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Localization of the four-dimensional $\mathcal{N} = 4$ SYM to a two-sphere and 1/8 BPS Wilson loops

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Abstract: We localize the four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory on S^4 to the two-dimensional constrained Hitchin/Higgs-Yang-Mills (cHYM) theory on S^2. We show that expectation values of certain 1/8 BPS supersymmetric Wilson loops on S^2 in the 4d $\mathcal{N} = 4$ SYM is captured by the 2d cHYM theory. We further argue that expectation values of Wilson loops in the cHYM theory agree with the prescription "two-dimensional bosonic Yang-Mills excluding instanton contributions". Hence, we support the recent conjecture by Drukker, Giombi, Ricci and Trancanelli on the 1/8 BPS Wilson loops on S^2 in the 4d $\mathcal{N} = 4$ SYM.

Keywords: Supersymmetric gauge theory, Wilson, 't Hooft and Polyakov loops

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1 Introduction

The dynamics of the $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions is probably the simplest among the four-dimensional gauge theories, but still this is a very rich and interesting theory from the theoretical perspective.

Integrable structures discovered in the $\mathcal{N} = 4$ SYM [1] or possible connection with the geometric Langlands program [2] are just a few examples of interesting mathematics represented by this maximally supersymmetric gauge theory. It is believed that the theory has the exact dual description — the type IIB ten-dimensional string theory in the $AdS_5 \times S^5$ background [3–5].

The basic observables in any gauge theory, the Wilson loop observables, in the $\mathcal{N} = 4$ theory can be generalized to preserve some amount of superconformal symmetry. The simplest operator of this kind is a circular Wilson loop which couples to one of the six adjoint scalar fields of the $\mathcal{N} = 4$ SYM. Such operator is called 1/2 BPS Wilson loop because it preserves one half of the 32 superconformal symmetries of the $\mathcal{N} = 4$ SYM. In the beautiful work [6] further elaborated in [7] it was conjectured that the expectation value of such operator can be computed in the Gaussian matrix model. In [8] it was shown that this conjecture follows from localization of the path integral to the supersymmetric configurations. From the dual string theory point of view, in a suitable limit of large N and large 't Hooft constant $\lambda = g^2_{YM}N$, these Wilson loop observables are usually described by a string worldsheet with a boundary located at the loop [9]. As was shown in the original paper [6], the large N and the large λ limit of the Gaussian matrix model nicely agrees with the solution to the minimal area problem in the dual string theory.

In [10–14] other kinds of Wilson loops, which preserve various amount of supersymmetry, have been studied. In particular [12–14] have constructed 1/16 BPS Wilson loops of arbitrary shape on a three-sphere S^3 in the Euclidean space-time $\mathbb{R}^4$ in the 4d $\mathcal{N} = 4$ SYM. Restricting these Wilson loops to the equator $S^2 \subset S^3$ one gets 1/8 BPS Wilson loops. In [12–14] a bold conjecture has been proposed: the expectation value of such Wilson loops is captured by the zero-instanton sector of the ordinary bosonic two-dimensional Yang-Mills living on the S^2. The coupling constant of the bosonic 2d YM is related to the coupling constant of the $\mathcal{N} = 4$ SYM as $g^2_{2d} = -g^2_{4d}/(2\pi r^2)$ where r is the radius of the S^2. This conjecture was further supported at the order λ^2 in [15, 16] for an expectation value of a single Wilson loop operator of arbitrary shape on S^2, but in [15] a discrepancy was found at the order λ^3 for a connected correlator of two circular concentric Wilson loops on S^2.

Clearly, to support or to refine the conjecture one needs a framework which allows to deduce this conjectural two-dimensional theory from the 4d $\mathcal{N} = 4$ SYM.

In this paper we use localization argument to explain how the dynamics of the Wilson loops on S^2 in the $d = 4$ $\mathcal{N} = 4$ SYM is captured by a two-dimensional theory. Usually, the localization involves two steps (compare with e.g. [8]): (i) finding out the configurations on

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which the theory localizes and evaluating the physical action on these configurations, (ii) computing the determinant for the fluctuations of all fields in the normal directions to the localization locus. We give the details of the step (i), leaving out the step (ii) for future research. Compared with [8], where the computation of the determinant was possible using the theory of indices for transversally elliptic operators on compact manifold, in the present case the complication is that the relevant operator in not transversally elliptic everywhere. However, this non-ellipticity is rather mild: the operator degenerates at the codimension two submanifold of the space-time — this is precisely the S^2 where the interesting Wilson loop operators are located. That gives a hope that these complications could be overpassed in a future.

In the localization we first use the circle action of the square of the relevant supersymmetry generator to reduce the theory from four-dimensions to three-dimensions. Next we study the supersymmetry equations on the resulting three-dimensional manifold with a boundary S^2. The interesting Wilson loop observables live at this two-dimensional boundary. The three-dimensional equations are quite complicated, but one can relate these equations and the extended Bogomolny equations which appeared in [2]. We do not study singular solutions to these equations, but in principle, this can be done, and it would correspond to the insertion of the 't Hooft operators running over the circles linked with the two-sphere.

Then we show that the moduli space of solutions to the supersymmetry equations is parametrized by the boundary data and that the three-dimensional action on the supersymmetric solutions is captured by the boundary term. This boundary term is effectively the action of the two-dimensional theory living on the boundary.

The resulting two-dimensional theory is the semi-topological Hitchin/Higgs-Yang-Mills theory (see e.g. [17–19]). We argue, though not totally rigorously, that the perturbative computation of the Wilson loop operators in the HYM theory agrees with the perturbative computation in the usual 2d bosonic YM, and that the unstable instantons in the HYM theory do not contribute to the partition function because of extra fermionic zero modes.

In other words, using the localization, we derive a Lagrangian formulation of the 2d theory which is supposed to capture Wilson loops on S^2 in the 4d $\mathcal{N} = 4$ SYM, and we support the prescription "the 0-instanton sector in the 2d bosonic Yang-Mills" suggested in [12–14].

We have not found the determinant of quantum fluctuations at the localization locus in this work, but there are good reasons to believe that this determinant in the $\mathcal{N} = 4$ theory is trivial like it happened in [8]. In this case, and if one shows rigorously that the 2d HYM theory is equivalent to the "zero-instanton sector" of the 2d bosonic Yang-Mills for correlation function of Wilson loop operators, the conjecture of [12–14] would be proved. It would be in a nice agreement with several recent computations on 1/8 BPS Wilson loops made in [20, 21]. However, then we will have a puzzle how to reconcile this result with the explicit Feynman diagram computations at the order λ^3 for a connected correlator of two Wilson loops done at [15] which were shown not to agree with the 2d YM conjecture. Perhaps, there are involved subtle issues related to the regularization of the conformal supersymmetric gauge theory and/or anomalies which require further studies in either approach to the problem.

Another scenario is that the HYM theory is corrected by the one-loop determinant. However, this correction must be constrained by the following results: (i) it could show

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up only at the order of λ^3 for the correlators of Wilson loops of arbitrary shape but not at smaller order [15, 16], (ii) in the large λ and the large N limit this correction has to vanish because the dual string computation agrees with the matrix model computation for the connected correlator which follows from the conjecture [20], (iii) this correction must not contribute to the expectation value of the Wilson loop operator on the equator on S^2 which was proved to be computed by the Hermitian matrix model [8] (and this Hermitian matrix model is implied by the conjecture). Future research is needed to resolve these interesting issues.

In section 2 we describe the geometry of the Wilson loops which we study together with the relevant supersymmetries and also set up various notations and conventions. A reader well familiar with constructions in [12–14] might wish to skip straight to the section 3 where we describe the actual localization computation. In section 4 we analyze the resulting two-dimensional theory.

2 The conventions and geometry

Let X_i, $i = 1 \dots 5$, be coordinates in $\mathbb{R}^5$ into which the S^4 is embedded as the hypersurface $\sum X_i^2 = r^2$. By x_i, $i = 1 \dots 4$, we denote the standard coordinates on the stereographic projection from S^4 to $\mathbb{R}^4$ which maps the North pole N of the S^4 with coordinates $\vec{X} = (0, 0, 0, 0, r)$ to the origin $\vec{x} = 0$ of the $\mathbb{R}^4$:

$$\begin{aligned} X_i &= \frac{x_i}{1 + \frac{x^2}{4r^2}}, \quad i = 1 \dots 4 \\ X_5 &= r \frac{1 - \frac{x^2}{4r^2}}{1 + \frac{x^2}{4r^2}}. \end{aligned} \tag{2.1}$$

We define the three-sphere $S^3 \subset S^4$ by the equation $X_5 = 0$. Equivalently, in the x_i coordinates on $\mathbb{R}^4$, this three-sphere is defined by the equation $x^2 = 4r^2$. Next, we define the two-sphere $S^2 \subset S^3$ by the additional equation $X_1 = 0$. In the x_i coordinates, the S^2 is described by the equations $\{x_1 = 0, x_2^2 + x_3^2 + x_4^2 = 4r^2\}$. We denote this S^2 as Σ.

We call the point P with $\vec{X}(P) = (0, r, 0, 0, 0)$ the North pole of Σ. (The points P and N are different points). In x^i coordinates, $\vec{x}(P) = (0, 2r, 0, 0)$. By y_i, $i = 1 \dots 4$, we denote the standard coordinates on the stereographic projection from S^4 to $\mathbb{R}^4$ which maps the point P to the origin of the $\mathbb{R}^4$:

$$\begin{aligned} X_i &= \frac{y_i}{1 + \frac{y^2}{4r^2}}, \quad i = 1, 3, 4 \\ X_5 &= \frac{-y_2}{1 + \frac{y^2}{4r^2}} \\ X_2 &= r \frac{1 - \frac{y^2}{4r^2}}{1 + \frac{y^2}{4r^2}}. \end{aligned} \tag{2.2}$$

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The SO(5) isometry group of S^4 can be broken to $\mathrm{SO}(2)_S \times \mathrm{SO}(3)_S$ where the $\mathrm{SO}(2)_S$ acts on (X_1, X_5) and the $\mathrm{SO}(3)_S$ acts on (X_2, X_3, X_4).[1] The two-sphere Σ is the fixed point set of the $\mathrm{SO}(2)_S$. Sometimes it is convenient to use the $\mathrm{SO}(2)_S \times \mathrm{SO}(3)_S$ spherical coordinates on S^4; we represent the S^4 as a warped $S^2 \times S^1$ fibration over an interval I. Let $\theta \in [0, \pi/2]$ be the coordinate on I. We also use notation $\xi = \pi/2 - \theta$. Let $\tau \in [0, 2\pi)$ be the standard coordinate on S^1 fibers and let $d\Omega_2^2$ be the standard unit metric on the S^2 fibers. Then the metric on S^4 of radius r is form

$$ds^2 = r^2(d\theta^2 + \sin^2\theta\, d\tau^2 + \cos^2\theta\, d\Omega_2^2) \tag{2.3}$$

At $\theta = 0$ the S^1 shrinks to zero and the S^2 is of maximal size, while at $\theta = \pi/2$ the S^2 shrinks to zero and the S^1 is of maximal size.

2.1 1/8 BPS Wilson loop operators

Following [12–14] we consider the Wilson loops located on the S^3 of the following form

$$W_R(C) = \mathrm{tr}_R\, \mathrm{Pexp} \oint \left(A_\mu + i\sigma^A_{\mu\nu} \frac{x^\nu}{2r} \Phi_A \right) dx^\mu, \tag{2.4}$$

specifically restricting our attention to the Wilson loops located on the equator $\Sigma = S^2 \subset S^3$. The definition (2.4) is given in the $\mathbb{R}^4$ stereographic coordinates x_i (2.1).[2]

The definition of such Wilson loops and the condition for supersymmetry was found in [12–14]. The Φ_A denotes three of six scalar fields of the $\mathcal{N} = 4$ super Yang-Mills theory. In our conventions the index A runs over $6, 7, 8$. The μ, ν are the space-times indices running over $1, \ldots 4$. The $\sigma^A_{\mu\nu}$ are the 't Hooft symbols: three 4×4 anti-self-dual matrices satisfying $\mathfrak{su}(2)$ commutation relations. Explicitly we choose

$$\sigma^{i+4}_{1i} = 1 \quad \sigma^{i+4}_{jk} = -\epsilon_{ijk} \quad \text{for} \quad i = 2, 3, 4, \tag{2.5}$$

where ϵ_{ijk} is the standard antisymmetric symbol with $\epsilon_{234} = 1$. The SO(6) R-symmetry group is broken to $\mathrm{SO}(3)_A \times \mathrm{SO}(3)_B$. Our conventions are that the $\mathrm{SO}(3)_A$ acts on the three scalars Φ_6, Φ_7, Φ_8 which couple to the Wilson loop (2.4). The $\mathrm{SO}(3)_B$ acts on the remaining scalars Φ_5, Φ_9, Φ_0. The Wilson loop (2.4) is explicitly invariant under the $\mathrm{SO}(3)_B$ symmetry, because the scalar fields Φ_5, Φ_9, Φ_0 do not appear in (2.4). In the case when the Wilson loop (2.4) is restricted to the two-sphere S^2 by the constraint $x_1 = 0$, it is also invariant under the diagonal SO(3) subgroup of the $\mathrm{SO}(3)_S \times \mathrm{SO}(3)_A$, i.e. under the simultaneous rotation of the coordinates x_i and the scalars Φ_{i+4}, $i = 2, 3, 4$.

The supersymmetries which are preserved by the Wilson loop (2.4) were found in [12–14]. To set all notations and conventions we repeat the derivation here.

[1] We shall use the subscript "S" to denote subgroups of the space-time symmetries, and the subscript "R" do denote subgroups of the R-symmetry. We also remark that the $\mathrm{SO}(3)_S$ subgroup of the SO(4) isometry group of $\mathbb{R}^4$ is not the left $\mathrm{SU}(2)_L$ subgroup in the decomposition $\mathrm{SO}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, but rather a diagonal embedding.

[2] Recall that in our conventions the equation of the S^3 is $\sum_{i=1}^4 x_i^2 = 4r^2$.

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2.2 Superconformal symmetries and conformal Killing spinors

The conformal Killing spinor on $\mathbb{R}^4$ is parameterized by two constant spinors which we call $\hat{\varepsilon}_s$ and $\hat{\varepsilon}_c$, where $\hat{\varepsilon}_s$ generates the usual Poincare supersymmetries, and $\hat{\varepsilon}_c$ generates the special superconformal symmetries

$$\varepsilon(x) = \hat{\varepsilon}_s + x^\rho \Gamma_\rho \hat{\varepsilon}_c. \tag{2.6}$$

The variation of the bosonic fields of the theory is

$$\delta A_M = \psi \Gamma_M \varepsilon, \tag{2.7}$$

where A_M, $M = 0, \ldots, 9$ is a collective notation for the gauge fields A_μ, $\mu = 1, \ldots, 4$, and the scalar fields Φ_A, $A = 5, \ldots, 9, 0$. The ψ denotes sixteen component fermionic fields of the $\mathcal{N} = 4$ theory written in the $d = 10$ $\mathcal{N} = 1$ SYM notations. The Γ_M, $M = 0, \ldots, 9$, are 16×16 gamma-matrices which act on the chiral spin representation S^+ of Spin(10). The spinors $\psi, \varepsilon, \hat{\varepsilon}_s$ are in the S^+ while $\hat{\varepsilon}_c$ is in the S^-. The variation of a generic Wilson loop (2.4) vanishes if and only if ε satisfies

$$\left(\Gamma_\mu + i\Gamma_A \sigma^A_{\mu\nu} \frac{x^\nu}{2r}\right)(\hat{\varepsilon}_s + x^\rho \Gamma_\rho \hat{\varepsilon}_c)\dot{x}^\mu = 0 \tag{2.8}$$

for any point $x \in S^3$ and the tangent vector $\dot{x}_\mu$ which is constrained by $\dot{x}_\mu x_\mu = 0$. The terms linear in x give the equation

$$x^\mu \dot{x}^\rho \left(\Gamma_\mu \Gamma_\rho \hat{\varepsilon}_c + i\Gamma_A \sigma^A_{\mu\rho} \frac{\hat{\varepsilon}_s}{2r}\right) = 0. \tag{2.9}$$

Since the vectors x^μ and $\dot{x}_\mu$ are constrained only by $x^\mu \dot{x}_\mu = 0$, we get

$$\Gamma_{\mu\rho}\hat{\varepsilon}_c + i\Gamma^A \sigma^A_{\mu\rho} \frac{\hat{\varepsilon}_s}{2r} = 0. \tag{2.10}$$

The constant and quadratic in x terms give the equation

$$\dot{x}^\mu \left(\Gamma_\mu \hat{\varepsilon}_s + i\Gamma_{A\lambda} \frac{\sigma^A_{\mu\nu}}{2r} x^\nu x^\lambda \hat{\varepsilon}_c\right) = 0. \tag{2.11}$$

Multiplying by non-degenerate matrix $x^\rho \Gamma_\rho$ we get

$$\dot{x}^\mu x^\rho \left(\Gamma_{\rho\mu}\hat{\varepsilon}_s + i\Gamma_\rho \Gamma_{A\lambda} \frac{\sigma^A_{\mu\nu}}{2r} x^\nu x^\lambda \hat{\varepsilon}_c\right) = 0. \tag{2.12}$$

Using $x^\mu x_\mu = 4r^2$ and $\dot{x}^\mu x_\mu = 0$ we get

$$\Gamma_{\mu\rho}\hat{\varepsilon}_s + i\Gamma_A \sigma^A_{\mu\rho}(2r)\hat{\varepsilon}_c = 0. \tag{2.13}$$

The equation (2.13) is actually equivalent to (2.10) and to

$$2r\hat{\varepsilon}_c = i\sigma^A_{\mu\rho}\Gamma_{A\mu\rho}\hat{\varepsilon}_s. \tag{2.14}$$

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If Wilson loop is restricted to S^2, then (2.14) amounts to three maximally orthogonal projections in the spinor representation space $S^+ \oplus S^-$. Each projection operator reduces the dimension of the space of solutions by half. Starting from the dimension 32 of $S^+ \oplus S^-$ we get $32/2^3 = 4$-dimensional space of solutions for $(\hat{\varepsilon}_s, \hat{\varepsilon}_c)$. For generic Wilson loops on S^3 the dimension of the space of solutions is further reduced by two, so there are only 2 supersymmetries left.

For explicit computation we use the following 16×16 gamma-matrices representing Clifford algebra on S^+:

$$\begin{aligned} \Gamma_M &= \begin{pmatrix} 0 & E_M^T \\ E_M & 0 \end{pmatrix}, \qquad M = 2 \dots 9 \\ \Gamma_1 &= \begin{pmatrix} 1_{8\times 8} & 0 \\ 0 & -1_{8\times 8} \end{pmatrix}, \\ \Gamma_0 &= \begin{pmatrix} i1_{8\times 8} & 0 \\ 0 & i1_{8\times 8} \end{pmatrix}, \end{aligned} \tag{2.15}$$

Here E_M, $M = 2 \dots 8$, are 8×8 matrices representing left multiplication of the octonions and $E_9 = 1_{8\times 8}$. (Let e_i, $i = 2, \dots, 9$, be the generators of the octonion algebra $\mathbb{O}$. We chose e_9 to be identity. Let c_{ij}^k be the structure constants of the left multiplication $e_i \cdot e_j = c_{ij}^k e_k$. Then $(E_i)_j^k = c_{ij}^k$. The multiplication table is defined by specifying the set of cyclic triples (ijk) such that $e_i e_j = e_k$. We define the cyclic triples to be $(234), (256), (357), (458), (836), (647), (728)$.)

Explicitly, the four linearly independent solutions of (2.14), i.e. supersymmetries of Wilson loops on the S^2 are the following

$$\begin{aligned} &\hat{\varepsilon}_1^s = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \otimes |1\rangle \quad \hat{\varepsilon}_2^s = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \otimes |1\rangle \qquad \hat{\varepsilon}_{\bar{1}}^s = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \otimes |1\rangle \qquad \hat{\varepsilon}_{\bar{2}}^s = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \otimes |1\rangle \\ &\hat{\varepsilon}_1^c = \frac{1}{2r} \begin{pmatrix} 0 \\ i \\ 0 \\ i \end{pmatrix} \otimes |1\rangle \quad \hat{\varepsilon}_2^c = \frac{1}{2r} \begin{pmatrix} -i \\ 0 \\ i \\ 0 \end{pmatrix} \otimes |1\rangle \quad \hat{\varepsilon}_{\bar{1}}^c = \frac{1}{2r} \begin{pmatrix} 0 \\ -i \\ 0 \\ i \end{pmatrix} \otimes |1\rangle \quad \hat{\varepsilon}_{\bar{2}}^c = \frac{1}{2r} \begin{pmatrix} i \\ 0 \\ i \\ 0 \end{pmatrix} \otimes |1\rangle \end{aligned} . \tag{2.16}$$

Sixteen components of the spinors are written in the 4×4 block notations, where

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.17}$$

In more generic case of Wilson loops on S^3, one gets only the two-dimensional space of solutions [12], which is spanned by $\varepsilon_1, \varepsilon_2$, but not by $\varepsilon_{\bar{1}}, \varepsilon_{\bar{2}}$.[3]

[3]We use indices $1, 2$ and $\bar{1}, \bar{2}$ only to enumerate the basis elements of the solutions to (2.14), but it is not supposed that $\varepsilon_{\bar{1}}$ or $\varepsilon_{\bar{2}}$ is the complex conjugate to ε_1 or ε_2.

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2.3 Anticommutation relations

Let $Q_1, Q_2, Q_{\bar{1}}, Q_{\bar{2}}$ be the four conformal supersymmetries generated by conformal Killing spinors (2.6) with $\hat{\varepsilon}_s, \hat{\varepsilon}_c$ given by (2.16). Let R_{AB} be the matrices in the fundamental representation of the SO(6) R-symmetry generators. On scalar fields the generators R_{AB} act as

$$(\delta_{R_{AB}}\Phi)_A = R_{AB}\Phi_B. \tag{2.18}$$

The fermionic symmetries anticommute according to (A.1), (A.4) as

$$\delta^2_\varepsilon \Phi_A = 2(\tilde{\varepsilon}\Gamma_{AB}\varepsilon)\Phi_B, \tag{2.19}$$

hence the R-symmetry part of the anticommutators is

$$Q_{\{\alpha}Q_{\beta\}} = 2(\tilde{\varepsilon}_{\{\alpha}\Gamma_{AB}\varepsilon_{\beta\}})R_{AB}. \tag{2.20}$$

For space-time rotations we have similar equation except for the sign. Let us consider a fixed point of the space-time rotation. Then, assuming that the $\mathrm{SO}(4)_S$ generators $R_{\mu\nu}$ act on tangent space $\mathbb{R}^4$ in the same way as the $\mathrm{SO}(6)_R$ generators R_{AB} act on the scalar target space $\mathbb{R}^6$, we get the space-time symmetry part of the anticommutators

$$Q_{\{\alpha}Q_{\beta\}} = -2(\tilde{\varepsilon}_{\{\alpha}\Gamma_{\mu\nu}\varepsilon_{\beta\}})R_{\mu\nu}, \tag{2.21}$$

where ε and $\tilde{\varepsilon}$ are taken at the fixed point set of the space-time rotation. To summarize,

$$Q_{\{\alpha}Q_{\beta\}} = 2(\tilde{\varepsilon}_{\{\alpha}\Gamma_{AB}\varepsilon_{\beta\}})R_{AB} - 2(\tilde{\varepsilon}_{\{\alpha}\Gamma_{\mu\nu}\varepsilon_{\beta\}})R_{\mu\nu}. \tag{2.22}$$

At a fixed point of space-time rotation, the $\mathrm{SO}(4)_S \times \mathrm{SO}(6)_R$ generators act on spinors in the S^+ representation of SO(10) as

$$\delta_{R_{MN}}\Psi = \frac{1}{4}R_{MN}\Gamma^{MN}\Psi. \tag{2.23}$$

Then there are the following anticommutation relations

$$\begin{aligned}
\{Q_1, Q_1\} &= \frac{2}{r}R_{05} - \frac{2}{r}iR_{59} & \{Q_{\bar{1}}, Q_{\bar{1}}\} &= \frac{2}{r}R_{05} + \frac{2}{r}iR_{59} \\
\{Q_2, Q_2\} &= -\frac{2}{r}R_{05} - \frac{2}{r}iR_{59} & \{Q_{\bar{2}}, Q_{\bar{2}}\} &= -\frac{2}{r}R_{05} + \frac{2}{r}iR_{59} \\
\{Q_1, Q_2\} &= \frac{2}{r}R_{09} & \{Q_{\bar{1}}, Q_{\bar{2}}\} &= -\frac{2}{r}R_{09} \\
\{Q_1, Q_{\bar{1}}\} &= -\frac{2}{r}R_{12} & \{Q_1, Q_{\bar{2}}\} &= 0 \\
\{Q_2, Q_{\bar{1}}\} &= 0 & \{Q_2, Q_{\bar{2}}\} &= -\frac{2}{r}R_{12}
\end{aligned} \tag{2.24}$$

These anticommutation relations can be packed into

$$\begin{aligned}
\{Q_\alpha, Q_\beta\} &= \frac{2}{r}(C\sigma^I)_{\alpha\beta}R_I \\
\{Q_{\bar{\alpha}}, Q_{\bar{\beta}}\} &= \frac{2}{r}(\overline{C\sigma^I})_{\bar{\alpha}\bar{\beta}}R_I \\
\{Q_\alpha, Q_{\bar{\beta}}\} &= \frac{2}{r}\delta_{\alpha\bar{\beta}}R_0,
\end{aligned} \tag{2.25}$$

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where σ^I, $I = 1, 2, 3$, are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.26}$$

The C denotes "the charge conjugation" matrix, $C = i\sigma_2$, the triplet of the $\mathrm{SO}(3)_B$ generators is denoted by R_I such that $(R_1, R_2, R_3) := (R_{05}, -R_{59}, -R_{09})$, and the $\mathrm{SO}(2)_S$ generator is called $R_0 := -R_{12}$.

The fermionic generators Q_α and $Q_{\bar\alpha}$ transform naturally in the representation $\mathbf{2}$ and $\bar{\mathbf{2}}$ of the $\mathrm{SO}(3)_B \simeq \mathrm{SU}(2)_B$, while $\mathrm{SO}(2)_S$ mixes them

$$\begin{aligned} [R_I Q_\alpha] &= -\frac{1}{2} i \sigma^I_{\alpha\beta} Q_\beta \quad & [R_0 Q_\alpha] &= -\frac{1}{2} i C_{\alpha\bar\beta} Q_{\bar\beta} \\ [R_I Q_{\bar\alpha}] &= \frac{1}{2} i \bar\sigma^I_{\bar\alpha\bar\beta} Q_{\bar\beta} \quad & [R_0 Q_{\bar\alpha}] &= \frac{1}{2} i C_{\bar\alpha\beta} Q_\beta. \end{aligned} \tag{2.27}$$

The relations (2.25) and (2.27) are the commutation relations of the Lie algebra $\mathfrak{su}(1|2)$ of the SU(1|2) subgroup of the superconformal group [12]. The bosonic part of $\mathfrak{su}(1|2)$ is $\mathfrak{so}(2)_S \times \mathfrak{so}(3)_B$, spanned by R_0, R_I, the fermionic part is four-dimensional, spanned by $Q_\alpha, Q_{\bar\alpha}$.

If we take a linear combination of the fermionic generators with complex coefficients $\varepsilon^\alpha, \varepsilon^{\bar\alpha}$

$$Q = \varepsilon^\alpha Q_\alpha + \varepsilon^{\bar\alpha} Q_{\bar\alpha}, \tag{2.28}$$

we will find that Q squares to a real generator of the $\mathrm{SO}(3)_B \times \mathrm{SO}(2)_S$ if $\varepsilon^{\bar\alpha}$ is actually the complex conjugate to ε^α. Such Q will be called Hermitian and will be used in the following for the localization computation. We shall also notice that if Q is Hermitian, i.e. if $\varepsilon^{\bar\alpha}$ is complex conjugate to ε^α, then the norm of the $\mathrm{SO}(2)_S$ generator and $\mathrm{SO}(3)_B$ generator in Q^2 is proportional to the norm of ε. Hence, a non-zero Hermitian Q always squares to a non-zero rotation generator in both $\mathrm{SO}(2)_S$ and $\mathrm{SO}(3)_B$.

2.3.1 The localization operator Q

For explicit localization computations we take Q to be

$$Q_\varepsilon = \frac{1}{2}(Q_1 + Q_{\bar 1}). \tag{2.29}$$

It corresponds to the conformal Killing spinor generated by

$$\hat\varepsilon_s = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes |1\rangle \quad \hat\varepsilon_c = \frac{1}{2r} \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix} \otimes |1\rangle. \tag{2.30}$$

By (2.25) we have

$$Q^2 = \frac{1}{r}(R_{05} - R_{12}). \tag{2.31}$$

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Clearly, since $[Q^2, Q] = 0$ we have

$$[R_{05} - R_{12}, Q] = 0 \quad \Rightarrow \quad (\Gamma_{05} - \Gamma_{12})\varepsilon_P = 0. \tag{2.32}$$

The last equality is written for the conformal Killing spinor ε associated with Q at the point P in coordinate patch y^μ (2.2). The rotation of (y_1, y_2) plane corresponds in the global coordinates to the rotation of (X_5, X_1) plane, or the vector field $\frac{\partial}{\partial\tau}$ in the polar coordinates (2.3). Geometrically, the equation (2.32) means that the conformal Killing spinor ε is invariant under simultaneous rotation of the (X_5, X_1) plane and (Φ_5, Φ_0) plane.

From the condition (2.14) on ε and (2.5) it follows that ε is also invariant under the diagonal rotations in the $\mathrm{SO}(3)_S \times \mathrm{SO}(3)_A$. Indeed, from (2.14) one gets

$$\Gamma_{j+4}\Gamma_{ki}\hat{\varepsilon}_s = \Gamma_{i+4}\Gamma_{jk}\hat{\varepsilon}_s \tag{2.33}$$

for pairwise distinct indices i, j, k running over $2, 3, 4$. Multiplying by $\Gamma_{j+4}\Gamma_{jk}$ both sides of this equation we get

$$\Gamma_{ji}\hat{\varepsilon}_s = -\Gamma_{j+4,i+4}\hat{\varepsilon}_s, \tag{2.34}$$

which shows that ε is invariant under simultaneous $\mathrm{SO}(3)_S$ rotation of (X_2, X_3, X_4) and the corresponding $\mathrm{SO}(3)_A$ rotation of (X_6, X_7, X_8) under the isomorphism $\mathbb{R}^3 \to \mathbb{R}^3 : X_i \mapsto X_{i+4}$.

2.3.2 Remark on 1/16 BPS Wilson loops on S^3

We shall remark that a generic supersymmetric Wilson loop on the S^3 is invariant only under the $\mathrm{OSp}(1|2, \mathbb{C})$ subgroup of the complexified superconformal group $\mathrm{PSU}(2, 2|4, \mathbb{C})$, see [12]. The fermionic part of $\mathrm{OSp}(1|2, \mathbb{C})$ is spanned by Q_α, i.e. by half of generators of $\mathrm{SU}(1|2, \mathbb{C})$. The bosonic part of $\mathfrak{osp}(1|2, \mathbb{C})$ is $\mathfrak{sp}(2, \mathbb{C}) \simeq \mathfrak{su}(2, \mathbb{C})$ spanned by R_I. The commutation relations are represented by the first equation in (2.25) and in (2.27). However, there is no real structure on $\mathrm{OSp}(1|2, \mathbb{C})$ such that the real version of $\mathrm{OSp}(1|2, \mathbb{C})$ could be embedded into the compact unitary supergroup $\mathrm{SU}(1|2, \mathbb{R})$.[4]

So there exists no fermionic element Q in $\mathrm{OSp}(1|2, \mathbb{C})$ such that Q^2 generates a unitary transformation. Since the localization method, which we are using in this work, requires the global transformation generated by Q^2 to be unitary, we cannot generalize our localization computation to the $\mathrm{OSp}(1|2, \mathbb{C})$ case, and, hence, we cannot treat generic Wilson loops on S^3 in the same way as generic Wilson loops on the $S^2 \subset S^3$. So we restrict the detailed study to the case of Wilson loops on $S^2 \subset S^3$.

2.3.3 Remark on 1/4 BPS circular Wilson loops

As discussed above, a Wilson loop (2.4) of an arbitrary shape on $\Sigma = S^2$ preserves 4 out of 32 superconformal symmetries, so it can be called $4/32 = 1/8$ BPS Wilson loop, but a circular Wilson loop on S^2 preserves 8 supersymmetries (1/4 BPS) [12–14, 22, 23], and the circular Wilson loop of maximal size preserves 16 supersymmetries (1/2 BPS). The Wilson

[4] If we use signature for $(5, 9, 0)$ directions $(+, +, -)$, then, since in this case gamma-matrices can be chosen real, we can get a real structure on $\mathrm{OSp}(1|2, \mathbb{R})$ by taking all generators to be real. However, in this case, Q^2 is always light-like generator of the bosonic part of $\mathrm{SO}(2, 1) \simeq \mathrm{SL}(2, \mathbb{R})$.

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loop on the equator of S^2 is the most familiar maximally supersymmetric superconformal Wilson loop, the study of which was initiated in [6, 7] and many consequent papers. There it was conjectured that expectation value of such operator can be computed in a Gaussian matrix model. In [7] an argument was given that the field theory localizes to matrix model, however that argument does not show that the matrix model is Gaussian. In [8] it was shown how to get the Gaussian matrix model from the localization computation.

In [23] it was conjectured that 1/4 BPS circular Wilson loops can also be computed using the Gaussian matrix model but with a rescaled coupling constant. Such 1/4 BPS circular Wilson loops can be considered as an intermediate case between maximally supersymmetric 1/2 BPS Wilson loops and 1/8 BPS Wilson loops of an arbitrary shape on S^2.

One may ask whether it is possible to localize the $\mathcal{N} = 4$ SYM field theory for 1/4 BPS circular Wilson loops straight to the Gaussian matrix model? We shall note that a new localization computation, different from localization computation for a generic Wilson loop on S^2, might be possible only for a single circular 1/4 BPS Wilson on S^2. But if we take two 1/4 BPS loops located at two distinct latitudes β_1 and β_2 on S^2, then each Wilson loop preserves eight supesymmetries, but only four supersymmetries are preserved by both loops simultaneously. These four common supersymmetries are actually the same as for a generic 1/8 BPS Wilson loop on S^2. Hence, if we want to compute the connected correlator of two latitudes on S^2, we are back to the case of generic 1/8 BPS loops on S^2, where the four-dimensional theory localizes to a certain two-dimensional theory on S^2. So to compute correlator of two 1/4 BPS circular Wilson loops we cannot localize the field theory straight to the two-matrix model [20], but we have to deal with an intermediate two-dimensional field theory on Σ.

2.4 Summary

We study supersymmetric Wilson loops on $\Sigma = S^2 \subset S^3$ given by (2.4). These Wilson loops are invariant under the SU(1|2) subgroup of the superconformal group, where $\mathrm{U}(1) = \mathrm{SO}(2)_S$ rotates (X_1, X_5) plane, and $\mathrm{SU}(2) = \mathrm{SU}(2)_B$ rotates (Φ_5, Φ_9, Φ_0). The Wilson loops are also invariant under the diagonal of $\mathrm{SO}(3)_S \times \mathrm{SO}(3)_A$, where $\mathrm{SO}(3)_S$ acts on (X_2, X_3, X_4) and $\mathrm{SO}(3)_A$ acts on (Φ_6, Φ_7, Φ_8), i.e. on scalar fields appearing in the definition of Wilson loop.

We choose Hermitian generator Q, generated by the conformal Killing spinor ε, as in (2.29). The spinor ε is invariant under the diagonal subgroup of $\mathrm{SO}(3)_S \times \mathrm{SO}(3)_A$ by (2.34) and the diagonal subgroup of $\mathrm{SO}(2)_S \times \mathrm{SO}(2)_B$ by (2.32), where the $\mathrm{SO}(2)_B \subset \mathrm{SO}(3)_B$ acts on (Φ_5, Φ_0)-plane.

3 Localization

3.1 Introduction

We want to show that the expectation value of the Wilson loops (2.4) on $\Sigma = S^2$ in four-dimensional $\mathcal{N} = 4$ Yang-Mills can be computed in a certain two-dimensional theory localized to Σ. The fermionic symmetry Q (2.29) is BRST-like generator of equivariantly

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cohomological field theory, thanks to the fact that Q squares to global unitary transformation and gauge transformation. This claim is true off-shell after adding to the theory the necessary auxiliary fields. The operator Q^2 is the off-shell symmetry of the action and of the Wilson loop observable that we study. By the well-known arguments, see e.g. [24, 25] for a general review and [8] for the technical details on applying localization to the 1/2-BPS supersymmetric circular Wilson loops, the theory localizes to the supersymmetric configurations $Q\Psi = 0$, where Ψ denotes fermionic fields of the theory. One can explain localization by deforming the action of the theory by Q-exact term $S_{YM} \to S(t) = S_{YM} + tQV$ with $V = (\Psi, \overline{Q\Psi})$ and sending t to infinity. Since the bosonic part of the deformed action is $S_{YM}^{bos} + t|Q\Psi|^2$, at the $t = +\infty$ limit the term $t|Q\Psi|^2$ dominates. So, at the $t = +\infty$ limit, in the path integral we shall integrate only over configurations solving $Q\Psi = 0$ with the measure coming from the one-loop determinant. On the other hand, the partition function and the expectation value of observables do not depend on the t-deformation. Indeed, let the partition function be $Z(t) = \int e^{S(t)}$. Then, if $S(t)$ is Q-closed and $\partial_t S(t)$ is Q-exact, we can integrate by parts in $\partial_t Z(t)$. If the space of fields is essentially compact (all fields decrease sufficiently fast at infinity) the boundary term vanishes and we obtain $\partial_t Z(t) = 0$.

In the present situation we use $V = (\Psi, \overline{Q\Psi})$. We recall, that Ψ is fermion of $\mathcal{N} = 4$ super Yang-Mills obtained by dimensional reduction of chiral sixteen-component spinor transforming in the S^+ irreducible spin representation of Spin(10). The other irreducible spin representation S^- of Spin(10) is dual to S^+. Therefore, there is a natural pairing $S^+ \otimes S^- \to \mathbb{C}$, so that if $\psi \in S^+$ and $\chi \in S^-$ are spinors of the opposite chirality, the bilinear (χ, ψ) is Spin(10)-invariant. (In components (χ, ψ) should be read as $\sum_{\alpha=1}^{16} \chi_\alpha \psi_\alpha$ with no complex conjugation operations).

In the Euclidean signature the representations S^+ and S^- of $\mathrm{Spin}(10, \mathbb{R})$ are unitary and complex conjugate to each other. Hence, if $\chi \in S^+$ and $\psi \in S^+$ are spinors of the same chirality, the bilinear $(\bar{\chi}, \psi) = \sum_{\alpha=1}^{16} \overline{\chi}_\alpha \psi_\alpha$ is invariant under $\mathrm{Spin}(10, \mathbb{R})$. So, because of our choice of Hermitian Q (2.29) and because Q squares to unitary global transformation in $\mathrm{SO}(2)_S \times \mathrm{SO}(2)_B$, the deformation term $V = (\overline{\Psi}, Q\Psi)$ is Q^2-invariant and can be used for the localization.

The localization from the four-dimensional $\mathcal{N} = 4$ SYM on S^4 to a two-dimensional theory on $\Sigma \subset S^4$ is done essentially in two steps. It is convenient to represent the S^4 as an $S^2 \times S^1$ warped fibration over an interval I as in (2.3).

1. We argue that $Q\Psi = 0$ implies the invariance under the $\mathrm{SO}(2)_S$ rotation, which acts by translation along the S^1 fiber: $\tau \to \tau + \mathrm{const}$. Hence, the $\mathcal{N} = 4$ SYM on S^4, for our purposes, reduces to a three-dimensional theory on the manifold D^3 represented as a warped S^2 fibration over I.

   The resulting three-dimensional theory on D^3 can be interpreted as a deformed version of certain cohomological field theory for extended Bogomolny equations which were introduced by Kapustin and Witten in [2]. The interesting observables, i.e. the Wilson loops (2.4), are located at the boundary $\Sigma = \partial D^3$.

2. We show that physical action S_{YM} for the reduced three-dimensional theory on D^3 can be represented as a total derivative term modulo the equations $Q\Psi = 0$. There-

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fore, at the supersymmetric configurations $Q\Psi = 0$, the value of the reduced physical action S_{YM} is determined by the boundary conditions at the Σ. The integral over the configurations satisfying $Q\Psi = 0$ reduces to an integral over the boundary conditions on Σ.

This is essentially the way how the two-dimensional theory appears. It turns out that the resulting two-dimensional theory is closely related to topological Higgs-Yang-Mills (or Hitchin-Yang-Mills) theory on Σ studied in [17–19].

It is possible to introduce point-like singularities to solutions of the reduced equations for the three-dimension theory on D^3 similar to the constructions in work [2] by Kapustin and Witten. Such point singularities in the reduced three-dimensional theory on D^3 are uplifted to the codimension one singularities in the four-dimensional theory on S^4 and they are precisely the conformal supersymmetric 't Hooft operators as explained in [2, 26].

In this paper we do not consider the equations with singularities and 't Hooft operators. We study correlation functions only for the initially introduced Wilson operators. However, we remark that our construction in principle might be used to study correlation functions of a set of Wilson operators on S^2 and a set of 't Hooft operators, which are located on the $\mathrm{U}(1)$ orbits linking with the S^2.

Also, it is possible to introduce codimension two singularity on the boundary of D^3. Such singularity corresponds to the disorder surface operator [27] inserted on the two-sphere $S^2 = \partial B^3$. This situation would be similar to the one studied in [28]. Again, in this work we aim to compute the expectation value only of Wilson loop operators on S^2 in absence of any extra singularities. We require all fields to be smooth and finite in the path integral.

Now we give more details on the geometry of our setup. The metric on D^3 in the first step above is

$$ds^2 = r^2(d\xi^2 + \sin\xi^2 d\Omega_2^2) \quad \text{where} \quad 0 \leq \xi \leq \pi/2, \tag{3.1}$$

which is the round metric on a half of a three-sphere. Topologically $D^3 = (S^4 \setminus \Sigma)/\mathrm{SO}(2)_S$ is a solid three-dimensional ball. Under the S^1-fiber-forgetful projection $\pi : S^4 \to D^3$ the $\Sigma \subset S^4$, where the interesting Wilson loops live, is mapped to the boundary of D^3. This boundary is located at $\xi = \pi/2$. The S^1 fiber shrinks to zero at Σ.

3.2 The supersymmetry equations

3.2.1 Choice of coordinates

To make the $\mathrm{SO}(2)_S \times \mathrm{SO}(3)_S$ isometry group of the S^4 explicit, we represent the metric as a warped product of the three-dimensional ball D^3 and the circle S^1. On the D^3 we introduce the $\mathbb{R}^3$ stereographic projection coordinates $\tilde{x}_i$, and we keep the notation τ for the coordinate on S^1. The metric in coordinates $\tilde{x}_i, \tau$ is then

$$ds^2(S^4) = ds^2(D^3 \times_w S^1) = \frac{d\tilde{x}_i d\tilde{x}_i}{(1 + \frac{\tilde{x}^2}{4r^2})^2} + r^2 \frac{(1 - \frac{\tilde{x}^2}{4r^2})^2}{(1 + \frac{\tilde{x}^2}{4r^2})^2} d\tau^2 \quad i = 2, 3, 4 \tag{3.2}$$

One can write $S^4 = D^3 \times_w S^1$ where $w(\tilde{x})$ is the warp function $w(\tilde{x}) = r^2 \cos^2\xi = r^2(1 - \tilde{x}^2/(4r^2))^2/(1 + \tilde{x}^2/(4r^2))^2$. The metric on D^3 is the standard round metric on the three-dimensional sphere.

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The $\mathbb{R}^4$ stereographic coordinates x_i $(i = 1 \dots 4)$ and the $D^3 \times_w S^1$ coordinates $(\tau, \tilde{x}_i)$ $(i = 2, 3, 4)$ are related in a simple way. At the slice $x_1 = \tau = 0$ we have $x_i = \tilde{x}_i$ $(i = 2, 3, 4)$. The generic relation between x_i and $(\tau, \tilde{x}_i)$ is the following. From (2.1) on gets

$$x_i = \frac{2}{1 + X_5/r} X_i, \quad i = 1 \dots 4 \tag{3.3}$$

The $\mathrm{SO}(2)_S$ orbits are labelled by (X_2, X_3, X_4). The τ is the coordinate along $\mathrm{SO}(2)_S$ orbits, and we have

$$\begin{aligned} X_1 &= R \sin\theta \sin\tau \\ X_5 &= R \sin\theta \cos\tau. \end{aligned} \tag{3.4}$$

So, from (3.3) we get the $\mathrm{SO}(2)_s$ orbits in the $\mathbb{R}^4$ coordinates x_i, and hence, the transformation from coordinates $(\tau, \tilde{x}_i)$ to coordinates (x_1, x_i)

$$\begin{aligned} x_i(\tau, \tilde{x}_i) &= \tilde{x}_i \frac{1 + \sin\theta}{1 + \sin\theta \cos\tau}, \qquad i = 2, \dots, 4 \\ x_1(\tau, \tilde{x}_i) &= R \frac{2 \sin\theta \sin\tau}{1 + \sin\theta \cos\tau} \end{aligned} \tag{3.5}$$

where

$$\sin\theta = \frac{1 - \frac{\tilde{x}^2}{4r^2}}{1 + \frac{\tilde{x}^2}{4r^2}}. \tag{3.6}$$

These $\mathrm{SO}(2)_S$ orbits are the usual circles in the $\mathbb{R}^4$ coordinates x_i. These circles link with the two-sphere $S^2 = \{x_i | x_2^2 + x_3^2 + x_4^2 = 4r^2, x_1 = 0\}$ and are labeled by points on $D^3 = \{\tilde{x}_i, \tilde{x}^2 < 4r^2\}$. For each $\tilde{x}_i$ the corresponding circle is located in the two-plane spanned by the vector $(1, 0, 0, 0)$ and the vector $(0, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4)$. The distance from the origin to the nearest point of the circle is $|\tilde{x}|$, the distance to the furthest point is $\frac{4r^2}{\tilde{x}^2}$, while its center has coordinates $x_1 = 0, x_i = \tilde{x}_i(\frac{1}{2} + \frac{r^2}{\tilde{x}^2})$, and its diameter is $(4r^2 - \tilde{x}^2)/|\tilde{x}|$.

3.2.2 Weyl invariance

The supersymmetry equations $Q\Psi = 0$ are Weyl invariant. Indeed, given that under Weyl transformation of metric $g_{\mu\nu} \to e^{2\Omega} g_{\mu\nu}$ the bosonic fields transform as $A_\mu \to A_\mu, \Phi_A \to \Phi_A e^{-\Omega}, K_i \to K_i e^{-2\Omega}$ and the conformal Killing spinor transform as $\varepsilon \to e^{\frac{1}{2}\Omega}\varepsilon$, one gets that $Q_\varepsilon \Psi \to e^{-\frac{3}{2}\Omega} Q_\varepsilon \Psi$ which is a correct dimension for fermions. Therefore, the localization procedure is essentially the same for two theories defined with respect to the metrics related by a smooth Weyl transformation. (We ask transformation to be smooth so that no conformal anomaly related to the infinity can appear.)

In the coordinates $(\tau, \tilde{x}_i)$ the $\mathrm{SO}(2)_S \times \mathrm{SO}(3)_S$ symmetry is simply represented, so we shall start from the metric in the form (3.2). Since $\tilde{x}$ is bounded $|\tilde{x}| < 2r$, the scale factor $(1 + \tilde{x}^2/(4r^2))$ is non-zero and smooth everywhere over the D^3. It is convenient to get rid of this factor in the equations by making Weyl transformation of the metric $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = (1 + \tilde{x}^2/(4r^2))^2 g_{\mu\nu}$. Under such rescaling the round spherical metric on D^3 becomes a flat metric. We refer to the D^3 equipped with the flat metric as the flat ball B^3.

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So we study the equations $Q\Psi = 0$ on the space $B^3 \times_{\tilde{w}} S^1$. The metric on this space is

$$ds^2(B^3 \times_{\tilde{w}} S^1) = d\tilde{x}_i d\tilde{x}_i + r^2 \left(1 - \frac{\tilde{x}^2}{4r^2}\right)^2 d\tau^2 \quad \text{where} \quad \tilde{x}_i^2 \leq 4r^2. \tag{3.7}$$

We still call the coordinates on $B^3 \subset \mathbb{R}^3$ as $\tilde{x}_i$, and the coordinate on S^1 as τ, with the new warp factor being

$$\tilde{w}(x) = r\left(1 - \frac{\tilde{x}^2}{4r^2}\right). \tag{3.8}$$

For fermions we use the following vielbein as an orthonormal basis in the cotangent bundle

$$(e_i) = (\tilde{w}(x)d\tau, d\tilde{x}_i), \quad i = 1 \ldots 4. \tag{3.9}$$

3.2.3 The diagonal $\mathrm{U}(1) \subset \mathrm{SO}(2)_S \times \mathrm{SO}(2)_B$ invariance

At $\tau = 0$ the coordinates $\tilde{x}_i$ and corresponding vielbein coincide with coordinates x_i. We take the conformal Killing spinor ε on the B^3

$$\varepsilon(\tilde{x}, \tau = 0) = \hat{\varepsilon}_s + \tilde{x}^i \Gamma_i \hat{\varepsilon}_c \tag{3.10}$$

to write the supersymmetry equations at $\tau = 0$. Then, of course, using the $\mathrm{U}(1) \subset \mathrm{SO}(2)_S \times \mathrm{SO}(2)_B$ invariance one can continue the equations to an arbitrary τ. The Killing spinor ε on the whole space $B^3 \times_{\tilde{w}} S^1$ is invariant under the diagonal $\mathrm{U}(1) \subset \mathrm{SO}(2)_S \times \mathrm{SO}(2)_B$, i.e. under simultaneous rotation of the (X_5, X_1) and the (Φ_5, Φ_0) planes. A convenient change of variables for this diagonal $\mathrm{U}(1)$ symmetry is to define the pair of "twisted" scalar fields[5]

$$\begin{aligned} \Phi_T &= \cos\tau \Phi_0 - \sin\tau \Phi_5 \\ \Phi_R &= \sin\tau \Phi_0 + \cos\tau \Phi_5. \end{aligned} \tag{3.11}$$

3.2.4 Conformal Killing spinor

The conformal Killing spinor ε satisfies equation

$$\nabla_\mu \varepsilon = \Gamma_\mu \tilde{\varepsilon}, \tag{3.12}$$

and the off-shell transformation of fermions is given by

$$Q\Psi = \frac{1}{2} F_{MN} \Gamma^{MN} \varepsilon - 2\Phi_A \tilde{\Gamma}^A \tilde{\varepsilon} + i\nu_i K_i. \tag{3.13}$$

The ε in components has explicit form

$$\varepsilon = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{2r} \begin{pmatrix} 0 \\ i \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{pmatrix} \tag{3.14}$$

[5] We remark that we are not making topological twisting of the theory. All computations are done for the usual physical $\mathcal{N} = 4$ SYM. We change variables for a convenience but we do not change the Lagrangian and the observables.

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and $\tilde{\varepsilon}$ is

$$\tilde{\varepsilon} = \frac{1}{2r} \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.15}$$

3.2.5 Off-shell closure

We also need 7 auxiliary spinors ν_i to write down the off-shell closure of the supersymmetry transformations (3.13) like in [29, 30]. It is easy to find such set of ν_i because only top 8 components of ε are non-zero. More invariantly, ε satisfies

$$(\Gamma^1 + i\Gamma^0)\varepsilon = 0, \tag{3.16}$$

i.e. it is chiral with respect to the SO(8) acting on the vector indices $2, \ldots, 9$. Then, as a set of 7 spinors ν_i, one can choose

$$\nu_i = \Gamma_{9i}\varepsilon \quad \text{for} \quad i = 2, \ldots 8. \tag{3.17}$$

Such spinors ν_i are again SO(8) chiral and have only 8 top components being non-zero.

3.2.6 Splitting of the supersymmetry equations: top and bottom

To compute the components of $Q\Psi$ it is convenient to split sixteen component spinors in S^+ into two eight-component spinors on which $\Gamma^1\Gamma^0$ acts by $+i$ or $-i$ respectively. (We will use interchangeably space-time index 1 or τ to denote direction along the coordinate τ in (3.7).) With our choice of gamma-matrices (2.15), if the eight-component spinors are called Ψ^t and Ψ^b, we have

$$\Psi = \begin{pmatrix} \Psi^t \\ \Psi^b \end{pmatrix}, \tag{3.18}$$

and

$$\varepsilon = \begin{pmatrix} \varepsilon^t \\ 0 \end{pmatrix} \quad \tilde{\varepsilon} = \begin{pmatrix} 0 \\ \tilde{\varepsilon}^b \end{pmatrix}. \tag{3.19}$$

Next, we represent the eight-component spinors Ψ^t and Ψ^b by the octonions $\mathbb{O}$. A spinor

$$\Psi^t = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \cdots \\ \Psi_8 \end{pmatrix} \tag{3.20}$$

will be written as

$$\Psi^t = \Psi_1^t e_9 + \Psi_2^t e_2 + \cdots + \Psi_8^t e_8, \tag{3.21}$$

where $e_9, e_2, \ldots, e_8$ are the basis elements of $\mathbb{O}$, see explanation after (2.15). Similarly,

$$\Psi^b = \Psi_1^b \tilde{e}_9 + \Psi_2^b \tilde{e}_2 + \cdots + \Psi_8^b \tilde{e}_8, \tag{3.22}$$

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where $\tilde{e}_9, \tilde{e}_2, \ldots, \tilde{e}_8$ are the basis elements in the second copy of $\mathbb{O}$ representing the bottom components of Ψ. In these notations

$$\varepsilon = e_9 - \frac{i}{2r}\tilde{x}_i e_{i+4} \tag{3.23}$$

and

$$\tilde{\varepsilon} = \frac{i}{2r}\tilde{e}_5. \tag{3.24}$$

3.3 Bottom equations and the circle invariance

Now we analyze the bottom components of the equations (3.13).

Taking into account the chiral structure of gamma-matrices (2.15) and spinors $\varepsilon, \tilde{\varepsilon}$ as in (3.19), we get

$$\begin{aligned} Q\Psi^b = \sum_{m=\hat{2}\ldots\hat{9}} (F_{\hat{0}\hat{m}}\Gamma^{\hat{0}\hat{m}} + F_{\hat{1}\hat{m}}\Gamma^{\hat{1}\hat{m}})\varepsilon - 2\Phi_0\tilde{\Gamma}^0\tilde{\varepsilon} = \\ -(iF_{\hat{0}\hat{m}} + F_{\hat{1}\hat{m}})E_{\hat{m}}\varepsilon + 2i\Phi_0\tilde{\varepsilon} = -(iF_{\hat{0}\hat{m}} + F_{\hat{1}\hat{m}})e_{\hat{m}}\left(e_9 - \frac{i}{2r}\tilde{x}_i e_{i+4}\right) + 2i\Phi_0\frac{i}{2r}\tilde{e}_5. \end{aligned} \tag{3.25}$$

We use indices with hat to denote vector components with respect to the orthonormal vielbein (3.9), e.g. $F_{\hat{1}\hat{m}} = \tilde{w}(x)^{-1}F_{\tau\hat{m}}$. For brevity we consider equations along the radial line $(\tau, \tilde{x}) = (0, \tilde{x}_2, 0, 0)$, and then, using the $\mathrm{SO}(2)_S$ and the $\mathrm{SO}(3)_S$ symmetry we can write the equations on the the whole space $B^3 \times_{\tilde{w}} S^1$. At $\tilde{x}_2 < 2r$, the six equations corresponding to the components $\hat{m} = 3, 4, 6, 7, 8, 9$ are linearly independent and imply

$$iF_{\hat{0}\hat{m}} + F_{\hat{1}\hat{m}} = 0 \quad \text{for} \quad \hat{m} = 3, 4, 6, 7, 8, 9. \tag{3.26}$$

We can make diagonal transformation in $\mathrm{SO}(2)_S \times \mathrm{SO}(2)_B$ like in (3.11) to transform (3.26) to an arbitrary τ

$$iF_{\hat{m}T} + \frac{1}{r(1 - \frac{\tilde{x}^2}{4r^2})}F_{\hat{m}\tau} = 0 \quad \hat{m} = 3, 4, 6, 7, 8, 9 \tag{3.27}$$

where we replaced index $\hat{1}$ by τ using the scaling function $\tilde{w}(\tilde{x})$, and where $F_{T\hat{m}} = [\Phi_T, \nabla_{\hat{m}}] = -\nabla_{\hat{m}}\Phi_T$. Next we consider the remaining two components in (3.25) for the basis elements e_2 and e_5. At $\tau = 0$ we have

$$\begin{aligned} iF_{\hat{0}\hat{2}} + F_{\hat{1}\hat{2}} - \frac{i}{2r}\tilde{x}_2(iF_{\hat{0}\hat{5}} + F_{\hat{1}\hat{5}}) = 0 \quad (\text{on } e_2) \\ iF_{\hat{0}\hat{5}} + F_{\hat{1}\hat{5}} + \frac{i}{2r}\tilde{x}_2(iF_{\hat{0}\hat{2}} + F_{\hat{1}\hat{2}}) - \frac{1}{r}\Phi_0 = 0 \quad (\text{on } e_5). \end{aligned} \tag{3.28}$$

Again we can make τ arbitrary by making the diagonal transformation $\mathrm{U}(1) \in \mathrm{SO}(2)_S \times \mathrm{SO}(2)_B$

$$\begin{aligned} (iF_{T\hat{2}} + \tilde{w}^{-1}F_{\tau\hat{2}}) - \frac{i}{2r}\tilde{x}_2(iF_{TR} + \tilde{w}^{-1}(F_{\tau R} - \Phi_T)) = 0 \\ (iF_{TR} + \tilde{w}^{-1}(F_{\tau R} - \Phi_T)) + \frac{i}{2r}\tilde{x}_2(iF_{T2} + \tilde{w}^{-1}F_{\tau 2}) + \frac{1}{r}\Phi_T = 0. \end{aligned} \tag{3.29}$$

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The first line plus the second multiplied by $i\tilde{x}_2/2r$ is

$$i\left(1-\frac{\tilde{x}^2}{4r^2}\right)F_{T2}+\frac{1}{r}F_{\tau 2}+i\frac{\tilde{x}_2}{2r^2}\Phi_T=0. \tag{3.30}$$

Introducing a rescaled field

$$\tilde{\Phi}_T=r\left(1-\frac{\tilde{x}^2}{4r^2}\right)\Phi_T, \tag{3.31}$$

the equation (3.31) is rewritten as

$$i\nabla_2\tilde{\Phi}_T+F_{2\tau}=0. \tag{3.32}$$

The remaining equation from (3.29) is then

$$i\left(1-\frac{\tilde{x}^2}{4r^2}\right)F_{TR}+\frac{1}{r}F_{\tau R}=0. \tag{3.33}$$

We can summarize the 8 equations (3.27), (3.31), (3.33) resulting from $Q\Psi^b=0$:

$$[\nabla_{\hat{m}},\nabla_\tau+i\tilde{\Phi}_T]=0 \quad \text{for} \quad \hat{m}=2,3,4,R,6,7,8,9. \tag{3.34}$$

One can introduce complexified connection $\nabla^{\mathbb{C}}_\tau=\nabla_\tau+i\tilde{\Phi}_T$ and interpret the equations (3.34), as vanishing of the electric field (the three equations $F^{\mathbb{C}}_{\tau\hat{i}}=0$, $i=2,3,4$) and covariant time independence of the remaining five scalars ($\nabla^{\mathbb{C}}_\tau\Phi_{R,6,7,8,9}=0$) in the conventions where τ is the time coordinate.

Since Q^2 generates translations along τ, we can interpret Q^2 as the Hamiltonian. The bottom equations (3.34) say that momenta of all fields vanish and that the theory localizes to some three-dimensional theory. This three-dimensional theory is defined on a three-dimensional ball B^3 whose boundary is the two-sphere Σ where interesting Wilson loop operators are located.

The supersymmetric configurations in this three-dimensional theory are determined by the top eight components of the equations $Q\Psi=0$, which we shall analyze now.

3.4 Top equations and the three-dimensional theory

Writing the top eight components of $Q\Psi$ explicitly we get

$$\begin{aligned}Q\Psi^t&=F_{\hat{0}\hat{1}}\Gamma^{\hat{0}\hat{1}}\varepsilon^t+\sum_{2\le m<n\le 9}F_{mn}\Gamma^{mn}\varepsilon^t-2\tilde{E}_A\Phi_A\tilde{\varepsilon}^b+\sum_{1\le I\le 8}iK_I\Gamma^{9I}\varepsilon^t=\\&=-iF_{\hat{0}\hat{1}}\varepsilon^t+(F_{9I}+iK_I)E_I\varepsilon^t-\sum_{2\le I<J\le 8}F_{IJ}E_IE_J\varepsilon^t-2\tilde{E}_A\Phi_A\tilde{\varepsilon}^b.\end{aligned} \tag{3.35}$$

In the following we use indices $I,J=2,\dots,8$ and $i,j,k,p,q=2,\dots,4$. In this section we put $r=1/2$ to avoid extra factors. We do not write tilde over x understanding that x^i ($i=2,3,4$) are the coordinates on the flat unit ball $B^3\subset\mathbb{R}^3$. The antisymmetric symbol ϵ_{ijk} is defined as $\epsilon_{234}=1$. The following multiplication table of octonions is helpful

$$\begin{aligned}&e_ie_j=\epsilon_{ijk}e_k-\delta_{ij}e_9\\&e_{i+4}e_i=e_5 \qquad e_ie_5=e_{i+4} \qquad e_5e_{i+4}=e_i\\&e_ke_{i+4}=-\epsilon_{kij}e_{j+4}-\delta_{ik}e_5 \quad e_{i+4}e_{j+4}=-\epsilon_{ijk}e_k-\delta_{ij}e_9 \quad e_{j+4}e_k=\delta_{jk}e_5-\epsilon_{jki}e_{i+4}\end{aligned} \tag{3.36}$$

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After some algebra we get the first term

$$Q\Psi^{t(1)} = -iF_{\hat{0}\hat{1}}\varepsilon = -iF_{\hat{0}\hat{1}}(e_9 - ix_j e_{j+4}), \tag{3.37}$$

the second term

$$\begin{aligned} Q\Psi^{t(2)} = (F_{9I} + iK_I)E_I\varepsilon = (F_{9I} + iK_I)e_I(e_9 - ix_j e_{j+4}) = \\ (F_{9i} + iK_i)(e_i + ix^j\epsilon_{ijk}e_{k+4} + ix^j\delta_{ij}e_5) + \\ (F_{95} + iK_5)(e_5 - ix_j e_j) + \\ (F_{9\,i+4} + iK_{i+4})(e_{i+4} + ix^j\epsilon_{ijk}e_k + ix^j\delta_{ij}e_9), \end{aligned} \tag{3.38}$$

the third term

$$\begin{aligned} Q\Psi^{t(3)} = -F_{I<J}E_IE_J\varepsilon = \\ = \left[-\frac{1}{2}(F_{ij} - F_{i+4\,j+4})\epsilon_{ijk}e_k + F_{i\,j+4}\epsilon_{ijk}e_{k+4} + F_{i\,i+4}e_5 - F_{5\,k+4}e_k - F_{k5}e_{k+4}\right] \\ + i\left[F_{ij}x_ie_{j+4} + \frac{1}{2}F_{ij}x_k\epsilon_{ijk}e_5\right. \\ + F_{i5}x_k\epsilon_{ikj}e_j - F_{i5}x_ie_9 - F_{i\,j+4}x_ie_j - F_{i\,j+4}x_je_i + F_{i\,i+4}x_ke_k + F_{i\,j+4}x_k\epsilon_{ijk}e_9 \\ \left. + F_{5,i+4}x_k\epsilon_{ikj}e_{j+4} - F_{5j+4}x_je_5 + F_{i+4\,j+4}x_ie_{j+4} - \frac{1}{2}F_{i+4j+4}\epsilon_{ijk}x_ke_5\right] \end{aligned} \tag{3.39}$$

and the fourth term

$$Q\Psi^{t(4)} = -2\tilde{E}_A\Phi_A\tilde{\varepsilon}^b = -2i(\Phi_9e_5 + \Phi_5e_9 + \Phi_{i+4}e_i). \tag{3.40}$$

Now we analyze the equations. We have eight complex, i.e. sixteen real, equations on eight real physical fields $A_{2,3,4}, \Phi_{R,6,7,8,9}$ and seven real auxiliary fields K_i. (The deformation term $t|Q\Psi|^2$ vanishes on the real integration contour if and only if both imaginary and complex part of $Q\Psi$ vanishes.) We shall see shortly that only 15 equations are independent. Seven auxiliary fields can be easily integrated out. Then we are left with eight equations. One of these eight equations gives real constraint on the complexified time connection:

$$[\nabla_\tau, \tilde{\Phi}_T] = 0. \tag{3.41}$$

This equation together with (3.34) completes our claim that the field configurations are all τ-invariant up to a gauge transformation.

What remains is the system of seven first order differential equations in three dimensional space on gauge field and five scalars. The equations are gauge invariant. Modulo gauge transformations, the system is elliptic in the interior of the three-dimensional ball B^3. The system is closely related to the extended three-dimensional Bogomolny equations studied in [26].

Now we shall give technical details on the equations. First we eliminate $\mathrm{Im}Q\Psi^t|e_9$ by adding to it $-x_i\mathrm{Re}Q\Psi|_{e_{i+4}}$

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$$\begin{aligned}\mathrm{Im}Q\Psi|e_9 - x_i \mathrm{Re}Q\Psi|_{e_{i+4}} &= -F_{\hat{0}\hat{1}} + F_{9\,i+4}x_i - F_{i5}x_i + F_{ij+4}x_k\epsilon_{ijk} - 2\Phi_5 \\ &\quad - (-F_{\hat{0}\hat{1}}x^2 + F_{9\,i+4}x_i - F_{i5}x_i + F_{i\,j+4}x_k\epsilon_{ijk}) = \\ &= -F_{\hat{0}\hat{1}}(1-x^2) - 2\Phi_5 = 2[\nabla_\tau \Phi_T] \end{aligned} \tag{3.42}$$

This is the real equation which completes the system of time-invariance equations (3.34).

Next we consider $\mathrm{Re}Q\Psi^t|_{e_9}$:

$$\mathrm{Re}Q\Psi^t|e_9 = -K_{i+4}x_i \tag{3.43}$$

This equation is one constraint on the auxiliary fields K_i. We are left with 14 more equations $\mathrm{Im}Q\Psi^t|_{e_I} = 0$ and $\mathrm{Re}Q\Psi^t|_{e_I} = 0$, $I = 2, \ldots, 8$. Using $\mathrm{Im}Q\Psi^t|_{e_I} = 0$ we shall solve for K_I in terms of the physical fields A and Φ, and we will see actually that the constraint (3.43) is automatically implied.

The seven equations $\mathrm{Im}Q\Psi^t|_{e_I} = 0$ imply

$$\begin{aligned} K_k &= F_{95}x_k - F_{9\,i+4}\epsilon_{ijk}x_j - F_{i5}x_j\epsilon_{ijk} + F_{i\,k+4}x_i + F_{k\,i+4}x_i - F_{i\,i+4}x_k + 2\Phi_{k+4} \\ K_5 &= -F_{9i}x_i - \frac{1}{2}F_{ij}x_k\epsilon_{ijk} + F_{5\,j+4}x_j + \frac{1}{2}F_{i+4\,j+4}x_k\epsilon_{ijk} + 2\Phi_9 \\ K_{k+4} &= -F_{9i}\epsilon_{ijk}x_j - F_{ik}x_i - F_{5\,i+4}x_j\epsilon_{ijk} - F_{i+4\,k+4}x_i. \end{aligned} \tag{3.44}$$

The seven components $\mathrm{Re}Q\Psi^t|_{e_I} = 0$ are

$$\begin{aligned} \mathrm{Re}Q\Psi^t|_{e_k} &= F_{9k} - \frac{1}{2}(F_{ij} - F_{i+4\,j+4})\epsilon_{ijk} - F_{5\,k+4} + K_5x_k - K_{i+4}x_j\epsilon_{ijk} \\ \mathrm{Re}Q\Psi^t|_{e_5} &= F_{95} + F_{i\,i+4} - K_ix_i \\ \mathrm{Re}Q\Psi^t|_{e_{k+4}} &= F_{9\,k+4} + F_{i\,j+4}\epsilon_{ijk} - F_{k5} + 2\Phi_5(1-x^2)^{-1}x_k - K_ix_j\epsilon_{ijk}. \end{aligned} \tag{3.45}$$

After plugging in (3.45) the expressions for K_I (3.44) we get

$$\begin{aligned} \mathrm{Re}Q\Psi^t|_{e_k} &= F_{9k}(1-x^2) - \frac{1}{2}F_{ij}\epsilon_{ijk}(1+x^2) + \frac{1}{2}F_{i+4\,j+4}\epsilon_{ijp}(\delta_{pk} - x^2\delta_{pk} + 2x_px_k) \\ &\quad - F_{5\,j+4}(\delta_{jk} + x^2\delta_{jk} - 2x_jx_k) + 2\Phi_9x_k \\ \mathrm{Re}Q\Psi^t|_{e_5} &= F_{95}(1-x^2) + F_{i\,j+4}(\delta_{ij} + \delta_{ij}x^2 - 2x_ix_j) - 2\Phi_{j+4}x_j \\ \mathrm{Re}Q\Psi^t|_{e_{k+4}} &= F_{9\,i+4}(\delta_{ik} + x_ix_k - x^2\delta_{ik}) - F_{i5}(\delta_{ik} - x_ix_k + x^2\delta_{ik}) + 2\Phi_5(1-x^2)^{-1}x_k \\ &\quad + F_{i\,j+4}(\epsilon_{ijk} - x_ix_p\epsilon_{jpk} - x_jx_p\epsilon_{ipk}) - 2\Phi_{i+4}\epsilon_{ijk}x_je_{k+4}. \end{aligned} \tag{3.46}$$

The above calculations are done at the slice $\tau = 0$. For an arbitrary τ the field Φ_5 should be replaced by Φ_R as in (3.11).

3.4.1 Simplification at the origin: extended Bogomolny equations

Let us analyze the equations $\mathrm{Re}Q\Psi^t|_{e_I} = 0$ using (3.46). At $x_i = 0$ the equations simplify to

$$-*(F - \Phi\wedge\Phi) - d_A\Phi_9 + [\Phi, \Phi_R] = 0 \tag{3.47}$$

$$*d_A\Phi - d_A\Phi_R - [\Phi, \Phi_9] = 0 \tag{3.48}$$

$$d_A*\Phi + [\Phi_9, \Phi_R] = 0. \tag{3.49}$$

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where we identified the three scalar fields Φ_{i+4} with the components of one-form Φ on $\mathbb{R}^3$, we set $\Phi = \Phi_{i+4}dx^i$, and $*$ is the Hodge operator on $\mathbb{R}^3$ equipped with the standard flat metric.

Let us combine the gauge field A and the one-form Φ into a complexified connection $A_{\mathbb{C}} = A + i\Phi$, and similarly combine the scalars Φ_R and Φ_9 into complexified scalar $\Phi_{\mathbb{C}} = \Phi_9 + i\Phi_R$.

Then the equations (3.47), (3.48) can be written as

$$-*\,\mathrm{Re}F_{\mathbb{C}} - \mathrm{Re}d_{A_{\mathbb{C}}}\Phi_{\mathbb{C}} = 0 \tag{3.50}$$

$$*\mathrm{Im}F_{\mathbb{C}} - \mathrm{Im}d_{A_{\mathbb{C}}}\Phi_{\mathbb{C}} = 0. \tag{3.51}$$

This pair of real equations can be combined into one complex equation

$$*\,\overline{F_{\mathbb{C}}} + d_{A_{\mathbb{C}}}\Phi_{\mathbb{C}} = 0. \tag{3.52}$$

The equation (3.52) was introduced by Kapustin and Witten in [2] and is called *extended Bogomolny equation*.

3.4.2 The three-dimensional equations in rescaled variables

Hence, we see that at the origin of $\mathbb{R}^3$, the equations (3.46) look exactly like the relatively familiar system of elliptic equations. Away from $x = 0$ the equations are deformed into something more complicated. We will try to make some simple rescaling of variables to convert the equations to more standard form.

For this purpose, we rescale the scalar fields and define $\tilde{\Phi}_j$, $j = 2, 3, 4$, by

$$\Phi_{i+4} = \tilde{\Phi}_j \left(\delta_{ij} + \frac{2x_i x_j}{1 - x^2}\right). \tag{3.53}$$

This change of variables is smooth in the interior of the ball B^3. In terms of $\tilde{\Phi}_i$ the first equation in (3.46) becomes

$$-\frac{1}{2}(1 + x^2)\epsilon_{ijk}(F_{ij} - [\tilde{\Phi}_i, \tilde{\Phi}_j]) - \nabla_k((1 - x^2)\Phi_9) + (1 + x^2)[\tilde{\Phi}_k, \Phi_R] = 0. \tag{3.54}$$

The second equation in (3.46) becomes

$$(1 - x^2)\epsilon_{ijk}\nabla_i\tilde{\Phi}_j - \nabla_k((1 - x^2)\Phi_R) - \frac{1 - x^2}{1 + x^2}\left((1 - x^2)\delta_{ik} + \frac{4x_i x_k}{1 - x^2}\right)[\tilde{\Phi}_i, \tilde{\Phi}_9] = 0. \tag{3.55}$$

Finally, the third equation in (3.46) becomes

$$(1 + x^2)\nabla_i\tilde{\Phi}_i + 2\frac{3 + x^2}{1 - x^2}x_i\tilde{\Phi}_i + (1 - x^2)[\Phi_9, \Phi_5] = 0. \tag{3.56}$$

3.4.3 The three-dimensional equations linearized

Let $\mathcal{M}$ denote the moduli space of smooth solutions to (3.54), (3.55), (3.56) with finite Yang-Mills action. In the localization computation we need to integrate over $\mathcal{M}$. Clearly, the zero configuration $A = \tilde{\Phi} = 0, \Phi_R = \Phi_9 = 0$ is a solution. Let us analyze the linearized

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problem near the zero configuration, in other words, let us find the fiber of the tangent space $T\mathcal{M}_0$. The linearized equations (3.54), (3.55), (3.56) are

$$(1+x^2)\,*_{\mathbb{R}^3} dA + d((1-x^2)\Phi_9) = 0 \tag{3.57}$$

$$(1-x^2)\,*_{\mathbb{R}^3} d\tilde{\Phi} - d((1-x^2)\Phi_R) = 0 \tag{3.58}$$

$$(1+x^2)d^*_{\mathbb{R}^3}\tilde{\Phi} + 2\frac{x^2+3}{1-x^2}(x,\tilde{\Phi}) = 0. \tag{3.59}$$

Here we by $*_{\mathbb{R}^3}$ we denoted the Hodge star operation with respect to the standard metric on $\mathbb{R}^3$. It is possible to absorb extra $(1 \pm x^2)$ factors in the Hodge star operation using a rescaled metric. We will use the metric

$$ds^2(S^3) = \frac{dx_i dx_i}{(1+x^2)^2}, \quad |x| < 1 \tag{3.60}$$

which is the metric on a half of the round S^3, and

$$ds^2(H_3) = \frac{dx_i dx_i}{(1-x^2)^2}, \quad |x| < 1 \tag{3.61}$$

which is a metric on hyperbolic space H^3 in Poincare coordinates. Then the first two equations in (3.57) turn into

$$*_{S^3} dA + d\tilde{\Phi}_9 = 0 \tag{3.62}$$

$$*_{H^3} d\tilde{\Phi} - d\tilde{\Phi}_R = 0, \tag{3.63}$$

where

$$\tilde{\Phi}_R = (1-x^2)\Phi_R \tag{3.64}$$

$$\tilde{\Phi}_9 = (1-x^2)\Phi_9. \tag{3.65}$$

The equation (3.62) implies that $\tilde{\Phi}_9$ is harmonic for the S^3 metric

$$\Delta_{S^3}\tilde{\Phi}_9 = 0, \tag{3.66}$$

and the equation (3.63) implies that $\tilde{\Phi}_5$ is harmonic for the H_3 metric

$$\Delta_{H^3}\tilde{\Phi}_R = 0. \tag{3.67}$$

We need to consider only such solutions that the fields Φ_R, Φ_9 are not singular at the boundary. (Singular solutions can be considered too, but they correspond to the disorder surface operator [28] inserted on the two-sphere $S^2 = \partial B^3$. In this work we aim to compute the expectation value of Wilson loop operators on S^2 in the absence of any surface operators. Hence we require Φ_R and Φ_9 fields to be finite at the S^2.) If Φ_R and Φ_9 fields are finite at $|x| = 1$, then $\tilde{\Phi}_R$ and $\tilde{\Phi}_9$ vanish there by (3.66), (3.67). Hence we have the Laplacian problem (3.66), (3.67) with Dirichlet boundary conditions

$$\tilde{\Phi}_R|_{\partial B^3} = \tilde{\Phi}_9|_{\partial B^3} = 0. \tag{3.68}$$

Since a harmonic function $Y(x)$ vanishing on the boundary must vanish (it can be shown integrating by parts $\int_B dY \wedge *dY = \int_{\partial B} Y \wedge *dY$), we conclude that there is no nontrivial finite solution for the fields Φ_R, Φ_9, so

$$\Phi_R = \Phi_9 = 0. \tag{3.69}$$

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Explicit solutions in spherical harmonics. One might worry that this argument fails for the H^3 because of the infinite boundary. However, the explicit solution of the Laplace equation on H^3 shows that all radial wave-functions, which are smooth in the interior of H^3, do not vanish at the boundary. In spherical coordinates, the H^3 metric is

$$ds^2 = \frac{d\xi^2 + \sin^2\xi d\Omega_2^2}{\cos^2\xi}, \tag{3.70}$$

where ξ is the radial coordinate $0 \leq \xi < \pi/2$ and $d\Omega_2^2$ is the standard metric on the unit two-sphere. Then

$$\Delta_{H^3} f = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j)f = \frac{\cos^3\xi}{\sin^2\xi}\partial_\xi\left(\frac{\sin^2\xi}{\cos\xi}\partial_\xi f\right) + \frac{\cos^2\xi}{\sin^2\xi}\Delta_{S^2} f. \tag{3.71}$$

If $f_s(\xi)$ is the radial wave-function for the angular momentum s on the S^2 then $\Delta_{S^2} f_s = -s(s+1)f_s$. So the equation (3.71) is a special case of the Laplace equation in the (p,q) polyspherical coordinates (see e.g. [31] p.499)

$$\frac{1}{\cos^p\xi\sin^q\xi}\frac{\partial}{\partial\xi}\left(\cos^p\xi\sin^q\xi\frac{\partial u}{\partial\xi}\right) - \left(\frac{r(r+p-1)}{\cos^2\xi} + \frac{s(s+q-1)}{\sin^2\xi} - l(l+p+q)\right)u = 0 \tag{3.72}$$

for $q=2, p=-1, r=0, l=0$. The solutions of (3.72) non-singular at $\xi=0$ are

$$u = \tan^s\xi F\left(\frac{s-l+r}{2}, \frac{s-l-r-p+1}{2}, s+\frac{q+1}{2}; -\tan^2\xi\right), \tag{3.73}$$

where $F(\alpha,\beta,\gamma;z)$ is the ${}_2F_1$ hypergeometric function. In our case we have

$$f_s(\xi) = \tan^s\xi F(s/2, s/2+1, s+3/2, -\tan^2\xi). \tag{3.74}$$

Using identity

$$F(\alpha,\beta,\gamma,z) = (1-z)^{-\alpha}F\left(\alpha,\gamma-\beta,\gamma;\frac{z}{z-1}\right) \tag{3.75}$$

we can rewrite (3.74) as

$$f_s(\xi) = \sin^s\xi F(s/2, s/2+1/2, s+3/2, \sin^2\xi). \tag{3.76}$$

The function $f_s(\xi)$ has asymptotic ξ^s at $\xi\to 0$ and a finite non-zero value at $\xi=\pi/2$:

$$\lim_{\xi\to\pi/2} f_s(\xi) = \frac{\Gamma(s+3/2)\Gamma(1)}{\Gamma(s/2+3/2)\Gamma(s/2+1)}. \tag{3.77}$$

This confirms our argument that there are no non-trivial solutions to the Laplace equation on H^3 with zero asymptotic at the boundary.

Now, given that Φ_R and Φ_9 vanish, the linearized equations (3.62), (3.63) turn into

$$dA = 0 \tag{3.78}$$

$$d\Phi = 0. \tag{3.79}$$

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That means that the complexified gauge connection $A_{\mathbb{C}} = A + i\Phi$ is flat. The third equation in (3.57) is effectively a partial gauge fixing condition on the imaginary part of $A_{\mathbb{C}}$. It is actually possible to rewrite this partial gauge fixing condition in terms of the d^* operator with respect to a rescaled metric. Namely, for the conformally flat metric on $\mathbb{R}^3$ of the form

$$g_{ij} = f(|x|)\delta_{ij} \tag{3.80}$$

the d_f^* operator acts on one-form $\tilde{\Phi}$ as

$$d_f^*\tilde{\Phi} = f^{-1}\left(\partial_i\tilde{\Phi}_i + \frac{1}{2}f^{-1}f'\tilde{\Phi}_i x_i/|x|\right), \tag{3.81}$$

where $f' = df(|x|)/dx$. Comparing (3.81) with (3.59) we get the scale factor

$$f(|x|) = \frac{(1+x^2)^2}{(1-x^2)^4}. \tag{3.82}$$

Hence, the partial gauge fixing equation (3.59) is rewritten as

$$d_f^*\tilde{\Phi} = 0. \tag{3.83}$$

Now we can find all solutions to the linearized problem as follows. From (3.79) we solve for $\tilde{\Phi}$ in terms of a scalar potential p

$$\tilde{\Phi} = dp. \tag{3.84}$$

The gauge fixing equation (3.83) implies then

$$d_f^* dp = 0, \tag{3.85}$$

i.e. that p is a harmonic function with respect to the metric (3.82). We can find explicitly the harmonic modes in spherical coordinates. The metric (3.80) is

$$ds^2 = \frac{d\xi^2 + \sin^2\xi d\Omega_2^2}{\cos^4\xi}, \tag{3.86}$$

so the Laplacian equation (3.85) on spherical mode $p_s(\xi)$ with angular momentum s is

$$\cot^2\xi\frac{\partial}{\partial\xi}\left(\tan^2\xi\frac{\partial p_s(\xi)}{\partial\xi}\right) - \frac{s(s+1)}{\sin^2\xi}p_s(\xi) = 0. \tag{3.87}$$

Again, this is the Laplacian equation in the (p,q) polyspherical coordinates (3.72) with $p=-2, q=2, r=0, l=0$. The solution regular at $\xi=0$ is

$$\begin{aligned} p_s(\xi) &= \tan^s\xi F(s/2, s/2+3/2, s+3/2, -\tan^2\xi) = \\ &= \sin^s\xi F(s/2, s/2, s+3/2, \sin^2\xi). \end{aligned} \tag{3.88}$$

The solution is finite at $\xi = \pi/2$ for any s, hence the components of $\tilde{\Phi}$ tangent to the boundary ∂B^3 are also finite. To find asymptotic of the normal component of $\tilde{\Phi}$ we need

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to know expansion of (3.88) at $\theta = \pi/2 - \xi$ at $\theta = 0$. For this purpose we rewrite (3.88) using identity on hypergeometric functions (see e.g. [32] p.160)

$$\begin{aligned} F(\alpha,\beta,\gamma,z) = &\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha,\beta,\alpha+\beta-\gamma+1,1-z) \\ &+ \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1,1-z). \end{aligned} \tag{3.89}$$

We get

$$\begin{aligned} p_s(\xi) = \sin^s(\xi)\Bigg(&\frac{\Gamma(s+3/2)\Gamma(3/2)}{\Gamma(s/2+3/2)^2}F(s/2,s/2,-1/2,\cos^2\xi) \\ &+ \frac{\Gamma(s+3/2)\Gamma(-3/2)}{\Gamma(s/2)^2}(\cos^2\xi)^{3/2}F(s/2+3/2,s/2+3/2,5/2,\cos^2\xi)\Bigg). \end{aligned} \tag{3.90}$$

Near $\theta = 0$ we obtain

$$p_s(\theta) = \cos^s\theta(A + B\sin^2\theta + C\sin^3\theta + \dots), \tag{3.91}$$

where A, B, C some constants. Therefore

$$\tilde{\Phi}_\theta = \frac{\partial p_s(\theta)}{\partial\theta} = (-As + B)\theta + O(\theta^2). \tag{3.92}$$

This means that the normal component of $\tilde{\Phi}$ at the boundary vanishes as the first power of θ or $(1 - x^2)$. Hence, the original scalars, related to $\tilde{\Phi}$ by (3.53), are all finite at the boundary S^2.

So all solutions of the linearized equations (3.57), (3.58), (3.59) modulo gauge transformations are parametrized by the scalar potential p (modulo zero modes of p), which is a harmonic function in the three-dimensional ball with respect to the metric (3.86). A harmonic functions p is uniquely defined by its boundary value on the S^2. Hence we see that that tangent space $T\mathcal{M}_0$ to the moduli space of solutions at the origin is isomorphic to the space of adjoint-valued scalar functions on the S^2 modulo zero modes.

3.4.4 Solution of non-abelian equations: complexified flat connections

Now we consider the full non-abelian equations (3.54), (3.55), (3.56). Looking back at our solution of the linearized problem (3.69), we shall suggest an ansatz $\Phi_R = \Phi_9 = 0$ for the exact solution. Then the remaining equations on the complexified connection $A_{\mathbb{C}} = A + i\tilde{\Phi}$ are

$$F_A - \tilde{\Phi}\wedge\tilde{\Phi} = 0 \tag{3.93}$$

$$d_A\tilde{\Phi} = 0 \tag{3.94}$$

$$d_A^{*_f}\tilde{\Phi} = 0, \tag{3.95}$$

which can be combined into the complexified flat curvature equation

$$F(A_{\mathbb{C}}) = 0 \tag{3.96}$$

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and a partial gauge-fixing equation using the metric (3.86)

$$d_A *_f \tilde{\Phi} = 0. \tag{3.97}$$

The first equation can be solved in terms of a scalar function $g_{\mathbb{C}} : B^3 \to G_{\mathbb{C}}$, which takes value in the complexified gauge group $G_{\mathbb{C}}$:

$$A_{\mathbb{C}} = g_{\mathbb{C}}^{-1} dg_{\mathbb{C}}. \tag{3.98}$$

The partial gauge-fixing condition can be complemented by a real gauge fixing $d^* A = 0$. That gives a non-linear analogue of the harmonic equation (3.85)

$$d_A *_f (g_{\mathbb{C}}^{-1} dg_{\mathbb{C}}) = 0. \tag{3.99}$$

The solutions of this second order differential equation are parameterized by the boundary value of $g_{\mathbb{C}}$. Hence, the tangent space of solutions to the full non-abelian equations constrained by $\Phi_R = \Phi_9 = 0$ coincides with the moduli space of the linearized problem.

We conclude, that the solutions of (3.99) represent completely moduli space $\mathcal{M}$ of smooth solutions of the supersymmetry equations (3.46) with finite action. Hence, the space of gauge orbits $\mathcal{M}/G_{\text{gauge}}$ can be parameterized by the boundary value of the $G_{\mathbb{C}}/G$-valued potential function $g_{\mathbb{C}}$.

Equivalently, we can parameterise $\mathcal{M}/G_{\text{gauge}}$ by the space of complex flat connections on Σ modulo the gauge transformations restricted on Σ

$$\{A_{\mathbb{C}}^{2d} | F_{A_{\mathbb{C}}} = 0\}. \tag{3.100}$$

Hence, the localization of the path integral of the four-dimensional $\mathcal{N} = 4$ SYM theory to the moduli space $\mathcal{M}/G_{\text{gauge}}$ can be represented by a path integral over the space of complex flat connections on the B^3 boundary S^2. The action of this two-dimensional theory is determined by evaluating the four-dimensional Yang-Mills functional on the field configurations representing $\mathcal{M}$.

We will show below that the $\mathcal{N} = 4$ Yang-Mill action S_{YM} restricted to the supersymmetric field configurations is a total derivative on B^3, hence it can be expressed in terms of a two-dimensional action on the boundary Σ.

We conclude that the outcome of the localization procedure is a two-dimensional path integral over the space of complex flat connections on Σ.

Now we will find the two-dimensional action S_{2d}. The measure of integration in the two-dimensional theory is then $\exp(-S_{2d})$ times the induced volume form from the four-dimensional theory on the moduli space $\mathcal{M}$.

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4 Two-dimensional theory

4.1 The physical action on the supersymmetric configurations

4.1.1 The physical action on $B^3 \times_{\tilde{w}} S^1$

The bosonic part of the $\mathcal{N} = 4$ Yang-Mills action on S^4 in coordinates (3.2) is[6]

$$S_{YM} = \frac{1}{2g_{YM}^2} \int_0^{2\pi} d\tau \int_{|x|<1} d^3x \sqrt{g} \Big(\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \Phi_A D^\mu \Phi_A + \frac{1}{2}[\Phi_A, \Phi_B]^2 + \frac{R}{6} \Phi_A^2 + K^2 \Big). \tag{4.2}$$

Here R denotes the scalar curvature, which for S^4 of radius 1/2 has value $R = 12/(1/2)^2 = 48$. First we make Weyl transformation and get the physical action on the space $B^3 \times_{\tilde{w}} S^1$ with the metric (3.7)

$$g_{\mu\nu}[S^4] = e^{2\Omega} g[\mathbb{R}^3 \times_{\tilde{w}} S^1] \tag{4.3}$$

$$\Phi_A[S^4] = e^{-\Omega} \Phi_A[\mathbb{R}^3 \times_{\tilde{w}} S^1] \tag{4.4}$$

$$K_I[S^4] = e^{-2\Omega} K_I[\mathbb{R}^3 \times_{\tilde{w}} S^1] \tag{4.5}$$

where

$$e^{2\Omega} = (1 + x^2)^{-2}. \tag{4.6}$$

In terms of the fields on $\mathbb{R}^3 \times_{\tilde{w}} S^1$ the bosonic action is

$$S_{YM} = \frac{1}{2g_{YM}^2} \int_0^{2\pi} d\tau \int_{|x|<1} d^3x \left(\frac{1}{2}(1 - x^2) \times \right.$$
$$\times \left(\frac{1}{2} F_{ij}^2 + g^{\tau\tau} F_{\tau i}^2 + g^{\tau\tau} (D_\tau \Phi_A)^2 + (D_i \Phi_A)^2 + \frac{2}{(1 - x^2)} \Phi_A^2 + \frac{1}{2}[\Phi_A \Phi_B]^2 + K^2 \right) \tag{4.7}$$
$$\left. + D_i \left(\frac{1 - x^2}{1 + x^2} x_i \Phi_A^2 \right) \right)$$

The last term is the total derivative which vanishes because the factor $(1 - x^2)$ vanishes at the integration boundary $|x| = 1$. The action on $\mathbb{R}^3 \times_{\tilde{w}} S^1$ can be also written starting from (4.2) and substituting the metric (3.7). The scalar curvature on $\mathbb{R}^3 \times_{\tilde{w}} S^1$ can be computed easily using a general formula for the scalar curvature on a warped product of two manifold $M \times_f N$, see e.g. [33]. If g_M and g_N are the metrics on M and N, and if $g_M \oplus f^2 g_N$ is the metric on $M \times_f N$, then

$$R_{M \times_f N} u = -\frac{4n}{n+1} \Delta_M u + R_M u + R_N u^{\frac{n-3}{n+1}}$$
$$\text{where} \quad n = \dim N, \quad u = f^{\frac{n+1}{2}}, \quad \Delta_M \text{ is Laplacian on M.} \tag{4.8}$$

[6] In all expressions for the action functionals below we do not explicitly write Lie algebra indices and the contractions of them by an invariant Killing form $\langle , \rangle$ on the Lie algebra (it exists uniquely up to an overall rescaling) but, of course, that is implicitly assumed. A pedantical reader might wish to substitute

$$\frac{1}{4g_{YM}^2} \int \sqrt{g} d^n x F_{\mu\nu} F^{\mu\nu} \mapsto \frac{1}{4g_{YM}^2} \int \sqrt{g} d^n x F_{\mu\nu}^a F_a^{\mu\nu}, \tag{4.1}$$

where a, b are the Lie algebra indices in an orthogonal basis, e.g. $F = F^a T_a$ where T^a are generators of the Lie algebra. For the SU(N) gauge group the conventional choice is such that $\mathrm{tr}_F \, T_a T_b = -\frac{1}{2} \delta_{ab}$.

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In the case $\mathbb{R}^3 \times_{\tilde{w}} S^1$ we get $n = \dim N = 1$, so $u = f = \frac{1}{2}(1-x^2)$. Then, for the radius 1/2, we get

$$R[\mathbb{R}^3 \times_{\tilde{w}} S^1] = -u^{-1}\Delta u = \frac{12}{1-x^2}, \tag{4.9}$$

which agrees with (4.2) and (4.7).

Next we rewrite the action in terms of the twisted scalars Φ_T, Φ_R and Φ_m, $m = 6,7,8,9$, (3.11)

$$\begin{aligned} S_{YM} = \frac{1}{2g_{YM}^2}\int_0^{2\pi} d\tau \int_{|x|<1} d^3x \frac{1}{2}(1-x^2)\Big(& g^{\tau\tau}F_{i\tau}^2 + (D_i\Phi_T)^2 \\ & + g^{\tau\tau}(D_\tau\Phi_R - \Phi_T)^2 + [\Phi_T,\Phi_R]^2 + g^{\tau\tau}(D_\tau\Phi_m)^2 + [\Phi_T,\Phi_m]^2 \\ & g^{\tau\tau}(D_\tau\Phi_T + \Phi_R)^2 + \frac{1}{2}F_{ij}^2 + (D_i\Phi_m)^2 + (D_i\Phi_R)^2 + \frac{1}{2}[\Phi_m,\Phi_n]^2 + [\Phi_R,\Phi_m]^2 \\ & + \frac{2}{(1-x^2)}(\Phi_m^2 + \Phi_T^2 + \Phi_R^2) + K_I^2\Big). \end{aligned} \tag{4.10}$$

4.1.2 The physical action reduced to the B^3

Then we restrict the action onto configurations invariant under the diagonal $\mathrm{U}(1)_S \subset \mathrm{SO}(2)_S \times \mathrm{SO}(2)_B$ using (3.34) and (3.41). We also assume that $\Phi_T = 0$ in the supersymmetric background, otherwise Φ_T has first order singularity near the S^2 which would mean insertion of surface operator. Removing the terms with ∇_τ and Φ_T from the action (4.10), we arrive to this three-dimensional action for the gauge field A_i and five scalars Φ_R, Φ_m, $m = 6,7,8,9$,

$$\begin{aligned} S_{YM}^{\mathrm{inv}}(B^3) = \frac{1}{2g_{YM}^2}2\pi\int_{|x|<1} d^3x \frac{1}{2}(1-x^2)\Big(& \frac{4}{(1-x^2)^2}\Phi_R^2 + \frac{1}{2}F_{ij}^2 + (D_i\Phi_m)^2 \\ & + (D_i\Phi_R)^2 + \frac{1}{2}[\Phi_m,\Phi_n]^2 + [\Phi_R,\Phi_m]^2 + \frac{2}{(1-x^2)}(\Phi_m^2 + \Phi_R^2) + K_I^2\Big). \end{aligned} \tag{4.11}$$

4.1.3 The boundary term

Now we show that modulo the supersymmetry equations the physical action (4.11) on the $\mathrm{U}(1)_S$ invariant configurations reduced to B^3 is a total derivative. We try the following ansatz

$$\begin{aligned} S_{\mathrm{susy}}^{\mathrm{inv}}(B^3) = \frac{1}{4g_{YM}^2}2\pi\int_{|x|<1} d^3x & \\ \Big(\Big(& -\frac{1}{2}(F_{ij} - [\Phi_{i+4}\Phi_{j+4}])\epsilon_{ijk} + K_5 x_k - K_{i+4}x_j\epsilon_{ijk}\Big)\times \\ & \Big(-\frac{1}{2}(F_{ij} - [\Phi_{i+4}\Phi_{j+4}])\epsilon_{ijk} - K_5 x_k + K_{i+4}x_j\epsilon_{ijk}\Big) \\ & + (\nabla_i\Phi_{i+4} - K_i x_i)(\nabla_j\Phi_{j+4} + K_j x_j) \\ & + ((\nabla_i\Phi_{j+4} - K_i x_j)\epsilon_{ijk})(\delta_{k\tilde{k}} - x_k x_{\tilde{k}})((\nabla_{\tilde{i}}\Phi_{\tilde{j}+4} + K_{\tilde{i}}x_{\tilde{j}})\epsilon_{\tilde{i}\tilde{j}\tilde{k}}) \\ & + (K_k - (x_i\nabla_i\Phi_{k+4} + x_i\nabla_k\Phi_{i+4} - x_k\nabla_i\Phi_{i+4} + 2\Phi_{k+4}))\times \\ & (K_k + (x_i\nabla_i\Phi_{k+4} + x_i\nabla_k\Phi_{i+4} - x_k\nabla_i\Phi_{i+4} + 2\Phi_{k+4})) \end{aligned}$$

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$$+\left(K_5+\frac{1}{2}x_k\epsilon_{ijk}(F_{ij}-[\Phi_{i+4}\Phi_{j+4}])\right)\left(K_5-\frac{1}{2}x_k\epsilon_{ijk}(F_{ij}-[\Phi_{i+4}\Phi_{j+4}])\right)$$
$$+(K_{k+4}+x_i(F_{ik}+[\Phi_{i+4}\Phi_{k+4}]))(K_{k+4}-x_i(F_{ik}+[\Phi_{i+4}\Phi_{k+4}]))$$
$$-(x_iK_{i+4})^2.) \quad (4.12)$$

Each term above corresponds to one of the top supersymmetry equations (3.43), (3.44) and (3.45) multiplied by a suitable factor to match the kinetic term of the reduced Yang-Mills action (4.11). Therefore at the supersymmetric configurations $S^{\rm inv}_{\rm susy}(B^3)$ vanishes. After some algebra, one can show that the actions (4.11) and (4.12) differ on a total derivative

$$S^{\rm inv}_{\rm susy}(B^3)=S^{\rm inv}_{YM}(B^3)+\frac{2\pi}{4g^2_{YM}}\int d^3x_{|x|<1}(\nabla_i((1-x^2)\Phi_{i+4}\nabla_j\Phi_{j+4}-\Phi_{j+4}\nabla_j\Phi_{i+4})$$
$$-4\nabla_j(x_ix_k\Phi_{k+4}\nabla_i\Phi_{j+4}-x_ix_j\Phi_i\nabla_{k+4}\Phi_{k+4})$$
$$-6\nabla_j(x_i\Phi_{i+4}\Phi_{j+4})) \quad (4.13)$$

Integrating the total derivative term we get a boundary action

$$S^{\rm inv}_{YM}(B^3) \;=\; S^{\rm inv}_{\rm susy}(B^3)\;+\;\frac{2\pi}{4g^2_{YM}}\int_{S^2:\,|x|=1}d\Omega\,(4\Phi_n(\nabla_n\Phi_n\;-\;\nabla_i\Phi_{i+4})\;+\;6\Phi_n^2), \quad (4.14)$$

where Φ_n is the normal component to the S^2 of the one-form Φ, i.e. $\Phi_n=n_i\Phi_{i+4}$, and ∇_n is the derivative in the normal direction, $n_i=x_i/|x|$. Using the equation (3.45) for $\mathrm{Re}Q\Psi^t|_{e_5}$ with K_i substituted from (3.44) we get a constraint on Φ_n on the boundary

$$\nabla_n\Phi_n-\nabla_i\Phi_{i+4}=-\Phi_n. \quad (4.15)$$

Hence, the boundary action (4.14) simplifies to

$$S^{\rm inv}_{YM}(B^3)=S^{\rm inv}_{\rm susy}(B^3)+\frac{\pi}{g^2_{YM}}\int_{S^2:|x|=1}d\Omega\,\Phi_n^2, \quad (4.16)$$

where $d\Omega$ is the standard volume form on S^2. On supersymmetric configuration $S^{\rm inv}_{\rm susy}(B^3)$ vanishes, thus the $\mathcal{N}=4$ Yang-Mills localizes to the two-dimensional theory on S^2 with the action

$$S_{2d}=\frac{\pi}{g^2_{YM}}\int_{S^2:|x|=1}d\Omega\,\Phi_n^2. \quad (4.17)$$

Equivalently we can express the action in terms of the tangent to S^2 components of Φ using the constraint (4.15)

$$S_{2d}=\frac{\pi}{g^2_{YM}}\int_{S^2:|x|=1}d\Omega\,(d_A^{*2d}\Phi_t)^2, \quad (4.18)$$

where Φ_t denotes an adjoined-valued one-form on Σ obtained from the components of Φ_i tangential to Σ. To get (4.18) we used (4.15) and the relation between the tangential components of Φ in $\mathbb{R}^3$ coordinates with the one-form Φ_t

$$\nabla_i\Phi_{i+4}-\nabla_n\Phi_n=d_A^{*2d}\Phi_t+2\Phi_n, \quad (4.19)$$

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from which one gets that

$$d_A^{*2d}\Phi_t = -\Phi_n \quad \text{on supersymmetric configurations.} \tag{4.20}$$

We recall that the scalar fields in (4.3)–(4.18) are the fields for the four-dimensional theory on $\mathbb{R}^3 \times_{\tilde{w}} S^1$. In terms of the original fields of the $\mathcal{N} = 4$ Yang-Mills on S^4 we have $\Phi[\mathbb{R}^3 \times_{\tilde{w}} S^1] = (1+x^2)^{-1}\Phi[S^4]$, so

$$S_{2d} = \frac{\pi}{4g_{YM}^2}\int_{S^2:|x|=1} d\Omega (d_A^{*2d}\Phi_t^{S^4})^2. \tag{4.21}$$

Above was assumed that the radius $r = \frac{1}{2}$. To restore r we need to insert a power of factor $(2r)$ to get the correct dimension

$$S_{2d} = (2r)^2\frac{\pi}{4g_{YM}^2}\int_{S^2:|x|=2r} \sqrt{g_{S^2}} d^2\sigma (d_A^{*2d}\Phi_t^{S^4})^2. \tag{4.22}$$

4.1.4 Relation to the constrained 2d complexified Yang-Mills

In this section Φ denotes the one-form on Σ previously called Φ_t. The Wilson loop operator (2.4) descends to the Wilson loop operator in the two-dimensional theory

$$W_R(C) = tr_R \,\mathrm{Pexp} \oint (A - i*\Phi) \tag{4.23}$$

We introduce another complexified connection

$$\tilde{A}_{\mathbb{C}} = A - i*\Phi, \tag{4.24}$$

so the Wilson loop operator (4.23) is the holonomy of $\tilde{A}_{\mathbb{C}}$

$$W_R(C) = \mathrm{tr}_R \,\mathrm{Pexp} \oint \tilde{A}_{\mathbb{C}}. \tag{4.25}$$

Let $F_{\tilde{A}_{\mathbb{C}}}$ be the curvature of $\tilde{A}_{\mathbb{C}}$, then

$$F_{\tilde{A}_{\mathbb{C}}} = d\tilde{A}_{\mathbb{C}} + \tilde{A}_{\mathbb{C}} \wedge \tilde{A}_{\mathbb{C}} = F_A - \Phi \wedge \Phi - i d_A * \Phi. \tag{4.26}$$

By (3.93) at the localized configurations we have $F_A - \Phi \wedge \Phi = 0$, then

$$d_A * \Phi = i F_{\tilde{A}_{\mathbb{C}}} \quad \text{for localized configurations.} \tag{4.27}$$

Then the action of the two-dimensional theory (4.22) is equivalent to the action of the bosonic Yang-Mills for complexified connection $\tilde{A}_{\mathbb{C}}$

$$S_{2d} = +\frac{1}{2g_{2d}^2}\int_{S^2} d\Omega \,(*_{2d} F_{\tilde{A}_{\mathbb{C}}})^2, \tag{4.28}$$

where the two-dimensional coupling constant is denoted g_{2d}

$$g_{2d}^2 = -\frac{g_{YM}^2}{2\pi r^2}. \tag{4.29}$$

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This relation agrees with the conjecture [12–14] given that g_{2d}^2 is properly defined in the 2d YM action.[7]

So the original four-dimensional problem has been reduced to complexified two-dimensional bosonic Yang-Mills theory (4.28) with the standard Wilson loop observables (4.25). The complexified connection $\tilde{A}_{\mathbb{C}} = A - i * \Phi$ is constrained by (3.93)

$$\mathrm{Re} F_{\tilde{A}_{\mathbb{C}}} = 0 \tag{4.30}$$

$$d_{\mathrm{Re}\tilde{A}_{\mathbb{C}}} * \mathrm{Im}\tilde{A}_{\mathbb{C}} = 0. \tag{4.31}$$

The two real constraints remove two real degrees of freedom from the four real degrees of freedom of complex one-form $\tilde{A}_{\mathbb{C}}$ (we do not subtract gauge symmetry in this counting). Therefore, the path integral is taken over a certain half-dimensional subspace of complexified connections $\tilde{A}_{\mathbb{C}}$.

We can interpret the path integral for the usual two-dimensional Yang-Mills for real connections as a contour integral in the space of complexified connections, where the contour is given by the constraint that the imaginary part of the connection vanishes: $\mathrm{Im}\tilde{A}_{\mathbb{C}} = 0$.

Our assertion is that the complexified theory (4.28) with constraints (4.30) is equivalent to the real theory by a change of the integration contour in the space of complexified connections.

Since perturbative correlation functions of holomorphic observables do not depend on deformation of the contour of integration, we conclude that the expectation value of Wilson loop observables (4.25) perturbatively coincides with the expectation values of Wilson loops in the ordinary two-dimensional Yang-Mills.

We shall look at the constrained complexified two-dimensional Yang-Mills theory from slightly broader viewpoint of so called topological Higgs-Yang-Mills theory [17–19] which deals with the moduli space of solutions to Hitchin equations.

4.2 Hitchin/Higgs-Yang-Mills theory

Here we will review Hitchin/Higgs-Yang-Mills theory [34, 35] following [2, 17–19]. Let Σ be a Riemann surface, A be a gauge field for the gauge group G (G is a compact Lie group) and Φ be a one-form taking value in the Lie algebra $\mathfrak{g}$ of G.

Let φ be a scalar field taking value in $\mathfrak{g}$. The field φ can be thought as an element of the Lie algebra $\mathfrak{g}_{\mathrm{gauge}}$ of the infinite-dimensional group of gauge transformations G_{gauge}. Let M be the space of fields (A, Φ). Using the invariant Killing form on $\mathfrak{g}$ we identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Then locally M is $T^*\Omega^1(\Sigma, \mathrm{ad}\,\mathfrak{g})$.

We notice (see [2, 17–19, 36]) that the space M can be equipped with a triplet of symplectic structures ω_i and a triplet of corresponding Hamiltonian moment maps μ_i for G_{gauge} acting on M.

[7] We write the action in terms of the scalar field $*_{2d}F$ which is the Hodge dual to the curvature two form F. In components one has $(*_{2d}F)^2 = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}$ which means that we use the same conventions for the normalization of the 2d YM action as for the 4d YM action (4.2), i.e. $S = \frac{1}{4g^2}\int d^n x\sqrt{g}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2g^2}\int d^n x\sqrt{g}\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu}$ for SU(N) gauge group.

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Explicitly we define the symplectic structure ω_i as follows. Let δ be the differential on M. Then[8]

$$\omega_1(\delta A_1, \delta\Phi_1; \delta A_2, \delta\Phi_2) = \int_\Sigma \delta A_1 \wedge \delta A_2 - \delta\Phi_1 \wedge \delta\Phi_2 \tag{4.32}$$

$$\omega_2(\delta A_1, \delta\Phi_1; \delta A_2, \delta\Phi_2) = \int_\Sigma \delta A_1 \wedge \delta\Phi_2 - \delta A_2 \wedge \delta\Phi_1 \tag{4.33}$$

$$\omega_3(\delta A_1, \delta\Phi_1; \delta A_2, \delta\Phi_2) = \int_\Sigma \delta A_1 \wedge *\delta\Phi_2 - \delta A_2 \wedge *\delta\Phi_1, \tag{4.34}$$

where $*$ is the Hodge star on Σ.

A functional $\mu : M \to \mathfrak{g}^*_{\text{gauge}}$ is called a moment map if

$$i_\phi \omega = \mu(\phi) \quad \text{for all} \quad \phi \in \mathfrak{g}_{\text{gauge}}, \tag{4.35}$$

where i_ϕ denotes a contraction with a vector field generated on M by an element $\phi \in \mathfrak{g}_{\text{gauge}}$.

The group G_{gauge} acts on M by the usual gauge transformations

$$\begin{aligned} \delta A &= -d_A \phi \\ \delta\Phi &= [\phi, \Phi]. \end{aligned} \tag{4.36}$$

One can check that the functionals

$$\mu_1(\phi) = \int (\phi, F - \Phi \wedge \Phi) \tag{4.37}$$

$$\mu_2(\phi) = \int (\phi, d_A \Phi) \tag{4.38}$$

$$\mu_3(\phi) = \int (\phi, d_A * \Phi) \tag{4.39}$$

are the moment maps for the symplectic structure $\omega_1, \omega_2, \omega_3$ correspondingly.

The space M has natural linear flat structure and the corresponding flat metric is

$$g(\delta A_1, \delta\Phi_1; \delta A_2, \delta\Phi_2) = \int \delta A_1 \wedge *\delta A_2 + \delta\Phi_1 \wedge *\delta\Phi_2. \tag{4.40}$$

Using the metric g on M, to each symplectic structure ω_i we can associate a complex structure I_i in the usual way $\omega(\cdot, \cdot) = g(I\cdot, \cdot)$.

Comparing

$$\int_\Sigma I(\delta A_1) \wedge *\delta A_2 + I(\delta\Phi_1) \wedge *\delta\Phi_2 \tag{4.41}$$

with (4.32)–(4.34) we get

$$I_1(\delta A) = *\delta A \qquad I_1(\delta\Phi) = - * \delta\Phi \tag{4.42}$$

$$I_2(\delta A) = *\delta\Phi \qquad I_2(\delta\Phi) = *\delta A \tag{4.43}$$

$$I_3(\delta A) = -\delta\Phi \qquad I_3(\delta\Phi) = \delta A \tag{4.44}$$

[8] Here the subscripts $1, 2$ enumerate arguments of the functional two-form ω, but not the coordinates on Σ.

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The following linear combinations span the holomorphic subspaces ($+i$-eigenspaces) of the corresponding complex structures:

$$\begin{aligned} I_1(A - i*A) &= i(A - i*A) \\ I_2(A - i*\Phi) &= i(A - i*\Phi) \\ I_3(A + i\Phi) &= i(A + i\Phi). \end{aligned} \tag{4.45}$$

One can also check that the complex structures satisfy $I_3 = I_2 I_1, I_1 = I_3 I_2, I_2 = I_1 I_3$. Hence the space M is the hyperKahler space.

We can use four-dimensional notations. Let us denote

$$\Phi_1 \equiv A_4 \quad \Phi_2 \equiv A_3, \tag{4.46}$$

then the three moment maps (4.37) correspond to the components of the self-dual part F_A^+ of the four-dimensional curvature F_A:

$$\begin{aligned} F - \Phi \wedge \Phi &= (F_{12} + F_{34}) dx^1 \wedge dx^2 \\ d_A \Phi &= (F_{13} - F_{24}) dx^1 \wedge dx^2 \\ d_A * \Phi &= (F_{14} + F_{23}) dx^1 \wedge dx^2 \end{aligned} \tag{4.47}$$

Clearly, the space $\mathbb{R}^4$ (or more generally $T^*\Sigma$) is hyperKahler, so it is equipped with $\mathbb{CP}^1$ family of complex structures. Let $z_1, \bar{z}_1, z_2, \bar{z}_2$ be complex coordinates with respect to some complex structure, e.g. $z_1 = x_1 + i x_2, z_2 = x_3 + i x_4$. Then, in terms of $A_{\bar{z}_1} = \frac{1}{2}(A_1 + i A_2)$, etc, we can write

$$\begin{aligned} F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} &= \frac{i}{2}(F_{12} + F_{34}) = \frac{i}{2}\mu_1 \\ F_{\bar{z}_1 \bar{z}_2} &= \frac{1}{4}(F_{13} - F_{24}) + \frac{i}{4}(F_{23} + F_{14}) = \frac{1}{4}(\mu_2 + i\mu_3) \end{aligned} \tag{4.48}$$

4.2.1 Constrained Higgs-Yang-Mill theory: cHYM and aYM

For the related story see [18, 19].

Consider the following path integral over ϕ and the space M of fields (A, Φ)

$$Z_{cHYM} = \int_{M|\mu_1=\mu_2=0} D\phi e^{i(\omega_3 - \mu_3(\phi)) - \frac{t_2}{2}\int \phi^2}. \tag{4.49}$$

The constraints $\mu_1 = \mu_2 = 0$ mean that we set to zero the complexified curvature $F_{A_{\mathbb{C}}} = 0$. After integrating out ϕ one gets the same action as (4.22).

In this work we did not compute the one-loop determinant associated with the localization, hence we do not have a rigorous and complete understanding of the resulting two-dimensional theory. However, the most natural assumption is that this determinant in the $\mathcal{N} = 4$ theory is trivial in the same way as in [8]. Let us assume that the proper treatment of the one-loop determinant and careful consideration of the fermions will lead to the constrained Hitchin-Yang-Mills theory (4.49), we call it cHYM theory.

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Later we will insert Wilson loop observables for the holomorphic part of the complexified connection with respect to the complex structure I_2. Explicitly such observables have form

$$W_R(C) = \mathrm{tr}_R \,\mathrm{Pexp} \oint_C (A - i * \Phi), \tag{4.50}$$

were C is a contour on Σ and R is representation of G.

We would like to look at the cHYM theory as a "hyperKahler rotation" of another theory

$$Z_{aYM} = \int_{M|\mu_2=\mu_3=0} D\phi e^{i(\omega_1 - \mu_1(\phi)) - \frac{t_2}{2}\int \phi^2}, \tag{4.51}$$

which is almost equivalent to the bosonic two-dimensional Yang-Mills, hence we refer to it as aYM theory. Let Σ be a Riemann sphere. The constraint $\mu_2 = \mu_3 = 0$ means $d_A^*\Phi = d_A\Phi = 0$. For a generic connection A, the only solution to these constraints is $\Phi = 0$. Then the path integral (4.51) reduces to the 2d bosonic Yang-Mills integral over A and ϕ written in the first order formalism as in [36].

We can insert Wilson loop observables (4.50) into the path integral. Since Φ vanishes because of the constraint, the Wilson loop (4.50) reduces to the ordinary Wilson loop of the connection A. Therefore, the expectation value of Wilson loops (4.50) naively is computed by the standard formulas of the two-dimensional Yang-Mills theory [36–38] modulo subtleties which are related to non-generic connections for which there are non-trivial solutions of the constraint $d_A^*\Phi = d_A\Phi = 0$. Such connections precisely correspond to unstable instantons, i.e. configurations with covariantly constant curvature F_A. It is well known that the partition function of bosonic two-dimensional Yang-Mills can be written as a sum of contributions from such unstable instantons [36, 39, 40]. A contribution of a classical solution with curvature F enters with a weight $\exp(-\frac{1}{2g^2}\rho(\Sigma)F^2)$ where $\rho(\Sigma)$ is the area of Σ. In the weak coupling limit such instanton contributions are exponentially suppressed and do not contribute to the perturbation theory. Hence, we conclude that perturbatively the aYM theory (4.51) is equivalent to the ordinary two-dimensional Yang-Mills.

However, at the non-perturbative level, the aYM theory is different from the usual 2d YM theory. Here we assume the gauge group $G = \mathrm{U}(N)$ and consider topologically trivial situation $c_1(E) = 0$ where E is the gauge bundle. And we take $\Sigma = S^2 \simeq \mathbb{CP}^1$. If A is a connection corresponding to an "unstable instanton", the holomorphic vector bundle E associated to A splits as a sum of nontrivial line bundles $\mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_N)$, for integers $n_1 + \cdots + n_N = 0$. Then the equation $d_A^*\Phi = d_A\Phi = 0$ has non-trivial solutions for Φ, and as well there are non-trivial zero modes for associated fermions.[9] One can see this by writing the one-form Φ as $\Phi = \Phi_z dz + \Phi_{\bar{z}}\overline{dz}$. The two real equations $\mu_2 = \mu_3 = 0$ are equivalent to the one complex equation $[\partial_{\bar{z}} + A_{\bar{z}}, \Phi_z] = 0$, which means that Φ_z is adjoined-valued holomorphic one-form. The field Φ_z represents a section of $\mathrm{Ad(E)} \otimes T^*_\Sigma$ where T^*_Σ denotes the holomorphic cotangent bundle on Σ. On $\Sigma = \mathbb{CP}^1$ one has $T^*_\Sigma \simeq \mathcal{O}(-2)$. If the bundle E splits as $\mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_N)$ then the N^2 dimensional bundle $\mathrm{Ad}(E)$ associated with the adjoint representation splits as $\oplus_{i,j=1\ldots N}\mathcal{O}(n_i - n_j)$. A bundle $\mathcal{O}(n)$

[9] We do not write the action for fermions in this section but assume that it is the natural as one can find in e.g. [17–19].

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has holomorphic sections only for non-negative n. Therefore, one concludes that if the connection A is generic, and hence the bundle E is a holomorphically trivial bundle with $n_1 = \cdots = n_N$, then there are no nontrivial holomorphic sections of $\mathrm{Ad}(E) \otimes T^*_\Sigma$, and thus Φ and the associated fermions must vanish. However, if the connection A corresponds to an "unstable instanton", and hence the bundle E is a holomorphically non-trivial bundle with some of $n_k \neq 0$, then there exist a non-zero holomorphic sections of $\mathrm{Ad}(E) \otimes T^*_\Sigma$ as well as there are some zero-modes for associated fermions. We assume that the path integral of aYM theory can be localized to the 2d instanton connections like in the case of the usual 2d YM theory, but that unstable instantons do not actually contribute because of the fermionic zero mode which appear for holomorphically non-trivial bundles E.

4.2.2 From cHYM to aYM perturbatively

Let us give more details supporting the claim that the perturbative expectation value of Wilson loop (4.50) in the cHYM theory (4.49) and the aYM theory (4.51) is the same.

aYM theory. First we consider the aYM theory (4.51). We write the constraints $\mu_2 = \mu_3 = 0$ using Lagrange multipliers. We introduce scalar auxiliary fields H_2, H_3 and their superpartners χ_2, χ_3. The superpartners of A and Ψ are fermionic adjoined valued one-forms on Σ. Then we consider the usual complex for equivariant cohomology

$$\begin{aligned} QA &= \psi_A & Q\chi_{2,3} &= H_{2,3} \\ Q\psi_A &= -d_A\phi & QH_{2,3} &= [\phi, \chi_{2,3}] \end{aligned} \tag{4.52}$$

with

$$Q\phi = 0. \tag{4.53}$$

The aYM theory (4.51) can be rewritten as

$$\begin{aligned} Z_{aYM} = \int D\phi\, DAD\psi_A\, D\Phi D\psi_\Phi DHD\chi \\ \times \exp\left(\int i(\psi_A \wedge \psi_A - \psi_\Phi \wedge \psi_\Phi - (F - \Phi \wedge \Phi))\phi - \frac{t_2}{2}\phi \wedge *\phi + S_c \right), \end{aligned} \tag{4.54}$$

where

$$\begin{aligned} S_c &= iQ\left(\int d_A\Phi \wedge \chi_2 + d_A * \Phi \wedge \chi_3 \right) = \\ &= i\int (d_A\psi_\Phi + [\psi_A, \Phi]) \wedge \chi_2 + (d_A * \psi_\Phi + [\psi_A, *\Phi]) \wedge \chi_3 + d_A\Phi \wedge H_2 + d_A * \Phi \wedge H_3. \end{aligned} \tag{4.55}$$

If we integrate out the Lagrange multipliers H_2, H_3 and Φ, and their fermionic partners χ_2, χ_3 and ψ_A, the resulting determinants cancel, while Φ becomes restricted to the slice $d_A\Phi = d^*_A\Phi = 0$, and similarly ψ_Φ is restricted to $d_A\psi_\Phi + [\psi_A, \Phi] = 0$ and $d_A * \psi_\Phi + [\psi_A, *\Phi] = 0$. Since $\Phi = 0$ we get $\psi_\Phi = 0$. Then what remains is

$$Z_{aYM} = \int DAD\psi_A D\phi \exp\left(\int i(\psi_A \wedge \psi_A - F\phi) - \frac{t_2}{2}\phi \wedge *\phi \right), \tag{4.56}$$

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which is the usual action of bosonic Yang-Mills in the first order formalism [36]. In this derivation we have been careless in assuming that $d_A\Phi = d_A^*\Phi = 0$ implies $\Phi = 0$, which is true for a generic connection but not for unstable instantons as discussed in 4.2.1. Therefore, here we only claim that aYM theory is equivalent to the YM up to the instanton corrections.

cHYM theory. Now consider the cHYM theory (4.49). First we write it in a slightly different way:

$$Z_{cHYM} = \int_{M|\mu_1=\mu_2=0} D\phi e^{i(\omega_3+i\omega_1-(\mu_3(\phi)+i\mu_1(\phi))-\frac{t_2}{2}\int\phi^2}. \tag{4.57}$$

Here we added to the action the term $\mu_1(\phi)$ and its supersymmetric extension ω_1. Since $\mu_1(\phi) = 0$ by constraint, classically this is the same theory as (4.49), and we assume the proper treatment of fermions makes this claim valid also on quantum level. The symplectic structure $\omega_1 - i\omega_3$ is the holomorphic $(2,0)$ two-form with respect to the second complex structure in (4.45).

Let us make a change of variables in the path integral from the fields (A, Φ) to the fields $(\tilde{A}_{\mathbb{C}}, \Phi)$ where

$$\tilde{A}_{\mathbb{C}} = A - i * \Phi. \tag{4.58}$$

Perturbatively we can rotate the integration contour for Φ to the imaginary axis, then $\tilde{A}_{\mathbb{C}}$ is real valued. The Jacobian for this change of variable is trivial.

The symplectic structure $\omega_1 - i\omega_3$ can be written as

$$\omega_1 - i\omega_3 = \int_\Sigma \delta\tilde{A}_{\mathbb{C}} \wedge \delta\tilde{A}_{\mathbb{C}}, \tag{4.59}$$

and the moment map $\mu_1 - i\mu_3$ is actually the curvature of $\tilde{A}_{\mathbb{C}}$

$$\mu_1 - i\mu_3 = F(\tilde{A}_{\mathbb{C}}) \tag{4.60}$$

One can see that if Σ is a sphere, then constraints $\mu_1 = 0, \mu_2 = 0$ determine Φ uniquely for each $\tilde{A}_{\mathbb{C}}$. Hence, the path integral (4.57) reduces to the integral over the fields $\tilde{A}_{\mathbb{C}}$ with the measure induced by the symplectic structure (4.59). That is the standard bosonic Yang-Mills theory in the first order formalism for the connection $\tilde{A}_{\mathbb{C}}$. The correlation function of Wilson loop operators (4.50) perturbatively are computed as in the usual bosonic two-dimensional Yang-Mills.

4.2.3 Remarks and outlook

In [36] Witten has related the physical two-dimensional Yang-Mills theory (4.51) with the topological two-dimensional Yang-Mills. The key point is that the path integral for the physical Yang-Mills theory can be represented as an integral of the equivariantly closed form with respect to the following operator Q

$$\begin{aligned} QA &= \psi \\ Q\psi &= -d_A\phi \\ Q\phi &= 0. \end{aligned} \tag{4.61}$$

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In other words, the $\omega_1 - \mu_1(\phi)$ is the equivariantly closed form constructed from the symplectic structure ω_1 and the Hamiltonian moment map μ_1 for the gauge group acting on the space of connections. Then localization method can be used to compute the integral of such equivariantly closed form [36, 41–43].

Though the Wilson loop observable is not Q-closed, its expectation value can be still solved exactly. That gives a hope that we can also find exact expectation value of Wilson loops (4.50) in the cHYM theory (4.51) and its sister aYM theory (4.49). See [17–19] for computation of correlation functions for the Q-closed observables $\operatorname{tr}\phi^n$.

Consider the aYM partition function (4.49). We can try to proceed in two directions. The first one is to try to use the localization method and relate the theory to some topological theory and computations with Q-equivariant cohomology. Though the Wilson loop operators are not Q-closed, we can try to solve for at least non-intersecting Wilson loops $\{C_1, \ldots, C_k\}$ by: (i) finding topological wave-function $\Psi(\mathrm{U}_1, \ldots, \mathrm{U}_k)$ on the boundary of the Riemann surface with Wilson loops deleted $\Sigma \setminus \{C_1 \cup \ldots C_k\}$, and (ii) then integrating over the space of holonomies $\{U_1, \ldots, U_k\}$. For the study of wave-functions in Higgs-Yang-Mills theory see [18, 19].

The second approach is to explicitly solve the constraint $\mu_1 = \mu_2 = 0$, which means that the complexified connection $A_{\mathbb{C}} = A + i\Phi$ is flat, in the form

$$A + i\Phi = g_{\mathbb{C}}^{-1} dg_{\mathbb{C}}, \tag{4.62}$$

where $g_{\mathbb{C}}$ takes value in the complexified gauge group $G_{\mathbb{C}}$. The gauge transformations for g taking value in the compact gauge group G

$$A + i\Phi \to g^{-1}(A + i\Phi)g + g^{-1}dg \tag{4.63}$$

can be represented by the right multiplications $g_{\mathbb{C}} \to g_{\mathbb{C}} g$. Hence the configurational space of the theory is the same as of gauged WZW model on the coset $G^{\mathbb{C}}/G$. We shall not proceed these ideas further in this work.

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A Supersymmetry closure

Let δ_ε be the supersymmetry transformation generated by a conformal Killing spinor ε.

The δ_ε^2 is represented on the fields as

$$\begin{aligned} \delta_\varepsilon^2 A_\mu &= -v^\nu F_{\nu\mu} - [v^B \Phi_B, D_\mu] \\ \delta_\varepsilon^2 \Phi_A &= -v^\nu D_\nu \Phi_A - [v^B \Phi_B, \Phi_A] - 2\varepsilon \tilde{\Gamma}_{AB} \tilde{\varepsilon} \Phi^B - 2\varepsilon\tilde{\varepsilon}\Phi_A, \end{aligned} \tag{A.1}$$

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where we introduced the vector field v

$$v^\mu \equiv \varepsilon\Gamma^\mu\varepsilon, \quad v^A \equiv \varepsilon\Gamma^A\varepsilon. \tag{A.2}$$

Therefore

$$\delta_\varepsilon^2 = -L_v - G_{v^M A_M} - R - \Omega. \tag{A.3}$$

Here L_v is the Lie derivative in the direction of the vector field v^μ. The transformation $G_{v^M A_M}$ is the gauge transformation generated by the parameter $v^M A_M$. On matter fields G acts as $G_u \cdot \Phi \equiv [u, \Phi]$, on gauge fields G acts as $G_u \cdot A_\mu = -D_\mu u$. The transformation R is the rotation of the scalar fields $(R\cdot\Phi)_A = R_{AB}\Phi^B$ with the generator $R_{AB} = 2\varepsilon\tilde{\Gamma}_{AB}\tilde{\varepsilon}$. Finally, the transformation Ω is the dilation transformation with the parameter $2(\varepsilon\tilde{\varepsilon})$.

On fermions the δ_ε^2 acts as

$$\delta_\varepsilon^2\Psi = -(\varepsilon\Gamma^N\varepsilon)D_N\Psi - \frac{1}{2}(\tilde{\varepsilon}\Gamma_{\mu\nu}\varepsilon)\Gamma^{\mu\nu}\Psi - \frac{1}{2}(\varepsilon\tilde{\Gamma}_{AB}\tilde{\varepsilon})\Gamma^{AB}\Psi - 3(\tilde{\varepsilon}\varepsilon)\Psi + \mathrm{eom}[\Psi]. \tag{A.4}$$

To achieve off-shell closure in the $\mathcal{N} = 4$ case we add seven auxiliary fields K_i with $i = 1, \ldots, 7$ and modify the transformations as

$$\begin{aligned} \delta_\varepsilon\Psi &= \frac{1}{2}\Gamma^{MN}F_{MN} + \frac{1}{2}\Gamma^{\mu A}\Phi_A D_\mu\varepsilon + K^i\nu_i \\ \delta_\varepsilon K_i &= -\nu_i\Gamma^M D_M\Psi. \end{aligned} \tag{A.5}$$

Here we introduced seven spinors ν_i. They depend on choice of the conformal Killing spinor ε and are required to satisfy the following relations:

$$\varepsilon\Gamma^M\nu_i = 0 \tag{A.6}$$

$$\frac{1}{2}(\varepsilon\Gamma_N\varepsilon)\tilde{\Gamma}^N_{\alpha\beta} = \nu^i_\alpha\nu^i_\beta + \varepsilon_\alpha\varepsilon_\beta \tag{A.7}$$

$$\nu_i\Gamma^M\nu_j = \delta_{ij}\varepsilon\Gamma_M\varepsilon. \tag{A.8}$$

The equation (A.6) ensures closure on A_M, the equation (A.7) ensures closure on Ψ.

After adding the auxiliary fields K_i, the term proportional to the equations of motion of the fermions in (A.4) is cancelled and the algebra is closed off-shell.

For the transformation $\delta_\varepsilon^2 K_i$ we get

$$\delta_\varepsilon^2 K_i = -(\varepsilon\Gamma^M\varepsilon)D_M K^i - (\nu_{[i}\Gamma^\mu D_\mu\nu_{j]})K^j - 4(\tilde{\varepsilon}\varepsilon)K_i. \tag{A.9}$$

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