

Rotating black holes, global symmetry and first order formalism

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ABSTRACT: In this paper we consider axisymmetric black holes in supergravity and address the general issue of defining a first order description for them. The natural setting where to formulate the problem is the De Donder-Weyl-Hamilton-Jacobi theory associated with the effective two-dimensional sigma-model action describing the axisymmetric solutions. We write the general form of the two functions S_m defining the first-order equations for the fields. It is invariant under the global symmetry group $G_{(3)}$ of the sigma-model. We also discuss the general properties of the solutions with respect to these global symmetries, showing that they can be encoded in two constant matrices belonging to the Lie algebra of $G_{(3)}$, one being the Nöther matrix of the sigma model, while the other is non-zero only for rotating solutions. These two matrices allow a $G_{(3)}$ -invariant characterization of the rotational properties of the solution and of the extremality condition. We also comment on extremal, under-rotating solutions from this point of view.

KEYWORDS: Black Holes, Supergravity Models

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1 Introduction

There has been a considerable progress in the knowledge of static black holes in supergravity, both from the point of view of finding solutions and of their classification [1–5], in four and higher dimensions.

A relevant role in these developments was played by the use of a first order formalism, corresponding to the introduction of a fake-superpotential [6–15] that was recognized to be strictly related to the Hamilton characteristic function in a mechanical problem where the evolution is in the radial variable τ [8, 10, 14]. The latter approach naturally applies to both extremal and non-extremal static, single center black holes.

As far as more general solutions, such as stationary and/or multicenter black holes [16–21], are concerned, a similar comprehensive study is still missing. In particular, the use of a first order formalism has not been much exploited except in very particular cases [22–24].

A peculiarity of static, spherically symmetric solutions is that one can exploit the symmetries to reduce the Lagrangian to a one-dimensional effective one, where the evolution variable is the radial one [25, 26]. However, when considering four dimensional solutions

with less symmetries, in particular stationary solutions where only the time-like Killing vector ∂_t is present, an effective three-dimensional Lagrangian can be obtained upon compactification along the time coordinate [27–36]. The fields in the effective Lagrangian now depend on the three space variables x^i , ($i = 1, 2, 3$). In particular, for stationary axisymmetric solutions, the presence of an angular Killing vector ∂_φ allows a further dimensional reduction to two dimensions.

The problem of extending the Hamilton-Jacobi (in the following, HJ) formalism from mechanical models, whose degrees of freedom depend on just one variable, to field theories where the degrees of freedom depend on two or more variables, was addressed and developed in generality from several points of view (a useful review is given by [37]).

Our main aim in the present paper is to apply such extended formalism to the study of black holes. We will adhere to the so-called De Donder-Weyl-Hamilton-Jacobi theory, hereafter referred to as DWHJ, which is the simplest extension of the classical HJ approach in mechanics. One important difference with respect to the case of classical mechanics consists in the replacement of the Hamilton principal function S (directly related to the fake-superpotential of static black holes) with a *Hamilton principal 1-form*, that is with a covariant vector S_i .

As it is usual in the three dimensional approach, by using Hodge-duality in three dimensions all the fields of the parent four dimensional theory are described by three dimensional scalars [27] and their interaction is given by gravity coupled to a σ -model. Correspondingly, the equations of motion give a set of conserved currents. A particularly interesting case is when the σ -model is a symmetric space $G_{(3)}/H^*$ (where H^* denotes a suitable non-compact maximal subgroup of $G_{(3)}$ [27]). Note that the effective geodesic Lagrangian is invariant under the three-dimensional isometry group $G_{(3)}$ (we will also refer to it as the three-dimensional *duality group*). One of the main results of our paper is to give a manifestly duality invariant expression for the Hamilton principal vector S_i , thus extending the results obtained for the Hamilton characteristic function \mathcal{W} in the static case [10].

For pure Einstein-Maxwell stationary configurations, the three-dimensional σ -model turns out to be $SU(1, 2)/U(1, 1)$. As is well known in General Relativity, in the presence of a time-like Killing vector Einstein-Maxwell theory is very efficiently described in terms of the so-called Ernst potentials \mathcal{E} , Ψ (see for example [38, 39]), which are complex functions of the $SU(1, 2)$ complex triplet of fields $\mathbb{U} = (W, V, U)$. We found particularly useful, outside the ergosphere, to parametrize the coset $SU(1, 2)/U(1, 1)$ with the homogeneous fields U, V, W , or more precisely with their inhomogeneous counterpart ($u = U/W, v = V/W$), corresponding to four real scalar degrees of freedom.

In the present paper we will give general results on stationary axisymmetric solutions of four dimensional supergravity and then focus on the first-order formulation of the Kerr-Newman solution and its extension in the presence of a NUT charge. Besides finding a duality invariant S_i , we will also express the conserved charges of the black hole [40] in terms of the conserved charges of the σ -model $G_{(3)}/H^*$. Actually, the Nöther charges associated with $G_{(3)}$ global symmetry do not include the angular momentum M_φ . The latter can nevertheless be expressed in terms of quantities which are intrinsic to the σ -

model. This is achieved by introducing a new $G_{(3)}$ -covariant constant matrix, besides the Nöther charge one Q , defined as follows:

$$Q_\psi = -\frac{3}{8\pi} \int_{S^2_\infty} \psi_{[i} J_{j]} dx^i \wedge dx^j, \tag{1.1}$$

J_i being the Nöther current with value in the algebra of $G_{(3)}$ and $\psi = \partial_\varphi$ the angular Killing vector. From straightforward application of the general four-dimensional expression for the angular momentum one finds that its squared value can be written in terms of the ratio of two $G_{(3)}$ invariants $\text{Tr}(Q_\psi^2)$ and $\text{Tr}(Q^2)$, and thus can be given a description which is invariant with respect to the global symmetry of the σ -model and is straightforwardly generalizable to more general models with $D = 4$ scalar fields. This analysis also provides a $G_{(3)}$ -invariant characterization of the extremality parameter (and thus of the extremality condition), see eq.s (3.40), (3.41), so that the cosmic-censor condition for Kerr black holes, $M_{ADM}^4 \geq M_\varphi^2$, can be recast for the generic regular axisymmetric solution, in a $G_{(3)}$ -invariant way as

$$[\text{Tr}(Q^2)]^2 \geq \frac{2}{k} \text{Tr}(Q_\psi^2),$$

k being a representation-dependent constant. In particular we show that in the extremal “ergo-free” solutions [20, 41–44], both matrices Q , Q_ψ are *nilpotent*, the former having a larger degree of nilpotency of the latter. The first-order formalism and the functions S_m for under-rotating solutions were derived in [23].

A description of the global symmetry properties of axisymmetric solutions should then include at least the *two independent, mutually orthogonal matrices* Q , Q_ψ belonging to the Lie algebra of the global symmetry group.

The paper is organized as follows.

In section 2 we present the extension of the HJ theory to field theory, following the DWHJ approach, and give a general formula to find the Hamilton principal 1-form.

In section 3 we focus on stationary axisymmetric black holes, whose description, following [27], is two dimensional. We review the construction of the two-dimensional effective Lagrangian and the expression of the characteristic physical quantities associated with the four-dimensional solution in terms of Nöther currents of the sigma-model. We also write the angular momentum in terms of the sigma-model Nöther currents and introduce, besides Q , the matrix Q_ψ , which allows to describe in a $G_{(3)}$ -invariant fashion the rotational properties of the solution. We also discuss the *under-rotating* extremal limit of a non-extremal solution in the $G_{(3)}$ -orbit of the Kerr-black hole. Then we find a manifestly (three-dimensional) duality invariant expression for the principal functions S_i ($i = 1, 2, 3$).

In section 4 we restrict our attention to the KN-Taub-NUT solution, making use of the Ernst potentials written in terms of the inhomogeneous fields (u, v) to parametrize the $SU(1, 2)/U(1, 1)$ coset and give the explicit form of the principal functions S_m ($m = 1, 2$) in terms of fields and two-dimensional coordinates.

We end in section 5 with some concluding remarks. Appendix A contains the explicit form of the algebra $SU(1, 2)$, while appendix B, extending the procedure of [45, 46] to the case where a NUT charge is present, shows how one can retrieve the KN-Taub-NUT

solution from Schwarzschild by use of duality and general coordinate transformations. In particular, appendix B.1 contains a manifestly H^* -invariant expression for the \mathcal{W} -function describing the RN-Taub-NUT solution in the universal model, which, to our knowledge, was not known so far, and then applies a known procedure [45, 46] to generate from it the KN-Taub-NUT solution by a set of duality and general coordinate transformations.

2 Hamilton-Jacobi formalism for field theory

In a previous work a formalism was developed to interpret the first-order description of static black holes in terms of Hamilton-Jacobi theory. In particular, the Hamilton characteristic function \mathcal{W} was shown to be related, for extremal solutions, to the “fake” superpotential: $\mathcal{W} = 2 e^{2U} W$ [8, 10]. The above construction works well in the static, spherically symmetric case where the metric only depends in a non-trivial way on the evolution radial variable τ so that the Einstein Lagrangian can be reduced to an effective one-dimensional Lagrangian. For more general black holes, with a lower number of isometries, we have to extend the Hamilton-Jacobi formalism to a more general setting. In particular, for stationary black holes corresponding to the existence of a Killing vector associated with time translations $\frac{\partial}{\partial t}$, the metric can be reduced to the following general form

$$ds^2 = e^{2\mathcal{U}}(dt + \omega)^2 - e^{-2\mathcal{U}} g_{ij} dx^i dx^j \tag{2.1}$$

where the fields $\mathcal{U}, \omega = \omega_i dx^i$ and the 3D metric tensor g_{ij} depend on the space coordinates $x^i, i = 1, 2, 3$.

In the static, spherically symmetric case, the HJ equations arise in a classical mechanical effective model where the evolution variable τ plays the role of time. A first-order formulation for a more general black-hole solution requires the extension of the Hamilton-Jacobi description from classical mechanics to a field theory depending on two or more variables (see, for example, [37] and references therein). In this setting the Hamilton-Jacobi description has to be generalized to the so-called De Donder-Weyl-Hamilton-Jacobi theory, hereafter referred to as DWHJ, which amounts to the following. Let $\mathcal{L}(z^a, v_i^a, x^i)$ be the Lagrangian density of the system, where z^a ($a = 1, \dots, n$) are the field variables which become functions of the x^i , $z^a = \xi^a(x)$, on the extremals, while $v_i^a = \partial_i z^a$. The canonical momenta are defined by $\pi_a^i = \frac{\partial \mathcal{L}}{\partial v_i^a}$, and the invariant Hamilton density function is

$$\mathcal{H} = \pi_a^i v_i^a - \mathcal{L}. \tag{2.2}$$

The DWHJ equation is a first-order partial differential equation for the functions $S^i(z, x)$:

$$\partial_i S^i(z, x) + \mathcal{H}(z, x, \pi) = 0, \tag{2.3}$$

where

$$\pi_a^i = \partial_a S^i(z, x). \tag{2.4}$$

The functions $S_i = \frac{1}{\sqrt{g}}g_{ij}S^j$ may be thought of as the components of a one-form $S^{(1)} \equiv S_i dx^i$.¹

In the field-theory case the issue of integrability is more involved than in mechanics since, even if a complete integral S^i can be found, solutions to the Euler-Lagrange equations can be constructed if the integrability conditions (which are trivial in mechanics):

$$\partial_{[i}v_{j]}^a = 0 \tag{2.5}$$

are satisfied. Taking into account that $v_i^a(\pi, z, x) = v_i^a\left(\frac{\partial S}{\partial z}, z, x\right)$, this imposes severe constraints on the solutions $S^i(z, x)$. From now on we will mainly focus on the two dimensional case, which is relevant when discussing axisymmetric black holes for which two Killing vectors exist, associated with time translations $\frac{\partial}{\partial t}$ and rotations about an axis $\psi = \frac{\partial}{\partial \varphi}$. Note however that the extension of the formalism from systems depending on two independent variables to systems with three or more independent variables is straightforward and does not bring anything conceptually new [37]. We will denote the independent variables for the two-dimensional case by x^m , $m = 1, 2$. The 3D metric in this case takes the form: $g_{ij}dx^i dx^j = \gamma_{mn}dx^m dx^n + \hat{\rho}^2 d\varphi^2$, where φ denotes the angle about the rotation axis, and the fields $\gamma_{mn}, \hat{\rho}$ depend on x^m .

If one introduces the two-form Lagrangian

$$\Omega_0 = -\mathcal{H}dx^m \wedge dx^n + \pi_a^m dz^a \wedge dx^n \epsilon_{mn} \tag{2.6}$$

then the Hamilton-Jacobi equations are given by the condition

$$d\Omega_0 = 0 \tag{2.7}$$

which implies that, locally, there exist two functions S^m in terms of which Ω_0 can be written in the following form:

$$\Omega_0 = dS^m \wedge dx^n \epsilon_{mn}, \tag{2.8}$$

so that²

$$\partial_m S^m = -\mathcal{H}, \tag{2.10}$$

$$\frac{\partial S^m}{\partial z^a} = \pi_a^m. \tag{2.11}$$

¹We observe that, in the presence of a gravitational field, which is the case we will deal with, (2.3) should be modified to contain the covariant divergence $\nabla_i S^i$. However, defining the contravariant vector density $S^i \equiv \sqrt{g}g^{ij}S_j$, S_j being a true covariant vector, makes it possible to trade the covariant derivatives for ordinary ones, so that the equations are formally the same as in flat space. In this case, however, by \mathcal{H} we mean the hamiltonian density including the factor $\sqrt{|g|}$.

²We denote with ∂_m the derivative with respect to explicit x^m dependence, while total derivative with respect to x^m is denoted by $\frac{d}{dx^m}$:

$$\frac{d}{dx^m} f(\xi, x) \equiv \partial_m \xi^a \frac{\partial f}{\partial \xi^a} + \partial_m f \tag{2.9}$$

2.1 Solving DWHJ equations

In the present section we discuss in a general setting a possible way to solve the DWHJ equations. Then, in the next sections we will apply this procedure to the study of axisymmetric black holes and their Taub-NUT extensions. We will give here a constructive recipe to find solutions to the field equations by solving the DWHJ equations, following a general procedure given in the literature (see for example [37] and references therein).

As already anticipated, in field theory the expression for S^m is strongly restricted by the integrability constraints (2.5). In particular, as opposed to the one-dimensional classical-mechanics case, it is not always possible to find an expression for S^m valid in an open neighborhood of the extremals $z^a = \xi^a(x)$ in the space of fields *and* coordinates. When this is possible, one says that the extremals $z^a = \xi^a(x)$ are *strongly* embedded in the wave fronts $S^m(z, x)$. In many cases, however, the solution S^m satisfies eqs. (2.10) and (2.11) only on the extremals $z^a = \xi^a(x)$. One then says that the extremals are *weakly* embedded in $S^m(z, x)$.

A possible solution which is weakly embedded in S^m is found by choosing one of the x^m , say x^1 , as the evolution variable:

$$S^m = (z^a - \xi^a(x)) \pi_a^m(\xi, x) + \delta_1^m \int^{x^1} dx^{1'} \mathcal{L}(\xi(x'), \partial_m \xi, x') + \mathcal{O}[(z^a - \xi^a(x))^2] \quad (2.12)$$

Indeed, from (2.12) we find, using (2.2)

$$\left. \frac{\partial S^m}{\partial z^a} \right|_{z=\xi} = \pi_a^m(\xi, x) \quad (2.13)$$

$$\left. \partial_m S^m \right|_{z=\xi} = -\partial_m \xi^a \pi_a^m + \mathcal{L}(\xi(x), \partial_m \xi, x) = -\mathcal{H}(\xi(x), \partial_m \xi, x) . \quad (2.14)$$

Eq. (2.12) can be understood as a linear approximation of the Taylor expansion of S^m in the neighborhood of the extremal.

3 The 2D effective Lagrangian and its field-theoretical DWHJ description

In the presence of a time-like Killing vector ∂_t , the vielbein V^a ($a = 0, 1, 2, 3$) of space-time can be put in the form

$$V^0 = e^{\mathcal{U}}(dt + \omega) = e^{\mathcal{U}}D^0; \quad V^i = e^{-\mathcal{U}}D^i \quad (3.1)$$

where D^i ($i = 1, 2, 3$) are 3D vielbein. The time-reduced 3-dimensional Lagrangian describing a stationary 4D black hole in the presence of a given number of scalars ϕ^r and

gauge fields A^Λ has the following form³

$$\begin{aligned} \frac{1}{\sqrt{g_{(3)}}} \mathcal{L}_{(3)} &= \frac{1}{2} \mathcal{R} - \frac{1}{2} G_{ab}(z) \partial_i z^a \partial^i z^b = \\ &= \frac{1}{2} \mathcal{R} - \left[\partial_i \mathcal{U} \partial^i \mathcal{U} + \frac{1}{2} G_{rs} \partial_i \phi^r \partial^i \phi^s + \frac{1}{2} \epsilon^{-2\mathcal{U}} \partial_i \mathbf{Z}^T \mathcal{M}_{(4)} \partial^i \mathbf{Z} + \right. \\ &\quad \left. + \frac{1}{4} \epsilon^{-4\mathcal{U}} (\partial_i a + \mathbf{Z}^T \mathbb{C} \partial_i \mathbf{Z}) (\partial^i a + \mathbf{Z}^T \mathbb{C} \partial^i \mathbf{Z}) \right], \end{aligned} \quad (3.2)$$

where $g_{(3)} \equiv \det(g_{(3)})$. Here, all the propagating degrees of freedom have been reduced to scalars by 3D Hodge-dualization [27]. In particular, the scalars $\mathbf{Z} = (\mathcal{Z}^\Lambda, \mathcal{Z}_\Lambda) = \{\mathcal{Z}^M\}$ include the electric components A_0^Λ of the 4D vector fields together with the Hodge dual of their magnetic components A_i^Λ ($i = 1, 2, 3$) and a is related to the Hodge-dual of the 3D graviphoton ω_i . More precisely,

$$A_{(4)}^\Lambda = A_0^\Lambda D^0 + A_{(3)}^\Lambda, \quad A_{(3)}^\Lambda \equiv A_i^\Lambda D^i, \quad (3.3)$$

$$\mathbf{F}_{(4)}^M = \begin{pmatrix} F_{(4)}^\Lambda \\ \mathcal{G}_{\Lambda(4)} \end{pmatrix} = d\mathcal{Z}^M \wedge D^0 + e^{-2\mathcal{U}} \mathbb{C}^{MN} \mathcal{M}_{(4)NP}^* d\mathcal{Z}^P, \quad (3.4)$$

$$da = -e^{4\mathcal{U}*} d\omega - \mathbf{Z}^T \mathbb{C} d\mathbf{Z}, \quad (3.5)$$

where $F_{(4)}^\Lambda = dA_{(4)}^\Lambda$, $\mathcal{G}_{\Lambda(4)} = -\frac{1}{2}^* \left(\frac{\partial \mathcal{L}}{\partial F_{(4)}^\Lambda} \right)$, and $\mathcal{M}_{(4)}(\phi)$ is the negative-definite symmetric, symplectic matrix depending on 4D scalar fields introduced in [47, 48].

The isometry group $G_{(3)}$ of the σ -model metric $G_{ab}(z)$ contains as non trivial subgroups the 4-dimensional U-duality group $G_{(4)}$ times the group $\text{SL}(2, \mathbb{R})$ (the Ehlers group) under which the degrees of freedom of the 4d metric transform. The simplest 3D model is the one originating from a pure 4D Einstein-Maxwell gravitational theory with a single time-like Killing vector. In this case $G_{(4)} = \text{U}(1)$ and the 3D σ -model has the homogeneous-symmetric target space $\frac{\text{SU}(1,2)}{\text{U}(1) \times \text{SU}(1,1)}$. Its field content consists of four scalars belonging to a pseudo-Riemannian version of the universal hypermultiplet, dubbed the universal *pseudo-hypermultiplet*. We will discuss in more detail the properties of this theory in the following section 4.

We will mainly focus our attention on stationary axisymmetric solutions admitting the two Killing vectors ∂_t and ∂_φ . In this case, as pointed out earlier, one may further reduce the 3D Lagrangian to two dimensions by compactification along φ . The fields now depend on the space coordinates x^m , $m = 1, 2$, and we assume that the three-dimensional space metric can be expressed in block-diagonal form as:

$$g_{(3)} = \begin{pmatrix} \lambda^2 h_{mn} & 0 \\ 0 & \hat{\rho}^2 \end{pmatrix}. \quad (3.6)$$

The resulting 2D Lagrangian takes the form [27]

$$\mathcal{L}_{(2)} = \sqrt{h} \hat{\rho} \left(\frac{\mathcal{R}_{(2)}}{2} - \frac{1}{2} G_{ab}(z) \partial_m z^a \partial^m z^b + \frac{\partial_m \hat{\rho} \partial^m \lambda}{\lambda \hat{\rho}} \right), \quad (3.7)$$

³For the $D = 4$ supergravity theory we use the units $\hbar = c = 8\pi G = 1$ and the normalization of the vector fields as in [5].

with $h \equiv \det(h_{mn})$. As shown in [27], the dynamics of the fields z^a is totally captured by the σ -model effective action:

$$S_{\text{eff}} = \int d^2x \sqrt{h} \frac{\hat{\rho}}{2} G_{ab}(z) \partial_m z^a \partial^m z^b, \quad (3.8)$$

where $\hat{\rho}(x^m)$ is a harmonic function in the subspace spanned by x^m .⁴ The metric on this space can be made conformally flat by a suitable choice of the x^m and the conformal factor absorbed in the definition of λ , so that the equations for z^a and $\hat{\rho}$ can be written in a flat 2D space (with $\mathcal{R}_{(2)} = 0$) spanned by x^m , with metric h_{mn} . As we shall show in section 4.1, in suitable coordinates, $\sqrt{h} \hat{\rho} = \sin \theta$.

The equation for λ can then be solved once the solutions to the σ -model are known [27].

We shall restrict our analysis to symmetric supergravities in which the scalar manifold $\mathcal{M}_{\text{scal}}$ of the $D = 3$ theory, spanned by the z^a , is homogeneous symmetric, i.e. of the form

$$\mathcal{M}_{\text{scal}} = \frac{G_{(3)}}{H^*}. \quad (3.9)$$

We shall use for this manifold the solvable Lie algebra parametrization by identifying the scalar fields z^a with parameters of a suitable solvable Lie algebra. Let us recall the main points [32]. The isometry group $G_{(3)}$ of the target space is the global symmetry group of the S_{eff} and H^* is a suitable non-compact semisimple maximal subgroup of it. The scalars $z^a = \{\mathcal{U}, a, \phi^r, \mathbf{Z}\}$ correspond to a *local* solvable parametrization, i.e. the corresponding patch, to be dubbed *physical patch* \mathcal{U} , is isometric to a solvable Lie group generated by a solvable Lie algebra *Solv*:

$$\mathcal{M}_{\text{scal}} \supset \mathcal{U} \equiv e^{\text{Solv}}, \quad (3.10)$$

Solv is defined by the Iwasawa decomposition of the Lie algebra \mathfrak{g} of $G_{(3)}$ with respect to its maximal compact subalgebra \mathfrak{h} . The solvable parametrization z^a can be defined by the following exponential map:

$$\mathbb{L}(z^a) = \exp(-aT_\bullet) \exp(\sqrt{2}\mathcal{Z}^M T_M) \exp(\phi^r T_r) \exp(2\mathcal{U}T_0), \quad (3.11)$$

where the generators T_0, T_\bullet, T_r, T_M satisfy the following commutation relations:

$$\begin{aligned} [T_0, T_M] &= \frac{1}{2} T_M; & [T_0, T_\bullet] &= T_\bullet; & [T_M, T_N] &= \mathbb{C}_{MN} T_\bullet, \\ [T_0, T_r] &= [T_\bullet, T_r] = 0; & [T_r, T_M] &= T_r^N{}_M T_N; & [T_r, T_s] &= -T_{rs}{}^{s'} T_{s'}, \end{aligned} \quad (3.12)$$

$T_r^N{}_M$ representing the symplectic representation of T_r on contravariant symplectic vectors $d\mathcal{Z}^M$. We can use for the generators of \mathfrak{g} a representation in which the generators of \mathfrak{h}^* , the Lie algebra of H^* , are invariant under the involution $\sigma : M \rightarrow -\eta M^\dagger \eta$, where $\eta \equiv (-1)^{2T_0}$. The vielbein P and connection \mathcal{W} 1-forms on the manifold are computed as the odd and even components, respectively, of the left-invariant one-form with respect to σ :

$$\mathbb{L}^{-1} d\mathbb{L} = P + \mathcal{W}, \quad (3.13)$$

⁴According to a general procedure in General Relativity one can perform a coordinate transformation such that the field $\hat{\rho}$ is chosen as one of the new harmonic coordinates, the second coordinate z being defined by $dz = -^* d\hat{\rho}$. Here * denotes Hodge-dualization in two dimensions. In these new variables $x^m = (\hat{\rho}, z)$, named Weyl-coordinates, the 2D metric is conformally flat $\gamma_{mn} = \lambda^2 \delta_{mn}$ [27, 38].

$P = \eta P^\dagger \eta = -\sigma(P)$, $\mathcal{W} = -\eta \mathcal{W}^\dagger \eta = \sigma(\mathcal{W})$. In terms of P the metric on the manifold reads:

$$dS_{(3)}^2 = G_{ab}(z) dz^a dz^b = k \text{Tr}(P^2), \quad (3.14)$$

where $k = 1/(2\text{Tr}(T_0^2))$ is a representation-dependent constant. It is also useful to introduce the hermitian, H^* -invariant matrix \mathcal{M} :

$$\mathcal{M}(z) \equiv \mathbb{L}\eta\mathbb{L}^\dagger = \mathcal{M}^\dagger, \quad (3.15)$$

in terms of which we can write the geodesic Lagrangian as:

$$\mathcal{L}_{(2)eff} = \frac{1}{2} \hat{\rho} \sqrt{h} G_{ab}(z) \partial_m z^a \partial^m z^b = \frac{k}{8} \hat{\rho} \sqrt{h} \text{Tr} [\mathcal{M}^{-1} \partial_m \mathcal{M} \mathcal{M}^{-1} \partial^m \mathcal{M}], \quad (3.16)$$

with a canonically conjugate momentum

$$\pi_a^m = \frac{\partial \mathcal{L}}{\partial \partial_m z^a} = \frac{k}{4} \hat{\rho} \sqrt{h} \text{Tr} [\mathcal{M}^{-1}(z) \partial_a \mathcal{M}(z) \mathcal{M}^{-1}(z) \partial_b \mathcal{M}(z)] \partial^m z^b. \quad (3.17)$$

The corresponding equations of motion are:

$$\partial_m \left(\sqrt{h} \hat{\rho} h^{mn} J_n \right) = 0, \quad (3.18)$$

where

$$J_m \equiv \frac{1}{2} \partial_m \xi^a \mathcal{M}^{-1} \partial_a \mathcal{M}. \quad (3.19)$$

3.1 Conserved quantities

Note that the quantity $\hat{\rho} J = \hat{\rho} J_m dx^m$ is a 1-form Nöther current of the two-dimensional effective theory with value in \mathfrak{g} implying that the integral:

$$Q = \frac{1}{4\pi} \int_{S_2} {}^* J = \frac{1}{2} \int \sqrt{h} h^{rr} \hat{\rho} J_r d\theta, \quad (3.20)$$

on a radius r sphere S_2 is an r -independent matrix in \mathfrak{g} .

From it we may derive the set of Nöther currents J_{Am} and the corresponding *constants of motion* Q_A characterizing the solution at radial infinity:

$$J_{Am} \equiv k \text{Tr} \left(T_A^\dagger J_m \right), \quad Q_A = k \text{Tr} \left(T_A^\dagger Q \right) = \frac{1}{4\pi} \int_{S_2} {}^* J_A = \frac{1}{2} \int \sqrt{h} \rho h^{rr} J_{Ar} d\theta, \quad (3.21)$$

which consist in the ADM mass m ($T_A = T_0$), the NUT charge ℓ ($T_A = T_\bullet$), the $D = 4$ scalar charges Σ_r ($T_A = T_r$) and the electric-magnetic charges Γ^M ($T_A = T_M$). The currents J_{Am} read:

$$\begin{aligned} J_{\bullet m} &= \frac{k}{2} \text{Tr}(T_\bullet^\dagger \mathcal{M}^{-1} \partial_m \mathcal{M}) = -\frac{1}{2} e^{-4\mathcal{U}} (\partial_m a + \mathbf{Z}^T \mathbb{C} \partial_m \mathbf{Z}), \\ J_{0m} &= \frac{k}{2} \text{Tr}(T_0^\dagger \mathcal{M}^{-1} \partial_m \mathcal{M}) = \partial_m \mathcal{U} + \frac{1}{2} e^{-2\mathcal{U}} \mathbf{Z}^T \mathcal{M} \partial_m \mathbf{Z} - a J_{\bullet m}, \\ J_{Mm} &= \frac{k}{2} \text{Tr}(T_M^\dagger \mathcal{M}^{-1} \partial_m \mathcal{M}) = \frac{1}{\sqrt{2}} e^{-2\mathcal{U}} \mathcal{M}_{(4)MN} \partial_m \mathcal{Z}^N + \sqrt{2} \mathbb{C}_{MN} \mathcal{Z}^N J_{\bullet m}, \\ J_{sm} &= \frac{k}{2} \text{Tr}(T_s^\dagger \mathcal{M}^{-1} \partial_m \mathcal{M}) = \frac{1}{\sqrt{2}} \mathbb{L}_{4s}{}^{s'} V_{4s''}{}^{s'} \partial_m \phi^{s''} + e^{-2\mathcal{U}} \mathbf{Z}^T T_s \mathcal{M}_{(4)} \partial_m \mathbf{Z} - \\ &\quad - T_{sMN} \mathcal{Z}^M \mathcal{Z}^N J_{\bullet m}, \end{aligned} \quad (3.22)$$

where $\mathbb{L}_{4s}^{\hat{s}'}$ is the coset representative of the symmetric scalar manifold in four-dimensions in the solvable parametrization, as a matrix in the adjoint representation of the solvable group, $V_{4s}^{\hat{s}'}$ is the vielbein of the same manifold and the hat denotes rigid indices.

The conserved quantities are then obtained as the flux of the currents across the 2-sphere at infinity, according to eq. (3.21):

$$\begin{aligned} m &= \frac{1}{4\pi} \int_{S_2} {}^{*3}J_0; \quad \ell = -\frac{1}{4\pi} \int_{S_2} {}^{*3}J_{\bullet}; \quad \Gamma^M = \frac{\sqrt{2}}{4\pi} \mathbb{C}^{MN} \int_{S_2} {}^{*3}J_N, \\ \Sigma_s &= \frac{1}{4\pi} \int_{S_2} {}^{*3}J_s. \end{aligned} \quad (3.23)$$

The other conserved quantity characterizing the axisymmetric solution is the angular momentum M_φ along the rotation axis Z . The expression of the angular momentum in terms of a conserved current can be found in standard textbooks (see for instance [40] and [49]). Here we would like to give an expression of it in terms of quantities which are intrinsic to the $D = 3$ effective action: the Killing vector field $\psi = \partial_\varphi$ and J_{\bullet} . To this end we start from the representation of M_φ as the integral over the sphere at infinity S_2^∞ of a suitable 2-form, as given in [40]:

$$M_\varphi = \frac{1}{16\pi} \int_{S_2^\infty} J^{(2)}; \quad J^{(2)} \equiv \sqrt{g} \epsilon_{\mu\nu\rho\sigma} \nabla^\rho \psi^\sigma dx^\mu \wedge dx^\nu. \quad (3.24)$$

The above integral can also be written in the form:

$$\begin{aligned} M_\varphi &= \frac{1}{8\pi} \int_{S_2^\infty} \sqrt{g} g^{\mu[t} \Gamma_{\mu\varphi}^{r]} d\theta d\varphi = \frac{1}{8\pi} \int_{S_2^\infty} \sqrt{g} g^{\mu[t} g^{r]\nu} \partial_{[\mu} g_{\nu]\varphi} d\theta d\varphi =, \\ &= \frac{1}{8\pi} \int_{S_2^\infty} \sqrt{g^{(3)}} \left[\frac{1}{2} g_{(3)}^{rr} g_{(3)}^{\varphi\varphi} \left(\partial_r \omega_\varphi g_{\varphi\varphi}^{(3)} - \omega_\varphi \partial_r g_{\varphi\varphi}^{(3)} + e^{4\mathcal{U}} \omega_\varphi^2 \partial_r \omega_\varphi + \right. \right. \\ &\quad \left. \left. + 4\omega_\varphi g_{\varphi\varphi}^{(3)} \partial_r \mathcal{U} \right) \right] d\theta d\varphi. \end{aligned} \quad (3.25)$$

Using the asymptotic behavior of the metric for axisymmetric solutions [49]:

$$\begin{aligned} \omega_\varphi &= \frac{2M_\varphi}{r} \sin^2(\theta) + O\left(\frac{1}{r^2}\right); \quad g_{rr}^{(3)} = 1 + O\left(\frac{1}{r^2}\right); \quad g_{\theta\theta}^{(3)} = r^2 \left(1 + O\left(\frac{1}{r}\right)\right), \\ g_{\varphi\varphi}^{(3)} &= r^2 \sin^2(\theta) \left(1 + O\left(\frac{1}{r}\right)\right); \quad e^{2\mathcal{U}} = 1 - \frac{2m}{r} + O\left(\frac{1}{r^2}\right), \end{aligned} \quad (3.26)$$

we see that only the first two terms in the integral (3.25) survive the asymptotic limit and yield contributions which are both proportional to M_φ , the second term contributing twice the first to the asymptotic limit. The first contribution in particular can be expressed in terms of ψ , J_{\bullet} , so that we can write:

$$\begin{aligned} M_\varphi &= -\frac{3}{8\pi} \int_{S_2^\infty} \psi_{[i} J_{\bullet j]} dx^i \wedge dx^j = -\frac{3}{4\pi} \int_{S_2^\infty} \psi_{[\theta} J_{\bullet\varphi]} d\theta d\varphi = \\ &= \frac{3}{8\pi} \int_{S_2^\infty} \psi_\varphi J_{\bullet\theta} d\theta d\varphi, \end{aligned} \quad (3.27)$$

where $\psi_\varphi = g_{\varphi\varphi}^{(3)}$.

3.1.1 $G_{(3)}$ -invariant characterization of the angular momentum

Let us define a new constant \mathfrak{g} -matrix as follows:

$$Q_\psi = -\frac{3}{8\pi} \int_{S_2^\infty} \psi_{[i} J_{j]} dx^i \wedge dx^j = \frac{3}{8\pi} \int_{S_2^\infty} \psi_\varphi J_\theta d\theta d\varphi \in \mathfrak{g}. \quad (3.28)$$

In the asymptotic limit $r \rightarrow \infty$ the components of J_m have the following behavior:

$$J_r = \frac{Q}{r^2} + O\left(\frac{1}{r^3}\right); \quad J_\theta = \frac{Q_\psi}{r^2} \sin\theta + O\left(\frac{1}{r^3}\right). \quad (3.29)$$

According to the general formula (3.27), the angular momentum can be written as:

$$M_\varphi = k \text{Tr}(T_\bullet^\dagger Q_\psi). \quad (3.30)$$

As pointed out earlier, $G_{(3)}$ is the global symmetry group of the three-dimensional effective theory. As an isometry group, its elements have a non-linear action on the coordinates:

$$g \in G_{(3)} : z^a \longrightarrow z_g^a = z_g^a(z), \quad (3.31)$$

where $z_g^a(z)$ are non-linear functions of the z^a , depending on the parameters of the transformation g . The same transformation, being a global symmetry, maps a solution $\xi^a(x)$ into another one of the same theory $\xi_g^a(x)$. The asymptotic limit $r \rightarrow \infty$, for the scalar fields, defines a single point $\xi_0 = (\xi_0^a)$ on the scalar manifold:

$$\lim_{r \rightarrow \infty} \xi^a(x) = \xi_0^a. \quad (3.32)$$

Since the action of $G_{(3)}$ on the scalar manifold is transitive, we can always map the point at infinity to the origin $O(\xi_0^a \equiv 0)$. Once we fix $\xi_0 = O$, we can only act on the solutions by means of the stability group H^* of the origin.

From the definition (3.15) we deduce the transformation property of the matrix $\mathcal{M}(z)$ under an isometry g :

$$\mathcal{M}(z) \longrightarrow \mathcal{M}(z_g) = g \mathcal{M}(z) g^\dagger, \quad (3.33)$$

where, with an abuse of notation, we have used the same symbol g to denote the matrix form of g in the representation of \mathcal{M} . The \mathfrak{g} -valued current $J_m = J_m(\xi(x))$ therefore transforms under an isometry g by conjugation:

$$J_m(\xi) \longrightarrow J_m(\xi_g) = (g^\dagger)^{-1} J_m(\xi) g^\dagger, \quad (3.34)$$

and so do the \mathfrak{g} -valued constant matrices Q and Q_ψ :

$$Q(\xi) \longrightarrow Q(\xi_g) = (g^\dagger)^{-1} Q(\xi) g^\dagger; \quad Q_\psi(\xi) \longrightarrow Q_\psi(\xi_g) = (g^\dagger)^{-1} Q_\psi(\xi) g^\dagger. \quad (3.35)$$

Generic axisymmetric stationary solutions are distinguished from the static ones by the following $G_{(3)}$ -invariant property:

$$\text{axisymmetric solutions} \quad \Rightarrow \quad Q_\psi \neq 0. \quad (3.36)$$

In particular for solutions in the same $G_{(3)}$ -orbit as the KN-Taub-NUT one, $\text{Tr}(Q_\psi^2) \neq 0$. In the universal model originating from Einstein-Maxwell supergravity in four dimensions, see section 4, $G_{(3)} = \text{SU}(1, 2)$, and we can evaluate on the KN-Taub-NUT solutions Q and Q_ψ explicitly. Using the covariant expression for the matrix \mathcal{M} in terms of U, V, W , given in appendix A and eq.s (4.21) introduced in section 4 we find:

$$\begin{aligned}
 Q &= \begin{pmatrix} 0 & 0 & (m - i\ell) \\ 0 & 0 & -\frac{q+ip}{\sqrt{2}} \\ (m + i\ell) & \frac{q-ip}{\sqrt{2}} & 0 \end{pmatrix}, \\
 Q_\psi &= \alpha \begin{pmatrix} 0 & 0 & (\ell + im) \\ 0 & 0 & -i(q + ip)/\sqrt{2} \\ (\ell - im) & -i(q - ip)/\sqrt{2} & 0 \end{pmatrix}.
 \end{aligned} \tag{3.37}$$

Then:

$$\text{Tr}(Q^2) = \frac{2}{k} \left(m^2 + \ell^2 - \frac{p^2 + q^2}{2} \right), \quad \text{Tr}(Q_\psi^2) = \frac{2\alpha^2}{k} \left(m^2 + \ell^2 - \frac{p^2 + q^2}{2} \right), \tag{3.38}$$

where $\alpha \equiv M_\varphi/m$ and $k = 1$ in the fundamental representation of $\text{SU}(1, 2)$, so that

$$\left(\frac{M_\varphi}{m} \right)^2 = \alpha^2 = \frac{\text{Tr}(Q_\psi^2)}{\text{Tr}(Q^2)}. \tag{3.39}$$

We wish to stress here that the above formula, although derived in the universal model, holds in all supergravity theories admitting the KN-Taub-NUT solution. This is a $G_{(3)}$ -invariant characterization of the angular momentum, which holds for all solutions in the same $G_{(3)}$ -orbit as the KN-Taub-NUT one. Using this result, we can write the extremality parameter in a $G_{(3)}$ -invariant fashion:

$$c^2 = m^2 + \ell^2 - \frac{p^2 + q^2}{2} - \alpha^2 = \frac{k}{2} \text{Tr}(Q^2) - \frac{\text{Tr}(Q_\psi^2)}{\text{Tr}(Q^2)}, \tag{3.40}$$

so that the extremality condition becomes:

$$c^2 = 0 \Leftrightarrow \text{Tr}(Q^2) = \frac{2}{k} \frac{\text{Tr}(Q_\psi^2)}{\text{Tr}(Q^2)}, \tag{3.41}$$

from which it is apparent that, as opposed to the static case, extremality does not imply nilpotency of Q , as noted in [36]. Eq. (3.41) provides a $G_{(3)}$ -invariant characterization of extremality. There is a class of extremal rotating solutions for which both sides of this equation vanish separately. These are the ‘‘ergo-free’’ (under-rotating) solutions constructed in [42–44] and further generalized in [20] within cubic supergravity models. Below we shall comment on some general $G_{(3)}$ -invariant properties of these solutions in terms of the matrices Q and Q_ψ .

3.1.2 Under-rotating solutions.

In [42–44] under-rotating solutions were constructed within the Kaluza-Klein theory originating from pure gravity in $D = 5$, as a limit of a dilatonic rotating black hole. In order to perform a similar limit in the context of supergravity, we need to consider a model which is larger than the universal one, but which contains it as a consistent truncation. The simplest choice is the $\mathcal{N} = 2$ t^3 -model in four dimensions, which consists of supergravity coupled to one vector multiplet, whose complex scalar field t parametrizes a special Kähler manifold with prepotential $\mathcal{F}(t) = t^3$. Upon time-like reduction to $D = 3$ we end up with an Euclidean sigma-model with target space $G_{2(2)}/[SL(2) \times SL(2)]$ and global symmetry group $G_{(3)} = G_{2(2)}$. Extremal solutions to this model were studied in [29, 36, 50].

We shall not enter into the mathematical details of the model but limit ourselves to illustrate the procedure for generating an extremal under-rotating solution from a non-extremal rotating one. The scalar fields originating from the $D = 4$ vector fields are four $(\mathcal{Z}^M) = (\mathcal{Z}^0, \mathcal{Z}^1, \mathcal{Z}_0, \mathcal{Z}_1)$, parametrizing the solvable generators $(T_M) = (T_0, T_1, T^0, T^1)$. Adopting a suitable representation of $G_{2(2)}$ for the generators (for example the fundamental real $\mathbf{7}$ representation), we can consider two commuting generators of Harrison transformations:

$$K_0 \equiv \frac{1}{2} (T_0 + T_0^\dagger); \quad K_1 \equiv \frac{1}{2} (T^1 + T^{1\dagger}), \quad (3.42)$$

and “boost” the Kerr solution with parameters m, α using the Harrison transformation:

$$\mathcal{O} \equiv e^{\log(\beta_1 m) K_0 + \log(\beta_2 m) K_1}, \quad (3.43)$$

The resulting solution is a non-extremal axion-dilaton rotating black hole with ADM-mass, electric-magnetic and scalar charges and angular momentum depending on the Kerr parameters m, α and encoded in the $\mathfrak{g}_{2(2)}$ -valued matrices:

$$Q = \mathcal{O}^{-1} Q^{(K)} \mathcal{O}; \quad Q_\psi = \mathcal{O}^{-1} Q_\psi^{(K)} \mathcal{O}, \quad (3.44)$$

$Q^{(K)}$ and $Q_\psi^{(K)}$ being the matrices corresponding to the original Kerr solution. We shall give the complete solution elsewhere, focussing here only on the characteristic quantities at radial infinity. Redefining $\alpha = \Omega m = M_\varphi/m$, these quantities read:

$$\begin{aligned} M_{ADM} &= \frac{1}{8} \left(m^2(\beta_1 + 3\beta_2) + \frac{1}{\beta_1} + \frac{3}{\beta_2} \right); \quad p^1 = \sqrt{3} \frac{m^2 \beta_2^2 - 1}{2\sqrt{2}\beta_2}; \quad q_0 = -\frac{m^2 \beta_1^2 - 1}{2\sqrt{2}\beta_1}, \\ \Sigma &= i \frac{\sqrt{3}(-m^2 \beta_2 \beta_1^2 + m^2 \beta_2^2 \beta_1 + \beta_1 - \beta_2)}{8\beta_1 \beta_2}; \quad M_\varphi = \frac{(\beta_1 \beta_2^3 m^4 + 3\beta_2(\beta_1 + \beta_2)m^2 + 1)\Omega}{8\sqrt{\beta_1 \beta_2^3/2}}, \end{aligned} \quad (3.45)$$

while $p^0 = q_1 = \ell = 0$. Taking the $m \rightarrow 0$ limit while keeping β_1, β_2 and Ω fixed, the above quantities remain finite:

$$\begin{aligned} M_{ADM} &= \frac{1}{8} \left(\frac{1}{\beta_1} + \frac{3}{\beta_2} \right); \quad p^1 = -\frac{\sqrt{3}}{2\sqrt{2}\beta_2}; \quad q_0 = \frac{1}{2\sqrt{2}\beta_1}; \quad \Sigma = i \frac{\sqrt{3}(\beta_1 - \beta_2)}{8\beta_1 \beta_2}; \\ M_\varphi &= \frac{\Omega}{8\sqrt{\beta_1 \beta_2^3/2}}. \end{aligned} \quad (3.46)$$

Inspection of the full solution shows that, as $m \rightarrow 0$, the ergo-sphere disappears and the three dimensional spatial part of the metric becomes conformally flat.

This limit corresponds to taking a singular Harrison transformation \mathcal{O} ($\log(\beta_1 m), \log(\beta_2 m) \rightarrow -\infty$) and at the same time a singular limit of the Kerr parameters ($m, \alpha \rightarrow 0$). As a result the matrices Q, Q_ψ remain finite but become *nilpotent*. In particular Q is a step-3 nilpotent matrix while Q_ψ is step 2. The fact that Q_ψ has a *lower* degree of nilpotency than Q is consistent with the fact that:

$$\lim_{m \rightarrow 0} \text{Tr}(Q^2) = 0; \quad \lim_{m \rightarrow 0} \frac{\text{Tr}(Q_\psi^2)}{\text{Tr}(Q^2)} = 0, \quad (3.47)$$

and the extremality condition (3.41) is satisfied. This is consistent with the classification of extremal solutions of [34, 36] in terms of suitable nilpotent subalgebras \mathfrak{N} of \mathfrak{g} . In this case the matrices Q and Q_ψ would correspond to characteristic generators of \mathfrak{N} .

3.2 A duality invariant expression for the DWHJ vector S_m

Let us now apply the construction of section 2 to our specific effective Lagrangian (3.16). The direct application of eq. (2.12) to our specific geodesic model is possible but lacks the property of being manifestly invariant under the isometry group $G_{(3)}$. However, the use of the $G_{(3)}$ -valued matrix \mathcal{M} introduced in (3.15) makes it possible to write an alternative expression for S^m which does exhibit manifest duality invariance (provided we transform both the off-shell fields z^a and their on-shell expression on a given background $\xi^a(x)$). The expression is the following:

$$S^m = -\frac{k}{4} \hat{\rho} \sqrt{h} \text{Tr} [\mathcal{M}^{-1}(z) \partial^m \mathcal{M}(\xi)] + \delta_r^m \int^r dr' \mathcal{L}(\xi(x'), \partial_m \xi, x'). \quad (3.48)$$

Indeed, from (3.48) we find:

$$\frac{\partial S^m}{\partial z^a} = \frac{k}{4} \hat{\rho} \sqrt{h} \text{Tr} \left[\mathcal{M}^{-1}(z) \frac{\partial \mathcal{M}}{\partial z^a} \mathcal{M}^{-1}(z) \partial^m \mathcal{M}(\xi) \right], \quad (3.49)$$

so that, for a weakly embedded solution $z = \xi$, we reproduce the on-shell expression of the conjugate momentum (3.17). Correspondingly we also find, using the field equations:

$$\partial_m S^m|_{z=\xi} = \left(\mathcal{L} - \frac{k}{4} \hat{\rho} \sqrt{h} \text{Tr} [\mathcal{M}^{-1}(z) \partial_m \mathcal{M}(\xi) \mathcal{M}^{-1}(\xi) \partial^m \mathcal{M}(\xi)] \right)_{z=\xi} = -\mathcal{H}|_{z=\xi}. \quad (3.50)$$

One may ask what the relation between the solution (3.48) and the general relation (2.12) is. The answer can be found by realizing that a Taylor-expansion of S^m given in (3.48) in powers of $z - \xi$, taking into account (3.11) and (3.15), exactly reproduces (2.12). It is important to stress that S_m , as defined above, is $G_{(3)}$ -invariant provided we simultaneously transform z^a and $\xi^a(x)$ in its expression, as it follows from the transformation property (3.33) of the matrix \mathcal{M} :

$$g \in G_{(3)} : S_m(z, \xi) \longrightarrow S_m(z_g, \xi_g) = S_m(z, \xi), \quad (3.51)$$

An important property of the DWHJ construction is that one can compute the conserved currents of the theory by varying S^m with respect to the parameters which it depends on [37]. In particular, we can reproduce the conserved Nöther currents $\hat{\rho}J_m$ of (3.19) by performing an infinitesimal isometry transformation on S^m , at fixed background $\xi^\alpha(x)$, and then by varying S^m with the corresponding symmetry parameters. If we set:

$$g = \mathbf{1} + \epsilon^\alpha \mathbf{T}_\alpha \tag{3.52}$$

the isometry transformed matrix is

$$\mathcal{M}(z_g) = g \cdot \mathcal{M}(z) \cdot g^\dagger \simeq \mathbf{1} + \epsilon^\alpha \left(\mathbf{T}_\alpha \cdot \mathcal{M} + \mathcal{M} \cdot \mathbf{T}_\alpha^\dagger \right). \tag{3.53}$$

On the g -transformed S^m we get:

$$\begin{aligned} \left. \frac{\partial S^m(z_g)}{\partial \epsilon^\alpha} \right|_{z=\xi} &= -\frac{k}{4} \hat{\rho} \sqrt{h} \left[(\mathcal{M}^{-1}(z) \partial^m \mathcal{M}(\xi))_i^j (T_\alpha)_j^i + \right. \\ &\quad \left. + (\mathcal{M}^{-1}(z) \partial^m \mathcal{M}(\xi))^j_i (T_\alpha)^i_j \right] = -2 \hat{\rho} \sqrt{h} \text{Tr}[T_\alpha^\dagger \cdot J^m]. \end{aligned} \tag{3.54}$$

4 Application to Einstein-Maxwell axisymmetric solutions

In the absence of four dimensional scalar fields ($\partial_i \phi = 0$, $\mathcal{M}_{(4)} \rightarrow -\mathbb{I}$), the geodesic part of the Lagrangian (3.2) reduces to

$$\begin{aligned} \frac{1}{\sqrt{g_{(3)}}} \mathcal{L}_{(3)} &= \partial_i \mathcal{U} \partial^i \mathcal{U} - \frac{1}{2} \epsilon^{-2\mathcal{U}} \partial_i \mathbf{Z}^T \partial^i \mathbf{Z} + \frac{1}{4} \epsilon^{-4\mathcal{U}} (\partial_i a + \mathbf{Z}^T \mathbb{C} \partial_i \mathbf{Z}) (\partial^i a + \mathbf{Z}^T \mathbb{C} \partial^i \mathbf{Z}) \\ &= \frac{1}{2} G_{ab}(z) \partial_i z^a \partial^i z^b. \end{aligned} \tag{4.1}$$

where $G_{ab}(z)$ is now the metric of the manifold:

$$\frac{\text{SU}(1,2)}{\text{U}(1) \times \text{SU}(1,1)}, \tag{4.2}$$

which is a pseudo-Kähler manifold, that is a non compact version of the Kähler manifold $CP(2)$.

As it is well known in General Relativity, a very simple and useful way to describe such theory is the use of the so-called Ernst potentials \mathcal{E} , Ψ [38, 39] defined as:

$$\mathcal{E} = e^{2\mathcal{U}} - |\Psi|^2 + i a; \quad \Psi = \frac{1}{\sqrt{2}} (\mathcal{Z}^0 + i \mathcal{Z}_0), \tag{4.3}$$

In terms of the Ernst potentials the metric (3.14) reads:

$$dS_{(3)}^2 = \frac{e^{-4\mathcal{U}}}{2} |d\mathcal{E} + 2 \bar{\Psi} d\Psi|^2 - 2 e^{-2\mathcal{U}} |d\Psi|^2. \tag{4.4}$$

The group $\text{SU}(1,2)$ acts non-linearly on the potentials \mathcal{E} , Ψ . However, one can introduce homogeneous complex coordinate fields (W, V, U) transforming in the $\mathbf{3}$ of $\text{SU}(1,2)$, in terms of which the Ernst potentials can be written as follows:

$$\mathcal{E} = \frac{U - W}{U + W}; \quad \Psi = \frac{V}{U + W} \tag{4.5}$$

Going to inhomogeneous variables $u = U/W$, $v = V/W$, they take the form

$$\mathcal{E} = \frac{u-1}{u+1}; \quad \Psi = \frac{v}{u+1}. \quad (4.6)$$

The scalar manifold $\frac{SU(1,2)}{U(1) \times SU(1,1)}$ can then be described in terms of the complex fields $z^a = (u, v)$ (where $a = 1, 2$).

We notice that the manifold (4.2) is a non-compact version of the minimal model $\frac{SU(1,2)}{U(1) \times SU(2)}$, which describes a particular case of a symmetric space of $N = 2$ special geometry in four dimensional supergravity. Accordingly, we can say that the variables (u, v) are “special coordinates” in terms of which the upper components of the corresponding holomorphic symplectic section (X^Λ, F_Λ) read:

$$X^\Lambda = \begin{pmatrix} W \\ V \\ U \end{pmatrix} = W \begin{pmatrix} 1 \\ v \\ u \end{pmatrix}, \quad (4.7)$$

while the lower components F_Λ are given in terms of the holomorphic homogeneous degree two prepotential $F(X^\Lambda)$, as $F_\Lambda = \frac{\partial F}{\partial X^\Lambda}$. The holomorphic prepotential in terms of the inhomogeneous coordinates reads:

$$\mathcal{F} = \frac{1}{W^2} F(X^\Lambda) = \frac{i}{4} (1 - u^2 - v^2), \quad (4.8)$$

and the Kähler potential \mathcal{K} has the following form:

$$\mathcal{K} = -\log [i (2(\mathcal{F} - \bar{\mathcal{F}}) - (z^a - \bar{z}^a)(\mathcal{F}_a + \bar{\mathcal{F}}_a))] = -\log [|u|^2 + |v|^2 - 1]. \quad (4.9)$$

The coordinate patch u, v is defined by the condition:

$$|u|^2 + |v|^2 > 1. \quad (4.10)$$

whose physical meaning will be given in the next subsection.

The σ -model metric in the special coordinates has the form:

$$dS_{(3)}^2 = 2 G_{a\bar{b}} dz^a d\bar{z}^b; \quad (4.11)$$

$$G_{a\bar{b}} = \partial_a \partial_{\bar{b}} \mathcal{K} = e^{2\mathcal{K}} \begin{pmatrix} (1 - |v|^2) & \bar{u}v \\ \bar{v}u & (1 - |u|^2) \end{pmatrix} = e^{2\mathcal{K}} (\delta_{a\bar{b}} - z_a \bar{z}_{\bar{b}}), \quad (4.12)$$

$$G^{\bar{a}b} = -e^{-\mathcal{K}} (\delta^{\bar{a}b} - \bar{z}^{\bar{a}} z^b).$$

where $z_a \equiv \epsilon_{ab} z^b$. The eigenvalues of $g_{a\bar{b}}$ are: $-1/(|u|^2 + |v|^2 - 1)$, $1/(|u|^2 + |v|^2 - 1)^2$ and, if $|u|^2 + |v|^2 > 1$, $g_{a\bar{b}}$ has the correct signature $(-, -, +, +)$.

4.1 Relation to known black-hole solutions

For stationary, axisymmetric, asymptotically flat solutions admitting the two Killing vectors ∂_t and ∂_φ , the most general case of complex scalar fields u, v corresponds to a Kerr-Newman solution with NUT-charge, whose metric reads [39]:

$$ds^2 = \frac{\tilde{\Delta}}{|\rho|^2} (dt + \omega)^2 - \frac{|\rho|^2}{\tilde{\Delta}} \left(\frac{\tilde{\Delta}}{\Delta} dr^2 + \tilde{\Delta} d\theta^2 + \Delta \sin^2 \theta d\varphi^2 \right) \quad (4.13)$$

where

$$\Delta = (r - m)^2 - c^2, \tag{4.14}$$

$$\tilde{\Delta} = \Delta - \alpha^2 \sin^2 \theta, \tag{4.15}$$

$$\rho = r + i(\alpha \cos \theta + \ell), \tag{4.16}$$

$$\omega = \left(\alpha \sin^2 \theta \frac{|\rho|^2 - \tilde{\Delta}}{\tilde{\Delta}} + 2\ell \cos(\theta) \right) d\varphi, \tag{4.17}$$

$$A^0 = [-qr + p(\ell + \alpha \cos \theta)] \frac{dt}{|\rho|^2} + \{ -p[(\alpha^2 + r^2 - \ell^2) \cos \theta + \alpha \ell \sin^2 \theta] + q[\alpha r \sin^2 \theta - 2\ell r \cos \theta] \} \frac{d\varphi}{|\rho|^2},$$

where $c^2 = m^2 + \ell^2 - \frac{1}{2}(q^2 + p^2) - \alpha^2$ as given in (3.40), in terms of the Boyer-Lindquist coordinates (r, θ) , of the electric and magnetic charges (q, p) and of the ADM-mass and NUT charge (m, ℓ) . The parameter α , as before, is related to the angular momentum M_φ of the solution by $\alpha = M_\varphi/m$. Here the metric field $U(r, \theta)$ is given by $e^{2U} = \frac{\tilde{\Delta}}{|\rho|^2}$. For this solution the fields λ , $\hat{\rho}$ and the flat 2D metric h_{mn} read:

$$\lambda^2 = \tilde{\Delta}; \quad \hat{\rho} = \sqrt{\tilde{\Delta}} \sin \theta; \quad h_{mn} \begin{pmatrix} 1/\tilde{\Delta} & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.18}$$

so that $\sqrt{h} \hat{\rho} = \sin(\theta)$. The latter expression holds, in suitable coordinates, for all axisymmetric solutions. The Ernst potentials are then:

$$\mathcal{E} = \frac{r - 2m + i(\alpha \cos \theta - \ell)}{r + i(\alpha \cos \theta + \ell)} \tag{4.19}$$

$$\Psi = \frac{-q + ip}{\sqrt{2}[r + i(\alpha \cos \theta + \ell)]}. \tag{4.20}$$

and the corresponding homogeneous coordinates can be chosen as:

$$\begin{aligned} U &= r - m + i\alpha \cos \theta \\ V &= \frac{1}{\sqrt{2}}(-q + ip) \\ W &= m + i\ell \end{aligned} \tag{4.21}$$

Let us observe that only an $SU(1, 1)$ subset of the $SU(1, 2)$ invariance is realized on the four dimensional fields, under which the ‘‘charges’’ (W, V) form a doublet while U is a singlet. The KN solution is retrieved by setting $\ell = 0$ in eq.s (4.19), (4.20), the RN electric-magnetic solution by further setting $\alpha = 0$ and finally the Schwarzschild solution is obtained from RN when $q = p = 0$.

Let us relate the explicit expressions for the Ernst potentials here with the σ -model description given above. The metric function $\tilde{\Delta}$ in (4.15) appears to be related to the $SU(1, 2)$ -invariant Kähler potential \mathcal{K} in (4.9):

$$\tilde{\Delta} = |U|^2 + |V|^2 - |W|^2 = |W|^2 e^{-\mathcal{K}} \tag{4.22}$$

According to the identification (4.21) the condition (4.10) acquires a precise physical meaning. In the static solutions ($\alpha = 0$) condition (4.10) is guaranteed as long as $r > r_+$, r_+ being the outer horizon

$$r_+ = m + \sqrt{m^2 + \ell^2 - \frac{p^2 + q^2}{2}}. \tag{4.23}$$

On the other hand, in the KN case ($\ell = 0$) it gives

$$r > m + \sqrt{m^2 - \frac{q^2 + p^2}{2} - \alpha^2 \cos^2 \theta} \equiv r_e \tag{4.24}$$

where $r_e > r_+$ defines the external boundary of the ergosphere, where the component g_{00} of the metric vanishes, while $r_+ = m + \sqrt{m^2 - \frac{q^2 + p^2}{2} - \alpha^2}$ is the radius of the outer event horizon. Then we see that the special-coordinate patch described by u, v is only valid outside the ergosphere.

If we cross the ergosphere surface $\tilde{\Delta} = 0$ we are bound to change the coordinate patch. The new patch can be described by the CP(2) riemannian space $SU(1, 2)/U(2)$, with Kaehler potential $\mathcal{K} = -\log(1 - |u|^2 - |v|^2)$.

The universal model considered here, and the KN-Taub-NUT solution thereof, can be embedded in more general supergravity models (for instance in all $\mathcal{N} = 2$ symmetric supergravity models, dimensionally reduced to $D = 3$) and thus it is interesting to consider the $G_{(3)}$ -invariant properties of this solution. In light of the discussion at the end of Sect 3, the description of such properties should take into account, aside from the Nöther charge matrix Q , also the constant matrix Q_ψ .

4.2 The DWHJ principal 1-form for the KN solution

Let us explicitly compute here the DWHJ principal functions S^r, S^θ for the KN solution.

We have:⁵

$$\partial_a S^m = \pi_a^m = \sin \theta G_{a\bar{b}}(z) h^{mn} \partial_n \bar{z}^{\bar{b}} \tag{4.26}$$

that is:

$$\pi_a^r = \sin \theta G_{a\bar{b}}(z) \Delta \partial_r \bar{z}^{\bar{b}} \tag{4.27}$$

$$\pi_a^\theta = \sin \theta G_{a\bar{b}}(z) \bar{\partial}_\theta \bar{z}^{\bar{b}}. \tag{4.28}$$

Eq. (4.26), recalling (2.12), admits the (weakly embedded) solution:

$$S^m = 2\Re [(z^a - \xi^a(x)) \pi_a^m(x)] + \delta_r^m \int^r d\hat{r} \mathcal{L}(\xi, \partial\xi, \hat{x}) \tag{4.29}$$

⁵We recall, from section 4.1, that the two-dimensional metric is

$$h_{mn} = \begin{pmatrix} 1/\Delta & 0 \\ 0 & 1 \end{pmatrix} \tag{4.25}$$

Using (4.12), if we denote by ξ^u, ξ^v the on-shell values of the fields u, v :

$$\begin{aligned}\xi^u &= \frac{r - m + i\alpha \cos \theta}{m + i\ell} \\ \xi^v &= \frac{-q + ip}{\sqrt{2}(m + i\ell)}\end{aligned}\tag{4.30}$$

we find

$$\begin{aligned}S^r(z, x) &= 2 \sin \theta (m^2 + \ell^2)^2 \frac{\Delta(x)}{\tilde{\Delta}^2(x)} \Re [(u - \xi^u)(1 - |\xi^v|^2) + (v - \xi^v)\xi^u \bar{\xi}^v] + \\ &\quad + \int^r d\hat{r} \mathcal{L}(\xi, \partial\xi, \hat{x})\end{aligned}\tag{4.31}$$

$$S^\theta(z, x) = -2 \alpha \sin^2 \theta \frac{(m^2 + \ell^2)^2}{\tilde{\Delta}^2(x)} \Im [(u - \xi^u)(1 - |\xi^v|^2) + (v - \xi^v)\xi^u \bar{\xi}^v]\tag{4.32}$$

5 Conclusions

In this paper we have addressed the issue of the first order description of generic (not necessarily extremal) axisymmetric solutions. This was done by working out the general form of the principal functions S_m associated with the corresponding effective 2D sigma-model in the DWHJ setting. We have also given a characterization of the general properties of such solutions with respect to the global symmetry group of the effective 2D sigma-model which describes them. This was done by introducing, aside from the Nöther charge matrix, a further characteristic constant matrix Q_ψ , in the Lie algebra of $G_{(3)}$, associated with the rotational motion of the black hole.

As a direction for further investigation it would be interesting to generalize this analysis to more general stationary solutions, including (non necessarily extremal) multicenter black holes. In this respect, as emphasized earlier, there is virtually no conceptual obstruction in generalizing the DWHJ construction and the general formula for S_m , which we have mainly used here within a 2D effective sigma-model, to the full 3D effective description of stationary solutions. It would moreover be interesting to analyze the axisymmetric solutions to symmetric supergravities from the point of view of the *integrability* of the corresponding effective 2D sigma-model, which we have not exploited here. This latter property being related to the presence in a gravity/supergravity theory, once dimensionally reduced to $D = 2$, of an infinite dimensional global symmetry group, generalizing the Geroch group of pure Einstein gravity (see for instance [51, 52]).

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A The $\mathfrak{su}(1, 2)$ -Algebra

Let us choose the $SU(1, 2)$ -invariant and the $H^* = U(1, 1)$ -invariant metrics η and $\bar{\eta}$, respectively, to be:

$$\eta = \text{diag}(-1, 1, 1); \quad \bar{\eta} = \text{diag}(-1, 1, -1), \quad (\text{A.1})$$

where the latter defines the coset generators. The solvable Lie algebra $Solv$ defining the Iwasawa decomposition of $\mathfrak{su}(1, 2)$ with respect to $\mathfrak{u}(2)$ is generated by:

$$\begin{aligned} Solv &= \text{span}(H_0, T_1, T_2, T_\bullet), \\ H_0 &= \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}; \quad T_1 = \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}; \quad T_2 = \begin{pmatrix} 0 & -\frac{i}{2} & 0 \\ \frac{i}{2} & 0 & -\frac{i}{2} \\ 0 & -\frac{i}{2} & 0 \end{pmatrix}, \\ T_\bullet &= \begin{pmatrix} -\frac{i}{2} & 0 & \frac{i}{2} \\ 0 & 0 & 0 \\ -\frac{i}{2} & 0 & \frac{i}{2} \end{pmatrix}. \end{aligned} \quad (\text{A.2})$$

The H^* algebra $\mathfrak{u}(1, 1)$ is generated by the compact component K_\bullet of T_\bullet , the non-compact components K_1, K_2 of T_1, T_2 , respectively, and the compact $D = 4$ duality generator K :

$$\begin{aligned} \mathfrak{u}(1, 1) &= \text{span}(K_1, K_2, K_\bullet, K), \\ K_\bullet &= T_\bullet - T_\bullet^\dagger = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}; \quad K_1 = T_1 + T_1^\dagger = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ K_2 &= T_2 + T_2^\dagger = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad K = \begin{pmatrix} -i & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -i \end{pmatrix}. \end{aligned} \quad (\text{A.3})$$

The $SU(1, 2)/U(1, 1)$ -coset representative describing the physical patch of the manifold is:

$$\mathbb{L} = e^{-aT_\bullet} e^{\sqrt{2}(Z^0 T_1 + Z_0 T_2)} e^{2UH_0}. \quad (\text{A.4})$$

The matrix $\mathcal{M} = \mathbb{L}\bar{\eta}\mathbb{L}^\dagger$ has the following simple form:

$$\mathcal{M} = \mathbb{L}\bar{\eta}\mathbb{L}^\dagger = \eta - \frac{2}{I_2} \eta \bar{\mathbb{U}} \mathbb{U}^T \eta, \quad (\text{A.5})$$

where

$$\mathbb{U} \equiv \begin{pmatrix} W \\ V \\ U \end{pmatrix}, \quad I_2 \equiv \mathbb{U}^T \eta \bar{\mathbb{U}} = |U|^2 + |V|^2 - |W|^2. \quad (\text{A.6})$$

B KN solution from Schwarzschild

In this appendix we give an alternative way to generate the Hamilton principal 1-form $S^{(1)}$ corresponding to the KN solution. It makes use of duality symmetry and general coordinate transformations starting from the Schwarzschild solution.

We will proceed in two steps. We first need an explicitly $SU(1,2)$ -duality invariant expression for the \mathcal{W}_3 of the RN solution in 3D. This can be achieved by using the generating technique of $SU(1,2)$ to generate solutions in 3D. In particular, starting from Schwarzschild field variables

$$\begin{aligned} U &= r - m \\ V &= 0 \\ W &= m, \end{aligned} \tag{B.1}$$

the action of the $SU(1,2)$ Harrison and Ehlers transformations generate electric, magnetic and in general also a NUT charge, thus leading to a RN-NUT solution. Next, as a second step we use a procedure first introduced by Clément [45, 46] allowing the generation of a KN solution from RN by an appropriate sequence of $SU(1,2)$ and coordinate transformations.

B.1 \mathcal{W}_3 for the RN-NUT solution

Let us recall that in the static case the prepotential \mathcal{W}_3 provides a first order description of $D = 3$ static solutions [10]:

$$\frac{d\bar{z}^{\bar{a}}}{d\tau} = g^{\bar{a}b} \partial_b \mathcal{W}_3 \tag{B.2}$$

satisfying the HJ equation

$$\partial_{\bar{a}} \mathcal{W}_3 g^{\bar{a}b} \partial_b \mathcal{W}_3 = c^2 \tag{B.3}$$

c being the extremality parameter.

Quite generally a static solution is completely defined by a point P of the scalar manifold representing the values of the scalars at radial infinity $\tau = 0$, and the tangent vector to the geodesic, which is an object transforming under H^* . Here H^* is the isotropy group of the coset G/H^* , G being the 3D isometry group. Since the action of G/H^* on P is transitive over the scalar manifold, we can always fix P to be the origin O at which all fields vanish, and study the geodesic solutions corresponding to various choices of the velocity vector at infinity. In this way we break G to the little group H^* of the origin and we expect the \mathcal{W}_3 describing the family of solutions with $P = O$ to be an H^* -invariant function.

In our case we have $G/H^* = \frac{SU(1,2)}{U(1) \times SU(1,1)}$ and we shall prove that the RN-NUT solutions are described by a solution to the HJ equation of the form:

$$\mathcal{W}_3 = -c \log \left(\frac{|U| + \sqrt{|W|^2 - |V|^2}}{|U| - \sqrt{|W|^2 - |V|^2}} \right) = -c \log \left(\frac{|u| + \sqrt{1 - |v|^2}}{|u| - \sqrt{1 - |v|^2}} \right). \tag{B.4}$$

The above function is clearly $H^* = U(1,1)$ -invariant since both $|U|$ and $|W|^2 - |V|^2$ are.

Let us recover the expression (B.4) for the \mathcal{W}_3 describing the most general static (non-extremal) black hole in our model, from the one-parameter $\mathcal{W}_3^{(S)}$ of the Schwarzschild solution by a *duality (isometric) continuation* of it on the whole σ -model. By duality continuation we mean *defining* the value of \mathcal{W}_3 out of the one-dimensional submanifold

on which $\mathcal{W}_3^{(S)}$ is defined by means of an isometry transformation on the σ -model. Of course here we are restricting to H^* transformations only and the resulting prepotential will be, by construction, H^* -invariant and still a solution to (B.3) being the latter duality invariant.

The geodesic corresponding to the Schwarzschild black hole is defined by the following prepotential:

$$\mathcal{W}_3^{(S)}(s) = -c \log \left(\frac{s+1}{s-1} \right), \quad (\text{B.5})$$

defined on the submanifold:

$$u = \bar{u} = s; \quad v = 0. \quad (\text{B.6})$$

It is straightforward to check that $\mathcal{W}_3^{(S)}(s)$ satisfies the HJ equation:

$$\partial_s \mathcal{W}_3^{(S)} \frac{\partial s}{\partial \bar{z}^{\bar{a}}} g^{\bar{a}b} \frac{\partial s}{\partial z^b} \partial_s \mathcal{W}_3^{(S)} = \frac{(s^2-1)^2}{4} \left(\partial_s \mathcal{W}_3^{(S)} \right)^2 = c^2, \quad (\text{B.7})$$

where we have written $s = (u + \bar{u})/2$ and $z^a = (u, v)$. Next we apply to the Schwarzschild fields a generic H^* -transformation h^* . The latter can be written as the product of a Harrison transformation, a Ehlers $U(1)_E$ -transformation and a second $U(1)$ -transformation (which corresponds to the $D = 4$ duality group). Referring to the notations of appendix A we have:

$$\begin{aligned} h^* &= H_{\text{arrison}} \cdot h_E \cdot h \\ H_{\text{arrison}} &= e^{a_1 K_1 + a_2 K_2} = \begin{pmatrix} \cosh(\mathbf{a}) & -e^{i\sigma} \sinh(\mathbf{a}) & 0 \\ -e^{-i\sigma} \sinh(\mathbf{a}) & \cosh(\mathbf{a}) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ h_E &= e^{\alpha K_\bullet} = \text{diag}(e^{-i\alpha}, 1, e^{i\alpha}); \quad h = e^{\beta K} = \text{diag}(e^{-i\beta}, e^{2i\beta}, e^{-i\beta}), \end{aligned} \quad (\text{B.8})$$

where we have written $a_1 + i a_2 = \mathbf{a} e^{i\sigma}$. If we apply h^* to the Schwarzschild fields described by $(W(s), V(s), U(s)) = (1, 0, s)$ we find:

$$\begin{pmatrix} W \\ V \\ U \end{pmatrix} = h^* \begin{pmatrix} 1 \\ 0 \\ s \end{pmatrix}, \quad (\text{B.9})$$

that is:

$$u = \frac{U}{W} = e^{2i\alpha} \frac{s}{\cosh(\mathbf{a})}; \quad v = \frac{U}{W} = -e^{-i\sigma} \tanh(\mathbf{a}). \quad (\text{B.10})$$

From the above relations we find s in terms of the duality-transformed variables u, v :

$$s = \frac{|u|}{\sqrt{1 - |v|^2}}. \quad (\text{B.11})$$

Then we define \mathcal{W}_3 by duality continuation of $\mathcal{W}_3^{(S)}$:

$$\mathcal{W}_3^{(RN)}(u, v, \bar{u}, \bar{v}) = \mathcal{W}_3^{(S)}(s(u, v, \bar{u}, \bar{v})) = -c \log \left(\frac{|u| + \sqrt{1 - |v|^2}}{|u| - \sqrt{1 - |v|^2}} \right), \quad (\text{B.12})$$

thus obtaining (B.4).

We may explicitly check our result by solving the corresponding first order equations (B.2)

$$\begin{aligned} \frac{d\bar{u}}{d\tau} &= c\bar{u} \frac{|u^2| - k^2}{|u|k}; \quad k^2 = 1 - |v|^2 > 0, \\ \frac{dv}{d\tau} &= 0. \end{aligned} \tag{B.13}$$

From the first we derive:

$$\frac{d|u|}{d\tau} = c \frac{|u^2| - k^2}{k} \Rightarrow |u| = k \frac{A e^{2c\tau} + 1}{1 - A e^{2c\tau}}, \tag{B.14}$$

where A is an arbitrary constant that we take equal to 1. The second equation is telling us that v also is an arbitrary complex constant which we can set to:

$$v = -\frac{q - ip}{\sqrt{2}m} e^{i\alpha} \Rightarrow k = c/m. \tag{B.15}$$

Being the phase of u a constant, the general solution can be written as follows:

$$u = k \frac{e^{2c\tau} + 1}{1 - e^{2c\tau}} e^{2i\alpha}. \tag{B.16}$$

Setting the arbitrary constant $A = 0$ and using the relation between τ and r :

$$\tau = \frac{1}{2c} \log \left(\frac{r - m - c}{r - m + c} \right), \tag{B.17}$$

we find:

$$u = \frac{r - m}{m} e^{2i\alpha}; \quad v = -\frac{q - ip}{\sqrt{2}m} e^{i\alpha}, \tag{B.18}$$

which defines the RN-Taub-NUT solution where m, p, q are the parameters of a RN solution and α is the effect of a Ehlers U(1)-transformation. The Nöther charge matrix reads:

$$Q = \frac{1}{2} \mathcal{M}^{-1} \frac{d}{d\tau} \mathcal{M} = \begin{pmatrix} 0 & 0 & e^{2i\alpha} m \\ 0 & 0 & -i e^{i\alpha} \frac{p - iq}{\sqrt{2}} \\ e^{-2i\alpha} m & e^{-i\alpha} \frac{q - ip}{\sqrt{2}} & 0 \end{pmatrix}. \tag{B.19}$$

The fields are obtained by the general formulas:

$$\mathcal{U} = \frac{1}{2} \log \left(\frac{|u|^2 + |v|^2 - 1}{|1 + u|^2} \right); \quad \Psi = \frac{v}{1 + u}; \quad a = -i \frac{u - \bar{u}}{|1 + u|^2}. \tag{B.20}$$

Using the generators of the solvable algebra of $\frac{SU(1,2)}{U(1) \times SU(1,1)}$ (see appendix) we can compute the physical charges in terms of the parameters of the solution. The ADM mass \hat{m} and NUT charge read:

$$\hat{m} = \text{Tr}(H_0^\dagger Q) = m \cos(2\alpha); \quad \ell = -\text{Tr}(G^\dagger Q) = -m \sin(2\alpha). \tag{B.21}$$

while the complex charge $\frac{\hat{q}+i\hat{p}}{\sqrt{2}}$ is:

$$\frac{\hat{q} + i\hat{p}}{\sqrt{2}} = -\text{Tr}((T_1 + iT_2)^\dagger Q) = \frac{q + ip}{\sqrt{2}} e^{i\alpha}. \quad (\text{B.22})$$

Using the above identifications, the matrix Q in (B.19) reduces to the Nöther charge matrix in the first of eq.s (3.37), identifying hatted with un-hatted quantities. This represents the fact that the Nöther charge matrix Q is the same for the KN-Taub-NUT and the RN-Taub-NUT solutions. The difference resides in the matrix Q_ψ which vanishes in the latter solution.

Since the Maxwell-Einstein theory is a consistent truncation of a generic $\mathcal{N} = 2$ model, the above procedure for constructing a manifestly H^* -invariant \mathcal{W}_3 for the generic solution in the same $G_{(3)}$ -orbit as the Schwarzschild one, from a *duality completion* of $\mathcal{W}_3^{(S)}$, applies to a generic $\mathcal{N} = 2$, $D = 4$ supergravity. In this case the Nöther charge Q of a generic representative of the Schwarzschild orbit, is a diagonalizable matrix in the space \mathfrak{K} , orthogonal complement of \mathfrak{H}^* in \mathfrak{g} (the point at infinity ξ_0 is always set to coincide with the origin O), and transforms under the adjoint action of H^* in a characteristic H^* -representation. In particular Q can be diagonalized using an H^* -transformation. The modulus s in $\mathcal{W}_3^{(S)}$ is a function of the eigenvalues of Q , and thus is an H^* -invariant function of the parameters Q_A of Q : $s = f(Q_A)$. These parameters also provide a parametrization of the coset $G_{(3)}/H^* \equiv e^{\mathfrak{K}}$ and, in the physical patch \mathcal{U} , can be expressed in terms of the scalar fields z^a , so that we can locally express s as a H^* -invariant function of z^a : $s = f(Q_A(z^a)) = s(z^a)$. A duality completion procedure, analogous to the one illustrated above, allows then to determine the following H^* -invariant expression for \mathcal{W}_3 for the Schwarzschild orbit:

$$\mathcal{W}_3 = -c \log \left(\frac{s(z^a) + 1}{s(z^a) - 1} \right). \quad (\text{B.23})$$

In the case of the universal model $s(z^a)$ was given in eq. (B.11).

B.2 The Clément generating technique

Having at our disposal a duality invariant \mathcal{W}_3 for the RN solution, we may now apply a procedure, introduced in [45, 46], to relate static and rotating black-hole solutions. In this way we shall arrive at the explicit expression of the U, V, W variables (4.21) of the KN(-NUT) solution. We shall apply to the RN set of homogeneous variables associated to (B.18), which for definiteness we choose to be

$$U = r - m, \quad V = -\frac{1}{\sqrt{2}}(q - ip), \quad W = m + i\ell \quad (\text{B.24})$$

the transformation $\Pi \cdot R \cdot \Pi$, where:

$$\Pi : \{U \rightarrow V, V \rightarrow U, W \rightarrow -W\} \quad (\text{B.25})$$

is a $SU(1, 2)$ involution, and R is the following 4D space-time coordinate transformation:

$$R : \begin{cases} d\varphi = d\varphi' + \gamma\Omega dt' \\ dt = \gamma dt' \end{cases}, \quad (\text{B.26})$$

relating the original reference frame to one rotating with constant angular velocity Ω . The constant time-rescaling factor γ will be fixed in the following to have the standard expression for the Ernst potentials of the KN solution.

The first involution Π gives rise to the following new potentials:

$$\begin{aligned}\mathcal{E}' &= \frac{U' - W'}{U' + W'} = \frac{-\frac{1}{\sqrt{2}}(q - ip) + m - i\ell}{-\frac{1}{\sqrt{2}}(q - ip) - m + i\ell}, \\ \Psi' &= \frac{V'}{U' + W'} = \frac{r - m}{-\frac{1}{\sqrt{2}}(q - ip) - m + i\ell}\end{aligned}\quad (\text{B.27})$$

One can readily see that the new solution corresponds to a Bertotti-Robinson space-time, with radius $R_{BR} \equiv |V - W| = \sqrt{(\frac{q}{\sqrt{2}} + m)^2 + (\frac{p}{\sqrt{2}} + \ell)^2}$ [45, 46].

The coordinate transformation R induces the following transformation of the 4D static metric and gauge fields:

$$R : \begin{cases} e^{2\tilde{u}'} &= \gamma^2 (e^{2u'} - e^{-2u'} \hat{\rho}^2 \Omega^2) \\ \tilde{\omega} &= \frac{\hat{\rho}^2 \Omega}{\gamma (e^{4u'} - \hat{\rho}^2 \Omega^2)} \\ \tilde{\hat{\rho}} &= \gamma \hat{\rho} \end{cases}\quad (\text{B.28})$$

where

$$e^{2u'} = \frac{|U|^2 + |V|^2 - |W|^2}{R_{BR}^2} \equiv \frac{\Delta}{R_{BR}^2}\quad (\text{B.29})$$

$$\tilde{a}' = a' = \frac{(\bar{V}W - V\bar{W})}{R_{BR}^2} = \frac{\sqrt{2}(q\ell - pm)}{R_{BR}^2}\quad (\text{B.30})$$

We have introduced here the $SU(1, 2)$ invariant $\tilde{\Delta}$, which, in the coordinates (B.24), is:

$$\tilde{\Delta} = (r - m)^2 - c_{RT}^2\quad (\text{B.31})$$

where $c_{RT}^2 \equiv |W|^2 - |V|^2 = m^2 + \ell^2 - \frac{1}{2}(q^2 + p^2)$ is the extremality parameter of the dyonic RN-NUT solution. Note that $c_{RT}^2 = \frac{k}{2} \text{Tr}[Q^2]$ (see eq. (3.38)).

The redefinition of the metric implies a transformation of the gauge field-strengths, that corresponds to the following transformation on the gradient of the Ernst potential Ψ (here $x^m = (r, \theta)$):

$$\partial_m \tilde{\Psi}' = \gamma \left[\partial_m \Psi' - \hat{\rho} \Omega e^{-2u'} ({}^{*(2)}\partial_m \bar{\Psi}') \right].\quad (\text{B.32})$$

The integration of eq. (B.32) is easily performed by observing that ${}^{*(2)}\partial_r \bar{\Psi}' = 0$ since $\Psi' = \Psi'(r)$ is only function of the radial variable. Further observing that $\partial_r \Psi' = -\frac{\gamma}{\sqrt{2} R_{BR}^2} [(q + \sqrt{2}m) + i(\sqrt{2}\ell + p)]$, the final result is

$$\begin{aligned}\tilde{\Psi}' &= \gamma \{ \Psi'(r) + i(V - W)\Omega \cos \theta \} \\ &= \frac{\gamma}{R_{BR}^2} \{ (r - m)(\bar{V} - \bar{W}) + i\alpha \cos \theta \}\end{aligned}\quad (\text{B.33})$$

together with

$$\begin{aligned}\tilde{\mathcal{E}}' &= e^{2\tilde{U}'} - |\tilde{\Psi}'|^2 + i\tilde{a}' \\ &= -\frac{\gamma^2}{R_{BR}^2} (c_{RT}^2 + \alpha^2) + \frac{i(\bar{V}W - V\bar{W})}{R_{BR}^2}\end{aligned}\quad (\text{B.34})$$

where we have defined $\alpha \equiv (\Omega R_{BR}^2)$.

We may give a simpler expression to the Ernst potentials by fixing the time rescaling γ as

$$\gamma^2 = \frac{c_{RT}^2}{c_{RT}^2 + \alpha^2}.\quad (\text{B.35})$$

With this redefinition we obtain

$$\tilde{\mathcal{E}}' = \frac{\tilde{U}' - \tilde{W}'}{\tilde{U}' + \tilde{W}'} = \frac{V + W}{V - W}\quad (\text{B.36})$$

$$\tilde{\Psi}' = \frac{\tilde{V}'}{\tilde{U}' + \tilde{W}'} = \frac{\gamma(U + i\alpha \cos \theta)}{V - W}.\quad (\text{B.37})$$

implying the following transformation on the homogeneous variables:

$$R \cdot \Pi : \begin{cases} \tilde{U}' = V \\ \tilde{V}' = \gamma(U + i\alpha \cos \theta) \\ \tilde{W}' = -W \end{cases}\quad (\text{B.38})$$

Performing again the transformation Π as given in (B.25), we finally obtain the KN (Taub-NUT) fields in terms of the corresponding variables of the RN (TaubNUT) solution:

$$\Pi \cdot R \cdot \Pi : \begin{cases} \tilde{U}'' = \gamma(U + i\alpha \cos \theta) \\ \tilde{V}'' = V \\ \tilde{W}'' = W \end{cases}\quad (\text{B.39})$$

corresponding to the potentials

$$\tilde{\mathcal{E}}'' = \frac{\gamma(U + i\alpha \cos \theta) - W}{\gamma(U + i\alpha \cos \theta) + W}\quad (\text{B.40})$$

$$\tilde{\Psi}'' = \frac{V}{\gamma(U + i\alpha \cos \theta) + W}.\quad (\text{B.41})$$

They coincide with the standard KN potentials (see, for example, [39], Chapter 21)

$$\mathcal{E}_{KN} = 1 - \frac{2m}{r + i\alpha \cos \theta}\quad (\text{B.42})$$

$$\Psi_{KN} = \frac{-\frac{1}{\sqrt{2}}(q - ip)}{r + i\alpha \cos \theta}.\quad (\text{B.43})$$

if we set, besides $\ell = 0$:

$$r \rightarrow \gamma(r - m) + m, \quad \alpha \rightarrow \gamma\alpha.\quad (\text{B.44})$$

For the KN solution, the field a appearing in (3.2) is given by the imaginary part of \mathcal{E} ,

$$a = 2\frac{m\alpha \cos \theta}{|\rho|^2}\quad (\text{B.45})$$

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