

M5/D4 brane partition function on a circle bundle

Dongsu Bak and Andreas Gustavsson

*Physics Department, University of Seoul,
13 Siripdae, Seoul 130-743 Korea*

E-mail: dsbak@uos.ac.kr, agbrev@gmail.com

ABSTRACT: We consider Abelian M5 brane on a six-manifold which we take as a circle bundle over a five-manifold M . We compute the zero mode part of the M5 brane partition function using Chern-Simons theory and Hamiltonian formulation respectively and find an agreement. We also show that the D4 brane on M shares exactly the same zero mode partition function again using the Hamiltonian formulation. For the oscillator modes we find that KK modes associated with the circle compactification are missing from the D4 brane. By making an infinitesimal noncommutative deformation we have instanton threshold bound states. We explicitly compute the instanton partition function up to instanton charge three, and show a perfect match with a corresponding contribution inside the M5 brane partition function, thus providing a very strong supporting evidence that D4 brane is identical with M5 brane which extends beyond the BPS sector. We comment on the modular properties of the M5 brane partition function when compactified on T^2 times a four-manifold. We briefly discuss a case of a singular fibration.

KEYWORDS: Non-Commutative Geometry, Chern-Simons Theories, Field Theories in Higher Dimensions, M-Theory

ARXIV EPRINT: [1209.4391](https://arxiv.org/abs/1209.4391)

Contents

1	Introduction	2
2	M5 brane from Chern-Simons theory	4
2.1	Zero mode part	7
2.2	Oscillators	13
3	The M5 brane partition function	14
3.1	Scalar field	15
3.2	Tensor gauge field	16
3.3	An alternative treatment of selfdual tensor gauge field	19
4	The D4 brane partition function	20
4.1	Scalar field	21
4.2	Gauge field	22
5	Oscillators	24
5.1	Selfdual tensor field	24
5.1.1	Two-torus	24
5.1.2	Six-manifold	25
5.1.3	Six-torus	26
5.2	Scalar fields	28
5.3	Fermions	28
6	Modular invariance of M5 brane on $T^2 \times M_4$	29
6.1	The zero mode part	30
6.2	Oscillator mode part	31
7	The M5/D4 partition function on T^6	33
8	Small instantons and KK modes	34
8.1	One instanton dynamics	36
8.2	Multi instanton dynamics	38
9	Singular fibration	42
A	Period matrix on $S^1 \times M_5$	44
B	Period matrix on $T^2 \times M_4$	46
C	Holomorphic factorization	48

1 Introduction

On the M5 brane worldvolume W there lives a selfdual two-form B , whose field strength $H = dB$ is selfdual. There is no covariant action one can write down for a selfdual two-form (without including auxiliary fields) because $H \wedge H = 0$. If one computes the partition function of a non-selfdual two-form, one finds for the zero mode part a sum of terms

$$\mathcal{Z}_{\text{zero}} = \sum_{a,b} \Theta \begin{bmatrix} a \\ b \end{bmatrix} (\tau) \overline{\Theta \begin{bmatrix} a \\ b \end{bmatrix} (\tau)}$$

Here τ is the period matrix on the intermediate Jacobian

$$J_W = H^3(W, \mathbb{R})/H^3(W, \mathbb{Z})$$

and a^i, b_i are characteristics each of them taking values 0 or $\frac{1}{2}$, and $i = 1, \dots, \frac{b_3}{2}$ where $b_3 = \dim H^3(W, \mathbb{R})$. $\Theta \begin{bmatrix} a \\ b \end{bmatrix} (\tau)$ denotes the Jacobi theta function, and one of these theta functions will correspond to the partition function of a selfdual two-form on W embedded in some eleven-manifold. In general it appears the M5 brane partition function is not completely diffeomorphism invariant. To see this, let us take the worldvolume as $W = T^2 \times \mathbb{C}\mathbb{P}^2$ [1].¹ We have precisely one selfdual harmonic two-form on $\mathbb{C}\mathbb{P}^2$ and the theory of selfdual two-form reduces to the theory of a compact chiral scalar field on T^2 . It is well-known that no modular invariant partition function exists for a compact chiral scalar on T^2 . What we can require is the M5 brane partition function be invariant only under diffeomorphisms that preserve the physical data which determine its characteristics. Such physical data may consist of the orientation of W and the spin structure of W [1]. How the spin structure dependence on the characteristics appears is complicated and we will not address the question in any satisfactory way. It appears that the framework where this question can be answered is within seven-dimensional Chern-Simons theory. In the first part of this paper we follow [1] to compute the partition function of a selfdual two-form on a generic six-manifold from Chern-Simons theory, but leave the determination of the characteristics aside perhaps for the future.

We next perform a direct computation of the M5 brane partition function starting from the M5 brane action. Previous works on direct computations include [2–4]. The separation of the partition function into a zero mode part and an oscillator mode part is somewhat intricate. We can illustrate this by taking a real compact 2d scalar on T^2 . Its full modular invariant partition function is given by

$$\mathcal{Z} = \sum_{a,b} \left| \frac{\Theta \begin{bmatrix} a \\ b \end{bmatrix} (\tau)}{\eta(\tau)} \right|^2$$

¹One may object that $\mathbb{C}\mathbb{P}^2$ has no spin structure. However the fermions on the M5 brane also carry an R-symmetry index and such fermions may live on $T^2 \times \mathbb{C}\mathbb{P}^2$. It is the eleven-dimensional space which must be a spin manifold, rather than the M5 brane worldvolume. One may also consider Taub-NUT space TN in place of $\mathbb{C}\mathbb{P}^2$ which also has a unique normalizable selfdual harmonic two-form. However this space is noncompact and so it might be less clear how to define the full partition function.

Here $\eta(\tau)$ denotes the Dedekind eta function. If we use the Hamiltonian formulation, then we are led to the following separation of this partition function into zero mode and oscillator mode parts, [5]

$$\mathcal{Z}_{\text{zero}} = \sum_{a,b} |\Theta_{[b]}^{[a]}(\tau)|^2$$

$$\mathcal{Z}_{\text{osc}} = \frac{1}{|\eta(\tau)|^2}$$

neither of which are modular invariant by themselves, and only the product of these is modular invariant. If on the other hand we use the Lagrangian formulation, then it is not so obvious how we shall define the measure in the path integral. One way is to define the measure so that we end up with the following separation [6]

$$\mathcal{Z}_{\text{zero}} = \tau_2 \sum_{a,b} |\Theta_{[b]}^{[a]}(\tau)|^2$$

$$\mathcal{Z}_{\text{osc}} = \frac{1}{\tau_2 |\eta(\tau)|^2}$$

again neither of which are modular invariant by themselves.²

One of our goals in this paper is to establish an equivalence between M5 brane partition function on a circle-bundle over some five-manifold M_5 , and a corresponding partition function for D4 brane on M_5 obtained by dimensional reduction along the fiber. To this end we need to preserve the separation between zero modes and oscillator modes in our comparison between M5 and D4 brane. This means that we can not work in the Lagrangian formulation to obtain the zero mode contribution of the D4 brane partition function as was done recently in [7, 8], and try to match this zero mode contribution with the zero mode contribution of the M5 brane when that is computed using the Hamiltonian formulation. In fact a mismatch is shown explicitly to arise between zero modes for D4 and M5 brane by using this approach in [8]. But this mismatch is entirely due to the fact that different formulations are used to compute zero modes. By using the Hamiltonian formulation for both M5 and D4 brane, we will see that their zero mode contributions respectively perfectly agree.

As we already indicated above, a careful treatment of zero mode and oscillator mode contributions is also crucial for modular invariance. In [2] it was shown that for the selfdual tensor field part of M5 brane on T^6 this gives rise to an $SL(6, \mathbb{Z})$ modular invariant partition function for the choice of 00-characteristics. For this we need to combine zero-mode and oscillator-mode contributions. In this paper we also show that the scalar fields part alone is $SL(6, \mathbb{Z})$ invariant by combining zero mode and oscillator mode contributions. For generic six manifolds of the form $T^2 \times M_4$ we study $SL(2, \mathbb{Z})$ mapping class group acting on the T^2 . Here we show that the M5 brane partition function with 00-characteristic is modular invariant up to a phase.

Considering now the oscillator modes of D4 on M_5 , we find that all KK-modes along the circle direction are missing. Thus, at this stage, we find a mismatch of the partition functions in their oscillator parts. But of course this is not the end of the story. Taking S^1

²A modular invariant combination is $\sqrt{\tau_2}|\eta(\tau)|^2$.

as an M-theory circle direction, a KK momentum along the circle direction is interpreted as a D0 brane in the type IIA string theory side. When D0 branes are bound to D4, their dynamics can be faithfully described by worldvolume degrees of freedom on the D4 brane. These are nothing but instantonic particles which satisfy the (anti) selfdual equation along the four spatial directions of D4. For our U(1) case, however, the corresponding instanton solutions become singular and, consequently, missed when one considers only regular configurations. In order to regularize them, we introduce a spatial noncommutativity which makes the size of instanton finite and, then, take the commutative limit in the end recovering the original symmetries of D4 brane.

One may wonder whether the D0 brane could escape the D4 brane when it becomes singular in the commutative limit. However in [9] it was argued that the Higgs and Coulomb branches are decoupled even when the instanton shrinks to zero size. This argument was further supported by an index calculation made in [10] for instanton number one.³

In this paper we show that these small instanton contributions match precisely with those of the missing KK modes, which is in accordance with the original idea of the proposal in [11, 12]. In particular one can be rather explicit for the one-instanton case. For two instantons, in addition to the Hilbert space of two separate instantons, one can find so-called threshold bound state in their moduli space which corresponds to a single $p_5 = 2$ KK mode of the M5 side [13]. This counting can be continued to higher instanton numbers [14].

Finally we discuss the case of singular fibration in which the circle size R becomes zero at some points of base manifold [15]. For this, in addition to the usual 5d sYM fields on D4 worldvolume, one may need some extra degrees localized at the singular points in order to have a full agreement. We illustrate this phenomenon rather explicitly in the example of $TN \times T^2$ where one has a codimension-four singularity if one takes the M-theory circle direction to be that of the Taub-NUT circle. We show that the extra degree needed is a chiral 2d scalar on T^2 .

Other works which are of some relevance to various aspects of our work include [16, 17].

In appendix A, we obtain the period matrix for six-manifolds W which are such types of circle-bundles in which the fiber-circle constitutes a one-cycle in W . So for instance this excludes the case of $W = S^3 \times S^3$. Even though S^3 is a circle-bundle there is no one-cycle in S^3 . In appendix B we obtain the period matrix for $W = T^2 \times M_4$ where for simplicity we assume there are no one-cycles on M_4 . In appendix C we present holomorphic factorization of the partition function of a nonchiral boson (applicable to 2d scalar and 6d two-form) at the free fermion radius.

2 M5 brane from Chern-Simons theory

We follow [1] and consider the action of a non-chiral two-form B such that only its selfdual part couples to a background three-form gauge potential C whose field strength is $G = dC$,

$$S[B, C] = -\frac{\lambda}{2} \int_W (|H|^2 + 2B \wedge G) \tag{2.1}$$

³We would like to thank Sethi for pointing out this problem to us, as well as informing us about his reference.

Here λ is a coupling constant, W denotes the six-dimensional world-volume of the M5 brane, $|H|^2 = H \wedge *H$ where we define $H = dB + C$. Our definition of Hodge dual is found in appendix A. This action is the six-dimensional analog of a gauged WZW action on a two-manifold to which there are many similarities. The action is not invariant under the gauge symmetry $\delta C = d\Lambda$, $\delta B = -\Lambda$ but it transforms as

$$\delta S[B, C] = \lambda \int_W \Lambda \wedge G$$

As shown in [1] this gauge anomaly is canceled by the term $\sim \int C \wedge G \wedge G$ in the M-theory effective action. For our purpose, the gauge anomaly will be helpful in order to match with Chern-Simons theory in seven dimensions, so for the moment we like to keep it, rather than canceling it. From the action, we derive the equation of motion $d * H = G$. We also have the Bianchi identity $dH = G$. On W we may decompose any three-form ω as $\omega = \omega^+ + \omega^-$ where $*\omega^+ = \pm\omega^\pm$. The equation of motion and the Bianchi identity are consistent with the selfduality equation $H^- = 0$ and it is in this sense that the action describes a selfdual three-form H^+ . Classically this selfduality equation is consistent for any value of λ . But as we will see, the situation changes in the quantum theory. If we introduce the inner product $(\omega, \eta) = \int_W \omega \wedge *\eta$ then we can write the M5 brane action in the form

$$S[B, C] = -\frac{\lambda}{2} ((dB, dB) + 2(C^+, C^-) + 4(dB, C^-))$$

Let us define the partition function of B in the background of C as

$$\mathcal{Z}(C) = \int \mathcal{D}B e^{iS[B, C]}$$

By differentiating under the integral sign, we then find that

$$\left(\frac{\delta}{\delta C^+} + i\lambda C^- \right) \mathcal{Z}(C) = 0$$

and

$$\left(d \frac{D}{DC^-} + 2i\lambda G \right) \mathcal{Z}(C) = 0$$

In the second relation we assume the equation of motion $dH^+ = G$ and we define a covariant derivatives as

$$\begin{aligned} \frac{D}{DC^-} &= \frac{\delta}{\delta C^-} - i\lambda C^+ \\ \frac{D}{DC^+} &= \frac{\delta}{\delta C^+} + i\lambda C^- \end{aligned}$$

Our functional derivatives are defined with respect to the inner product as $\frac{\delta}{\delta \omega}(\omega, \eta) = \eta$.

Let us now consider Chern-Simons theory at level k on a seven-manifold U bounded by W

$$S = \frac{k}{4\pi} \int_U C \wedge dC$$

As usual we can make a variation and read off the symplectic potential from the boundary term. If we assume that $\partial U = W$, then we have the symplectic potential

$$A = -\frac{k}{4\pi} \int_W C \wedge \delta C \tag{2.2}$$

which we can also write as

$$A = \frac{k}{4\pi} (-(C^-, \delta C^+) + (C^+, \delta C^-))$$

Here we have noted that the inner product of two selfdual or two antiselfdual three-forms vanish identically. Its curvature is the symplectic two-form

$$\Omega = \frac{k}{2\pi} (\delta C^+, \delta C^-)$$

In components we have

$$\Omega_{+-} = \frac{k}{2\pi}$$

and the inverse is

$$\Omega^{-+} = \frac{2\pi}{k}$$

We define covariant derivatives

$$\frac{D}{DC^\pm} = \frac{\delta}{\delta C^\pm} - iA_{C^\pm}$$

where

$$A_{C^\pm} = \mp \frac{k}{4\pi} C^\mp$$

consistent with

$$\lambda = \frac{k}{4\pi}$$

Gauge transformations act on a wave function $\psi(C)$ and the gauge potential as $\psi \rightarrow e^{i\Lambda}\psi$, $A_{C^\pm} \rightarrow A_{C^\pm} + \frac{\delta\Lambda}{\delta C^\pm}$. Under a gauge variation $\delta C = d\Lambda$ the Chern-Simons action varies by

$$\delta S = \frac{k}{4\pi} \int_W \Lambda \wedge G$$

This variation is identical with the gauge variation of the M5 brane action if we choose the coupling constant as $\lambda = \frac{k}{4\pi}$.

We impose holomorphic polarization of the wave function ψ ,

$$\frac{D}{DC^+} \psi = 0$$

In temporal gauge we have the Gauss law constraint $d(C^+ + C^-) = 0$. After quantization this becomes $d(P(C^+) + P(C^-))\psi = 0$ where the prequantum operators are given by

$$P(C^+) = -\frac{2\pi i}{k} \frac{D}{DC^-} + C^+$$

$$P(C^-) = \frac{2\pi i}{k} \frac{D}{DC^+} + C^-$$

The general solution to the holomorphic polarization condition is of the form

$$\psi(C^+, C^-) = \exp\left(\frac{ik}{4\pi}(C^+, C^-)\right)\phi(C^-)$$

and so

$$\begin{aligned} P(C^+) &= -\frac{2\pi i}{k}D_- + C^+ \\ P(C^-) &= C^- \end{aligned}$$

From this we get the quantum version (2.3) of Gauss law as

$$\left(d\frac{D}{DC^-} + \frac{ik}{2\pi}G\right)\psi = 0 \tag{2.3}$$

We see the same equations from the M5 brane action if we choose the coupling $\lambda = \frac{k}{4\pi}$. This implies that we shall identify the wave function ψ with the M5 brane partition function \mathcal{Z} .

2.1 Zero mode part

Let us now compute the zero mode part of the wave function ψ . We expand the harmonic part of C in a basis of harmonic three-forms on W . We can choose this as a symplectic basis. If we let a_i and b^i denote three-cycles in W , then α_i and β^i which denote the Poincaré dual harmonic three-forms which we normalize as

$$\begin{aligned} \frac{1}{2\pi} \int_{a_j} \alpha_i &= \delta_i^j \\ \frac{1}{2\pi} \int_{b^j} \beta^i &= \delta_j^i \end{aligned}$$

will constitute a symplectic basis on W in the sense that

$$\begin{pmatrix} \int_W \alpha_i \wedge \alpha_j & \int_W \alpha_i \wedge \beta^l \\ \int_W \beta^k \wedge \alpha_j & \int_W \beta^k \wedge \beta^l \end{pmatrix} = 4\pi^2 \begin{pmatrix} 0 & \delta_i^l \\ -\delta_j^k & 0 \end{pmatrix}$$

We will also define the following linear combinations

$$\begin{aligned} \omega_i &= \alpha_i + \tau_{ij}\beta^j \\ \bar{\omega}_i &= \alpha_i + \bar{\tau}_{ij}\beta^j \end{aligned}$$

by demanding these are selfdual and antiselfdual harmonic three-forms respectively. The period matrix τ_{ij} can then be extracted by integrating ω_i over the b^j -cycles,

$$\frac{1}{2\pi} \int_{b^j} \omega_i = \tau_{ij}$$

If we define

$$\begin{aligned} \tau &= -\tau_1 - \tau_2 \\ \bar{\tau} &= -\tau_1 + \tau_2 \end{aligned}$$

then we have

$$\int_W \omega_i \wedge \bar{\omega}_j = 8\pi^2 (\tau_2)_{ij}$$

We expand the harmonic part of the C -field as

$$C = x^i \alpha_i + y_i \beta^i$$

In terms of complex coordinates that we define as

$$\begin{aligned} a_i &= \bar{\tau}_{ij} x^j - y_i \\ \bar{a}_i &= \tau_{ij} x^j - y_i \end{aligned}$$

we have

$$C = \frac{1}{2} (\tau_2)^{ij} (a_i \omega_j - \bar{a}_i \bar{\omega}_j)$$

Large gauge transformations leave the holonomy

$$H(C) = e^{i \int_C C}$$

invariant for any choice of three-cycle C . Since the connection is constrained to be flat by the Chern-Simons action, the exponent defines an element in J_W which is a torus $T^{b_3(W)}$ whose coordinates are (x^i, y_j) subject to torus identifications

$$\begin{aligned} x^i &\sim x^i + 1 \\ y_j &\sim y_j + 1 \end{aligned}$$

Large gauge variations act on J_W as

$$\begin{aligned} \delta x^i &= n^i \\ \delta y_j &= m_j \end{aligned} \tag{2.4}$$

for integers n^i, m_j . The Chern-Simons Lagrangian becomes⁴

$$L = \frac{k\pi}{2} (\tau_2)^{ij} (-a_i \dot{\bar{a}}_j + \bar{a}_i \dot{a}_j)$$

and we get the symplectic potential and symplectic two-form as

$$A = \frac{k\pi}{2} (\tau_2)^{ij} (-a_i \delta \bar{a}_j + \bar{a}_i \delta a_j) \tag{2.5}$$

$$\Omega = -k\pi (\tau_2)^{ij} \delta a_i \wedge \delta \bar{a}_j \tag{2.6}$$

Canonical commutation relations are

$$[a_i, \bar{a}_j] = \frac{i}{k\pi} (\tau_2)_{ij} \tag{2.7}$$

⁴While a non-Abelian generalization of the M5 brane Lagrangian has not been found, one can probably more easily write down a non-Abelian generalization of the reduced Chern-Simons quantum mechanics Lagrangian on the space of zero modes (the intermediate Jacobian). Here on the intermediate Jacobian we have just complex-valued scalar fields a_i which depend only on time.

and prequantum operators are

$$P(a_i) = \frac{i}{k\pi}(\tau_2)_{ij}D_{\bar{a}_j} + a_i$$

$$P(\bar{a}_i) = -\frac{i}{k\pi}(\tau_2)_{ij}D_{a_j} + \bar{a}_i$$

which one may check realize the algebra (2.7).

Large gauge transformations act on the potential and the wave function as

$$A' = A + \delta\Lambda(n, m)$$

$$\psi' = e^{i\Lambda(n, m)}\psi$$

where we define

$$A'(a, \bar{a}) = A(a + \bar{\tau}n - m, a + \tau n - m)$$

$$\psi'(a, \bar{a}) = \psi(a + \bar{\tau}n - m, a + \tau n - m)$$

For the sake of convenience (and not for pedagogical reasons), we will switch back and forth between real and complex coordinates. As we want to see how various quantities depend on the various choices we make, we will now assume the symplectic potential is chosen as

$$A = 2\pi k x^i \delta y_i + \delta\mu(x^i, y_j)$$

with an arbitrary function μ . Such a gauge transformation of the symplectic potential corresponds to a canonical transformation of the phase space variables x^i, y_j . We impose holomorphic polarization on the wave function

$$D_{a_i}\psi(a, \bar{a}) = 0$$

This condition is solved by

$$\psi(a, \bar{a}) = e^{i(K(a, \bar{a}) + \mu(a, \bar{a}))}\phi(\bar{a})$$

where

$$K = \pi k \tau_{ij} x^i x^j + f(\bar{a})$$

is the Kahler potential. In the Kahler potential we have the freedom of adding an arbitrary antiholomorphic function f .

We can read off the gauge parameter associated to a large gauge transformation (2.4) from the variation of the symplectic potential

$$\delta A = A(x + n, y + m) - A(x, y) = \delta\Lambda(n, m)$$

This gives us the gauge parameter as

$$\Lambda(n, m) = 2\pi k n^i y_i + \mu(x + n, y + m) - \mu(x, y) + c(n, m)$$

where $c(n, m)$ is a further closed term that we can always add. From

$$e^{i\Lambda(n,m)}\psi(x, y) = \psi(x + n, y + m)$$

we find that μ cancels out and we are left with the condition that

$$e^{2\pi i k n^i y_i} e^{i c(n,m)} e^{i K(a,\bar{a})} \phi(\bar{a}) = e^{i K(a+\delta a, \bar{a}+\delta \bar{a})} \phi(\bar{a} + \delta \bar{a})$$

Here

$$K(a + \delta a, \bar{a} + \delta \bar{a}) - K(a, \bar{a}) = \pi k \tau_{ij} n^i n^j + 2\pi k \tau_{ij} n^i x^j + f(\bar{a} + \delta \bar{a}) - f(\bar{a})$$

and so we have, by noting that

$$n^i (y_i - \tau_{ij} x^j) = \bar{a}_i$$

that

$$\phi(\bar{a} + \delta \bar{a}) = e^{i c(n,m)} e^{f(\bar{a}+\delta \bar{a})-f(\bar{a})} e^{-i \pi k \tau_{ij} n^i n^j + 2\pi i k n^i \bar{a}_i} \phi(\bar{a})$$

Here

$$\delta \bar{a}_i = \tau_{ij} n^j - m_i$$

and $c(n, m)$ can be partially fixed by requiring that $\phi(\bar{a})$ satisfies the ‘cocycle condition’ of large gauge transformations

$$\phi(\bar{a} + \tau(n + n') - m - m') = \phi((\bar{a} + \tau n' - m') + \tau n - m) \quad (2.8)$$

Here, on the right-hand side our notation is supposed to mean that we first compute the gauge transformation $\phi(\bar{a}' + \tau n - m)$ in terms of $\phi(\bar{a}')$ and subsequently we express $\phi(\bar{a}')$ as a gauge transformation of $\phi(\bar{a})$. Let us first consider the case $f(\bar{a}) = 0$. We then find

$$c(n, m) = \pi k \tau_{ij} n^i m^j$$

is the minimal choice that makes the cocycle condition satisfied. But we can add further linear terms to this

$$c(n, m) = \pi k \tau_{ij} n^i m^j + 2\pi (c_i n^i + d^i m_i)$$

and still satisfy the cocycle condition. If we have non-vanishing f , we still get the same result for $c(n, m)$ due to the exponential nature of e^{if} which automatically makes the cocycle condition satisfied.

If we choose $f = 0$ then we can identify the gauge transformation of ϕ with the gauge transformation of a Jacobi theta function with certain characteristics, related to c_i and d^i as

$$\phi(\bar{a}) = \Theta \left[\begin{array}{c} \frac{c}{k} \\ -d \end{array} \right] (-k\tau | k\bar{a})$$

Let us now consider a three-cycle $C_{N,M}$ in W characterized by periods

$$\int_{C_{N,M}} \omega_i = M_i + \tau_{ij} N^j$$

and define the holonomy

$$H(N, M) = e^{ik \int_{C_{N,M}} C}$$

over this three-cycle. When we insert the zero mode expansion for C we get

$$H(N, M) = e^{i\pi k((M+\tau N)_i(\tau_2)^{ij} a_j - (M+\bar{\tau} N)_i(\tau_2)^{ij} \bar{a}_j)}$$

Since we do not make any canonical transformation of the phase space variable C when we define the holonomy, we shall use the symplectic potential that we obtain as the boundary term of the Chern-Simons action, that is (2.5). In this gauge we solve the polarization condition by

$$\psi(a, \bar{a}) = e^{-\frac{i\pi k}{2}(\tau_2)^{ij} a_i \bar{a}_j} \phi(\bar{a})$$

where $\phi(\bar{a})$ depends holomorphically on \bar{a} . The gauge variation of ψ now induces the gauge variation

$$\phi(\bar{a} + \tau n - m) = e^{\frac{i\pi k}{2}(\tau_2)^{ij}[(\bar{\tau} n - m)_i(\tau n - m)_j + 2(\bar{\tau} n - m)_i \bar{a}_j]} e^{ic(n,m)} \phi(\bar{a}) \quad (2.9)$$

As before we can partially fix $c(n, m)$ by demanding that we satisfy the cocycle condition (2.8). We then again find that

$$c(n, m) = k\pi m_i n^i + 2\pi(c^i m_i + d_i n^i)$$

up to some constants c^i and d_i that we have to fix by other means. On the wave function ϕ the prequantum operators reduce to the quantum operators

$$\begin{aligned} Q(a_i) &= -\frac{i}{\pi k}(\tau_2)_{ij} \partial_{\bar{a}_j} \\ Q(\bar{a}_i) &= \bar{a}_i \end{aligned}$$

which also realize the algebra (2.7). We use the BCH formula

$$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B$$

to express the holonomy in the form

$$H(N, M) = e^{-\frac{i\pi k}{2}(\tau_2)^{ij}(M+\bar{\tau} N)_i(M+\tau N)_j} e^{-i\pi k(M+\bar{\tau} N)_i(\tau_2)^{ij} \bar{a}_j} e^{(M+\tau N)_i \partial_{\bar{a}_i}}$$

When we act with the holonomy on $\phi(\bar{a})$ we then get

$$H(N, M)\phi(\bar{a}) = e^{-\frac{i\pi k}{2}(\tau_2)^{ij}(M+\bar{\tau} N)_i(M+\tau N)_j} e^{-i\pi k(M+\bar{\tau} N)_i(\tau_2)^{ij} \bar{a}_j} \phi(\bar{a} + M + \tau N)$$

Finally we use the gauge transformation of ϕ as given in (2.9) and find that most exponential factors cancel out, and we are left with

$$H(N, M)\phi(\bar{a}) = e^{2\pi i(c^i M_i + d_i N^i)} e^{ik\pi M_i N^i} \phi(\bar{a})$$

We now see that the eigenvalue of the holonomy corresponds to the characteristics c^i and d_i . Using the BCH formula we can show that the holonomies obey the algebra

$$H(N, M)H(N', M') = e^{\pi ik(M'_i N^i + M_i N'^i)} H(N + N', M + M') \quad (2.10)$$

We can be ignorant about the signs in the exponent since the difference is always of the form $e^{2\pi i k M_i N^i} = 1$. For k even this formula admits that we take all holonomies to be $H(N, M) = 1$. For k odd this formula forbids us to take all $H(N, M) = 1$ and the best we can do is to allow for some (or all) of them to be -1 .

Let us now turn to holonomies on J_W ,

$$W(N, M) = e^{i \int_{c_{N,M}} A}$$

As a preliminary attempt we define these by choosing the gauge potential as

$$A = 2\pi k x^i \delta y_i + 2\pi i (c^i \delta y_i + d_i \delta x^i)$$

since this makes $W(N, M)$ invariant under (2.4) while any other choice obtained by adding for example adding an exact term on the form $\xi \delta(x^i y_i)$ would not give us an invariant holonomy under (2.4) for generic values on ξ . We will return to this issue more fully below. Let us choose a closed loop on J_W as a straight line

$$\begin{aligned} c_{N,M} &= \{(x^i, y_j) = (N^i \theta, M_j \theta) | 0 \leq \theta \leq 1\} \\ &= \{\bar{a}_i = (\tau_{ij} N^i - M_i) \theta | 0 \leq \theta \leq 1\} \end{aligned}$$

The holonomy can now be evaluated to

$$W(N, M) = e^{2\pi i (c^i M_i + d_i N^i)} e^{i k \pi M_i N^i}$$

and we can read off the characteristics from

$$\begin{aligned} W(e^i, 0) &= e^{2\pi i c^i} \\ W(0, e_j) &= e^{2\pi i d_j} \end{aligned}$$

where $e^i = (0, \dots, 1, \dots, 0)$ with 1 at the i -th entry. However this does not bring us any closer to what the characteristics really should be.

Our definition of the holonomy $W(N, M)$ is not satisfactory since it is gauge choice dependent. We also can not see if using the definition of holonomy as proposed in [18] can help us here. If we can find a gauge invariant definition of $W(N, M)$ we may expect that $W(N, M) = H(N, M)$ for that gauge invariant definition since clearly $H(N, M)$ is gauge invariant. Indeed there might exist a gauge invariant definition of $W(N, M)$. To allow for the most general possible extension of manifold and symplectic two-form, we shall include all the oscillator modes and we are led to consider seven-dimensional Chern-Simons theory to compute the holonomy following [1]. By including the oscillators, we have to consider the symplectic potential on the form $A = \frac{k}{4\pi} \int_W C \wedge \delta C$. We then consider a one-cycle C in J_W around which we want to compute the holonomy for this symplectic potential. Let us parameterize the one-cycle by $\theta \in [0, 1]$ and consider a one-parameter family of C -fields that we denote as $C_C(\theta)$. Time coordinate is being replaced by θ coordinate. The differential δC in field space becomes a differential in the seven dimensional space $S^1 \times W$ where S^1 is the one-cycle parametrized by θ . The holonomy can be expressed as

$$H(C) = \exp i \int_C A = \exp \frac{ik}{4\pi} \int_{S^1 \times W} C_C \wedge dC_C$$

However this formula is only valid if A is globally defined along C . For the generic case we instead define the holonomy by finding an extension X whose boundary is $\partial X = S^1 \times W$ and define the holonomy as

$$H(C) = \exp \frac{ik}{4\pi} \int_X G \wedge G$$

where $G = dC_C$. This is still a simplification of the real situation in M theory since in general G is an element in a shifted integral cohomology [1]. Nevertheless, the point is that we need to find an extension X over which the spin structure of $S^1 \times W$ and the gauge field C_C can be extended. How one can make such an extension may depend on the spin structure on W . For example, if we have periodic (Ramond) fermions on a circle we can not extend the circle to a disk. This means that the holonomy will depend on the spin structure on W .

One may consider $W = T^6$ as an example. Here $\Theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the only fully modular invariant choice. But this need not necessarily be the partition function of the M5 brane of a given spin structure if that spin structure is not fully modular invariant. The M5 brane partition function need only be invariant under modular transformations that preserve the spin structure.

2.2 Oscillators

The inner product of two wave functions is defined as⁵

$$\begin{aligned} (\psi, \psi) &= \int \mathcal{D}C \overline{\psi(C)} \psi(C) \\ &= \int \mathcal{D}C^+ \mathcal{D}C^- \exp \left(\frac{ik}{2\pi} (C^+, C^-) \right) \overline{\phi(C^-)} \phi(C^-) \end{aligned}$$

We can expand $C = C_{\text{harmonic}} + d\chi$ in a harmonic and an exact piece since by Gauss law $dC = 0$. We note that χ is real in both Euclidean as well as in Minkowski signature since Wick rotation can only affect the components C_{0MN} which in our gauge choice are zero. The fact that χ is real means that it can be gauged away by a gauge transformation

$$\delta C = d\lambda$$

by taking the real gauge parameter as $\lambda = -\chi$. This is thus different from the quantization of three-dimensional Chern-Simons theory [19]. The generator of gauge transformations is given by the Gauss law constraint, which we here express for an infinitesimal gauge parameter λ in the form

$$\int \lambda \wedge \left(d \frac{\delta}{\delta C^-} - \frac{ik}{2\pi} G \right) \phi(C^-) = 0$$

Upon integration by parts, this equation makes sense only if λ is such that $(d\lambda)^+ = 0$ and $C^+ = 0$. In that case this equation reduces to

$$\phi(C^- + (d\lambda)^-) = \phi(C^-)$$

⁵In Euclidean signature we define the bar as usual complex conjugation. In this signature we also have $C^+ \wedge C^- = -iC^+ \wedge *C^-$ and $\overline{C^\pm} = C^\pm$. The exponent $\exp \frac{ik}{2\pi} (C^+, C^-)$ is now real with respect to our complex conjugation in Euclidean signature.

whose solution can be taken as

$$\phi(C^-) = \phi(C_{\text{harmonic}}^-)$$

with a trivial oscillator contribution. The oscillator mode contribution to the inner product of wave functions is now given by

$$(\psi, \psi)_{\text{osc}} = \int \mathcal{D}C \delta(dC) \exp\left(\frac{ik}{4\pi} \int |C|^2\right)$$

where we have noticed that $(C^+, C^-) = \frac{1}{2}(C, C)$. We replace

$$\delta(dC) = \int \mathcal{D}\chi \exp\left(\frac{ik}{2\pi} \int dC \wedge \chi\right)$$

Completing the square and shifting $C \rightarrow C + d\chi$ and noting the measure is gauge invariant, $\mathcal{D}(C + d\chi) = \mathcal{D}C$, we get

$$(\psi, \psi)_{\text{osc}} = \int \mathcal{D}\chi \exp\left(-\frac{ik}{4\pi} \int |d\chi|^2\right)$$

The path integral is suffering of a gauge redundancy $\delta\chi = d\lambda$ and so needs to be gauge fixed. It is clear that this path integral is precisely equal to the path integral of a non-chiral two-form gauge field on W , but which here has been obtained from the Chern-Simons theory on $I \times W$. The path integral was over C but this is constrained to be flat $dC = 0$, and the exact part can therefore be re-expressed as an integral over χ . We note that the sign gets correct. We express the path integral in Minkowski signature, and we have the phase factor e^{iS} from which we read off the action

$$S = -\frac{k}{4\pi} \int |d\chi|^2$$

which corresponds to the action (2.1) in a vanishing background C field if we make the identifications $B = \chi$.

3 The M5 brane partition function

In this section we perform a direct and explicit computation of the full Abelian M5 brane partition function using Hamiltonian formulation. For the selfdual three-form we recover the result we got above from Chern-Simons theory, but here we will also supplement this with contributions from the five scalar fields and the fermions.

In order to apply the Hamiltonian formulation we assume that the M5 brane worldvolume M_6 is a circle-bundle over a base-manifold M_5 . We choose the following parametrization for the metric on M_6

$$ds_{M_5}^2 = \beta dt^2 + G_{mn}(dx^m - V^m dt)(dx^n - V^n dt) \tag{3.1}$$

and we assume that β is a constant. We will assume that ∂_t is a Killing vector field, which means that $\partial_t V^m = 0$ and $\partial_t G_{mn} = 0$. We will associate t with time and x^m with spatial

directions on M_5 . We will promote β to a complex holomorphic parameter, thus allowing for Minkowskian and Euclidian signatures.

As the M5 brane action we will take

$$S = S_B + S_\phi + S_\psi$$

where

$$\begin{aligned} S_B &= -\frac{\lambda}{12} \int d^6x \sqrt{-g} H^{MNP} H_{MNP} \\ S_\phi &= -\lambda \int d^6x \sqrt{-g} \partial^M \phi^A \partial_M \phi^A \\ S_\psi &= i\lambda \int d^6x \sqrt{-g} \bar{\psi} \Gamma^M \partial_M \psi \end{aligned}$$

Selfduality fixes the coupling constant to be

$$\lambda = \frac{1}{4\pi}$$

3.1 Scalar field

Let us consider the scalar field action

$$S = -\lambda \int_{M_6} d^6x \sqrt{-g} g^{MN} h_M h_N \tag{3.2}$$

for one of the five real scalar fields. Here we define the field strength of a zero form scalar field ϕ as

$$h_M = \partial_M \phi$$

The momentum conjugate ϕ is

$$\pi = -2\lambda \sqrt{-\beta} \sqrt{G} h^t$$

The Hamiltonian is

$$H = \lambda \sqrt{-\beta} \int_{M_5} d^5x \sqrt{G} (-h^t h_t + h_{6D}^m h_m)$$

We note the following metric identities⁶

$$\begin{aligned} h_t &= \beta h^t - V^m h_m \\ h_{6D}^m &= h^m + V^m h^t \end{aligned}$$

and we see that the Hamiltonian can be expressed in terms of the five-dimensional metric as

$$H = \lambda \sqrt{-\beta} \int_{M_5} d^5x \sqrt{G} (-\beta (h^t)^2 + h^m h_m + 2h^t V^m h_m)$$

⁶Let us derive them here: $h_t = g_{tt} h^t + g_{tm} g^{mn} h_n + g^{tm} g^{mt} h_t$ gives $h_t (1 - g_{tm} g^{mt}) = g_{tt} h^t + g_{tm} g^{mn} h_n$. We then miraculously notice that g_{tt} , $1 - g_{tm} g^{mt}$ and $g_{tm} g^{mn}$ all contain the common factor $\beta + V^2$ which can be factored out and the identity follows. Next we compute $h_{6D}^{mn} = g^{mt} h_t + g^{mn} h_n = \frac{1}{\beta} V^m h_t + G^{mn} h_n + \frac{1}{\beta} V^m V^n h_n$ and substitute $h_t = g_{tt} h^t - V^n h_n$ and we find that two terms $\sim V^m V^n h_n$ cancel and the second identity follows.

To get a better handle on the scalar field partition function, and in particular its modular invariance, we temporarily introduce a regulator and assume that the scalar field is compact

$$\phi(x) \sim \phi(x) + 2\pi r$$

with a radius r that we in the end will take to infinity. The important point is that the regulator r does not mix with the geometry of space-time, so we can introduce r as a separate quantity on which the partition function may depend, in order to extract its dependence on the geometry. Using this together periodicity of the scalar field together with the canonical commutation relation

$$[\phi(x), \pi(y)] = i\delta^5(x - y)$$

we conclude that

$$P = -2\lambda\sqrt{-\beta} \int d^5y \sqrt{G} h^t$$

is integer quantized as

$$P = \frac{n}{r}$$

where n is integer. Moreover, we expand h^t in a basis of harmonic zero forms. But such harmonics are necessarily constants. So h^t is a constant. It must therefore be given by

$$h^t = -\frac{n}{2\lambda\sqrt{-\beta}(\text{Vol})r}$$

where $\text{Vol} = \int d^5x \sqrt{G}$. The Hamiltonian is now given by

$$H = \frac{\lambda\sqrt{-\beta} n^2}{\text{Vol} r^2} + \dots$$

where the $+\dots$ terms will be irrelevant in the limit $r \rightarrow \infty$. In that limit the discrete sum can be replaced by an integral over $q = n/r$, and we get

$$\mathcal{Z}_{\text{zero}} = \int_{-\infty}^{\infty} dq e^{-2\pi i H} = \left(\frac{\text{Vol}}{2i\lambda\sqrt{-\beta}} \right)^{\frac{1}{2}} \tag{3.3}$$

Of course the integral is evaluated by taking β real and positive, and then we make the analytic continuation.

3.2 Tensor gauge field

The tensor gauge field can be treated in an analogous way as the compact scalar field. The action for the non-chiral tensor gauge field is given by

$$S = -\frac{\lambda}{12} \int d^6x \sqrt{-g} H^{MNP} H_{MNP}$$

We define the field strength of the two-form gauge potential B_{MN} as

$$H_{MNP} = \partial_M B_{NP} + \partial_N B_{PM} + \partial_P B_{MN}$$

The momentum conjugate to B_{MN} is

$$E^{MN} = -\frac{\lambda\sqrt{-\beta}}{2}\sqrt{G}H^{tMN}$$

and we have five primary constraints

$$E^{mt} = 0$$

for $m = 1, \dots, 5$. These we must supplement by equally many gauge fixing constraints. We will choose the temporal gauge

$$B_{mt} = 0$$

The Hamiltonian is

$$H = \frac{\lambda\sqrt{-\beta}}{12} \int d^5x \sqrt{G} (-3H^{tmn}H_{tmn} + H^{mnp}H_{mnp})$$

We note the following metric identities⁷

$$\begin{aligned} H_{0mn} &= \beta H^t{}_{mn} - V^p H_{mnp} \\ H_{6D}^{mnp} &= H^{mnp} + 3V^{[m} H^{t|np]} \end{aligned}$$

and we see that the Hamiltonian can be expressed in terms of the five-dimensional metric as

$$H = \frac{\lambda\sqrt{-\beta}}{12} \int d^5x \sqrt{G} (-3\beta H^{tmn}H^t{}_{mn} + 6V^p H^{tmn}H_{mnp} + H_{mnp}H^{mnp})$$

The holonomies are periodic

$$\int_{\Sigma} B \sim \int_{\Sigma} B + 2\pi$$

Let us introduce the covariant momentum variable $e^{mn} = \frac{E^{mn}}{\sqrt{G}}$ whose indices we can covariantly lower by the five-dimensional metric to define a momentum two-form $e = \frac{1}{2}e_{mn}dx^m \wedge dx^n$. From the canonical commutation relations

$$[B_{mn}(x), E^{pq}(y)] = i\delta_{mn}^{pq}\delta^5(x-y)$$

we see that

$$\left[\int_{\Sigma_j} B, \int_{B_5} e \wedge *\Omega^i \right] = \frac{i}{2} \int_{\Sigma_j} \Omega^i$$

Here we denote by Ω^i the basis elements of harmonic two-forms on M_5 and we have a metric

$$\mathcal{G}^{ij} = \int_{M_5} \Omega^i \wedge *\Omega^j$$

If we expand

$$e = e_i \Omega^i$$

⁷The derivation is analogous to what we did for the scalar field.

then we get

$$\left[\int_{\Sigma_j} B, e^i \right] = \frac{i}{2} \delta_j^i$$

where we define $e^i = \mathcal{G}^{ij} e_j$. Then

$$2e^i = n^i$$

must be integer quantized momenta conjugate to the 2π -periodic holonomies $\int_{\Sigma_i} B$. We also expand $H = m^i \tilde{\Omega}_i$ where m^i are integer quantized magnetic charges and $\tilde{\Omega}_i$ are dual three-forms to Ω^i as defined in appendix A. By taking

$$\lambda = \frac{1}{4\pi}$$

we get the zero mode part of the Hamiltonian as

$$H_{\text{zero}} = \sqrt{-\beta} 2\pi \mathcal{G}_{ij} \left(n^i n^j + \frac{1}{4} m^i m^j \right) - 2\pi \mathcal{L}_{ij} n^i m^j$$

where \mathcal{G}_{ij} denotes the inverse of \mathcal{G}^{ij} and \mathcal{L}_{ij} is defined in Appedix A. The zero mode partition function is defined as

$$\mathcal{Z}_{\text{zero}} = \sum_{n^i, m^i} e^{-2\pi i H_{\text{zero}}}$$

By identifying the period matrix as

$$\tau_{ij} = 2\pi \left(-\mathcal{L}_{ij} + \sqrt{-\beta} \mathcal{G}_{ij} \right)$$

the zero mode partition function can be written as⁸

$$\mathcal{Z}_{\text{zero}} = \sum_{a^i, b_i} \Theta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (-\tau) \overline{\Theta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (-\tau)}$$

Here the Jacobi theta functions are given by

$$\Theta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (\tau) = \sum_{n^i \in \mathbb{Z}} \exp \left(\pi i (n^i + a^i) \tau_{ij} (n^j + a^j) + 2\pi i n^i b_i \right)$$

and the characteristics a^i and b_i are running over $0, \frac{1}{2}$.

So far we have computed the zero mode part of the partition function of a non-selfdual three-form H , and we have seen that it is holomorphically factorizable in a certain sense, as a finite sum of products of chiral and antichiral parts. On the Chern-Simons theory side, this corresponds to the inner product of wave functions. The wave function that corresponds to the partition function of selfdual three-form is now given by

$$\mathcal{Z} = \Theta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (-\tau | 0)$$

for some certain characteristics.

This computation shows that we shall choose the corresponding Chern-Simons level to be $k = 1$ but it might be interesting to ask whether other values of k can be implemented in M5 brane theory as well.

⁸Complex conjugation works as usual only in Euclidean signature where β is real and positive. But we can extend analytically to arbitrary complex β while always prescribing the conjugation rule $\sqrt{-\beta} = -\sqrt{-\beta}$. Using this prescription we can use theta function formalism also in Minkowski signature which is the natural signature for Hamiltonian quantization.

3.3 An alternative treatment of selfdual tensor gauge field

For the tensor gauge field we may use a selfdual Lagrangian. Holomorphic factorization works nicely for the zero modes if we define the period matrix on the space of harmonic three-forms. For the oscillator modes this method requires an extension of the period matrix to include the oscillator modes. However this extended period matrix would be an infinite-dimensional matrix which could lead to additional subtleties in the holomorphic factorization. The alternative route is to work directly with a selfdual Lagrangian from which we can compute the contribution of the the selfdual oscillator modes using conventional quantization methods, that is Hamiltonian quantization or path integral quantization.

Let us first return to the nonselfdual Lagrangian. For the zero modes we have two set of integers, m^i and n^i . Using notations introduced in appendix A, these integers were defined as

$$H_{mnp} = m^i \tilde{\Omega}_{i,mnp}$$

$$e^{mn} = \frac{n^i}{2} \Omega_i^{mn}$$

Let us now decompose the three-form into selfdual and antiselfdual parts,

$$H = H^+ + H^-$$

where

$$H^{\pm,tmn} = \pm \frac{\sqrt{-g}}{6} \epsilon^{tmnpqr} H_{pqr}^{\pm}$$

We accordingly decompose the two sets of integers as

$$m^i = m^{i+} + m^{i-}$$

$$2n^i = m^{i+} - m^{i-}$$

whose solutions are

$$m^{i\pm} = n^i \pm \frac{m^i}{2}$$

We can express this as saying that

$$m^{i+} = p^i + a^i$$

where $p^i \in \mathbb{Z}$ and each a^i is either 0 or $\frac{1}{2}$.

We now turn to the selfdual Lagrangian. We define a six-dimensional vielbein $e^A = e_M^A dx^M$ which we decompose into $e^A = (e^t, e^\alpha)$ where $\alpha = 1, \dots, 5$. For the three-form field strength we have

$$H_{MNP} = e_M^A e_N^B e_P^C H_{ABC}$$

where we define

$$\partial_A = e_A^M \partial_M$$

and

$$H_{ABC} = 3\partial_{[A} B_{BC]}$$

To see this one notices that the ordinary derivative can be replaced covariant derivatives in the three-form, and the vielbein is covariantly constant. Then we consider the following six-dimensional Lagrangian ($\lambda = \frac{1}{4\pi}$)

$$L = -\frac{\lambda}{12} \int d^5x \sqrt{G} \epsilon^{\alpha\beta\gamma\delta\epsilon} H_{\alpha\beta\gamma} \partial_t B_{\delta\epsilon} - \frac{\lambda \sqrt{-\beta}}{6} \int d^5x \sqrt{G} H^{\alpha\beta\gamma} H_{\alpha\beta\gamma}$$

As we showed above, the Dirac charge quantization for the selfdual field strength is given by

$$\frac{1}{2\pi} \int_{c_i} H = \mathbb{Z} + a^i$$

over spatial three-cycles c_i . Moreover, the selfdual holonomies

$$X_i(t) = \exp^{i \int_{\Sigma_i} B^+}$$

are periodic or antiperiodic according to

$$X_i(t + 2\pi) = (-1)^{b_i} X_i(t)$$

The b_i correspond to the characteristics b_i in the theta function. In two-dimensions the holonomies correspond to fermions via bosonization.

4 The D4 brane partition function

If we choose the metric on the M5 brane worldvolume on the form

$$ds_{M5}^2 = \tilde{G}_{\mu\nu} dx^\mu dx^\nu + R^2(dx^5 + v_\mu dx^\mu)^2$$

then the corresponding D4 brane action will be given by

$$S = S_{\text{YM}} + S_{\phi'} + S_{\psi'}$$

where

$$\begin{aligned} S_{\text{YM}} &= -\frac{1}{4g^2} \int d^5x \sqrt{-\tilde{G}} F^{\mu\nu} F_{\mu\nu} + \frac{1}{32\pi^2} \int d^5x \tilde{G} \epsilon^{\mu\nu\kappa\tau\sigma} F_{\mu\nu} F_{\kappa\tau} v_\sigma \\ S_{\phi'} &= -\frac{1}{2g^2} \int d^5x \sqrt{-\tilde{G}} \partial^\mu \phi'^A \partial_\mu \phi'^A \\ S_{\psi'} &= \frac{i}{g^2} \int d^5x \sqrt{-\tilde{G}} \bar{\psi}' \Gamma^\mu \partial_\mu \psi' \end{aligned}$$

and the coupling constant is

$$g^2 = 4\pi^2 R$$

4.1 Scalar field

The scalar field action is given by

$$S = -\frac{1}{2g^2} \int d^5x \sqrt{-\tilde{G}} \tilde{G}^{\mu\nu} \partial_\mu \phi' \partial_\nu \phi'$$

where

$$g^2 = 4\pi^2 R$$

This action is obtained by dimensional reduction of (3.2) and by defining

$$\phi' = 2\pi R \phi$$

We introduce a regulator r' such that

$$\phi' \sim \phi' + 2\pi r'$$

but of course this regulator is different from the M5 brane regulator r . These are related as

$$r' = 2\pi R r$$

The quickest way to compute the zero mode contribution to the partition function is by lending the result from the M5 brane. To this end we rescale the scalar field so that the action gets the factor $-\lambda$ in front in place of $\frac{1}{2g^2}$. We then find the periodicity gets modified to

$$\phi' \sim \phi' + 2\pi r''$$

with

$$r'' = \frac{1}{\sqrt{2\pi R}} 2\pi r'$$

Then we can borrow the M5 brane result and we get the zero mode part of the Hamiltonian as

$$H_{D4} = \frac{\lambda \sqrt{-\beta}}{\text{Vol}'} \frac{n^2}{r''^2}$$

where we use $\text{Vol}' = \int d^4x \sqrt{g}$ in place of $\text{Vol} = \int d^5x \sqrt{G}$. These volume factors are related by

$$\text{Vol} = 2\pi R \text{Vol}'$$

To compare with the M5 brane Hamiltonian, we substitute $r'' = \sqrt{2\pi R} r$. Then

$$H_{D4} = \frac{\lambda \sqrt{-\beta}}{\text{Vol}' 2\pi R} \frac{n^2}{r^2}$$

which now agrees with the zero mode part of the M5 brane Hamiltonian. Consequently also $\mathcal{Z}_{\text{zero},\phi'}$ agrees with $\mathcal{Z}_{\text{zero},\phi}$ of the M5 brane.

4.2 Gauge field

The momentum conjugate to A_i is

$$E^i = -\frac{1}{4\pi^2 R} \sqrt{-\tilde{G}} F^{ti} + \frac{1}{8\pi^2} \tilde{G} \epsilon^{tijk} F_{jk} v_l$$

We choose temporal gauge

$$A_t = 0$$

We first show that we can express the conjugate momentum in a six-dimensional form as

$$E^i = -\frac{1}{2\pi} \sqrt{-g} H_{6D}^{5ti}$$

To see this, we first note that

$$H_{6D}^{5\mu\nu} = \frac{1}{R^2} \tilde{G}^{\mu\kappa} \tilde{G}^{\nu\lambda} H_{5\kappa\lambda} - v_\lambda H_{6D}^{\mu\nu\lambda}$$

Then using selfduality, we have

$$H_{6D}^{5\mu\nu} = \frac{1}{R^2} \tilde{G}^{\mu\kappa} \tilde{G}^{\nu\lambda} H_{5\kappa\lambda} - \frac{1}{2} \sqrt{-g} \epsilon^{\mu\nu\lambda\kappa\tau 5} v_\lambda H_{\kappa\tau 5}$$

and then

$$R \sqrt{-\tilde{G}} H_{6D}^{ti5} = \frac{1}{2\pi} \left(\frac{1}{R} \sqrt{-\tilde{G}} F^{ti} + \frac{1}{2} \tilde{G} \epsilon^{tijk} v_j F_{kl} \right)$$

where we define

$$H_{\mu\nu 5} = \frac{1}{2\pi} F_{\mu\nu}$$

and the identity follows.

We next comment on the integer quantization of the momentum which follows by the fact that it is conjugate to 2π -periodic holonomies. For YM we then have

$$\left[\int_0^{2\pi} dx^1 A_1, E^1 \right] = 2\pi i$$

and for the six-dimensional theory we have

$$\left[\int_0^{2\pi} dx^1 \int_0^{2\pi} dx^5 B_{15}, 2\pi E^{15} \right] = 2\pi^2 i$$

where we write $2\pi E^{15}$ due to integration over x^5 as necessary to get δ^4 quantities on the r.h.s. in both cases. From these relations together with 2π -periodicity of holonomies, we conclude that

$$\begin{aligned} E^1 &= 2\pi n^1 \\ 2\pi E^{15} &= 2\pi^2 n^{15} \end{aligned}$$

where n^1 and n^{15} are integer quantized. By identifying these integers as $n^i = n^{i5}$ for $i = 1, 2, 3, 4$, we conclude that

$$2\pi E^{i5} = \pi E^i$$

and thus we should expect the conjugate momentum of our ‘would be’ six-dimensional theory, should be given by

$$E^{mn} = -\frac{1}{4\pi}\sqrt{-g}H_{6D}^{tmn}$$

or in other words,

$$E^{mn} = -\lambda\sqrt{-g}H_{6D}^{tmn}$$

The Hamiltonian we derive from the 5D YM is given by

$$\mathcal{H}_{\text{YM}} = E^i F_{ti} - \mathcal{L}_{\text{YM}}$$

The first term is

$$E^i F_{ti} = -\sqrt{-g}H_{6D}^{ti5}H_{ti5}$$

For the second term we first notice that

$$-\frac{1}{4}\sqrt{-g}H_{6D}^{5\mu\nu}H_{5\mu\nu} = \mathcal{L}_{\text{YM}}$$

and furthermore

$$-\frac{1}{4}\sqrt{-g}H_{6D}^{5\mu\nu}H_{5\mu\nu} = -\sqrt{-g}\left(\frac{1}{4}H_{6D}^{5ij}H_{5ij} + \frac{1}{2}H_{6D}^{5ti}H_{5ti}\right)$$

Thus

$$\mathcal{H}_{\text{YM}} = \sqrt{-g}\left(\frac{1}{4}H_{6D}^{5ij}H_{5ij} - \frac{1}{2}H_{6D}^{5ti}H_{5ti}\right)$$

The six-dimensional Hamiltonian is given by

$$H_{\text{M5}} = \frac{\lambda R}{6} \int d^5x \sqrt{-\tilde{G}} H_{6D}^{mnp} H_{mnp}$$

but by using the selfduality constraint this Hamiltonian can also be expressed as

$$\begin{aligned} H_{\text{M5}} &= -\frac{\lambda R}{2} \int d^5x \sqrt{-\tilde{G}} H_{6D}^{tnp} H_{tnp} \\ &= R \int d^4x \sqrt{-\tilde{G}} \left(-\frac{1}{2} H_{6D}^{tij} H_{tij} + \frac{1}{4} H_{6D}^{5ij} H_{5ij} \right) \end{aligned}$$

To get to the second line we have assumed that the fields are independent of the coordinate on the fiber, so that integration along the fiber just gives a factor of 2π . We also used selfduality again, now in the form of $H_{6D}^{tij} H_{tij} = -H_{6D}^{5ij} H_{5ij}$. We now have

$$H_{\text{M5}} = H_{\text{YM}}$$

under the assumption of fields being independent on the fiber coordinate. In particular then, the zero mode partition functions are identical.

5 Oscillators

5.1 Selfdual tensor field

In the spirit that M5 brane theory is a higher dimensional analog of selfdual scalar field in two dimensions, we begin by computing the oscillator partition function for a two-dimensional selfdual scalar field in a way that is easily generalized to a selfdual two-form gauge potential in six dimensions.

5.1.1 Two-torus

Let us consider a real scalar field on T^2 with metric

$$ds^2 = \beta dt^2 + G_{xx}(dx - V^x dt)^2 = G_{xx}(dx + \tau dt)(dx + \bar{\tau} dt)$$

where

$$\begin{aligned} \tau &= -V^x + \frac{\sqrt{-\beta}}{\sqrt{G_{xx}}} \\ \bar{\tau} &= -V^x - \frac{\sqrt{-\beta}}{\sqrt{G_{xx}}} \end{aligned}$$

The action is

$$S = -\frac{1}{2} \int dt dx \sqrt{-g} g^{MN} \partial_M \phi \partial_N \phi$$

As the orthonormal basis on the base-manifold whose coordinate is $x \sim x + 2\pi$ we take $\varphi_n = \frac{1}{2\pi} e^{inx}$ for $n \geq 0$. The complex conjugates are $\varphi^n = \varphi_{-n}$. The quadratic equation of motion reads (for $n \geq 0$)

$$\left(\frac{1}{\beta} (\partial_t + V^x \partial_x)^2 + G^{xx} \partial_x^2 \right) e^{iEt+inx} = 0$$

and it has solutions

$$\begin{aligned} E_n^+ &= \tau n \\ E_n^- &= \bar{\tau} n \end{aligned}$$

In Minkowski signature $E_n^+ \geq 0$ and $E_n^- \leq 0$. The Hamiltonian is

$$H = \frac{1}{2} \sum_{n \in \mathbb{Z}_+} n \left[\tau \left(a_n^\dagger a_n + \frac{1}{2} \right) - \bar{\tau} \left(b_n^\dagger b_n + \frac{1}{2} \right) \right]$$

where

$$\begin{aligned} [a_m, a_n^\dagger] &= \delta_{m,n} \\ [b_m, b_n^\dagger] &= \delta_{m,n} \end{aligned}$$

The partition function becomes

$$Z_{\text{osc}} = \frac{1}{\eta(\tau)\overline{\eta(\tau)}}$$

where

$$\eta(\tau) = e^{-\frac{i\pi\tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i\tau n})$$

is the Dedekind eta function.

We note that the partition function can be expressed as a determinant

$$Z_{\text{osc}} = \frac{2\pi\sqrt{\beta}}{\sqrt{\det' \Delta}}$$

where in this determinant we shall put $\partial_t = im$ where $m \in \mathbb{Z}$ due to our convention where the time coordinate interval over which we compute the partition function is $[0, 2\pi]$. The physical time interval is $2\pi\sqrt{|\beta|}$.

5.1.2 Six-manifold

We now proceed to generalize this to six dimensions. If we impose temporal and Lorentz gauge

$$\begin{aligned} i_t B &= 0 \\ d^\dagger B &= 0 \end{aligned}$$

with respect to six-dimensional metric, then the Maxwell equation of motion becomes

$$\Delta_6 B = 0$$

where Δ_6 denotes the Laplacian on M_6 . The energies are obtained by solving the equation

$$\Delta_6(e^{iE_\alpha t} \varphi_\alpha) = 0$$

where φ_α denote an orthonormal basis of two-forms on M_5 . Such a basis can be conveniently chosen such that

$$\begin{aligned} \Delta_5 \varphi_\alpha &= \lambda_\alpha \varphi_\alpha \\ (\varphi_\alpha, \varphi_\beta) &= \delta_\alpha^\beta \end{aligned}$$

where the inner product is given by $(\varphi_\alpha, \varphi_\beta) = \int_{M_5} \varphi_\alpha \wedge *_5 \bar{\varphi}_\beta$ and λ_α denote eigenvalues and bar denotes complex conjugation.

Obviously the eigenvalues λ_α may be degenerate. What is less obvious is that the partition function can be complex-valued. To see this, let us assume a Cartesian product $M_6 = T^2 \times M_4$. In this case we get

$$B = \phi(t, x) \varphi(x^i)$$

and we wish to solve the equation of motion

$$(\Delta_2 \phi) \varphi + \phi \Delta_4 \varphi = 0$$

Let us assume that we solve this equation by requiring

$$\Delta_4 \varphi = 0$$

Then $d\varphi = 0$ and we get

$$dB = d\phi \wedge \varphi$$

Requiring this to be selfdual means that either both φ and $d\phi$ are selfdual or both anti-selfdual. This shows that the partition function will now contain a factor

$$\mathcal{Z}_{\text{osc}}(\tau, g_{ij}) = \frac{1}{\eta(\tau)^{b^+} \bar{\eta}(\tau)^{b^-}}$$

where b^\pm denote the number of selfdual and anti-selfdual harmonics on M_4 respectively, and $\eta(\tau)$ is the contribution from one selfdual scalar field on T^2 . Only if $b^+ - b^- = 0$ do we have a real-valued oscillator partition function. The oscillator mode contribution for a selfdual tensor gauge field is given by the holomorphic factor $\frac{1}{\eta(\tau)^{b^+}}$. We do not know how this factorization extends beyond the zero modes on M_4 in this case. One way to compute the oscillator mode contribution to the partition function of a selfdual tensor gauge field could be to directly work with a selfdual action for example the one we wrote down in section 3.3.

5.1.3 Six-torus

Let us now instead turn to the case when M_6 is flat T^6 . Since this is $T^2 \times T^4$ and $b^+(T^4) = b^-(T^4) = 3$ the oscillator partition function will be real. This means that to compute the oscillator mode partition function for a selfdual tensor gauge field on T^6 we can simply take the square root of the oscillator mode partition function of a non-selfdual tensor gauge field on T^6 . The Laplacian is

$$\Delta_6 = -g^{MN} \partial_M \partial_N$$

and this gets split into

$$\Delta_6 = -\frac{1}{\beta} \partial_t^2 - \frac{2}{\beta} V^m \partial_t \partial_m + \Delta_5$$

Let us define

$$f_{\vec{p}}^{mn} = \frac{1}{(2\pi)^{\frac{5}{2}}} e^{ip_m x^m} dx^m \wedge dx^n$$

which satisfy

$$\begin{aligned} \Delta_5 f_{\vec{p}}^{mn} &= \lambda_{\vec{p}} f_{\vec{p}}^{mn} \\ (f_{\vec{p}}^{mn}, f_{\vec{q}}^{pq}) &= \sqrt{G} G^{mn,pq} \delta_{\vec{p},\vec{q}} \end{aligned}$$

where

$$\begin{aligned} \lambda_{\vec{p}} &= p^2 \\ p &:= \sqrt{G^{mn} p_m p_n} \end{aligned}$$

We wish to consider a basis $f_{\vec{p}}^{mn}$ with the property that $(f_{\vec{p}}^{mn})^*$ are linearly independent of $f_{\vec{p}}^{mn}$. This can be achieved by restricting ourselves to a half-space $\vec{p} \in \mathbb{Z}_+^5$. This half-space can be selected by first picking a co-dimension one hyperplane in \mathbb{R}^5 which only intersects the origin but no other lattice points in \mathbb{Z}^5 . Such a hyperplane may be defined by one normal vector which we may take for example as $\vec{n} = (1, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11})$. Then we may define the half-space as $\mathbb{Z}_+^5 = \{\vec{p} \in \mathbb{Z}^5 | \vec{n} \cdot \vec{p} > 0\}$. The energies are

$$E_{\vec{p}}^{\pm} = -V^m p_m \pm \sqrt{-\beta p}$$

Let us define a projector

$$(\Pi_{\vec{p}})_m^n = \frac{1}{2}(\delta_m^n - \hat{p}_m \hat{p}^n)$$

Then

$$\tilde{f}_{\vec{p}}^{mn} = (\Pi_{\vec{p}})_{pq}^{mn} f_{\vec{p}}^{pq}$$

satisfy Lorentz gauge $p_m f_{\vec{p}}^{mn} = 0$ (we drop the tilde for notation convenience). Moreover

$$(f_{\vec{p}}^{mn}, f_{\vec{q}}^{pq}) = \sqrt{G} \Pi^{mn,pq} \delta_{\vec{p},\vec{q}}$$

We also define selfduality projectors

$$(S_{\vec{p}})_{pq}^{mn} = \frac{1}{2} \left(\Pi_{pq}^{mn} + \frac{1}{2} \sqrt{G} \epsilon^{mn}{}_{pqr} \hat{p}^r \right)$$

and define

$$\tilde{f}_{\vec{p}}^{mn} = (S_{\vec{p}})_{pq}^{mn} f_{\vec{p}}^{pq}$$

which obey (again dropping the tilde)

$$(f_{\vec{p}}^{mn}, f_{\vec{q}}^{pq}) = \sqrt{G} S^{mn,pq} \delta_{\vec{p},\vec{q}}$$

We expand

$$B(x) = \sum_{\vec{p} \in \mathbb{Z}_+^5} (a_{mn,\vec{p}} e^{iE_{\vec{p}}^+ t} + b_{mn,\vec{p}} e^{iE_{\vec{p}}^- t}) f_{\vec{p}}^{mn}(\vec{x}) + \text{c.c.}$$

and this will now satisfy the Maxwell equation of motion, be in Lorentz gauge and give a selfdual field strength. We see that there are three independent polarizations $a_{mn,\vec{p}}$ for a given momentum \vec{p} . Let us denote these independent ones as $a_{\vec{p}}^i$ for $i = 1, 2, 3$.

It is a general result that the Hamiltonian of a quadratic Lagrangian is given by

$$H = \sum_{\vec{p} \in \mathbb{Z}_+^5, i} \left[E_{\vec{p}}^+ \left(a_{\vec{p}}^{i\dagger} a_{\vec{p}}^i + \frac{1}{2} \right) - E_{\vec{p}}^- \left(b_{\vec{p}}^{i\dagger} b_{\vec{p}}^i + \frac{1}{2} \right) \right]$$

where

$$\begin{aligned} [a_{\vec{p}}^i, a_{\vec{q}}^{j\dagger}] &= \delta^{ij} \delta_{\vec{p},\vec{q}} \\ [b_{\vec{p}}^i, b_{\vec{q}}^{j\dagger}] &= \delta^{ij} \delta_{\vec{p},\vec{q}} \end{aligned}$$

Since the Hamiltonian shall be non-negative in Minkowski signature, we shall take E^+ to be non-negative, and E^- to be non-positive solutions to the quadratic equation of motion. The oscillator partition function now becomes

$$\mathcal{Z}_{\text{osc}} = \frac{1}{\eta(g^{MN})^3 \overline{\eta(g^{MN})}^3} \quad (5.1)$$

where we define a generalized Dedekind eta function and its conjugate as

$$\begin{aligned} \eta(g^{MN}) &= e^{-i\pi \sum_{\vec{p} \in \mathbb{Z}_+^5} E_{\vec{p}}^+} \prod_{\vec{p} \in \mathbb{Z}_+^5} (1 - e^{2\pi i E_{\vec{p}}^+}) \\ \overline{\eta(g^{MN})} &= e^{i\pi \sum_{\vec{p} \in \mathbb{Z}_+^5} E_{\vec{p}}^-} \prod_{\vec{p} \in \mathbb{Z}_+^5} (1 - e^{-2\pi i E_{\vec{p}}^-}) \end{aligned}$$

If we pick T^2 with modular parameter τ embedded in T^6 then Z_{osc} contains as a factor $\frac{1}{\eta(\tau)^3 \overline{\eta(\tau)}^3}$. This is an important observation to show modular invariance of the full partition function on T^6 .

5.2 Scalar fields

For the five scalar fields we get the contribution

$$\frac{1}{\eta(g^{MN})^5 \overline{\eta(g^{MN})}^5}$$

where E^\pm are computed from Δ_6 acting on zero forms $e^{iEt} \phi(x^m)$. For a flat T^6 we get that same eta functions as those for the selfdual tensor field since for flat space we have the same energies.

5.3 Fermions

In order to preserve supersymmetry on T^6 we shall impose periodic boundary conditions on the fermions on all the 6 one-cycles of T^6 . For the spatial directions this amounts to integer momenta. To get periodic fermions in the time direction we must insert $(-1)^F$. Thus the fermionic partition function is

$$\text{tr}((-1)^F e^{2\pi i H}) = e^{-8i\pi \sum_{\vec{p} \in \mathbb{Z}^5} (E_{\vec{p}}^+ - E_{\vec{p}}^-)} \prod_{\vec{p} \in \mathbb{Z}_+^5 + \vec{0}} (1 - e^{2\pi i E_{\vec{p}}^+})^8 (1 - e^{-2\pi i E_{\vec{p}}^-})^8$$

Now for this periodic boundary conditions there is a zero mode $\vec{p} = 0$ which kills the whole partition function so that

$$\mathcal{Z} = 0$$

Obviously this partition function is modular invariant. We may also relax period boundary condition in time direction while preserving supersymmetry since supersymmetry variations are at a fixed time and do not see such a boundary condition in time. In that case we get the partition function as

$$\text{tr}(e^{2\pi i H}) = e^{-8i\pi \sum_{\vec{p} \in \mathbb{Z}^5} (E_{\vec{p}}^+ - E_{\vec{p}}^-)} \prod_{\vec{p} \in \mathbb{Z}_+^5 + \vec{0}} (1 + e^{2\pi i E_{\vec{p}}^+})^8 (1 + e^{-2\pi i E_{\vec{p}}^-})^8$$

but now we can only expect this to be modular invariant under modular transformations that act on $T^5 \subset T^6$ and leave the time direction unaffected. So we shall expect this partition function to be only $SL(5, \mathbb{Z})$ invariant.

6 Modular invariance of M5 brane on $T^2 \times M_4$

So far we have been studying M5 brane on circle fibration over a five-manifold. If we continue the program of circle fibrations, the next step will be to consider a two-torus fibered over some four-manifold. Since the modular group of a two-torus is $SL(2, \mathbb{Z})$ we can ask whether the M5 brane partition function is $SL(2, \mathbb{Z})$ invariant. Such a strategy was used in [2] for the case of a flat T^6 to prove full $SL(6, \mathbb{Z})$ modular invariance on T^6 by first proving $SL(2, \mathbb{Z})$ modular invariance on a T^2 embedded in T^6 . But we may generalize. The most general geometry for which $SL(2, \mathbb{Z})$ modular invariance can be studied is where we have two commuting Killing vectors corresponding to a two-torus T^2 in the six-manifold M_6 . If we take one circle direction of that T^2 to be associated with time, the natural choice of metric will be

$$ds^2 = \beta dt^2 + G_{mn}(dx^m - V^m dt)(dx^n - V^n dt)$$

on which we make a subsequent decomposition of the same form,

$$G_{mn} dy^m dy^n = \gamma (dy^5)^2 + g_{ij}(dy^i - U^i dy^5)(dy^j - V^j dy^5)$$

Thus, by substituting $dy^m = dx^m - V^m dt$, the full M5 brane metric is

$$ds^2 = \beta dt^2 + \gamma(dx - V^5 dt)^2 + g_{ij}(dx^i - V^i dt - U^i dx)(dx^j - V^j dt - U^i dx)$$

The first two terms can be combined into the standard metric on a two-torus

$$\gamma(dx + \tau dt)(dx + \bar{\tau} dt)$$

if we define

$$\tau = -V^5 + \frac{\sqrt{-\beta}}{\sqrt{\gamma}}$$

The metric is invariant under simultaneous exchange of coordinates on T^2 ,

$$\begin{aligned} t' &= x \\ x' &= -t \end{aligned}$$

and the following change of parameters in the metric,

$$\begin{aligned} V'^i &= U^i \\ U'^i &= -V^i \\ \tau' &= -\frac{1}{\tau} \\ \bar{\tau}' &= -\frac{1}{\bar{\tau}} \\ \gamma' &= \tau \bar{\tau} \gamma \end{aligned}$$

We will postpone a study of modular invariance on generic such fiber-bundles to future work. Here we will assume the six-manifold M_6 is on the form of a Cartesian product

$$M_6 = T^2 \times M_4$$

This corresponds to putting $V^i = U^i = 0$ above. The metric on M_6 is then

$$ds_{M_6}^2 = ds_{M_2}^2 + ds_{M_4}^2$$

where

$$\begin{aligned} ds_{M_2}^2 &= \gamma(dx + \tau dt)(dx + \bar{\tau} dt) \\ ds_{M_4}^2 &= g_{ij} dx^i dx^j \end{aligned}$$

The Laplace operator becomes

$$\Delta_6 = \Delta_2 + \Delta_4$$

6.1 The zero mode part

In the appendix B we obtain the period matrix for $T^2 \times M_4$ for the case that $\dim H^1(M_4) = 0$. This period matrix has been previously obtained in [22]. We have the following results for the period matrix and its inverse,

$$\begin{aligned} \tau_{IJ} &= \tau_1(Q^{-1})_{IJ} + \tau_2 G_{IJ} \\ (\tau^{-1})^{IJ} &= \tilde{\tau}_1 Q^{IJ} + \tilde{\tau}_2 Q^{IK} Q^{JL} G_{KL} \end{aligned}$$

Here τ is the period matrix on T^2 and Q^{IJ} and G^{IJ} is the intersection form and the metric associated with M_4 as defined in the appendix B. We define⁹

$$\begin{aligned} \tilde{\tau}_1 &= -\frac{\tau_1}{\tau_1^2 - \tau_2^2} \\ \tilde{\tau}_2 &= -\frac{\tau_2}{\tau_1^2 - \tau_2^2} \end{aligned}$$

The partition function is¹⁰

$$\mathcal{Z}_{\text{zero}} = \Theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau_{IJ})$$

Two ingredients are involved to see the modular property of this partition function under S -transformation. The first ingredient is Poisson resummation,

$$\sum_{n^I \in \mathbb{Z}} e^{i\pi n^I \tau_{IJ} n^J} = \frac{1}{\sqrt{\det \tau_{IJ}}} \sum_{m_I \in \mathbb{Z}} e^{i\pi m_I (\tau^{-1}) m_J}$$

The second ingredient is to make use the intersection form (whose entries are integers) to define dual integers

$$m^I = Q^{IJ} m_J$$

⁹We remind that our notation is $\tau = -\tau_1 - \tau_2$ and $\bar{\tau} = -\tau_1 + \tau_2$ in both Minkowski and Euclidean signatures. We then define $\tilde{\tau} = -\frac{1}{\tau} = -\frac{\bar{\tau}}{\tau\bar{\tau}}$.

¹⁰For simplicity we consider only zero characteristics. It is also true that this choice is the most modular invariant choice [20, 21]

which results in

$$m_I(\tau^{-1})m_J = m^I \tilde{\tau}_{IJ} m^J$$

where we define

$$\tilde{\tau}_{IJ} = \tilde{\tau}_1(Q^{-1})_{IJ} + \tilde{\tau}_2 G_{IJ}$$

To compute the determinant we define

$$m = m_I \Omega^I$$

and

$$m^\pm = m \pm *m$$

we then consider the following equality (noting that $** = 1$ and defining the inner product as usual). Then

$$m_I Z^{IJ} m_J = \frac{1}{4} (\tau(m^+, m^+) + \bar{\tau}(m^-, m^-))$$

We can compute the determinant as a product of exponents. These exponents factorize into selfdual parts, so we can compute the determinant of each factor. One such factor is

$$\begin{aligned} \frac{1}{\sqrt{\det Z^{IJ}}} &= \prod_{m^+} \int dm^+ \exp \frac{1}{4} \tau(m^+, m^+) \\ &= \left(\frac{1}{\sqrt{\tau}} \right)^{b_+} \end{aligned}$$

The product runs over the selfdual harmonic two-forms, and b_+ is the number of such two-forms.

No contribution relevant for S-duality comes from harmonic one-forms on M_4 because these do not give a dependence on τ . They involve various combinations that contain $dt \wedge dx$, but no single dx nor single dt , so there is no τ dependence.

We find that the zero mode partition function transforms like

$$\begin{aligned} \mathcal{Z}_{\text{zero}} \left(-\frac{1}{\tau}, g_{ij} \right) &= \sqrt{\tau}^{b_+} \sqrt{\bar{\tau}}^{b_-} \mathcal{Z}_{\text{zero}}(\tau, g_{ij}) \\ \mathcal{Z}_{\text{zero}}(\tau + 1, g_{ij}) &= \mathcal{Z}_{\text{zero}}(\tau, g_{ij}) \end{aligned}$$

A corresponding result was obtained in [22] for non-Abelian gauge group. However in this reference it seems the second ingredient of using the intersection form Q^{IJ} to define dual integer numbers is missing.

6.2 Oscillator mode part

Oscillators contribute with a generalized eta function. Zero modes have already been analyzed, and the determinant of the six-dimensional Laplacian acting on two-forms, equals the generalized eta function up to a zero mode factor.

We have learned how to extract \mathcal{Z}_{osc} from the determinant. Let us now consider the determinant that is relevant to this situation

$$\det \Delta_6 = \det(\Delta_2 + \Delta_4)$$

We will ignore the issue of gauge fixing and ghost determinants. Instead we will map this determinant directly to \mathcal{Z}_{osc} . Only massless modes from two-dimensional point of view can contribute an $\text{SL}(2, \mathbb{Z})$ anomaly. So it is enough to consider zero modes of Δ_4 . These are the harmonic two-forms on M_4 . Thus the determinant we will be interested in reads

$$(\det \Delta_2)^{b_2}$$

From this result we can immediately conclude that the τ dependence of the oscillator partition function is

$$\mathcal{Z}_{\text{osc}}(\tau, g_{ij}) = \frac{1}{\eta(\tau)^{b_+} \bar{\eta}(\tau)^{b_-}}$$

and it transforms as

$$\begin{aligned} \mathcal{Z}_{\text{osc}}\left(-\frac{1}{\tau}, g_{ij}\right) &= \left(\frac{1}{\sqrt{\tau}}\right)^{b_+} \left(\frac{1}{\sqrt{\bar{\tau}}}\right)^{b_-} \mathcal{Z}_{\text{osc}}(\tau, g_{ij}) \\ \mathcal{Z}_{\text{osc}}(\tau + 1, g_{ij}) &= e^{-\frac{i\pi}{12}(b_+ - b_-)} \mathcal{Z}_{\text{osc}}(\tau, g_{ij}) \end{aligned}$$

The total partition function is thus invariant under S -transformation $\tau \rightarrow -\frac{1}{\tau}$ and transforms at most by a phase under T -transformation $\tau \rightarrow \tau + 1$. The partition function is modular invariant whenever $b_+ = b_-$. This shows that the partition function is modular invariant on T^6 but not on $T^2 \times TN$ where $b_+ \neq b_-$. (On TN we have $b_+ = 1$ and $b_- = 0$).

For the five scalar fields we also find modular invariance on T^6 , despite the oscillator mode contribution by itself is not modular invariant. But we do have a zero mode contribution also for the non-compact scalar fields, as we showed in section 3.1. For the case that $W = T^2 \times M_4$ (whereof T^6 is a special case) we find that we can express the zero mode factor as

$$\begin{aligned} \mathcal{Z}_{\text{zero}} &= \left(\frac{\gamma}{-\beta}\right)^{\frac{1}{4}} \\ &= \frac{1}{\sqrt{\tau_2}} \end{aligned}$$

It transforms under S -transformation into

$$\mathcal{Z}_{\text{zero}}\left(-\frac{1}{\tau}, g_{ij}\right) = |\tau| \mathcal{Z}_{\text{zero}}(\tau)$$

For a single scalar field the oscillator mode partition function transforms as

$$\mathcal{Z}_{\text{osc}}\left(-\frac{1}{\tau}, g_{ij}\right) = \frac{1}{|\tau|} \mathcal{Z}_{\text{osc}}(\tau)$$

showing that $\mathcal{Z}_{\text{zero}} \mathcal{Z}_{\text{osc}}$ is invariant under S -transformation. The same holds true for T -transformation on T^6 .

For the fermions on T^6 and if we assume antiperiodic boundary conditions in all directions, we find a zero mode that kills the whole partition function. Obviously 0 is $\text{SL}(6, \mathbb{Z})$ modular invariant. If we choose periodic boundary conditions on the circle in

T^6 that we associate with time, then we shall only expect to find $SL(5, \mathbb{Z})$ invariance. And indeed this amount of symmetry is manifest in our expression for the oscillator mode partition function, as was also noted in [2] in the context of tensor gauge field on T^6 .

Thus the M5 brane partition function with zero characteristics on a flat T^6 is $SL(6, \mathbb{Z})$ modular invariant, and on $T^2 \times M_4$ it is $SL(2, \mathbb{Z})$ modular invariant up to a phase. Under dimensional reduction to four dimensions we find a partition function which is a modular form [23]. It is natural that we get different modular properties of the partition functions in four and six dimensions since we truncate away all the KK modes in four dimensions. Nevertheless we do find a modular property in four dimensions. This experience could motivate us to ask if some modular property could also be found in five dimensions.

The oscillator parts of both D4 and M5 are on the form of generalized Dedekind eta functions. Let us consider a flat six-torus. Then the frequencies E_{n_m} that appear in the generalized Dedekind eta function are the positive roots to the equation of motion

$$g^{MN} n_M n_N = 0$$

where we solve this equation for $n_t = E_{n_m}$ and pick the positive solutions (in Minkowski signature). For the D4 brane E_{n_i} are positive solutions to

$$\tilde{G}^{\mu\nu} n_\mu n_\nu = 0$$

where $n_t = E_{n_i}$. We use the index notation $M = (\mu, 5)$ and $\mu = (t, i)$ for $i = 1, \dots, 4$. Since

$$g^{MN} n_M n_N = \tilde{G}^{\mu\nu} (n_\mu - v_\mu n_5)(n_\nu - v_\nu n_5) + \frac{1}{R^2} n_5^2$$

we see that

$$g^{MN} n_M n_N = \tilde{G}^{\mu\nu} n_\mu n_\nu$$

only for vanishing KK momentum $n_5 = 0$.

If we pick factors in $\mathcal{Z}_{\text{osc}}^{\text{M5}}$ with $n_i = 0$ and perform the product over all non-vanishing Kaluza-Klein momenta $n_5 = 1, 2, 3, \dots$, then these give rise to a τ -dependent factor

$$\frac{1}{\eta(\tau)^3 \overline{\eta(\tau)}^3}$$

Here the powers 3 correspond to the six harmonics b_{ij} on T^4 , separated into selfdual and anti-selfdual parts. These harmonics are responsible for the 3-fold degeneracy of E_{n_5} . For the D4 brane we put $n_5 = 0$ and this τ -dependent factor does not arise in $\mathcal{Z}_{\text{osc}}^{\text{D4}}$.

7 The M5/D4 partition function on T^6

Multiplying together the various contributions, what have found on T^6 is the following M5 brane partition function

$$\mathcal{Z}^{\text{M5}} = \mathcal{Z}_{\text{zero}}^{\text{M5}} \mathcal{Z}_{\text{osc}}^{\text{M5}}$$

where

$$\mathcal{Z}_{\text{zero}}^{\text{M5}} = \Theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (-\tau) \left(\frac{i\text{Vol}}{\sqrt{-\beta}} \right)^{\frac{5}{2}}$$

$$\mathcal{Z}_{\text{osc}}^{\text{M5}} = \prod_{\vec{p} \neq 0} \left(\frac{1 + e^{2\pi i E_{\vec{p}}^+}}{1 - e^{2\pi i E_{\vec{p}}^+}} \right)^4 \left(\frac{1 + e^{-2\pi i E_{\vec{p}}^-}}{1 - e^{-2\pi i E_{\vec{p}}^-}} \right)^4$$

and where $\vec{p} = (p_m) = (p_i, p_5)$ for $i = 1, \dots, 4$. For the D4 brane we have found that

$$\mathcal{Z}_{\text{zero}}^{\text{D4}} = \mathcal{Z}_{\text{zero}}^{\text{M5}}$$

$$\mathcal{Z}_{\text{osc}}^{\text{D4}} = \prod_{p_i \neq 0} \left(\frac{1 + e^{2\pi i E_{p_i}^+}}{1 - e^{2\pi i E_{p_i}^+}} \right)^4 \left(\frac{1 + e^{-2\pi i E_{p_i}^-}}{1 - e^{-2\pi i E_{p_i}^-}} \right)^4$$

What is missing so far from the D4 brane are the KK modes with $p_5 \neq 0$, to which we will turn our attention in the next section. Let us here just note that a particularly simple KK sector in the M5 brane oscillators is obtained by taking $p_i = 0$ with $p_5 \neq 0$. We note that

$$E_{p_i=0, p_5}^{\pm} = \pm \sqrt{-\beta} \sqrt{G^{55}} |p_5| - V^5 p_5$$

and so by defining

$$q = e^{2\pi i (\sqrt{-\beta} \sqrt{G^{55}} - V^5)}$$

$$\tilde{q} = e^{-2\pi i (-\sqrt{-\beta} \sqrt{G^{55}} - V^5)}$$

we find that this sector gives the following contribution

$$\mathcal{Z}_{\text{osc}, p_i=0}^{\text{M5}} = \prod_{p_5=1}^{\infty} \left(\frac{1 + q^{p_5}}{1 - q^{p_5}} \right)^8 \left(\frac{1 + \tilde{q}^{p_5}}{1 - \tilde{q}^{p_5}} \right)^8 \tag{7.1}$$

8 Small instantons and KK modes

Up to now we have shown that the naive D4 brane partition function on T^5 does not match with that for the M5 brane on T^6 . The zero mode's part has the precise agreement while there is a clear mismatch in the oscillator part of the partition functions. If the proposed D4/M5 on S^1 correspondence is correct, this implies that one is missing certain degrees of freedom from the D4 brane side. It is clear from the previous discussion so far that the missing part are all those spectra of KK modes along the circle direction S^1 . Here we would like to show that this KK part of the partition function is precisely generated by small instantons corresponding to D0 branes of the type IIA string theory.

Taking our S^1 as an M-theory circle, the M5 becomes D4 by the type IIA reduction and the KK circle momenta are corresponding to D0 branes from the view point of the type IIA theory. If D0 branes are away from the D4 to the five transverse directions, then their Hilbert space is not to do with that of the D4 brane. When D0 branes are located on top of the D4, they can be bound to the D4 brane forming a (threshold) bound state,

which can be faithfully captured by the D4 brane worldvolume dynamics we are dealing with. This is one of the basic construct of the D4/M5 on S^1 correspondence proposed in refs. [11, 12]. These D0 branes are captured by the well known instanton dynamics of D4 branes satisfying the selfdual (or anti-selfdual) equation

$$F = (-) *_4 F$$

where the hodge dual $*_4$ is taken over the four spatial direction of the worldvolume directions of D4 branes. The instanton number is counted by an integer n_5

$$n_5 = \frac{1}{8\pi^2} \int \text{tr} F \wedge F$$

For the $SU(N)$ part of N D4 branes with $N \geq 2$, regular solutions of the above self dual equation can be found explicitly. It is well known, however, that for the case of $U(1)$, the corresponding instanton solutions become singular classically. Hence one has to introduce some regularization in order to use the conventional techniques including the moduli space approximation. Here we shall introduce the noncommutativity parameters for the spatial part of the coordinates for the sake of the regularization and then, in the end, we take the commutative limit by sending the noncommutative parameters to zero. In this regularization, the instanton size squared turns out to be of order of the noncommutativity parameters. Since we are interested in the small size limit, the physics of global aspects decouples from the local dynamics and, for instance, the boundary condition on their wave function can be imposed separately afterwards.

Hence by an appropriate coordinate transformation, we make the D4 brane worldvolume metric to the standard flat Minkowski form,

$$ds_5^2 = -dz_0^2 + dz_i dz_i$$

With these coordinates, the 6d metric including circle fibration now takes the form

$$ds_6^2 = ds_5^2 + R^2(dx^5 + \tilde{v}_\mu dz^\mu)^2$$

We shall introduce the spatial noncommutativity

$$[z_i, z_j] = i\theta_{ij}$$

where the noncommutativity parameters θ_{ij} can be fully parameterized by

$$\theta_{ij} = \xi^a \eta_{ij}^a + \zeta^a \bar{\eta}_{ij}^a$$

where η_{ij}^a and $\bar{\eta}_{ij}^a$ $a = 1, 2, 3$ are the selfdual and anti-selfdual 't Hooft tensors respectively.

We shall focus on the Maxwell part of the D4 brane action that is given by

$$S = -\frac{1}{16\pi^2 R} \int d^5 z F_{\mu\nu} F^{\mu\nu} + \frac{1}{32\pi^2} \int d^5 z \epsilon^{\mu\nu\lambda\delta\rho} F_{\mu\nu} F_{\lambda\delta} \tilde{v}_\rho \quad (8.1)$$

and we are interested in the instanton solutions satisfying

$$F_{ij} = \frac{1}{2} \epsilon_{ijkl} F^{kl}$$

and their moduli space dynamics. The instanton solutions of the selfdual equation and their modular space dynamics can be systematically studied by the so called Atiyah-Drinfeld-Hitchin-Manin (ADHM) method of the construction [24], for which the noncommutative deformation can be incorporated in addition [25]. We shall not introduce all the details here but just necessary part for the present purpose. The ADHM equations for selfdual instantons reads¹¹

$$\begin{aligned} [Z, Z^\dagger] + [W, W^\dagger] + II^\dagger - J^\dagger J &= \zeta^3 \equiv \zeta \\ [Z, W] + IJ &= \zeta^1 + i\zeta^2 \end{aligned}$$

where X and W are complex $n_5 \times n_5$ matrices and I^\dagger and J are $N \times n_5$ complex matrices. As before N is for the N parallel $D4$'s and we are here only concerned about the $N = 1$ case. We note the anti-selfdual part of the noncommutativity parameters enters for the ADHM equations for selfdual instantons. Since there are $U(n_5)$ gauge symmetry for the n_5 instanton dynamics, one has to mode out the degrees of the above ADHM equations by an appropriate $U(n_5)$ gauge fixing conditions. Thus the moduli space dimension as parameterized by the matrices is given by $4n_5^2 + 4n_5N$ moded out by $3n_5^2$ conditions imposed by the ADHM equations together with n_5^2 gauge fixing conditions. Therefore the resulting dimension becomes $4n_5N$ or $4n_5$ for our interest of $N = 1$ case. The moduli space metric is induced from the flat metric

$$ds^2 = \text{tr}(dIdI^\dagger + dJ^\dagger dJ + dZdZ^\dagger + dWdW^\dagger)$$

by a hyper-Kahler quotient procedure [13].

8.1 One instanton dynamics

From now on, we shall focus on $U(1)$ case with $N = 1$, set $\xi_1 + i\xi_2 = \zeta_1 + i\zeta_2 = 0$ and take $\zeta > 0$. Then for one instanton case with $n_5 = 1$, the gauge fixed solution reads

$$I_{1 \times 1} = \sqrt{\zeta}, \quad J_{1 \times 1} = 0$$

while $Z_{1 \times 1}$ and $W_{1 \times 1}$ are not constrained at all. Thus we conclude that one instanton moduli space is given simply by flat R^4 with

$$ds_{n_5=1}^2 = |dZ|^2 + |dW|^2 = dX_i dX_i$$

where X_i is the position on R^4 . This describes a $D0$ with a definite finite size freely moving along the worldvolume directions of the $D4$ brane. (One can show that there is a finite binding energy between $D0$ and $D4$ due the nonvanishing noncommutativity parameter ζ .)

For the details of its Hilbert space and moduli dynamics, one may be more explicit for this simplest case. Of course via the ADHM construction, one may find explicit nonsingular one instanton solution $A_i(\vec{z})$ with $A_0 = 0$ [25, 26] but we do not need here its explicit form

¹¹The anti-selfdual instanton corresponds to replacing ζ^a by ξ^a in the ADHM equations. Therefore, to have ADHM constructions for both instantons and anti-instantons, we need to have both ζ^a and ξ^a non-vanishing.

since the translational symmetry of the system will be enough to show some details as will be clear below. With the choice of $A_0 = 0$, the configuration $A_i(\vec{z} - \vec{X})$ is explicitly describing the moduli space R^4 . Here X_i are the so-called collective coordinates for the moduli (solution) space interpreted as a position of D0 along the worldvolume directions of the D4. In general the zero mode that is the variation of the solution tangent to the moduli space requires an extra gauge transformation

$$\delta_j A_i = \frac{\partial}{\partial X_j} A_i + \frac{\partial}{\partial z_i} \lambda_j$$

where λ_j will be chosen such that the constraints $\frac{\partial}{\partial z_i} \delta_j A_i = 0$ hold, which means that the zero modes are orthogonal to gauge transformation modes $\delta A_i = D_i \Lambda$. For the present one instanton case, the constraints are trivially solved with $\lambda_j = 0$ by assuming the gauge fixing condition $\partial_i A_i = 0$. Thus for the explicit moduli space dynamics, we insert the ansatz

$$A_i(\vec{z} - \vec{X}(t)), \quad A_0 = 0$$

to the action (8.1) in order to obtain the low energy effective action for the one-instanton dynamics. This will be valid up to quadratic order of the velocity \dot{X}_i . For the evaluation we further note

$$\begin{aligned} \frac{1}{16\pi^2} \int d^4 z F_{ij} F_{ij} &= n_5 = 1 \\ \frac{1}{16\pi^2} \int d^4 z F_{ij} F_{im} &= \frac{n_5}{4} \delta_{jm} \end{aligned}$$

where for the latter we have used the translational symmetry of the one-instanton solution. The resulting effective Lagrangian for the one-instanton dynamics reads

$$L_{\text{eff}}^{n_5=1} = -\frac{n_5}{R} + n_5 \tilde{v}_0 + \frac{n_5}{2R} \dot{X}_i \dot{X}_i + n_5 \tilde{v}_i \dot{X}^i$$

For anti-instantons we instead find (with $n_5 = -1$)

$$L_{\text{eff}}^{n_5=-1} = \frac{n_5}{R} + n_5 \tilde{v}_0 - \frac{n_5}{2R} \dot{X}_i \dot{X}_i + n_5 \tilde{v}_i \dot{X}^i$$

The canonical momentum has an extra contribution from the first-order time derivative term,

$$p_i = \pm \frac{n_5}{R} \dot{X}_i + n_5 \tilde{v}_i$$

with upper sign for instanton and lower sign for anti-instanton respectively. The Hamiltonian becomes

$$H = \pm \frac{n_5}{R} - n_5 \tilde{v}_0 \pm \frac{R}{2n_5} (p_i - n_5 \tilde{v}_i)(p_i - n_5 \tilde{v}_i)$$

Here $\pm \frac{n_5}{R}$ is the mass of the nonrelativistic motion of an instanton and of an anti-instanton respectively. We note that the sign is precisely such that this mass is always positive. One then recognizes that this is the nonrelativistic version of the relativistic counterpart

$$-(p_0 - n_5 \tilde{v}_0)^2 + (p_i - n_5 \tilde{v}_i)^2 + \frac{n_5^2}{R^2} = 0 \tag{8.2}$$

in Minkowski signature if we make the following identifications

$$H = -p_0^- > 0$$

for the instanton and the anti-instanton respectively. Here p_0^\pm denote the positive and negative roots to the quadratic equation of motion (8.2) (for a sufficiently small \tilde{v}_0). Since one has the 5d Poincaré symmetry of D4 in the commutative limit, one may argue that the above should be the precise spectrum; Its 6d version takes the form

$$g^{MN} p_M p_N = 0$$

with the instanton number n_5 is identified with the KK momentum p_5 along the circle direction. Thus we conclude that the required spectra for single KK mode are precisely reproduced from the single instanton and anti-instanton dynamics respectively.

The supersymmetric extension of this analysis for the one-instanton case is rather straightforward. We shall describe simply the result here. The one-instanton is half BPS (preserving half of the 16 supersymmetries) and the 8 fermionic zero modes form a multiplet consisting of 8 bosonic and 8 fermionic states. This agrees with the ground state degeneracy of one-D0 and one-D4 bound states. Among the 8 bosonic states, **5** correspond to the KK parts of 5 scalar while the remaining **3** are the KK modes from the selfdual two form gauge field. The fermionic part are also matching precisely with the KK modes from **8** fermionic degrees. Thus we conclude that the dynamics of one instanton sector is precisely matching with that of the one KK mode of the M-theory circle.

8.2 Multi instanton dynamics

We now turn to the cases of higher instanton numbers. The case of two U(1) noncommutative instantons has been analyzed in [13] which leads to the metric on the 8 dimensional moduli space

$$ds_{n_5=2}^2 = ds_{\text{com}}^2 + ds_{\text{rel}}^2$$

where the center of mass part is \mathbb{R}^4 with metric

$$ds_{\text{com}}^2 = dX_i dX_i$$

while the relative part of the metric is given by the Eguchi-Hanson metric [27]

$$ds_{\text{rel}}^2 = \frac{r^2}{\sqrt{r^4 + 4\zeta^2}} \left(dr^2 + \frac{r^2}{4} \sigma_3^2 \right) + \frac{1}{4} \sqrt{r^4 + 4\zeta^2} (\sigma_1^2 + \sigma_2^2)$$

In this metric σ_a are the standard SU(2) left invariant one forms given by

$$\begin{aligned} \sigma_1 &= -\sin \psi d\theta + \cos \psi \sin \theta d\varphi \\ \sigma_2 &= \cos \psi d\theta + \sin \psi \sin \theta d\varphi \\ \sigma_3 &= d\psi + \cos \theta d\varphi \end{aligned}$$

and the angular variables are ranged over

$$0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi, \quad 0 \leq \psi \leq 2\pi$$

due to these angular ranges, the geometry involves S^3/\mathbb{Z}_2 whereas a full S^3 requires the range $0 \leq \psi \leq 4\pi$. As expected, this metric is nonsingular at $r = 0$. In the commutative limit of $\zeta \rightarrow 0$, one may ignore any interactions between instantons completely leading to the metric for

$$\mathbb{R}^4 \times \mathbb{R}^4/\mathbb{Z}_2$$

or

$$T^4 \times T^4/\mathbb{Z}_2$$

including the global boundary conditions, where \mathbb{Z}_2 is interpreted as the permutation symmetry of identical particles. Thus the Hilbert space for two KK modes is realized by the two instanton dynamics.

Their quantum mechanical Hamiltonian is given by the Laplacian operators of the geometry acting on the wave functions with internal structure characterized by n -forms. It is well known that the Eguchi-Hanson metric allows a unique normalizable selfdual harmonic two-form state

$$\omega_2 = -\frac{2r^3}{(r^4 + 4\zeta^2)^{\frac{3}{2}}} dr \wedge \sigma_3 + \frac{1}{(r^4 + 4\zeta^2)^{\frac{1}{2}}} \sigma_1 \wedge \sigma_2$$

This is interpreted as a threshold bound state two U(1) instantons. As noticed in [13], this threshold bound state corresponds to the state of the charge two ($n_5 = 2$) KK mode.

For the maximally supersymmetric case, the counting of relevant two-instanton states goes as follows. As explained each D0 brane has 2^4 states, eight bosonic and eight fermionic. Now for two D0 branes, considering them as identical particles, the exchange symmetry has to be incorporated in this counting of internal states. There are $\frac{1}{2} \times 8 \times 9$ possible states with both D0 branes in the bosonic states. There are also $\frac{1}{2} \times 8 \times 7$ states with both D0 branes in (different) fermionic states (due to the exclusion principle). The remaining possibilities include 8×8 states with one in a bosonic state and one in a fermionic state. Hence the two (separate) D0 branes involve in total 128 states. But as just described above, two D0 branes can also form a threshold bound state and be bound to the D4 brane, which is then identified as a single $n_5 = 2$ KK mode. This involves 16 states as a single entity. Thus the total number of states with $n_5 = 2$ is counted as $d_2 = 16 + 128 = 144$.

Threshold bound states for $n_5 > 2$ have not yet been constructed. However their existence was proven in [14]. For $n_5 = 3$ there is thus a threshold bound state which corresponds to a single instanton particle carrying instanton number $n_5 = 3$ and which has 16 internal states (eight bosonic plus eight fermionic). We can also have two separate instanton particles, one with instanton number $n_5 = 1$ and the other being a threshold bound state with instanton number $n_5 = 2$. Since these two instanton particles are distinct they contribute with $16 \times 16 = 256$ internal states. Finally we can have three separate instantons each carrying instanton number $n_5 = 1$. As each of these instantons can be either bosonic or fermionic they add up to $120 + 288 + 224 + 56 = 688$ internal states. (The four terms correspond to instanton types BBB , BBF , BFF , FFF where B and F refers to 8-bosonic and 8-fermionic states.) Thus for $n_5 = 3$ we have in total $d_3 = 16 + 256 + 688 = 960$ states.

In general we get the total number of degrees of freedom d_{n_5} at a given instanton number n_5 from the following generating partition function as

$$2^8 \sum_{n=0}^{\infty} d_n q^n = 2^8 \prod_{n_5=1}^{\infty} \left(\frac{1+q^{n_5}}{1-q^{n_5}} \right)^8 \tag{8.3}$$

where the overall 2^8 is the counting of states for D4 brane ground states. This formula can be derived from the counting of BPS states in string theory and using some chains of U dualities [14, 28]. We have explicitly checked that the coefficient d_1 , d_2 and d_3 are correctly generated by this generating function.

The counting is precisely in parallel with the required KK states which is the missing part of M5 brane partition function in the naive D4 brane computation without incorporating instanton contributions. We note that $\mathcal{Z}_{\text{osc}, p_i=0}^{\text{M5}}$ as obtained in eq. (7.1) contains as one factor the generating function of instantons (8.3). The second factor in eq. (7.1) which involves a product over \tilde{q} shall be interpreted as the contribution from anti-instantons.

Instantons are BPS configurations, which means that static instantons do not interact among themselves. Static anti-instantons likewise do not interact among themselves. However an instanton interacts with an anti-instanton since such a configuration is not BPS. Also if we give some small momenta to instantons, they start to interact and feel the metric of the moduli space. However as the instanton and anti-instanton sizes become small in the commutative limit, the moduli space becomes flat and these become non-interacting particles. In the same fashion instanton and anti-instanton interactions disappear in this limit. Their contribution to the D4 brane partition function then factorizes. First from the instantons we have the factor

$$\mathcal{Z}^{\text{inst}} = \prod_{n_i \in \mathbb{Z}} \mathcal{Z}_{n_i}^{\text{inst}}$$

where n_i labels momenta of an instanton or a threshold-bound state of instantons which we call a KK particle with $n_5 = 1$ or with some nonvanishing n_5 respectively. Using the counting of degrees of freedom as obtained above up to instanton number $n_5 = 3$, we have the following contribution from KK particles with a specific momentum n_i to the D4 brane partition function

$$\begin{aligned} \mathcal{Z}_{n_i}^{\text{inst}} = & 1 + 16 e^{2\pi i E_{n_i,1}^+} + 16 e^{2\pi i E_{n_i,2}^+} + 128 e^{4\pi i E_{n_i,1}^+} \\ & + 16 e^{2\pi i E_{n_i,3}^+} + 256 e^{2\pi i (E_{n_i,1}^+ + E_{n_i,2}^+)} + 688 e^{6\pi i E_{n_i,1}^+} \\ & + \dots \end{aligned}$$

where E_{n_i, n_5}^+ refers to $-p_0^- > 0$ where p_0^\pm are the two roots to the equation (8.2). Second from the anti-instantons we have the factor

$$\mathcal{Z}^{\text{anti-inst}} = \prod_{n_i} \mathcal{Z}_{n_i}^{\text{anti-inst}}$$

As we have seen, the anti-instanton is associated with $E_{n_i, n_5}^+ = -p_0^- > 0$ for n_5 negative. We then find that

$$\begin{aligned} \mathcal{Z}_{n_i}^{\text{anti-inst}} &= 1 + 16 e^{2\pi i E_{n_i, -1}^+} + 16 e^{2\pi i E_{n_i, -2}^+} + 128 e^{4\pi i E_{n_i, -1}^+} \\ &\quad + 16 e^{2\pi i E_{n_i, -3}^+} + 256 e^{2\pi i (E_{n_i, -1}^+ + E_{n_i, -2}^+)} + 688 e^{6\pi i E_{n_i, -1}^+} \\ &\quad + \dots \end{aligned}$$

We now claim that $n_5 \neq 0$ contributions to the M5 brane partition function contain generating functions for these instanton and anti-instanton threshold bound states. More precisely, we claim that

$$\begin{aligned} \mathcal{Z}_{n_i}^{\text{inst}} &= \prod_{n_5=1}^{\infty} \left(\frac{1 + e^{2\pi i E_{n_i, n_5}^+}}{1 - e^{2\pi i E_{n_i, n_5}^+}} \right)^8 \\ \mathcal{Z}_{n_i}^{\text{anti-inst}} &= \prod_{n_5=-\infty}^{-1} \left(\frac{1 + e^{2\pi i E_{n_i, n_5}^+}}{1 - e^{2\pi i E_{n_i, n_5}^+}} \right)^8 \end{aligned}$$

It is rather easy to check explicitly that these generating functions reproduce the correct number of degrees of freedom for each threshold bound state up to total instanton number $n_5 = 3$. By noting the relation

$$E_{n_i, n_5}^+ = -E_{-n_i, -n_5}^-$$

it is easy to see that we have

$$\mathcal{Z}^{\text{M5}}(T^6) = \mathcal{Z}^{\text{D4}}(T^5) \mathcal{Z}^{\text{inst}}(T^5) \mathcal{Z}^{\text{anti-inst}}(T^5)$$

where the various factors on the right-hand side correspond to instanton numbers $n_5 = 0$, $n_5 > 0$ and $n_5 < 0$ respectively. We may also phrase this relation as

$$\mathcal{Z}^{\text{M5}}(T^6) = \lim_{\zeta, \xi \rightarrow 0} \mathcal{Z}_{\zeta, \xi}^{\text{D4}}(T^5)$$

since the non-commutative D4 already has those instanton sectors in the theory so they shall not be supplemented.

There is now little doubt that the M5 brane is really the same thing as (noncommutative) D4 in the above sense. In particular the M5 brane partition function encodes the number of internal degrees of freedom of each D0 brane threshold bound state.

One final note is the fact that the zero mode part of the partition function include also so called large instanton configurations which are satisfying the same (anti) selfdual equation of instantons. However as we showed explicitly, these large instantons are nothing to do with D0 branes. Namely, without involving any D0 branes, we have shown that the zero mode part of D4 and M5 has already an agreement with each other.

Since the essential properties of small instanton contribution we are using are only local ones, one expects that the above discussion can be extended to the case of general five manifold with circle fibration. But the relevant ADHM construction and the corresponding regularization is lacking currently. We shall leave this issue for future investigations.

9 Singular fibration

So far we have considered circle fibers with constant radius R . We may also consider situations where R depends on the D4 brane worldvolume coordinate. At some points one may also allow $R = 0$ in which case we have a singular fibration. From the M5 brane worldvolume point of view such singular fibrations are smooth and pose no further problems in the computation of the partition function, but from the D4 brane point of view the computation of the partition function becomes much more difficult when R is not constant. Also we may need to add new degrees of freedom at the locus of the singularity $R = 0$. As a singular fibration we may consider the M5 brane worldvolume on the form $W = TN \times T^2$ with TN being a Taub-NUT space that supports a single harmonic selfdual two-form Ω^+ . Such a fibration has been considered also in [15, 29]. Then the zero mode part of the M5 brane partition function will involve $\Theta(\tau)$ where τ is the period matrix on T^2 . There is no fully modular invariant theta function except for $\Theta\left[\begin{smallmatrix} \frac{1}{2} \\ \frac{1}{2} \end{smallmatrix}\right](\tau)$ that vanishes,¹² so the M5 brane partition function will depend on the choice of spin structure. As symplectic basis of harmonic three-forms on W we take

$$\begin{aligned}\alpha &= \Omega^+ \wedge dx \\ \beta &= \Omega^+ \wedge dy\end{aligned}$$

and the period matrix of W becomes the τ parameter of T^2 . On the five-dimensional base-manifold $\mathbb{R}^3 \times T^2$ we have sYM theory with coupling constant $g^2 \sim R$. It appears that the zero mode contribution from sYM can not give us a τ dependence of the partition function because the only harmonic two-form is the one on T^2 itself, while to have a τ dependence we would need a harmonic two-form that has only one component on T^2 . Such a harmonic two-form is not normalizable on $\mathbb{R}^3 \times T^2$ and must be excluded. Instead we have to add a chiral scalar action localized at the singularity $\{0\} \times T^2$ of $TN \times T^2$. The chiral scalar of course gives us a partition function $\Theta(\tau)$ on T^2 .

Another way to see this is by studying the gauge anomaly cancelation. The graviphoton term

$$S_{\text{sYM}} = \frac{1}{8\pi^2} \int_{\mathbb{R}^3 \times T^2} A \wedge dA \wedge w$$

with

$$w = -\frac{1}{2r^3} \epsilon_{ijk} x^i dx^j \wedge dx^k$$

has the anomalous gauge variation

$$\begin{aligned}\delta S_{\text{sYM}} &= \frac{1}{8\pi^2} \int_{T^2 \times R_+} d\lambda \wedge F \int_{S^2} w \\ &= \frac{1}{4\pi} \int_{T^2} \lambda F\end{aligned}$$

under the variation

$$\delta A = d\lambda$$

¹²That is not so if we couple M5 brane to a background C field though.

This gauge anomaly can be canceled by a gauged scalar field theory

$$S_{WZW} = \frac{1}{8\pi} \int_{T^2} (|d\phi + A|^2 + 2\phi F)$$

supported on $\{0\} \times T^2$ whose gauge variation is

$$\delta S_{WZW} = -\frac{1}{4\pi} \int_{T^2} \lambda F$$

under

$$\begin{aligned} \delta\phi &= -\lambda \\ \delta A &= d\lambda \end{aligned}$$

Moreover, 2π periodicity of the holonomy $\int A$ implies that ϕ is also 2π periodic in order to make $d\phi + A$ a gauge invariant quantity. Now this together with the normalization we found for S_{WZW} tells us that we have a coupling constant which corresponds to the free fermion radius. This action corresponds to Chern-Simons theory in three dimensions at level $k = 1$. Such a Chern-Simons theory may be derived from seven dimensions by expanding

$$C = \Omega^+ \wedge A$$

We then get

$$\frac{k}{4\pi} \int_{\mathbb{R} \times W} C \wedge dC = \frac{k}{4\pi} \int_{TN} \Omega^+ \wedge \Omega^+ \int_{\mathbb{R} \times T^2} A \wedge dA$$

The selfdual harmonic two-form was found in [30, 31]. We will normalize it so that $\Omega^+ = \frac{1}{4\pi}\omega$ where

$$\omega = \frac{r}{r+1} \sigma_1 \wedge \sigma_2 + \frac{1}{(r+1)^2} dr \wedge \sigma_3$$

using the notation and conventions of [30]. Then we have

$$\begin{aligned} \int_c \Omega^+ &= 1 \\ \int \Omega^+ \wedge \Omega^+ &= 1 \end{aligned}$$

where the three-cycle c is spanning the r, ψ plane where the coordinate range is $\psi \in [0, 4\pi]$. We then find that A is a connection one-form whose holonomy is 2π periodic, induced from C , and we find that

$$\frac{k}{4\pi} \int A \wedge dA$$

with standard normalization, induced from Chern-Simons action for C .

Another generalization we did not consider in this paper is to include six-manifolds which have no one-cycles at all. As we can see from our appendix A, once we have a one-cycle, there is a natural choice of symplectic basis of three-forms and corresponding three-cycles. If we have no one-cycle in our six-manifold we may of course still find a

symplectic basis though the method in appendix A does not apply to such cases. For example we may consider the case when the six-manifold is $W = S^3 \times S^3$. On S^3 we do not have a one-cycle even though it can be viewed as a Hopf bundle over a base manifold S^2 . Nevertheless we can define a symplectic basis for this situation by taking as basis three-forms certain rescaled volume forms on each of the two S^3 respectively. One may also consider a situation where the two Hopf fibers make up a skew torus, in which case the period matrix τ (which is just a complex number) can take any value in the upper half-plane. Also we have been ignorant about curvature corrections and essentially our presentation in the present paper is correct only for a flat T^6 . On curved space we need to add a curvature term for the scalar fields to maintain conformal invariance. Moreover superconformal invariance puts severe constraints on the possible six-manifolds one may consider [32]. However one may circumvent this constraint from superconformal invariance by considering a situation where one has a background flux on S^3 which spontaneously breaks conformal symmetry, while preserving maximal supersymmetry. Such a situation was considered in [33] where it was shown that the zero mode parts of the M5 brane partition function on $T^3 \times S^3$ matches with the corresponding zero mode part of the D4 brane partition function on $T^3 \times S^2$ respectively. It would be interesting to extend this analysis to the oscillator modes.

Acknowledgments

DB would like to thank Seok Kim for helpful discussions. This work was supported in part by NRF SRC-CQUeST-2005-0049409 and NRF Mid-career Researcher Program 2011-0013228.

A Period matrix on $S^1 \times M_5$

We assume a six-manifold on the form $M_6 = S^1 \times M_5$. On the space of harmonic two-forms Ω^i on M_5 , we define a metric

$$\mathcal{G}^{ij} = \int \Omega^i \wedge * \Omega^j$$

where $*$ denotes the Hodge star on M_5 . We define a basis of dual harmonic three-forms

$$\tilde{\Omega}_i = 2\pi \mathcal{G}_{ij} * \Omega^j$$

where \mathcal{G}_{ij} is the inverse of \mathcal{G}^{ij} . Then we have a symplectic basis

$$\begin{aligned} \alpha_i &= \tilde{\Omega}_i \\ \beta^i &= \Omega^i \wedge dt \end{aligned}$$

We assume $\int dt = 2\pi$ and that $\int_{C_j} \Omega^i = \delta_j^i$ when integrated over dual two-cycle. The α_i and β^i then have periods 2π when integrated over their dual three-cycles in M_6 . We also have the symplectic property

$$\int \alpha_i \wedge \beta^j = 4\pi^2 \delta_i^j$$

One may object that we should not need a metric to define a symplectic basis. As we will see in the next section, if $M_6 = T^2 \times M_4$ then this symplectic basis can be rewritten in terms of the intersection form on M_4 and the metric does not enter the definition of the symplectic basis.

We define the period matrix τ_{ij} and its conjugate $\bar{\tau}_{ij}$ by requiring that

$$\begin{aligned}\omega_i &= \alpha_i + \tau_{ij}\beta^j \\ \bar{\omega}_i &= \alpha_i + \bar{\tau}_{ij}\beta^j\end{aligned}$$

are selfdual and antiselfdual respectively with respect to the six-dimensional Hodge star that we define with respect to the six-dimensional metric tensor

$$g^{MN} = \begin{pmatrix} \frac{1}{\beta} & \frac{1}{\beta}V^n \\ \frac{1}{\beta}V^m & G^{mn} + \frac{1}{\beta}V^mV^n \end{pmatrix}$$

We will define this six-dimensional Hodge-star when acting on three-forms as

$$*_6 dx^M \wedge dx^N \wedge dx^P = \frac{1}{6} \sqrt{-g} \epsilon^{MNP}{}_{M'N'P'} dx^{M'} \wedge dx^{N'} \wedge dx^{P'}$$

This definition is related to the conventional Hodge-star as $*_6 = \frac{\sqrt{-\beta}}{\sqrt{|\beta|}} *_6, \text{conventional}$. Our Hodge-star depends holomorphically on β which we may promote into a holomorphic coordinate. Our Hodge star now squares according to

$$*_6 *_6 = 1$$

when acting on three-forms in six dimensions, and so we shall by selfduality/antiselfduality mean that

$$\begin{aligned}*_6 \omega &= \omega \\ *_6 \bar{\omega} &= -\bar{\omega}\end{aligned}$$

It turns out that the solution to these conditions can be expressed like

$$\tau_{ij} = -(\tau_1)_{ij} - (\tau_2)_{ij}$$

where

$$\alpha_i = (\tau_1)_{ij}\beta^j + (\tau_2)_{ij} *_6 \beta^j$$

If we expand

$$*_6 \beta^i = C^{ij}\alpha_j + A^i{}_j \beta^j$$

then

$$\begin{aligned}X_{ij} &= (C^{-1})_{ik} A^k{}_j \\ Y_{ij} &= (C^{-1})_{ij}\end{aligned}$$

From the symplectic properties, it follows that

$$C^{ij} = \frac{1}{4\pi^2} \int *_{6}\beta^i \wedge \beta^j$$

$$A^i{}_j = -\frac{1}{4\pi^2} \int *_{6}\beta^i \wedge \alpha_j$$

By brute force computation we get

$$\int *_{6}\beta^i \wedge \beta^j = \frac{2\pi}{\sqrt{-\beta}} \mathcal{G}^{ij}$$

$$\int *_{6}\beta^i \wedge \alpha_j = \frac{2\pi}{\sqrt{-\beta}} \mathcal{G}_{jk} 2\pi \mathcal{L}^{ki}$$

Here we define

$$\mathcal{K}^i{}_j = \frac{1}{2} \int d^5x \sqrt{G} \Omega_{mn}^i \tilde{\Omega}_j^{mnp} V_p$$

$$\mathcal{L}^{ij} = \frac{1}{4} \int d^5x G \epsilon^{mnpqr} \Omega_{mn}^i \Omega_{pq}^j V_r$$

and we have the relation

$$\mathcal{K}^{ij} = 2\pi \mathcal{L}^{ij}$$

however the latter form \mathcal{L}^{ij} shows manifest symmetry in ij . Then we get the period matrix as

$$\tau_{ij} = 2\pi(-\mathcal{L}_{ij} + \sqrt{-\beta} \mathcal{G}_{ij})$$

B Period matrix on $T^2 \times M_4$

For simplicity we assume that M_4 has no one-cycles. Then the harmonic two-forms on M_4 must be the same as the harmonic two-forms on $M_5 = S^1 \times M_4$. Let us here denote these harmonic two-forms as

$$\Omega^I = \frac{1}{2} \Omega_{mn}^I dx^m \wedge dx^n = \frac{1}{2} \Omega_{ij}^I dx^i \wedge dx^j$$

where $x^m = (x, x^i)$ for $i = 1, 2, 3, 4$ and x denotes the coordinate on the spatial S^1 in M_5 . We define the intersection form and the metric on $H^2(M_4, \mathbb{Z})$ as

$$Q^{IJ} = \int_{M_4} \Omega^I \wedge \Omega^J$$

$$G^{IJ} = \int_{M_4} \Omega^I \wedge *_4 \Omega^J$$

We then find that

$$*_4 \Omega^I = X^I{}_J \Omega^J$$

where

$$X^I{}_J = G^{IK} (Q^{-1})_{KJ}$$

By $*_4*_4 = 1$ when acting on two-forms, we get

$$G^{IK}(Q^{-1})_{KJ} = Q^{IK}G_{KJ}$$

We define a symplectic basis on M_6 as

$$\begin{aligned}\alpha_I &= (Q^{-1})_{IJ}\Omega^J \wedge dx \\ \beta^I &= \Omega^I \wedge dt\end{aligned}$$

and indeed this is consistent with our choice of symplectic basis on $S^1 \times M_5$ in our previous section. To see this we first notice that

$$\mathcal{G}^{IJ} = 2\pi R G^{IJ}$$

and then

$$2\pi \int_{M_5} *_5\Omega^K \wedge \Omega^L = \mathcal{G}^{KI}(Q^{-1})_{IJ} \int_{M_5} \Omega^J \wedge \Omega^L \wedge dx$$

Since Ω^I is an arbitrary element of $H^2(M_4, \mathbb{Z})$ we conclude that

$$2\pi \mathcal{G}_{IK} *_5 \Omega^K = (Q^{-1})_{IJ}\Omega^J \wedge dx$$

We define the period matrix as

$$\tau_{IJ} = -(\tau_1)_{IJ} - (\tau_2)_{IJ}$$

where

$$\alpha_I = (\tau_1)_{IJ}\beta^J + Y_{IJ} *_6 \beta^J$$

We have

$$*_6\beta^I = *_4\Omega^I \wedge *_2dt$$

and we have

$$\begin{aligned}*_4\Omega^I &= G^{IK}(Q^{-1})_{KJ}\Omega^J \\ *_2dt &= \frac{\sqrt{\gamma}}{\sqrt{-\beta}}(-dx + V^x dt)\end{aligned}$$

We define the two-dimensional period matrix as

$$\tau = -\tau_1 - \tau_2$$

where

$$\alpha = \tau_1\beta + \tau_2 *_2 \beta$$

and where

$$\begin{aligned}\beta &= dt \\ \alpha &= dx\end{aligned}$$

From this definition we get

$$\tau = -V^x + \frac{\sqrt{-\beta}}{\sqrt{\gamma}}$$

We get the six-dimensional period matrix as

$$\tau_{IJ} = \tau_1(Q^{-1})_{IJ} + \tau_2 G_{IJ}$$

C Holomorphic factorization

The Hamiltonian is

$$\begin{aligned} H &= -\tau_2 \left(n^2 + \frac{1}{4} m^2 \right) - \tau_1 n m \\ &= \frac{1}{2} \left(\left(\frac{m}{2} + n \right)^2 \tau - \left(\frac{m}{2} - n \right)^2 \bar{\tau} \right) \end{aligned}$$

The partition function is

$$\sum e^{2\pi i H} = \sum e^{i\pi\tau \left(n + \frac{m}{2}\right)^2} e^{-i\pi\bar{\tau} \left(n - \frac{m}{2}\right)^2}$$

Following page 115 in [6] we define

$$\begin{aligned} q &= e^{2\pi i\tau} \\ \bar{q} &= e^{-2\pi i\bar{\tau}} \end{aligned}$$

and compute

$$\sum |\Theta \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (-\tau)|^2 = \sum \left(q^{\frac{1}{2}n^2} \bar{q}^{\frac{1}{2}m^2} + q^{\frac{1}{2}(n+\frac{1}{2})^2} \bar{q}^{\frac{1}{2}(m+\frac{1}{2})^2} \right) \frac{1}{2} (1 + (-1)^{n+m})$$

As the summand vanishes unless $n + m = 2p$ is even, we can substitute

$$m = 2p - n$$

and we get a sum over n, p

$$\sum \left(q^{\frac{1}{2}n^2} \bar{q}^{\frac{1}{2}(2p-n)^2} + q^{\frac{1}{2}(n+\frac{1}{2})^2} \bar{q}^{\frac{1}{2}(2p-n+\frac{1}{2})^2} \right)$$

We substitute

$$n = p - q$$

and we get

$$\begin{aligned} \sum \left(q^{\frac{1}{2}(p-q)^2} \bar{q}^{\frac{1}{2}(p+q)^2} + q^{\frac{1}{2}(p-q+\frac{1}{2})^2} \bar{q}^{\frac{1}{2}(p+q+\frac{1}{2})^2} \right) &= \sum q^{\frac{1}{2}(\frac{m}{2}-q)^2} \bar{q}^{\frac{1}{2}(\frac{m}{2}+q)^2} \\ &= \sum e^{2\pi i H} \end{aligned}$$

since in the second line we may break the sum over m into sum over $m = 2p$ and a sum over $m = 2p + 1$.

References

- [1] E. Witten, *Five-brane effective action in M-theory*, *J. Geom. Phys.* **22** (1997) 103 [[hep-th/9610234](#)] [[INSPIRE](#)].
- [2] L. Dolan and C.R. Nappi, *A modular invariant partition function for the five-brane*, *Nucl. Phys. B* **530** (1998) 683 [[hep-th/9806016](#)] [[INSPIRE](#)].

- [3] M. Henningson, B.E.W. Nilsson and P. Salomonson, *Holomorphic factorization of correlation functions in $(4k+2)$ -dimensional $(2k)$ form gauge theory*, *JHEP* **09** (1999) 008 [[hep-th/9908107](#)] [[INSPIRE](#)].
- [4] M. Henningson, *The quantum Hilbert space of a chiral two form in $d = (5+1)$ -dimensions*, *JHEP* **03** (2002) 021 [[hep-th/0111150](#)] [[INSPIRE](#)].
- [5] J. Polchinski, *String theory. Volume 1: An introduction to the bosonic string*, Cambridge University Press, Cambridge U.K. (1998) [[INSPIRE](#)].
- [6] P.H. Ginsparg, *Applied conformal field theory*, [hep-th/9108028](#) [[INSPIRE](#)].
- [7] A. Gustavsson, *A preliminary test of Abelian $D4$ - $M5$ duality*, *Phys. Lett.* **B 706** (2011) 225 [[arXiv:1111.6339](#)] [[INSPIRE](#)].
- [8] L. Dolan and Y. Sun, *Partition functions for Maxwell theory on the five-torus and for the fivebrane on $S^1 \times T^5$* , [arXiv:1208.5971](#) [[INSPIRE](#)].
- [9] E. Witten, *Some comments on string dynamics*, in Proceedings of the *STRINGS 95: Future Perspectives in String Theory Conference*, Los Angeles U.S.A., 13–18 Mar 1995, World Scientific (1996), pp. 501–523 [[hep-th/9507121](#)] [[INSPIRE](#)].
- [10] S. Sethi and M. Stern, *A comment on the spectrum of H monopoles*, *Phys. Lett.* **B 398** (1997) 47 [[hep-th/9607145](#)] [[INSPIRE](#)].
- [11] N. Lambert, C. Papageorgakis and M. Schmidt-Sommerfeld, *$M5$ -branes, $D4$ -branes and quantum $5D$ super-Yang-Mills*, *JHEP* **01** (2011) 083 [[arXiv:1012.2882](#)] [[INSPIRE](#)].
- [12] M.R. Douglas, *On $D = 5$ super Yang-Mills theory and $(2,0)$ theory*, *JHEP* **02** (2011) 011 [[arXiv:1012.2880](#)] [[INSPIRE](#)].
- [13] K.-M. Lee, D. Tong and S. Yi, *The moduli space of two $U(1)$ instantons on noncommutative R^4 and $R^3 \times S_1$* , *Phys. Rev.* **D 63** (2001) 065017 [[hep-th/0008092](#)] [[INSPIRE](#)].
- [14] H.-C. Kim, S. Kim, E. Koh, K. Lee and S. Lee, *On instantons as Kaluza-Klein modes of $M5$ -branes*, *JHEP* **12** (2011) 031 [[arXiv:1110.2175](#)] [[INSPIRE](#)].
- [15] E. Witten, *Geometric Langlands from six dimensions*, [arXiv:0905.2720](#) [[INSPIRE](#)].
- [16] E. Hatefi, A.J. Nurmagambetov and I.Y. Park, *N^3 entropy of $M5$ branes from dielectric effect*, *Nucl. Phys.* **B 866** (2013) 58 [[arXiv:1204.2711](#)] [[INSPIRE](#)].
- [17] M.-C. Tan, *Five-branes in M -theory and a two-dimensional geometric Langlands duality*, *Adv. Theor. Math. Phys.* **14** (2010) 179 [[arXiv:0807.1107](#)] [[INSPIRE](#)].
- [18] O. Alvarez, *Topological quantization and cohomology*, *Commun. Math. Phys.* **100** (1985) 279 [[INSPIRE](#)].
- [19] M. Bos and V.P. Nair, *Coherent state quantization of Chern-Simons theory*, *Int. J. Mod. Phys.* **A 5** (1990) 959 [[INSPIRE](#)].
- [20] G. Bonelli, *On the supersymmetric index of the M -theory five-brane and little string theory*, *Phys. Lett.* **B 521** (2001) 383 [[hep-th/0107051](#)] [[INSPIRE](#)].
- [21] A. Gustavsson, *On the holomorphically factorized partition function for Abelian gauge theory in six-dimensions*, *Int. J. Mod. Phys.* **A 17** (2002) 383 [[hep-th/0008161](#)] [[INSPIRE](#)].
- [22] M. Henningson, *A class of six-dimensional conformal field theories*, *Phys. Rev. Lett.* **85** (2000) 5280 [[hep-th/0006231](#)] [[INSPIRE](#)].

- [23] E. Witten, *On S duality in Abelian gauge theory*, *Selecta Math.* **1** (1995) 383 [[hep-th/9505186](#)] [[INSPIRE](#)].
- [24] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Y.I. Manin, *Construction of instantons*, *Phys. Lett. A* **65** (1978) 185 [[INSPIRE](#)].
- [25] N. Nekrasov and A.S. Schwarz, *Instantons on noncommutative R^4 and $(2,0)$ superconformal six-dimensional theory*, *Commun. Math. Phys.* **198** (1998) 689 [[hep-th/9802068](#)] [[INSPIRE](#)].
- [26] D. Bak, *Exact multivortex solutions in noncommutative Abelian Higgs theory*, *Phys. Lett. B* **495** (2000) 251 [[hep-th/0008204](#)] [[INSPIRE](#)].
- [27] T. Eguchi and A.J. Hanson, *Selfdual solutions to Euclidean gravity*, *Annals Phys.* **120** (1979) 82 [[INSPIRE](#)].
- [28] A. Sen, *U duality and intersecting Dirichlet branes*, *Phys. Rev. D* **53** (1996) 2874 [[hep-th/9511026](#)] [[INSPIRE](#)].
- [29] F. Ohlsson, *$(2,0)$ theory on Taub-NUT: a note on WZW models on singular fibrations*, [arXiv:1205.0694](#) [[INSPIRE](#)].
- [30] J.P. Gauntlett and D.A. Lowe, *Dyons and S duality in $N = 4$ supersymmetric gauge theory*, *Nucl. Phys. B* **472** (1996) 194 [[hep-th/9601085](#)] [[INSPIRE](#)].
- [31] K.-M. Lee, E.J. Weinberg and P. Yi, *Electromagnetic duality and SU(3) monopoles*, *Phys. Lett. B* **376** (1996) 97 [[hep-th/9601097](#)] [[INSPIRE](#)].
- [32] H. Linander and F. Ohlsson, *$(2,0)$ theory on circle fibrations*, *JHEP* **01** (2012) 159 [[arXiv:1111.6045](#)] [[INSPIRE](#)].
- [33] A. Gustavsson, *Deconstructing graviphoton from mass-deformed ABJM*, *JHEP* **06** (2012) 174 [[arXiv:1203.5883](#)] [[INSPIRE](#)].