-d^{-n_N}}{1-q^{-n_N}} a_{n_N}} \cdots e^{\sum_{n_1>0} \frac{1}{n_1} \frac{c^{-n_1}-d^{-n_1}}{1-q^{-n_1}} a_{n_1}}). \tag{3.51}$$

Here, the a_{-n_k}'s and a_{n_k}'s act nontrivially only on $|\emptyset\rangle_k$ and $_k\langle\emptyset|$, respectively; the rescaling-invariant parameters $c = e^{-\beta(m\to\infty)}$, $d = e^{-2\beta(m\to\infty)}$, $q = e^{-i\beta\sqrt{\epsilon_1\epsilon_2}}$, $\quad t = e^{-i\beta(\epsilon_1+\epsilon_2+\sqrt{\epsilon_1\epsilon_2})}$; while $a_{m_k>0}|\emptyset\rangle_k = 0$, $_k\langle\emptyset|a_{m_k<0} = 0$, and $[a_{m_k}, a_{n_k}] = m_k \frac{1-q^{|m_k|}}{1-t^{|m_k|}} \delta_{m_k+n_k,0}$.

Nonetheless, $d \to 0$ so $\mathscr{G} \to 1$ whence we can write (3.48) as

$$\boxed{Z_{\text{inst, SU}(N)}^{\text{pure}, 5d}(\epsilon_1, \epsilon_2, \vec{a}, \beta, \Lambda) = \langle G_{\text{SU}(N)}|G_{\text{SU}(N)}\rangle} \tag{3.52}$$

Also, note that (i) for $n_k > 0$, we have $(c^{-n_k}-c^{n_k}) = (d^{-n_k}-d^{n_k}) \to \infty$, i.e., $(c^{-n_k}-d^{-n_k}) = (c^{n_k}-d^{n_k})$; (ii) since c and d are rescaling-invariant, so must $(c^{n_k}-d^{n_k})$; (iii) for $(c^{n_k}-d^{n_k})$ to be well-defined as $m \to \infty$, it would mean that m in $(c^{n_k}-d^{n_k})$ ought to be replaced by a physically equivalent parameter such that $(c^{n_k}-d^{n_k})$ continues to be well-defined in this limit (although its dependence on β and independence of $\epsilon_{1,2}$ should remain) — such a parameter is the energy scale Λ, which is fixed as $m \to \infty$; (iv) (2.15) and (2.17) mean that the n_k^{th} power of Λ ought to accompany the state of energy level n_k in $|G_{\text{SU}(N)}\rangle$ that is created from $|\emptyset\rangle_k$. Altogether, this means that we can also write (3.50) and (3.51) as

$$\boxed{|G_{\text{SU}(N)}\rangle = (e^{-\sum_{n_1>0} \frac{1}{n_1} \frac{(\beta\Lambda)^{n_1}}{1-q^{n_1}} a_{-n_1}} \cdots e^{-\sum_{n_N>0} \frac{1}{n_N} \frac{(\beta\Lambda)^{n_N}}{1-q^{n_N}} a_{-n_N}}) \cdot (|\emptyset\rangle_1 \otimes \cdots \otimes |\emptyset\rangle_N)} \tag{3.53}$$

and

$$\boxed{\langle G_{\text{SU}(N)}| = (_N\langle\emptyset| \otimes \cdots \otimes {}_1\langle\emptyset|) \cdot (e^{\sum_{n_N>0} \frac{1}{n_N} \frac{(\beta\Lambda)^{n_N}}{1-q^{-n_N}} a_{n_N}} \cdots e^{\sum_{n_1>0} \frac{1}{n_1} \frac{(\beta\Lambda)^{n_1}}{1-q^{-n_1}} a_{n_1}})} \tag{3.54}$$

where

$$\boxed{[a_{m_k}, a_{n_k}] = m_k \frac{1-q^{|m_k|}}{1-t^{|m_k|}} \delta_{m_k+n_k,0}, \quad a_{m_k>0}|\emptyset\rangle_k = 0} \tag{3.55}$$

and

$$\boxed{q = e^{-i\beta\sqrt{\epsilon_1\epsilon_2}}, \quad t = e^{-i\beta(\epsilon_1+\epsilon_2+\sqrt{\epsilon_1\epsilon_2})}} \tag{3.56}$$

Clearly, in arriving at the boxed relations (3.52), (3.53), (3.54), (3.55) and (3.56), we have just derived a 5d AGT correspondence for pure SU(N)!

Moreover, (3.46), and the fact that $|G_{\text{SU}(N)}\rangle$ is a sum over states of all possible energy levels (see (3.53)), mean that $|G_{\text{SU}(N)}\rangle$ is a *coherent state* in a Verma module over $\mathcal{W}_N^q$, a *q*-deformed affine $\mathcal{W}_N$-algebra. Thus, we have an SU(N) generalization of the result for SU(2) in [15]!

An equivariant index of the Dirac operator on instanton moduli space and a Whittaker function on a *q*-deformed affine $\mathcal{W}_N$-algebra. Recall at this point from section 2.1 that since the BPS spectrum $\mathcal{H}_{\text{BPS}}^\Omega$ in (2.3) arises from the *topological* sector

of a (non-dynamically) **g**-gauged supersymmetric sigma model on $\Sigma_t = \mathbf{S}^1 \times \mathbb{I}_t$, it would coincide with the spectrum of the resulting (non-dynamically) **g**-gauged supersymmetric quantum mechanical instanton as $\Sigma_t \to$ point.[9] As such, $Z'_{\mathrm{BPS},m}(\epsilon_1, \epsilon_2, \vec{a}, \beta \to 0)$ in (2.12) can also be regarded as a partition function associated with a (non-dynamically) **g**-gauged supersymmetric quantum mechanical instanton with target $\mathcal{U}(\mathcal{M}_{\mathrm{SU}(N),m})$. In turn, this means that Z_{BPS} in (2.4) which underlies $Z^{\mathrm{pure},5d}_{\mathrm{inst},\mathrm{SU}(N)}$ in (3.52), can be regarded as a partition function associated with a (non-dynamically) **g**-gauged supersymmetric quantum mechanical instanton with target $L\mathcal{U}(\mathcal{M}_{\mathrm{SU}(N)})$, the loop space of $\mathcal{U}(\mathcal{M}_{\mathrm{SU}(N)})$. Consequently, according to [30], $Z^{\mathrm{pure},5d}_{\mathrm{inst},\mathrm{SU}(N)}$ ought to be given by Z_{inst} in (2.11) but with the underlying integrand multiplied by $\widehat{A}^{\mathbf{g}}_\beta(\mathcal{U}(\mathcal{M}_{\mathrm{SU}(N)}))$, the **g**-equivariant A-roof genus of $\mathcal{U}(\mathcal{M}_{\mathrm{SU}(N)})$ with expansion parameter β. Hence, we can also write (3.52) as (cf. [31])

$$\boxed{\sum_{m=0}^{\infty} \Lambda^{2mN} \oint_{\mathcal{U}(\mathcal{M}_{\mathrm{SU}(N),m})} \widehat{A}^{\mathbf{g}(\epsilon_i, \vec{a})}_\beta(\mathcal{U}(\mathcal{M}_{\mathrm{SU}(N),m})) = \mathscr{W}_{G_{\mathrm{SU}(N)}}}$$
(3.57)

where $\oint_{\mathcal{M}} \widehat{A}^{\mathbf{g}}_\beta(\mathcal{M})$ is the **g**-equivariant index of the Dirac operator on $\mathcal{M}$, and $\mathscr{W}_{G_{\mathrm{SU}(N)}}$ is a Whittaker function on $\mathcal{W}^q_N$.

Rederiving the 5d AGT correspondence for pure $\mathrm{SU}(N)$ in the topological string limit. In the topological string limit of $\epsilon_1 = -\epsilon_2 = \hbar$, Omega-deformation on the r.h.s. of (2.1) effectively vanishes. As explained in section 2.2, the partially gauged chiral CFT behind (2.6) would be ungauged, whence the N complex chiral fermions on the I-brane would (i) realize a level one affine algebra $\mathfrak{su}(N)_{\mathrm{aff},1}$; (ii) generate, via their convenient packing into a single complex chiral fermion Ψ and its bosonization, a Heisenberg algebra.

From (3.56), we find that if $\epsilon_1 = -\epsilon_2 = \hbar$, then $q = t$. In turn, (3.55) becomes $[a_{m_k}, a_{n_k}] = m_k \delta_{m_k+n_k,0}$, which is also a Heisenberg algebra. Therefore, point (ii) above would mean that we can regard the N-tensor product of vacuum states $|\emptyset\rangle_1 \otimes \cdots \otimes |\emptyset\rangle_N$ as a *single* vacuum state $|p\rangle$, whence there is no longer a distinction between the individual vacuum states $|\emptyset\rangle_k$ and the corresponding operators a_{n_k} that act nontrivially on them. As such, we can write (3.49)–(3.51) in this topological string limit as

$$\mathscr{G}^\hbar_{\mathrm{SU}(N)} = e^{\sum_{n>0} \frac{N}{n} \frac{d^n(t^n - q^n)}{q^n(1-q^n)} a_{-n}}, \qquad |G^\hbar_{\mathrm{SU}(N)}\rangle = e^{-\sum_{n>0} \frac{1}{n} \frac{N(c^n - d^n)}{1-q^n} a_{-n}} |p\rangle,$$
$$\langle G^\hbar_{\mathrm{SU}(N)}| = \langle p| e^{\sum_{n>0} \frac{1}{n} \frac{N(c^{-n} - d^{-n})}{1-q^{-n}} a_n}.$$
(3.58)

Here, $a_{n>0}|p\rangle = 0$, $\langle p|a_{n<0} = 0$, and $[a_m, a_n] = m\delta_{m+n,0}$.

As before, $d \to 0$ so $\mathscr{G} \to 1$. Also, note that (i) for $n > 0$, we have $N(c^{-n} - c^n) = N(d^{-n} - d^n) \to \infty$, i.e., $N(c^{-n} - d^{-n}) = N(c^n - d^n)$; (ii) since c and d are rescaling-invariant, so must $N(c^n - d^n)$; (iii) for $N(c^n - d^n)$ to be well-defined as $m \to \infty$, it would mean that m in $N(c^n - d^n)$ ought to be replaced by a physically equivalent parameter such that $N(c^n - d^n)$ continues to be well-defined in this limit (although its dependence on N, β and independence of $\epsilon_{1,2}$ should remain) — such a parameter is the energy scale

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[9]Here, $\mathbf{g} \in \mathrm{U}(1) \times \mathrm{U}(1) \times T$, where $T \subset \mathrm{SU}(N)$ is a Cartan subgroup.

Λ, which is fixed as $m \to \infty$; (iv) (2.15) and (2.17) mean that the nth power of Λ^N ought to accompany the state of energy level n in $|G_{\text{SU}(N)}^{\hbar}\rangle$. Altogether, this means that we can also write (3.48) in this topological string limit as

$$\boxed{Z_{\text{inst, SU}(N)}^{\text{pure}, 5d, \hbar}(\hbar, \vec{a}, \beta, \Lambda') = \langle G_{\text{SU}(N)}^{\hbar}|G_{\text{SU}(N)}^{\hbar}\rangle} \tag{3.59}$$

where

$$\langle G_{\text{SU}(N)}^{\hbar}| = \langle p| \exp\left(\sum_{n>0} \frac{1}{n} \frac{(\beta\Lambda')^n}{1-q^{-n}} \mathcal{J}_n\right), \quad |G_{\text{SU}(N)}^{\hbar}\rangle = \exp\left(-\sum_{n>0} \frac{1}{n} \frac{(\beta\Lambda')^n}{1-q^n} \mathcal{J}_{-n}\right)|p\rangle,$$
$$q = e^{\beta\hbar} \tag{3.60}$$

$\Lambda' = \Lambda^N$, and[10]

$$\boxed{[\mathcal{J}_m, \mathcal{J}_n] = m\, \delta_{m+n,0}, \qquad \mathcal{J}_0|p\rangle = p \sim \sqrt{\vec{a}^2/\hbar^2}, \qquad \mathcal{J}_m|p\rangle = 0 \quad \text{for} \quad m > 0} \tag{3.61}$$

Clearly, in arriving at the boxed relations (3.59), (3.60) and (3.61), we have just rederived a 5d AGT correspondence for pure $\text{SU}(N)$ in the topological string limit. The above three relations coincide with (2.34)–(2.36), as expected.

Level N representation of the Ding-Iohara algebra and the double loop algebra of $\text{SU}(N)$. Just like $|G_{\text{SU}(N)}\rangle$, $|G_{\text{SU}(N)}^{\hbar}\rangle$ is a coherent state or Whittaker vector in a level N module over a Ding-Iohara algebra with $c, d \to 0$, albeit at $q = t$. Comparing (3.59) with (2.33), we find a correspondence between $|G_{\text{SU}(N)}^{\hbar}\rangle$ and $|\text{cir}'\rangle$; in turn, (2.39) means that *a Whittaker vector in a level N module over a Ding-Iohara algebra with $c, d \to 0$ and $q = t$, is also a Whittaker vector in a module over a universal central extension of the double loop algebra of $\text{SU}(N)$!*

4 A 5d "fully-ramified" AGT correspondence and relativistic integrable systems

4.1 An M-theoretic derivation of a 4d "fully-ramified" pure AGT correspondence: a review

Let us now review the M-theoretic derivation of a 4d "fully-ramified" AGT correspondence for pure $\text{SU}(N)$ in [8].

To this end, note that the "fully-ramified" version of the AGT correspondence for pure $\text{SU}(N)$ can be obtained by considering the following duality setup which is a slight variant of (2.1):

$$\underbrace{\mathbb{R}^4|_{\epsilon_1, \epsilon_2} \times \Sigma_t}_{N \text{ M5-branes} + 4d \text{ full defect}} \times \mathbb{R}^5|_{\epsilon_3; x_{6,7}} \quad \Longleftrightarrow \quad \mathbb{R}^5|_{\epsilon_3; x_{4,5}} \times \underbrace{\mathcal{C} \times TN_N^{R\to 0}|_{\epsilon_3; x_{6,7}}}_{1 \text{ M5-branes} + 4d \text{ full defect}} . \tag{4.1}$$

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[10]In the following, we have $\mathcal{J}_0|p\rangle = p$, where $p \sim \sqrt{\vec{a}^2/\hbar^2}$ is the energy level of $|p\rangle$, because the Heisenberg algebra is realized by N complex chiral fermions on the I-brane $\mathcal{C}_{\text{eff}} = \mathbf{S}^2$.

Here, the worldvolume 4d full defects are effected by an appropriate orbifold background (see [8, section 2.3]); we have a common half-BPS boundary condition at the tips of $\mathbb{I}_t \subset \Sigma_t = \mathbf{S}^1 \times \mathbb{I}_t$; the radius of $\mathbf{S}^1$ is β; $\mathbb{I}_t \ll \beta$; $\mathcal{C}$ is a priori the same as Σ_t; the 4d world-volume full defect on the l.h.s. of (4.1) wraps Σ_t and the z-plane in $\mathbb{R}^4|_{\epsilon_1,\epsilon_2} \simeq \mathbb{C}_z|_{\epsilon_1} \times \mathbb{C}_w|_{\epsilon_2}$; the *dual* 4d worldvolume full defect on the r.h.s. of (4.1) wraps $\mathcal{C}$ and the x_8-x_9 directions in $TN_N^{R\to 0}|_{\epsilon_3;x_{6,7}}$, where the x_9-direction is spanned by the $\mathbf{S}^1$-fiber of $TN_N^{R\to 0}|_{\epsilon_3;x_{6,7}}$, while the x_6-x_7-x_8-directions are spanned by its $\mathbb{R}^3|_{\epsilon_3;x_{6,7}}$ base; and the ϵ_i's are parameters of the Omega-deformation along the indicated planes described in (2.2).

The spectrum of spacetime BPS states on the l.h.s. of (4.1). Let us now determine the spectrum of spacetime BPS states on the l.h.s. of (4.1) that defines a "fully-ramified" generalization of the partition function in (2.4). As explained in detail in [8, section 6.1], we can express the Hilbert space $\mathcal{H}_{\mathrm{BPS}}^{\Omega}$ of spacetime BPS states on the l.h.s. of (4.1) as

$$\mathcal{H}_{\mathrm{BPS}}^{\Omega} = \bigoplus_{a'} \mathcal{H}_{\mathrm{BPS},a'}^{\Omega} = \bigoplus_{a'} \mathrm{IH}_{\mathrm{U}(1)^2 \times T}^* \mathcal{U}(\mathcal{M}_{\mathrm{SU}(N),T,a'}), \tag{4.2}$$

where $\mathrm{IH}_{\mathrm{U}(1)^2 \times T}^* \mathcal{U}(\mathcal{M}_{\mathrm{SU}(N),T,a'})$ is the $\mathrm{U}(1)^2 \times T$-equivariant intersection cohomology of the Uhlenbeck compactification $\mathcal{U}(\mathcal{M}_{\mathrm{SU}(N),T,a'})$ of the (singular) moduli space $\mathcal{M}_{\mathrm{SU}(N),T,a'}$ of "fully-ramified" $\mathrm{SU}(N)$-instantons on $\mathbb{R}^4$ with "fully-ramified" instanton number a'. Here, the positive number $a' = a + \mathrm{Tr}\,\alpha\mathfrak{m}$, where a is the ordinary instanton number; Tr is a quadratic form on $\mathfrak{su}(N)$; $\alpha \in \mathfrak{t}$ is the holonomy parameter that is the commutant of the Cartan subgroup $T \subset \mathrm{SU}(N)$; $\mathfrak{t}$ is the Lie algebra of T; $\mathfrak{m} \in \Lambda_{\mathrm{cochar}}$ is the "magnetic charge"; and $\Lambda_{\mathrm{cochar}}$ is the cocharacter lattice of $\mathrm{SU}(N)$.

The spectrum of spacetime BPS states on the r.h.s. of (4.1). Let us next ascertain the corresponding spectrum of spacetime BPS states on the r.h.s. of (4.1). According to [8, section 6.1], the spacetime BPS states would be furnished by the "fully-ramified" I-brane theory in the following type IIA configuration:

$$\text{IIA}: \qquad \underbrace{\mathbb{R}^5|_{\epsilon_3;x_{4,5}} \times \mathcal{C} \times \mathbb{R}^3|_{\epsilon_3;x_{6,7}}}_{\text{I-brane on } \mathcal{C} = N\mathrm{D6} \cap 1\mathrm{D4} \cap 3\mathrm{d\,full\,defect}} \qquad . \tag{4.3}$$

Here, we have a stack of N coincident D6-branes whose worldvolume is given by $\mathbb{R}^5|_{\epsilon_3;x_{4,5}} \times \mathcal{C}$; a single D4-brane whose worldvolume is given by $\mathcal{C} \times \mathbb{R}^3|_{\epsilon_3;x_{6,7}}$; and a 3d worldvolume full defect which wraps $\mathcal{C}$ and the x_8-direction in $\mathbb{R}^3|_{\epsilon_2;x_{6,7}} = \mathbb{R} \times \mathbb{R}^2|_{\epsilon_3}$.

As explained in detail in [8, section 6.1], from (4.3), we find that in place of (2.7), the Hilbert space $\mathcal{H}_{\mathrm{BPS}}^{\Omega'}$ of space-time BPS states on the r.h.s. of (4.1) would be given by

$$\mathcal{H}_{\mathrm{BPS}}^{\Omega'} = \widehat{\mathfrak{su}(N)}_{\mathrm{aff},k}, \tag{4.4}$$

where $\widehat{\mathfrak{su}(N)}_{\mathrm{aff},k}$ is an integrable module over the affine Lie algebra $\mathfrak{su}(N)_{\mathrm{aff},k}$ of level k.

A "fully-ramified" AGT correspondence for pure SU(N). Clearly, the physical duality of the compactifications in (4.1) will mean that $\mathcal{H}_{\mathrm{BPS}}^{\Omega}$ in (4.2) is equivalent to $\mathcal{H}_{\mathrm{BPS}}^{\Omega'}$ in (4.4), i.e.,

$$\bigoplus_{a'} \mathrm{IH}_{\mathrm{U}(1)^2 \times T}^* \mathcal{U}(\mathcal{M}_{\mathrm{SU}(N),T,a'}) = \widehat{\mathfrak{su}(N)}_{\mathrm{aff},k}, \tag{4.5}$$

where the level k and central charge c of $\widehat{\mathfrak{su}(N)}_{\mathrm{aff},k}$ are given by

$$k = -N - \frac{\epsilon_2}{\epsilon_1}, \qquad c = \frac{\epsilon_1}{\epsilon_2}(N^3 - N) + N^2 - 1. \tag{4.6}$$

Consequently, in place of (2.18), we have

$$Z_{\mathrm{inst}}(\Lambda, \epsilon_1, \epsilon_2, \vec{a}, T) = \langle \mathrm{coh}_T | \mathrm{coh}_T \rangle, \tag{4.7}$$

where

$$|\mathrm{coh}_T\rangle = \bigoplus_{a'} \mathcal{G}^{a'} |\Psi_{a',T}\rangle. \tag{4.8}$$

Here, $\vec{a}$ is the Coulomb moduli of the underlying pure SU(N) theory along $\mathbb{R}^4|_{\epsilon_1,\epsilon_2}$ on the l.h.s. of (4.1); $|\mathrm{coh}_T\rangle \in \widehat{\mathfrak{su}(N)}_{\mathrm{aff},k}$; $\mathcal{G}^{a'}$ is some real constant depending on the energy scale Λ; and $|\Psi_{a',T}\rangle \in \mathrm{IH}^*_{\mathrm{U}(1)^2 \times T}\mathcal{U}(\mathcal{M}_{\mathrm{SU}(N),T,a'})$ is also a state in $\widehat{\mathfrak{su}(N)}_{\mathrm{aff},k}$ with energy level $n_{a'}$ determined by the "fully-ramified" instanton number a'.

Like $|\mathrm{coh}\rangle$ and $\langle\mathrm{coh}|$, $|\mathrm{coh}_T\rangle$ and $\langle\mathrm{coh}_T|$ ought to be a state and its dual associated with the puncture at $z = 0$ and $z = \infty$ in $\mathcal{C} = \mathbf{S}^2$, respectively. Furthermore, as the r.h.s. of (4.8) is a sum over states of all possible energy levels, it must mean that $|\mathrm{coh}_T\rangle$ is actually a *coherent state*.

In arriving at the relations (4.5), (4.6), (4.7) and (4.8), we have just derived a "fully-ramified" AGT correspondence for pure SU(N).

4.2 A 5d "fully-ramified" pure AGT correspondence and a relativistic periodic Toda integrable system

The "fully-ramified" 4d pure Nekrasov instanton partition function in the NS limit and a quantum affine Toda system. Let us consider the NS limit $\epsilon_2 = 0$ in our derivation of the 4d "fully-ramified" AGT correspondence for pure SU(N) in the last subsection. From (4.6), we find that in place of (4.7), the "fully-ramified" 4d pure Nekrasov instanton partition function is now

$$Z_{\mathrm{inst}}(\Lambda, \epsilon_1, 0, \vec{a}, T) = \langle \mathrm{coh}_T^{\mathrm{crit}} | \mathrm{coh}_T^{\mathrm{crit}} \rangle, \tag{4.9}$$

where $\vec{a}$ is the Coulomb moduli of the underlying pure SU(N) theory along $\mathbb{R}^4|_{\epsilon_1,\epsilon_2=0}$ on the l.h.s. of (4.1); $T \subset \mathrm{SU}(N)$ is the maximal torus that SU(N) reduces to along $\mathbb{R}^2|_{\epsilon_1} \subset \mathbb{R}^4|_{\epsilon_1,\epsilon_2=0}$; the coherent state $|\mathrm{coh}_T^{\mathrm{crit}}\rangle \in \widehat{\mathfrak{su}(N)}_{\mathrm{aff},\mathrm{crit}}$; and $\widehat{\mathfrak{su}(N)}_{\mathrm{aff},\mathrm{crit}}$ is an integrable module over the affine Lie algebra $\mathfrak{su}(N)_{\mathrm{aff},\mathrm{crit}}$ at the *critical level* $k = -N$.

Via the dimension-one currents J_{a_i} that generate $\mathfrak{su}(N)_{\mathrm{aff},\mathrm{crit}}$ on $\mathcal{C} = \mathbf{S}^2$, one can define the Segal-Sugawara operators

$$S^{(s_i)}(z) = (k + h^\vee)T^{(s_i)}(z), \quad s_i = i + 1, \quad i = 1, 2, \ldots, N - 1, \tag{4.10}$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{su}(N)$, the spin-s_i operators $T^{(s_i)}$ are just higher spin generalizations of the holomorphic stress tensor $T^{(2)}$, and

$$S^{(s_i)}(z) =: d^{a_1 a_2 a_3 \ldots a_{s_i}}(k)(J_{a_1} J_{a_2} \ldots J_{a_{s_i}})(z) :, \tag{4.11}$$

where $d^{a_1 a_2 a_3 \ldots a_{s_i}}(k)$ is a completely symmetric traceless $\mathfrak{su}(N)$-invariant tensor of rank s_i (which depends on the level k); in other words, the $N - 1$ number of $S^{(s_i)}$'s are Casimir operators.

From (4.10), one can see that the $S^{(s_i)}$'s generate in their OPE's with all other operators of the quantum CFT on $\mathcal{C}$, $(k+h^\vee)$ times the field transformations generated by the $T^{(s_i)}$'s. Therefore, at the critical level $k = -N = -h^\vee$, the $S^{(s_i)}$'s generate *no* field transformations at all: their OPE's amongst themselves, and with all other field operators, are regular, and they are said to span the central elements of (the universal enveloping algebra of) the affine $SU(N)$-algebra at the critical level. Hence, on any correlation function of operators, the $S^{(s_i)}$'s effectively act as c-numbers. In particular, this means that the r.h.s. of (4.9) — which can be interpreted as a two-point correlation function of coherent state operators — is a simultaneous eigenfunction of the commuting $S^{(s_i)}$'s. As the $S^{(s_i)}$'s generically act as order-s_i differential operators in their action on a correlation function of primary state operators (see for example [19, section 15.7]), and since a coherent state can be obtained by applying creation operators on a primary state, i.e., a coherent state operator can be derived from a primary state operator, our discussion hitherto would mean that $Z_{\text{inst}}(\Lambda, \epsilon_1, 0, \vec{a}, T)$ ought to be a simultaneous eigenfunction of $N-1$ commuting differential operators $D_1, D_2, \ldots D_{N-1}$ derived from the Casimirs of an affine $SU(N)$-algebra.

Indeed, note that the coherent state $|\text{coh}_T^{\text{crit}}\rangle$ is also known as a Whittaker vector in representation theory whence its inner product on the r.h.s. of (4.9) is a Whittaker function associated with $\mathfrak{su}(N)_{\text{aff,crit}}$; in turn, according to [32, section 2], this would mean that $Z_{\text{inst}}(\Lambda, \epsilon_1, 0, \vec{a}, T)$ must be a simultaneous eigenfunction of $N - 1$ quantum Toda Hamiltonians $\mathcal{D}_{\text{Toda}}^{(1)}, \mathcal{D}_{\text{Toda}}^{(2)}, \ldots, \mathcal{D}_{\text{Toda}}^{(N-1)}$ that are associated with an affine $SU(N)$-algebra, i.e.,

$$\mathcal{D}_{\text{Toda}}^{(l)} \cdot Z_{\text{inst}}(\Lambda, \epsilon_1, 0, \vec{a}, T) = \mathcal{E}_{\text{Toda}}^{(l)} Z_{\text{inst}}(\Lambda, \epsilon_1, 0, \vec{a}, T), \qquad (4.12)$$

where the $\mathcal{D}_{\text{Toda}}^{(l)}$'s are commuting Casimir differential operators; the $\mathcal{E}_{\text{Toda}}^{(l)}$'s are complex eigenvalues; and $l = 1, 2, \ldots, N - 1$. Furthermore, it has also been shown in [33] that the spectral curve of the quantum affine Toda system defined by (4.12), is just Σ_{SW}, the 4d Seiberg-Witten curve.

Note that this is also consistent with [11, Corollary 3.7(2)], where $\mathcal{Z}_{G,B}^{\text{aff}}$ and ϵ in loc. cit. correspond respectively to $Z_{\text{inst}}(G, \epsilon_1, 0, \vec{a}, T)$ and ϵ_2 here.

The 5d Seiberg-Witten curve. Now recall that the I-brane consisting of N D6-branes and a single D4-brane wrapping $\mathcal{C}$ can be regarded as an N-fold cover of $\mathcal{C}$ which, in the 4d case, can be interpreted as Σ_{SW} (see explanations leading up to (2.19)). What is its corresponding interpretation in the 5d case?

Similar to section 2.2, we have on $\mathcal{C}$, a (chiral) $SU(N)$ WZW model (which underlies $\mathfrak{su}(N)_{\text{aff,crit}}$). As such, according to our explanations in the paragraph leading up to (2.39), one can equivalently study the 4d theory with gauge group $LSU(N)$ instead of $SU(N)$. This amounts to replacing [34] the 4d Higgs field ϕ with a first-order differential operator on a circle: $\phi \to \partial_t + A_t + i\varphi$, where A_t and φ are an $SU(N)$ gauge field and a real scalar along the circle with coordinate t (in the adjoint representation of $SU(N)$). Correspondingly, we ought to also make this replacement in Σ_{SW} of (2.19), where A_t and φ are here an $SU(N)$

gauge field and a real scalar along a circle C_β in $\mathcal{C}$ of radius β (in the adjoint representation of SU(N)) that originate from the SU(N) gauge theory on the N D6-branes in (4.3).

Note at this point that we can also parameterize Σ_{SW} as

$$\Sigma_{SW}: \ y^2 = P(x)^2 - \Lambda^{2N}, \tag{4.13}$$

where $x, y \in \mathbb{C}$, and $P(x) = \det(\phi - x)$. Thus, in the 5d case, the curve ought to be given by

$$\Sigma_{SW}^{\text{5d}}: \ y^2 = \det(\partial_t + A_t + i\varphi - x)^2 - \Lambda^{2N}. \tag{4.14}$$

The determinant $\det(\partial_t + A_t + i\varphi - x)$ can be computed via zeta-regularization as

$$\det(\partial_t + A_t + i\varphi - x) = \prod_i \frac{1}{\beta} \sinh(2\pi\beta a_i - x), \tag{4.15}$$

where the a_i's are the eigenvalues of $\partial_t + A_t + i\varphi$ when $\beta \to 0$. Substituting this in (4.14), we get, after an appropriate change of variables,

$$\boxed{\Sigma_{SW}^{\text{5d}}: \ z + \frac{\Lambda^{2N}}{z} = \prod_i \frac{1}{\beta} \sinh(2\pi\beta a_i - x)} \tag{4.16}$$

Using the fact that $\sum_i a_i = 0$, one can, after another change of variables, show that Σ_{SW}^{5d} is the spectral curve of a relativistic periodic Toda integrable system [34].

The "fully-ramified" 5d pure Nekrasov instanton partition function in the NS limit and a relativistic periodic Toda integrable system. Hence, the 5d interpretation of Σ_{SW} is a spectral curve Σ_{SW}^{5d} of a relativistic periodic Toda integrable system or equivalently, a q-deformed quantum affine Toda system [32, section 6]. In the limit $q \to 1$, a q-deformed quantum affine Toda system would reduce to the earlier-mentioned quantum affine Toda system [32, section 6]. In the limit $q \to 1$, it has also been rigorously established in [35] that the $N-1$ commuting difference operators $\mathcal{D}_{q-\text{Toda}}^{(1)}, \mathcal{D}_{q-\text{Toda}}^{(2)}, \ldots, \mathcal{D}_{q-\text{Toda}}^{(N-1)}$ which define a q-deformed quantum affine Toda system, would reduce to the $N-1$ commuting differential operators $\mathcal{D}_{\text{Toda}}^{(1)}, \mathcal{D}_{\text{Toda}}^{(2)}, \ldots, \mathcal{D}_{\text{Toda}}^{(N-1)}$ which define the earlier-mentioned quantum affine Toda system. If we conveniently identify $q = e^\beta$, where β is the radius of the fifth dimension circle, then in the limit $q \to 1$, we also have $\Sigma_{SW}^{\text{5d}} \to \Sigma_{SW}$ and $Z_{\text{inst}, \text{SU}(N)}^{\text{pure}, 5d}(\Lambda, \epsilon_1, 0, \vec{a}, T, \beta) \to Z_{\text{inst}}(\Lambda, \epsilon_1, 0, \vec{a}, T)$, where $Z_{\text{inst}, \text{SU}(N)}^{\text{pure}, 5d}(\Lambda, \epsilon_1, 0, \vec{a}, T, \beta)$ is the "fully-ramified" 5d pure Nekrasov instanton partition function in the $\epsilon_2 = 0 \neq \epsilon_1$ NS limit. These statements hold vice-versa, so this correspondence can be summarized in the diagram below.

$$\{Z_{\text{inst}}; \Sigma_{SW}; \mathcal{D}_{\text{Toda}}^{(l)}\} \xleftarrow[q \to 1]{q \nrightarrow 1} \{Z_{\text{inst}}^{\text{pure}, 5d}; \Sigma_{SW}^{\text{5d}}; \mathcal{D}_{q-\text{Toda}}^{(l)}\}$$

From the diagram, it is clear that the 5d version of (4.12) must be

$$\boxed{\mathcal{D}_{q-\text{Toda}}^{(l)} \cdot Z_{\text{inst}, \text{SU}(N)}^{\text{pure}, 5d}(\Lambda, \epsilon_1, 0, \vec{a}, T, \beta) = \mathcal{E}_{q-\text{Toda}}^{(l)} Z_{\text{inst}, \text{SU}(N)}^{\text{pure}, 5d}(\Lambda, \epsilon_1, 0, \vec{a}, T, \beta)} \tag{4.17}$$

where the $\mathcal{E}^{(l)}_{q-\text{Toda}}$'s are complex eigenvalues, and $l = 1, 2, \ldots, N - 1$. In other words, $Z^{\text{pure}, 5d}_{\text{inst}, \text{SU}(N)}(\Lambda, \epsilon_1, 0, \vec{a}, T, \beta)$ must be a simultaneous eigenfunction of the $N - 1$ commuting difference operators $\mathcal{D}^{(1)}_{q-\text{Toda}}, \mathcal{D}^{(2)}_{q-\text{Toda}}, \ldots, \mathcal{D}^{(N-1)}_{q-\text{Toda}}$ which define a relativistic periodic Toda integrable system.

A 5d "fully-ramified" AGT correspondence for pure SU(N) in the NS limit. Note that the $\mathcal{D}^{(l)}_{q-\text{Toda}}$'s correspond to the central elements of a quantum affine SU(N)-algebra at the critical level, while the $\mathcal{D}^{(1)}_{\text{Toda}}$'s correspond to the central elements of (the universal enveloping algebra of) an affine SU(N)-algebra at the critical level. Thus, from (4.17), (4.12) and (4.9), one can conclude that

$$\boxed{Z^{\text{pure}, 5d}_{\text{inst}, \text{SU}(N)}(\Lambda, \epsilon_1, 0, \vec{a}, T, \beta) = \langle \text{coh}^{\text{crit}}_{T,\beta} | \text{coh}^{\text{crit}}_{T,\beta} \rangle} \tag{4.18}$$

where

$$\boxed{|\text{coh}^{\text{crit}}_{T,\beta}\rangle \in \widehat{U_q}(\mathfrak{su}(N)_{\text{aff}})_{\text{crit}}, \quad q = e^\beta} \tag{4.19}$$

and $\widehat{U_q}(\mathfrak{su}(N)_{\text{aff}})_{\text{crit}}$ is the integrable module over $U_q(\mathfrak{su}(N)_{\text{aff}})_{\text{crit}}$, a quantum affine SU($N$)-algebra at the critical level. Like $|\text{coh}^{\text{crit}}_T\rangle$ and $\langle\text{coh}^{\text{crit}}_T|$ in (4.9), $|\text{coh}^{\text{crit}}_{T,\beta}\rangle$ and $\langle\text{coh}^{\text{crit}}_{T,\beta}|$ can be regarded as a state and its dual associated with the puncture at $z = 0$ and $z = \infty$ in $\mathcal{C} = \mathbf{S}^2$, respectively. Furthermore, as $|\text{coh}^{\text{crit}}_{T,\beta}\rangle$ is a 5d analog of $|\text{coh}^{\text{crit}}_T\rangle$, i.e., $|\text{coh}^{\text{crit}}_{T,\beta}\rangle$ contains higher excited states beyond those present in the coherent state $|\text{coh}^{\text{crit}}_T\rangle$ (see explanations after (2.33)), it must mean that $|\text{coh}^{\text{crit}}_{T,\beta}\rangle$ is also a *coherent state*. This is consistent with the mathematical result in [32, section 4] that the simultaneous eigenfunction of the $\mathcal{D}^{(l)}_{q-\text{Toda}}$'s is necessarily a Whittaker function on $U_q(\mathfrak{su}(N)_{\text{aff}})_{\text{crit}}$.

Clearly, in arriving at the boxed relations (4.18) and (4.19), we have just derived a 5d "fully-ramified" AGT correspondence for pure SU(N) in the NS limit!

4.3 A 5d "fully-ramified" AGT correspondence with matter and a relativistic elliptic Calogero-Moser system

The "fully-ramified" 4d Nekrasov instanton partition function with single adjoint matter in the NS limit and an elliptic Calogero-Moser system. Like the U(1) case mentioned in the sections before, a pure SU(N) theory can also be interpreted as the $m \to \infty$, $q' = e^{2\pi i\tau'} \to 0$ limit of an SU(N) theory with an adjoint hypermultiplet matter of mass m and complexified gauge coupling τ', where $m^N e^{2\pi i\tau'} = \Lambda^N$ remains fixed. Moreover, note that in this limit, the Hamiltonian and spectral curve of the elliptic Calogero-Moser system associated with the aforementioned theory with adjoint matter, reduce to the Hamiltonian and spectral curve of the quantum affine Toda system [36].[11]

The above two points mean that in the reverse limit, we have, in (4.12), the transformation (i) $Z_{\text{inst}}(\Lambda, \epsilon_1, 0, \vec{a}, T) \to Z^{\text{adj}}_{\text{inst}}(q', \epsilon_1, 0, \vec{a}, m, T)$, where $Z^{\text{adj}}_{\text{inst}}(q', \epsilon_1, 0, \vec{a}, m, T)$ is the "fully-ramified" 4d Nekrasov instanton partition function in the NS limit $\epsilon_2 = 0$ of the SU(N) theory on $\mathbb{R}^4|_{\epsilon_1, \epsilon_2 = 0}$ with an adjoint hypermultiplet matter of mass m, $\vec{a}$ are its Coulomb moduli, and $T \subset \text{SU}(N)$ is the maximal torus that SU(N) reduces to along

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[11]See also [37, 38].

$\mathbb{R}^2|_{\epsilon_1} \subset \mathbb{R}^4|_{\epsilon_1, \epsilon_2=0}$; (ii) $\mathcal{D}_{\text{Toda}}^{(l)} \to \mathcal{D}_{\text{CM}}^{(l)}$, where the $\mathcal{D}_{\text{CM}}^{(l)}$'s are differential operators that correspond to the quantum Hamiltonians of the elliptic Calogero-Moser system. In other words, we can write

$$\mathcal{D}_{\text{CM}}^{(l)} \cdot Z_{\text{inst}}^{\text{adj}}(q', \epsilon_1, 0, \vec{a}, m, T) = \mathcal{E}_{\text{CM}}^{(l)} Z_{\text{inst}}^{\text{adj}}(q', \epsilon_1, 0, \vec{a}, m, T), \tag{4.20}$$

where the $\mathcal{E}_{\text{CM}}^{(l)}$'s are complex eigenvalues, and $l = 1, 2, \ldots, N - 1$. In the case where $N = 2$, the eigenvalue $\mathcal{E}_{\text{CM}}^{(1)}$ has been determined explicitly in [5, section 4.3].

The "fully-ramified" 5d Nekrasov instanton partition function with single adjoint matter in the NS limit and a relativistic elliptic Calogero-Moser system.
Likewise, in the above reverse limit, (i) $Z_{\text{inst, SU}(N)}^{\text{pure, 5}d}(\Lambda, \epsilon_1, 0, \vec{a}, T, \beta) \to Z_{\text{inst, SU}(N)}^{\text{adj, 5}d}(q', \epsilon_1, 0, \vec{a}, m, T, \beta)$, where $Z_{\text{inst, SU}(N)}^{\text{pure, 5}d}(\Lambda, \epsilon_1, 0, \vec{a}, T, \beta)$ is the "fully-ramified" 5d pure Nekrasov instanton partition function while $Z_{\text{inst, SU}(N)}^{\text{adj, 5}d}(q', \epsilon_1, 0, \vec{a}, m, T, \beta)$ is the "fully-ramified" 5d Nekrasov instanton partition function with single adjoint matter of mass m, both in the NS limit; (ii) $\mathcal{D}_{q-\text{Toda}}^{(l)} \to \mathcal{D}_{q-\text{CM}}^{(l)}$ [32, section 7], where the $\mathcal{D}_{q-\text{Toda}}^{(l)}$'s and $\mathcal{D}_{q-\text{CM}}^{(l)}$'s are commuting difference operators which define a relativistic periodic Toda and elliptic Calogero-Moser system, respectively. In the $q = e^\beta \to 1$ limit, where β is the radius of the fifth dimension circle, (i) $Z_{\text{inst, SU}(N)}^{\text{pure, 5}d}(\Lambda, \epsilon_1, 0, \vec{a}, T, \beta) \to Z_{\text{inst}}(\Lambda, \epsilon_1, 0, \vec{a}, T)$; (ii) $Z_{\text{inst, SU}(N)}^{\text{adj, 5}d}(q', \epsilon_1, 0, \vec{a}, m, T, \beta) \to Z_{\text{inst}}(q', \epsilon_1, 0, \vec{a}, m, T) = Z_{\text{inst}}^{\text{adj}}$; (iii) $\{\mathcal{D}_{q-\text{Toda}}^{(l)}, \mathcal{D}_{q-\text{CM}}^{(l)}\} \to \{\mathcal{D}_{\text{Toda}}^{(l)}, \mathcal{D}_{\text{CM}}^{(l)}\}$. These statements hold vice-versa, so this correspondence can be summarized in the diagram below.

$$\{Z_{\text{inst}}^{\text{pure, 5}d}; \mathcal{D}_{q-\text{Toda}}^{(l)}\} \underset{m \to \infty}{\overset{m \nrightarrow \infty}{\rightleftarrows}} \{Z_{\text{inst}}^{\text{adj, 5}d}; \mathcal{D}_{q-\text{CM}}^{(l)}\}$$

$$q \nrightarrow 1 \Bigg\updownarrow q \to 1 \qquad\qquad q \nrightarrow 1 \Bigg\updownarrow q \to 1$$

$$\{Z_{\text{inst}}; \mathcal{D}_{\text{Toda}}^{(l)}\} \underset{m \to \infty}{\overset{m \nrightarrow \infty}{\rightleftarrows}} \{Z_{\text{inst}}^{\text{adj}}; \mathcal{D}_{\text{CM}}^{(l)}\}$$

From the diagram, and (4.12), (4.17) and (4.20), it is clear that we ought to have

$$\boxed{\mathcal{D}_{q-\text{CM}}^{(l)} \cdot Z_{\text{inst, SU}(N)}^{\text{adj, 5}d}(q', \epsilon_1, 0, \vec{a}, m, T, \beta) = \mathcal{E}_{q-\text{CM}}^{(l)} Z_{\text{inst, SU}(N)}^{\text{adj, 5}d}(q', \epsilon_1, 0, \vec{a}, m, T, \beta)} \tag{4.21}$$

where the $\mathcal{E}_{q-\text{CM}}^{(l)}$'s are complex eigenvalues, and $l = 1, 2, \ldots, N - 1$. In other words, $Z_{\text{inst, SU}(N)}^{\text{adj, 5}d}(q', \epsilon_1, 0, \vec{a}, m, T, \beta)$ must be a simultaneous eigenfunction of the $N - 1$ commuting difference operators $\mathcal{D}_{q-\text{CM}}^{(1)}, \mathcal{D}_{q-\text{CM}}^{(2)}, \ldots, \mathcal{D}_{q-\text{CM}}^{(N-1)}$ which define a relativistic elliptic Calogero-Moser integrable system.

A 5d "fully-ramified" AGT correspondence for SU(N) with single adjoint matter in the NS limit. Now recall that we can obtain the result for the pure SU(N) case as a $m \to \infty$ limit of the SU(N) case with a single adjoint hypermultiplet matter of mass m — see (3.45). This means that we can also write (4.18) as

$$Z^{\text{pure},\,5d}_{\text{inst},\,\text{SU}(N)}(\Lambda, \epsilon_1, 0, \vec{a}, T, \beta) = \langle \mathbf{0} | \Phi^q_{m \to \infty}(1) | \mathbf{0} \rangle_{\mathbf{S}^2}, \tag{4.22}$$

where $\Phi^q_{m \to \infty}$ is a $m \to \infty$ limit of a vertex operator of $\widehat{U_q}(\mathfrak{su}(N)_{\text{aff}})_{\text{crit}}$, an integrable module over a quantum affine SU(N)-algebra at the critical level. As usual, the states $\langle \mathbf{0} |$ and $| \mathbf{0} \rangle$ correspond to lowest energy vertex operators $V^q(\infty)$ and $V^q(0)$ of this module inserted at $z = \infty$ and 0, respectively. In turn, the arguments in footnote 5 (which also apply to the present "fully-ramified", $N > 1$, 5d case) mean that we can write $Z^{\text{adj},\,5d}_{\text{inst}}$ as the one-point correlation function on $\mathbf{T}^2$:

$$\boxed{Z^{\text{adj},\,5d}_{\text{inst},\,\text{SU}(N)}(q', \epsilon_1, 0, \vec{a}, m, T, \beta) = \langle\, \Phi^q_m(1) \,\rangle_{\mathbf{T}^2}} \tag{4.23}$$

where $\Phi^q_m(z)$ is a vertex operator of $\widehat{U_q}(\mathfrak{su}(N)_{\text{aff}})_{\text{crit}}$ (which depends on m), whence it defines the following map

$$\boxed{\Phi^q_m : \widehat{U_q}(\mathfrak{su}(N)_{\text{aff}})_{\text{crit}} \to \widehat{U_q}(\mathfrak{su}(N)_{\text{aff}})_{\text{crit}}} \tag{4.24}$$

Clearly, in arriving at the boxed relations (2.34) and (2.35), we have just derived a 5d "fully-ramified" AGT correspondence for SU(N) with a single adjoint matter in the NS limit!

Elliptic Macdonald polynomials and quantum affine SU(N)-algebra at the critical level. Note that the $\mathcal{D}^{(l)}_{q-\text{CM}}$'s are known to be an elliptic generalization of the Macdonald operators [39]. Thus, their simultaneous eigenfunctions, i.e., $Z^{\text{adj},\,5d}_{\text{inst},\,\text{SU}(N)}$, ought to be given by an elliptic generalization of the Macdonald polynomials. In turn, (4.23) and (4.24) mean that *these elliptic Macdonald polynomials can be understood as one-point correlation functions of a 2d QFT on $\mathbf{T}^2$ whose underlying symmetry is generated by $U_q(\mathfrak{su}(N)_{\text{aff}})_{\text{crit}}$, a quantum affine* SU($N$)*-algebra at the critical level!*

This observation is in fact consistent with various results in the mathematical literature as follows. Firstly, note that (4.21) and (4.23) mean that the $\mathcal{D}^{(l)}_{q-\text{CM}}$'s correspond to central elements of $U_q(\mathfrak{su}(N)_{\text{aff}})_k$ (for some level k) — incidentally, this has also been pointed out in [40] — whence according to [41], they are affine Macdonald operators whose simultaneous eigenfunctions are the affine Macdonald polynomials constructed in [42]. In turn, these affine Macdonald polynomials are given by traces of intertwiners of $U_q(\mathfrak{su}(N)_{\text{aff}})_k$ [42], i.e., they are given by one-point correlation functions of vertex operators of $\widehat{U_q}(\mathfrak{su}(N)_{\text{aff}})_k$ on $\mathbf{T}^2$. This coincides with our aforementioned observation, after one notes the terminological correspondence between the elliptic and affine Macdonald operators and polynomials.

5 An M-theoretic derivation of a 6d AGT correspondence

5.1 An M-theoretic derivation of a 6d AGT correspondence with matter

A 6d AGT correspondence for U(1) with $N_f = 2$ fundamental matter. Let $n = 1 = N$ in figure 2(3), and glue together the two ends of the interval $\mathbb{I}_t \subset \Sigma_t = \mathbf{S}^1 \times \mathbb{I}_t$,

where β therein is the radius of $\mathbf{S}^1$. Now, Σ_t is a two-torus $\mathbf{S}^1 \times \mathbf{S}_t^1$, and the planes at the ends labeled by m_1 and m_4 would be coincident, intersecting Σ_t along a one-cycle in the $\mathbf{S}^1$-direction. On the 2d side, $\mathcal{C}_{\text{eff}}$ also becomes a two-torus $\mathbf{T}^2$ after we correspondingly glue together the punctures at $z = \infty$ and $z = 0$, whence there are left two vertex operators $V_{\vec{j}_2}$ and $V_{\vec{j}_3}$. Let $\beta \twoheadrightarrow 0$, and let the radius of $\mathbf{S}_t^1$ be R_6. Then, one can define a 6d generalization $Z_{\text{inst}}^{\text{lin},\,6\text{d}}$ of the 4d Nekrasov instanton partition function $Z_{\text{inst}}^{\text{lin}}$ along the $\mathbb{R}^4|_{\epsilon_1,\epsilon_2} \times \mathbf{S}^1 \times \mathbf{S}_t^1$ worldvolume of the M5-brane lying in the $X^9|_{\epsilon_i}$-directions, where from (3.6), we can write

$$Z_{\text{inst, U}(1)}^{\text{lin},\,6\text{d}}(q_1, \epsilon_1, \epsilon_2, \vec{m}, \beta, R_6) = \langle\, \tilde{\Phi}_v^w(z_1)\, \tilde{\Phi}_u^v(z_2)\, \rangle_{\mathbf{T}^2} \tag{5.1}$$

up to some constant factor, where the $N_f = 2$ fundamental matter masses are given by $\vec{m} = (m_1, m_2)$, and

$$w = e^{-\beta m_1}, \quad v = e^{-2\beta(m_1 + m_2)}, \quad u = e^{-\beta m_2} \tag{5.2}$$

From a straightforward 6d generalization of our explanations in section 2.2 about how a 5d state would have a projection onto a circle C_β in $\mathcal{C}_{\text{eff}}$ which has radius β, we find that the vertex operator $\tilde{\Phi}_d^c(z)$ — which is a 6d analog of those in (3.23) — ought to be associated with a 6d state which has a projection onto two transverse circles C_β and C_{R_6} in $\mathbf{T}^2$ of radius β and R_6, respectively, which intersect at the point z.

In the 5d case, the vertex operator was given in (3.24). What about the 6d case? What explicit form should $\tilde{\Phi}_d^c(z)$ take? To answer this, first note that because $\tilde{\Phi}_d^c(z)$ is supposed to be associated with a state that has a projection onto the transverse circles C_β and C_{R_6} which *intersect*, the commutation relation (3.26) — which is associated with the circle C_β — must be modified to depend on R_6. Since this modified commutation relation should reduce to (3.26) in the limit $R_6 \to 0$, a consistent modification would be

$$[\tilde{a}_m, \tilde{a}_n] = m(1 - p^{|m|})\frac{1 - q^{|m|}}{1 - t^{|m|}}\delta_{m+n,0}, \quad \tilde{a}_{m>0}|\tilde{\emptyset}\rangle = 0 \tag{5.3}$$

where

$$q = e^{-i\beta\sqrt{\epsilon_1\epsilon_2}}, \quad t = e^{-i\beta(\epsilon_1 + \epsilon_2 + \sqrt{\epsilon_1\epsilon_2})}, \quad p = e^{-\frac{1}{R_6}} \tag{5.4}$$

In (5.3), we have an $|m|^{\text{th}}$ power of p because it is also a parameter associated with the radius of a circle in $\mathbf{T}^2$, i.e., it is on equal footing with q and t.

Second, because the 6d state has a projection onto *two* transverse circles C_β and C_{R_6}, it would mean that we can write an arbitrary 6d state as

$$|\Psi\rangle = \tilde{b}_{-\mu_1}\tilde{b}_{-\mu_2}\ldots\tilde{a}_{-\lambda_1}\tilde{a}_{-\lambda_2}\ldots|\tilde{\emptyset}\rangle, \tag{5.5}$$

where the $\tilde{b}_k$'s are operators associated with the circle C_{R_6}, which a priori obey the same commutation relation as the $\tilde{a}_k$'s, i.e.,

$$[\tilde{b}_m, \tilde{b}_n] = m(1 - p^{|m|})\frac{1 - q^{|m|}}{1 - t^{|m|}}\delta_{m+n,0}, \qquad \tilde{b}_{m>0}|\tilde{\emptyset}\rangle = 0, \tag{5.6}$$

and

$$[\tilde{a}_m, \tilde{b}_n] = 0 \tag{5.7}$$

Here, the λ_i's and μ_i's define the partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$.

Notice that when $\beta \to \infty$, there will not be any Omega-deformation in the directions transverse to C_β — recall that Omega-deformation is effected as one traverses C_β completely, but this cannot happen if its radius is infinitely large. What this means is that the commutation relation (5.6) — which is associated with the circle C_{R_6} that is transverse to C_β — ought to be independent of the ϵ_i-dependent parameters q and t when $\beta \to \infty$. When $\beta \to \infty$, $q, t \to \infty$ (see (5.4) and footnote 1); in other words, the factor of $1 - q^{|m|}/1 - t^{|m|}$ in (5.6) would tend to $(qt^{-1})^{|m|}$ as $\beta \to \infty$. Therefore, we must divide the r.h.s. of (5.6) by $(qt^{-1})^{|m|}$ to get

$$[\tilde{b}_m, \tilde{b}_n] = \frac{m(1 - p^{|m|})}{(qt^{-1})^{|m|}} \frac{1 - q^{|m|}}{1 - t^{|m|}} \delta_{m+n,0}, \qquad \tilde{b}_{m>0}|\tilde{\emptyset}\rangle = 0. \tag{5.8}$$

Also, when $R_6 \to 0$, the $\tilde{b}_k$ operators should not enter our story at all; in particular, the commutation relations ought to be physically invalid when $R_6 \to 0$. Since a physically valid commutation relation would either be equal to zero or some finite constant, as $R_6 \to 0$, i.e., as $p \to 0$, the r.h.s. of (5.8) should blow up. In other words, we ought to modify (5.8) to

$$[\tilde{b}_m, \tilde{b}_n] = \frac{m(1 - p^{|m|})}{(qt^{-1}p)^{|m|}} \frac{1 - q^{|m|}}{1 - t^{|m|}} \delta_{m+n,0}, \qquad \tilde{b}_{m>0}|\tilde{\emptyset}\rangle = 0 \tag{5.9}$$

(where again, we have an $|m|^{\text{th}}$ power of p because it is also a parameter associated with the radius of a circle in $\mathbf{T}^2$, i.e., it is on equal footing with q and t).

Third, from (5.5), it is clear that we can write

$$\tilde{\Phi}_d^c(z) = \tilde{\Phi}_d^{c, \tilde{b}_k}(z) \cdot \tilde{\Phi}_d^{c, \tilde{a}_k}(z) \tag{5.10}$$

where $\tilde{\Phi}_d^{c, \tilde{a}_k(\tilde{b}_k)}$ is a vertex operator which involves the $\tilde{a}_k$'s ($\tilde{b}_k$'s). To obtain the explicit expression for $\tilde{\Phi}_d^c$, note that according to [43, Theorem 1.2], the Fock space $\tilde{\mathcal{F}}$ spanned by the states $|\Psi\rangle$ in (5.5) (bearing in mind the commutation relations (5.3), (5.9) and (5.7)), affords a representation of an elliptic generalization of the Ding-Iohara algebra we first encountered in section 3.2. Since the vertex operator $\tilde{\Phi}_d^c$ defines a map $\tilde{\mathcal{F}}_d \to \tilde{\mathcal{F}}_c$ from one Fock space associated with parameter d to another associated with parameter c, by comparing the currents of the Ding-Iohara algebra in [14, Fact 2.6] with the currents of the aforementioned elliptic Ding-Iohara algebra in [43, Theorem 1.2], it is clear that $\tilde{\Phi}_d^{c, \tilde{a}_k}$ can be obtained from (3.24) by replacing the factor of $(1 - q^{\pm n})$ therein with $(1 - p^{|n|})(1 - q^{\pm n})$, i.e.,

$$\tilde{\Phi}_d^{c, \tilde{a}_k}(z) = \exp\left(\sum_{n>0} \frac{1}{n} \frac{c^{-n} - d^{-n}}{(1 - p^n)(1 - q^{-n})} \tilde{a}_n z^{-n}\right) \exp\left(-\sum_{n>0} \frac{1}{n} \frac{c^n - (t/q)^n d^n}{(1 - p^n)(1 - q^n)} \tilde{a}_{-n} z^n\right) \tag{5.11}$$

Likewise, $\tilde{\Phi}_d^{c, \tilde{b}_k}$ can be obtained from (3.24) by replacing (i) the a_n/n and $-a_{-n}/n$ terms therein with $-\tilde{b}_{-n}/n$ and $\tilde{b}_n/n$, respectively; (ii) the factor of $(1 - q^{\pm n})$

therein with $(1 - p^{|n|})(1 - q^{\pm n})/p^{|n|}$; (iii) the factor of $(1 - q^n)/(t/q)^n$ therein with $(1 - p^{|n|})(1 - q^n)/(t/q)^n (q/t)^{|n|} p^{|n|}$:

$$\tilde{\Phi}_d^{c, \tilde{b}_k}(z) = \exp\left(-\sum_{n>0} \frac{1}{n} \frac{p^n (c^{-n} - d^{-n})}{(1 - p^n)(1 - q^{-n})} \tilde{b}_{-n} z^{-n} \right) \exp\left(\sum_{n>0} \frac{1}{n} \frac{p^n (c^n - d^n)}{(1 - p^n)(1 - q^n)} \tilde{b}_n z^n \right) \tag{5.12}$$

One can readily check from (5.11), (5.12) and (5.10) that when $R_6 \to 0$ (whence $p \to 0$), $(\tilde{\Phi}_d^{c, \tilde{b}_k}, \tilde{\Phi}_d^{c, \tilde{a}_k}) \to (1, \Phi_d^c)$ so $\tilde{\Phi}_d^c \to \Phi_d^c$, where Φ_d^c is the 5d vertex operator in (3.24), as expected.

Clearly, in arriving at the boxed relations (5.1), (5.2), (5.3), (5.4), (5.7), (5.9), (5.10), (5.11) and (5.12), we have just derived a 6d AGT correspondence for U(1) with $N_f = 2$ fundamental matter!

A 6d AGT correspondence for SU(N) with $N_f = 2N$ fundamental matter. Let us now proceed to consider an SU(N) theory with $N_f = 2N$ fundamental matter. Note that from the 6d generalization of figure 3, we effectively have N D6-branes and 1 D4-brane wrapping $\mathcal{C}_{\text{eff}} = \mathbf{T}^2$ — that is, we effectively have an $N \times 1 = N$-fold cover of $\mathcal{C}_{\text{eff}}$. This means that we ought to make the following replacements to get the corresponding result for the SU(N) case.

Firstly, the vacuum state $|\tilde{\emptyset}\rangle$ and its dual $\langle\tilde{\emptyset}|$ ought to be replaced by their N-tensor product:

$$|\tilde{\mathbf{0}}\rangle = |\tilde{\emptyset}\rangle^{\otimes N} \quad \text{and} \quad \langle\tilde{\mathbf{0}}| = \langle\tilde{\emptyset}|^{\otimes N}. \tag{5.13}$$

Secondly, the vertex operators $\tilde{\Phi}_v^w$ and $\tilde{\Phi}_u^v$ in (5.1) ought to also be replaced by their N-tensor product

$$\tilde{\Phi}_{\mathbf{v}}^{\mathbf{w}}(z_1) = \tilde{\Phi}_{v_N}^{w_N} \cdots \tilde{\Phi}_{v_1}^{w_1}(z_1) \qquad \text{and} \qquad \tilde{\Phi}_{\mathbf{u}}^{\mathbf{v}}(z_2) = \tilde{\Phi}_{u_1}^{v_1} \cdots \tilde{\Phi}_{u_N}^{v_N}(z_2) \tag{5.14}$$

where $\mathbf{c} = (c_1, c_2, \ldots, c_N)$.

Thus, we can express the 6d Nekrasov instanton partition function for SU(N) with $N_f = 2N$ fundamental matter as

$$Z_{\text{inst, SU}(N)}^{\text{lin, 6d}}(q_1, \epsilon_1, \epsilon_2, \vec{a}, \vec{m}, \beta, R_6) = \langle \tilde{\Phi}_{\mathbf{v}}^{\mathbf{w}}(z_1) \tilde{\Phi}_{\mathbf{u}}^{\mathbf{v}}(z_2) \rangle_{\mathbf{T}^2} \tag{5.15}$$

where

$$\tilde{\Phi}_{\mathbf{d}}^{\mathbf{c}} : \tilde{\mathcal{F}}_{d_1} \otimes \tilde{\mathcal{F}}_{d_2} \otimes \cdots \otimes \tilde{\mathcal{F}}_{d_N} \longrightarrow \tilde{\mathcal{F}}_{c_1} \otimes \tilde{\mathcal{F}}_{c_2} \otimes \cdots \otimes \tilde{\mathcal{F}}_{c_N}. \tag{5.16}$$

Note that in the U(1) case, the dependence of $\tilde{\Phi}_d^c$ on the Coulomb modulus actually drops out where it is replaced by a linear combination of the masses $m_{1,2}$ which appears in the parameter v in (5.2). In other words, in the SU(N) case, $\mathbf{v} = (v_1, \ldots, v_N)$ would be associated with the Coulomb moduli $\vec{a} = (a_1, \ldots, a_N)$ (where $\sum_i a_i = 0$), while $\mathbf{w} = (w_1, \ldots, w_N)$ and $\mathbf{u} = (u_1, \ldots, u_N)$ would continue to be associated with the $2N$ masses $\vec{m} = (m_1, m_2, \ldots, m_{2N})$. As the parameters $\mathbf{w}, \mathbf{v}, \mathbf{u}$ must reduce to w, v, u in (5.2) when $N = 1$, we conclude that

$$w_i = e^{-\beta m_i}, \quad v_i = e^{-\beta a_i}, \quad u_i = e^{-\beta m_{N+i}} \tag{5.17}$$

Since $\tilde{\tilde{\mathcal{F}}}_{c,d}$ is a module over the elliptic Ding-Iohara algebra, (5.16) would mean that $\tilde{\Phi}_{\mathbf{d}}^{\mathbf{c}}$ is a vertex operator of an N-tensor product representation of the elliptic Ding-Iohara algebra. In turn, this means that

$$\boxed{\tilde{\Phi}_{\mathbf{d}}^{\mathbf{c}} : \widehat{\mathcal{W}_N^{q,p}} \to \widehat{\mathcal{W}_N^{q,p}}} \tag{5.18}$$

where $\widehat{\mathcal{W}_N^{q,p}}$ is a Verma module over $\mathcal{W}_N^{q,p}$, an elliptic affine $\mathcal{W}_N$-algebra.[12]

Clearly, in arriving at the boxed relations (5.14), (5.15), (5.17) and (5.18), we have just derived a 6d AGT correspondence for SU(N) with $N_f = 2N$ fundamental matter!

An equivariant elliptic genus on instanton moduli space and a two-point correlation function of a 2d CFT with elliptic affine $\mathcal{W}_N$-algebra symmetry. Recall from our explanation in section 3.3 that $Z_{\text{inst, SU}(N)}^{\text{pure, 5d}}$ in (3.52) can be regarded as a partition function associated with a (non-dynamically) $\mathbf{g}$-gauged supersymmetric quantum mechanical instanton with target $L\mathcal{U}(\mathcal{M}_{\text{SU}(N)})$, the loop space of $\mathcal{U}(\mathcal{M}_{\text{SU}(N)})$.[13] In the same way, $Z_{\text{inst, SU}(N)}^{\text{lin, 6d}}$ can be regarded as a partition function associated with a (non-dynamically) $\mathbf{g}$-gauged supersymmetric quantum mechanical instanton with target $LL\mathcal{U}(\mathcal{M}_{\text{SU}(N)})$, the double loop space of $\mathcal{U}(\mathcal{M}_{\text{SU}(N)})$. Consequently, according to [30], $Z_{\text{inst, SU}(N)}^{\text{lin, 6d}}$ ought to be given by some version of the elliptic genus of $\mathcal{U}(\mathcal{M}_{\text{SU}(N)})$, whence we can also write (5.15) as (cf. [44])

$$\boxed{\sum_{k=0}^{\infty} q_1^k \, \chi_{\mathbf{g}(\epsilon_i, \vec{a})}(\mathcal{U}(\mathcal{M}_{\text{SU}(N),k}))(\vec{m}, \tau) = \langle \tilde{\Phi}_{\mathbf{v}}^{\mathbf{w}}(z_1) \tilde{\Phi}_{\mathbf{u}}^{\mathbf{v}}(z_2) \rangle_{\mathbf{T}^2}} \tag{5.19}$$

where $\chi_{\mathbf{g}}(\mathcal{M})(\vec{m}, \tau)$ is the $\mathbf{g}$-equivariant elliptic genus of $\mathcal{M}$ which depends on the $2N$ fundamental matter masses $\vec{m}$ and the complex structure τ of the two-torus Σ_t whose independent one-cycles have radii β and R_6.

6 A 6d "fully-ramified" AGT correspondence and an elliptized integrable system

6.1 A 6d "fully-ramified" AGT correspondence with matter and an elliptized integrable system
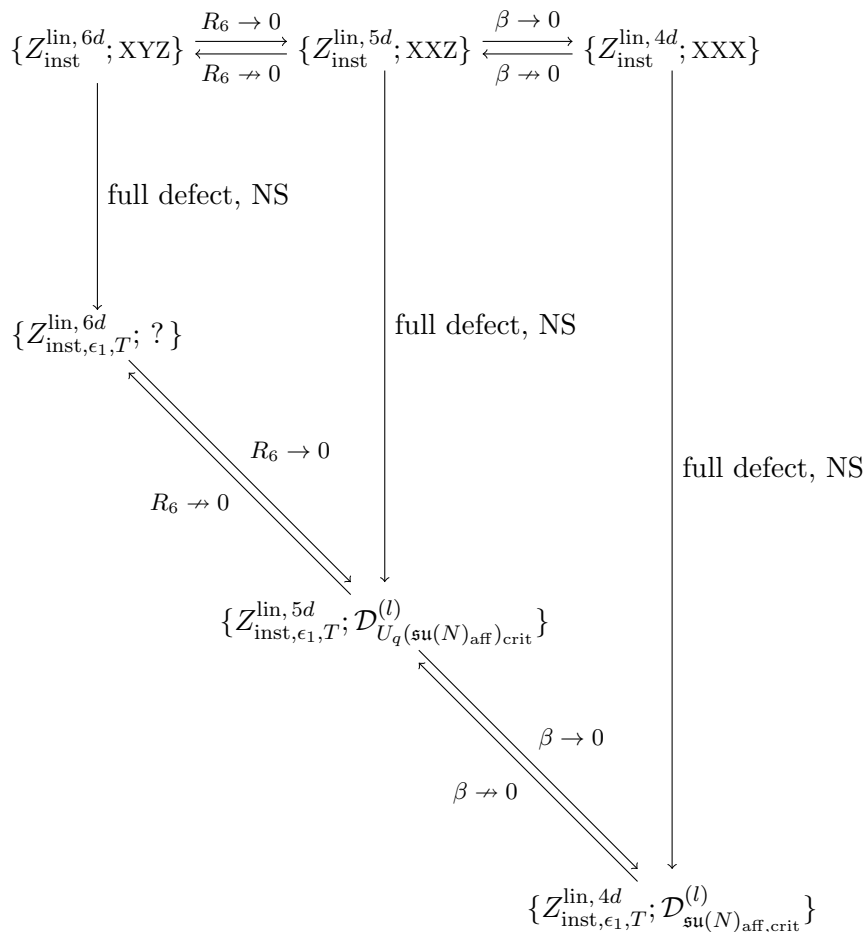
The "fully-ramified" 6d Nekrasov instanton partition function with $N_f = 2N$ fundamental matter in the NS limit and an elliptized integrable system. Let $Z_{\text{inst, SU}(N)}^{\text{lin, 5d(6d)}}(q_1, \epsilon_1, 0, \vec{a}, \vec{m}, \beta(, R_6), T)$ denote the "fully-ramified" 5d(6d) Nekrasov instanton partition function in the NS limit $\epsilon_2 = 0$ of the SU(N) theory on $\mathbb{R}^4|_{\epsilon_1, \epsilon_2 = 0} \times \mathbf{S}^1(\times \mathbf{S}_t^1)$ with $N_f = 2N$ fundamental matter of masses $\vec{m}$, Coulomb moduli $\vec{a}$, complexified gauge coupling τ' where $q_1 = e^{2\pi i \tau'}$, and $\mathbf{S}^1$ (and $\mathbf{S}_t^1$) radius β (and R_6), such that

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[12]This is true because the elliptic Ding-Iohara algebra acts on the N-tensor space $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_N$ as $\mathcal{W}_N^{q,p}$, an elliptic analog of $\mathcal{W}_N^q$, a q-deformed affine $\mathcal{W}_N$-algebra. I would like to thank Y. Saito for an explanation of this point.

[13]Here, $\mathbf{g} \in$ U(1) $\times$ U(1) $\times T$, where $T \subset$ SU(N) is a Cartan subgroup.

the SU(N) gauge group reduces to its maximal torus T along $\mathbb{R}^2|_{\epsilon_1} \subset \mathbb{R}^4|_{\epsilon_1,\epsilon_2=0}$. Let $Z^{\mathrm{lin},\,4\mathrm{d}}_{\mathrm{inst},\,\mathrm{SU}(N)}(q_1,\epsilon_1,0,\vec{a},\vec{m},T)$ denote the same thing in 4d over $\mathbb{R}^4|_{\epsilon_1,\epsilon_2=0}$.

Note that the integrable system associated with the "unramified", non-Omega-deformed Nekrasov instanton partition function $Z^{\mathrm{lin},\,6\mathrm{d}}_{\mathrm{inst},\,\mathrm{SU}(N)}(q_1,0,0,\vec{a},\vec{m},\beta,R_6,\mathrm{SU}(N))$, $Z^{\mathrm{lin},\,5\mathrm{d}}_{\mathrm{inst},\,\mathrm{SU}(N)}(q_1,0,$

$0,\vec{a},\vec{m},\beta,\mathrm{SU}(N))$ and $Z^{\mathrm{lin},\,4\mathrm{d}}_{\mathrm{inst},\,\mathrm{SU}(N)}(q_1,0,0,\vec{a},\vec{m},\mathrm{SU}(N))$, is an XYZ, XXZ and SU(2) XXX spin chain, respectively [45, 46]. Upon adding a full defect and turning on Omega-deformation in the NS limit, (i) according to [8, section 7.2], $Z^{\mathrm{lin},\,4\mathrm{d}}_{\mathrm{inst},\,\mathrm{SU}(N)}(q_1,\epsilon_1,0,\vec{a},\vec{m},T) = Z^{\mathrm{lin},\,4d}_{\mathrm{inst},\epsilon_1,T}$ becomes a simultaneous eigenfunction of commuting differential operators defined by the central elements of (the universal enveloping algebra of) an affine SU(N)-algebra at the critical level; (ii) according to (4.18)–(4.19) and the discussion leading up to them, and our discussion in section 3.1 (which also applies to the 5d case with a defect), $Z^{\mathrm{lin},\,5\mathrm{d}}_{\mathrm{inst},\,\mathrm{SU}(N)}(q_1,0,0,\vec{a},\vec{m},\beta,\mathrm{SU}(N)) = Z^{\mathrm{lin},\,5d}_{\mathrm{inst},\epsilon_1,T}$ becomes a simultaneous eigenfunction of commuting difference operators defined by the central elements of a quantum affine SU(N)-algebra at the critical level; (iii) $Z^{\mathrm{lin},\,6\mathrm{d}}_{\mathrm{inst},\,\mathrm{SU}(N)}(q_1,$

$0,0,\vec{a},\vec{m},\beta,R_6,\mathrm{SU}(N)) = Z^{\mathrm{lin},\,6d}_{\mathrm{inst},\epsilon_1,T}$ becomes a simultaneous eigenfunction of a set of commuting operators defined by the central elements of some algebra. These results are summarized in the diagram below.

$$\{Z^{\mathrm{lin},\,6d}_{\mathrm{inst}};\mathrm{XYZ}\} \xrightleftharpoons[R_6 \nrightarrow 0]{R_6 \to 0} \{Z^{\mathrm{lin},\,5d}_{\mathrm{inst}};\mathrm{XXZ}\} \xrightleftharpoons[\beta \nrightarrow 0]{\beta \to 0} \{Z^{\mathrm{lin},\,4d}_{\mathrm{inst}};\mathrm{XXX}\}$$

full defect, NS

$$\{Z^{\mathrm{lin},\,6d}_{\mathrm{inst},\epsilon_1,T};\,?\,\}$$

full defect, NS

$R_6 \to 0$

$R_6 \nrightarrow 0$

full defect, NS

$$\{Z^{\mathrm{lin},\,5d}_{\mathrm{inst},\epsilon_1,T};\mathcal{D}^{(l)}_{U_q(\mathfrak{su}(N)_{\mathrm{aff}})_{\mathrm{crit}}}\}$$

$\beta \to 0$

$\beta \nrightarrow 0$

$$\{Z^{\mathrm{lin},\,4d}_{\mathrm{inst},\epsilon_1,T};\mathcal{D}^{(l)}_{\mathfrak{su}(N)_{\mathrm{aff,crit}}}\}$$

We shall now argue that $Z_{\text{inst},\epsilon_1,T}^{\text{lin},6d}$ is a simultaneous eigenfunction of a set of commuting operators defined by the central elements of an elliptic affine algebra. To this end, first note that the XXX and XXZ integrable systems have underlying dynamical symmetries generated [45, 46] by $Y(\mathfrak{g})$ and $U_q(\mathfrak{g}_{\text{aff}})$,[14] which are a Yangian and quantum affine algebra, respectively, such that $Y(\mathfrak{g})$ acts [47] on the irreducible integrable representations of $\mathfrak{g}_{\text{aff}}$. In other words, we can associate $\mathfrak{g}_{\text{aff}}$ and $U_q(\mathfrak{g}_{\text{aff}})$ with the symmetries of the XXX and XXZ integrable systems. This is reflected in the fact that $Z_{\text{inst},\epsilon_1,T}^{\text{lin},4d}$ and $Z_{\text{inst},\epsilon_1,T}^{\text{lin},5d}$ are simultaneous eigenfunctions of commuting operators $\mathcal{D}_{\mathfrak{su}(N)_{\text{aff,crit}}}^{(l)}$ and $\mathcal{D}_{U_q(\mathfrak{su}(N)_{\text{aff}})_{\text{crit}}}^{(l)}$ which are defined purely by an affine and quantum affine Lie algebra, respectively, as shown in the diagram above.

Second, note that the XYZ integrable system has an underlying dynamical symmetry generated [48] by $E_{\tau,\eta}(\mathfrak{g})$,[15] which is an elliptic quantum group first defined by Felder in [49]. Further note that the representations of $E_{\tau,\eta}(\mathfrak{g})$ are isomorphic to the representations of $U_{p,q}(\mathfrak{g}_{\text{aff}})$ [50], an elliptic affine G-algebra first defined by Konno in [51] that is a two-parameter generalization of $U_q(\mathfrak{g}_{\text{aff}})$, i.e., $U_{0,q}(\mathfrak{g}_{\text{aff}}) = U_q(\mathfrak{g}_{\text{aff}})$. In sum, this means that we can associate $U_{p,q}(\mathfrak{g}_{\text{aff}})$ with the symmetry of the XYZ integrable system.

Third, let us identify p with R_6 and q with e^β. Then, as $R_6 \to 0$, we have $U_{p,q}(\mathfrak{g}_{\text{aff}}) \to U_q(\mathfrak{g}_{\text{aff}})$, and when $\beta \to 0$, we have $U_q(\mathfrak{g}_{\text{aff}}) \to U(\mathfrak{g}_{\text{aff}})$, the universal enveloping algebra of $\mathfrak{g}_{\text{aff}}$. These statements hold vice-versa, so that with regard to the central elements of these algebras at the critical level, we have

$$\mathcal{D}_{U_{p,q}(\mathfrak{g}_{\text{aff}})_{\text{crit}}}^{(l)} \xleftarrow[R_6 \nrightarrow 0]{R_6 \to 0} \mathcal{D}_{U_q(\mathfrak{g}_{\text{aff}})_{\text{crit}}}^{(l)} \xleftarrow[\beta \nrightarrow 0]{\beta \to 0} \mathcal{D}_{\mathfrak{g}_{\text{aff,crit}}}^{(l)}$$

where the $\mathcal{D}_{U_{p,q}(\mathfrak{g}_{\text{aff}})_{\text{crit}}}^{(l)}$'s are a set of commuting operators defined by the central elements of $U_{p,q}(\mathfrak{g}_{\text{aff}})_{\text{crit}}$, just as the $\mathcal{D}_{U_q(\mathfrak{g}_{\text{aff}})_{\text{crit}}}^{(l)}$'s and $\mathcal{D}_{\mathfrak{g}_{\text{aff,crit}}}^{(l)}$'s are sets of commuting operators defined by the central elements of $U_q(\mathfrak{g}_{\text{aff}})_{\text{crit}}$ and $U(\mathfrak{g}_{\text{aff}})_{\text{crit}}$, respectively.

Altogether, the preceding three paragraphs and diagram mean that

$$\boxed{\mathcal{D}_{U_{p,q}(\mathfrak{su}(N)_{\text{aff}})_{\text{crit}}}^{(l)} \cdot Z_{\text{inst,SU}(N)}^{\text{lin},6d}(q_1, \epsilon_1, 0, \vec{a}, \vec{m}, \beta, R_6, T) = \mathcal{E}_{U_{p,q}}^{(l)} Z_{\text{inst,SU}(N)}^{\text{lin},6d}(q_1, \epsilon_1, 0, \vec{a}, \vec{m}, \beta, R_6, T)}$$
(6.1)

where the $\mathcal{E}_{U_{p,q}}^{(l)}$'s are complex eigenvalues; $l = 1, 2, \ldots, N-1$; and the $N-1$ commuting operators $\mathcal{D}_{U_{p,q}(\mathfrak{su}(N)_{\text{aff}})_{\text{crit}}}^{(1)}, \ldots, \mathcal{D}_{U_{p,q}(\mathfrak{su}(N)_{\text{aff}})_{\text{crit}}}^{(N-1)}$ can be identified with the generators of a classical two-parameter-deformed affine $\mathcal{W}_N$-algebra [52]. As claimed, $Z_{\text{inst},\epsilon_1,T}^{\text{lin},6d}$ is a simultaneous eigenfunction of a set of commuting operators defined by the central elements of $U_{p,q}(\mathfrak{su}(N)_{\text{aff}})_{\text{crit}}$, an elliptic affine SU($N$)-algebra at the critical level, whence (6.1) defines an ellipticized (i.e. two-parameter deformed) integrable system.

A 6d "fully-ramified" AGT correspondence for SU(N) with $N_f = 2N$ fundamental matter in the NS limit. According to section 3.1 and its obvious generalization

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[14] Here, the Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$, and $\mathfrak{g}_{\text{aff}}$ is its affine generalization.

[15] Here, the Lie algebra is again $\mathfrak{g} = \mathfrak{su}(2)$.

to the "fully-ramified" 6d case, (6.1) and (5.1) would mean that

$$Z_{\text{inst, SU}(N)}^{\text{lin, 6d}}(q_1, \epsilon_1, 0, \vec{a}, \vec{m}, \beta, R_6, T) = \langle \tilde{\Phi}_{\vec{a}}^{\vec{m}_1}(z_1) \tilde{\Phi}_{\vec{m}_2}^{\vec{a}}(z_2) \rangle_{\mathbf{T}^2} \tag{6.2}$$

where $\vec{m}_1 = (m_1, \ldots, m_N)$ and $\vec{m}_2 = (m_{N+1}, \ldots, m_{2N})$ are the masses of the $N_f = 2N$ fundamental matter, and $\tilde{\Phi}_{\vec{a}}^{\vec{m}_i}$ is a $(\vec{m}_i, \vec{a})$-dependent vertex operator of $\widehat{U_{p,q}(\mathfrak{su}(N)_{\text{aff}})_{\text{crit}}}$, an integrable module over $U_{p,q}(\mathfrak{su}(N)_{\text{aff}})_{\text{crit}}$, i.e.,

$$\tilde{\Phi}_{\vec{a}}^{\vec{m}_i} : \widehat{U_{p,q}(\mathfrak{su}(N)_{\text{aff}})_{\text{crit}}} \to \widehat{U_{p,q}(\mathfrak{su}(N)_{\text{aff}})_{\text{crit}}} \tag{6.3}$$

Clearly, in arriving at the boxed relations (6.2) and (6.3), we have just derived a 6d "fully-ramified" AGT correspondence for SU(N) with $N_f = 2N$ fundamental matter!

Acknowledgments

References

[1] L.F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, *Lett. Math. Phys.* **91** (2010) 167 [arXiv:0906.3219] [INSPIRE].

[2] D. Gaiotto, *Asymptotically free N = 2 theories and irregular conformal blocks*, arXiv:0908.0307 [INSPIRE].

[3] N. Wyllard, A_{N-1} *conformal Toda field theory correlation functions from conformal N = 2* SU(N) *quiver gauge theories*, *JHEP* **11** (2009) 002 [arXiv:0907.2189] [INSPIRE].

[4] A. Mironov and A. Morozov, *On AGT relation in the case of* U(3), *Nucl. Phys.* **B 825** (2010) 1 [arXiv:0908.2569] [INSPIRE].

[5] L.F. Alday and Y. Tachikawa, *Affine* SL(2) *conformal blocks from 4d gauge theories*, *Lett. Math. Phys.* **94** (2010) 87 [arXiv:1005.4469] [INSPIRE].

[6] O. Schiffmann and E. Vasserot, *Cherednik algebras, W algebras and the equivariant cohomology of the moduli space of instantons on* $\mathbb{A}^2$, arXiv:1202.2756.

[7] D. Maulik and A. Okounkov, *Quantum groups and quantum cohomology*, arXiv:1211.1287 [INSPIRE].

[8] M.-C. Tan, *M-theoretic derivations of 4d-2d dualities: from a geometric Langlands duality for surfaces, to the AGT correspondence, to integrable systems*, *JHEP* **07** (2013) 171 [arXiv:1301.1977] [INSPIRE].

[9] N. Wyllard, *Instanton partition functions in N = 2* SU(N) *gauge theories with a general surface operator and their W-algebra duals*, *JHEP* **02** (2011) 114 [arXiv:1012.1355] [INSPIRE].

[10] H. Kanno and Y. Tachikawa, *Instanton counting with a surface operator and the chain-saw quiver*, *JHEP* **06** (2011) 119 [arXiv:1105.0357] [INSPIRE].

[11] A. Braverman, *Instanton counting via affine Lie algebras. 1. Equivariant J functions of (affine) flag manifolds and Whittaker vectors*, math/0401409 [ɪɴSPIRE].

[12] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, in *The Unity of Mathematics*, *Progr. Math.* **244** (2006) 525 [hep-th/0306238] [ɪɴSPIRE].

[13] J.-T. Ding and K. Iohara, *Generalization and deformation of Drinfeld quantum affine algebras*, *Lett. Math. Phys.* **41** (1997) 181 [q-alg/9608002] [ɪɴSPIRE].

[14] H. Awata et al., *Notes on Ding-Iohara algebra and AGT conjecture*, arXiv:1106.4088 [ɪɴSPIRE].

[15] H. Awata and Y. Yamada, *Five-dimensional AGT conjecture and the deformed Virasoro algebra*, *JHEP* **01** (2010) 125 [arXiv:0910.4431] [ɪɴSPIRE].

[16] E. Witten, *Topological σ-models*, *Commun. Math. Phys.* **118** (1988) 411 [ɪɴSPIRE].

[17] M.-C. Tan, *Equivariant cohomology of the chiral de Rham complex and the half-twisted gauged σ-model*, *Adv. Theor. Math. Phys.* **13** (2009) 897 [hep-th/0612164] [ɪɴSPIRE].

[18] S.V. Ketov, *Conformal field theory*, World Scientific Press, Singapore (1997) [ɪɴSPIRE].

[19] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal field theory*, Springer-Verlag, New York U.S.A. (1999) [ɪɴSPIRE].

[20] P. Bouwknegt and K. Schoutens, *W symmetry in conformal field theory*, *Phys. Rept.* **223** (1993) 183 [hep-th/9210010] [ɪɴSPIRE].

[21] N. Nekrasov and S. Shadchin, *ABCD of instantons*, *Commun. Math. Phys.* **252** (2004) 359 [hep-th/0404225] [ɪɴSPIRE].

[22] M. Atiyah and R. Bott, *The moment map and equivariant cohomology*, *Topology* **23** (1984) 1 [ɪɴSPIRE].

[23] W. Lerche, *Introduction to Seiberg-Witten theory and its stringy origin*, *Nucl. Phys. Proc. Suppl.* **55B** (1997) 83 [*Fortsch. Phys.* **45** (1997) 293] [hep-th/9611190] [ɪɴSPIRE].

[24] B.A. Khesin and R. Wendt, *The geometry of infinite-dimensional groups*, Springer, New York U.S.A. (2008).

[25] D. Gaiotto, *N = 2 dualities*, *JHEP* **08** (2012) 034 [arXiv:0904.2715] [ɪɴSPIRE].

[26] A. Gorsky and N. Nekrasov, *Relativistic Calogero-Moser model as gauged WZW theory*, *Nucl. Phys.* **B 436** (1995) 582 [hep-th/9401017] [ɪɴSPIRE].

[27] E. Carlsson and A. Okounkov, *Exts and vertex operators*, arXiv:0801.2565.

[28] B. Feigin, A. Hoshino, J. Shibahara, J. Shiraishi and S. Yanagida, *Kernel function and quantum algebras*, arXiv:1002.2485.

[29] H. Itoyama, T. Oota and R. Yoshioka, *2d-4d connection between q-Virasoro/W block at root of unity limit and instanton partition function on ALE space*, *Nucl. Phys.* **B 877** (2013) 506 [arXiv:1308.2068] [ɪɴSPIRE].

[30] L. Baulieu, A. Losev and N. Nekrasov, *Chern-Simons and twisted supersymmetry in various dimensions*, *Nucl. Phys.* **B 522** (1998) 82 [hep-th/9707174] [ɪɴSPIRE].

[31] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2004) 831 [hep-th/0206161] [ɪɴSPIRE].

[32] P. Etingof, *Whittaker functions on quantum groups and q-deformed Toda operators*, math/9901053.

[33] E.J. Martinec and N.P. Warner, *Integrable systems and supersymmetric gauge theory*, [*Nucl. Phys.* **B 459** (1996) 97](Nucl.Phys.B459(1996)97) [[hep-th/9509161](hep-th/9509161)] [INSPIRE].

[34] N. Nekrasov, *Five dimensional gauge theories and relativistic integrable systems*, [*Nucl. Phys.* **B 531** (1998) 323](Nucl.Phys.B531(1998)323) [[hep-th/9609219](hep-th/9609219)] [INSPIRE].

[35] J.-T. Ding and P. Etingof, *Center of a quantum affine algebra at the critical level*, submitted to *Math. Res. Lett.* (1994) [[hep-th/9403064](hep-th/9403064)] [INSPIRE].

[36] E. D'Hoker and D. Phong, *Seiberg-Witten theory and Calogero-Moser systems*, [*Prog. Theor. Phys. Suppl.* **135** (1999) 75](Prog.Theor.Phys.Suppl.135(1999)75) [[hep-th/9906027](hep-th/9906027)] [INSPIRE].

[37] H. Itoyama and A. Morozov, *Integrability and Seiberg-Witten theory: curves and periods*, [*Nucl. Phys.* **B 477** (1996) 855](Nucl.Phys.B477(1996)855) [[hep-th/9511126](hep-th/9511126)] [INSPIRE].

[38] H. Itoyama and A. Morozov, *Prepotential and the Seiberg-Witten theory*, [*Nucl. Phys.* **B 491** (1997) 529](Nucl.Phys.B491(1997)529) [[hep-th/9512161](hep-th/9512161)] [INSPIRE].

[39] S.N.M. Ruijsenaars, *Elliptic integrable systems of Calogero-Moser type: a survey*, www.math.kobe-u.ac.jp/publications/rlm18/13.pdf.

[40] K. Hasegawa, *Ruijsenaars' commuting difference operators as commuting transfer matrices*, [q-alg/9512029](q-alg/9512029).

[41] P. Etingof, *Central elements for quantum affine algebras and affine Macdonald's operators*, [q-alg/9412007](q-alg/9412007).

[42] P.I. Etingof and A. Kirillov Jr., *On the affine analog of Jack's and MacDonald's polynomials*, [hep-th/9403168](hep-th/9403168) [INSPIRE].

[43] Y. Saito, *Elliptic Ding-Iohara algebra and the free field realization of the elliptic Macdonald operator*, arXiv:1301.4912.

[44] T.J. Hollowood, A. Iqbal and C. Vafa, *Matrix models, geometric engineering and elliptic genera*, [*JHEP* **03** (2008) 069](JHEP03(2008)069) [[hep-th/0310272](hep-th/0310272)] [INSPIRE].

[45] A. Gorsky, S. Gukov and A. Mironov, *Multiscale $N = 2$ SUSY field theories, integrable systems and their stringy/brane origin. 1*, [*Nucl. Phys.* **B 517** (1998) 409](Nucl.Phys.B517(1998)409) [[hep-th/9707120](hep-th/9707120)] [INSPIRE].

[46] A. Gorsky, S. Gukov and A. Mironov, *SUSY field theories, integrable systems and their stringy/brane origin. 2*, [*Nucl. Phys.* **B 518** (1998) 689](Nucl.Phys.B518(1998)689) [[hep-th/9710239](hep-th/9710239)] [INSPIRE].

[47] D. Uglov, *Yangian actions on higher level irreducible integrable modules of affine $gl(N)$*, math/9802048.

[48] N. Cirilo-Antonio, *Algebraic Bethe ansatz for deformed Gaudin model*, http://www.fctec.ualg.pt/qisg/slides/Antonio.pdf.

[49] G. Felder, *Elliptic quantum groups*, [hep-th/9412207](hep-th/9412207) [INSPIRE].

[50] H. Konno, part of work in progress.

[51] H. Konno, *q-deformation of the coset conformal field theory and the fusion RSOS model*, talks given at the *XII*[th] *International Congress of Mathematical Physics*, Brisbane Australia July 13–19 1997.

[52] H. Konno, private communication.