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Foundation and generalization of the expansion by regions

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Abstract: The "expansion by regions" is a method of asymptotic expansion developed by Beneke and Smirnov in 1997. It expands the integrand according to the scaling prescriptions of a set of regions and integrates all expanded terms over the whole integration domain. This method has been applied successfully to many complicated loop integrals, but a general proof for its correctness has still been missing.

This paper shows how the expansion by regions manages to reproduce the exact result correctly in an expanded form and clarifies the conditions on the choice and completeness of the considered regions. A generalized expression for the full result is presented that involves additional overlap contributions. These extra pieces normally yield scaleless integrals which are consistently set to zero, but they may be needed depending on the choice of the regularization scheme.

While the main proofs and formulae are presented in a general and concise form, a large portion of the paper is filled with simple, pedagogical one-loop examples which illustrate the peculiarities of the expansion by regions, explain its application and show how to evaluate contributions within this method.

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1 Introduction

When loop integrals involve many different scales from masses and kinematical parameters, it can be hard or even impossible to evaluate them exactly. The integrand may be simplified before integration by exploiting hierarchies of parameters and expanding in powers of small parameter ratios. When these expansions are done naively, neglecting their breakdown in certain parts of the integration domain, new singularities may be generated and important contributions to the full result can be missed. A proper treatment requires sophisticated methods of *asymptotic expansions*. One of them is the so-called "strategy of regions" or "expansion by regions" developed by M. Beneke and V.A. Smirnov [1]. The recipe for applying the expansion by regions to a loop integral reads as follows [1–4]:

1. Divide the space of the loop momenta into various regions and, in every region, expand the integrand into a Taylor series with respect to the parameters that are considered small there.

2. Integrate the integrand, expanded in the appropriate way in every region, over the *whole integration domain* of the loop momenta.

3. Set to zero any scaleless integral.

The sum of these contributions yields the full result of the loop integral in an expanded form. This recipe will be illustrated in the examples of the following sections.

However, despite the successful application of the expansion by regions in many loop calculations, each of the steps in the recipe above raises questions:

- In step 2 all expanded terms are integrated over the whole integration domain, neglecting the domains of convergence of the expansions. This generally leads to new singularities which obviously must be cancelled between the contributions of the various regions. How is this cancellation ensured?

- In the original integral each point of the integration domain contributes exactly once. According to step 2 each point of the integration domain contributes once per region. How can this double- or multiple-counting of contributions be correct?

- How do we have to choose the regions in step 1? And how do we know that the chosen set of regions is complete?

- Although it seems natural to eliminate scaleless integrals when using dimensional or analytic regularization, we can ask: What is the role of the scaleless integrals in step 3?

These questions will be addressed in the current paper.

The developers of the expansion by regions have introduced their method using examples and "heuristic motivations" (e.g. related to analogies between the regions and degrees of freedom in effective theories). They have shown the validity of the method for some types of expansions (in particular Euclidean-type limits like off-shell large-momentum expansion or large-mass expansion) through the agreement with existing and proven expansions by subgraphs [4]. They admit, however, that they cannot give a general mathematical proof of their prescriptions [1] and that "it is not guaranteed that expansion by regions works in all situations" [4].

A practical problem in the application of the expansion by regions is the correct choice of the regions. In the original paper [1] the authors determine the relevant regions from the structure of the poles of the propagators in the loops. They close the contour of the integration over the zero-component of the loop momentum and study its scaling at the residues depending on the size of the spatial components. In general, relevant regions can often be found by looking at the structure of the integrand and at singularities which arise in the given parameter limit. It does not matter to consider more regions than necessary: The irrelevant regions will simply produce scaleless contributions which are set to zero. The tricky point here is to avoid double-counting of regions which look different but yield equivalent expansions.

Alternatively, the expansion by regions has been applied to the alpha-parameter representation of loop integrals [3, 4]. The double-counting of equivalent regions should not occur here, but there exist at least some regions (in particular the "potential region" in threshold expansions) which cannot easily be identified in the language of the alpha parameters. Based on this approach, an algorithm for finding the regions in alpha-parameter representations has recently been developed [5]. In the present paper I stick exclusively to the version of the expansion by regions which is applied directly to loop integrals because I find this original variant of the method more natural. However, the formalism which I present is general and can be applied to the asymptotic expansion of many types of integrals, including alpha-parameter representations.

An important check for the completeness of the regions is the cancellation of singularities. The sum of all contributions must not be more singular than the original integral, and if additional singularities persist, then probably the contribution from another region is missing. This check is, however, not sufficient to guarantee the completeness of regions, because some regions or a subset of them can also yield non-singular contributions such that their absence could remain undetected.

Another possibility of finding all relevant regions for an integral in a given parameter limit is the one which I have successfully used in many calculations (e.g. [6–9]): Evaluate the full integral with the propagators raised to generic powers in terms of as many Mellin-Barnes representations [10] as needed. Extract the leading-order asymptotic expansion by closing those Mellin-Barnes integrals which involve the expansion parameter and taking the residue of the first pole next to the contour. This yields a sum of terms reproducing the exact result up to higher powers of the expansion parameter than those already present. The terms often still contain Mellin-Barnes integrals and may be too complicated to evaluate them further. But each term is characterized by a homogeneous dependence on the expansion parameter raised to some power which is a function of the space-time dimension $d = 4 - 2\epsilon$ and of the propagator powers. An examination of each term's dependence on the expansion parameter is usually able to tell the corresponding region needed for producing this contribution. If the asymptotic expansion by Mellin-Barnes representations is performed correctly, the prescription described here yields all the relevant regions for a subsequent application of the expansion by regions.

The correspondence between the contributions of the expansion by regions and the asymptotic expansion via Mellin-Barnes representations is, of course, as heuristic as the derivations and justifications known so far for the expansion by regions. In the present paper I follow a different approach: I start from a general integral for which an expansion is required and transform the expression step by step, in a mathematically well-defined way, into a sum of expansions which can be identified with the ones originating from the expansion by regions. The resulting expression contains additional *overlap contributions* which are absent in the usual prescription for the expansion by regions. While these extra pieces normally yield scaleless integrals which are consistently set to zero according to step 3 above, there are cases — depending on the choice of the regularization scheme — where these overlap contributions are present and required for the correct result. Such overlap contributions have also been introduced in effective-theory treatments as "zero-bin subtractions" (see e.g. [11, 12]).

The basic idea of the formalism which I develop in this paper goes back to a one-dimensional toy example [13] (see also section 3.2 of [4]). I do not treat this one-dimensional toy integral, but start directly with a simple d-dimensional loop integral in section 2 before presenting a first version of the formalism for general integrals in section 3. A generalized version of the formalism is elaborated in section 5, where one restriction of the first version is relaxed. Examples which illustrate the application of the formalism and explain the evaluation of loop integrals with the expansion by regions are shown in sections 4, 6 and 7. The last of these examples demonstrates the relevance of overlap contributions under certain circumstances. The main statements and formulae are summarized and discussed in the conclusions of section 8. Details from the evaluation of the expanded loop integrals have been shifted to appendix A. Finally appendix B treats an example with a finite integration boundary which illustrates a subtlety raised in section 3.

The order of the general parts (sections 3, 5 and 8) and illustrating examples (sections 2, 6 and 7) has been chosen from a pedagogic viewpoint in order to facilitate the understanding especially for readers who are not very familiar with the expansion by regions. More experienced readers may skip the examples and concentrate on the general sections for a quick study of the main statements. The examples, however, also show how the general formalism can actually be applied to loop integrals and how its conditions on the regions are checked. Later examples use notations and conventions introduced in earlier examples.

2 Example: off-shell large-momentum expansion

The two-point one-loop integral of the first example is defined by the expression

$$F = \int \mathrm{D}k \, I \tag{2.1}$$

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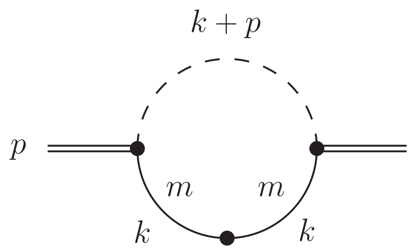

Figure 1. Two-point loop integral for off-shell large-momentum expansion.

with the integrand $I = I_1 I_2$, the propagators

$$I_1 = \frac{1}{\big((k+p)^2\big)^{n_1}} = \frac{1}{(k^2 + 2k\cdot p + p^2)^{n_1}} \quad \text{and} \quad I_2 = \frac{1}{(k^2 - m^2)^{n_2}} \tag{2.2}$$

and the integration measure

$$\int \mathrm{D}k \equiv \mu^{2\epsilon}\, e^{\epsilon\gamma_{\mathrm{E}}} \int \frac{\mathrm{d}^d k}{i\pi^{d/2}}\,, \tag{2.3}$$

where $d = 4 - 2\epsilon$ is the space-time dimension, μ is the scale of dimensional regularization, and $\gamma_{\mathrm{E}} \approx 0.577216$ is Euler's constant. The usual infinitesimal imaginary part in the Feynman propagators is understood and has been dropped in the notation for brevity.

The integral F depends on the external momentum p and the mass m. We are interested in the off-shell large-momentum limit

$$|p^2| \gg m^2$$

and look for an asymptotic expansion in powers of m^2/p^2. The integral also depends on the propagator powers n_1 and n_2, and we focus particularly on the case $n_1 = 1$, $n_2 = 2$ which is depicted in figure 1 and for which the integral is finite for both small and large k. Exact results for this integral can easily be obtained and expanded in m^2/p^2 for comparison with the asymptotic expansion.

In the following two subsections we first treat this loop integral according to the recipe formulated in section 1 for the expansion by regions before proving the correctness of this approach by independent mathematical transformations of the integral.

2.1 Expansion by regions in the large-momentum limit

The first propagator in (2.2) is characterized by the large momentum p, whereas the second propagator is characterized by the small mass m. It is therefore natural to assume that the two regions of relevance to this problem are

- the *hard region* (h), where $k \sim p$,
- and the *soft region* (s), where $k \sim m$.

By $k \sim p$ we mean that all components of the loop momentum k are of the order of ("scale like") the (overall size of) the momentum p, and similarly for $k \sim m$. Instead of dividing the integration domain into subdomains with explicit boundaries, the regions simply define scaling prescriptions for the loop momentum on the basis of which we are able to perform expansions of the integrand.

For the hard region we consider $|k^2| \gg m^2$ and expand the second propagator in (2.2), while the first propagator remains unchanged:

$$I_1 \to T^{(h)} I_1 = I_1\,, \qquad I_2 \to T^{(h)} I_2 \equiv \sum_j T_j^{(h)} I_2 = \sum_{j=0}^{\infty} \frac{(n_2)_j}{j!} \frac{(m^2)^j}{(k^2)^{n_2+j}}\,, \tag{2.4}$$

where $T^{(h)}$ is the expansion operator of the hard region, and $T_j^{(h)}$ generates its j-th order expansion term. For the soft region we consider $|k^2| \ll |p^2|$ and $|2k \cdot p| \ll |p^2|$, permitting the expansion of the first propagator, while the second remains unchanged:

$$\begin{aligned} &I_1 \to T^{(s)} I_1 \equiv \sum_j T_j^{(s)} I_1 \equiv \sum_{j_1,j_2} T_{j_1,j_2}^{(s)} I_1 = \sum_{j_1,j_2=0}^{\infty} \frac{(n_1)_{j_{12}}}{j_1!\,j_2!} \frac{(-k^2)^{j_1}\,(-2k\cdot p)^{j_2}}{(p^2)^{n_1+j_{12}}}\,, \\ &I_2 \to T^{(s)} I_2 = I_2\,, \end{aligned} \tag{2.5}$$

where

$$j_{\alpha\beta\cdots} \equiv j_\alpha + j_\beta + \ldots \tag{2.6}$$

is introduced as a shorthand notation, which is also used for other symbols. Here $T_j^{(s)}$ generates the j-th order expansion term of the soft region, and it is *a priori* not clear how j relates to j_1 and j_2 because k^2 and $2k\cdot p$ involve different powers of the soft momentum k. So we postpone this question until after the loop integration when we will rearrange the summation over j_1 and j_2. Both the hard and the soft expansions have employed the generic Taylor expansion

$$\frac{1}{(X+y_1+\ldots+y_m)^n} = \sum_{j_1,\ldots,j_m=0}^{\infty} \frac{(n)_{j_1+\ldots+j_m}}{j_1!\cdots j_m!} \frac{(-y_1)^{j_1}\cdots(-y_m)^{j_m}}{X^{n+j_1+\ldots+j_m}} \quad \text{if } |y_j| \ll |X|\,\forall j\,, \tag{2.7}$$

with the Pochhammer symbol $(\alpha)_j \equiv \Gamma(\alpha+j)/\Gamma(\alpha)$.

According to step 2 of the recipe in section 1, each expanded term has to be integrated over the *whole integration domain*. The j-th order contribution of the hard region reads

$$F_j^{(h)} = \int \mathrm{D}k\, T_j^{(h)} I = \frac{(n_2)_j}{j!}\,(m^2)^j \int \frac{\mathrm{D}k}{\left((k+p)^2\right)^{n_1} (k^2)^{n_2+j}}\,, \tag{2.8}$$

and the soft-region contribution with indices j_1, j_2 is given by

$$F_{j_1,j_2}^{(s)} = \int \mathrm{D}k\, T_{j_1,j_2}^{(s)} I = \frac{(n_1)_{j_{12}}}{j_1!\,j_2!} \frac{(-1)^{j_{12}}}{(p^2)^{n_1+j_{12}}} \int \mathrm{D}k\, \frac{(k^2)^{j_1}\,(2k\cdot p)^{j_2}}{(k^2-m^2)^{n_2}}\,. \tag{2.9}$$

These new integrals are simpler than the original integral (2.1): The hard contributions (2.8) are massless integrals and functions only of p^2. And the soft contributions (2.9), once each scalar product in the numerator has been separated into $k \cdot p = p_\mu k^\mu$, are massive tadpole tensor integrals of rank j_2 and functions only of m^2. These are all one-scale integrals, and it is already clear at this step from a dimensional analysis that the hard and soft contributions are *homogeneous functions* of m^2/p^2 with

$$F_j^{(h)} \propto (m^2)^j \, |p^2|^{\frac{d}{2}-n_{12}-j} \,, \qquad F_{j_1,j_2}^{(s)} \propto (m^2)^{\frac{d}{2}-n_2+j_1+j_2/2} \, |p^2|^{-n_1-j_1-j_2/2} \,. \tag{2.10}$$

We also see that the expansions generate new singularities: The hard contributions (2.8) will become infrared-singular (for $k \to 0$) due to the increasing number of k^2-terms in the denominator, and the soft contributions (2.9) will become ultraviolet-singular (for $k \to \infty$) with more and more powers of the loop momentum in the numerator. In the particular case $n_1 = 1$, $n_2 = 2$, the original integral is finite, but all terms from the hard region are infrared-singular and all terms from the soft region are ultraviolet-singular, even the leading-order contributions with $j = 0$ and $j_1 = j_2 = 0$.

The contributions (2.8) and (2.9) can easily be evaluated. The hard-region integrals are straightforward by standard methods. They yield

$$\begin{aligned} F_j^{(h)} = \mu^{2\epsilon} \, e^{\epsilon\gamma_\mathrm{E}} \, e^{-i\pi n_{12}} \, (-p^2 - i0)^{2-n_{12}-\epsilon} \left(\frac{m^2}{p^2}\right)^j \frac{\Gamma(2-n_1-\epsilon)}{\Gamma(n_1)\,\Gamma(n_2)} \\ \times \frac{\Gamma(n_{12}-2+\epsilon+j)\,\Gamma(2-n_2-\epsilon-j)}{j!\,\Gamma(4-n_{12}-2\epsilon-j)} \end{aligned} \tag{2.11}$$

and can be summed up to the all-order hard contribution

$$\begin{aligned} F^{(h)} = \sum_{j=0}^{\infty} F_j^{(h)} &= \mu^{2\epsilon} \, e^{\epsilon\gamma_\mathrm{E}} \, e^{-i\pi n_{12}} \, (-p^2 - i0)^{2-n_{12}-\epsilon} \\ &\times \frac{\Gamma(n_{12}-2+\epsilon)\,\Gamma(2-n_1-\epsilon)\,\Gamma(2-n_2-\epsilon)}{\Gamma(n_1)\,\Gamma(n_2)\,\Gamma(4-n_{12}-2\epsilon)} \\ &\times {}_2F_1\!\left(n_{12}-2+\epsilon, n_{12}-3+2\epsilon; n_2-1+\epsilon; \frac{m^2}{p^2}\right), \end{aligned} \tag{2.12}$$

where ${}_2F_1$ is the hypergeometric function and "$-i0$" indicates the side of the branch cut for the analytic continuation in the case $p^2 > 0$.

The soft-region contributions can e.g. be solved by tensor reduction (for $j_2 > 0$). The integrals are only non-zero for even j_2. We can identify $j = j_1 + \frac{j_2}{2}$ as the number of additional powers of m^2/p^2 compared to the leading order and rewrite

$$\sum_{j_1,j_2=0}^{\infty} F_{j_1,j_2}^{(s)} \equiv \sum_{j=0}^{\infty} F_j^{(s)} \,. \tag{2.13}$$

An evaluation of the soft contributions (2.9) in closed form without the need for tensor reduction is shown in appendix A.1.1. The result reads

$$\begin{aligned} F_j^{(s)} = \mu^{2\epsilon} \, e^{\epsilon\gamma_\mathrm{E}} \, e^{-i\pi n_{12}} \, (m^2)^{2-n_2-\epsilon} \, (-p^2 - i0)^{-n_1} \left(\frac{m^2}{p^2}\right)^j \frac{\Gamma(2-n_1-\epsilon)}{\Gamma(n_1)\,\Gamma(n_2)} \\ \times \frac{\Gamma(n_1+j)\,\Gamma(n_2-2+\epsilon-j)}{j!\,\Gamma(2-n_1-\epsilon-j)} \end{aligned} \tag{2.14}$$

and, summed up to all orders,

$$F^{(s)} = \sum_{j=0}^{\infty} F_j^{(s)} = \mu^{2\epsilon}\, e^{\epsilon\gamma_{\rm E}}\, e^{-i\pi n_{12}}\, (m^2)^{2-n_2-\epsilon}\, (-p^2-i0)^{-n_1}\, \frac{\Gamma(n_2-2+\epsilon)}{\Gamma(n_2)} \times {}_2F_1\!\left(n_1, n_1-1+\epsilon; 3-n_2-\epsilon; \frac{m^2}{p^2}\right). \tag{2.15}$$

The asymptotic expansion of the original integral (2.1) in powers of m^2/p^2 is obtained by adding the contributions of the hard and the soft region. Which of the terms $F_j^{(h)}$ and $F_j^{(s)}$ are of the same order in m^2/p^2 depends on n_2 (and ϵ). For the special choice $n_1 = 1$, $n_2 = 2$, we have

$$\begin{aligned} F_j^{(h)} &= \frac{1}{p^2}\left(\frac{m^2}{p^2}\right)^j \left(\frac{\mu^2}{-p^2-i0}\right)^\epsilon \frac{e^{\epsilon\gamma_{\rm E}}\,\Gamma(1+\epsilon)\,\Gamma(1-\epsilon)\,\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\,\frac{(2\epsilon)_j}{j!}\,, \\ F_j^{(s)} &= \frac{1}{p^2}\left(\frac{m^2}{p^2}\right)^j \left(\frac{\mu^2}{m^2}\right)^\epsilon e^{\epsilon\gamma_{\rm E}}\,\Gamma(\epsilon)\,\frac{(\epsilon)_j}{(1-\epsilon)_j}\,. \end{aligned} \tag{2.16}$$

All terms $F_j^{(h)}$ are infrared-singular and all terms $F_j^{(s)}$ are ultraviolet-singular, but only the leading-order terms $F_0^{(h)}$ and $F_0^{(s)}$ exhibit an explicit $1/\epsilon$ singularity:

$$\begin{aligned} F_0^{(h)} &= \frac{1}{p^2}\left[-\frac{1}{\epsilon} + \ln\!\left(\frac{-p^2-i0}{\mu^2}\right)\right] + \mathcal{O}(\epsilon)\,, \qquad & F_j^{(h)} &= -\frac{2}{p^2}\left(\frac{m^2}{p^2}\right)^j \frac{1}{j} + \mathcal{O}(\epsilon)\,, \\ F_0^{(s)} &= \frac{1}{p^2}\left[\frac{1}{\epsilon} + \ln\!\left(\frac{\mu^2}{m^2}\right)\right] + \mathcal{O}(\epsilon)\,, & F_j^{(s)} &= \frac{1}{p^2}\left(\frac{m^2}{p^2}\right)^j \frac{1}{j} + \mathcal{O}(\epsilon)\,, \end{aligned} \tag{2.17}$$

where the column to the right is valid for $j \geq 1$. The $1/\epsilon$ poles cancel each other such that the complete expansion terms

$$F_j = F_j^{(h)} + F_j^{(s)} \tag{2.18}$$

are all as finite in the limit $\epsilon \to 0$ as the original integral:

$$F_0 = \frac{1}{p^2}\,\ln\!\left(\frac{-p^2-i0}{m^2}\right) + \mathcal{O}(\epsilon)\,, \qquad F_j = -\frac{1}{p^2}\left(\frac{m^2}{p^2}\right)^j \frac{1}{j} + \mathcal{O}(\epsilon)\,,\ j \geq 1\,. \tag{2.19}$$

But remember that in F_0 an infrared $1/\epsilon$ pole has been cancelled by an ultraviolet pole, which looks unnatural. We will come back to this point in the next section.

The known exact result is reproduced by summing up all orders of the asymptotic expansion, either from (2.19) or by expanding the summed contributions (2.12) and (2.15) from the regions about $\epsilon = 0$:

$$F = \sum_{j=0}^{\infty} F_j = \frac{1}{p^2}\left[\ln\!\left(\frac{-p^2-i0}{m^2}\right) + \ln\!\left(1 - \frac{m^2}{p^2}\right)\right] + \mathcal{O}(\epsilon)\,. \tag{2.20}$$

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For general n_1 and n_2, the original integral can also be evaluated directly by standard methods, yielding

$$F = \mu^{2\epsilon}\, e^{\epsilon\gamma_{\rm E}}\, e^{-i\pi n_{12}}\, (m^2)^{2-n_{12}-\epsilon}\, \frac{\Gamma(2-n_1-\epsilon)\,\Gamma(n_{12}-2+\epsilon)}{\Gamma(n_2)\,\Gamma(2-\epsilon)} \times {}_2F_1\left(n_1, n_{12}-2+\epsilon; 2-\epsilon; \frac{p^2+i0}{m^2}\right) . \quad (2.21)$$

By applying the transformation formula for the hypergeometric function which inverses its argument (see e.g. [14]), the sum of $F^{(h)}$ (2.12) and $F^{(s)}$ (2.15) is exactly recovered from (2.21). Another check is obtained by evaluating the full integral with the help of a Mellin-Barnes representation and extracting the complete series of residues with rising powers of m^2/p^2. By doing so, the expansion terms (2.11) and (2.14) are reproduced.

As a final remark for this section I want to emphasize that — although the evaluation of the integrals originating from the expansion by regions *to all orders* in the expansion parameter can be quite tedious — it is usually rather easy to extract just the leading-order contributions from all regions. This method is therefore particularly well suited for obtaining the leading order or the first few terms of an asymptotic expansion.

2.2 Proof of the large-momentum expansion

While the previous section has introduced the use of the expansion by region, we restart in this section from the original integral and transform it in a mathematically well-defined way until we arrive at the expanded expressions which have already been employed and evaluated in the previous section.

We want to use the expansions of the hard (2.4) and soft (2.5) region which, obviously, do not converge towards the original integrand throughout the complete integration domain. But we can divide the integration domain into a hard domain D_h and a soft domain D_s,

$$D_h = \left\{k \in \mathbb{R}^d : |k^2| \geq \Lambda^2\right\}, \qquad D_s = \left\{k \in \mathbb{R}^d : |k^2| < \Lambda^2\right\}, \quad (2.22)$$

with some intermediate scale Λ^2 chosen such that $m^2 \ll \Lambda^2 \ll |p^2|$. These two domains are non-intersecting and cover the complete integration domain:

$$D_h \cap D_s = \emptyset\,, \qquad D_h \cup D_s = \mathbb{R}^d\,. \quad (2.23)$$

When the loop momentum k is restricted to one of these domains, the corresponding expansion *converges absolutely* and the integrand is identical to its series expansion as long as the latter is summed to all orders:

$$I = T^{(h)} I \equiv \sum_j T_j^{(h)} I \quad \text{for } k \in D_h\,,$$
$$I = T^{(s)} I \equiv \sum_j T_j^{(s)} I \quad \text{for } k \in D_s\,. \quad (2.24)$$

For the soft expansion, the summation index j symbolically represents a proper combination of the indices j_1 and j_2 in (2.5). The hard expansion $T^{(h)}$ only requires $|k^2| \gg m^2$ which is certainly fulfilled for $k \in D_h$. But the statement in (2.24) about the soft expansion $T^{(s)}$ is less trivial: While one of the conditions, $|k^2| \ll |p^2|$, surely holds for $k \in D_s$, the other condition, $|2k \cdot p| \ll |p^2|$, can still be violated. We have to remember, though, that the expansions are always performed under the loop integral $\int \mathrm{D}k$. By tensor reduction, each power of $(k \cdot p)^2$ in the numerator of the soft-region integrals (2.9) gives a contribution proportional to $k^2 p^2$, while odd powers of $k \cdot p$ vanish under the integration. This is still true if we restrict the integration by the Lorentz-invariant condition $|k^2| < \Lambda^2$. So we can safely count $|2k \cdot p| \sim |k^2 p^2|^{1/2}$, and the condition $|2k \cdot p| \ll |p^2|$ holds under the integral over the soft domain D_s.

Alternatively, for ensuring the convergence of $T^{(s)}$ within D_s, one may choose a reference frame in which either the zero-component or the spatial components of the vector p vanish, depending on the sign of p^2, and define the boundaries of D_h and D_s appropriately in this reference frame. Or, for $p^2 < 0$, one may perform a Wick rotation and define the boundaries of the domains as relations between positive-definite norms of Euclidean vectors.

A consequence of the absolute convergence of the expansions is that the summation $\sum_j$ commutes with the integration and can safely be pulled out of the integral if the latter is restricted to the corresponding domain:

$$\int_{k \in D_h} \mathrm{D}k\, I = \sum_j \int_{k \in D_h} \mathrm{D}k\, T_j^{(h)} I\,, \qquad \int_{k \in D_s} \mathrm{D}k\, I = \sum_j \int_{k \in D_s} \mathrm{D}k\, T_j^{(s)} I\,. \tag{2.25}$$

After these preliminaries we can start transforming the original integral by splitting the integration into the two domains and performing the appropriate expansions in each of them:

$$\begin{aligned} F = \int \mathrm{D}k\, I &= \int_{k \in D_h} \mathrm{D}k\, I + \int_{k \in D_s} \mathrm{D}k\, I \\ &= \sum_j \int_{k \in D_h} \mathrm{D}k\, T_j^{(h)} I + \sum_j \int_{k \in D_s} \mathrm{D}k\, T_j^{(s)} I\,. \end{aligned} \tag{2.26}$$

With dimensional regularization at hand, we can also perform the integration of each expanded term over the complete integration domain, but we have to compensate for this by subtracting the surplus from the added domain:

$$\begin{aligned} \int_{k \in D_h} \mathrm{D}k\, T_j^{(h)} I &= \int \mathrm{D}k\, T_j^{(h)} I - \int_{k \in D_s} \mathrm{D}k\, T_j^{(h)} I\,, \\ \int_{k \in D_s} \mathrm{D}k\, T_j^{(s)} I &= \int \mathrm{D}k\, T_j^{(s)} I - \int_{k \in D_h} \mathrm{D}k\, T_j^{(s)} I\,. \end{aligned} \tag{2.27}$$

Without the indication of a restriction, the integrations are understood as being performed over the complete integration domain $\mathbb{R}^d$. Relation (2.25) also holds if the integrand I is not the original one, but a term from a previous expansion, i.e. each expansion can be

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applied to any integrand if the integral is restricted to the corresponding domain:

$$\begin{aligned}\int_{k\in D_h} \mathrm{D}k\, T_j^{(s)} I &= \sum_i \int_{k\in D_h} \mathrm{D}k\, T_i^{(h)} T_j^{(s)} I\,,\\ \int_{k\in D_s} \mathrm{D}k\, T_j^{(h)} I &= \sum_i \int_{k\in D_s} \mathrm{D}k\, T_i^{(s)} T_j^{(h)} I\,. \end{aligned} \tag{2.28}$$

Let us have a look at the newly generated double expansions. The order in which the hard and soft expansions are applied is irrelevant because the doubly expanded integrand is the same in both cases (written here with two separate indices for the soft expansion to be specific):

$$T_i^{(h)} T_{j_1,j_2}^{(s)} I = T_{j_1,j_2}^{(s)} T_i^{(h)} I = \frac{(n_2)_i}{i!}\,\frac{(n_1)_{j_{12}}}{j_1!\,j_2!}\,\frac{(m^2)^i\,(-1)^{j_{12}}}{(p^2)^{n_1+j_{12}}}\,\frac{(2k\cdot p)^{j_2}}{(k^2)^{n_2+i-j_1}}\,. \tag{2.29}$$

In such cases of *commuting expansions*, we label multiple expansions by a comma-separated list in the round brackets:

$$T_i^{(h)} T_j^{(s)} = T_j^{(s)} T_i^{(h)} \equiv T_{i,j}^{(h,s)}\,. \tag{2.30}$$

After an appropriate relabelling of the summation indices, the two contributions with double expansions can be added together,

$$\sum_i \sum_j \int_{k\in D_s} \mathrm{D}k\, T_{i,j}^{(h,s)} I + \sum_j \sum_i \int_{k\in D_h} \mathrm{D}k\, T_{i,j}^{(h,s)} I = \sum_{i,j} \int \mathrm{D}k\, T_{i,j}^{(h,s)} I\,, \tag{2.31}$$

arriving at an integral over the complete integration domain. The non-trivial point here is that we have to exchange the order of the two summations in one of the contributions. While e.g. in the first term with k restricted to the soft domain D_s, the summation $\sum_j$ of the soft expansion is absolutely convergent, we cannot easily claim convergence for the summation $\sum_i$ of the hard expansion when $k \in D_s$. However, we are not summing expanded integrands here, but integrals. And the only scale involved in the integrals over the doubly-expanded integrand $T_{i,j}^{(h,s)} I$ (2.29) originates from the boundary of the integration domain, as all occurrences of the momentum p in the scalar products $k \cdot p$ in the numerator can be pulled out of the integral. In fact, for dimensional reasons, we know that

$$\int_{k\in D_s} \mathrm{D}k\, T_{i,j_1,j_2}^{(h,s)} I \propto |p^2|^{-n_1}\,(\Lambda^2)^{\frac{d}{2}-n_2} \left(\frac{m^2}{\Lambda^2}\right)^i \left(\frac{\Lambda^2}{|p^2|}\right)^{j_1+j_2/2}\,, \tag{2.32}$$

because the only dimensionful parameter in the definition of the domain D_s is Λ^2. By the same reasoning, (2.32) holds if the integral is restricted to $k \in D_h$ instead. As the boundary has been chosen to obey $m^2 \ll \Lambda^2 \ll |p^2|$, both the summations over i and over j_1, j_2 in (2.32) converge absolutely, and their order can be exchanged at will.

We are now able to collect all pieces contributing to the integral F. Writing

$$F_j^{(h)} = \int \mathrm{D}k\, T_j^{(h)} I\,, \qquad F_j^{(s)} = \int \mathrm{D}k\, T_j^{(s)} I\,, \qquad F_{i,j}^{(h,s)} = \int \mathrm{D}k\, T_{i,j}^{(h,s)} I \tag{2.33}$$

for the individual contributions and

$$F^{(h)} = \sum_j F_j^{(h)}\,, \qquad F^{(s)} = \sum_j F_j^{(s)}\,, \qquad F^{(h,s)} = \sum_{i,j} F_{i,j}^{(h,s)} \tag{2.34}$$

for the summed-up series, we obtain

$$F = F^{(h)} + F^{(s)} - F^{(h,s)} \tag{2.35}$$

for the original integral after the above transformations. Note that all integrations involved in (2.35) are performed over the whole integration domain $\mathbb{R}^d$. So all restrictions to the two individual domains D_h and D_s drop out and the final terms in (2.35) are individually independent of the separating scale Λ^2. Thus the exact position of the boundary between the domains is irrelevant, and we could have defined the domains e.g. in the following way:

$$D_h = \left\{k \in \mathbb{R}^d :\, |k^2| \gg m^2\right\}, \qquad D_s = \left\{k \in \mathbb{R}^d :\, |k^2| \lesssim m^2\right\}, \tag{2.36}$$

where "$\lesssim$" is understood as the negation of "$\gg$", such that $D_h \cap D_s = \emptyset$ and $D_h \cup D_s = \mathbb{R}^d$ hold. In later examples we will not introduce specific boundaries between convergence domains, but use rather "sloppy" specifications of the domains as above. It is understood, however, that exact positions of the boundaries exist and could be specified if needed.

The first two terms in the final identity (2.35) correspond exactly to the contributions from the hard (2.8) and soft (2.9) regions prescribed by the expansion by regions and evaluated in the previous section. But now we have obtained a third term, subtracted from the first two. This additional *overlap contribution* $F^{(h,s)}$ is absent in the recipe formulated in section 1. Let us have a look at its terms:

$$F_{i,j_1,j_2}^{(h,s)} = \int \mathrm{D}k\, T_{i,j_1,j_2}^{(h,s)} I = \frac{(n_2)_i}{i!}\,\frac{(n_1)_{j_{12}}}{j_1!\,j_2!}\,\frac{(m^2)^i\,(-1)^{j_{12}}}{(p^2)^{n_1+j_{12}}} \int \mathrm{D}k\, \frac{(2k\cdot p)^{j_2}}{(k^2)^{n_2+i-j_1}} = 0\,. \tag{2.37}$$

These are *scaleless integrals*, which must consistently be set to zero when using dimensional regularization. In fact, each of the integrals in (2.37) can be transformed by tensor reduction into an integral $\int \mathrm{D}k\,(k^2)^{-n}$ with some power n, and these massless tadpole integrals exhibit both ultraviolet and infrared singularities in such a way that they cancel each other. (Individual parts of the integration have different convergence domains in the complex plane of the space-time dimension d, but analytic continuation permits to combine the pieces.) The integral

$$\int \frac{\mathrm{D}k}{(k^2)^2} = \frac{1}{\epsilon_{\mathrm{UV}}} - \frac{1}{\epsilon_{\mathrm{IR}}} = 0 \tag{2.38}$$

is the only case in this class of integrals where ultraviolet poles $1/\epsilon_{\mathrm{UV}}$ or infrared poles $1/\epsilon_{\mathrm{IR}}$ appear. In the special case $n_1 = 1$, $n_2 = 2$, where the original integral F is finite, the integrals of the overlap contribution (2.37) exhibit exactly those ultraviolet singularities to cancel the ones in the soft contribution $F^{(s)}$ and those infrared singularities to cancel the ones in the hard contribution $F^{(h)}$. Although the overlap contribution $F^{(h,s)}$ is scaleless and vanishes, it is this term which makes the complete result (2.35) separately ultraviolet-finite

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and infrared-finite. One can check explicitly (see appendix A.1.2) that the summed-up overlap contributions (2.37) with ultraviolet and infrared $1/\epsilon$ poles separated cancel the corresponding poles in (2.17).

Note that the overlap contribution terms (2.37) are the same as the doubly expanded terms arising in an expansion by subgraphs [4].

Having checked that the overlap contribution $F^{(h,s)}$ is scaleless and vanishes, the original integral is reproduced by

$$F = F^{(h)} + F^{(s)} . \tag{2.39}$$

This is exactly the sum of contributions which has been evaluated in section 2.1, where we have assumed that we need these two regions (hard and soft) and evaluated them according to the recipe of the expansion by regions. Now we have obtained the same answer by mathematically transforming the original integral. Let us recapitulate what we had to check on our way:

- For the two regions we had to find domains (D_h and D_s) where their expansions converge absolutely. These domains have to be non-intersecting ($D_h \cap D_s = \emptyset$) and cover the complete integration domain ($D_h \cup D_s = \mathbb{R}^d$).
- In the double expansion the order of the two expansions has to be irrelevant ("commuting expansions").
- The overlap contribution from the double expansion involves only scaleless integrals.

These three points had to be proven explicitly for the example integral. The rest of the transformations used in this section is general and applies to any other integral with a set of regions and domains obeying analogous conditions. We will work this out in the next section.

Note that we did not have to evaluate any of the integrals in $F^{(h)}$, $F^{(s)}$ or $F^{(h,s)}$ in order to prove the identity (2.35). It is sufficient to study the expansions at the integrand level. And even for the form (2.39), where the scaleless overlap contribution has been dropped, a look at the expanded integrand has been enough (although in other cases it can be more involved to show that the overlap contributions are scaleless).

Remember how important it is within the framework of the expansion by regions that scaleless integrals can be set to zero. In our example it is dimensional regularization which regularizes scaleless integrals in such a way that they vanish. In some cases (see in particular the examples in sections 6 and 7) this is not sufficient, and we have to use analytic regularization as well. In the absence of such nicely behaved regularization schemes, however, interesting patterns appear (see section 7.3 and appendix B) where overlap contributions play an important role.

3 General formalism with commuting expansions

In this section the proof of section 2.2 is generalized to arbitrary integrals, with some restrictions. Consider the following situation:

- We want to expand the *integral*

$$F = \int \mathrm{D}k\, I\,, \tag{3.1}$$

where the integrand I is integrated over the integration domain D. This can be a one-loop integral (with $D = \mathbb{R}^d$), a multi-loop integral ($D = \mathbb{R}^{n \cdot d}$) or any arbitrary integral.

- We have identified a set

$$R = \{x_1, \ldots, x_N\} \tag{3.2}$$

of N *regions* x_i. Each region x is characterized by an *expansion*

$$T^{(x)} \equiv \sum_j T_j^{(x)} \tag{3.3}$$

which, when applied to the integrand, replaces the latter by a series of expanded terms. The summation index j can also represent a set of indices $j_1, j_2, \ldots$, but we only write one index per expansion. These expansion operators also have to be defined when they are applied to terms resulting from previous expansions (when multiple expansions are generated). In such cases it may happen that a certain expansion is an identity transformation and the set of summation indices represented by j is empty.

- For each region x there is a *domain* $D_x \subset D$ such that the expansion $T^{(x)}$ *converges absolutely* when the integration variable is restricted to $k \in D_x$.

Let us assume that the regions, expansions and domains fulfill the following *conditions*:

1. The domains are non-intersecting and cover the complete integration domain:

$$D_x \cap D_{x'} = \emptyset \;\forall x \neq x'\,, \qquad \bigcup_{x \in R} D_x = D\,. \tag{3.4}$$

2. All expansions *commute* with each other:

$$T^{(x)} T^{(x')} = T^{(x')} T^{(x)} \;\forall x, x' \in R\,. \tag{3.5}$$

3. The original integral itself and all integrals over expanded terms — whether restricted to some convergence domain D_x or not — are regularized.

4. The series expansions $T^{(x)}$ converge absolutely (or are properly regularized) even when the expanded terms are integrated over the whole integration domain D instead of just their convergence domain D_x.

Condition 1 ensures that any integral over the complete integration domain can be split into integrals restricted to the domains D_x:

$$\int \mathrm{D}k \equiv \int_{k \in D} \mathrm{D}k = \sum_{x \in R} \int_{k \in D_x} \mathrm{D}k\,. \tag{3.6}$$

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The consequence of this condition is that we might have to invent "auxiliary" regions (which do not contribute to the final result) in order to cover the complete integration domain D with the convergence domains D_x.

Condition 2 is to be understood at the operator level of the expansions: Whatever integrand the double expansion $T^{(x)}T^{(x')}$ is applied to, whether the original integrand I or a term resulting from previous expansions, the order in which the expansions are performed must be irrelevant, i.e. in all cases the same (multiple) series of doubly expanded integrand terms is established. This condition, however, cannot be fulfilled in all cases where the expansion by regions has been applied successfully. In section 5 this restriction will be relaxed and the treatment of non-commuting expansions will be presented.

Condition 3 implies that we use a regularization prescription which provides a mathematically well-defined meaning to all integrals occurring in the formalism described in this section. Usually this is dimensional regularization, eventually combined with analytic regularization, but other schemes are possible.

Finally condition 4 requires a certain mechanism that makes series expansions converge even outside their convergence domains. For loop integrals with dimensional regularization this is usually the case. More generally, this mostly works for integrals where the boundaries either lie at zero or at infinity such that they do not introduce new scales. Then the integration over the complete domain D, although formally divergent, is regularized and determined only by the scaleful parameters within the convergence domain, keeping the series expansions as convergent as with integrals restricted to the convergence domains D_x. It is possible to apply the expansion by regions to other integrals, e.g. with domains D involving finite boundaries, where condition 4 is possibly violated. But then the summation of the series expansions has to be done with care, and it might be necessary to combine certain terms which are individually divergent. This is a subtle issue in the expansion by region and in the formalism presented here. Its consequences will be pointed out at the relevant steps later in this section, and appendix B presents an example illustrating this behaviour.

Let us introduce the notations to be used in this section. They have partially already been defined for the example in section 2. Multiple expansions (replacing some integrand first by its expansion terms according to one region and repeating this step with the resulting terms for other regions) are denoted by

$$T^{(x_1,x_2,\ldots)}_{j_1,j_2,\ldots} \equiv T^{(x_1)}_{j_1} T^{(x_2)}_{j_2} \cdots \,, \qquad T^{(x_1,x_2,\ldots)} \equiv \sum_{j_1,j_2,\ldots} T^{(x_1,x_2,\ldots)}_{j_1,j_2,\ldots} \,, \tag{3.7}$$

if the expansions are commuting, i.e. their order is irrelevant. The j-th order expanded integral according to the region x is denoted by

$$F^{(x)}_j \equiv \int \mathrm{D}k \, T^{(x)}_j I \,, \tag{3.8}$$

and its summation to all orders by

$$F^{(x)} \equiv \sum_j F^{(x)}_j = \sum_j \int \mathrm{D}k \, T^{(x)}_j I \,, \tag{3.9}$$

where the integrals are performed over the complete integration domain D. Analogous notations are used for multiple expansions:

$$F^{(x_1,x_2,\ldots)}_{j_1,j_2,\ldots} \equiv \int \mathrm{D}k\, T^{(x_1,x_2,\ldots)}_{j_1,j_2,\ldots} I\,, \qquad F^{(x_1,x_2,\ldots)} \equiv \sum_{j_1,j_2,\ldots} F^{(x_1,x_2,\ldots)}_{j_1,j_2,\ldots}\,. \tag{3.10}$$

The restriction of an integration to a domain D_x is indicated by a lower index in square brackets:

$$F_{[x]} \equiv \int_{k\in D_x} \mathrm{D}k\, I\,, \qquad F^{(x',\ldots)}_{j,\ldots\,[x]} \equiv \int_{k\in D_x} \mathrm{D}k\, T^{(x',\ldots)}_{j,\ldots} I\,, \qquad \text{etc.} \tag{3.11}$$

If the integration is performed over the combination of several domains, we write

$$F_{[x_1+\ldots+x_n]} \equiv \sum_{i=1}^{n} F_{[x_i]}\,, \qquad F^{(x',\ldots)}_{j,\ldots\,[x_1+\ldots+x_n]} \equiv \sum_{i=1}^{n} F^{(x',\ldots)}_{j,\ldots\,[x_i]}\,. \tag{3.12}$$

The absolute convergence of each expansion $T^{(x)}$ within the corresponding domain D_x implies that we can safely replace any integrand I' (the original integrand I or the result of previous expansions) by its expanded series if the integration variable is restricted to $k \in D_x$:

$$I' = T^{(x)} I' = \sum_j T^{(x)}_j I' \quad \text{for } k \in D_x\,. \tag{3.13}$$

Absolute convergence also implies that we can pull the summation of such an expansion out of an adequately restricted integral, using the notation of (3.8)–(3.11):

$$F_{[x]} = F^{(x)}_{[x]} = \sum_j F^{(x)}_{j\,[x]}\,, \qquad F^{(x',\ldots)}_{j',\ldots\,[x]} = \sum_j F^{(x,x',\ldots)}_{j,j',\ldots\,[x]}\,. \tag{3.14}$$

Now we can start with the original integral (3.1) and split the integration into the N domains corresponding to the N regions, according to (3.6). In each domain we replace the integral by its series expansion according to (3.14):

$$F = \sum_{x\in R} F_{[x]} = \sum_{x\in R} F^{(x)}_{[x]}\,. \tag{3.15}$$

The right-hand side of (3.15) involves a sum of N series expansions with each integral restricted to the corresponding convergence domain. This is a special case (with $n = 1$) of the expression

$$\sum_{\{x'_1,\ldots,x'_n\}\subset R} F^{(x'_1,\ldots,x'_n)}_{[x'_1+\ldots+x'_n]}\,, \tag{3.16}$$

where the sum runs over all subsets of n distinct regions out of the N regions in R ($1 \le n \le N$). Each integrand is multiply expanded according to these n regions, and the integrals are performed over the combination of the n corresponding domains. Let us postpone for a few lines the question whether the expression (3.16) is a convergent series expansion, despite the fact that (for $n > 1$) the integrations are performed over larger domains than the convergence domain of each of the n individual expansions.

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If $n < N$, i.e. if the integrations in (3.16) are not performed over the complete integration domain D yet, the regularization of the integrals (see condition 3 above) allows to extend all these integrations to $k \in D$ when compensating for this by subtracting the integrations over the additional domains:

$$\begin{aligned} F^{(x'_1,\ldots,x'_n)}_{[x'_1+\ldots+x'_n]} &= \sum_{j_1,\ldots,j_n} F^{(x'_1,\ldots,x'_n)}_{j_1,\ldots,j_n\,[x'_1+\ldots+x'_n]} \\ &= \sum_{j_1,\ldots,j_n} \left(F^{(x'_1,\ldots,x'_n)}_{j_1,\ldots,j_n} - \sum_{x'_{n+1}\in R\setminus\{x'_1,\ldots,x'_n\}} F^{(x'_1,\ldots,x'_n)}_{j_1,\ldots,j_n\,[x'_{n+1}]} \right) . \end{aligned} \tag{3.17}$$

The subtraction terms are integrals performed over one domain $D_{x'_{n+1}}$ each (where x'_{n+1} runs over all regions which are absent in the subset $\{x'_1, \ldots, x'_n\}$). These subtraction terms can be replaced by their expansions according to (3.14):

$$F^{(x'_1,\ldots,x'_n)}_{j_1,\ldots,j_n\,[x'_{n+1}]} = \sum_{j_{n+1}} F^{(x'_1,\ldots,x'_n,x'_{n+1})}_{j_1,\ldots,j_n,j_{n+1}\,[x'_{n+1}]} , \tag{3.18}$$

where we have already used condition 2 above that the expansions commute. Now we sum the individual terms in (3.17) separately and write

$$F^{(x'_1,\ldots,x'_n)}_{[x'_1+\ldots+x'_n]} = F^{(x'_1,\ldots,x'_n)} - \sum_{x'_{n+1}\in R\setminus\{x'_1,\ldots,x'_n\}} F^{(x'_1,\ldots,x'_n,x'_{n+1})}_{[x'_{n+1}]} . \tag{3.19}$$

This is a non-trivial step. Even if the complete expression (3.17) is a finite series expansion, this does not necessarily mean that all summations in (3.19) are individually convergent. Depending on the parameters involved, especially the boundaries of the complete domain D and the boundaries between the individual domains D_x, we might have to regularize the summations in (3.17). We may e.g. think of truncating the summations by some upper limit ($j_i \leq j_{\max}\ \forall i$), thus limiting the accuracy of the expanded expressions, but dealing only with finite sums. This truncation is removed (by $j_{\max} \to \infty$) only in the end when the summations have been combined into convergent ones (cf. condition 4 above).

For reproducing (3.16) we finally we have to sum (3.19) over all subsets of n regions $\{x'_1, \ldots, x'_n\}$. The subtraction terms yield summations over subsets of $(n + 1)$ regions, where each term appears $(n + 1)$ times with different integration domains $D_{x'_{n+1}}$:

$$\begin{aligned} \sum_{\{x'_1,\ldots,x'_n\}\subset R} \; \sum_{x'_{n+1}\in R\setminus\{x'_1,\ldots,x'_n\}} F^{(x'_1,\ldots,x'_n,x'_{n+1})}_{[x'_{n+1}]} &= \sum_{\{x'_1,\ldots,x'_{n+1}\}\subset R} \; \sum_{i=1}^{n+1} F^{(x'_1,\ldots,x'_{n+1})}_{[x'_i]} \\ &= \sum_{\{x'_1,\ldots,x'_{n+1}\}\subset R} F^{(x'_1,\ldots,x'_{n+1})}_{[x'_1+\ldots+x'_{n+1}]} . \end{aligned} \tag{3.20}$$

This requires not only that the expansion commute (condition 2), but also that the order of the series summations in (3.20) can be exchanged, which in turn requires their absolute convergence. For the example presented in section 2.2 this has been shown explicitly. I cannot provide a rigorous proof for this convergence issue in the general case treated here, but I am convinced that even divergent series in intermediate steps of this derivation (which then have to be regularized) are not problematic when condition 4 ensures the convergence of those sums which remain in the final result. Note that this problem only arises when the series expansions are summed up to all orders. If an approximation with limited accuracy is sought and the series expansions are truncated at some finite order of the expansion parameter, then convergence problems of individual (intermediary or final) terms are absent.

Combining all terms from (3.19) and (3.20), the expression (3.16) yields

$$\sum_{\{x'_1,\ldots,x'_n\}\subset R} F^{(x'_1,\ldots,x'_n)}_{[x'_1+\ldots+x'_n]} = \sum_{\{x'_1,\ldots,x'_n\}\subset R} F^{(x'_1,\ldots,x'_n)} - \sum_{\{x'_1,\ldots,x'_{n+1}\}\subset R} F^{(x'_1,\ldots,x'_{n+1})}_{[x'_1+\ldots+x'_{n+1}]} . \tag{3.21}$$

The first term on the right-hand side consists of integrals performed over the complete domain D. We want to keep such terms for the final result. The second term is exactly the same as the one on the left-hand side, but with n replaced by $(n+1)$ and with opposite sign. Thus (3.21) represents a recursion formula which can be iterated from $n = 1$, as in (3.15), up to $n = N - 1$. This allows us to write the original integral in the following form:

$$F = \sum_{x\in R} F^{(x)} - \sum_{\{x'_1,x'_2\}\subset R} F^{(x'_1,x'_2)} + \ldots - (-1)^n \sum_{\{x'_1,\ldots,x'_n\}\subset R} F^{(x'_1,\ldots,x'_n)} + \ldots - (-1)^N F^{(x_1,\ldots,x_N)} . \tag{3.22}$$

This is the master identity for the expansion by regions in the formalism presented in this section. It involves only integrations over the complete domain D, so at least the series expansions in this final result are all convergent if condition 4 holds.

The master identity (3.22) involves single and multiple expansions, according to the N regions $x_1, \ldots, x_N$ and all of their combinations, with alternating signs. The recipe for the expansion by regions presented in section 1 only knows about the first term on the right-hand side of (3.22), where a single-expanded integrand according to each region is integrated over the complete domain. This means that all other terms with multiple expansions must vanish in "normal" situations for the recipe to be valid.

Indeed, if the regularization of a loop integral and the regions are chosen properly, then each of the terms $F_j^{(x)}$ (i.e. the single-expanded terms present in the known recipe) is a single-scale integral yielding a *homogeneous* function of the expansion parameter with a *unique scaling*[1] (cf. section 2.1). Every further expansion of such a single-scale integral according to a different region (which would yield a different scaling with the expansion parameter) makes the integral scaleless such that it is set to zero. This is why usually the terms with multiple expansions do not contribute to the asymptotic expansion of loop integrals.

But the identity (3.22) is more general: It is independent of the chosen regularization scheme, as long as all individual terms are mathematically well-defined (as required by

[1] This means that each term in the series of expanded integrals depends on the expansion parameter by a simple power which is different for each region. If two regions share the same dependence on the expansion parameter, then their overlap contribution does not have to be scaleless. Note that single-scale integrals as understood above may exhibit a non-trivial dependence on additional parameters or ratios of $\mathcal{O}(1)$ which are considered neither small nor large in the expansion.

conditions 3 and 4). The identity (3.22) can also be applied to other types of integrals or to loop integrals which use other regularizations than the standard dimensional and analytic ones. In such cases the *overlap contributions*, i.e. the terms in (3.22) with expansions according to more than one region, may become relevant.

While these overlap contributions arise naturally out of the formalism presented here, they have already been noted in the context of effective theories and are called "zero-bin subtractions" there (see e.g. [11, 12]). The overlap contributions in (3.22) have exactly the same form of multiple expansions as what is — usually only to leading order — introduced in the literature under the name zero-bin subtractions. The identity (3.22) clarifies the whole picture of subtractions which are needed in the general case.

One example where overlap contributions (or zero-bin subtractions) are relevant is provided in [12] where Δ-regulators are introduced which push all propagator denominators artificially off-shell by some amount. These Δ-regulators introduce new scales into the integrals. Therefore the contributions from each region are not homogeneous functions of the expansion parameter and the overlap contributions are not scaleless. Note that the authors of [12] only consider the overlap of each collinear region $n, \bar{n}$ with the soft region s, in the language of this section $F^{(n,s)}$ and $F^{(\bar{n},s)}$. The identity (3.22) would have told them that the full result is (assuming that this set of regions is complete)

$$F = F^{(n)} + F^{(\bar{n})} + F^{(s)} - F^{(n,s)} - F^{(\bar{n},s)} - F^{(n,\bar{n})} + F^{(n,\bar{n},s)} \,. \tag{3.23}$$

But the soft expansion of their integral is identical to the combination of the two collinear expansions, such that $F^{(s)} = F^{(n,\bar{n})} = F^{(n,s)} = F^{(\bar{n},s)} = F^{(n,\bar{n},s)}$. Cancelling the last two terms in (3.23), the result can be written as

$$F = \left(F^{(n)} - F^{(n,s)}\right) + \left(F^{(\bar{n})} - F^{(\bar{n},s)}\right) + F^{(s)} \,, \tag{3.24}$$

as they did, or by omitting the (irrelevant) soft region:

$$F = F^{(n)} + F^{(\bar{n})} - F^{(n,\bar{n})} \,. \tag{3.25}$$

The master identity (3.22) is an exact relation for the original integral F. It involves the summation of the single and multiple series expansions to all orders according to the definitions (3.9) and (3.10). If instead only the leading terms of the right-hand side in (3.22) are taken into account, then an approximation for the integral F on the left-hand side is obtained. The expansion parameter by whose powers higher-order terms are suppressed is related to the parameter hierarchies exploited in the expansions $T^{(x)}$. While intermediate expressions in the derivation of this formula involve the boundaries between the convergence domains, the final result (3.22) is independent of these boundaries, and so are the series expansions in the master identity.

A leading-order asymptotic expansion of the integral F can also be obtained directly without ever touching infinite series expansions. If, in the steps (3.15) and (3.18) above, the integrands are simply replaced by their leading-order expansion terms according to the regions x and x'_{n+1}, respectively,

$$F \to \sum_{x \in R} F^{(x)}_{0\,[x]} \,, \qquad F^{(x'_1,\ldots,x'_n)}_{0,\ldots,0\,[x'_{n+1}]} \to F^{(x'_1,\ldots,x'_n,x'_{n+1})}_{0,\ldots,0,0\,[x'_{n+1}]} \,, \tag{3.26}$$

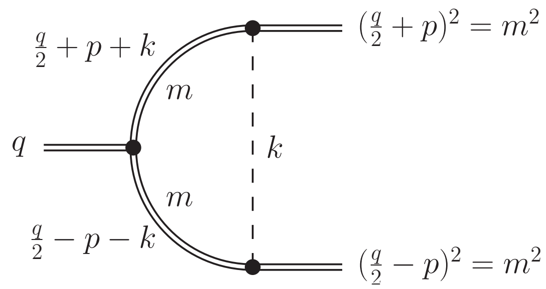

Figure 2. Loop integral for the threshold expansion.

then higher-order terms are neglected which are suppressed either by powers of the expansion parameter or by other small parameter ratios involving the boundaries between the convergence domains. Remember that these expansions are only introduced when they are absolutely convergent because the integration variable is restricted to the corresponding domain, D_x or $D_{x'_{n+1}}$, respectively. In the course of the derivation above, with all summations replaced by their leading terms, the contributions are combined such that any dependence on the boundaries cancels out. Finally the leading-order approximation

$$F_0 = \sum_{x\in R} F_0^{(x)} - \sum_{\{x'_1,x'_2\}\subset R} F_{0,0}^{(x'_1,x'_2)} + \ldots - (-1)^n \sum_{\{x'_1,\ldots,x'_n\}\subset R} F_{0,\ldots,0}^{(x'_1,\ldots,x'_n)} + \ldots - (-1)^N F_{0,\ldots,0}^{(x_1,\ldots,x_N)} \tag{3.27}$$

is obtained, which reproduces the original integral F up to terms suppressed by powers of the expansion parameter. The leading contributions from the different regions may start at different powers of the expansion parameter such that only some of the terms in (3.27) are actually leading-order contributions while others are suppressed.

The leading-order expression (3.27), although it may be derived from the all-order result (3.22), has a value of its own because it still holds when the validity of condition 3 above cannot be verified for higher-order terms or when condition 4 is violated because the summations do not converge individually. Appendix B illustrates such a behaviour of the expansion by regions with an example involving a finite integration boundary and non-converging series expansions.

4 Example: threshold expansion

Before generalizing the formalism of section 3 to non-commuting expansions, let us apply it to another example. The threshold expansion of the one-loop three-point integral presented here is also the first example treated in the original paper [1]. It is illustrated in figure 2. We choose the loop-momentum parametrization which the authors of [1] only use for the soft/ultrasoft region because this parametrization is better adapted for what we want to demonstrate here:

$$
\begin{aligned}
&F = \int \mathrm{D}k\, I\,, \quad \text{with } I = I_1 I_2 I_3 \quad \text{and} \\
&I_1 = \frac{1}{\left((\frac{q}{2}+p+k)^2 - m^2\right)^{n_1}} = \frac{1}{(k^2 + q\cdot k + 2p\cdot k)^{n_1}}\,, \\
&I_2 = \frac{1}{\left((\frac{q}{2}-p-k)^2 - m^2\right)^{n_2}} = \frac{1}{(k^2 - q\cdot k + 2p\cdot k)^{n_2}}\,, \qquad I_3 = \frac{1}{(k^2)^{n_3}}\,.
\end{aligned} \tag{4.1}
$$

We will only evaluate this integral for $n_1 = n_2 = n_3 = 1$, but we need the general propagator powers for the analytic regularization of some contributions. In the expressions (4.1) for the propagators the on-shell conditions $(\frac{q}{2} \pm p)^2 = m^2$ have been used. These also imply $q \cdot p = 0$ and $p^2 = m^2 - q^2/4$. We are interested in the threshold regime $q^2 \approx (2m)^2$, where

$$
q^2 \gg |p^2|\,. \tag{4.2}
$$

Let us choose as a specific reference frame the centre-of-mass system of the momentum q, where

$$
(q^\mu) = (q_0, \vec{0}\,) \quad \text{and} \quad (p^\mu) = (0, \vec{p}\,)\,. \tag{4.3}
$$

The propagators then read

$$
\begin{aligned}
&I_1 = \frac{1}{(k_0^2 - \vec{k}^2 + q_0 k_0 - 2\vec{p}\cdot\vec{k})^{n_1}}\,, \qquad I_2 = \frac{1}{(k_0^2 - \vec{k}^2 - q_0 k_0 - 2\vec{p}\cdot\vec{k})^{n_2}}\,, \\
&I_3 = \frac{1}{(k_0^2 - \vec{k}^2)^{n_3}}\,,
\end{aligned} \tag{4.4}
$$

and we are looking for an expansion in powers of $|\vec{p}\,|/q_0 \ll 1$. The loop-momentum components k_0 and $\vec{k}$ are multiplied with prefactors of different orders of magnitude in the propagators I_1 and I_2. Thus it is natural that we also get a region which distinguishes between k_0 and $\vec{k}$. The three regions needed for this example are

- the *hard region* (h), characterized by $k_0 \sim \vec{k} \sim q_0$, with the expansion

$$
T^{(h)} I_{1,2} = \sum_{j=0}^{\infty} \frac{(n_{1,2})_j}{j!} \frac{(2\vec{p}\cdot\vec{k})^j}{(k_0^2 - \vec{k}^2 \pm q_0 k_0)^{n_{1,2}+j}}\,, \qquad T^{(h)} I_3 = I_3\,, \tag{4.5}
$$

  converging absolutely within $D_h = \left\{k \in D : |\vec{k}| \gg |\vec{p}\,| \vee |k_0| \gg |\vec{p}\,|\right\}$,

- the *soft region* (s), characterized by $k_0 \sim \vec{k} \sim \vec{p}$, with the expansion

$$
T^{(s)} I_{1,2} = \sum_{j_1,j_2,j_3=0}^{\infty} \frac{(n_{1,2})_{j_{123}}}{j_1!\, j_2!\, j_3!} \frac{(-k_0^2)^{j_1} (\vec{k}^2)^{j_2} (2\vec{p}\cdot\vec{k})^{j_3}}{(\pm q_0 k_0)^{n_{1,2}+j_{123}}}\,, \qquad T^{(s)} I_3 = I_3\,, \tag{4.6}
$$

  converging absolutely within $D_s = \left\{k \in D : |\vec{k}| \lesssim |k_0| \lesssim |\vec{p}\,|\right\}$,

- and the *potential region* (p), characterized by $k_0 \sim \vec{p}^2/q_0$ and $\vec{k} \sim \vec{p}$, with the expansion

$$T^{(p)} I_{1,2} = \sum_{j=0}^{\infty} \frac{(n_{1,2})_j}{j!} \frac{(-k_0^2)^j}{(-\vec{k}^2 \pm q_0 k_0 - 2\vec{p}\cdot\vec{k})^{n_{1,2}+j}} ,$$
$$T^{(p)} I_3 = \sum_{j=0}^{\infty} \frac{(n_3)_j}{j!} \frac{(-k_0^2)^j}{(-\vec{k}^2)^{n_3+j}} , \tag{4.7}$$

  converging absolutely within $D_p = \{k \in D : |k_0| \ll |\vec{k}| \lesssim |\vec{p}|\}$,

where $D = \mathbb{R}^d$ is the complete integration domain. We do not have to specify the exact positions of the boundaries between the convergence domains, and the relation "$\lesssim$" is understood as the negation of "$\gg$" like in (2.36). So condition 1 of the formalism in section 3 holds:

$$D_h \cap D_s = D_h \cap D_p = D_s \cap D_p = \emptyset , \qquad D_h \cup D_s \cup D_p = D . \tag{4.8}$$

All expansions commute with each other (condition 2), and the multiple expansions read:

$$\begin{aligned}
(h,s):&\quad T^{(h,s)} I_{1,2} = T^{(s)} I_{1,2} , && T^{(h,s)} I_3 = I_3 ,\\
(h,p):&\quad T^{(h,p)} I_{1,2} = \sum_{j_1,j_2=0}^{\infty} \frac{(n_{1,2})_{j_{12}}}{j_1!\, j_2!} \frac{(-k_0^2)^{j_1}\,(2\vec{p}\cdot\vec{k})^{j_2}}{(-\vec{k}^2 \pm q_0 k_0)^{n_{1,2}+j_{12}}} , && T^{(h,p)} I_3 = T^{(p)} I_3 ,\\
(s,p):&\quad T^{(s,p)} I_{1,2} = T^{(s)} I_{1,2} , && T^{(s,p)} I_3 = T^{(p)} I_3 ,\\
(h,s,p):&\quad T^{(h,s,p)} I_{1,2} = T^{(s)} I_{1,2} , && T^{(h,s,p)} I_3 = T^{(p)} I_3 .
\end{aligned} \tag{4.9}$$

Appendix A.2.1 demonstrates that the expansions (4.5)–(4.7) converge absolutely when the integration variable k is restricted to the corresponding domain. The domains D_h, D_s and D_p have been chosen as large as possible within the convergence domains of the expansions. They contain parts which do not correspond to the scaling of the loop momentum components specified for each region. We may e.g. have $|\vec{k}| \ll |k_0|$ within D_s, which contradicts $\vec{k} \sim k_0 \sim \vec{p}$. But by choosing such enlarged domains we avoid the introduction of additional, artificial regions for covering the complete integration domain.

The expanded integrals read

$$F^{(h)} = \sum_{j_1,j_2=0}^{\infty} \frac{(n_1)_{j_1}\,(n_2)_{j_2}}{j_1!\, j_2!} \times \int \frac{\mathrm{D}k\,(2\vec{p}\cdot\vec{k})^{j_{12}}}{(k_0^2 - \vec{k}^2 + q_0 k_0)^{n_1+j_1}\,(k_0^2 - \vec{k}^2 - q_0 k_0)^{n_2+j_2}\,(k_0^2 - \vec{k}^2)^{n_3}} ,$$

$$
\begin{aligned}
F^{(s)} &= \sum_{j_1,\dots,j_6=0}^{\infty} \frac{(n_1)_{j_{123}}\,(n_2)_{j_{456}}}{j_1!\cdots j_6!} \\
&\quad \times \int \frac{\mathrm{D}k\,(-k_0^2)^{j_{14}}\,(\vec{k}^2)^{j_{25}}\,(2\vec{p}\cdot\vec{k})^{j_{36}}}{(q_0k_0+i0)^{n_1+j_{123}}\,(-q_0k_0+i0)^{n_2+j_{456}}\,(k_0^2-\vec{k}^2)^{n_3}}\,, \\
F^{(p)} &= \sum_{j_1,j_2,j_3=0}^{\infty} \frac{(n_1)_{j_1}\,(n_2)_{j_2}\,(n_3)_{j_3}}{j_1!\,j_2!\,j_3!} \\
&\quad \times \int \frac{\mathrm{D}k\,(-k_0^2)^{j_{123}}}{(-\vec{k}^2+q_0k_0-2\vec{p}\cdot\vec{k})^{n_1+j_1}\,(-\vec{k}^2-q_0k_0-2\vec{p}\cdot\vec{k})^{n_2+j_2}\,(-\vec{k}^2)^{n_3+j_3}}\,, \\
F^{(h,s)} &= F^{(s)}\,, \\
F^{(h,p)} &= \sum_{j_1,\dots,j_5=0}^{\infty} \frac{(n_1)_{j_{12}}\,(n_2)_{j_{34}}\,(n_3)_{j_5}}{j_1!\cdots j_5!} \\
&\quad \times \int \frac{\mathrm{D}k\,(-k_0^2)^{j_{135}}\,(2\vec{p}\cdot\vec{k})^{j_{24}}}{(-\vec{k}^2+q_0k_0)^{n_1+j_{12}}\,(-\vec{k}^2-q_0k_0)^{n_2+j_{34}}\,(-\vec{k}^2)^{n_3+j_5}}\,, \\
F^{(s,p)} &= \sum_{j_1,\dots,j_7=0}^{\infty} \frac{(n_1)_{j_{123}}\,(n_2)_{j_{456}}\,(n_3)_{j_7}}{j_1!\cdots j_7!} \\
&\quad \times \int \frac{\mathrm{D}k\,(-k_0^2)^{j_{147}}\,(\vec{k}^2)^{j_{25}}\,(2\vec{p}\cdot\vec{k})^{j_{36}}}{(q_0k_0+i0)^{n_1+j_{123}}\,(-q_0k_0+i0)^{n_2+j_{456}}\,(-\vec{k}^2)^{n_3+j_7}}\,, \\
F^{(h,s,p)} &= F^{(s,p)}\,.
\end{aligned} \tag{4.10}
$$

As we will see in a moment when evaluating the expressions in (4.10), the integrals there are well-defined through dimensional and analytic regularization, and the summations are absolutely convergent. So also conditions 3 and 4 of section 3 hold, and the original integral (4.1) can be expressed through the master identity (3.22):

$$
F = F^{(h)} + F^{(s)} + F^{(p)} - F^{(h,s)} - F^{(h,p)} - F^{(s,p)} + F^{(h,s,p)}\,. \tag{4.11}
$$

Now, independent of what the contributions $F^{(s)}$ and $F^{(s,p)}$ are, we can see that they drop out because the hard expansion $T^{(h)}$ does not change the integrand if the latter has been expanded via $T^{(s)}$ before, i.e. $F^{(h,s)} = F^{(s)}$ and $F^{(h,s,p)} = F^{(s,p)}$, and therefore

$$
\big(F^{(s)} - F^{(h,s)}\big) - \big(F^{(s,p)} - F^{(h,s,p)}\big) = 0\,. \tag{4.12}
$$

So all terms including the soft expansion in (4.11) do not contribute to the result.

Examining these terms nevertheless, we recognize them as scaleless contributions: Scaling the loop momentum by $k_0 \to \lambda k_0$ and $\vec{k} \to \lambda\vec{k}$ in the integrals of $F^{(s)}$ and $F^{(s,p)}$ in (4.10), we notice that each of these integrals is identical to itself times $\lambda^{4-n_{12}-2n_3+j_{1245}-2\epsilon}$. Within dimensional regularization ($\epsilon \notin \mathbb{Z}$) and analytic regularization ($n_i \notin \mathbb{Z}$), this factor is different from 1, so the integrals are scaleless, they either vanish or diverge. Without analytic regularization, the integrals of $F^{(s)}$ and $F^{(s,p)}$ are regularized by the $(3-2\epsilon)$-dimensional $\vec{k}$-integration, but the one-dimensional k_0-integration can still be divergent when it is considered separately and evaluated before the $\vec{k}$-integration. In particular, the k_0-integration is then singular because the integration contour is pinched between two poles both at $k_0 = 0$, but on different sides of the contour.[2] So, strictly speaking, we need analytic regularization (through non-integer powers of the first two propagators) or some other additional regularization here in order to make the integrals well-defined. Then the contributions $F^{(s)}$, $F^{(h,s)}$, $F^{(s,p)}$ and $F^{(h,s,p)}$ are scaleless and must be set to zero.

A similar argument shows that the contribution $F^{(h,p)}$ in (4.10) is scaleless as well: Scaling the loop momentum components by $k_0 \to \lambda^2 k_0$ and $\vec{k} \to \lambda\vec{k}$, each of these integrals is found to be identical to itself times $\lambda^{5-2n_{123}+2j_{135}-j_{24}-2\epsilon}$. So also $F^{(h,p)}$ vanishes.

Finally we obtain

$$F = F^{(h)} + F^{(p)}\,. \tag{4.13}$$

The evaluation of these two remaining contributions is sketched in the appendices A.2.2 and A.2.3. For $n_1 = n_2 = n_3 = 1$ the results read [1]

$$\begin{aligned} F^{(h)} &= -\frac{2}{q^2}\left(\frac{4\mu^2}{q^2}\right)^{\epsilon} e^{\epsilon\gamma_{\mathrm{E}}}\,\Gamma(\epsilon)\sum_{j=0}^{\infty}\left(-\frac{4p^2}{q^2}\right)^{j}\frac{(1+\epsilon)_j}{j!\,(1+2\epsilon+2j)}\,, \\ F^{(p)} &= \frac{1}{\sqrt{q^2\,(p^2-i0)}}\left(\frac{\mu^2}{p^2-i0}\right)^{\epsilon}\frac{e^{\epsilon\gamma_{\mathrm{E}}}\,\Gamma(\frac{1}{2}+\epsilon)\,\sqrt{\pi}}{2\epsilon}\,. \end{aligned} \tag{4.14}$$

Note that the potential region only has a leading-order contribution, its higher-order integrals vanish, cf. appendix A.2.3. The series expansion of the hard contribution $F^{(h)}$ is absolutely convergent for $|p^2| \ll q^2$ as required by condition 4 in section 3. The hard contribution can be summed up and expressed through a hypergeometric function:

$$F^{(h)} = -\frac{2}{q^2}\left(\frac{4\mu^2}{q^2}\right)^{\epsilon}\frac{e^{\epsilon\gamma_{\mathrm{E}}}\,\Gamma(\epsilon)}{1+2\epsilon}\,{}_2F_1\left(1+\epsilon,\frac{1}{2}+\epsilon;\frac{3}{2}+\epsilon;-\frac{4p^2}{q^2}\right). \tag{4.15}$$

The original integral (4.1) can alternatively be solved by introducing a Mellin-Barnes representation. For $n_1 = n_2 = n_3 = 1$ the Mellin-Barnes integral reads

$$F = \frac{1}{q^2}\left(\frac{4\mu^2}{q^2}\right)^{\epsilon}\frac{e^{\epsilon\gamma_{\mathrm{E}}}}{\epsilon}\int_{-i\infty}^{i\infty}\frac{\mathrm{d}z}{2i\pi}\left(\frac{4p^2-i0}{q^2}\right)^{z}\frac{\Gamma(-z)\,\Gamma(1+\epsilon+z)}{-\frac{1}{2}-\epsilon-z}\,, \tag{4.16}$$

where the pole at $z = -\frac{1}{2}-\epsilon$ lies to the right of the integration contour. An asymptotic expansion in powers of p^2/q^2 is obtained by closing the z-contour to the right. The residues of the poles of $\Gamma(-z)$ reproduce the hard-region expansion $F^{(h)}$ in (4.14), whereas the residue at $z = -\frac{1}{2}-\epsilon$ yields the potential contribution $F^{(p)}$.

[2] In [1] the authors argue that the pinching of poles must be ignored in the soft region because these poles have already been taken into account through the potential region. In the formalism developed here we cannot use this argument because we have to take the integral as it arises from the expansions, and all possible double-counting is eliminated via the subtractions of the overlap contributions.

The Mellin-Barnes representation (4.16) also permits to perform an asymptotic expansion in the opposite case, $|p^2| \gg q^2$, by closing the contour to the left. Here only one series of residues, from the poles of $\Gamma(1+\epsilon+z)$, contributes and is expressed through a single hypergeometric function:

$$F = \frac{1}{2p^2} \left(\frac{\mu^2}{p^2 - i0} \right)^{\epsilon} e^{\epsilon\gamma_{\rm E}} \, \Gamma(\epsilon) \, {}_2F_1\left(1+\epsilon, \frac{1}{2}; \frac{3}{2}; \frac{-q^2}{4p^2 - i0}\right). \tag{4.17}$$

Through analytic continuation of this result to $q^2 \gg |p^2|$ by inversing the argument of the hypergeometric function (see e.g. [14]), the sum of $F^{(h)}$ (4.15) and $F^{(p)}$ (4.14) is recovered.

5 Formalism for non-commuting expansions

Let us return to the general case. When introducing the general formalism in section 3, we required (condition 2) that all expansions commute with each other. As we will see in examples, this condition cannot always be fulfilled by a proper choice of the regions. In the following paragraphs a generalized formalism is developed which relaxes this condition to some extent.

We start with the same situation as described at the beginning of section 3: The integral $F = \int \mathrm{D}k \, I$ with integration domain D shall be expanded. We have a set R of N regions, $R = \{x_1, \ldots, x_N\}$. Each region x is characterized by an expansion $T^{(x)} \equiv \sum_j T_j^{(x)}$ which converges absolutely within the domain D_x. Conditions 1, 3 and 4 hold, i.e. the domains are non-intersecting and cover the complete integration domain D, all integrals over expanded terms are regularized, and the series expansions $F^{(x)}$ (with integrals over the complete domain D) converge absolutely.

Condition 2 is relaxed to the new condition 2a:

2a. All expansions corresponding to regions within some subset $R_{\rm c} \subset R$ commute with each other and with expansions of any other region in R:

$$T^{(x)} T^{(x')} = T^{(x')} T^{(x)} \;\forall x \in R_{\rm c}\,, \; x' \in R\,. \tag{5.1}$$

In other words, two expansions can be interchanged if at least one of them belongs to a region from the subset $R_{\rm c}$. Without loss of generality let the subset $R_{\rm c}$ contain the first $N_{\rm c}$ regions from R:

$$R_{\rm c} = \{x_1, \ldots, x_{N_{\rm c}}\} \subset R\,, \qquad 0 \le N_{\rm c} \le N\,. \tag{5.2}$$

Let

$$R_{\rm nc} = R \setminus R_{\rm c} = \{x_{N_{\rm c}+1}, \ldots, x_N\} \tag{5.3}$$

be the subset of regions with non-commuting expansions. Then two expansions are non-commuting only if their corresponding regions both belong to $R_{\rm nc}$. We do not specify how small or large the set $R_{\rm nc}$ can be within R, but the formalism developed in the following will only provide useful statements if there are still regions left within $R_{\rm c}$. Obviously the case $N_\mathrm{c} = N - 1$ is equivalent to $N_\mathrm{c} = N$ because a single region within R_nc would still provide an expansion commuting with all others, as specified in condition 2a above.

The notations introduced in section 3 are used here as well, with one addition: According to (3.7), a multiple expansion $T^{(x_1,x_2,\ldots)} \equiv \sum_{j_1,j_2,\ldots} T^{(x_1,x_2,\ldots)}_{j_1,j_2,\ldots}$ implies that all these expansions $T^{(x_i)}_{j_i}$ commute with each other, i.e. at most one of them is a non-commuting expansion of a region from R_nc. Whenever two non-commuting expansions are applied successively, the order of their application has to be specified. We define

$$T^{(x_2' \leftarrow x_1')}_{j_2,j_1} \equiv T^{(x_2')}_{j_2} T^{(x_1')}_{j_1} , \qquad T^{(x_2' \leftarrow x_1')} \equiv \sum_{j_1,j_2} T^{(x_2' \leftarrow x_1')}_{j_2,j_1} , \qquad x_1', x_2' \in R_\mathrm{nc} , \tag{5.4}$$

as the operator which first expands according to the region x_1', then according to the region x_2', indicated by the arrow in the superscript. Such a pair of non-commuting expansions may be combined with further commuting ones corresponding to regions from R_c. They are specified in a comma-separated list because the order of their application with respect to each other and to the pair of non-commuting expansions is irrelevant:

$$T^{(x_2' \leftarrow x_1', x_3', \ldots)}_{j_2,j_1,j_3,\ldots} \equiv T^{(x_2')}_{j_2} T^{(x_1')}_{j_1} T^{(x_3')}_{j_3} \cdots = T^{(x_3')}_{j_3} \cdots T^{(x_2')}_{j_2} T^{(x_1')}_{j_1} , \qquad x_3', \ldots \in R_\mathrm{c} . \tag{5.5}$$

Exactly as in section 3, we start by splitting the integration into the N domains and expanding each restricted integral accordingly, see (3.15):

$$F = \sum_{x \in R} F^{(x)}_{[x]} . \tag{5.6}$$

The right-hand side of (5.6) is a special case (for $n = 1$) of the expression

$$\sum_{\{x_1',\ldots,x_n'\} \subset R}^{\langle R_\mathrm{c}+1 \rangle} F^{(x_1',\ldots,x_n')}_{[x_1'+\ldots+x_n']} , \tag{5.7}$$

where each integrand is multiply expanded according to n regions and integrated over the combination of the n corresponding domains. The sum runs over subsets of n distinct regions, as in section 3. But the superscript $\langle R_\mathrm{c}+1 \rangle$ indicates that the sum is restricted to such subsets which contain at most one region from R_nc with a non-commuting expansion:

$$\sum_{\{x_1',\ldots,x_n'\} \subset R}^{\langle R_\mathrm{c}+1 \rangle} \equiv \sum_{\substack{\{x_1',\ldots,x_n'\} \subset R, \\ \{x_1',\ldots,x_n'\} \cap R_\mathrm{nc} = \emptyset \text{ or } \{x\}}} \equiv \sum_{\substack{\{x_1',\ldots,x_{n-1}'\} \subset R_\mathrm{c} , \\ x_n' \in R \setminus \{x_1',\ldots,x_{n-1}'\}}} . \tag{5.8}$$

This restriction ensures that all expansions $T^{(x_1')}, \ldots, T^{(x_n')}$ in (5.7) commute with each other and that the multiple expansion can be written in the form (5.7). Obviously such a restricted sum over subsets of regions excludes any pair of regions from R_nc with non-commuting expansions.

Following the steps of section 3 in (3.17)–(3.19), we extend the integrations in (5.7) to the complete integration domain, compensating for this by subtraction terms:

$$F^{(x_1',\ldots,x_n')}_{[x_1'+\ldots+x_n']} = F^{(x_1',\ldots,x_n')} - \sum_{x_{n+1}' \in R \setminus \{x_1',\ldots,x_n'\}} F^{(x_1',\ldots,x_n')}_{[x_{n+1}']} . \tag{5.9}$$

Note that, in contrast to (3.19), the subtraction terms in (5.9) have not yet been expanded according to the additional region x'_{n+1}. We can perform this expansion $T^{(x'_{n+1})}$, but we have to distinguish two cases here: If $\{x'_1, \ldots, x'_n\} \subset R_\mathrm{c}$ or $x'_{n+1} \in R_\mathrm{c}$, then all expansions $T^{(x'_1)}, \ldots, T^{(x'_{n+1})}$ commute with each other and we proceed as in section 3:

$$F^{(x'_1,\ldots,x'_n)}_{[x'_{n+1}]} = F^{(x'_1,\ldots,x'_n,x'_{n+1})}_{[x'_{n+1}]} \,. \tag{5.10}$$

But if $x'_{n+1} \in R_\mathrm{nc}$ and also $\{x'_1, \ldots, x'_n\}$ already contains one region from R_nc (let us assume without loss of generality that $x'_1 \in R_\mathrm{nc}$ and $\{x'_2, \ldots, x'_n\} \subset R_\mathrm{c}$), then

$$F^{(x'_1,\ldots,x'_n)}_{[x'_{n+1}]} = F^{(x'_{n+1}\leftarrow x'_1,x'_2\ldots,x'_n)}_{[x'_{n+1}]} \,, \tag{5.11}$$

because the expansion $T^{(x'_1)}$ has been performed before $T^{(x'_{n+1})}$. Summing these terms according to (5.7) and (5.9), we get two contributions, one in analogy to (3.20) in section 3, the other involving pairs of non-commuting expansions:

$$\begin{aligned}
&\sum^{\langle R_\mathrm{c}+1\rangle}_{\{x'_1,\ldots,x'_n\}\subset R} \; \sum_{x'_{n+1}\in R\setminus\{x'_1,\ldots,x'_n\}} F^{(x'_1,\ldots,x'_n)}_{[x'_{n+1}]} \\
&= \sum^{\langle R_\mathrm{c}+1\rangle}_{\{x'_1,\ldots,x'_{n+1}\}\subset R} \underbrace{\sum_{i=1}^{n+1} F^{(x'_1,\ldots,x'_{n+1})}_{[x'_i]}}_{F^{(x'_1,\ldots,x'_{n+1})}_{[x'_1+\ldots+x'_{n+1}]}} + \sum_{x'_1\in R_\mathrm{nc}} \; \sum_{\substack{x'_2\in R_\mathrm{nc},\\ x'_2\neq x'_1}} \; \sum_{\{x'_3,\ldots,x'_{n+1}\}\subset R_\mathrm{c}} F^{(x'_2\leftarrow x'_1,x'_3\ldots,x'_{n+1})}_{[x'_2]} \,.
\end{aligned} \tag{5.12}$$

We arrive at the following recursion formula for the expression (5.7):

$$\begin{aligned}
\sum^{\langle R_\mathrm{c}+1\rangle}_{\{x'_1,\ldots,x'_n\}\subset R} F^{(x'_1,\ldots,x'_n)}_{[x'_1+\ldots+x'_n]} &= \sum^{\langle R_\mathrm{c}+1\rangle}_{\{x'_1,\ldots,x'_n\}\subset R} F^{(x'_1,\ldots,x'_n)} - \sum^{\langle R_\mathrm{c}+1\rangle}_{\{x'_1,\ldots,x'_{n+1}\}\subset R} F^{(x'_1,\ldots,x'_{n+1})}_{[x'_1+\ldots+x'_{n+1}]} \\
&\quad - \sum_{\substack{x'_1,x'_2\in R_\mathrm{nc},\\ x'_1\neq x'_2}} \; \sum_{\{x'_3,\ldots,x'_{n+1}\}\subset R_\mathrm{c}} F^{(x'_2\leftarrow x'_1,x'_3\ldots,x'_{n+1})}_{[x'_2]} \,.
\end{aligned} \tag{5.13}$$

The first two terms on the right-hand side of (5.13) look similar to the ones in (3.21), but their sums are restricted to combinations of expansions which commute with each other. The first term involves integrals performed over the complete domain D and will be kept for the final result. The second term is reinserted on the left-hand side when the recursion formula (5.13) is iterated from $n = 1$ as in (5.6) to $n = \min\{N_\mathrm{c} + 1, N - 1\}$. The recursion ends when $n = N_\mathrm{c}+1$ because this is the maximal number of regions which can be combined into a subset with commuting expansions, i.e. subsets containing all regions in R_c plus one region out of R_nc each. In the last step of the recursion, when $n = N_\mathrm{c} + 1$, the second term on the right-hand side of (5.13) is absent.

Finally the third term on the right-hand side of (5.13) sums over pairs of regions with non-commuting expansions plus an arbitrary number of commuting expansions. The integrals in this term are performed over the convergence domain $D_{x'_2}$ of the non-commuting expansion applied last. Also these terms are kept unchanged for the final result, so no combinations of more than two non-commuting expansions appear in this formalism.

The recursion results in the following expression for the original integral F:

$$\begin{aligned}
F = &\sum_{x\in R} F^{(x)} - \sum_{\{x'_1,x'_2\}\subset R}^{\langle R_c+1\rangle} F^{(x'_1,x'_2)} + \ldots - (-1)^n \sum_{\{x'_1,\ldots,x'_n\}\subset R}^{\langle R_c+1\rangle} F^{(x'_1,\ldots,x'_n)} \\
&+ \ldots + (-1)^{N_c} \sum_{x'\in R_{nc}} F^{(x',x_1,\ldots,x_{N_c})} \\
&+ \sum_{\substack{x'_1,x'_2\in R_{nc},\\ x'_1\neq x'_2}} \Bigg(-F^{(x'_2\leftarrow x'_1)}_{[x'_2]} + \ldots - (-1)^n \sum_{\{x''_1,\ldots,x''_n\}\subset R_c} F^{(x'_2\leftarrow x'_1,x''_1,\ldots,x''_n)}_{[x'_2]} \\
&\qquad\qquad + \ldots - (-1)^{N_c} F^{(x'_2\leftarrow x'_1,x_1,\ldots,x_{N_c})}_{[x'_2]} \Bigg) .
\end{aligned} \tag{5.14}$$

This is the generalized master identity for the expansion by regions with non-commuting expansions. The terms in the first two lines of (5.14) look familiar when compared to the ones from section 3 in (3.22). The first term is exactly the same: single expansions according to all regions, integrated over the complete integration domain, corresponding to the known recipe of the expansion by regions.

Also the following terms in the first two lines of (5.14) are similar to the identity from section 3: Multiple expansions integrated again over the complete domain. Here only those combinations of expansions appear which commute with each other, but otherwise these terms are the same as in (3.22). Whether they are scaleless or relevant contributions depends on the same properties of the regularization and choice of regions as described in section 3.

The terms in the last two lines of (5.14), however, are disturbing. Each of them is an integral performed over the convergence domain $D_{x'_2}$ of a non-commuting expansion. We would not expect such terms to appear from the expansion by regions. And we cannot be sure that the series expansions in these terms converge separately, because such restricted integrals are not covered by condition 4 from section 3. So we have to clarify why and under which circumstances these extra terms vanish.

There is a general argument why these extra terms should cancel among themselves if we only have a small number of non-commuting regions such that the boundaries of their convergence domains can be varied independently. The complete integral F must be independent of the boundaries between the convergence domains D_x. But all integrals in the large round brackets of (5.14) are only performed over the subdomain $D_{x'_2}$. If the boundary of $D_{x'_2}$ is varied infinitesimally without changing the boundaries of the other regions from R_{nc}, then the expression in the brackets and therefore the complete result changes — unless the sum of all the integrands within the brackets vanishes close to the boundary. And because the position of the boundary is quite arbitrary within some range, we may expect that the sum of the integrands vanishes all over $D_{x'_2}$ in order to make the complete result independent of the boundary.

In the following we use a different, more formal way to show that the extra terms cancel. We impose an additional condition on the expansions:

5. For every combination of two non-commuting expansions, there is a region from R_{c} whose expansion does not further change the doubly expanded integrand:

$$\forall x_1', x_2' \in R_{\mathrm{nc}},\ x_1' \neq x_2',\ \exists\, x \in R_{\mathrm{c}} : T^{(x_2' \leftarrow x_1', x)} = T^{(x_2' \leftarrow x_1')}\,. \tag{5.15}$$

In every example with non-commuting regions which I have studied so far, this is indeed the case. A few such examples are presented in sections 6 and 7 and in appendix B.2 of this paper.

If condition 5 holds, then, for every pair $x_1', x_2' \in R_{\mathrm{nc}}$, the extra terms in the round brackets of (5.14) can be grouped in pairs of two, with and without this particular region $x \in R_{\mathrm{c}}$ involved:

$$\begin{aligned} &\left(-F^{(x_2' \leftarrow x_1')}_{[x_2']} + F^{(x_2' \leftarrow x_1', x)}_{[x_2']}\right) + \ldots \\ &\quad - (-1)^n \sum_{\{x_1'',\ldots,x_n''\} \subset R_{\mathrm{c}} \setminus \{x\}} \left(F^{(x_2' \leftarrow x_1', x_1'',\ldots,x_n'')}_{[x_2']} - F^{(x_2' \leftarrow x_1', x, x_1'',\ldots,x_n'')}_{[x_2']}\right) + \ldots \end{aligned} \tag{5.16}$$

With the equality of the expansions imposed in (5.15), these combinations of integrals vanish at the integrand level, as

$$F^{(x_2' \leftarrow x_1', x)}_{[x_2']} = F^{(x_2' \leftarrow x_1')}_{[x_2']}\,, \qquad F^{(x_2' \leftarrow x_1', x, x_1'',\ldots,x_n'')}_{[x_2']} = F^{(x_2' \leftarrow x_1', x_1'',\ldots,x_n'')}_{[x_2']}\,, \tag{5.17}$$

and therefore all extra terms in the round brackets of (5.14) cancel with each other and vanish.

We conclude: If condition 5 holds, the master identity for the expansion by regions with non-commuting expansions reduces to

$$\begin{aligned} F = \sum_{x \in R} F^{(x)} - \sum_{\{x_1', x_2'\} \subset R}^{\langle R_{\mathrm{c}}+1 \rangle} F^{(x_1', x_2')} + \ldots - (-1)^n \sum_{\{x_1',\ldots,x_n'\} \subset R}^{\langle R_{\mathrm{c}}+1 \rangle} F^{(x_1',\ldots,x_n')} \\ + \ldots + (-1)^{N_{\mathrm{c}}} \sum_{x' \in R_{\mathrm{nc}}} F^{(x', x_1, \ldots, x_{N_{\mathrm{c}}})}\,, \end{aligned} \tag{5.18}$$

which reproduces the identity (3.22) in section 3 with the difference that the sums over sets of expansions are now restricted to such combinations which commute with each other.

Note that, as in section 3, it is possible to restrict the derivation of this identity to the leading-order term in each expansion, yielding the leading-order approximation F_0 (3.27) for the integral F with the same restrictions on the sums over regions as in (5.18).

6 Example: Sudakov form factor

A toy example for non-commuting expansions can be found in appendix B.2, where it is discussed in the context of divergence problems with the series expansions. Let us

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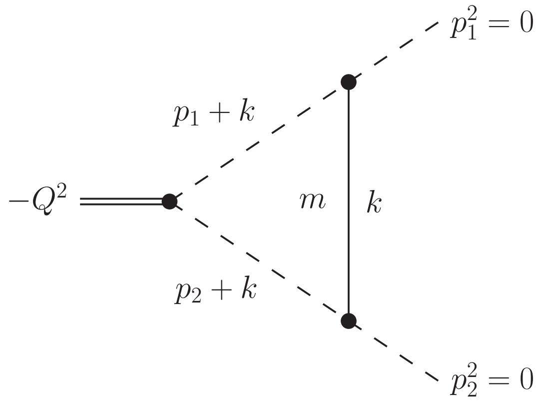

Figure 3. Vertex correction to the Sudakov form factor.

study here as a "real" loop integral with non-commuting expansions the one-loop vertex correction to the Sudakov form factor: a three-point function with one massive exchanged particle where one leg is off shell by a large amount, whereas the other two are on shell (and massless, for simplicity):

$$
\begin{aligned}
&F = \int \mathrm{D}k\, I\,, \quad \text{with } I = I_1 I_2 I_3 \quad \text{and} \\
&I_1 = \frac{1}{\big((k+p_1)^2\big)^{n_1}} = \frac{1}{(k^2 + 2p_1 \cdot k)^{n_1}}\,, \\
&I_2 = \frac{1}{\big((k+p_2)^2\big)^{n_2}} = \frac{1}{(k^2 + 2p_2 \cdot k)^{n_2}}\,, \qquad I_3 = \frac{1}{(k^2 - m^2)^{n_3}}\,,
\end{aligned}
\tag{6.1}
$$

with $p_1^2 = p_2^2 = 0$ and $(p_1 - p_2)^2 = -2p_1 \cdot p_2 = -Q^2 < 0$. This integral is illustrated in figure 3. We consider the integral in the Sudakov limit

$$
Q^2 \gg m^2\,,
\tag{6.2}
$$

so we want to expand it in powers of m^2/Q^2. We are particularly interested in the case $n_1 = n_2 = n_3 = 1$, but we will use the propagator powers, especially n_1 and n_2, as analytic regulators.

In the Sudakov limit the integral (6.1) is characterized by the two light-like directions p_1 and p_2. It is convenient to parametrize any momentum k through the light-cone coordinates

$$
k^\pm = \frac{2}{Q}\, p_{1,2} \cdot k
\tag{6.3}
$$

and the perpendicular components $k_\perp$ such that

$$
k = k^- \frac{p_1}{Q} + k^+ \frac{p_2}{Q} + k_\perp\,, \qquad \text{with } p_{1,2} \cdot k_\perp = 0\,, \qquad k_\perp^2 = -\vec{k}_\perp^2\,.
\tag{6.4}
$$

In terms of these light-cone coordinates, scalar products are given by

$$
k \cdot \ell = \frac{k^+ \ell^- + k^- \ell^+}{2} - \vec{k}_\perp \cdot \vec{\ell}_\perp\,, \qquad k^2 = k^+ k^- - \vec{k}_\perp^2\,,
\tag{6.5}
$$

the propagators read

$$I_1 = \frac{1}{(k^+k^- - \vec{k}_\perp^2 + Qk^+)^{n_1}}\,, \qquad I_2 = \frac{1}{(k^+k^- - \vec{k}_\perp^2 + Qk^-)^{n_2}}\,,$$
$$I_3 = \frac{1}{(k^+k^- - \vec{k}_\perp^2 - m^2)^{n_3}}\,, \tag{6.6}$$

and the integration measure (2.3) factorizes into

$$\int \mathrm{D}k = \frac{\mu^{2\epsilon}\, e^{\epsilon\gamma_\mathrm{E}}}{2i\pi^{d/2}} \int \mathrm{d}^{d-2}\vec{k}_\perp \int_{-\infty}^{\infty} \mathrm{d}k^+\, \mathrm{d}k^-\,. \tag{6.7}$$

When looking for the relevant regions of this integral, let us start with the known hard and collinear types:

- the *hard region* (h), characterized by $k^+ \sim k^- \sim \vec{k}_\perp \sim Q$, with the expansion

$$T^{(h)} I_{1,2} = I_{1,2}\,, \qquad T^{(h)} I_3 = \sum_{j=0}^{\infty} \frac{(n_3)_j}{j!} \frac{(m^2)^j}{(k^+k^- - \vec{k}_\perp^2)^{n_3+j}}\,, \tag{6.8}$$

  converging absolutely within $D_h = \{k \in D : \vec{k}_\perp^2 \gg m^2\}$,

- the *1-collinear region* ($1c$), characterized by $k^+ \sim m^2/Q$, $k^- \sim Q$ and $\vec{k}_\perp \sim m$, with the expansion

$$T^{(1c)} I_{1,3} = I_{1,3}\,, \qquad T^{(1c)} I_2 = \sum_{j_1,j_2=0}^{\infty} \frac{(n_2)_{j_{12}}}{j_1!\, j_2!} \frac{(-k^+k^-)^{j_1}\, (\vec{k}_\perp^2)^{j_2}}{(Qk^-)^{n_2+j_{12}}}\,, \tag{6.9}$$

  converging absolutely within

$$D_{1c} = \left\{k \in D : |k^+| \ll Q \wedge \vec{k}_\perp^2 \ll Q|k^-| \wedge \vec{k}_\perp^2 \lesssim m^2\right\},$$

- and the *2-collinear region* ($2c$), characterized by $k^+ \sim Q$, $k^- \sim m^2/Q$ and $\vec{k}_\perp \sim m$, with the expansion

$$T^{(2c)} I_1 = \sum_{j_1,j_2=0}^{\infty} \frac{(n_1)_{j_{12}}}{j_1!\, j_2!} \frac{(-k^+k^-)^{j_1}\, (\vec{k}_\perp^2)^{j_2}}{(Qk^+)^{n_1+j_{12}}}\,, \qquad T^{(2c)} I_{2,3} = I_{2,3}\,, \tag{6.10}$$

  converging absolutely within

$$D_{2c} = \left\{k \in D \setminus D_{1c} : |k^-| \ll Q \wedge \vec{k}_\perp^2 \ll Q|k^+| \wedge \vec{k}_\perp^2 \lesssim m^2\right\},$$

where $D = \mathbb{R}^d$ is the complete integration domain. The domains D_h, D_{1c}, D_{2c} have been chosen to cover as much of each expansion's convergence domain as possible, but in such a way that they are non-intersecting. In particular, in the domain with $\vec{k}_\perp^2 \lesssim m^2$ and $\vec{k}_\perp^2/Q \ll |k^\pm| \ll Q$, both collinear expansions $T^{(1c)}$ and $T^{(2c)}$ converge. Here this domain is attributed to D_{1c} and therefore explicitly excluded from D_{2c}, but this is an arbitrary choice, and the boundary between D_{1c} and D_{2c} can be chosen differently.

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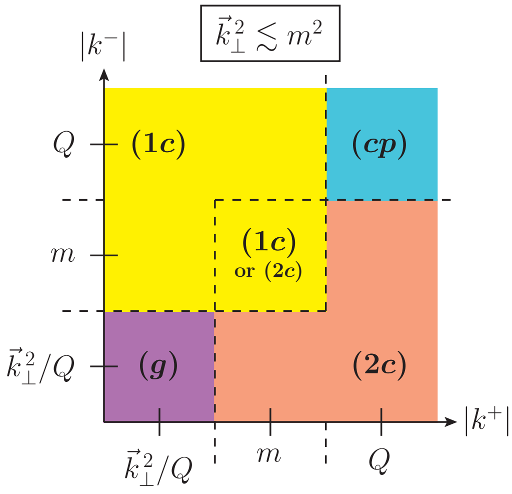

Figure 4. Convergence domains of the regions for the Sudakov form factor in the case $\vec{k}_\perp^2 \lesssim m^2$.

For the convergence of $T^{(h)}$ we assume that $|k^+k^- - \vec{k}_\perp^2| \gg m^2$ can be obtained whenever $\vec{k}_\perp^2 \gg m^2$, through an appropriate bending of the integration contours away from the real axes where necessary to avoid cancellations between the terms k^+k^- and $\vec{k}_\perp^2$. Such convergence issues have been checked explicitly for the threshold expansion in appendix A.2.1.[3]

Before we search for further relevant regions, let us state why we do not need a soft region here. The soft scaling is $k^+ \sim k^- \sim \vec{k}_\perp \sim m$, and the corresponding expansion is equivalent to the double collinear expansion, $T^{(1c,2c)}$, expanding both propagators I_1 and I_2 according to (6.9) and (6.10).[4] So a soft, i.e. double collinear expansion is involved anyway, even if we do not add a soft region. Adding a soft region would not change the result within our formalism. As $T^{(s)} = T^{(1c,2c)} = T^{(1c,s)} = T^{(2c,s)} = T^{(1c,2c,s)}$, all additional terms would cancel each other:

$$(-1)^n \left(F^{(s,x_1,\ldots,x_n)} - F^{(1c,s,x_1,\ldots,x_n)} - F^{(2c,s,x_1,\ldots,x_n)} + F^{(1c,2c,s,x_1,\ldots,x_n)} \right) = 0 \,, \tag{6.11}$$

for any set of other regions with $x_i \notin \{1c, 2c, s\}$.

To determine which further regions we need, we take a look at which parts of the integration domain $D = \mathbb{R}^d$ are already covered by D_h, D_{1c} and D_{2c}. The domain with $\vec{k}_\perp^2 \gg m^2$ is identical to D_h. For $\vec{k}_\perp^2 \lesssim m^2$, the partitioning of the $|k^+|$–$|k^-|$-plane is schematically shown in figure 4. Two corners of this plane are not covered by D_{1c} and D_{2c}: The domain where both $|k^+|$ and $|k^-|$ are small, of the order $\vec{k}_\perp^2/Q$. And the opposite corner where both $|k^\pm|$ are large, of the order Q. We associate these two missing domains

[3] Based on the same reasoning, we could have included into D_h the domain with $|k^+k^-| \gg m^2$ and $\vec{k}_\perp^2 \lesssim m^2$. But this only works as long as the term k^+k^- is present in the denominator of the third propagator. We also need a region where k^+ and k^- are both small (see below), expanding I_3 by eliminating k^+k^- from its denominator. A second expansion according to the hard region is then impossible within this domain where $\vec{k}_\perp^2 \lesssim m^2$. So we choose to exclude this domain from D_h and rather establish an additional region especially dedicated to large $k^\pm$.

[4] See the expanded integral (A.47) in appendix A.3.1 for details.

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to convergence domains of additional regions and establish the corresponding expansions. To the list of regions we add

- the *Glauber region* (g), characterized by $k^+ \sim k^- \sim m^2/Q$ and $\vec{k}_\perp \sim m$, with the expansion

$$T^{(g)} I_{1,2} = \sum_{j=0}^{\infty} \frac{(n_{1,2})_j}{j!} \frac{(-k^+k^-)^j}{(-\vec{k}_\perp^2 + Qk^\pm)^{n_{1,2}+j}}\,,$$
$$T^{(g)} I_3 = \sum_{j=0}^{\infty} \frac{(n_3)_j}{j!} \frac{(-k^+k^-)^j}{(-\vec{k}_\perp^2 - m^2)^{n_3+j}}\,, \tag{6.12}$$

  converging absolutely within $D_g = \big\{k \in D:\, Q|k^\pm| \lesssim \vec{k}_\perp^2 \lesssim m^2\big\}$,

- and the *collinear-plane region* (cp), characterized by $k^+ \sim k^- \sim Q$ and $\vec{k}_\perp \sim m$, with the expansion

$$T^{(cp)} I_{1,2} = \sum_{j=0}^{\infty} \frac{(n_{1,2})_j}{j!} \frac{(\vec{k}_\perp^2)^j}{(k^+k^- + Qk^\pm)^{n_{1,2}+j}}\,,$$
$$T^{(cp)} I_3 = \sum_{j_1,j_2=0}^{\infty} \frac{(n_3)_{j_{12}}}{j_1!\, j_2!} \frac{(\vec{k}_\perp^2)^{j_1}\,(m^2)^{j_2}}{(k^+k^-)^{n_3+j_{12}}}\,, \tag{6.13}$$

  converging absolutely within $D_{cp} = \big\{k \in D:\, |k^\pm| \gtrsim Q \,\wedge\, \vec{k}_\perp^2 \lesssim m^2\big\}$.

While the Glauber region has some physical meaning and can yield relevant contributions for other integrals or with other regularization schemes, the collinear-plane region is completely artificial and just needed to cover the integration domain with convergence domains.

We now have five regions with non-intersecting convergence domains which fulfill

$$D_h \cup D_{1c} \cup D_{2c} \cup D_g \cup D_{cp} = D\,. \tag{6.14}$$

Most of the corresponding expansions commute with each other, as one can easily verify, with one exception: The Glauber and collinear-plane expansions do not commute, their double expansions read

$$T^{(cp\leftarrow g)} I = \sum_{j_1,\ldots,j_5=0}^{\infty} \frac{(n_1)_{j_{12}}\,(n_2)_{j_{34}}\,(n_3)_{j_5}}{j_1!\cdots j_5!} \frac{(-k^+k^-)^{j_{135}}\,(\vec{k}_\perp^2)^{j_{24}}}{(Qk^+)^{n_1+j_{12}}\,(Qk^-)^{n_2+j_{34}}\,(-\vec{k}_\perp^2 - m^2)^{n_3+j_5}}\,,$$
$$T^{(g\leftarrow cp)} I = \sum_{j_1,\ldots,j_6=0}^{\infty} \frac{(n_1)_{j_{12}}\,(n_2)_{j_{34}}\,(n_3)_{j_{56}}}{j_1!\cdots j_6!} \frac{(-k^+k^-)^{j_{13}}\,(\vec{k}_\perp^2)^{j_{245}}\,(m^2)^{j_6}}{(Qk^+)^{n_1+j_{12}}\,(Qk^-)^{n_2+j_{34}}\,(k^+k^-)^{n_3+j_{56}}}\,. \tag{6.15}$$

So we have a situation where the formalism for non-commuting expansions developed in section 5 is needed. The sets of commuting and non-commuting expansions are $R_{\rm c} = \{h, 1c, 2c\}$ and $R_{\rm nc} = \{g, cp\}$, respectively. Condition 5 (p. 29) is easily satisfied, there are always several regions whose expansions do not further change the doubly expanded integrands:

$$T^{(cp\leftarrow g,1c)} = T^{(cp\leftarrow g,2c)} = T^{(cp\leftarrow g)}\,,$$
$$T^{(g\leftarrow cp,h)} = T^{(g\leftarrow cp,1c)} = T^{(g\leftarrow cp,2c)} = T^{(g\leftarrow cp)}\,. \tag{6.16}$$

Therefore the extra terms in the identity (5.14) vanish:

$$
\begin{aligned}
&- F_{[cp]}^{(cp\leftarrow g)} + F_{[cp]}^{(cp\leftarrow g,h)} + F_{[cp]}^{(cp\leftarrow g,1c)} + F_{[cp]}^{(cp\leftarrow g,2c)} \\
&\qquad - F_{[cp]}^{(cp\leftarrow g,h,1c)} - F_{[cp]}^{(cp\leftarrow g,h,2c)} - F_{[cp]}^{(cp\leftarrow g,1c,2c)} + F_{[cp]}^{(cp\leftarrow g,h,1c,2c)} \\
&\qquad\qquad = (-1+2-1)\, F_{[cp]}^{(cp\leftarrow g)} + (1-2+1)\, F_{[cp]}^{(cp\leftarrow g,h)} = 0
\end{aligned} \tag{6.17}
$$

and

$$
\begin{aligned}
&- F_{[g]}^{(g\leftarrow cp)} + F_{[g]}^{(g\leftarrow cp,h)} + F_{[g]}^{(g\leftarrow cp,1c)} + F_{[g]}^{(g\leftarrow cp,2c)} \\
&\qquad - F_{[g]}^{(g\leftarrow cp,h,1c)} - F_{[g]}^{(g\leftarrow cp,h,2c)} - F_{[g]}^{(g\leftarrow cp,1c,2c)} + F_{[g]}^{(g\leftarrow cp,h,1c,2c)} \\
&\qquad\qquad = (-1+3-3+1)\, F_{[g]}^{(g\leftarrow cp)} = 0\,.
\end{aligned} \tag{6.18}
$$

We are left with the terms of the identity (5.18),

$$
\begin{aligned}
F = {} & F^{(h)} + F^{(1c)} + F^{(2c)} + F^{(g)} + F^{(cp)} - \Big(F^{(h,1c)} + F^{(h,2c)} + F^{(h,g)} + F^{(h,cp)} \\
& + F^{(1c,2c)} + F^{(1c,g)} + F^{(1c,cp)} + F^{(2c,g)} + F^{(2c,cp)} \Big) \\
& + F^{(h,1c,2c)} + F^{(h,1c,g)} + F^{(h,1c,cp)} + F^{(h,2c,g)} + F^{(h,2c,cp)} + F^{(1c,2c,g)} + F^{(1c,2c,cp)} \\
& - \Big(F^{(h,1c,2c,g)} + F^{(h,1c,2c,cp)} \Big)\,,
\end{aligned} \tag{6.19}
$$

summing over all combinations of regions where the expansions commute with each other. The evaluation of these contributions is described in appendix A.3. The first subsection, appendix A.3.1, shows that the contributions $F^{(g)}$ and $F^{(cp)}$ from the Glauber and collinear-plane regions are scaleless, and the same holds for all overlap contributions with multiple expansions. Omitting these scaleless contributions, the integral F can be expressed as follows:

$$
F = F^{(h)} + F^{(1c)} + F^{(2c)}\,. \tag{6.20}
$$

Remember that we have used analytic regularization, and we actually need it for having all scaleless contributions well-defined.

If we had introduced a soft region and added $F^{(s)}$ to the result (6.20), according to the recipe in section 1 for the expansion by regions, then nothing would have changed because $F^{(s)} = F^{(1c,2c)} = 0$.

We also note that even if the collinear-plane region did not yield scaleless integrals, all contributions involving the collinear-plane expansion in (6.19) would cancel each other. This is because adding the hard expansion does not change the collinear-plane expansion, $T^{(h,cp)} = T^{(cp)}$ (unless applied after $T^{(g)}$), which can be seen in the expressions (A.51) in appendix A.3.1. So the contributions involving $T^{(cp)}$ can be grouped into pairs

$$
(-1)^n \left(F^{(cp,x_1,\ldots,x_n)} - F^{(h,cp,x_1,\ldots,x_n)} \right) = 0\,, \quad \text{with } \{x_1,\ldots,x_n\} \subset \{1c, 2c\}\,, \tag{6.21}
$$

which cancel each other.

The contributions remaining in the identity (6.20) read

$$
F^{(h)} = \sum_{j=0}^{\infty} \frac{(n_3)_j}{j!}\, (m^2)^j \int \frac{\mathrm{D}k}{(k^2 + 2p_1\cdot k)^{n_1}\, (k^2 + 2p_2\cdot k)^{n_2}\, (k^2)^{n_3+j}}\,,
$$

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$$F^{(1c)} = \sum_{j=0}^{\infty} \frac{(n_2)_j}{j!} \int \frac{\mathrm{D}k\,(-k^2)^j}{(k^2+2p_1\cdot k)^{n_1}\,(2p_2\cdot k)^{n_2+j}\,(k^2-m^2)^{n_3}}\,,$$
$$F^{(2c)} = \sum_{j=0}^{\infty} \frac{(n_1)_j}{j!} \int \frac{\mathrm{D}k\,(-k^2)^j}{(2p_1\cdot k)^{n_1+j}\,(k^2+2p_2\cdot k)^{n_2}\,(k^2-m^2)^{n_3}}\,, \tag{6.22}$$

based on the expansions $T^{(h)}$ (6.8), $T^{(1c)}$ (6.9) and $T^{(2c)}$ (6.10), but rewriting the expressions in a Lorentz-invariant way, for the collinear regions using (A.41). The evaluation of the hard contribution $F^{(h)}$ is straightforward with Feynman parameters (see appendix A.3.2). The collinear contributions are calculated in appendix A.3.3. The integrals yield

$$\begin{aligned}
F^{(h)} &= \frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_\mathrm{E}}\,e^{-i\pi n_{123}}}{\Gamma(n_1)\,\Gamma(n_2)\,\Gamma(n_3)}\,(Q^2)^{2-n_{123}-\epsilon} \sum_{j=0}^{\infty}\left(-\frac{m^2}{Q^2}\right)^j \frac{\Gamma(n_3+j)\,\Gamma(n_{123}-2+\epsilon+j)}{j!\,\Gamma(4-n_{123}-2\epsilon-j)} \\
&\qquad \times \Gamma(2-n_{13}-\epsilon-j)\,\Gamma(2-n_{23}-\epsilon-j)\,, \\
F^{(1c)} &= \frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_\mathrm{E}}\,e^{-i\pi n_{123}}}{\Gamma(n_1)\,\Gamma(n_2)\,\Gamma(n_3)}\,(m^2)^{2-n_{13}-\epsilon}\,(Q^2)^{-n_2} \sum_{j=0}^{\infty}\left(-\frac{m^2}{Q^2}\right)^j \\
&\qquad \times \frac{\Gamma(n_2+j)\,\Gamma(n_1-n_2-j)\,\Gamma(n_{13}-2+\epsilon-j)\,\Gamma(2-n_1-\epsilon+j)}{j!\,\Gamma(2-n_2-\epsilon-j)}\,, \\
F^{(2c)} &= \frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_\mathrm{E}}\,e^{-i\pi n_{123}}}{\Gamma(n_1)\,\Gamma(n_2)\,\Gamma(n_3)}\,(m^2)^{2-n_{23}-\epsilon}\,(Q^2)^{-n_1} \sum_{j=0}^{\infty}\left(-\frac{m^2}{Q^2}\right)^j \\
&\qquad \times \frac{\Gamma(n_1+j)\,\Gamma(n_2-n_1-j)\,\Gamma(n_{23}-2+\epsilon-j)\,\Gamma(2-n_2-\epsilon+j)}{j!\,\Gamma(2-n_1-\epsilon-j)}\,.
\end{aligned} \tag{6.23}$$

Two remarks on these results are in order. First, all terms in the series expansions in (6.23) are homogeneous functions of m^2 and Q^2, where the leading-order terms scale as

$$\begin{aligned}
F_0^{(h)} &\propto (Q^2)^{\frac{d}{2}-n_{123}}\,, \\
F_0^{(1c)} &\propto (m^2)^{\frac{d}{2}-n_{13}}\,(Q^2)^{-n_2}\,, \\
F_0^{(2c)} &\propto (m^2)^{\frac{d}{2}-n_{23}}\,(Q^2)^{-n_1}\,.
\end{aligned} \tag{6.24}$$

These scalings could have been obtained directly from the integrals (6.22), taking into account in each region the scaling of the integration measure and of each propagator with m^2 and Q^2. Using such scaling arguments, the contributions from the Glauber and collinear-plane regions would scale as

$$\begin{aligned}
F_0^{(g)} &\propto (m^2)^{\frac{d}{2}+1-n_{123}}\,(Q^2)^{-1}\,, \\
F_0^{(cp)} &\propto (m^2)^{\frac{d}{2}-1}\,(Q^2)^{1-n_{123}}\,,
\end{aligned} \tag{6.25}$$

if they were not scaleless. Each region's scaling has a unique dependence on the regularization parameters d and n_1, n_2, n_3 by which it differs from the other regions. The overlap contributions originate from multiple expansions with different scalings, so we could have expected all overlap contributions to be scaleless even without calculating them.

Second, the two collinear contributions $F^{(1c)}$ and $F^{(2c)}$ in (6.23) are individually divergent when n_1 and n_2 tend to integer numbers, in particular for the case $n_1 = n_2 = n_3 = 1$ we are interested in. The Gamma functions $\Gamma(n_1 - n_2 - j)$ and $\Gamma(n_2 - n_1 - j)$, respectively, provide single poles in the variable $(n_1 - n_2)$ once the expansion order j is large enough (starting at $j = 0$ for $n_1 = n_2$). So here analytic regularization is not only needed to make scaleless contributions well-defined, but also to prevent contributing regions from yielding ill-defined integrals.

Let us have a look at the particular case $n_1 = n_2 = n_3 = 1$. We cannot simply set all propagator powers to integer values, but we have to understand this as a limit. We use the symmetric values $n_1 = 1 + \delta$ and $n_2 = 1 - \delta$ and expand both collinear contributions in a Laurent expansion about $\delta = 0$ up to the finite order δ^0. This choice of the limit has the advantage that the overall dimension of the integral is not changed, but any other way to approach the point $n_1 = n_2 = 1$ would yield the same result for the sum $F^{(1c)} + F^{(2c)}$, though not for the individual terms.

The hard contribution is not singular in the limit $\delta \to 0$, so we can directly set $n_1 = n_2 = n_3 = 1$ here. According to (A.54) and (A.56) in appendix A.3.2,

$$\begin{aligned} F^{(h)} &= -\frac{1}{Q^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \frac{e^{\epsilon\gamma_{\mathrm{E}}}\,\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)}\,{}_2F_1\left(2\epsilon,1;1+\epsilon;\frac{m^2}{Q^2}\right) \\ &= -\frac{1}{Q^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\left[\frac{1}{\epsilon^2} - \frac{2}{\epsilon}\ln\left(1-\frac{m^2}{Q^2}\right) + \ln^2\left(1-\frac{m^2}{Q^2}\right) - 2\,\mathrm{Li}_2\left(\frac{m^2}{Q^2}\right) - \frac{\pi^2}{12}\right] \\ &\quad + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{6.26}$$

The original integral F (6.1) is finite in four dimensions (i.e. for $\epsilon = 0$) when $n_1 = n_2 = n_3 = 1$, because there are no ultraviolet divergences and the infrared (soft and collinear) divergences are absent due to the finite mass m of the exchanged particle. The hard contribution $F^{(h)}$, however, involves infrared singularities (single and double poles in ϵ) because the expansion, assuming large loop momenta, yields massless integrals. These singularities in (6.26) have to be cancelled by the contributions from the collinear regions.

The collinear contributions are evaluated for $n_1 = 1 + \delta$ and $n_2 = 1 - \delta$. The results obtained in (A.62) and (A.65) in appendix A.3.3 read

$$\begin{aligned} F^{(1c)} &= -\frac{1}{Q^2}\left(\frac{\mu^2}{m^2}\right)^{\epsilon}\left(\frac{Q^2}{m^2}\right)^{\delta} \frac{e^{\epsilon\gamma_{\mathrm{E}}}\,\Gamma(\epsilon+\delta)\,\Gamma(1-\epsilon-\delta)}{\Gamma(1-\epsilon+\delta)}\,\frac{\Gamma(2\delta)}{\Gamma(1+\delta)} \\ &\quad \times {}_2F_1\left(\epsilon-\delta,1-\delta;1-2\delta;\frac{m^2}{Q^2}\right) \\ &= -\frac{1}{2Q^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\left(\frac{1}{\delta}\left[\frac{1}{\epsilon} + \ln\left(\frac{Q^2}{m^2}\right) - \ln\left(1-\frac{m^2}{Q^2}\right)\right] - \frac{1}{\epsilon^2} + \frac{2}{\epsilon}\ln\left(1-\frac{m^2}{Q^2}\right)\right. \\ &\quad \left. + \frac{1}{2}\ln^2\left(\frac{Q^2}{m^2}\right) + \ln\left(\frac{Q^2}{m^2}\right)\ln\left(1-\frac{m^2}{Q^2}\right) - \ln^2\left(1-\frac{m^2}{Q^2}\right) + \mathrm{Li}_2\left(\frac{m^2}{Q^2}\right) + \frac{5\pi^2}{12}\right) \\ &\quad + \mathcal{O}(\delta) + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{6.27}$$

The 2-collinear contribution is obtained by replacing $\delta \to -\delta$:

$$F^{(2c)} = -\frac{1}{2Q^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\left(-\frac{1}{\delta}\left[\frac{1}{\epsilon} + \ln\left(\frac{Q^2}{m^2}\right) - \ln\left(1 - \frac{m^2}{Q^2}\right)\right] - \frac{1}{\epsilon^2} + \frac{2}{\epsilon}\ln\left(1 - \frac{m^2}{Q^2}\right)\right.$$
$$\left. + \frac{1}{2}\ln^2\left(\frac{Q^2}{m^2}\right) + \ln\left(\frac{Q^2}{m^2}\right)\ln\left(1 - \frac{m^2}{Q^2}\right) - \ln^2\left(1 - \frac{m^2}{Q^2}\right) + \mathrm{Li}_2\left(\frac{m^2}{Q^2}\right) + \frac{5\pi^2}{12}\right)$$
$$+ \mathcal{O}(\delta) + \mathcal{O}(\epsilon)\,. \tag{6.28}$$

When the two collinear contributions are added, the $1/\delta$-singularities drop out. The total collinear contribution is finite in the limit $\delta \to 0$, i.e. in the case $n_1 = n_2 = n_3 = 1$:

$$F^{(1c)} + F^{(2c)} = -\frac{1}{Q^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\left[-\frac{1}{\epsilon^2} + \frac{2}{\epsilon}\ln\left(1 - \frac{m^2}{Q^2}\right) + \frac{1}{2}\ln^2\left(\frac{Q^2}{m^2}\right)\right.$$
$$\left. + \ln\left(\frac{Q^2}{m^2}\right)\ln\left(1 - \frac{m^2}{Q^2}\right) - \ln^2\left(1 - \frac{m^2}{Q^2}\right) + \mathrm{Li}_2\left(\frac{m^2}{Q^2}\right) + \frac{5\pi^2}{12}\right]$$
$$+ \mathcal{O}(\epsilon)\,. \tag{6.29}$$

The total collinear contribution exhibits the same single and double poles in ϵ as the hard contribution $F^{(h)}$ (6.26), just with opposite signs. So the sum of hard and collinear contributions is finite in the limit $\epsilon \to 0$:

$$F = F^{(h)} + F^{(1c)} + F^{(2c)}$$
$$= -\frac{1}{Q^2}\left[\frac{1}{2}\ln^2\left(\frac{Q^2}{m^2}\right) + \ln\left(\frac{Q^2}{m^2}\right)\ln\left(1 - \frac{m^2}{Q^2}\right) - \mathrm{Li}_2\left(\frac{m^2}{Q^2}\right) + \frac{\pi^2}{3}\right] + \mathcal{O}(\epsilon)\,. \tag{6.30}$$

This result agrees with a direct evaluation of the original integral F (6.1) for $n_1 = n_2 = n_3 = 1$ via Feynman parameters without expanding in m^2/Q^2.

For general propagator powers $n_{1,2,3}$ the following Mellin-Barnes representation of the full integral F (6.1) is found:

$$F = \frac{\mu^{2\epsilon}\, e^{\epsilon\gamma_{\mathrm{E}}}\, e^{-i\pi n_{123}}}{\Gamma(n_1)\,\Gamma(n_2)\,\Gamma(n_3)}\,(Q^2)^{2-n_{123}-\epsilon}\int_{-i\infty}^{i\infty}\frac{\mathrm{d}z}{2i\pi}\left(\frac{m^2}{Q^2}\right)^{z}\,\Gamma(n_3+z)\,\Gamma(n_{123}-2+\epsilon+z)$$
$$\times\,\frac{\Gamma(-z)\,\Gamma(2-n_{13}-\epsilon-z)\,\Gamma(2-n_{23}-\epsilon-z)}{\Gamma(4-n_{123}-2\epsilon-z)}\,. \tag{6.31}$$

An asymptotic expansion for $m^2 \ll Q^2$ is obtained by closing the Mellin-Barnes integral to the right and picking up the residues of the poles from the three $\Gamma(\ldots - z)$ functions in the numerator. If the difference $(n_1 - n_2)$ is non-integer, then each of these three Gamma functions provides a series of single poles, and the three contributions $F^{(h)}$, $F^{(1c)}$ and $F^{(2c)}$ in (6.23) are exactly reproduced. Otherwise, especially in the case $n_1 = n_2 = 1$, the poles of two of the Gamma functions coincide, providing a series of double poles. For this reason the contributions $F^{(1c)}$ and $F^{(2c)}$ cannot be individually finite in the case $n_1 = n_2 = 1$, but their finite sum, $F^{(1c)} + F^{(2c)}$, corresponds to the series of residues from the double poles.

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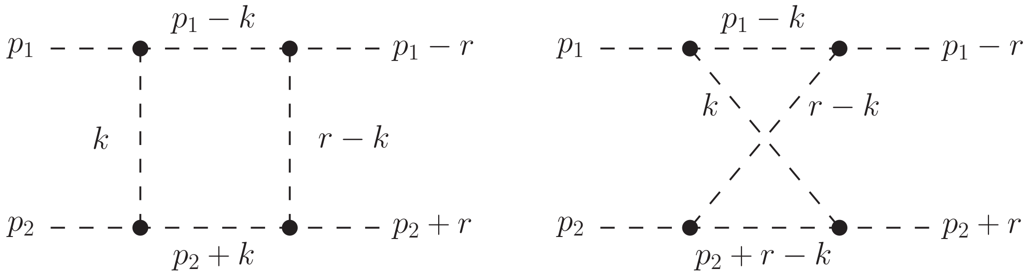

Figure 5. One-loop corrections to the forward-scattering amplitude with small momentum exchange r.

If we go one step further and set $\epsilon = 0$ in (6.31) in addition to $n_1 = n_2 = n_3 = 1$, then a series of triple poles arises which yields the finite result (6.30).

The residues of the Mellin-Barnes integrand (6.31) are in one-to-one correspondence with the contributions from the (h), $(1c)$ and $(2c)$ regions. This correspondence links singularities arising in contributions from individual regions to coinciding poles of the Mellin-Barnes integrand.

7 Example: forward scattering with small momentum exchange

Let us look at a last example which permits us to study the dependence of individual contributions on different regularization schemes, in particular with and without analytic regularization. It also is an example where overlap contributions turn out to be relevant under certain circumstances. We evaluate a set of one-loop corrections to the forward-scattering amplitude of two light-like particles with a large centre-of-mass energy, but small momentum exchange between them.

The two contributing diagrams are displayed in figure 5. For reasons to be seen later we symmetrize the loop integral under commutation of the momenta of the exchanged particles, $k \leftrightarrow (r-k)$, before specifying general propagator powers $n_{1,2,3,4}$. The loop integral reads

$$F = \frac{1}{2}\int \frac{\mathrm{D}k}{(k^2)^{n_1}\left((r-k)^2\right)^{n_2}}\left(\frac{1}{\left((p_1-k)^2\right)^{n_3}}+\frac{1}{\left((p_1-r+k)^2\right)^{n_3}}\right) \times\left(\frac{1}{\left((p_2+k)^2\right)^{n_4}}+\frac{1}{\left((p_2+r-k)^2\right)^{n_4}}\right). \quad (7.1)$$

We are interested in the case $n_1 = n_2 = n_3 = n_4 = 1$, but we will partly use the propagator powers as analytic regulators.

All internal and external lines are massless. The on-shell conditions for the external particles, $p_1^2 = (p_1 - r)^2 = p_2^2 = (p_2 + r)^2 = 0$, imply $2p_1 \cdot r = -2p_2 \cdot r = r^2$. The centre-of-mass energy shall be much larger than the momentum exchange, $Q^2 = (p_1 + p_2)^2 = 2p_1 \cdot p_2 \gg |r^2|$. We use the light-cone coordinates specified in (6.3) and (6.4) such that

$r^\pm = 2p_{1,2}\cdot r/Q = \pm r^2/Q$ and $r^2 = r^+r^- - \vec{r}_\perp^{\,2}$. The only independent kinematical parameters are Q^2 and $\vec{r}_\perp^{\,2}$ with

$$Q^2 \gg \vec{r}_\perp^{\,2}\,. \tag{7.2}$$

The dependent parameters $r^\pm$ are determined by

$$r^\pm = \mp\frac{\vec{r}_\perp^{\,2} - r^+r^-}{Q} = \mp\frac{\vec{r}_\perp^{\,2}}{Q} + \mathcal{O}\biggl(\frac{(\vec{r}_\perp^{\,2})^2}{Q^3}\biggr)\,. \tag{7.3}$$

In terms of the expansion parameter $\vec{r}_\perp^{\,2}/Q^2$, the exchanged momentum r thus has the scaling of the Glauber region. The loop integral is written using light-cone coordinates:

$$\begin{aligned}
F &= \frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_\mathrm{E}}}{2i\pi^{d/2}}\int \mathrm{d}^{d-2}\vec{k}_\perp \int_{-\infty}^{\infty}\mathrm{d}k^+\,\mathrm{d}k^-\,I\,,\\
I &= \frac{1}{2}\,\frac{1}{(k^+k^- - \vec{k}_\perp^{\,2})^{n_1}}\,\frac{1}{\bigl((k^+-r^+)(k^- - r^-) - (\vec{k}_\perp - \vec{r}_\perp)^2\bigr)^{n_2}}\\
&\quad\times\biggl(\frac{1}{\bigl(k^+(k^- - Q) - \vec{k}_\perp^{\,2}\bigr)^{n_3}} + \frac{1}{\bigl(k^+(k^- - r^- + Q) - r^+k^- - \vec{k}_\perp^{\,2} + 2\vec{r}_\perp\cdot\vec{k}_\perp\bigr)^{n_3}}\biggr)\\
&\quad\times\biggl(\frac{1}{\bigl((k^+ + Q)k^- - \vec{k}_\perp^{\,2}\bigr)^{n_4}} + \frac{1}{\bigl((k^+ - r^+ - Q)k^- - k^+r^- - \vec{k}_\perp^{\,2} + 2\vec{r}_\perp\cdot\vec{k}_\perp\bigr)^{n_4}}\biggr)\,,
\end{aligned} \tag{7.4}$$

where relation (7.3) has been used to cancel certain terms in the denominators.

7.1 Regions for the forward-scattering integral

One finds that the same regions as in the previous example of section 6 are relevant here, with m replaced by $|\vec{r}_\perp|$ for the scaling prescriptions. The convergence domains D_x ($x = h, 1c, 2c, g, cp$) are similar, but slightly more involved. They are illustrated in figure 6, where the partitioning of the $|k^+|$–$|k^-|$-plane is shown for the cases $|\vec{k}_\perp| \gg |\vec{r}_\perp|$ (left diagram) and $|\vec{k}_\perp| \lesssim |\vec{r}_\perp|$ (right diagram).

In the present example we will study all expansions only to leading order in the expansion parameter $\vec{r}_\perp^{\,2}/Q^2$. In detail the regions we need are

- the *hard region* (h), characterized by $k^+ \sim k^- \sim \vec{k}_\perp \sim Q$, with the (leading-order) expansion

$$\begin{aligned}
T_0^{(h)} I &= \frac{1}{2}\,\frac{1}{(k^+k^- - \vec{k}_\perp^{\,2})^{n_1+n_2}}\\
&\quad\times\biggl(\frac{1}{\bigl(k^+(k^- - Q) - \vec{k}_\perp^{\,2}\bigr)^{n_3}} + \frac{1}{\bigl(k^+(k^- + Q) - \vec{k}_\perp^{\,2}\bigr)^{n_3}}\biggr)\\
&\quad\times\biggl(\frac{1}{\bigl((k^+ + Q)k^- - \vec{k}_\perp^{\,2}\bigr)^{n_4}} + \frac{1}{\bigl((k^+ - Q)k^- - \vec{k}_\perp^{\,2}\bigr)^{n_4}}\biggr)\,,
\end{aligned} \tag{7.5}$$

  where the all-order expansion converges absolutely within

$$D_h = \biggl\{k\in D:\,|\vec{k}_\perp| \gg |\vec{r}_\perp| \,\wedge\, |k^\pm| \gg \frac{\vec{r}_\perp^{\,2}}{Q}\biggr\}\,, \tag{7.6}$$

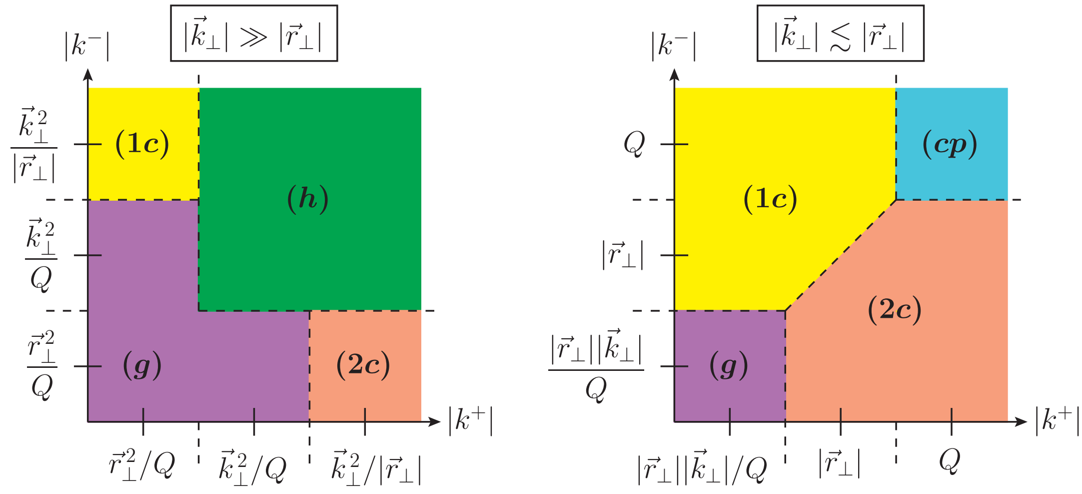

Figure 6. Convergence domains of the regions for the forward scattering with small momentum exchange.

- the *1-collinear region* (1c), characterized by $k^+ \sim \vec{r}_\perp^{\,2}/Q$, $k^- \sim Q$ and $\vec{k}_\perp \sim \vec{r}_\perp$, with the expansion

$$T_0^{(1c)} I = \frac{1}{2}\,\frac{1}{(k^+k^- - \vec{k}_\perp^2)^{n_1}}\,\frac{1}{\left((k^+ - r_0^+)k^- - (\vec{k}_\perp - \vec{r}_\perp)^2\right)^{n_2}}$$
$$\times\left(\frac{1}{\left(k^+(k^- - Q) - \vec{k}_\perp^2\right)^{n_3}} + \frac{1}{\left(k^+(k^- + Q) - r_0^+ k^- - \vec{k}_\perp^2 + 2\vec{r}_\perp\cdot\vec{k}_\perp\right)^{n_3}}\right)$$
$$\times\left(\frac{1}{(Qk^-)^{n_4}} + \frac{1}{(-Qk^-)^{n_4}}\right), \tag{7.7}$$

converging absolutely within

$$D_{1c} = \left\{k \in D : \left(|\vec{k}_\perp| \gg |\vec{r}_\perp| \wedge |k^+| \lesssim \frac{\vec{r}_\perp^{\,2}}{Q} \wedge |k^-| \gg \frac{\vec{k}_\perp^2}{Q}\right)\right.$$
$$\left.\vee\left(|\vec{k}_\perp| \lesssim |\vec{r}_\perp| \wedge |k^+| \ll Q \wedge |k^-| \gg \frac{|\vec{r}_\perp|\,|\vec{k}_\perp|}{Q} \wedge |k^+| \leq |k^-|\right)\right\}, \tag{7.8}$$

- the *2-collinear region* (2c), characterized by $k^+ \sim Q$, $k^- \sim \vec{r}_\perp^{\,2}/Q$ and $\vec{k}_\perp \sim \vec{r}_\perp$, with the expansion

$$T_0^{(2c)} I = \frac{1}{2}\,\frac{1}{(k^+k^- - \vec{k}_\perp^2)^{n_1}}\,\frac{1}{\left(k^+(k^- - r_0^-) - (\vec{k}_\perp - \vec{r}_\perp)^2\right)^{n_2}}$$
$$\times\left(\frac{1}{(-Qk^+)^{n_3}} + \frac{1}{(Qk^+)^{n_3}}\right)$$
$$\times\left(\frac{1}{\left((k^+ + Q)k^- - \vec{k}_\perp^2\right)^{n_4}} + \frac{1}{\left((k^+ - Q)k^- - k^+r_0^- - \vec{k}_\perp^2 + 2\vec{r}_\perp\cdot\vec{k}_\perp\right)^{n_4}}\right), \tag{7.9}$$

converging absolutely within

$$D_{2c} = \left\{ k \in D : \left(|\vec{k}_\perp| \gg |\vec{r}_\perp| \wedge |k^-| \lesssim \frac{\vec{r}_\perp^{\,2}}{Q} \wedge |k^+| \gg \frac{\vec{k}_\perp^{\,2}}{Q} \right) \right.$$
$$\left. \vee \left(|\vec{k}_\perp| \lesssim |\vec{r}_\perp| \wedge |k^-| \ll Q \wedge |k^+| \gg \frac{|\vec{r}_\perp|\,|\vec{k}_\perp|}{Q} \wedge |k^+| > |k^-| \right) \right\}, \tag{7.10}$$

- the *Glauber region* (g), characterized by $k^+ \sim k^- \sim \vec{r}_\perp^{\,2}/Q$ and $\vec{k}_\perp \sim \vec{r}_\perp$, with the expansion

$$T_0^{(g)} I = \frac{1}{2} \frac{1}{(-\vec{k}_\perp^{\,2})^{n_1}} \frac{1}{\big(-(\vec{k}_\perp - \vec{r}_\perp)^2\big)^{n_2}}$$
$$\times \left(\frac{1}{\big(-Qk^+ - \vec{k}_\perp^{\,2}\big)^{n_3}} + \frac{1}{\big(Qk^+ - \vec{k}_\perp^{\,2} + 2\vec{r}_\perp \cdot \vec{k}_\perp\big)^{n_3}} \right)$$
$$\times \left(\frac{1}{\big(Qk^- - \vec{k}_\perp^{\,2}\big)^{n_4}} + \frac{1}{\big(-Qk^- - \vec{k}_\perp^{\,2} + 2\vec{r}_\perp \cdot \vec{k}_\perp\big)^{n_4}} \right), \tag{7.11}$$

converging absolutely within

$$D_g = \left\{ k \in D : \left(|\vec{k}_\perp| \gg |\vec{r}_\perp| \wedge |k^+| \lesssim \frac{\vec{k}_\perp^{\,2}}{Q} \wedge |k^-| \lesssim \frac{\vec{r}_\perp^{\,2}}{Q} \right) \right.$$
$$\vee \left(|\vec{k}_\perp| \gg |\vec{r}_\perp| \wedge |k^-| \lesssim \frac{\vec{k}_\perp^{\,2}}{Q} \wedge |k^+| \lesssim \frac{\vec{r}_\perp^{\,2}}{Q} \right)$$
$$\left. \vee \left(|\vec{k}_\perp| \lesssim |\vec{r}_\perp| \wedge |k^\pm| \lesssim \frac{|\vec{r}_\perp|\,|\vec{k}_\perp|}{Q} \right) \right\}, \tag{7.12}$$

- and the *collinear-plane region* (cp), characterized by $k^+ \sim k^- \sim Q$ and $\vec{k}_\perp \sim \vec{r}_\perp$, with the expansion

$$T_0^{(cp)} I = \frac{1}{2} \frac{1}{(k^+k^-)^{n_1+n_2}} \left(\frac{1}{\big(k^+(k^- - Q)\big)^{n_3}} + \frac{1}{\big(k^+(k^- + Q)\big)^{n_3}} \right)$$
$$\times \left(\frac{1}{\big((k^+ + Q)k^-\big)^{n_4}} + \frac{1}{\big((k^+ - Q)k^-\big)^{n_4}} \right), \tag{7.13}$$

converging absolutely within

$$D_{cp} = \left\{ k \in D : |\vec{k}_\perp| \lesssim |\vec{r}_\perp| \wedge |k^\pm| \gtrsim Q \right\}, \tag{7.14}$$

where $D = \mathbb{R}^d$ is the complete integration domain and $r_0^\pm = \mp \vec{r}_\perp^{\,2}/Q$ is the leading-order approximation of $r^\pm$ according to (7.3).

Note that without analytic regularization, i.e. setting $n_1 = n_2 = n_3 = n_4 = 1$, the Glauber contribution $F_0^{(g)} = \int \mathrm{D}k\, T_0^{(g)} I$ from (7.11) is only convergent because the integral has been symmetrized. Integrals over individual terms are divergent for $|k^+| \to \infty$ or $|k^-| \to \infty$ like $\mathrm{d}k^\pm/k^\pm$, but the leading term in each of the round brackets of (7.11) is cancelled, making the integrals convergent at infinity like $\mathrm{d}k^\pm/(k^\pm)^2$.

The convergence domains D_x ($x = h, 1c, 2c, g, cp$) are non-intersecting and cover the complete integration domain,

$$D_h \cup D_{1c} \cup D_{2c} \cup D_g \cup D_{cp} = D \,. \tag{7.15}$$

It might look strange that the domains of the collinear and Glauber regions extend into the zone with $|\vec{k}_\perp| \gg |\vec{r}_\perp|$ (cf. left diagram of figure 6). But this is necessary if we do not want to invent new artificial regions, because the hard convergence domain cannot cover the complete $|k^+|$–$|k^-|$-plane with $|\vec{k}_\perp| \gg |\vec{r}_\perp|$. The definitions of the convergence domains given above ensure that for every region x and every point $k \in D_x$ there is at least one large term left by $T^{(x)}$ in the denominator against which the smaller terms according to the scaling of the region x can be expanded.

The multiple expansions are determined from the scaling prescriptions for the regions given above; they are listed in appendix A.4.1. All expansions commute with each other with the exception of $T^{(g)}$ and $T^{(cp)}$:

$$\begin{aligned} T_0^{(cp\leftarrow g)} I = \frac{1}{2} \frac{1}{(-\vec{k}_\perp^2)^{n_1}} \frac{1}{\left(-(\vec{k}_\perp - \vec{r}_\perp)^2\right)^{n_2}} \left(\frac{1}{(-Qk^+)^{n_3}} + \frac{1}{(Qk^+)^{n_3}} \right) \\ \times \left(\frac{1}{(Qk^-)^{n_4}} + \frac{1}{(-Qk^-)^{n_4}} \right), \end{aligned} \tag{7.16}$$

$$\begin{aligned} T_0^{(g\leftarrow cp)} I = T_0^{(cp)} I = \frac{1}{2} \frac{1}{(k^+k^-)^{n_1+n_2}} \left(\frac{1}{\left(k^+(k^- - Q)\right)^{n_3}} + \frac{1}{\left(k^+(k^- + Q)\right)^{n_3}} \right) \\ \times \left(\frac{1}{\left((k^+ + Q)k^-\right)^{n_4}} + \frac{1}{\left((k^+ - Q)k^-\right)^{n_4}} \right). \end{aligned} \tag{7.17}$$

The expansion $T^{(g)}$ does not alter the integrand when it is applied after $T^{(cp)}$, because the Glauber domain D_g also contains a part with $|\vec{k}_\perp| \gg |\vec{r}_\perp|$ (so possibly $|\vec{k}_\perp| \gtrsim Q$) and $|k^+| \sim \vec{k}_\perp^2/Q$ or $|k^-| \sim \vec{k}_\perp^2/Q$. Thus we cannot rely on $|k^\pm| \ll Q$ throughout D_g and have to leave $(k^\pm \pm Q)$ unexpanded in $T^{(g\leftarrow cp)}$ (7.17). When, however, $T^{(g)}$ is applied to the original integrand (7.4) or to the integrand expanded according to any other region except (cp), the term k^+k^- in the denominators is always accompanied by $\vec{k}_\perp^2$. So we can use $|k^+k^-| \ll \vec{k}_\perp^2$, which always holds within D_g, in order to perform the expansion $T^{(g)}$ as stated in (7.11). All other multiple expansions converge within the convergence domain of the last expansion, whatever expansions have been applied before.

With the notations from section 5, the sets of regions with commuting and non-commuting expansions are $R_\mathrm{c} = \{h, 1c, 2c\}$ and $R_\mathrm{nc} = \{g, cp\}$, respectively, as in the previous example of section 6. Condition 5 from p. 29 holds because the doubly expanded integrand (7.16) is unchanged from further applications of $T^{(1c)}$ or $T^{(2c)}$, and the expression (7.17) is invariant under the hard expansion $T^{(h)}$:

$$\begin{aligned} T^{(cp\leftarrow g,1c)} = T^{(cp\leftarrow g,2c)} = T^{(cp\leftarrow g)} \,, \\ T^{(g\leftarrow cp,h)} = T^{(g\leftarrow cp)} \,. \end{aligned} \tag{7.18}$$

So the identity (5.18) holds and has the same form as the identity (6.19) of the previous example. Here we restrict ourselves to the leading-order contributions:

$$\begin{aligned} F_0 = {} & F_0^{(h)} + F_0^{(1c)} + F_0^{(2c)} + F_0^{(g)} + F_0^{(cp)} - \Big(F_0^{(h,1c)} + F_0^{(h,2c)} + F_0^{(h,g)} + F_0^{(h,cp)} \\ & + F_0^{(1c,2c)} + F_0^{(1c,g)} + F_0^{(1c,cp)} + F_0^{(2c,g)} + F_0^{(2c,cp)}\Big) \\ & + F_0^{(h,1c,2c)} + F_0^{(h,1c,g)} + F_0^{(h,1c,cp)} + F_0^{(h,2c,g)} + F_0^{(h,2c,cp)} + F_0^{(1c,2c,g)} + F_0^{(1c,2c,cp)} \\ & - \Big(F_0^{(h,1c,2c,g)} + F_0^{(h,1c,2c,cp)}\Big) . \end{aligned} \tag{7.19}$$

Then, as explained at the end of section 3, we just have to ensure that these leading-order integrals are properly regularized (condition 3 of p. 14). And condition 4, concerning the convergence of the expansions outside their corresponding domains, is irrelevant.

For the same reasons as in (6.21) we see that all contributions involving the collinear-plane expansion $T^{(cp)}$ cancel each other.

Using scaling arguments, i.e. considering the scaling of the integration measure and the propagators in each region, we predict how the leading-order contributions of the individual regions depend on $\vec{r}_\perp^{\,2}$ and Q^2 (this will be confirmed by explicit evaluations later):

$$\begin{aligned} F_0^{(h)} &\propto (Q^2)^{\frac{d}{2}-n_{1234}} , \\ F_0^{(1c)} &\propto (\vec{r}_\perp^{\,2})^{\frac{d}{2}-n_{123}} (Q^2)^{-n_4} , \\ F_0^{(2c)} &\propto (\vec{r}_\perp^{\,2})^{\frac{d}{2}-n_{124}} (Q^2)^{-n_3} , \\ F_0^{(g)} &\propto (\vec{r}_\perp^{\,2})^{\frac{d}{2}+1-n_{1234}} (Q^2)^{-1} , \\ F_0^{(cp)} &\propto (\vec{r}_\perp^{\,2})^{\frac{d}{2}-1} (Q^2)^{1-n_{1234}} . \end{aligned} \tag{7.20}$$

When the propagator powers n_i are used as analytic regulators, each region exhibits a unique dependence on $\vec{r}_\perp^{\,2}$ and Q^2, so we expect that all overlap contributions must be scaleless. Without analytic regulators, however, i.e. for $n_1 = n_2 = n_3 = n_4 = 1$, the scalings read

$$\begin{aligned} F_0^{(h)} &\propto (Q^2)^{-2-\epsilon} , \\ F_0^{(1c)} &\propto (\vec{r}_\perp^{\,2})^{-1-\epsilon} (Q^2)^{-1} , \\ F_0^{(2c)} &\propto (\vec{r}_\perp^{\,2})^{-1-\epsilon} (Q^2)^{-1} , \\ F_0^{(g)} &\propto (\vec{r}_\perp^{\,2})^{-1-\epsilon} (Q^2)^{-1} , \\ F_0^{(cp)} &\propto (\vec{r}_\perp^{\,2})^{1-\epsilon} (Q^2)^{-3} . \end{aligned} \tag{7.21}$$

We notice that the hard and collinear-plane contributions, $F_0^{(h)}$ and $F_0^{(cp)}$, are suppressed with respect to the "true" leading-order contributions $F_0^{(1c)}$, $F_0^{(2c)}$ and $F_0^{(g)}$. And the leading collinear and Glauber contributions share exactly the same dependence on $\vec{r}_\perp^{\,2}$ and Q^2. So there is no reason why the overlap contributions $F_0^{(1c,2c)}$, $F_0^{(1c,g)}$, $F_0^{(2c,g)}$ and $F_0^{(1c,2c,g)}$ should be scaleless; we must be prepared for relevant contributions from these multiple expansions.

Altogether we expect that the leading-order result for the complete integral F (7.4), with or without analytic regulators (in any case for $n_i \approx 1\,\forall i$ and $\epsilon \approx 0$) is given by

$$F_0 = F_0^{(1c)} + F_0^{(2c)} + F_0^{(g)} - \left(F_0^{(1c,2c)} + F_0^{(1c,g)} + F_0^{(2c,g)}\right) + F_0^{(1c,2c,g)}\,, \tag{7.22}$$

where omitted contributions are either suppressed or scaleless, which is checked explicitly in appendix A.4.

7.2 Evaluation with analytic regulators

The integrals contributing to (7.19) are evaluated in appendix A.4.1 using the propagator powers $n_{1,2,3,4}$ as analytic regulators. It is shown there that the Glauber contribution $F_0^{(g)}$, the collinear-plane contribution $F_0^{(cp)}$ and all overlap contributions are scaleless. For the non-vanishing contributions $F_0^{(h)}$, $F_0^{(1c)}$ and $F_0^{(2c)}$, the scaling with $\vec{r}_\perp^{\,2}$ and Q^2 predicted in (7.20) is confirmed. In the limit $n_i \to 1\,\forall i$, i.e. when all propagator powers tend to 1, and close to $d = 4$ dimensions ($\epsilon \approx 0$), the hard contribution $F_0^{(h)}$ is suppressed with respect to the collinear contributions by one power of $\vec{r}_\perp^{\,2}/Q^2$. So the general expression (7.22) for the leading-order result is confirmed. When analytic regularization is employed, all but the first two terms are scaleless, and the leading-order result reads

$$F_0 = F_0^{(1c)} + F_0^{(2c)}\,. \tag{7.23}$$

These two collinear contributions are given in (A.77) and (A.79) of appendix A.4.1. They both involve a factor which is singular for $n_3 = n_4$, but the limits $n_1 \to 1$ and $n_2 \to 1$ are well-defined. Let us evaluate the contributions for $n_1 = n_2 = 1$, $n_3 = 1+\delta_3$ and $n_4 = 1+\delta_4$:

$$\begin{aligned} F_0^{(1c)} &= \frac{1}{\vec{r}_\perp^{\,2}\,Q^2}\left(\frac{\mu^2}{\vec{r}_\perp^{\,2}}\right)^{\epsilon} (\vec{r}_\perp^{\,2})^{-\delta_3}\,(Q^2)^{-\delta_4}\,e^{-i\pi\delta_3}\,(e^{-i\pi\delta_4}-1)\,\Gamma(\delta_3-\delta_4) \\ &\qquad\qquad\times \frac{e^{\epsilon\gamma_{\rm E}}\,\Gamma(1+\epsilon+\delta_3)\,\Gamma^2(-\epsilon-\delta_3)}{\Gamma(1+\delta_3)\,\Gamma(-2\epsilon-\delta_3-\delta_4)}\,, \\ F_0^{(2c)} &= \frac{1}{\vec{r}_\perp^{\,2}\,Q^2}\left(\frac{\mu^2}{\vec{r}_\perp^{\,2}}\right)^{\epsilon} (\vec{r}_\perp^{\,2})^{-\delta_4}\,(Q^2)^{-\delta_3}\,e^{-i\pi\delta_4}\,(e^{-i\pi\delta_3}-1)\,\Gamma(\delta_4-\delta_3) \\ &\qquad\qquad\times \frac{e^{\epsilon\gamma_{\rm E}}\,\Gamma(1+\epsilon+\delta_4)\,\Gamma^2(-\epsilon-\delta_4)}{\Gamma(1+\delta_4)\,\Gamma(-2\epsilon-\delta_3-\delta_4)}\,. \end{aligned} \tag{7.24}$$

We are interested in the case $n_1 = n_2 = n_3 = n_4 = 1$, i.e. in the point $\delta_3 = \delta_4 = 0$. The two collinear contributions individually depend on the direction in the δ_3–δ_4-plane by which we approach this point. One possible choice is the antisymmetric case of approaching the desired limit on the line $\delta_3 = -\delta_4 \equiv \delta$:

$$\begin{aligned} F_0^{(1c)}\big|_{\delta_{3,4}=\pm\delta} &= \frac{1}{\vec{r}_\perp^{\,2}\,Q^2}\left(\frac{\mu^2}{\vec{r}_\perp^{\,2}}\right)^{\epsilon}\left(\frac{Q^2}{\vec{r}_\perp^{\,2}}\right)^{\delta}(1-e^{-i\pi\delta})\,\Gamma(2\delta) \\ &\qquad\times \frac{e^{\epsilon\gamma_{\rm E}}\,\Gamma(1+\epsilon+\delta)\,\Gamma^2(-\epsilon-\delta)}{\Gamma(1+\delta)\,\Gamma(-2\epsilon)} \\ &= \frac{1}{2}\,\frac{i\pi}{\vec{r}_\perp^{\,2}\,Q^2}\left(\frac{\mu^2}{\vec{r}_\perp^{\,2}}\right)^{\epsilon} \frac{e^{\epsilon\gamma_{\rm E}}\,\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)} + \mathcal{O}(\delta)\,, \end{aligned}$$

$$F_0^{(2c)}\big|_{\delta_{3,4}=\pm\delta} = \frac{1}{\vec{r}_\perp^{\,2}\, Q^2} \left(\frac{\mu^2}{\vec{r}_\perp^{\,2}}\right)^{\epsilon} \left(\frac{Q^2}{\vec{r}_\perp^{\,2}}\right)^{-\delta} (1-e^{i\pi\delta})\,\Gamma(-2\delta) \times \frac{e^{\epsilon\gamma_{\rm E}}\,\Gamma(1+\epsilon-\delta)\,\Gamma^2(-\epsilon+\delta)}{\Gamma(1-\delta)\,\Gamma(-2\epsilon)}$$

$$= \frac{1}{2}\,\frac{i\pi}{\vec{r}_\perp^{\,2}\, Q^2} \left(\frac{\mu^2}{\vec{r}_\perp^{\,2}}\right)^{\epsilon} \frac{e^{\epsilon\gamma_{\rm E}}\,\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)} + \mathcal{O}(\delta)\,. \qquad (7.25)$$

In the limit $\delta \to 0$ both collinear contributions are equal. Note that each collinear contribution is individually finite, which is due to a cancellation of $1/\delta$ singularities between the two diagrams shown in figure 5, such that the factor $(1-e^{\mp i\pi\delta})$ cancels the singularity from $\Gamma(\pm 2\delta)$. The total result with $n_1 = n_2 = n_3 = n_4 = 1$ reads

$$F_0\big|_{n_i=1} = \lim_{\delta\to 0} F_0^{(1c)}\big|_{\delta_{3,4}=\pm\delta} + \lim_{\delta\to 0} F_0^{(2c)}\big|_{\delta_{3,4}=\pm\delta} = \frac{i\pi}{\vec{r}_\perp^{\,2}\, Q^2} \left(\frac{\mu^2}{\vec{r}_\perp^{\,2}}\right)^{\epsilon} \frac{e^{\epsilon\gamma_{\rm E}}\,\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)}\,. \qquad (7.26)$$

Another choice for the analytic regularization consists in taking the limit $\delta_3 \to 0$ first:

$$F_0^{(1c)}\big|_{\delta_3=0} = \frac{1}{\vec{r}_\perp^{\,2}\, Q^2} \left(\frac{\mu^2}{\vec{r}_\perp^{\,2}}\right)^{\epsilon} (Q^2)^{-\delta_4}\,(e^{-i\pi\delta_4}-1)\,\Gamma(-\delta_4)\,\frac{e^{\epsilon\gamma_{\rm E}}\,\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon-\delta_4)}\,,$$

$$F_0^{(2c)}\big|_{\delta_3=0} = 0\,. \qquad (7.27)$$

The total result is then reproduced entirely by the 1-collinear contribution:

$$F_0\big|_{n_i=1} = \lim_{\delta_4\to 0} F_0^{(1c)}\big|_{\delta_3=0} = \frac{i\pi}{\vec{r}_\perp^{\,2}\, Q^2} \left(\frac{\mu^2}{\vec{r}_\perp^{\,2}}\right)^{\epsilon} \frac{e^{\epsilon\gamma_{\rm E}}\,\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)}\,. \qquad (7.28)$$

Other choices are possible, but they all lead to the same total result. In fact, the analytic regularization parameters can be expressed through $\delta_{3,4} = \Delta \pm \delta$, isolating the singularity in the variable δ. The total result remains invariant upon subtracting the $1/\delta$ terms from each collinear contribution because these subtraction terms cancel each other:

$$\Big[F_0^{(1c)} + F_0^{(2c)}\Big]_{\delta_{3,4}=\Delta\pm\delta} = \frac{1}{\vec{r}_\perp^{\,2}\, Q^2} \left(\frac{\mu^2}{\vec{r}_\perp^{\,2}}\right)^{\epsilon} (\vec{r}_\perp^{\,2}\, Q^2)^{-\Delta}\,\frac{e^{\epsilon\gamma_{\rm E}}\,e^{-i\pi\Delta}}{\Gamma(-2\epsilon-2\Delta)}$$
$$\times \Bigg[\left(\frac{Q^2}{\vec{r}_\perp^{\,2}}\right)^{\delta} (e^{-i\pi\Delta}-e^{-i\pi\delta})\,\Gamma(2\delta)\,\frac{\Gamma(1+\epsilon+\Delta+\delta)\,\Gamma^2(-\epsilon-\Delta-\delta)}{\Gamma(1+\Delta+\delta)}$$
$$- \frac{e^{-i\pi\Delta}-1}{2\delta}\,\frac{\Gamma(1+\epsilon+\Delta)\,\Gamma^2(-\epsilon-\Delta)}{\Gamma(1+\Delta)}\Bigg] + (\delta\to-\delta)\,. \qquad (7.29)$$

Here the expression in the square brackets has a well-defined finite limit for $\delta \to 0$ and $\Delta \to 0$, independent of the way in which this limit is approached, which reproduces the known result:

$$F_0\big|_{n_i=1} = \lim_{\delta,\Delta\to 0}\Big[F_0^{(1c)} + F_0^{(2c)}\Big]_{\delta_{3,4}=\Delta\pm\delta} = \frac{i\pi}{\vec{r}_\perp^{\,2}\, Q^2} \left(\frac{\mu^2}{\vec{r}_\perp^{\,2}}\right)^{\epsilon} \frac{e^{\epsilon\gamma_{\rm E}}\,\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)}\,. \qquad (7.30)$$

It is evident that $\lim_{n_i\to 1} F_0$ must be independent of the order in which the propagator powers $n_{1,2,3,4}$ are sent to 1, because expression (7.23) is derived from the identity (7.19) which is valid for any regularization scheme (as long as the conditions of the formalism presented in sections 3 and 5 hold).

The result obtained with the expansion by regions can be checked by evaluating the original integral F (7.1) with the help of a Mellin-Barnes representation:

$$\begin{aligned} F = & \frac{\mu^{2\epsilon}\, e^{\epsilon\gamma_{\rm E}}\, e^{-i\pi n_{1234}}}{\Gamma(n_1)\,\Gamma(n_2)\,\Gamma(n_3)\,\Gamma(n_4)\,\Gamma(4-n_{1234}-2\epsilon)} \\ & \times \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2i\pi}\,(-r^2)^z \left[(-Q^2-i0)^{2-n_{1234}-\epsilon-z} + (Q^2+r^2)^{2-n_{1234}-\epsilon-z}\right] \\ & \qquad \times \Gamma(n_1+z)\,\Gamma(n_2+z)\,\Gamma(n_{1234}-2+\epsilon+z) \\ & \qquad \times \Gamma(-z)\,\Gamma(2-n_{123}-\epsilon-z)\,\Gamma(2-n_{124}-\epsilon-z)\,. \end{aligned} \tag{7.31}$$

The asymptotic expansion for $|r^2| \ll Q^2$ is obtained by closing the Mellin-Barnes integral to the right and extracting the residues of the three Gamma functions in the last line of (7.31). The (possibly) leading terms originate from the residues at $z=0$, $z=2-n_{123}-\epsilon$ and $z=2-n_{124}-\epsilon$. Using $-r^2 \approx \vec{r}_\perp^{\,2}$ and $Q^2+r^2 \approx Q^2$, they reproduce the contributions from the hard (A.68), 1-collinear (A.77) and 2-collinear (A.79) regions, respectively.

7.3 Evaluation without analytic regulators

Now we would like to see how the contributions from the previous section 7.2 change when we evaluate them without using analytic regulators. This means that we set all propagator powers to the fixed values $n_1=n_2=n_3=n_4=1$ before performing the integrals.

The evaluation without analytic regularization of the integrals contributing to the complete expression (7.19) is described in appendix A.4.2. The scaling of the contributions $F_0^{(h)}$ (A.91), $F_0^{(1c)}$ (A.97), $F_0^{(2c)}$ (A.99) and $F_0^{(g)}$ (A.101) with the parameters $\vec{r}_\perp^{\,2}$ and Q^2 as predicted in (7.21) is confirmed by explicit calculation. This implies that the hard contribution $F_0^{(h)}$ is suppressed by one power of $\vec{r}_\perp^{\,2}/Q^2$ with respect to the collinear and Glauber contributions and does not contribute to the leading-order result F_0.

The collinear-plane contribution $F_0^{(cp)}$ (A.107) and all overlap contributions involving the collinear-plane region are found to be scaleless via dimensional regularization, even without analytic regulators.[5] Also the overlap contributions involving the hard region yield scaleless integrals (A.111).

These results confirm the identity (7.22) for the complete leading-order result F_0, made up of the contributions from the 1-collinear, 2-collinear and Glauber regions including all their overlap contributions. The corresponding integrals are evaluated

[5] A subtle point concerning the regularization of poles pinching the integration contours at $k^\pm=0$ is treated in appendix A.4.2.

in (A.97), (A.99), (A.101), (A.103) and (A.106). They all yield identical results,

$$\begin{aligned} F_0^{(1c)} &= F_0^{(2c)} = F_0^{(g)} = F_0^{(1c,2c)} = F_0^{(1c,g)} = F_0^{(2c,g)} = F_0^{(1c,2c,g)} \\ &= \frac{i\pi}{\vec{r}_\perp^{\,2}\, Q^2} \left(\frac{\mu^2}{\vec{r}_\perp^{\,2}}\right)^{\epsilon} \frac{e^{\epsilon\gamma_{\rm E}}\,\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)} . \end{aligned} \tag{7.32}$$

This is very different from the contributions with analytic regularization determined in section 7.2. In particular, the terms (7.32) involve finite and relevant overlap contributions. The complete result, to leading order in $\vec{r}_\perp^{\,2}/Q^2$, reads

$$\begin{aligned} F_0 &= F_0^{(1c)} + F_0^{(2c)} + F_0^{(g)} - \left(F_0^{(1c,2c)} + F_0^{(1c,g)} + F_0^{(2c,g)}\right) + F_0^{(1c,2c,g)} \\ &= \Big(1+1+1-(1+1+1)+1\Big) \frac{i\pi}{\vec{r}_\perp^{\,2}\, Q^2} \left(\frac{\mu^2}{\vec{r}_\perp^{\,2}}\right)^{\epsilon} \frac{e^{\epsilon\gamma_{\rm E}}\,\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)} \\ &= \frac{i\pi}{\vec{r}_\perp^{\,2}\, Q^2} \left(\frac{\mu^2}{\vec{r}_\perp^{\,2}}\right)^{\epsilon} \frac{e^{\epsilon\gamma_{\rm E}}\,\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)} . \end{aligned} \tag{7.33}$$

The total result obtained without analytic regularization agrees with the result from the previous section, expressions (7.26), (7.28) or (7.30), where analytic regulators have been used. But the partitioning of the contributions among the regions is completely different. If we had omitted the overlap contributions from (7.33), the result would have been three times too large.

At first sight it might seem to be just by chance that the cancellations between the various contributions in (7.33) leave exactly once the individual contribution. But this is a general feature whenever a certain set of regions (with commuting expansions) yields identical contributions, including all overlap contributions from these regions. Assume that

$$\begin{aligned} F_0^{(x'_1,\dots,x'_n)} &\equiv \int \mathrm{D}k\, T_0^{(x'_1)} \cdots T_0^{(x'_n)}\, I \\ &= \hat{F}_0 \quad \forall x'_1,\dots,x'_n \in \hat{R} \equiv \{x_1,\dots,x_{\hat{N}}\}\,, \quad 1 \le n \le \hat{N} \le N\,. \end{aligned} \tag{7.34}$$

Then the total (leading-order) contribution from these regions, including the multiple expansions, reads

$$\begin{aligned} \sum_{n=1}^{\hat{N}} (-1)^{n-1} \sum_{\{x'_1,\dots,x'_n\}\subset\hat{R}} F_0^{(x'_1,\dots,x'_n)} &= \int \mathrm{D}k \left[1 - \prod_{i=1}^{\hat{N}} \left(1 - T_0^{(x_i)}\right)\right] I \\ &= \left[1 - \prod_{i=1}^{\hat{N}} (1-1)\right] \hat{F}_0 = \hat{F}_0\,, \end{aligned} \tag{7.35}$$

where we have used $\int \mathrm{D}k\,(1-1)\,I = (1-1)F = 0 = (1-1)\hat{F}_0$. The individual contribution $\hat{F}_0$ is left exactly once from all the cancellations.

The Mellin-Barnes representation (7.31) is also valid without analytic regularization. For $n_1 = n_2 = n_3 = n_4 = 1$ it reads

$$\begin{aligned} F = \frac{\mu^{2\epsilon}\, e^{\epsilon\gamma_{\rm E}}}{\Gamma(-2\epsilon)} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2i\pi}\, (-r^2)^z \Big[(-Q^2 - i0)^{-2-\epsilon-z} + (Q^2 + r^2)^{-2-\epsilon-z}\Big] \\ \times \Gamma^2(1+z)\,\Gamma(2+\epsilon+z)\,\Gamma(-z)\,\Gamma^2(-1-\epsilon-z)\,. \end{aligned} \tag{7.36}$$

Closing the Mellin-Barnes integral to the right, the leading-order asymptotic expansion in r^2/Q^2 is extracted from the residue at $z = -1 - \epsilon$. The z-integral has a double pole there, and its residue yields the leading-order result (7.33), using $-r^2 \approx \vec{r}_\perp^{\,2}$ and $Q^2 + r^2 \approx Q^2$.

8 Conclusions

Let us summarize the main statements of the general formalism introduced in section 3 and generalized in section 5.

The integral $F = \int \mathrm{D}k\, I$ with integration domain D shall be expanded in some limit. We have defined a set R of N regions, $R = \{x_1, \ldots, x_N\}$. Each region x is characterized by an expansion $T^{(x)} \equiv \sum_j T_j^{(x)}$ which converges absolutely when the integration variable is restricted to the corresponding domain D_x. The conditions for the applicability of the formalism read:

1. The convergence domains are non-intersecting, $D_x \cap D_{x'} = \emptyset \;\forall x \neq x'$, and cover the complete integration domain, $\bigcup_{x \in R} D_x = D$.

2a. All expansions corresponding to regions within some subset $R_\mathrm{c} \subset R$ commute with each other and with expansions of any other region in R:

$$T^{(x)} T^{(x')} = T^{(x')} T^{(x)} \;\forall x \in R_\mathrm{c}\,,\; x' \in R\,.$$

We write $R_\mathrm{c} = \{x_1, \ldots, x_{N_\mathrm{c}}\}$ with $0 \leq N_\mathrm{c} \leq N$ and $R_\mathrm{nc} = R \setminus R_\mathrm{c} = \{x_{N_\mathrm{c}+1}, \ldots, x_N\}$.

3. The original integral F and all integrals over expanded terms,

$$F_{j_1, j_2, \ldots}^{(x_1', x_2', \ldots)} = \int \mathrm{D}k\, T_{j_1}^{(x_1')} T_{j_2}^{(x_2')} \cdots I\,,$$

are well-defined within the chosen regularization scheme, even if these integrals are not restricted to any convergence domain D_x.

4. The series expansions

$$F^{(x_1', x_2', \ldots)} = \sum_{j_1, j_2, \ldots} F_{j_1, j_2, \ldots}^{(x_1', x_2', \ldots)}\,,$$

converge absolutely, although the expanded terms have been integrated over the complete integration domain D.

5. For every combination of two non-commuting expansions, there is a region from R_c whose expansion does not further change the doubly expanded integrand:

$$\forall x_1', x_2' \in R_\mathrm{nc},\; x_1' \neq x_2',\; \exists\, x \in R_\mathrm{c} : T^{(x)} T^{(x_2')} T^{(x_1')} = T^{(x_2')} T^{(x_1')}\,.$$

Note that this condition requires $N_\mathrm{c} \geq 1$.

If all these conditions hold, then the original integral is reproduced exactly, i.e. to all orders in the expansion, through the master identity (5.18):

$$F = \sum_{x\in R} F^{(x)} - \sum^{\langle R_c+1\rangle}_{\{x'_1,x'_2\}\subset R} F^{(x'_1,x'_2)} + \ldots - (-1)^n \sum^{\langle R_c+1\rangle}_{\{x'_1,\ldots,x'_n\}\subset R} F^{(x'_1,\ldots,x'_n)} + \ldots + (-1)^{N_c} \sum_{x'\in R_{nc}} F^{(x',x_1,\ldots,x_{N_c})} \,, \tag{8.1}$$

where the summations with superscript $\langle R_c + 1\rangle$ are defined in (5.8). They are restricted to such subsets of regions with at most one region from R_{nc}. This means that the identity (8.1) sums over exactly those combinations of regions whose expansions commute with each other.

The first term on the right-hand side of (8.1) corresponds to the known recipe of the expansion by regions presented in section 1. The following terms represent overlap contributions originating from multiple expansions. These overlap contributions may be relevant under certain circumstances, see the example from section 7 evaluated without analytic regularization. In many cases, however, the usual recipe of the expansion by regions is recovered, especially when dimensional and analytic regularization are used. If the contributions $F^{(x)}$ with single expansions yield homogeneous functions of the expansion parameter in each order of the expansion and if this dependence on the expansion parameter is unique for each region, then we expect all overlap contributions to be scaleless, and only the usual contributions from each region survive.

This means that in situations where the expansion by regions has normally been employed the known recipe [1–4] remains correct. The original authors of the method have always stressed how important the homogeneity of the contributions is. Where non-homogeneous contributions from individual regions appear, the formalism presented in this paper may still be applied if its conditions hold, and overlap contributions might arise. But it is usually preferable to change the choice of regions or regulators in order to obtain only homogeneous contributions and get rid of overlap contributions. In any case, one can always check whether overlap contributions are relevant by simply evaluating them to see if they vanish.

The original recipe of the expansion by regions [1–4] can be understood such that *any* scaleless integral must be set to zero, whether it is well-defined through regularization or not. However, this is not the approach of the formalism presented here. Throughout the whole paper only such scaleless integrals are dropped which are mathematically well-defined through some regularization (and analytic continuation) and which can explicitly be shown to vanish (see appendix A.1.2 and other appendices). Wherever an integral is ill-defined, a suitable regularization is added. Vanishing scaleless integrals thus are not a requirement for this formalism, but simply a well-understood property of dimensional regularization and analytic continuation which is used here.

The leading-order asymptotic expansion of the integral F is obtained by replacing each series expansion in (8.1) with its leading-order term. Some of the leading-order terms obtained in this way may be suppressed with respect to others and must be dropped from the leading-order result. For the leading-order approximation condition 4 above does not apply, and condition 3 only needs to hold for the leading-order integrals.

The examples presented in this paper illustrate the application of the formalism to various integrals. This implies choosing the regions with corresponding expansions, covering the integration domain with the convergence domains and checking the conditions listed above. Most example integrals have alternatively been evaluated and expanded with the help of a Mellin-Barnes representation. For these simple one-loop examples, the asymptotic expansion via Mellin-Barnes integrals can be easier than the application of the expansion by regions. My experience with more elaborate multi-loop integrals depending on several small and large parameters, however, shows that the asymptotic expansion via multiple Mellin-Barnes integrals is often more difficult than the expansion at the integrand level within the strategy of regions.

What is the practical use of the formalism presented in this paper, apart from the generalization of the expansion by regions to cases where overlap contributions are needed? In principle, it should be possible also for more complicated integrals to prove their asymptotic expansion by checking the conditions listed above. Most of these conditions can indeed be checked in a straightforward way once the list of regions, expansions and convergence domains is established. On the other hand, the first obstacle in getting there is finding all relevant regions such that the convergence domains of their expansions cover the complete integration domain. And this step might be quite tedious for complicated multi-loop integrals. It can be easier to evaluate successfully the contributions from the regions than to prove, using this formalism, that the evaluation is correct and complete. In particular, the formalism may need extra regions with scaleless contributions in order to cover the integration domain with convergence domains or to ensure the commutation of the expansions.

Even with the formalism at hand, we will certainly not always want to prove its applicability, if only for practical limitations. But I hope this paper will convince readers that the expansion by regions is a well-founded method of asymptotic expansion and that its applicability can at least in principle be proven case by case. The standard recipe of the expansion by regions remains valid if overlap contributions are absent. So the formalism and its conditions provide hints on the proper choice of regions and regularization schemes which ensure the homogeneous and unique dependence of the individual contributions on the expansion parameter. Finally, the present paper shows that the expansion by regions, in its generalized form, can be applied to very different kinds of integrals, even such with finite boundaries or scaleful regularization parameters where the contributions do not exhibit a homogeneous dependence on the expansion parameter. Appendix B illustrates such an example and points out possible convergence problems arising there.

Having obtained such knowledge on the foundation and generalization of the expansion by regions, this method is ready to be employed trustfully to various kinds of asymptotic expansions.

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A Technical details of the evaluations

A.1 Large-momentum expansion

This appendix presents details of the evaluation of the off-shell large-momentum expansion in section 2.

A.1.1 Large-momentum expansion: soft contributions

The soft-region integral

$$\int \mathrm{D}k \, \frac{(k^2)^{j_1} \, (2k \cdot p)^{j_2}}{(k^2 - m^2 + i0)^{n_2}} \tag{A.1}$$

in (2.9) can be treated as an analytic function of the summation indices j_1 and j_2. We first assume $\mathrm{Re}\, j_{1,2} < 0$ and later analytically continue the result to positive values of $j_{1,2}$. Using alpha parameters

$$\frac{1}{(A + i0)^n} = \frac{e^{-i\pi n/2}}{\Gamma(n)} \int_0^\infty \mathrm{d}\alpha \, \alpha^{n-1} \, e^{i\alpha(A+i0)} \,, \tag{A.2}$$

the integral (A.1) is written as

$$\frac{e^{-i\pi(n_2 - j_{12})/2}}{\Gamma(n_2)\,\Gamma(-j_1)\,\Gamma(-j_2)} \int_0^\infty \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \, \mathrm{d}\alpha_3 \, \alpha_1^{n_2-1} \, \alpha_2^{-j_1-1} \, \alpha_3^{-j_2-1} \, e^{i\alpha_1(-m^2+i0)} \times \int \mathrm{D}k \, e^{i(\alpha_{12}k^2 + 2\alpha_3 \, p\cdot k + i0)} \,, \tag{A.3}$$

with the shorthand notation (2.6). The loop momentum is integrated via

$$\int \frac{\mathrm{d}^d k}{i\pi^{d/2}} \, e^{i(\alpha k^2 + 2p\cdot k + i0)} = e^{-i\pi d/4} \, \alpha^{-d/2} \, \exp\left[i \left(-\frac{p^2}{\alpha} + i0 \right) \right] , \tag{A.4}$$

yielding the alpha-parameter representation

$$\mu^{2\epsilon} \, e^{\epsilon\gamma_{\mathrm{E}}} \, \frac{e^{-i\pi(n_2 - j_{12} + 2 - \epsilon)/2}}{\Gamma(n_2)\,\Gamma(-j_1)\,\Gamma(-j_2)} \int_0^\infty \mathrm{d}\alpha_1 \, \alpha_1^{n_2-1} \, e^{i\alpha_1(-m^2+i0)} \int_0^\infty \mathrm{d}\alpha_2 \, \alpha_2^{-j_1-1} \, \alpha_{12}^{-2+\epsilon} \times \int_0^\infty \mathrm{d}\alpha_3 \, \alpha_3^{-j_2-1} \, \exp\left[i\alpha_3^2 \left(-\frac{p^2}{\alpha_{12}} + i0 \right) \right] \tag{A.5}$$

of the soft-region integral. After substituting $\alpha_3 = t^{1/2}$ and evaluating the t-integral by reversing (A.2), the α_2-integral is solved using

$$\int_0^\infty \mathrm{d}\alpha\, \alpha^{\nu-1}\, (A + B\alpha)^{-\rho} = \frac{\Gamma(\nu)\,\Gamma(\rho-\nu)}{\Gamma(\rho)}\, B^{-\nu}\, A^{\nu-\rho}\,, \tag{A.6}$$

followed by the α_1-integral via reversing (A.2). The result reads

$$\mu^{2\epsilon}\, e^{\epsilon\gamma_{\mathrm{E}}}\, e^{-i\pi(n_2-j_1-j_2/2)}\, (m^2)^{2-n_2-\epsilon+j_1+j_2/2}\, (-p^2+i0)^{j_2/2} \\ \times \frac{\Gamma\!\left(-\frac{j_2}{2}\right)\Gamma\!\left(2-\epsilon+j_1+\frac{j_2}{2}\right)\Gamma\!\left(n_2-2+\epsilon-j_1-\frac{j_2}{2}\right)}{2\,\Gamma(n_2)\,\Gamma(-j_2)\,\Gamma\!\left(2-\epsilon+\frac{j_2}{2}\right)}\,. \tag{A.7}$$

With the help of the doubling formula for the Gamma function (see e.g. [14]) we rewrite

$$\frac{\Gamma\!\left(-\frac{j_2}{2}\right)}{\Gamma(-j_2)} = \frac{2^{1+j_2}\,\sqrt{\pi}}{\Gamma\!\left(\frac{1-j_2}{2}\right)}\,. \tag{A.8}$$

Now we can safely perform the analytic continuation to $j_{1,2} \geq 0$. The factor $\Gamma\!\left(\frac{1-j_2}{2}\right)$ in the denominator makes the result vanish for odd j_2, as it should due to the Lorentz structure of tadpole tensor integrals. For even j_2 we can write

$$\frac{1}{\Gamma\!\left(\frac{1-j_2}{2}\right)} = \frac{(-1)^{j_2/2}}{\Gamma\!\left(\frac{1}{2}\right)}\left(\tfrac{1}{2}\right)_{j_2/2} = \frac{(-1)^{j_2/2}}{\sqrt{\pi}}\,\frac{j_2!}{2^{j_2}\left(\frac{j_2}{2}\right)!}\,, \tag{A.9}$$

where the relation

$$\Gamma(\alpha-j) = (-1)^j\,\frac{\Gamma(\alpha)\,\Gamma(1-\alpha)}{\Gamma(1-\alpha+j)} = (-1)^j\,\frac{\Gamma(\alpha)}{(1-\alpha)_j} \tag{A.10}$$

for integer j has been used. Inserting the result into (2.9) we obtain, for even j_2,

$$F^{(s)}_{j_1,j_2} = \mu^{2\epsilon}\, e^{\epsilon\gamma_{\mathrm{E}}}\, e^{-i\pi n_{12}}\, (m^2)^{2-n_2-\epsilon}\, (-p^2-i0)^{-n_1}\left(\frac{m^2}{p^2}\right)^{j_1+j_2/2} \\ \times \frac{\Gamma\!\left(2-\epsilon+j_1+\frac{j_2}{2}\right)\Gamma\!\left(n_2-2+\epsilon-j_1-\frac{j_2}{2}\right)}{\Gamma(n_1)\,\Gamma(n_2)}\, \frac{(-1)^{j_2/2}\,\Gamma(n_1+j_{12})}{j_1!\left(\frac{j_2}{2}\right)!\,\Gamma\!\left(2-\epsilon+\frac{j_2}{2}\right)} \tag{A.11}$$

and $F^{(s)}_{j_1,j_2} = 0$ for odd j_2. Rewriting the sum over j_1, j_2 as a sum over $j = j_1 + \frac{j_2}{2}$ and $j_2' = \frac{j_2}{2}$ according to (2.13), we calculate

$$\sum_{j_2'=0}^{j} \frac{(-1)^{j_2'}\,\Gamma(n_1+j+j_2')}{j_2'!\,(j-j_2')!\,\Gamma(2-\epsilon+j_2')} \\ = \frac{1}{j!\,\Gamma(2-n_1-\epsilon-j)}\int_0^1 \mathrm{d}x\, x^{n_1+j-1}\,(1-x)^{1-n_1-\epsilon-j} \underbrace{\sum_{j_2'=0}^{j}\binom{j}{j_2'}(-x)^{j_2'}}_{(1-x)^j} \\ = \frac{\Gamma(n_1+j)\,\Gamma(2-n_1-\epsilon)}{j!\,\Gamma(2-n_1-\epsilon-j)\,\Gamma(2-\epsilon+j)}$$

and obtain the result (2.14) for $F^{(s)}_j$.

A.1.2 Large-momentum expansion: overlap contributions

This appendix is dedicated to the extraction of ultraviolet and infrared $1/\epsilon$ poles from the scaleless overlap contributions (2.37) in the particular case $n_1 = 1$, $n_2 = 2$ where the complete integral F is finite. We have to evaluate

$$F^{(h,s)} = \sum_{i,j_1,j_2=0}^{\infty} F^{(h,s)}_{i,j_1,j_2} = \sum_{i,j_1,j_2=0}^{\infty} (1+i)\,\frac{j_{12}!}{j_1!\,j_2!}\,\frac{(m^2)^i\,(-1)^{j_{12}}}{(p^2)^{1+j_{12}}}\int \mathrm{D}k\,\frac{(2k\cdot p)^{j_2}}{(k^2)^{2+i-j_1}}\,, \tag{A.12}$$

which only consists of vanishing scaleless integrals, but involves contributions of the form $(1/\epsilon_{\mathrm{UV}} - 1/\epsilon_{\mathrm{IR}})$ which we want to extract. The loop integral can be solved in analogy to the previous appendix A.1.1 by assuming $\operatorname{Re} j_2 < 0$ before analytically continuing the result to $j_2 \geq 0$:

$$\begin{aligned}
\int \mathrm{D}k\,\frac{(2k\cdot p)^{j_2}}{(k^2+i0)^n} &= \frac{e^{-i\pi(n-j_2)/2}}{\Gamma(n)\,\Gamma(-j_2)}\int_0^\infty \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\alpha_1^{n-1}\,\alpha_2^{-j_2-1}\int \mathrm{D}k\,e^{i(\alpha_1 k^2+2\alpha_2\,p\cdot k+i0)}\\
&= \mu^{2\epsilon}\,e^{\epsilon\gamma_{\mathrm{E}}}\,\frac{e^{-i\pi(n-j_2+2-\epsilon)/2}}{\Gamma(n)\,\Gamma(-j_2)}\int_0^\infty \mathrm{d}\alpha_1\,\alpha_1^{n-3+\epsilon}\int_0^\infty \mathrm{d}\alpha_2\,\alpha_2^{-j_2-1}\,\exp\!\left[i\alpha_2^2\left(-\frac{p^2}{\alpha_1}+i0\right)\right]\\
&= \mu^{2\epsilon}\,e^{\epsilon\gamma_{\mathrm{E}}}\,e^{-i\pi(n+2-j_2/2-\epsilon)/2}\,(-p^2+i0)^{j_2/2}\,\frac{\Gamma\!\left(-\frac{j_2}{2}\right)}{2\,\Gamma(n)\,\Gamma(-j_2)}\int_0^\infty \mathrm{d}\alpha_1\,\alpha_1^{n-3-j_2/2+\epsilon}\,.
\end{aligned} \tag{A.13}$$

The prefactor is rewritten with the help of (A.8) and (A.9), vanishing for odd values of j_2, as it should. The α_1-integral is again scaleless. For extracting the singularities at $\alpha_1 \to 0$ and $\alpha_1 \to \infty$ we have to separate the integration at some intermediate scale λ. Consider the following integral for integer j:

$$\int_0^\infty \mathrm{d}\alpha\,\alpha^{j-1+\epsilon} = \int_0^\lambda \mathrm{d}\alpha\,\alpha^{j-1+\epsilon} + \int_\lambda^\infty \mathrm{d}\alpha\,\alpha^{j-1+\epsilon} = \underbrace{\frac{\lambda^{j+\epsilon}}{j+\epsilon}}_{\operatorname{Re}\epsilon>-j} - \underbrace{\frac{\lambda^{j+\epsilon}}{j+\epsilon}}_{\operatorname{Re}\epsilon<-j} = 0\,. \tag{A.14}$$

The first term is ultraviolet-divergent for $\operatorname{Re}\epsilon \leq -j$ while the second term is infrared-divergent for $\operatorname{Re}\epsilon \geq -j$. The two terms are combined using analytic continuation, and the integral vanishes. We are interested in the separate $1/\epsilon$ singularities, which only occur for $j = 0$:

$$\int_0^\infty \mathrm{d}\alpha\,\alpha^{j-1+\epsilon} = \delta_{j,0}\left(\frac{1}{\epsilon_{\mathrm{UV}}} - \frac{1}{\epsilon_{\mathrm{IR}}}\right). \tag{A.15}$$

Plugging (A.15) into (A.13) we obtain

$$\int \mathrm{D}k\,\frac{(2k\cdot p)^{j_2}}{(k^2)^n} = \delta_{n-2,\,j_2/2}\left(\frac{1}{\epsilon_{\mathrm{UV}}} - \frac{1}{\epsilon_{\mathrm{IR}}}\right)(p^2)^{j_2/2}\,\frac{j_2!}{\Gamma(n)\left(\frac{j_2}{2}\right)!} \tag{A.16}$$

for even j_2 and zero otherwise. Inserting this into (A.12) with $n = 2+i-j_1$, eliminating j_1 with the Kronecker delta in (A.16) and replacing $j_2 = 2j_2'$ yields

$$F^{(h,s)} = \frac{1}{p^2}\left(\frac{1}{\epsilon_{\mathrm{UV}}} - \frac{1}{\epsilon_{\mathrm{IR}}}\right)\sum_{i=0}^{\infty}\left(\frac{m^2}{p^2}\right)^i (1+i)\,(-1)^i \sum_{j_2'=0}^{i}\frac{(-1)^{j_2'}\,(i+j_2')!}{j_2'!\,(i-j_2')!\,(1+j_2')!}\,. \tag{A.17}$$

The sum over j_2' can be evaluated in the following way:

$$\begin{aligned}\sum_{j_2'=0}^{i} \frac{(-1)^{j_2'}\,(i+j_2')!}{j_2'!\,(i-j_2')!\,(1+j_2')!} &= \lim_{\delta\to 0}\frac{1}{i!}\sum_{j_2'=0}^{i}\binom{i}{j_2'}(-1)^{j_2'}\frac{\Gamma(1+i+j_2')}{\Gamma(2+j_2'+\delta)} \\ &= \lim_{\delta\to 0}\frac{1}{i!\,\Gamma(1-i+\delta)}\int_0^1 \mathrm{d}x\, x^i\,(1-x)^{\delta-i}\underbrace{\sum_{j_2'=0}^{i}\binom{i}{j_2'}(-x)^{j_2'}}_{(1-x)^i} \\ &= \lim_{\delta\to 0}\frac{\Gamma(1+\delta)}{\Gamma(1-i+\delta)\,\Gamma(2+i+\delta)} \\ &= \frac{1}{\Gamma(1-i)\,(1+i)!} = \delta_{i,0}\,.\end{aligned}$$

For fixed $i \geq 1$ all terms in (A.17) cancel each other in the sum over j_2', resulting in $1/\Gamma(1-i)=0$. The only remaining contribution is from $i=0$, and we obtain

$$F^{(h,s)} = \frac{1}{p^2}\left(\frac{1}{\epsilon_{\rm UV}} - \frac{1}{\epsilon_{\rm IR}}\right). \tag{A.18}$$

Combining the complete result $F = F^{(h)} + F^{(s)} - F^{(h,s)}$ (2.35), the poles in $F^{(h,s)}$ cancel the corresponding ones in (2.17): $1/\epsilon_{\rm UV}$ cancels the ultraviolet pole in $F_0^{(s)}$ and $1/\epsilon_{\rm IR}$ cancels the infrared pole in $F_0^{(h)}$.

A.2 Threshold expansion

Details of the threshold expansion in section 4 are dealt with in this appendix.

A.2.1 Threshold expansion: check of convergence

Let us check that the expansions (4.5)–(4.7) of the threshold expansion in section 4 really converge within their corresponding domains:

- In the hard domain $D_h = \left\{k \in D : |\vec{k}| \gg |\vec{p}| \vee |k_0| \gg |\vec{p}|\right\}$, we either have $|\vec{k}| \gg |\vec{p}|$, then $|\vec{p}\cdot\vec{k}| \leq |\vec{p}|\,|\vec{k}| \ll \vec{k}^2$. If this is not the case, then there is $|\vec{k}| \lesssim |\vec{p}| \ll |k_0|$, which implies $|\vec{p}\cdot\vec{k}| \lesssim |\vec{p}|^2 \ll |q_0 k_0|$. So $|\vec{p}\cdot\vec{k}|$ is small compared to either $\vec{k}^2$ or $|q_0 k_0|$ in (4.5), and these two terms are present in the denominators even if $T^{(h)}$ is applied to $T^{(p)} I_{1,2}$ where k_0^2 has been eliminated from there before, cf. $T^{(h,p)} I_{1,2}$ in (4.9).

  It remains to be shown that $|\vec{p}\cdot\vec{k}|$ is also small compared to the combination of terms $|k_0^2 - \vec{k}^2 \pm q_0 k_0|$, which could be spoiled by a cancellation among the individual terms. Keeping track of the infinitesimal imaginary part in the propagators expanded with $T^{(h)}$, we look for zeros of the denominators $(k_0^2 - \vec{k}^2 \pm q_0 k_0 + i0)$ and $(k_0^2 - \vec{k}^2 + i0)$ in the complex k_0-plane where the integration is performed along the real axis. These zeros exclusively lie in the upper left and lower right quadrants of the complex plane. For $|\vec{k}| \gg |\vec{p}|$ zeros on different sides of the integration contour are well separated from each other. So we can bypass all zeros by bending the contour of the k_0-integration away from the real axis near the zeros. For $|\vec{k}| \lesssim |\vec{p}|$ the zeros may pinch the contour near $k_0 = 0$. But then D_h requires $|k_0| \gg |\vec{p}|$, where only two zeros, far apart from each other, are found. The same argument also works for the case when $T^{(h)}$ is applied to $T^{(p)} I_{1,2}$. Therefore we can always choose the integration contour within D_h such that $|\vec{p}\cdot\vec{k}| \ll |k_0^2 - \vec{k}^2 \pm q_0 k_0|$ in (4.5) and $|\vec{p}\cdot\vec{k}| \ll |-\vec{k}^2 \pm q_0 k_0|$ for $T^{(h,p)} I_{1,2}$ in (4.9).

- In the soft domain $D_s = \{k \in D : |\vec{k}| \lesssim |k_0| \lesssim |\vec{p}|\}$, we have $\vec{k}^2 \lesssim k_0^2 \lesssim |\vec{p}|\,|k_0| \ll |q_0 k_0|$ and $|\vec{p}\cdot\vec{k}| \le |\vec{p}|\,|\vec{k}| \ll |q_0 k_0|$, so each of the factors in the numerator of (4.6) is small compared to the denominator.

- In the potential domain $D_p = \{k \in D : |k_0| \ll |\vec{k}| \lesssim |\vec{p}|\}$, there is $k_0^2 \ll \vec{k}^2$. For the expansion $T^{(p)} I_{1,2}$ in (4.7), we additionally need $k_0^2 \ll |-\vec{k}^2 \pm q_0 k_0 - 2\vec{p}\cdot\vec{k}|$. Again the two zeros of $(-\vec{k}^2 \pm q_0 k_0 - 2\vec{p}\cdot\vec{k} + i0)$ exclusively lie in the upper left and lower right quadrants of the complex k_0-plane. The k_0-contour is pinched at $k_0 = 0$ in the limit $|\vec{k}| \to 0$, but as D_p requires $|k_0| \ll |\vec{k}|$, we can discard this case when checking the convergence of $T^{(p)}$. Otherwise the two zeros only pinch the k_0-contour when $\vec{k}^2 + 2\vec{p}\cdot\vec{k} \to 0$, which, for finite $|\vec{k}|$, requires a certain angular correlation between $\vec{k}$ and $\vec{p}$. So the pinching can be avoided if the contour of the angular $\vec{k}$-integration is bent into its complex plane in order to bypass the zeros of $\vec{k}^2 + 2\vec{p}\cdot\vec{k}$. The unpinched k_0-contour may bypass the zeros in the k_0-plane by bending away from the real axis. If $T^{(p)}$ is applied to $T^{(h)} I_{1,2}$, then the k_0-contour is only pinched in the limit $|\vec{k}| \to 0$. So the integration contours within D_p can always be chosen such that $|k_0^2| \ll |-\vec{k}^2 \pm q_0 k_0 - 2\vec{p}\cdot\vec{k}|$ in (4.7) and $|k_0^2| \ll |-\vec{k}^2 \pm q_0 k_0|$ for $T^{(h,p)} I_{1,2}$ in (4.9).

A.2.2 Threshold expansion: hard contributions

In the hard-region integrals of $F^{(h)}$ (4.10), the k_0-integration converges for positive n_1, n_2, n_3. We do not need the analytic regularization here and set $n_1 = n_2 = n_3 = 1$ from the start:

$$F^{(h)} = \sum_{j_1,j_2=0}^{\infty} \int \frac{\mathrm{D}k\,(2\vec{p}\cdot\vec{k})^{j_{12}}}{(k_0^2 - \vec{k}^2 + q_0 k_0 + i0)^{1+j_1}\,(k_0^2 - \vec{k}^2 - q_0 k_0 + i0)^{1+j_2}\,(k_0^2 - \vec{k}^2 + i0)}\,. \tag{A.19}$$

The first two propagators are combined with Feynman parameters, adding the third propagator afterwards:

$$F^{(h)} = -\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_{\mathrm{E}}}}{i\pi^{d/2}} \sum_{j_1,j_2=0}^{\infty} \frac{(2+j_{12})!}{j_1!\,j_2!}\,(-1)^{j_{12}} \int_0^1 \mathrm{d}x\,\mathrm{d}y\,x^{j_1}\,(1-x)^{j_2}\,y^{1+j_{12}}$$
$$\times \int_{-\infty}^{\infty} \mathrm{d}k_0 \int \mathrm{d}^{d-1}\vec{k}\,\frac{(2\vec{p}\cdot\vec{k})^{j_{12}}}{\left(\vec{k}^2 - k_0^2 - (2x-1)y q_0 k_0 - i0\right)^{3+j_{12}}}\,. \tag{A.20}$$

The $\vec{k}$-integral is factorized into a radial $|\vec{k}|$-integration and an angular integration. For the latter we use

$$\int \mathrm{d}\Omega_{d-1} = \int_{-1}^{1} \mathrm{d}\cos\theta\,(1-\cos^2\theta)^{(d-4)/2} \int \mathrm{d}\Omega_{d-2}\,, \qquad \int \mathrm{d}\Omega_{d-2} = \frac{2\pi^{(d-2)/2}}{\Gamma\left(\frac{d-2}{2}\right)}\,, \tag{A.21}$$

to distinguish the direction of the external momentum $\vec{p}$, writing $\vec{p}\cdot\vec{k} = |\vec{p}|\,|\vec{k}|\cos\theta$. The angular integral vanishes for odd j_{12}, and for even j_{12} the $\vec{k}$-integral yields

$$\begin{aligned}(2|\vec{p}\,|)^{j_{12}}\,\frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}\int_0^1 \mathrm{d}\cos^2\theta\,(\cos^2\theta)^{(j_{12}-1)/2}\,(1-\cos^2\theta)^{-\epsilon}\\ \times\int_0^\infty \mathrm{d}\vec{k}^2\,\frac{(\vec{k}^2)^{(1+j_{12})/2-\epsilon}}{\left(\vec{k}^2-k_0^2-(2x-1)yq_0k_0-i0\right)^{3+j_{12}}}\\ =\pi^{1-\epsilon}\,(4\vec{p}^{\,2})^{j_{12}/2}\,\frac{\Gamma\left(\frac{1+j_{12}}{2}\right)\Gamma\left(\frac{3+j_{12}}{2}+\epsilon\right)}{(2+j_{12})!}\,\frac{1}{\left(-k_0^2-(2x-1)yq_0k_0-i0\right)^{(3+j_{12})/2+\epsilon}}\,.\end{aligned} \tag{A.22}$$

The k_0-integral over the last factor in (A.22) is performed with a shift $k_0 \to k_0' = k_0 + \left(x-\frac{1}{2}\right)yq_0$:

$$\begin{aligned}\int_0^\infty \mathrm{d}k_0'^{\,2}\,\frac{(k_0'^{\,2})^{-1/2}}{\left((-1-i0)\,k_0'^{\,2}+\left(x-\frac{1}{2}\right)^2y^2q_0^2\right)^{(3+j_{12})/2+\epsilon}}\\ =\frac{\sqrt{\pi}\,\Gamma\left(1+\epsilon+\frac{j_{12}}{2}\right)}{\Gamma\left(\frac{3+j_{12}}{2}+\epsilon\right)}\,\underbrace{(-1-i0)^{-1/2}}_{=i}\left(\left(x-\tfrac{1}{2}\right)^2y^2q_0^2\right)^{-1-\epsilon-j_{12}/2}.\end{aligned} \tag{A.23}$$

By introducing $j = j_{12}/2$, the summation over j_1, j_2 with even j_{12} is rewritten as

$$\begin{aligned}F^{(h)} = -\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_{\mathrm{E}}}}{\sqrt{\pi}\,(q_0^2)^{1+\epsilon}}\int_0^1\frac{\mathrm{d}y}{y^{1+2\epsilon}}\sum_{j=0}^{\infty}\left(\frac{4\vec{p}^{\,2}}{q_0^2}\right)^j\frac{\Gamma\left(\frac{1}{2}+j\right)\Gamma(1+\epsilon+j)}{(2j)!}\\ \times\int_0^1\frac{\mathrm{d}x}{\left|x-\frac{1}{2}\right|^{2+2\epsilon+2j}}\underbrace{\sum_{j_1=0}^{2j}\binom{2j}{j_1}x^{j_1}\,(1-x)^{2j-j_1}}_{=[x+(1-x)]^{2j}=1}\,.\end{aligned} \tag{A.24}$$

The x- and y-integrations are easily solved. By substituting

$$(2j)! = \frac{2^{2j}}{\sqrt{\pi}}\,\Gamma\left(\tfrac{1}{2}+j\right)j! \tag{A.25}$$

the result $F^{(h)}$ in (4.14) is obtained.

A.2.3 Threshold expansion: potential contributions

The potential-region integrals of $F^{(p)}$ (4.10) diverge at $k_0 \to \pm\infty$ if $j_{123} > 0$. For evaluating the potential contribution to all orders in j_1, j_2, j_3 we therefore need to keep general complex propagator powers n_1 and n_2 as analytic regulators. The first two propagators of $F^{(p)}$ (4.10) are combined with Feynman parameters:

$$\begin{aligned}F^{(p)} = \frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_{\mathrm{E}}}}{i\pi^{d/2}}\,\frac{e^{-i\pi n_{123}}}{\Gamma(n_1)\,\Gamma(n_2)}\sum_{j_1,j_2,j_3=0}^{\infty}\frac{\Gamma(n_{12}+j_{12})\,(n_3)_{j_3}}{j_1!\,j_2!\,j_3!}\int_0^1\mathrm{d}x\,x^{n_1+j_1-1}\,(1-x)^{n_2+j_2-1}\\ \times\int\frac{\mathrm{d}^{d-1}\vec{k}}{(\vec{k}^2)^{n_3+j_3}}\int_{-\infty}^{\infty}\mathrm{d}k_0\,\frac{(k_0^2)^{j_{123}}}{\left(\vec{k}^2+2\vec{p}\cdot\vec{k}+(1-2x)q_0k_0-i0\right)^{n_{12}+j_{12}}}\,.\end{aligned} \tag{A.26}$$

Partitioning the k_0-integral into the two intervals $(-\infty, 0)$ and $(0, \infty)$, we obtain

$$\int_0^\infty \mathrm{d}k_0 \frac{k_0^{2j_{123}}}{\left((\vec{k}^2 + 2\vec{p}\cdot\vec{k} - i0) + (1-2x-i0)q_0 k_0\right)^{n_{12}+j_{12}}} + \int_0^\infty \mathrm{d}(-k_0) \frac{(-k_0)^{2j_{123}}}{\left((\vec{k}^2 + 2\vec{p}\cdot\vec{k} - i0) + (2x-1-i0)q_0(-k_0)\right)^{n_{12}+j_{12}}}$$
$$= \frac{(2j_{123})!\,\Gamma(n_{12}-1-j_{12}-2j_3)}{\Gamma(n_{12}+j_{12})} \frac{1}{(\vec{k}^2 + 2\vec{p}\cdot\vec{k} - i0)^{n_{12}-1-j_{12}-2j_3}} \frac{1}{q_0^{1+2j_{123}}} \times \left(\frac{1}{(1-2x-i0)^{1+2j_{123}}} - \frac{1}{(1-2x+i0)^{1+2j_{123}}}\right). \quad \text{(A.27)}$$

Under the x-integral, this term extracts the residue at $x = \frac{1}{2}$:

$$\int_0^1 \mathrm{d}x\, x^{n_1+j_1-1}(1-x)^{n_2+j_2-1}\left(\frac{1}{(1-2x-i0)^{1+2j_{123}}} - \frac{1}{(1-2x+i0)^{1+2j_{123}}}\right)$$
$$= -2i\pi\ \mathrm{Res}\left.\frac{x^{n_1+j_1-1}(1-x)^{n_2+j_2-1}}{(1-2x)^{1+2j_{123}}}\right|_{x=\frac{1}{2}}$$
$$= \frac{i\pi}{4^{j_{123}}(2j_{123})!}\left(\frac{\partial}{\partial t}\right)^{2j_{123}} \left.\left(\tfrac{1}{2}+t\right)^{n_1+j_1-1}\left(\tfrac{1}{2}-t\right)^{n_2+j_2-1}\right|_{t=0}, \quad \text{(A.28)}$$

with $t = x - \frac{1}{2}$. The $\vec{k}$-integration

$$\int \frac{\mathrm{d}^{d-1}\vec{k}}{(\vec{k}^2 + 2\vec{p}\cdot\vec{k} - i0)^{n_{12}-1-j_{12}-2j_3}\,(\vec{k}^2)^{n_3+j_3}} \quad \text{(A.29)}$$

is obviously scaleless if $n_1 = n_2 = 1$ and any of the j_1, j_2, j_3 is larger than zero (cf. [1]). But exactly in this case the Gamma function in the numerator of (A.27) would be ill-defined, so we still need analytic regulators and continue the calculation with general n_1, n_2, n_3. Combining the two denominators with Feynman parameters, the $\vec{k}$-integral yields

$$\frac{\Gamma(n_{123}-1-j_{123})}{\Gamma(n_{12}-1-j_{12}-2j_3)\,\Gamma(n_3+j_3)} \int_0^1 \mathrm{d}y\, y^{n_{12}-2-j_{12}-2j_3}(1-y)^{n_3+j_3-1} \times \int \frac{\mathrm{d}^{d-1}\vec{k}}{\left((\vec{k}+y\vec{p})^2 - y^2\vec{p}^2 - i0\right)^{n_{123}-1-j_{123}}}$$
$$= \frac{\pi^{(d-1)/2}\,\Gamma\left(n_{123}-\frac{5}{2}+\epsilon-j_{123}\right)}{\Gamma(n_{12}-1-j_{12}-2j_3)\,\Gamma(n_3+j_3)} (-\vec{p}^2 - i0)^{\frac{5}{2}-n_{123}-\epsilon+j_{123}} \times \int_0^1 \mathrm{d}y\, y^{3-n_{12}-2n_3-2\epsilon+j_{12}}(1-y)^{n_3+j_3-1}$$
$$= \frac{\pi^{(d-1)/2}\,\Gamma\left(n_{123}-\frac{5}{2}+\epsilon-j_{123}\right)\Gamma(4-n_{12}-2n_3-2\epsilon+j_{12})}{\Gamma(n_{12}-1-j_{12}-2j_3)\,\Gamma(4-n_{123}-2\epsilon+j_{123})} \times (p^2 - i0)^{\frac{5}{2}-n_{123}-\epsilon+j_{123}}. \quad \text{(A.30)}$$

This cancels the singularity from (A.27) when $(n_{12} - 1 - j_{12} - 2j_3)$ is zero or a negative integer. The complete contribution

$$F^{(p)} = \frac{\mu^{2\epsilon}\, e^{\epsilon\gamma_E}}{\sqrt{q^2}\,(p^2 - i0)^{n_{123}-5/2+\epsilon}} \frac{e^{-i\pi n_{123}}\sqrt{\pi}}{\Gamma(n_1)\,\Gamma(n_2)}$$
$$\times \sum_{j_1,j_2,j_3=0}^{\infty} \left(\frac{p^2}{4q^2}\right)^{j_{123}} \frac{(n_3)_{j_3}\,\Gamma\big(n_{123} - \frac{5}{2} + \epsilon - j_{123}\big)\,\Gamma(4 - n_{12} - 2n_3 - 2\epsilon + j_{12})}{j_1!\, j_2!\, j_3!\,\Gamma(4 - n_{123} - 2\epsilon + j_{123})}$$
$$\times \left(\frac{\partial}{\partial t}\right)^{2j_{123}} \big(\tfrac{1}{2} + t\big)^{n_1+j_1-1} \big(\tfrac{1}{2} - t\big)^{n_2+j_2-1}\Big|_{t=0} \tag{A.31}$$

is regularized dimensionally. We can switch off the analytic regularization at this point and set $n_1 = n_2 = n_3 = 1$:

$$F^{(p)} = -\frac{e^{\epsilon\gamma_E}\sqrt{\pi}}{\sqrt{q^2\,(p^2 - i0)}} \left(\frac{\mu^2}{p^2 - i0}\right)^{\epsilon} \sum_{j_1,j_2,j_3=0}^{\infty} \left(\frac{p^2}{4q^2}\right)^{j_{123}} \frac{\Gamma\big(\frac{1}{2} + \epsilon - j_{123}\big)\,\Gamma(-2\epsilon + j_{12})}{j_1!\, j_2!\,\Gamma(1 - 2\epsilon + j_{123})}$$
$$\times \left(\frac{\partial}{\partial t}\right)^{2j_{123}} \big(\tfrac{1}{2} + t\big)^{j_1} \big(\tfrac{1}{2} - t\big)^{j_2}\Big|_{t=0}. \tag{A.32}$$

Now it is obvious that the potential region gets no contributions from $j_{123} > 0$: The maximal power of the polynomial in t in the last line of (A.32) is $(j_1 + j_2)$, so this polynomial vanishes upon the $2(j_1 + j_2 + j_3)$ derivatives with respect to t. The only non-vanishing contribution originates from $j_1 = j_2 = j_3 = 0$ and is reported in (4.14).

The leading-order potential contribution (from $j_1 = j_2 = j_3 = 0$) is well-defined without analytic regularization. So for checking the result we can set $n_1 = n_2 = n_3 = 1$ from the start. We solve the k_0-integration by closing its contour in the imaginary infinity and picking up one of the two poles:

$$F_0^{(p)} = \frac{\mu^{2\epsilon}\, e^{\epsilon\gamma_E}}{i\pi^{d/2}} \int \frac{\mathrm{d}^{d-1}\vec{k}}{\vec{k}^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}k_0}{(q_0 k_0 - \vec{k}^2 - 2\vec{p}\cdot\vec{k} + i0)\,(q_0 k_0 + \vec{k}^2 + 2\vec{p}\cdot\vec{k} - i0)}$$
$$= -\frac{1}{q_0}\frac{\mu^{2\epsilon}\, e^{\epsilon\gamma_E}}{\pi^{d/2-1}} \int \frac{\mathrm{d}^{d-1}\vec{k}}{(\vec{k}^2 + 2\vec{p}\cdot\vec{k} - i0)\,\vec{k}^2}$$
$$= -\frac{1}{\sqrt{q^2\,(p^2 - i0)}} \left(\frac{\mu^2}{p^2 - i0}\right)^{\epsilon} \frac{e^{\epsilon\gamma_E}\sqrt{\pi}\,\Gamma(\frac{1}{2} + \epsilon)\,\Gamma(-2\epsilon)}{\Gamma(1 - 2\epsilon)}, \tag{A.33}$$

using (A.30) with $n_1 = n_2 = n_3 = 1$ and $j_1 = j_2 = j_3 = 0$. This agrees with the result obtained from (A.32).

A.3 Sudakov form factor

This appendix sketches the evaluation of the contributions to the vertex correction in the Sudakov limit from section 6.

A.3.1 Sudakov form factor: scaleless contributions

First we have a look at the contributions from the Glauber and collinear-plane regions and at the overlap contributions which all turn out to be scaleless within dimensional and analytic regularization.

Glauber contribution. According to the expansion $T^{(g)}$ (6.12) the Glauber contribution reads

$$F^{(g)} = \frac{\mu^{2\epsilon}\, e^{\epsilon\gamma_{\rm E}}}{2i\pi^{d/2}} \sum_{j_1,j_2,j_3=0}^{\infty} \frac{(n_1)_{j_1}\,(n_2)_{j_2}\,(n_3)_{j_3}}{j_1!\,j_2!\,j_3!}\,(-1)^{j_{123}} \int \frac{{\rm d}^{d-2}\vec{k}_\perp}{(-\vec{k}_\perp^2 - m^2 + i0)^{n_3+j_3}}$$
$$\times \int_{-\infty}^{\infty} \frac{{\rm d}k^+\,(k^+)^{j_{123}}}{(-\vec{k}_\perp^2 + Qk^+ + i0)^{n_1+j_1}} \int_{-\infty}^{\infty} \frac{{\rm d}k^-\,(k^-)^{j_{123}}}{(-\vec{k}_\perp^2 + Qk^- + i0)^{n_2+j_2}}\,. \tag{A.34}$$

The $k^\pm$-integrals can be rewritten via $t^\pm = Qk^\pm - \vec{k}_\perp^2$:

$$\int_{-\infty}^{\infty} \frac{{\rm d}k^\pm\,(k^\pm)^{j_{123}}}{(-\vec{k}_\perp^2 + Qk^\pm + i0)^{n_{1,2}+j_{1,2}}} = \frac{1}{Q^{1+j_{123}}} \int_{-\infty}^{\infty} {\rm d}t^\pm\, \frac{(t^\pm + \vec{k}_\perp^2)^{j_{123}}}{(t^\pm + i0)^{n_{1,2}+j_{1,2}}}$$
$$= \frac{1}{Q^{1+j_{123}}} \sum_{i=0}^{j_{123}} \binom{j_{123}}{i} (\vec{k}_\perp^2)^{j_{123}-i} \int_{-\infty}^{\infty} \frac{{\rm d}t^\pm}{(t^\pm + i0)^{n_{1,2}+j_{1,2}-i}}\,. \tag{A.35}$$

These $t^\pm$-integrals are scaleless, however. Within analytic regularization, we may assume that ${\rm Re}\, n_1 > 1 + j_{23}$ and ${\rm Re}\, n_2 > 1 + j_{13}$ (separately for every term under the sum $\sum_{j_1,j_2,j_3}$), which is required for the convergence of the $t^\pm$-integrals and also allows to close the integration contour in the upper half of the complex $t^\pm$-plane. The only pole at $t^\pm = -i0$ and the branch cut slightly below the negative real axis are then located outside of the closed contour. So both $t^\pm$-integrals and therefore the $k^\pm$-integrals vanish. The $\vec{k}_\perp$-integral is regularized dimensionally. We get

$$F^{(g)} = 0\,. \tag{A.36}$$

Collinear-plane contribution. From $T^{(cp)}$ (6.13) we get

$$F^{(cp)} = \frac{\mu^{2\epsilon}\, e^{\epsilon\gamma_{\rm E}}}{2i\pi^{d/2}} \sum_{j_1,\dots,j_4=0}^{\infty} \frac{(n_1)_{j_1}\,(n_2)_{j_2}\,(n_3)_{j_{34}}}{j_1!\cdots j_4!}\,(m^2)^{j_4} \int {\rm d}^{d-2}\vec{k}_\perp\,(\vec{k}_\perp^2)^{j_{123}}$$
$$\times \int_{-\infty}^{\infty} \frac{{\rm d}k^+\,{\rm d}k^-}{\big(k^+(k^-+Q) + i0\big)^{n_1+j_1} \big((k^++Q)k^- + i0\big)^{n_2+j_2} (k^+k^- + i0)^{n_3+j_{34}}}\,. \tag{A.37}$$

Here the $\vec{k}_\perp$-integral is scaleless:

$$\int {\rm d}^{d-2}\vec{k}_\perp\,(\vec{k}_\perp^2)^{j_{123}} = \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \int_0^\infty {\rm d}|\vec{k}_\perp|\,|\vec{k}_\perp|^{1+2j_{123}-2\epsilon} = 0\,. \tag{A.38}$$

In the $|\vec{k}_\perp|$-integral the singularities at $|\vec{k}_\perp| \to 0$ and $|\vec{k}_\perp| \to \infty$ are regularized dimensionally (by ϵ) and cancel each other. The k^+-k^--integral is regularized analytically. It yields the contribution

$$2i\pi\, \frac{e^{-i\pi n_{123}}\,(-1)^{j_{1234}}}{(Q^2)^{n_{123}+j_{1234}-1}}\, \frac{\Gamma(n_{123}+j_{1234}-1)\,\Gamma(1-n_{13}-j_{134})\,\Gamma(1-n_{23}-j_{234})}{\Gamma(n_1+j_1)\,\Gamma(n_2+j_2)\,\Gamma(2-n_{123}-j_{1234})}\,, \tag{A.39}$$

which is non-zero and well-defined for $n_{1,2,3} \notin \mathbb{Z}$. Altogether we have

$$F^{(cp)} = 0\,. \tag{A.40}$$

Hard-collinear overlap contributions. For a Lorentz-invariant integration, the 1-collinear expansion $T^{(1c)}$ (6.9) can be rewritten as

$$T^{(1c)} I_2 = \sum_{j_1,j_2=0}^{\infty} \frac{(n_2)_{j_{12}}}{j_1!\, j_2!} \frac{(-k^+k^-)^{j_1}\, (\vec{k}_\perp^2)^{j_2}}{(Qk^-)^{n_2+j_{12}}} = \sum_{j=0}^{\infty} \frac{(n_2)_j}{j!} \frac{(-k^2)^j}{(2p_2\cdot k)^{n_2+j}}\,, \tag{A.41}$$

where terms in the numerator with the same $j = j_1 + j_2$ have been combined via the binomial theorem. Putting (A.41) together with $T^{(h)}$ (6.8), the overlap contribution reads

$$F^{(h,1c)} = \sum_{j_1,j_2=0}^{\infty} \frac{(n_2)_{j_1}\,(n_3)_{j_2}}{j_1!\, j_2!} (-1)^{j_1}\,(m^2)^{j_2} \times \int \frac{\mathrm{D}k}{(k^2+2p_1\cdot k)^{n_1}\,(2p_2\cdot k)^{n_2+j_1}\,(k^2)^{n_3+j_2-j_1}}\,. \tag{A.42}$$

We evaluate this loop integral for general propagator powers $n_{1,2,3}$ with the help of alpha parameters (A.2):

$$\int \frac{\mathrm{D}k}{(k^2+2p_1\cdot k)^{n_1}\,(2p_2\cdot k)^{n_2}\,(k^2)^{n_3}} = \left(\prod_{i=1}^{3} \frac{e^{-i\pi n_i/2}}{\Gamma(n_i)} \int_0^\infty \mathrm{d}\alpha_i\, \alpha_i^{n_i-1}\right) \int \mathrm{D}k\, \exp\Big[i\Big(\alpha_{13}k^2 + 2k\cdot(\alpha_1 p_1 + \alpha_2 p_2)\Big)\Big]\,. \tag{A.43}$$

Performing the loop integration via (A.4), the α_2-integral through reversing (A.2) and then the α_1-integral using (A.6), a scaleless α_3-integral remains:

$$\mu^{2\epsilon}\, e^{\epsilon\gamma_{\mathrm{E}}} \frac{e^{-i\pi(n_{13}+2n_2+2-\epsilon)/2}}{(Q^2)^{n_2}} \frac{\Gamma(n_1-n_2)\,\Gamma(2-n_1-\epsilon)}{\Gamma(n_1)\,\Gamma(n_3)\,\Gamma(2-n_2-\epsilon)} \int_0^\infty \mathrm{d}\alpha_3\, \alpha_3^{n_{13}-3+\epsilon} = 0\,. \tag{A.44}$$

Plugging this result into (A.42) with $n_2 \to n_2 + j_1$ and $n_3 \to n_3 + j_2 - j_1$, we obtain

$$F^{(h,1c)} = 0\,. \tag{A.45}$$

The two collinear regions are symmetric to each other via exchanging $p_1 \leftrightarrow p_2$ and $n_1 \leftrightarrow n_2$. Therefore we also have

$$F^{(h,2c)} = 0\,. \tag{A.46}$$

Collinear-collinear overlap contribution. Rewriting both collinear expansions $T^{(1c)}$ (6.9) and $T^{(2c)}$ (6.10) via (A.41), the collinear-collinear overlap contribution reads

$$\begin{aligned} F^{(1c,2c)} &= \sum_{j_1,j_2=0}^{\infty} \frac{(n_1)_{j_1}\,(n_2)_{j_2}}{j_1!\, j_2!} (-1)^{j_{12}} \int \frac{\mathrm{D}k\,(k^2)^{j_{12}}}{(2p_1\cdot k)^{n_1+j_1}\,(2p_2\cdot k)^{n_2+j_2}\,(k^2-m^2)^{n_3}} \\ &= \sum_{j_1,j_2=0}^{\infty} \frac{(n_1)_{j_1}\,(n_2)_{j_2}}{j_1!\, j_2!} (-1)^{j_{12}} \sum_{i=0}^{j_{12}} \binom{j_{12}}{i} (m^2)^{j_{12}-i} \\ &\quad\times \int \frac{\mathrm{D}k}{(2p_1\cdot k)^{n_1+j_1}\,(2p_2\cdot k)^{n_2+j_2}\,(k^2-m^2)^{n_3-i}}\,, \end{aligned} \tag{A.47}$$

where $(k^2)^{j_{12}} = \big((k^2-m^2)+m^2\big)^{j_{12}}$ in the numerator is expanded according to the binomial theorem. Again using alpha parameters (A.2),

$$\int \frac{\mathrm{D}k}{(2p_1\cdot k)^{n_1}\,(2p_2\cdot k)^{n_2}\,(k^2-m^2)^{n_3}} = \left(\prod_{i=1}^{3} \frac{e^{-i\pi n_i/2}}{\Gamma(n_i)} \int_0^\infty \mathrm{d}\alpha_i\, \alpha_i^{n_i-1}\right) e^{i\alpha_3(-m^2)} \times \int \mathrm{D}k\, \exp\Big[i\Big(\alpha_3 k^2 + 2k\cdot(\alpha_1 p_1 + \alpha_2 p_2)\Big)\Big]. \tag{A.48}$$

Performing the integrations over k, α_1 and α_3, a scaleless α_2-integral remains,

$$\mu^{2\epsilon}\, e^{\epsilon\gamma_\mathrm{E}}\, \frac{e^{-i\pi(3n_1+n_2+2n_3)/2}}{(Q^2)^{n_1}\,(m^2)^{n_{13}-2+\epsilon}}\, \frac{\Gamma(n_{13}-2+\epsilon)}{\Gamma(n_2)\,\Gamma(n_3)} \int_0^\infty \mathrm{d}\alpha_2\, \alpha_2^{n_2-n_1-1} = 0\,, \tag{A.49}$$

which is regularized analytically. Therefore

$$F^{(1c,2c)} = 0\,. \tag{A.50}$$

Other overlap contributions. The remaining overlap contributions in (6.19) are combinations of expansions which already individually lead to scaleless integrals. So it is obvious that these contributions are scaleless as well:

$$\begin{aligned}
F^{(h,1c,2c)} &= \sum_{j_1,j_2,j_3=0}^{\infty} \frac{(n_1)_{j_1}\,(n_2)_{j_2}\,(n_3)_{j_3}}{j_1!\,j_2!\,j_3!}\,(-1)^{j_{12}}\,(m^2)^{j_3} \\
&\quad\times \int \frac{\mathrm{D}k}{(2p_1\cdot k)^{n_1+j_1}\,(2p_2\cdot k)^{n_2+j_2}\,(k^2)^{n_3+j_3-j_{12}}} = 0\,, \\
F^{(h,g)} &= \frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_\mathrm{E}}}{2i\pi^{d/2}} \sum_{j_1,\dots,j_4=0}^{\infty} \frac{(n_1)_{j_1}\,(n_2)_{j_2}\,(n_3)_{j_{34}}}{j_1!\cdots j_4!}\,(-1)^{j_{123}}\,(m^2)^{j_4} \int \frac{\mathrm{d}^{d-2}\vec{k}_\perp}{(-\vec{k}_\perp^2)^{n_3+j_{34}}} \\
&\quad\times \int_{-\infty}^{\infty} \frac{\mathrm{d}k^+\,(k^+)^{j_{123}}}{(-\vec{k}_\perp^2+Qk^+ + i0)^{n_1+j_1}} \int_{-\infty}^{\infty} \frac{\mathrm{d}k^-\,(k^-)^{j_{123}}}{(-\vec{k}_\perp^2+Qk^- + i0)^{n_2+j_2}} = 0\,, \\
F^{(1c,g)} &= \frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_\mathrm{E}}}{2i\pi^{d/2}} \sum_{j_1,\dots,j_4=0}^{\infty} \frac{(n_1)_{j_1}\,(n_2)_{j_{23}}\,(n_3)_{j_4}}{j_1!\cdots j_4!}\,\frac{(-1)^{j_{124}}}{Q^{n_2+j_{23}}} \int \frac{\mathrm{d}^{d-2}\vec{k}_\perp\,(\vec{k}_\perp^2)^{j_3}}{(-\vec{k}_\perp^2-m^2)^{n_3+j_4}} \\
&\quad\times \int_{-\infty}^{\infty} \frac{\mathrm{d}k^+\,(k^+)^{j_{124}}}{(-\vec{k}_\perp^2+Qk^+ + i0)^{n_1+j_1}} \int_{-\infty}^{\infty} \frac{\mathrm{d}k^-}{(k^- + i0)^{n_2+j_3-j_{14}}} = 0\,, \\
F^{(2c,g)} &= 0 \quad [\text{via symmetry related to } F^{(1c,g)}]\,, \\
F^{(h,1c,g)} &= \frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_\mathrm{E}}}{2i\pi^{d/2}} \sum_{j_1,\dots,j_5=0}^{\infty} \frac{(n_1)_{j_1}\,(n_2)_{j_{23}}\,(n_3)_{j_{45}}}{j_1!\cdots j_5!}\,\frac{(-1)^{j_{1234}}\,(m^2)^{j_5}}{Q^{n_2+j_{23}}} \int \frac{\mathrm{d}^{d-2}\vec{k}_\perp}{(-\vec{k}_\perp^2)^{n_3+j_{45}-j_3}} \\
&\quad\times \int_{-\infty}^{\infty} \frac{\mathrm{d}k^+\,(k^+)^{j_{124}}}{(-\vec{k}_\perp^2+Qk^+ + i0)^{n_1+j_1}} \int_{-\infty}^{\infty} \frac{\mathrm{d}k^-}{(k^- + i0)^{n_2+j_3-j_{14}}} = 0\,, \\
F^{(h,2c,g)} &= 0 \quad [\text{via symmetry related to } F^{(h,1c,g)}]\,,
\end{aligned}$$

$$F^{(1c,2c,g)} = \frac{\mu^{2\epsilon}\, e^{\epsilon\gamma_{\rm E}}}{2i\pi^{d/2}} \sum_{j_1,\ldots,j_5=0}^{\infty} \frac{(n_1)_{j_{12}}\,(n_2)_{j_{34}}\,(n_3)_{j_5}}{j_1!\cdots j_5!}\, \frac{(-1)^{j_{135}}}{Q^{n_{12}+j_{1234}}} \int \frac{\mathrm{d}^{d-2}\vec{k}_\perp\,(\vec{k}_\perp^2)^{j_{24}}}{(-\vec{k}_\perp^2-m^2)^{n_3+j_5}}$$
$$\times \int_{-\infty}^{\infty} \frac{\mathrm{d}k^+}{(k^+ + i0)^{n_1+j_2-j_{35}}} \int_{-\infty}^{\infty} \frac{\mathrm{d}k^-}{(k^- + i0)^{n_2+j_4-j_{15}}} = 0\,,$$
$$F^{(h,1c,2c,g)} = \frac{\mu^{2\epsilon}\, e^{\epsilon\gamma_{\rm E}}}{2i\pi^{d/2}} \sum_{j_1,\ldots,j_6=0}^{\infty} \frac{(n_1)_{j_{12}}\,(n_2)_{j_{34}}\,(n_3)_{j_{56}}}{j_1!\cdots j_6!}\, \frac{(-1)^{j_{12345}}\,(m^2)^{j_6}}{Q^{n_{12}+j_{1234}}}$$
$$\times \int \frac{\mathrm{d}^{d-2}\vec{k}_\perp}{(-\vec{k}_\perp^2)^{n_3+j_{56}-j_{24}}} \int_{-\infty}^{\infty} \frac{\mathrm{d}k^+}{(k^+ + i0)^{n_1+j_2-j_{35}}} \int_{-\infty}^{\infty} \frac{\mathrm{d}k^-}{(k^- + i0)^{n_2+j_4-j_{15}}} = 0\,,$$
$$F^{(h,cp)} = F^{(cp)} = 0 \quad \text{[same expansion]}\,,$$
$$F^{(1c,cp)} = \frac{\mu^{2\epsilon}\, e^{\epsilon\gamma_{\rm E}}}{2i\pi^{d/2}} \sum_{j_1,\ldots,j_5=0}^{\infty} \frac{(n_1)_{j_1}\,(n_2)_{j_{23}}\,(n_3)_{j_{45}}}{j_1!\cdots j_5!}\, \frac{(-1)^{j_2}\,(m^2)^{j_5}}{Q^{n_2+j_{23}}} \int \mathrm{d}^{d-2}\vec{k}_\perp\,(\vec{k}_\perp^2)^{j_{134}}$$
$$\times \int_{-\infty}^{\infty} \frac{\mathrm{d}k^+\,\mathrm{d}k^-}{\big(k^+(k^-+Q)+i0\big)^{n_1+j_1}\,(k^-+i0)^{n_2+j_{23}}\,(k^+k^-+i0)^{n_3+j_{45}-j_2}} = 0\,,$$
$$F^{(2c,cp)} = 0 \quad \text{[via symmetry related to } F^{(1c,cp)}]\,,$$
$$F^{(h,1c,cp)} = F^{(1c,cp)} = 0 \quad \text{[same expansion]}\,,$$
$$F^{(h,2c,cp)} = F^{(2c,cp)} = 0 \quad \text{[same expansion]}\,,$$
$$F^{(1c,2c,cp)} = \frac{\mu^{2\epsilon}\, e^{\epsilon\gamma_{\rm E}}}{2i\pi^{d/2}} \sum_{j_1,\ldots,j_6=0}^{\infty} \frac{(n_1)_{j_{12}}\,(n_2)_{j_{34}}\,(n_3)_{j_{56}}}{j_1!\cdots j_6!}\, \frac{(-1)^{j_{13}}\,(m^2)^{j_6}}{Q^{n_{12}+j_{1234}}} \int \mathrm{d}^{d-2}\vec{k}_\perp\,(\vec{k}_\perp^2)^{j_{245}}$$
$$\times \int_{-\infty}^{\infty} \frac{\mathrm{d}k^+\,\mathrm{d}k^-}{(k^+ + i0)^{n_1+j_2-j_3}\,(k^- + i0)^{n_2+j_4-j_1}\,(k^+k^- + i0)^{n_3+j_{56}}} = 0\,,$$
$$F^{(h,1c,2c,cp)} = F^{(1c,2c,cp)} = 0 \quad \text{[same expansion]}\,. \tag{A.51}$$

These contributions are all well-defined and scaleless through dimensional and analytic regularization.

A.3.2 Sudakov form factor: hard contribution

The hard-region integrals in $F^{(h)}$ (6.22),

$$\int \frac{\mathrm{D}k}{(k^2+2p_1\cdot k)^{n_1}\,(k^2+2p_2\cdot k)^{n_2}\,(k^2)^{n_3+j}}\,, \tag{A.52}$$

are easily evaluated using Feynman parameters, yielding

$$\frac{\Gamma(n_{123}+j)}{\Gamma(n_1)\,\Gamma(n_2)\,\Gamma(n_3+j)} \int_0^1 \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\delta(1-x_{123})\,x_1^{n_1-1}\,x_2^{n_2-1}\,x_3^{n_3+j-1}$$
$$\times \int \frac{\mathrm{D}k}{\big((k+x_1p_1+x_2p_2)^2 - x_1x_2Q^2\big)^{n_{123}+j}}$$

$$= \mu^{2\epsilon}\, e^{\epsilon\gamma_E}\, e^{-i\pi n_{123}}\, (-1)^j\, (Q^2)^{2-n_{123}-\epsilon-j}\, \frac{\Gamma(n_{123}-2+\epsilon+j)}{\Gamma(n_1)\,\Gamma(n_2)\,\Gamma(n_3+j)}$$
$$\times \underbrace{\int_0^1 \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3\, \delta(1-x_{123})\, x_1^{1-n_{23}-\epsilon-j}\, x_2^{1-n_{13}-\epsilon-j}\, x_3^{n_3+j-1}}_{\Gamma(2-n_{23}-\epsilon-j)\,\Gamma(2-n_{13}-\epsilon-j)\,\Gamma(n_3+j)/\Gamma(4-n_{123}-2\epsilon-j)}\,. \quad \text{(A.53)}$$

The complete contribution from the hard region is shown in (6.23). In the case $n_1 = n_2 = n_3 = 1$ this is

$$F^{(h)} = -\frac{1}{Q^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon} e^{\epsilon\gamma_E} \sum_{j=0}^{\infty}\left(-\frac{m^2}{Q^2}\right)^j \frac{\Gamma(1+\epsilon+j)\,\Gamma^2(-\epsilon-j)}{\Gamma(1-2\epsilon-j)}$$
$$= -\frac{1}{Q^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \frac{e^{\epsilon\gamma_E}\,\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)} \underbrace{\sum_{j=0}^{\infty}\left(\frac{m^2}{Q^2}\right)^j \frac{(2\epsilon)_j}{(1+\epsilon)_j}}_{{}_2F_1(2\epsilon,1;1+\epsilon;\,m^2/Q^2)}\,. \quad \text{(A.54)}$$

For the hypergeometric function we use the representations

$${}_2F_1(\alpha,\beta;\gamma;z) = \sum_{j=0}^{\infty} z^j\, \frac{(\alpha)_j\,(\beta)_j}{j!\,(\gamma)_j} = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)} \int_0^1 \mathrm{d}t\, \frac{t^{\beta-1}\,(1-t)^{\gamma-\beta-1}}{(1-tz)^{\alpha}} \quad \text{(A.55)}$$

and extract the singularity of the t-integral at $t \to 1$:

$${}_2F_1(2\epsilon,1;1+\epsilon;z) = \frac{\Gamma(1+\epsilon)}{\Gamma(\epsilon)}\left(\int_0^1 \mathrm{d}t\,(1-t)^{\epsilon-1}\left[(1-tz)^{-2\epsilon}-(1-z)^{-2\epsilon}\right]\right.$$
$$\left. + (1-z)^{-2\epsilon}\int_0^1 \mathrm{d}t\,(1-t)^{\epsilon-1}\right)$$
$$= -2\epsilon^2 \int_0^1 \frac{\mathrm{d}t}{1-t}\, \ln\!\left(\frac{1-tz}{1-z}\right) + (1-z)^{-2\epsilon} + \mathcal{O}(\epsilon^3)$$
$$= 1 - 2\epsilon\,\ln(1-z) + \epsilon^2\left(\ln^2(1-z) - 2\,\mathrm{Li}_2(z)\right) + \mathcal{O}(\epsilon^3)\,, \quad \text{(A.56)}$$

where Li_2 is the dilogarithm function. Expanding also the prefactor in (A.54), the result reported in (6.26) is produced.

A.3.3 Sudakov form factor: collinear contributions

The 1-collinear contribution $F^{(1c)}$ in (6.22) reads

$$F^{(1c)} = \sum_{j=0}^{\infty} \frac{(n_2)_j}{j!}\,(-1)^j \sum_{i=0}^{j}\binom{j}{i}\,(m^2)^{j-i}$$
$$\times \int \frac{\mathrm{D}k}{(k^2+2p_1\cdot k)^{n_1}\,(2p_2\cdot k)^{n_2+j}\,(k^2-m^2)^{n_3-i}}\,, \quad \text{(A.57)}$$

where $(k^2)^j = \left((k^2-m^2)+m^2\right)^j$ in the numerator has been expanded according to the binomial theorem. We evaluate this loop integral for general propagator powers $n_{1,2,3}$ using alpha parameters (A.2):

$$\int \frac{\mathrm{D}k}{(k^2+2p_1\cdot k)^{n_1}\,(2p_2\cdot k)^{n_2}\,(k^2-m^2)^{n_3}} = \left(\prod_{i=1}^{3} \frac{e^{-i\pi n_i/2}}{\Gamma(n_i)} \int_0^\infty \mathrm{d}\alpha_i\, \alpha_i^{n_i-1}\right) e^{i\alpha_3(-m^2)} \times \int \mathrm{D}k\, \exp\Big[i\Big(\alpha_{13}k^2 + 2k\cdot(\alpha_1 p_1 + \alpha_2 p_2)\Big)\Big]. \quad \text{(A.58)}$$

The loop and alpha-parameter integrations are performed with (A.4), (A.2) and (A.6). They yield

$$\mu^{2\epsilon}\, e^{\epsilon\gamma_{\mathrm{E}}}\, e^{-i\pi n_{123}}\, (m^2)^{2-n_{13}-\epsilon}\, (Q^2)^{-n_2}\, \frac{\Gamma(n_1-n_2)\,\Gamma(n_{13}-2+\epsilon)\,\Gamma(2-n_1-\epsilon)}{\Gamma(n_1)\,\Gamma(n_3)\,\Gamma(2-n_2-\epsilon)}\,. \quad \text{(A.59)}$$

Plugging this result with $n_2 \to n_2 + j$ and $n_3 \to n_3 - i$ into (A.57), we obtain

$$F^{(1c)} = \mu^{2\epsilon}\, e^{\epsilon\gamma_{\mathrm{E}}}\, e^{-i\pi n_{123}}\, (m^2)^{2-n_{13}-\epsilon}\, (Q^2)^{-n_2}\, \frac{\Gamma(2-n_1-\epsilon)}{\Gamma(n_1)\,\Gamma(n_2)} \sum_{j=0}^{\infty} \left(\frac{m^2}{Q^2}\right)^j \times \frac{\Gamma(n_2+j)\,\Gamma(n_1-n_2-j)}{j!\,\Gamma(2-n_2-\epsilon-j)} \sum_{i=0}^{j} \binom{j}{i} (-1)^i\, \frac{\Gamma(n_{13}-2+\epsilon-i)}{\Gamma(n_3-i)}\,. \quad \text{(A.60)}$$

The sum over i is evaluated as follows:

$$\begin{aligned}
&\sum_{i=0}^{j} \binom{j}{i} (-1)^i\, \frac{\Gamma(n_{13}-2+\epsilon-i)}{\Gamma(n_3-i)} \\
&\quad = \frac{1}{\Gamma(2-n_1-\epsilon)} \int_0^1 \mathrm{d}x\, x^{n_{13}-3+\epsilon}\, (1-x)^{1-n_1-\epsilon} \underbrace{\sum_{i=0}^{j} \binom{j}{i} \left(-\frac{1}{x}\right)^i}_{(-1)^j\, x^{-j}\,(1-x)^j} \\
&\quad = (-1)^j\, \frac{\Gamma(n_{13}-2+\epsilon-j)\,\Gamma(2-n_1-\epsilon+j)}{\Gamma(2-n_1-\epsilon)\,\Gamma(n_3)}\,,
\end{aligned} \quad \text{(A.61)}$$

producing the result which is reported in (6.23). The 2-collinear contribution $F^{(2c)}$ can be obtained from $F^{(1c)}$ by simply exchanging $n_1 \leftrightarrow n_2$.

As explained in section 6, we need analytic regularization for the collinear contributions, and we choose to calculate the case $n_1 = 1+\delta$, $n_2 = 1-\delta$ and $n_3 = 1$ in a Laurent expansion about $\delta = 0$ up to the finite order δ^0. Using the results from above, we obtain

$$\begin{aligned}
F^{(1c)} &= -\frac{1}{Q^2} \left(\frac{\mu^2}{m^2}\right)^{\epsilon} \left(\frac{Q^2}{m^2}\right)^{\delta} \frac{e^{\epsilon\gamma_{\mathrm{E}}}}{\Gamma(1+\delta)\,\Gamma(1-\delta)} \sum_{j=0}^{\infty} \left(-\frac{m^2}{Q^2}\right)^j \\
&\qquad \times \frac{\Gamma(1-\delta+j)\,\Gamma(2\delta-j)\,\Gamma(\epsilon+\delta-j)\,\Gamma(1-\epsilon-\delta+j)}{j!\,\Gamma(1-\epsilon+\delta-j)} \\
&= -\frac{1}{Q^2} \left(\frac{\mu^2}{m^2}\right)^{\epsilon} \left(\frac{Q^2}{m^2}\right)^{\delta} \frac{e^{\epsilon\gamma_{\mathrm{E}}}\,\Gamma(\epsilon+\delta)\,\Gamma(1-\epsilon-\delta)}{\Gamma(1-\epsilon+\delta)}\, \frac{\Gamma(2\delta)}{\Gamma(1+\delta)} \\
&\qquad \times \underbrace{\sum_{j=0}^{\infty} \left(\frac{m^2}{Q^2}\right)^j \frac{(1-\delta)_j\,(\epsilon-\delta)_j}{j!\,(1-2\delta)_j}}_{{}_2F_1(\epsilon-\delta,1-\delta;1-2\delta;\,m^2/Q^2)}\,.
\end{aligned} \quad \text{(A.62)}$$

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Using the integral representation (A.55) the hypergeometric function is expanded about $\delta = 0$ like in (A.56):

$$
{}_2F_1(\epsilon-\delta,1-\delta;1-2\delta;z) = (1-z)^{-\epsilon}\Bigg(1+\delta\,\ln(1-z) \\
+\delta\int_0^1\frac{\mathrm{d}t}{t}\left[1-\left(1+\frac{tz}{1-z}\right)^{-\epsilon}\right]\Bigg)+\mathcal{O}(\delta^2)\,. \tag{A.63}
$$

The 1-collinear contribution, expanded in δ, but valid for any ϵ, reads

$$
F^{(1c)} = -\frac{1}{2Q^2}\left(\frac{\mu^2}{m^2}\right)^{\epsilon} e^{\epsilon\gamma_E}\,\Gamma(\epsilon)\left(1-\frac{m^2}{Q^2}\right)^{-\epsilon}\Bigg(\frac{1}{\delta}+\ln\!\left(\frac{Q^2}{m^2}\right)+\ln\!\left(1-\frac{m^2}{Q^2}\right) \\
+\int_0^1\frac{\mathrm{d}t}{t}\left[1-\left(1+t\,\frac{m^2}{Q^2-m^2}\right)^{-\epsilon}\right]-\gamma_E+\psi(\epsilon)-2\,\psi(1-\epsilon)\Bigg) \\
+\mathcal{O}(\delta)\,, \tag{A.64}
$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function. We expand additionally about $\epsilon = 0$, keeping the same prefactor $(\mu^2/Q^2)^\epsilon$ as in the hard contribution:

$$
F^{(1c)} = -\frac{1}{2Q^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\Bigg(\frac{1}{\delta}\left[\frac{1}{\epsilon}+\ln\!\left(\frac{Q^2}{m^2}\right)-\ln\!\left(1-\frac{m^2}{Q^2}\right)\right]-\frac{1}{\epsilon^2}+\frac{2}{\epsilon}\,\ln\!\left(1-\frac{m^2}{Q^2}\right) \\
+\frac{1}{2}\,\ln^2\!\left(\frac{Q^2}{m^2}\right)+\ln\!\left(\frac{Q^2}{m^2}\right)\ln\!\left(1-\frac{m^2}{Q^2}\right)-\ln^2\!\left(1-\frac{m^2}{Q^2}\right)+\mathrm{Li}_2\!\left(\frac{m^2}{Q^2}\right)+\frac{5\pi^2}{12}\Bigg) \\
+\mathcal{O}(\delta)+\mathcal{O}(\epsilon)\,. \tag{A.65}
$$

Using the symmetry between the two collinear regions upon $p_1 \leftrightarrow p_2$ and $n_1 \leftrightarrow n_2$, the 2-collinear contribution $F^{(2c)}$ can be obtained from (A.65) by replacing $\delta \to -\delta$.

A.4 Forward scattering

A.4.1 Forward scattering: evaluation with analytic regulators

In this appendix the evaluation of the contributions (7.19) to the forward-scattering integral (7.1) or (7.4) in section 7 is sketched, using the propagator powers $n_{1,2,3,4}$ as analytic regulators.

Hard contribution. The integral over $T_0^{(h)}I$ (7.5) is written in a Lorentz-invariant way:

$$
F_0^{(h)} = \frac{1}{2}\int\frac{\mathrm{D}k}{(k^2)^{n_{12}}}\left(\frac{1}{(k^2-2p_1\cdot k)^{n_3}}+\frac{1}{(k^2+2p_1\cdot k)^{n_3}}\right) \\
\times\left(\frac{1}{(k^2+2p_2\cdot k)^{n_4}}+\frac{1}{(k^2-2p_2\cdot k)^{n_4}}\right) \\
= F_{+;0}^{(h)}+F_{-;0}^{(h)}\,, \\
F_{\pm;0}^{(h)} = \int\frac{\mathrm{D}k}{(k^2)^{n_{12}}\,(k^2-2p_1\cdot k)^{n_3}\,(k^2\pm 2p_2\cdot k)^{n_4}}\,, \tag{A.66}
$$

where terms including the factor $(k^2+2p_1\cdot k)^{-n_3}$ have been transformed via $k\to -k$ under the integral. The two terms $F^{(h)}_{\pm;0}$ directly correspond to the diagrams in figure 5 (p. 38). They are evaluated in a straightforward way using Feynman parameters and yield

$$F^{(h)}_{\pm;0} = \mu^{2\epsilon}\, e^{\epsilon\gamma_{\rm E}}\, e^{-i\pi n_{1234}}\,(\mp Q^2 - i0)^{2-n_{1234}-\epsilon} \times \frac{\Gamma(n_{1234}-2+\epsilon)\,\Gamma(2-n_{123}-\epsilon)\,\Gamma(2-n_{124}-\epsilon)}{\Gamma(n_3)\,\Gamma(n_4)\,\Gamma(4-n_{1234}-2\epsilon)}\,. \tag{A.67}$$

The complete result reads

$$F^{(h)}_{0} = \mu^{2\epsilon}\, e^{\epsilon\gamma_{\rm E}}\, e^{-i\pi n_{1234}}\,(Q^2)^{2-n_{1234}-\epsilon}\,(1+e^{i\pi(n_{1234}+\epsilon)}) \times \frac{\Gamma(n_{1234}-2+\epsilon)\,\Gamma(2-n_{123}-\epsilon)\,\Gamma(2-n_{124}-\epsilon)}{\Gamma(n_3)\,\Gamma(n_4)\,\Gamma(4-n_{1234}-2\epsilon)}\,. \tag{A.68}$$

Collinear contributions. We also write the integral over $T^{(1c)}_0 I$ (7.7) as a Lorentz-invariant integral using the d-dimensional vector $r_\perp = r + (p_1 - p_2)\, r^2/Q^2$ satisfying $p_{1,2}\cdot r_\perp = 0$ and $r_\perp^2 = -\vec{r}_\perp^{\,2}$. With this we express

$$\vec{r}_\perp\cdot\vec{k}_\perp = -r\cdot k + \frac{r^+k^- + r^-k^+}{2} = -r_\perp\cdot k\,, \tag{A.69}$$

using (6.5), and obtain the integral

$$F^{(1c)}_0 = \frac{1}{2}\int \frac{{\rm D}k}{(k^2)^{n_1}\,\big((k-r_\perp)^2 - 2(r_\perp^2/Q^2)\,p_2\cdot k\big)^{n_2}} \times \left(\frac{1}{(k^2-2p_1\cdot k)^{n_3}} + \frac{1}{\big(k^2 - 2r_\perp\cdot k + 2p_1\cdot k - 2(r_\perp^2/Q^2)\,p_2\cdot k\big)^{n_3}}\right) \times \left(\frac{1}{(2p_2\cdot k)^{n_4}} + \frac{1}{(-2p_2\cdot k)^{n_4}}\right). \tag{A.70}$$

Let us first evaluate only the contribution from the first term in each round bracket of (A.70), which is separately well-defined through analytic regularization. Using alpha parameters (A.2),

$$\begin{aligned}\frac{1}{2}\int &\frac{{\rm D}k}{(k^2)^{n_1}\,\big((k-r_\perp)^2 - 2(r_\perp^2/Q^2)\,p_2\cdot k\big)^{n_2}\,(k^2-2p_1\cdot k)^{n_3}\,(2p_2\cdot k)^{n_4}} \\ &= \frac{1}{2}\left(\prod_{i=1}^{4}\frac{e^{-i\pi n_i/2}}{\Gamma(n_i)}\int_0^\infty {\rm d}\alpha_i\,\alpha_i^{n_i-1}\right) e^{i\alpha_2(r_\perp^2+i0)} \\ &\quad\times \int {\rm D}k\, \exp\left[i\left(\alpha_{123}k^2 - 2k\cdot\left[\alpha_2 r_\perp + \alpha_3 p_1 + \left(\frac{r_\perp^2}{Q^2}\,\alpha_2 - \alpha_4\right)p_2\right] + i0\right)\right].\end{aligned} \tag{A.71}$$

After performing the loop integral with (A.4) and the α_4-integral via (A.2) we obtain

$$\frac{1}{2}\,\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_{\rm E}}\,e^{-i\pi(n_{123}+2-\epsilon)/2}}{\Gamma(n_1)\,\Gamma(n_2)\,\Gamma(n_3)\,(Q^2+i0)^{n_4}}\int_0^\infty {\rm d}\alpha_1\,{\rm d}\alpha_2\,{\rm d}\alpha_3\,\alpha_1^{n_1-1}\,\alpha_2^{n_2-1}\,\alpha_3^{n_3-n_4-1}\,\alpha_{123}^{n_4-2+\epsilon} \times \exp\left(i\,\frac{\alpha_1\alpha_2}{\alpha_{123}}\,(r_\perp^2+i0)\right). \tag{A.72}$$

We can solve this integral by multiplying it with unity in the form of

$$1 = \int_0^\infty d\eta \, \delta\Big(\eta - \sum_{j\in S} \alpha_j\Big), \qquad \text{with } \emptyset \neq S \subset \{1,2,3\}\,, \tag{A.73}$$

where the sum of alpha parameters in the delta function runs over an arbitrary non-empty subset of $\alpha_{1,2,3}$. Then, changing the order of integration, all alpha parameters are rescaled as $\alpha_i \to \eta\alpha_i$ under the η-integral and the delta function is rewritten as $(1/\eta)\,\delta(1-\sum_{j\in S}\alpha_j)$. Performing the η-integral first using (A.2) yields

$$\frac{1}{2}\,\mu^{2\epsilon}\, e^{\epsilon\gamma_E}\, e^{-i\pi n_{1234}}\, (-r_\perp^2 - i0)^{2-n_{123}-\epsilon}\, (-Q^2 - i0)^{-n_4}\, \frac{\Gamma(n_{123}-2+\epsilon)}{\Gamma(n_1)\,\Gamma(n_2)\,\Gamma(n_3)} \\ \times \int_0^\infty d\alpha_1\, d\alpha_2\, d\alpha_3\, \delta\Big(1 - \sum_{j\in S}\alpha_j\Big)\, \alpha_1^{1-n_{23}-\epsilon}\, \alpha_2^{1-n_{13}-\epsilon}\, \alpha_3^{n_3-n_4-1}\, \alpha_{123}^{n_{1234}-4+2\epsilon}\,. \tag{A.74}$$

Now the three-fold integral is of the form

$$\int_0^\infty d\alpha_1 \cdots d\alpha_n\, \delta\Big(1 - \sum_{j\in S}\alpha_j\Big)\, \frac{\alpha_1^{\nu_1-1}\cdots\alpha_n^{\nu_n-1}}{(\alpha_1+\ldots+\alpha_n)^{\nu_1+\ldots+\nu_n}} = \frac{\Gamma(\nu_1)\cdots\Gamma(\nu_n)}{\Gamma(\nu_1+\ldots+\nu_n)}\,, \tag{A.75}$$

valid for any $\emptyset \neq S \subset \{1,\ldots,n\}$, and we obtain

$$\frac{1}{2}\,\mu^{2\epsilon}\, e^{\epsilon\gamma_E}\, e^{-i\pi n_{1234}}\, (\vec{r}_\perp^{\,2})^{2-n_{123}-\epsilon}\, (-Q^2 - i0)^{-n_4}\, \Gamma(n_3 - n_4) \\ \times \frac{\Gamma(n_{123}-2+\epsilon)\,\Gamma(2-n_{13}-\epsilon)\,\Gamma(2-n_{23}-\epsilon)}{\Gamma(n_1)\,\Gamma(n_2)\,\Gamma(n_3)\,\Gamma(4-n_{1234}-2\epsilon)} \tag{A.76}$$

for the contribution from the first term in each factor of (A.70). Analytic regularization is needed for this contribution, otherwise $\Gamma(n_3 - n_4)$ would be ill-defined.

The contribution from the first $\times$ second term in (A.70), i.e. with $(-2p_2\cdot k)^{-n_4}$ instead of $(2p_2\cdot k)^{-n_4}$, can be calculated from (A.76) by replacing $p_2 \to -p_2$ and therefore $Q^2 \to -Q^2$. (The combination p_2/Q^2 remains invariant under this transformation.) The remaining two contributions, involving the second term in the first round bracket of (A.70), are obtained from the previous two contributions via the replacement $k \to r_\perp + (r_\perp^2/Q^2)\,p_2 - k$ and $n_1 \leftrightarrow n_2$ under the loop integral, which leaves the expression (A.76) invariant. The complete result reads

$$F_0^{(1c)} = \mu^{2\epsilon}\, e^{\epsilon\gamma_E}\, e^{-i\pi n_{1234}}\, (\vec{r}_\perp^{\,2})^{2-n_{123}-\epsilon}\, (Q^2)^{-n_4}\, (1 + e^{i\pi n_4})\, \Gamma(n_3 - n_4) \\ \times \frac{\Gamma(n_{123}-2+\epsilon)\,\Gamma(2-n_{13}-\epsilon)\,\Gamma(2-n_{23}-\epsilon)}{\Gamma(n_1)\,\Gamma(n_2)\,\Gamma(n_3)\,\Gamma(4-n_{1234}-2\epsilon)}\,. \tag{A.77}$$

The 2-collinear contribution,

$$F_0^{(2c)} = \frac{1}{2}\int \frac{Dk}{(k^2)^{n_1}\,\big((k-r_\perp)^2 + 2(r_\perp^2/Q^2)\,p_1\cdot k\big)^{n_2}} \left(\frac{1}{(-2p_1\cdot k)^{n_3}} + \frac{1}{(2p_1\cdot k)^{n_3}}\right) \\ \times \left(\frac{1}{(k^2 + 2p_2\cdot k)^{n_4}} + \frac{1}{\big(k^2 - 2r_\perp\cdot k + 2(r_\perp^2/Q^2)\,p_1\cdot k - 2p_2\cdot k\big)^{n_4}}\right), \tag{A.78}$$

results from (A.70) by exchanging $p_1 \leftrightarrow -p_2$ and $n_3 \leftrightarrow n_4$. It is thus obtained from (A.77) via $n_3 \leftrightarrow n_4$ and reads

$$F_0^{(2c)} = \mu^{2\epsilon}\, e^{\epsilon\gamma_{\rm E}}\, e^{-i\pi n_{1234}}\, (\vec{r}_\perp^{\,2})^{2-n_{124}-\epsilon}\, (Q^2)^{-n_3}\, (1+e^{i\pi n_3})\, \Gamma(n_4-n_3) \\ \times \frac{\Gamma(n_{124}-2+\epsilon)\,\Gamma(2-n_{14}-\epsilon)\,\Gamma(2-n_{24}-\epsilon)}{\Gamma(n_1)\,\Gamma(n_2)\,\Gamma(n_4)\,\Gamma(4-n_{1234}-2\epsilon)}\,. \tag{A.79}$$

Glauber contribution. The integral over $T_0^{(g)} I$ (7.11) reads

$$F_0^{(g)} = \frac{1}{2}\,\frac{\mu^{2\epsilon}\, e^{\epsilon\gamma_{\rm E}}}{2i\pi^{d/2}} \int \frac{\mathrm{d}^{d-2}\vec{k}_\perp}{(-\vec{k}_\perp^{\,2})^{n_1}\, \big(-(\vec{k}_\perp-\vec{r}_\perp)^2\big)^{n_2}} \\ \times \int_{-\infty}^{\infty} \mathrm{d}k^+ \left(\frac{1}{(-Qk^+ - \vec{k}_\perp^{\,2}+i0)^{n_3}} + \frac{1}{(Qk^+ - \vec{k}_\perp^{\,2} + 2\vec{r}_\perp\cdot\vec{k}_\perp + i0)^{n_3}}\right) \\ \times \int_{-\infty}^{\infty} \mathrm{d}k^- \left(\frac{1}{(Qk^- - \vec{k}_\perp^{\,2}+i0)^{n_4}} + \frac{1}{(-Qk^- - \vec{k}_\perp^{\,2} + 2\vec{r}_\perp\cdot\vec{k}_\perp + i0)^{n_4}}\right). \tag{A.80}$$

Analytic regularization makes the $k^\pm$-integrations over the individual terms well-defined and allows to assume $\mathrm{Re}\, n_{3,4} > 1$. Then we can close the integration contours at $i\infty$ or $-i\infty$, choosing separately for each term the side where no pole and no branch cut lies within the closed contour. So all k^+- and k^--integrals vanish as scaleless integrals, cf. (A.35). The $\vec{k}_\perp$-integral is regularized dimensionally. Therefore,

$$F_0^{(g)} = 0\,. \tag{A.81}$$

Collinear-plane contribution. Integrating over $T_0^{(cp)} I$ (7.13) yields

$$F_0^{(cp)} = \frac{1}{2}\,\frac{\mu^{2\epsilon}\, e^{\epsilon\gamma_{\rm E}}}{2i\pi^{d/2}} \int \mathrm{d}^{d-2}\vec{k}_\perp \int_{-\infty}^{\infty} \frac{\mathrm{d}k^+\,\mathrm{d}k^-}{(k^+k^-+i0)^{n_{12}}} \\ \times \left(\frac{1}{\big(k^+(k^- - Q)+i0\big)^{n_3}} + \frac{1}{\big(k^+(k^- + Q)+i0\big)^{n_3}}\right) \\ \times \left(\frac{1}{\big((k^+ + Q)k^- + i0\big)^{n_4}} + \frac{1}{\big((k^+ - Q)k^- + i0\big)^{n_4}}\right). \tag{A.82}$$

As the integrand is independent of $\vec{k}_\perp$, the $\vec{k}_\perp$-integration is scaleless through dimensional regularization, cf. (A.38). The $k^\pm$-integrals are regularized analytically. So

$$F_0^{(cp)} = 0\,. \tag{A.83}$$

Hard-collinear overlap contributions. The hard-1-collinear overlap contribution reads

$$F_0^{(h,1c)} = \frac{1}{2} \int \frac{\mathrm{D}k}{(k^2)^{n_{12}}} \left(\frac{1}{(k^2 - 2p_1\cdot k)^{n_3}} + \frac{1}{(k^2+2p_1\cdot k)^{n_3}}\right) \\ \times \left(\frac{1}{(2p_2\cdot k)^{n_4}} + \frac{1}{(-2p_2\cdot k)^{n_4}}\right). \tag{A.84}$$

It corresponds to $F_0^{(1c)}$ (A.70) with $r_\perp = 0$. According to (A.72), we get

$$\begin{aligned} F_0^{(h,1c)} &= \frac{\mu^{2\epsilon}\, e^{\epsilon\gamma_E}\, e^{-i\pi(n_{123}+2-\epsilon)/2}}{\Gamma(n_1)\,\Gamma(n_2)\,\Gamma(n_3)} \Big((Q^2+i0)^{-n_4} + (-Q^2+i0)^{-n_4} \Big) \\ &\qquad \times \int_0^\infty \mathrm{d}\alpha_1\, \mathrm{d}\alpha_2\, \mathrm{d}\alpha_3\, \alpha_1^{n_1-1}\, \alpha_2^{n_2-1}\, \alpha_3^{n_3-n_4-1}\, \alpha_{123}^{n_4-2+\epsilon} \\ &= \frac{\mu^{2\epsilon}\, e^{\epsilon\gamma_E}\, e^{-i\pi(n_{123}+2-\epsilon)/2}}{(Q^2)^{n_4}} \left(1+e^{-i\pi n_4}\right) \frac{\Gamma(2-n_{124}-\epsilon)}{\Gamma(n_3)\,\Gamma(2-n_4-\epsilon)} \int_0^\infty \mathrm{d}\alpha_3\, \alpha_3^{n_{123}-3+\epsilon} \\ &= 0\,. \end{aligned} \tag{A.85}$$

This integral is scaleless by dimensional regularization. Analytic regularization is only required if the α_3-integration is performed first. Similarly, via the symmetry $p_1 \leftrightarrow -p_2$ and $n_3 \leftrightarrow n_4$,

$$F_0^{(h,2c)} = 0\,. \tag{A.86}$$

Collinear-collinear overlap contribution. The 1-collinear-2-collinear overlap contribution reads

$$\begin{aligned} F_0^{(1c,2c)} = \frac{1}{2} \int \frac{\mathrm{D}k}{(k^2)^{n_1} \left((k-r_\perp)^2\right)^{n_2}} &\left(\frac{1}{(-2p_1\cdot k)^{n_3}} + \frac{1}{(2p_1\cdot k)^{n_3}} \right) \\ &\times \left(\frac{1}{(2p_2\cdot k)^{n_4}} + \frac{1}{(-2p_2\cdot k)^{n_4}} \right). \end{aligned} \tag{A.87}$$

The first term of each round bracket yields the contribution

$$\begin{aligned} &\frac{1}{2} \left(\prod_{i=1}^{4} \frac{e^{-i\pi n_i/2}}{\Gamma(n_i)} \int_0^\infty \mathrm{d}\alpha_i\, \alpha_i^{n_i-1} \right) e^{i\alpha_2(r_\perp^2+i0)} \\ &\qquad\qquad \times \int \mathrm{D}k\, \exp\Big(i\big[\alpha_{12}k^2 - 2k\cdot(\alpha_2 r_\perp + \alpha_3 p_1 - \alpha_4 p_2) + i0\big] \Big) \\ &= \frac{1}{2} \frac{\mu^{2\epsilon}\, e^{\epsilon\gamma_E}\, e^{-i\pi(n_{123}+2-\epsilon)/2}}{\Gamma(n_1)\,\Gamma(n_2)\,\Gamma(n_3)\,(Q^2+i0)^{n_4}} \int_0^\infty \mathrm{d}\alpha_3\, \alpha_3^{n_3-n_4-1} \\ &\qquad\qquad \times \int_0^\infty \mathrm{d}\alpha_1\, \mathrm{d}\alpha_2\, \alpha_1^{n_1-1}\, \alpha_2^{n_2-1}\, \alpha_{12}^{n_4-2+\epsilon} \exp\left(i\, \frac{\alpha_1\alpha_2}{\alpha_{12}} \left(r_\perp^2 + i0\right) \right). \end{aligned} \tag{A.88}$$

The $\alpha_{1,2}$-integrations are regularized dimensionally, while the α_3-integral is scaleless by analytic regularization. The remaining contributions from (A.87) are obtained via $p_1 \to -p_1$ and/or $p_2 \to -p_2$ and vanish as well. Therefore,

$$F_0^{(1c,2c)} = 0\,. \tag{A.89}$$

Other overlap contributions. All other overlap contributions in (7.19) are expansions of scaleless integrals evaluated above, so they are scaleless as well:

$$\begin{aligned} F_0^{(h,1c,2c)} = \frac{1}{2} \int \frac{\mathrm{D}k}{(k^2)^{n_{12}}} &\left(\frac{1}{(-2p_1\cdot k)^{n_3}} + \frac{1}{(2p_1\cdot k)^{n_3}} \right) \\ &\times \left(\frac{1}{(2p_2\cdot k)^{n_4}} + \frac{1}{(-2p_2\cdot k)^{n_4}} \right) = 0\,, \end{aligned}$$

$$
\begin{aligned}
F_0^{(h,g)} &= \frac{1}{2}\,\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_{\mathrm{E}}}}{2i\pi^{d/2}}\int\frac{\mathrm{d}^{d-2}\vec{k}_\perp}{(-\vec{k}_\perp^2)^{n_{12}}}\\
&\quad\times\int_{-\infty}^{\infty}\mathrm{d}k^+\left(\frac{1}{(-Qk^+-\vec{k}_\perp^2+i0)^{n_3}}+\frac{1}{(Qk^+-\vec{k}_\perp^2+i0)^{n_3}}\right)\\
&\quad\times\int_{-\infty}^{\infty}\mathrm{d}k^-\left(\frac{1}{(Qk^--\vec{k}_\perp^2+i0)^{n_4}}+\frac{1}{(-Qk^--\vec{k}_\perp^2+i0)^{n_4}}\right)=0\,,\\
F_0^{(1c,g)} &= \frac{1}{2}\,\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_{\mathrm{E}}}}{2i\pi^{d/2}}\,\frac{1}{Q^{n_4}}\int\frac{\mathrm{d}^{d-2}\vec{k}_\perp}{(-\vec{k}_\perp^2)^{n_1}\left(-(\vec{k}_\perp-\vec{r}_\perp)^2\right)^{n_2}}\\
&\quad\times\int_{-\infty}^{\infty}\mathrm{d}k^+\left(\frac{1}{(-Qk^+-\vec{k}_\perp^2+i0)^{n_3}}+\frac{1}{(Qk^+-\vec{k}_\perp^2+2\vec{r}_\perp\cdot\vec{k}_\perp+i0)^{n_3}}\right)\\
&\quad\times\int_{-\infty}^{\infty}\mathrm{d}k^-\left(\frac{1}{(k^-+i0)^{n_4}}+\frac{1}{(-k^-+i0)^{n_4}}\right)=0\,,\\
F_0^{(2c,g)} &= 0\quad[\text{via symmetry related to } F_0^{(1c,g)}]\,,\\
F_0^{(h,1c,g)} &= \frac{1}{2}\,\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_{\mathrm{E}}}}{2i\pi^{d/2}}\,\frac{1}{Q^{n_4}}\int\frac{\mathrm{d}^{d-2}\vec{k}_\perp}{(-\vec{k}_\perp^2)^{n_{12}}}\\
&\quad\times\int_{-\infty}^{\infty}\mathrm{d}k^+\left(\frac{1}{(-Qk^+-\vec{k}_\perp^2+i0)^{n_3}}+\frac{1}{(Qk^+-\vec{k}_\perp^2+i0)^{n_3}}\right)\\
&\quad\times\int_{-\infty}^{\infty}\mathrm{d}k^-\left(\frac{1}{(k^-+i0)^{n_4}}+\frac{1}{(-k^-+i0)^{n_4}}\right)=0\,,\\
F_0^{(h,2c,g)} &= 0\quad[\text{via symmetry related to } F_0^{(h,1c,g)}]\,,\\
F_0^{(1c,2c,g)} &= \frac{1}{2}\,\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_{\mathrm{E}}}}{2i\pi^{d/2}}\,\frac{1}{Q^{n_{34}}}\int\frac{\mathrm{d}^{d-2}\vec{k}_\perp}{(-\vec{k}_\perp^2)^{n_1}\left(-(\vec{k}_\perp-\vec{r}_\perp)^2\right)^{n_2}}\\
&\quad\times\int_{-\infty}^{\infty}\mathrm{d}k^+\left(\frac{1}{(-k^++i0)^{n_3}}+\frac{1}{(k^++i0)^{n_3}}\right)\\
&\quad\times\int_{-\infty}^{\infty}\mathrm{d}k^-\left(\frac{1}{(k^-+i0)^{n_4}}+\frac{1}{(-k^-+i0)^{n_4}}\right)=0\,,\\
F_0^{(h,1c,2c,g)} &= \frac{1}{2}\,\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_{\mathrm{E}}}}{2i\pi^{d/2}}\,\frac{1}{Q^{n_{34}}}\int\frac{\mathrm{d}^{d-2}\vec{k}_\perp}{(-\vec{k}_\perp^2)^{n_{12}}}\int_{-\infty}^{\infty}\mathrm{d}k^+\left(\frac{1}{(-k^++i0)^{n_3}}+\frac{1}{(k^++i0)^{n_3}}\right)\\
&\quad\times\int_{-\infty}^{\infty}\mathrm{d}k^-\left(\frac{1}{(k^-+i0)^{n_4}}+\frac{1}{(-k^-+i0)^{n_4}}\right)=0\,,\\
F_0^{(h,cp)} &= F_0^{(cp)}=0\quad[\text{same expansion}]\,,\\
F_0^{(1c,cp)} &= \frac{1}{2}\,\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_{\mathrm{E}}}}{2i\pi^{d/2}}\,\frac{1}{Q^{n_4}}\int\mathrm{d}^{d-2}\vec{k}_\perp\int_{-\infty}^{\infty}\frac{\mathrm{d}k^+\,\mathrm{d}k^-}{(k^+k^-+i0)^{n_{12}}}\\
&\quad\times\left(\frac{1}{\left(k^+(k^--Q)+i0\right)^{n_3}}+\frac{1}{\left(k^+(k^-+Q)+i0\right)^{n_3}}\right)\\
&\quad\times\left(\frac{1}{(k^-+i0)^{n_4}}+\frac{1}{(-k^-+i0)^{n_4}}\right)=0\,,
\end{aligned}
$$

$$
\begin{aligned}
F_0^{(2c,cp)} &= 0 \quad [\text{via symmetry related to } F_0^{(1c,cp)}]\,, \\
F_0^{(h,1c,cp)} &= F_0^{(1c,cp)} = 0 \quad [\text{same expansion}]\,, \\
F_0^{(h,2c,cp)} &= F_0^{(2c,cp)} = 0 \quad [\text{same expansion}]\,, \\
F_0^{(1c,2c,cp)} &= \frac{1}{2}\,\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_{\rm E}}}{2i\pi^{d/2}}\,\frac{1}{Q^{n_{34}}}\int \mathrm{d}^{d-2}\vec{k}_\perp \int_{-\infty}^{\infty} \frac{\mathrm{d}k^+\,\mathrm{d}k^-}{(k^+k^- + i0)^{n_{12}}} \\
&\quad \times \left(\frac{1}{(-k^+ + i0)^{n_3}} + \frac{1}{(k^+ + i0)^{n_3}}\right)\left(\frac{1}{(k^- + i0)^{n_4}} + \frac{1}{(-k^- + i0)^{n_4}}\right) = 0\,, \\
F_0^{(h,1c,2c,cp)} &= F_0^{(1c,2c,cp)} = 0 \quad [\text{same expansion}]\,. \qquad \text{(A.90)}
\end{aligned}
$$

All these integrals are well-defined via dimensional and analytic regularization.

A.4.2 Forward scattering: evaluation without analytic regulators

Here the evaluation of the contributions (7.19) to the forward-scattering integral in section 7 is repeated with analytic regularization switched off. This means that the propagator powers are set to the fixed values $n_1 = n_2 = n_3 = n_4 = 1$ from the start.

Hard contribution. The evaluation of the hard contribution in appendix A.4.1 is also valid without analytic regularization. From (A.68) we get

$$
F_0^{(h)} = \frac{1}{(Q^2)^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}(1 + e^{i\pi\epsilon})\,\frac{e^{\epsilon\gamma_{\rm E}}\,\Gamma(2+\epsilon)\,\Gamma^2(-1-\epsilon)}{\Gamma(-2\epsilon)}\,. \qquad \text{(A.91)}
$$

Collinear contributions. The 1-collinear contribution without analytic regulators reads

$$
\begin{aligned}
F_0^{(1c)} &= \frac{1}{2Q}\,\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_{\rm E}}}{2i\pi^{d/2}}\int \mathrm{d}^{d-2}\vec{k}_\perp \int_{-\infty}^{\infty} \frac{\mathrm{d}k^+\,\mathrm{d}k^-}{(k^+k^- - \vec{k}_\perp^2 + i0)\,\big((k^+ - r_0^+)k^- - (\vec{k}_\perp - \vec{r}_\perp)^2 + i0\big)} \\
&\quad \times \left(\frac{1}{k^+(k^- - Q) - \vec{k}_\perp^2 + i0} + \frac{1}{k^+(k^- + Q) - r_0^+ k^- - \vec{k}_\perp^2 + 2\vec{r}_\perp\cdot\vec{k}_\perp + i0}\right) \\
&\quad \times \left(\frac{1}{k^- + i0} + \frac{1}{-k^- + i0}\right). \qquad \text{(A.92)}
\end{aligned}
$$

The factor in the last line of (A.92) vanishes everywhere under the k^--integral except for the pole at $k^- = 0$ which lies below the real axis in the first term and above the real axis in the second term. The only remaining contribution arises from the closed integration contour around this pole, i.e. from the residue at $k^- = 0$, which allows us to express

$$
\frac{1}{k^- + i0} + \frac{1}{-k^- + i0} = \frac{1}{k^- + i0} - \frac{1}{k^- - i0} = -2i\pi\,\delta(k^-)\,. \qquad \text{(A.93)}
$$

The 1-collinear contribution becomes

$$
\begin{aligned}
F_0^{(1c)} &= -\frac{1}{2Q}\,\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_{\rm E}}}{\pi^{1-\epsilon}}\int \frac{\mathrm{d}^{d-2}\vec{k}_\perp}{\vec{k}_\perp^2\,(\vec{k}_\perp - \vec{r}_\perp)^2} \\
&\quad \times \int_{-\infty}^{\infty} \mathrm{d}k^+ \left(\frac{1}{-Qk^+ - \vec{k}_\perp^2 + i0} + \frac{1}{Qk^+ - \vec{k}_\perp^2 + 2\vec{r}_\perp\cdot\vec{k}_\perp + i0}\right). \qquad \text{(A.94)}
\end{aligned}
$$

The k^+-integral converges at $|k^+| \to \infty$ because the leading behaviour for large $|k^+|$ is cancelled between the two terms, such that the integrand falls off like $1/(k^+)^2$ towards infinity. We close the integration contour at $+i\infty$ or at $-i\infty$, taking into account the residue of the one pole within the contour. In general, we get

$$\int_{-\infty}^{\infty} \mathrm{d}k^+ \left(\frac{1}{-Qk^+ + A + i0} + \frac{1}{Qk^+ + B + i0} \right) = -\frac{2i\pi}{Q} , \tag{A.95}$$

for real A, B and $Q > 0$, which is in particular independent of A and B. The $\vec{k}_\perp$-integral is solved using a Feynman parameter,

$$\begin{aligned} \int \frac{\mathrm{d}^{d-2}\vec{k}_\perp}{\vec{k}_\perp^2 \, (\vec{k}_\perp - \vec{r}_\perp)^2} &= \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^{d-2}\vec{k}_\perp}{\left((\vec{k}_\perp - x\,\vec{r}_\perp)^2 + x(1-x)\,\vec{r}_\perp^{\,2}\right)^2} \\ &= \pi^{1-\epsilon}\,\Gamma(1+\epsilon) \int_0^1 \mathrm{d}x \left(x(1-x)\,\vec{r}_\perp^{\,2}\right)^{-1-\epsilon} \\ &= \frac{\pi^{1-\epsilon}}{(\vec{r}_\perp^{\,2})^{1+\epsilon}} \, \frac{\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)} . \end{aligned} \tag{A.96}$$

The complete 1-collinear contribution reads

$$F_0^{(1c)} = \frac{i\pi}{\vec{r}_\perp^{\,2}\, Q^2} \left(\frac{\mu^2}{\vec{r}_\perp^{\,2}} \right)^{\epsilon} \frac{e^{\epsilon\gamma_\mathrm{E}}\,\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)} . \tag{A.97}$$

The 2-collinear contribution,

$$\begin{aligned} F_0^{(2c)} = \frac{1}{2Q}\, \frac{\mu^{2\epsilon}\, e^{\epsilon\gamma_\mathrm{E}}}{2i\pi^{d/2}} \int \mathrm{d}^{d-2}\vec{k}_\perp \int_{-\infty}^{\infty} & \frac{\mathrm{d}k^+\,\mathrm{d}k^-}{(k^+k^- - \vec{k}_\perp^2 + i0)\left(k^+(k^- - r_0^-) - (\vec{k}_\perp - \vec{r}_\perp)^2 + i0\right)} \\ \times \left(\frac{1}{(k^+ + Q)k^- - \vec{k}_\perp^2 + i0} + \frac{1}{(k^+ - Q)k^- - k^+ r_0^- - \vec{k}_\perp^2 + 2\vec{r}_\perp \cdot \vec{k}_\perp + i0} \right) & \\ \times \left(\frac{1}{-k^+ + i0} + \frac{1}{k^+ + i0} \right) , & \end{aligned} \tag{A.98}$$

is obtained from $F_0^{(1c)}$ (A.92) by replacing $k^+ \leftrightarrow -k^-$ and $r_0^+ \to -r_0^- = r_0^+$. So the same result is reproduced,

$$F_0^{(2c)} = F_0^{(1c)} = \frac{i\pi}{\vec{r}_\perp^{\,2}\, Q^2} \left(\frac{\mu^2}{\vec{r}_\perp^{\,2}} \right)^{\epsilon} \frac{e^{\epsilon\gamma_\mathrm{E}}\,\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)} . \tag{A.99}$$

Glauber contribution. The Glauber-region integral (A.80) for $n_1 = n_2 = n_3 = n_4 = 1$ reads

$$\begin{aligned} F_0^{(g)} &= \frac{1}{2}\, \frac{\mu^{2\epsilon}\, e^{\epsilon\gamma_\mathrm{E}}}{2i\pi^{d/2}} \int \frac{\mathrm{d}^{d-2}\vec{k}_\perp}{\vec{k}_\perp^2\,(\vec{k}_\perp - \vec{r}_\perp)^2} \\ &\quad \times \int_{-\infty}^{\infty} \mathrm{d}k^+ \left(\frac{1}{-Qk^+ - \vec{k}_\perp^2 + i0} + \frac{1}{Qk^+ - \vec{k}_\perp^2 + 2\vec{r}_\perp \cdot \vec{k}_\perp + i0} \right) \\ &\quad \times \int_{-\infty}^{\infty} \mathrm{d}k^- \left(\frac{1}{Qk^- - \vec{k}_\perp^2 + i0} + \frac{1}{-Qk^- - \vec{k}_\perp^2 + 2\vec{r}_\perp \cdot \vec{k}_\perp + i0} \right) . \end{aligned} \tag{A.100}$$

The $k^\pm$-integrals converge because the leading behaviour for $|k^\pm| \to \infty$ is cancelled between the two terms of each round bracket. All integrals needed for (A.100) are known from (A.95) and (A.96). The result is

$$F_0^{(g)} = \frac{i\pi}{\vec{r}_\perp^{\,2}\, Q^2} \left(\frac{\mu^2}{\vec{r}_\perp^{\,2}}\right)^\epsilon \frac{e^{\epsilon\gamma_{\rm E}}\,\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)}\,. \tag{A.101}$$

Collinear-collinear overlap contribution.

$$F_0^{(1c,2c)} = \frac{1}{2Q^2}\,\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_{\rm E}}}{2i\pi^{d/2}} \int \mathrm{d}^{d-2}\vec{k}_\perp \int_{-\infty}^{\infty} \frac{\mathrm{d}k^+\,\mathrm{d}k^-}{(k^+k^- - \vec{k}_\perp^{\,2} + i0)\,(k^+k^- - (\vec{k}_\perp - \vec{r}_\perp)^2 + i0)}$$
$$\times \underbrace{\left(\frac{1}{-k^+ + i0} + \frac{1}{k^+ + i0}\right)}_{-2i\pi\,\delta(k^+)} \underbrace{\left(\frac{1}{k^- + i0} + \frac{1}{-k^- + i0}\right)}_{-2i\pi\,\delta(k^-)}, \tag{A.102}$$

where the two round brackets are replaced according to (A.93). Using (A.96), the 1-collinear–2-collinear overlap contribution yields

$$F_0^{(1c,2c)} = \frac{i\pi}{\vec{r}_\perp^{\,2}\, Q^2} \left(\frac{\mu^2}{\vec{r}_\perp^{\,2}}\right)^\epsilon \frac{e^{\epsilon\gamma_{\rm E}}\,\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)}\,. \tag{A.103}$$

Collinear-Glauber overlap contributions. The 1-collinear-Glauber overlap contribution

$$F_0^{(1c,g)} = \frac{1}{2Q}\,\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_{\rm E}}}{2i\pi^{d/2}} \int \frac{\mathrm{d}^{d-2}\vec{k}_\perp}{\vec{k}_\perp^{\,2}\,(\vec{k}_\perp - \vec{r}_\perp)^2} \int_{-\infty}^{\infty} \mathrm{d}k^- \left(\frac{1}{k^- + i0} + \frac{1}{-k^- + i0}\right)$$
$$\times \int_{-\infty}^{\infty} \mathrm{d}k^+ \left(\frac{1}{-Qk^+ - \vec{k}_\perp^{\,2} + i0} + \frac{1}{Qk^+ - \vec{k}_\perp^{\,2} + 2\vec{r}_\perp\cdot\vec{k}_\perp + i0}\right), \tag{A.104}$$

its symmetric counterpart $F_0^{(2c,g)}$ (obtained via $k^+ \leftrightarrow -k^-$), and the threefold overlap contribution

$$F_0^{(1c,2c,g)} = \frac{1}{2Q^2}\,\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_{\rm E}}}{2i\pi^{d/2}} \int \frac{\mathrm{d}^{d-2}\vec{k}_\perp}{\vec{k}_\perp^{\,2}\,(\vec{k}_\perp - \vec{r}_\perp)^2} \int_{-\infty}^{\infty} \mathrm{d}k^+ \left(\frac{1}{-k^+ + i0} + \frac{1}{k^+ + i0}\right)$$
$$\times \int_{-\infty}^{\infty} \mathrm{d}k^- \left(\frac{1}{k^- + i0} + \frac{1}{-k^- + i0}\right) \tag{A.105}$$

are evaluated using the relations (A.93), (A.95) and (A.96). They yield identical results,

$$F_0^{(1c,g)} = F_0^{(2c,g)} = F_0^{(1c,2c,g)} = \frac{i\pi}{\vec{r}_\perp^{\,2}\, Q^2} \left(\frac{\mu^2}{\vec{r}_\perp^{\,2}}\right)^\epsilon \frac{e^{\epsilon\gamma_{\rm E}}\,\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)}\,. \tag{A.106}$$

Contributions involving the collinear-plane region. The contributions $F_0^{(cp)}$ (A.82) and $F_0^{(h,cp)}$, $F_0^{(1c,cp)}$, $F_0^{(2c,cp)}$, $F_0^{(h,1c,cp)}$, $F_0^{(h,2c,cp)}$, $F_0^{(1c,2c,cp)}$, $F_0^{(h,1c,2c,cp)}$ (A.90) all involve the scaleless integral $\int \mathrm{d}^{d-2}\vec{k}_\perp$ (with no further integrand) which vanishes through dimensional regularization. This is still true without analytic regularization, when the propagator powers are fixed to $n_1 = n_2 = n_3 = n_4 = 1$. But the $k^\pm$-integrals in these contributions are problematic without analytic regulators. The integrands fall off sufficiently fast at $|k^\pm| \to \infty$, but the collinear-plane expansion makes poles above and below the real $k^\pm$-axis collapse at $k^\pm = 0$. So the integration contour is pinched between poles which are infinitesimally close to each other. These $k^\pm$-integrals are regularized analytically in the evaluation of appendix A.4.1, but here they are singular.

We may regularize the pinching of poles in the $k^\pm$-integrals without resorting to analytic regularization by keeping a finite (instead of infinitesimal) imaginary part $i\delta$ (with $\delta > 0$) in all propagators, e.g.

$$\begin{aligned} F_0^{(cp)} = \frac{1}{2}\,\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_{\rm E}}}{2i\pi^{d/2}} \int \mathrm{d}^{d-2}\vec{k}_\perp \int_{-\infty}^{\infty} \frac{\mathrm{d}k^+\,\mathrm{d}k^-}{(k^+k^- + i\delta)^2} \\ \times \left(\frac{1}{k^+(k^- - Q) + i\delta} + \frac{1}{k^+(k^- + Q) + i\delta}\right) \\ \times \left(\frac{1}{(k^+ + Q)k^- + i\delta} + \frac{1}{(k^+ - Q)k^- + i\delta}\right). \end{aligned} \tag{A.107}$$

When δ is fixed at some finite positive value, the poles above and below the integration contours are separated from each other and the collinear-plane integral is well-defined. The result from the $k^\pm$-integrations is singular in the limit $\delta \to 0$, but it is multiplied by the scaleless $\vec{k}_\perp$-integral, so the complete contribution vanishes. This is true for all contributions involving the collinear-plane region:

$$\begin{aligned} F_0^{(cp)} = F_0^{(h,cp)} = F_0^{(1c,cp)} = F_0^{(2c,cp)} = F_0^{(h,1c,cp)} = F_0^{(h,2c,cp)} = F_0^{(1c,2c,cp)} \\ = F_0^{(h,1c,2c,cp)} = 0\,. \end{aligned} \tag{A.108}$$

For consistency, the same regularization by a finite imaginary part $i\delta$ should be applied to the contributions from the other regions as well, but there the limit $\delta \to 0$ (with $\delta > 0$) is regular when it is performed before the limit $d \to 4$ from dimensional regularization.

In addition, as we have already noted after the identity (7.19), all contributions involving the collinear-plane expansion $T^{(cp)}$ cancel each other in pairs at the integrand level, according to (6.21), because the hard expansion $T^{(h)}$ does not change the integrand when it is applied in addition to $T^{(cp)}$:

$$\begin{aligned} \big(F_0^{(cp)} - F_0^{(h,cp)}\big) - \big(F_0^{(1c,cp)} - F_0^{(h,1c,cp)}\big) - \big(F_0^{(2c,cp)} - F_0^{(h,2c,cp)}\big) \\ + \big(F_0^{(1c,2c,cp)} - F_0^{(h,1c,2c,cp)}\big) = 0\,. \end{aligned} \tag{A.109}$$

So even if individual collinear-plane contributions were singular, their sum would vanish at the integrand level. Also, the collinear-plane contribution $F_0^{(cp)}$ is parametrically of order $(\vec{r}_\perp^{\,2})^{1-\epsilon}\,(Q^2)^{-3}$ (7.21) and thus suppressed by two powers of $\vec{r}_\perp^{\,2}/Q^2$ with respect to $F_0^{(1c)}$, $F_0^{(2c)}$ and $F_0^{(g)}$. So it does not contribute to the leading-order result F_0.

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Overlap contributions involving the hard region. The overlap contributions involving the hard expansion and any of the collinear or Glauber regions read

$$\begin{aligned}
F_0^{(h,1c)} &= \frac{1}{2Q}\,\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_E}}{2i\pi^{d/2}} \int \mathrm{d}^{d-2}\vec{k}_\perp \int_{-\infty}^{\infty} \frac{\mathrm{d}k^+\,\mathrm{d}k^-}{(k^+k^- - \vec{k}_\perp^2 + i0)^2} \left(\frac{1}{k^- + i0} + \frac{1}{-k^- + i0}\right) \\
&\quad\times \left(\frac{1}{k^+(k^- - Q) - \vec{k}_\perp^2 + i0} + \frac{1}{k^+(k^- + Q) - \vec{k}_\perp^2 + i0}\right), \\
F_0^{(h,g)} &= \frac{1}{2}\,\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_E}}{2i\pi^{d/2}} \int \frac{\mathrm{d}^{d-2}\vec{k}_\perp}{(\vec{k}_\perp^2)^2} \int_{-\infty}^{\infty} \mathrm{d}k^+ \left(\frac{1}{-Qk^+ - \vec{k}_\perp^2 + i0} + \frac{1}{Qk^+ - \vec{k}_\perp^2 + i0}\right) \\
&\quad\times \int_{-\infty}^{\infty} \mathrm{d}k^- \left(\frac{1}{Qk^- - \vec{k}_\perp^2 + i0} + \frac{1}{-Qk^- - \vec{k}_\perp^2 + i0}\right), \\
F_0^{(h,1c,2c)} &= \frac{1}{2Q^2}\,\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_E}}{2i\pi^{d/2}} \int \mathrm{d}^{d-2}\vec{k}_\perp \int_{-\infty}^{\infty} \frac{\mathrm{d}k^+\,\mathrm{d}k^-}{(k^+k^- - \vec{k}_\perp^2 + i0)^2} \\
&\quad\times \left(\frac{1}{-k^+ + i0} + \frac{1}{k^+ + i0}\right)\left(\frac{1}{k^- + i0} + \frac{1}{-k^- + i0}\right), \\
F_0^{(h,1c,g)} &= \frac{1}{2Q}\,\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_E}}{2i\pi^{d/2}} \int \frac{\mathrm{d}^{d-2}\vec{k}_\perp}{(\vec{k}_\perp^2)^2} \int_{-\infty}^{\infty} \mathrm{d}k^+ \left(\frac{1}{-Qk^+ - \vec{k}_\perp^2 + i0} + \frac{1}{Qk^+ - \vec{k}_\perp^2 + i0}\right) \\
&\quad\times \int_{-\infty}^{\infty} \mathrm{d}k^- \left(\frac{1}{k^- + i0} + \frac{1}{-k^- + i0}\right), \\
F_0^{(h,1c,2c,g)} &= \frac{1}{2Q^2}\,\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_E}}{2i\pi^{d/2}} \int \frac{\mathrm{d}^{d-2}\vec{k}_\perp}{(\vec{k}_\perp^2)^2} \int_{-\infty}^{\infty} \mathrm{d}k^+ \left(\frac{1}{-k^+ + i0} + \frac{1}{k^+ + i0}\right) \\
&\quad\times \int_{-\infty}^{\infty} \mathrm{d}k^- \left(\frac{1}{k^- + i0} + \frac{1}{-k^- + i0}\right),
\end{aligned} \tag{A.110}$$

omitting the symmetric integrals $F_0^{(h,2c)}$ and $F_0^{(h,2c,g)}$ obtained from the integrals above via $(1c) \to (2c)$ with $k^+ \leftrightarrow -k^-$. The $k^\pm$-integrals are evaluated using (A.93) and (A.95). All contributions in (A.110) identically yield

$$\begin{aligned}
F_0^{(h,1c)} &= F_0^{(h,2c)} = F_0^{(h,g)} = F_0^{(h,1c,2c)} = F_0^{(h,1c,g)} = F_0^{(h,2c,g)} = F_0^{(h,1c,2c,g)} \\
&= \frac{i\pi}{Q^2}\,\frac{\mu^{2\epsilon}\,e^{\epsilon\gamma_E}}{\pi^{1-\epsilon}} \int \frac{\mathrm{d}^{d-2}\vec{k}_\perp}{(\vec{k}_\perp^2)^2} = 0\,,
\end{aligned} \tag{A.111}$$

which is a scaleless integral through dimensional regularization.

B Expansion by regions with a finite boundary

This appendix illustrates the behaviour of the expansion by regions and the formalism of sections 3 and 5 when a finite integration boundary is involved. Consider the one-dimensional integral

$$F = \int_0^B \mathrm{d}k\, I \quad \text{with} \quad I = \frac{1}{(k+m)^\alpha\,(k+M)^\beta} \quad \text{and} \quad 0 < m \ll M \ll B\,. \tag{B.1}$$

B.1 Direct evaluation with a finite boundary

The integration domain $D = [0, B]$ in (B.1) involves the finite, non-zero boundary B, which introduces an additional scale into the problem. Let us define two regions,

- the *soft region* (s), characterized by $k \sim m$, with the expansion

$$T^{(s)} \equiv \sum_{j_1=0}^{\infty} T^{(s)}_{j_1}\,, \qquad T^{(s)}_{j_1} I = \frac{(\beta)_{j_1}\,(-1)^{j_1}}{j_1!\,M^{\beta+j_1}}\,\frac{k^{j_1}}{(k+m)^{\alpha}}\,, \tag{B.2}$$

  converging absolutely for $k \in D_s = [0, \lambda]$ with $m \ll \lambda \ll M$,

- and the *hard region* (h), characterized by $k \sim M$, with the expansion

$$T^{(h)} \equiv \sum_{j_2=0}^{\infty} T^{(h)}_{j_2}\,, \qquad T^{(h)}_{j_2} I = \frac{(\alpha)_{j_2}\,(-m)^{j_2}}{j_2!}\,\frac{1}{k^{\alpha+j_2}\,(k+M)^{\beta}}\,, \tag{B.3}$$

  converging absolutely for $k \in D_h = (\lambda, B]$.

We have $D_s \cap D_h = \emptyset$, $D_s \cup D_h = D$ and

$$\begin{aligned} T^{(s)}T^{(h)} = T^{(h)}T^{(s)} \equiv T^{(s,h)} &\equiv \sum_{j_1,j_2=0}^{\infty} T^{(s,h)}_{j_1,j_2}\,, \\ T^{(s,h)}_{j_1,j_2} I &= \frac{(\beta)_{j_1}\,(\alpha)_{j_2}\,(-1)^{j_1}\,(-m)^{j_2}}{j_1!\,j_2!\,M^{\beta+j_1}}\,\frac{1}{k^{\alpha+j_2-j_1}}\,. \end{aligned} \tag{B.4}$$

Also the integrals

$$\begin{aligned} F^{(s)}_{j_1} &= \frac{(\beta)_{j_1}\,(-1)^{j_1}}{j_1!\,M^{\beta+j_1}} \int_0^B \frac{\mathrm{d}k\,k^{j_1}}{(k+m)^{\alpha}}\,, \\ F^{(h)}_{j_2} &= \frac{(\alpha)_{j_2}\,(-m)^{j_2}}{j_2!} \int_0^B \frac{\mathrm{d}k}{k^{\alpha+j_2}\,(k+M)^{\beta}}\,, \\ F^{(s,h)}_{j_1,j_2} &= \frac{(\beta)_{j_1}\,(\alpha)_{j_2}\,(-1)^{j_1}\,(-m)^{j_2}}{j_1!\,j_2!\,M^{\beta+j_1}} \int_0^B \frac{\mathrm{d}k}{k^{\alpha+j_2-j_1}} \end{aligned} \tag{B.5}$$

are well-defined if we use the denominator power α as analytic regulator, choosing its value for each of the integrals $F^{(h)}_{j_2}$ and $F^{(s,h)}_{j_1,j_2}$ such that they converge at $k \to 0$. So the conditions 1, 2 and 3 of the formalism in section 3 hold. But we have a problem with the convergence of the summations as required in condition 4: The series $\sum_{j_1=0}^{\infty} F^{(s)}_{j_1}$ and $\sum_{j_1=0}^{\infty} F^{(s,h)}_{j_1,j_2}$ (for fixed j_2) diverge, because these two integrals involve contributions from $k \sim B$ producing powers $(B/M)^{j_1}$ (with $B/M \gg 1$).[6]

With $F^{(s)}$ and $F^{(s,h)}$ being divergent and therefore condition 4 violated, we cannot naively claim the master identity (3.22), here $F = F^{(s)} + F^{(h)} - F^{(s,h)}$, to be true. However,

[6] Note, however, that the series $\sum_{j_2=0}^{\infty} F^{(h)}_{j_2}$ and $\sum_{j_2=0}^{\infty} F^{(s,h)}_{j_1,j_2}$ (for fixed j_1) are convergent, although contributions $(m/k)^{j_2}$ with $k \to 0$ are involved. But the analytic regularization prevents contributions from the zero-boundary $k = 0$ and makes the integrals yield results scaling only with $k \sim M$ and $k \sim B$.

the difference $F^{(s)} - F^{(s,h)}$ is finite when the terms are combined before summing over j_1. This can be seen by writing

$$\int_0^B \frac{\mathrm{d}k\, k^{j_1}}{(k+m)^\alpha} = \int_0^\infty \frac{\mathrm{d}k\, k^{j_1}}{(k+m)^\alpha} - \int_B^\infty \frac{\mathrm{d}k\, k^{j_1}}{(k+m)^\alpha}$$
$$= \frac{j_1!\,\Gamma(\alpha-1-j_1)}{\Gamma(\alpha)\, m^{\alpha-1-j_1}} - \sum_{j_2=0}^\infty \frac{(\alpha)_{j_2}\,(-m)^{j_2}}{j_2!} \int_B^\infty \frac{\mathrm{d}k}{k^{\alpha+j_2-j_1}}\,, \tag{B.6}$$

using the power α as analytic regulator also at $k \to \infty$ and expanding the second term safely for $m \ll B \le k$. The two contributions combined yield

$$F^{(s)}_{j_1} - \sum_{j_2=0}^\infty F^{(s,h)}_{j_1,j_2} = \frac{(\beta)_{j_1}\,\Gamma(\alpha-1-j_1)}{\Gamma(\alpha)\, m^{\alpha-1}\, M^\beta} \Big(-\frac{m}{M}\Big)^{j_1}$$
$$- \frac{(\beta)_{j_1}\,(-1)^{j_1}}{j_1!\, M^{\beta+j_1}} \sum_{j_2=0}^\infty \frac{(\alpha)_{j_2}\,(-m)^{j_2}}{j_2!} \bigg(\int_0^B \frac{\mathrm{d}k}{k^{\alpha+j_2-j_1}} + \int_B^\infty \frac{\mathrm{d}k}{k^{\alpha+j_2-j_1}}\bigg)\,. \tag{B.7}$$

The sum of the two integrals in the second line of (B.7) is zero (using analytic regularization), because

$$\int_0^B \frac{\mathrm{d}k}{k^{\alpha+j_2-j_1}} = \frac{B^{1-\alpha+j_1-j_2}}{1-\alpha+j_1-j_2} \quad \text{for } \mathrm{Re}\,\alpha < 1 + j_1 - j_2\,,$$
$$\int_B^\infty \frac{\mathrm{d}k}{k^{\alpha+j_2-j_1}} = -\frac{B^{1-\alpha+j_1-j_2}}{1-\alpha+j_1-j_2} \quad \text{for } \mathrm{Re}\,\alpha > 1 + j_1 - j_2\,, \tag{B.8}$$

or simply because the two integrals add up to a scaleless integral from 0 to ∞.

The upper limit of the integrals $F^{(h)}_{j_2}$ in (B.5) can be extended to ∞ like in (B.6), and the second term is evaluated in analogy to (B.8). In total the result is

$$F = \sum_{j_1=0}^\infty \bigg(F^{(s)}_{j_1} - \sum_{j_2=0}^\infty F^{(s,h)}_{j_1,j_2}\bigg) + \sum_{j_2=0}^\infty F^{(h)}_{j_2}$$
$$= \frac{1}{m^{\alpha-1}\, M^\beta}\, \frac{1}{\alpha-1} \sum_{j_1=0}^\infty \Big(\frac{m}{M}\Big)^{j_1} \frac{(\beta)_{j_1}}{(2-\alpha)_{j_1}}$$
$$+ \frac{1}{M^{\alpha+\beta-1}}\, \frac{\Gamma(1-\alpha)}{\Gamma(\beta)} \sum_{j_2=0}^\infty \Big(\frac{m}{M}\Big)^{j_2} \frac{\Gamma(\alpha+\beta-1+j_2)}{j_2!}$$
$$- \frac{1}{B^{\alpha+\beta-1}} \sum_{j_1,j_2=0}^\infty \Big(-\frac{M}{B}\Big)^{j_1} \Big(-\frac{m}{B}\Big)^{j_2} \frac{(\beta)_{j_1}\,(\alpha)_{j_2}}{j_1!\, j_2!\,(\alpha+\beta-1+j_{12})}\,. \tag{B.9}$$

The series expansions in (B.9) converge and reproduce the exact result of the integral F in (B.1). Note that due to the finite integration boundary at $k = B$ the individual integrals $F^{(s)}_{j_1}$ and $F^{(h)}_{j_2}$ in (B.5) do not yield homogeneous functions of m, M and B, and therefore the overlap contributions $F^{(s,h)}_{j_1,j_2}$ are not scaleless. Additional expansions of $F^{(s)}_{j_1}$ and $F^{(h)}_{j_2}$ as in (B.6) are needed to arrive at the form (B.9) where all summation terms are homogeneous functions of m, M and B.

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Individual terms in the result (B.9) are singular when α is a positive integer or when $(\alpha+\beta-1)$ is zero or a negative integer. One can check, however, that all these singularities cancel between the terms in (B.9) such that the total result is finite as it should be.

The difficulties with the convergence of the expansions $F^{(s)}$ and $F^{(s,h)}$ may be avoided if one is only interested in a leading-order expansion. According to (3.27), the leading-order approximation to F is given by

$$F_0 = F_0^{(s)} + F_0^{(h)} - F_{0,0}^{(s,h)} = \frac{1}{M^\beta}\int_0^B \frac{\mathrm{d}k}{(k+m)^\alpha} + \int_0^B \frac{\mathrm{d}k}{k^\alpha\,(k+M)^\beta} - \frac{1}{M^\beta}\int_0^B \frac{\mathrm{d}k}{k^\alpha}\,, \tag{B.10}$$

with the zeroth-order terms from (B.5). It does not require the convergence of the full series. The first two contributions to the leading-order approximation F_0 are not homogeneous functions of m, M and B; they may be further approximated using (B.6) and keeping only the leading term of the additional expansion. This leads to a different leading-order approximation with homogeneous contributions:

$$\begin{aligned}\tilde{F}_0 &= \frac{1}{M^\beta}\int_0^\infty \frac{\mathrm{d}k}{(k+m)^\alpha} + \int_0^\infty \frac{\mathrm{d}k}{k^\alpha\,(k+M)^\beta} - \int_B^\infty \frac{\mathrm{d}k}{k^{\alpha+\beta}} - \frac{1}{M^\beta}\int_0^\infty \frac{\mathrm{d}k}{k^\alpha} \\ &= \frac{1}{m^{\alpha-1}\,M^\beta}\,\frac{1}{\alpha-1} + \frac{1}{M^{\alpha+\beta-1}}\,\frac{\Gamma(1-\alpha)\,\Gamma(\alpha+\beta-1)}{\Gamma(\beta)} - \frac{1}{B^{\alpha+\beta-1}}\,\frac{1}{\alpha+\beta-1}\,,\end{aligned} \tag{B.11}$$

which corresponds to the leading-order term (with $j_1 = j_2 = 0$) in (B.9).

B.2 Evaluation by extending to an infinite boundary

There is an alternative way to treat the integral (B.1) with finite boundaries in the expansion by regions, which, however, requires the formalism for non-commuting expansions introduced in section 5.

Let us write the original integral as

$$F = \int_0^\infty \mathrm{d}k\,\hat{I} \quad \text{with} \quad \hat{I} = \frac{\theta(B-k)}{(k+m)^\alpha\,(k+M)^\beta}\,, \tag{B.12}$$

with infinite upper boundary of the integration domain $\hat{D} = [0,\infty)$, expressing the restriction $k \le B$ through the Heaviside step function θ. With a third factor and the new scale B in the integrand $\hat{I}$, we add a third region. Let us keep $D_s = [0,\lambda]$, but redefine $D_h = (\lambda, \Lambda]$ with $m \ll \lambda \ll M \ll \Lambda \ll B$. So, in the soft and hard regions, we have $k \le \Lambda \ll B$, which allows us to use

$$\theta(B-k) = 1\,, \quad k < B\,, \tag{B.13}$$

there.[7] So the soft and hard expansions of the new integrand, $T^{(s)}$ and $T^{(h)}$, as well as their combined double expansion $T^{(s,h)}$ exactly correspond to the known expressions (B.2), (B.3) and (B.4). Additionally we now have

[7]Formally this is the expansion $\theta(B-k) = \theta(B) - k\,\delta(B) + \frac{k^2}{2}\,\delta'(B) + \ldots \equiv 1$ with $\theta(B) = 1$ and $\delta^{(j)}(B) = 0$ for $B > 0$ to all orders j.

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- the *ultrahard region* (uh), characterized by $k \sim B$, with the expansion

$$T^{(uh)} \equiv \sum_{j_1,j_2=0}^{\infty} T^{(uh)}_{j_1,j_2}\,, \qquad T^{(uh)}_{j_1,j_2}\hat{I} = \frac{(\beta)_{j_1}\,(\alpha)_{j_2}\,(-M)^{j_1}\,(-m)^{j_2}}{j_1!\,j_2!}\,\frac{\theta(B-k)}{k^{\alpha+\beta+j_{12}}}\,, \tag{B.14}$$

converging absolutely for $k \in D_{uh} = (\Lambda, \infty)$.

We have chosen the new convergence domains such that $D_s \cap D_h = D_s \cap D_{uh} = D_h \cap D_{uh} = \emptyset$ and $D_s \cup D_h \cup D_{uh} = \hat{D}$. The ultrahard expansion commutes with the hard expansion,

$$\begin{aligned} T^{(h)}T^{(uh)} = T^{(uh)}T^{(h)} \equiv T^{(h,uh)} &\equiv \sum_{j_1,j_2=0}^{\infty} T^{(h,uh)}_{j_1,j_2}\,, \\ T^{(h,uh)}_{j_1,j_2}\hat{I} = \frac{(\beta)_{j_1}\,(\alpha)_{j_2}\,(-M)^{j_1}\,(-m)^{j_2}}{j_1!\,j_2!}\,&\frac{1}{k^{\alpha+\beta+j_{12}}}\,, \end{aligned} \tag{B.15}$$

but not with the soft expansion:

$$\begin{aligned} T^{(uh)}T^{(s)} &\equiv T^{(uh\leftarrow s)} = T^{(s,h)}\,, \\ T^{(s)}T^{(uh)} &\equiv T^{(s\leftarrow uh)} = T^{(h,uh)}\,. \end{aligned} \tag{B.16}$$

Therefore we need the formalism for non-commuting expansions as developed in section 5. The subset of regions with non-commuting expansions is $R_{\rm nc} = \{s, uh\}$, and condition 2a (p. 25) holds with $R_{\rm c} = \{h\}$. Also condition 5 (p. 29) is fulfilled because the hard expansion does not further change the integrands which are already doubly expanded with $T^{(s)}$ and $T^{(uh)}$:

$$T^{(uh\leftarrow s,h)} = T^{(uh\leftarrow s)}\,, \qquad T^{(s\leftarrow uh,h)} = T^{(s\leftarrow uh)}\,. \tag{B.17}$$

The integrals needed within this formalism read

$$\begin{aligned} F^{(s)}_{j_1} &= \frac{(\beta)_{j_1}\,(-1)^{j_1}}{j_1!\,M^{\beta+j_1}} \int_0^\infty \frac{\mathrm{d}k\,k^{j_1}}{(k+m)^\alpha} = \frac{1}{m^{\alpha-1}\,M^\beta}\left(\frac{m}{M}\right)^{j_1} \frac{1}{\alpha-1}\,\frac{(\beta)_{j_1}}{(2-\alpha)_{j_1}}\,, \\ F^{(h)}_{j_2} &= \frac{(\alpha)_{j_2}\,(-m)^{j_2}}{j_2!} \int_0^\infty \frac{\mathrm{d}k}{k^{\alpha+j_2}\,(k+M)^\beta} \\ &= \frac{1}{M^{\alpha+\beta-1}}\left(\frac{m}{M}\right)^{j_2} \frac{\Gamma(1-\alpha)}{\Gamma(\beta)}\,\frac{\Gamma(\alpha+\beta-1+j_2)}{j_2!}\,, \\ F^{(uh)}_{j_1,j_2} &= \frac{(\beta)_{j_1}\,(\alpha)_{j_2}\,(-M)^{j_1}\,(-m)^{j_2}}{j_1!\,j_2!} \int_0^\infty \mathrm{d}k\,\frac{\theta(B-k)}{k^{\alpha+\beta+j_{12}}} \\ &= -\frac{1}{B^{\alpha+\beta-1}}\left(-\frac{M}{B}\right)^{j_1}\left(-\frac{m}{B}\right)^{j_2} \frac{(\beta)_{j_1}\,(\alpha)_{j_2}}{j_1!\,j_2!\,(\alpha+\beta-1+j_{12})}\,, \\ F^{(s,h)}_{j_1,j_2} &= \frac{(\beta)_{j_1}\,(\alpha)_{j_2}\,(-1)^{j_1}\,(-m)^{j_2}}{j_1!\,j_2!\,M^{\beta+j_1}} \int_0^\infty \frac{\mathrm{d}k}{k^{\alpha+j_2-j_1}} = 0\,, \\ F^{(h,uh)}_{j_2,j_1} &= \frac{(\beta)_{j_1}\,(\alpha)_{j_2}\,(-M)^{j_1}\,(-m)^{j_2}}{j_1!\,j_2!} \int_0^\infty \frac{\mathrm{d}k}{k^{\alpha+\beta+j_{12}}} = 0\,, \end{aligned} \tag{B.18}$$

i.e. all combinations of expansions which commute with each other. All integrals in (B.18) are well-defined via the analytic regulator α (condition 3), and the summations over j_1, j_2 converge absolutely for $m \ll M \ll B$ (condition 4).

According to (5.14), the integral can be expressed as

$$
\begin{aligned}
F = F^{(s)} + F^{(h)} + F^{(uh)} - F^{(s,h)} - F^{(h,uh)} \\
- F^{(uh\leftarrow s)}_{[uh]} + F^{(uh\leftarrow s,h)}_{[uh]} - F^{(s\leftarrow uh)}_{[s]} + F^{(s\leftarrow uh,h)}_{[s]} \,.
\end{aligned}
\tag{B.19}
$$

As condition 5 holds, the extra terms in the second line of (B.19) cancel with each other. So according to (5.18),

$$
F = F^{(s)} + F^{(h)} + F^{(uh)} - F^{(s,h)} - F^{(h,uh)} \,. \tag{B.20}
$$

The first three terms with single expansions are series of homogeneous functions of m, M and B, so the overlap contributions in the last two terms yield scaleless integrals and vanish, see (B.18). The final result shortens to

$$
F = F^{(s)} + F^{(h)} + F^{(uh)} \,, \tag{B.21}
$$

reproducing the result (B.9) from the previous calculation, when the expressions in (B.18) are added together.

Let us have a short look at the extra terms in (B.19) which we have dropped:

$$
\begin{aligned}
F^{(uh\leftarrow s)}_{j_2,j_1\,[uh]} = F^{(uh\leftarrow s,h)}_{j_2,j_1\,[uh]} &= \frac{(\beta)_{j_1}\,(\alpha)_{j_2}\,(-1)^{j_1}\,(-m)^{j_2}}{j_1!\,j_2!\,M^{\beta+j_1}} \int_\Lambda^\infty \frac{\mathrm{d}k}{k^{\alpha+j_2-j_1}} \\
&= -\frac{1}{\Lambda^{\alpha-1}\,M^\beta} \left(-\frac{\Lambda}{M}\right)^{j_1} \left(-\frac{m}{\Lambda}\right)^{j_2} \frac{(\beta)_{j_1}\,(\alpha)_{j_2}}{j_1!\,j_2!\,(1-\alpha+j_1-j_2)} \,, \\
F^{(s\leftarrow uh)}_{j_1,j_2\,[s]} = F^{(s\leftarrow uh,h)}_{j_1,j_2\,[s]} &= \frac{(\beta)_{j_1}\,(\alpha)_{j_2}\,(-M)^{j_1}\,(-m)^{j_2}}{j_1!\,j_2!} \int_0^\lambda \frac{\mathrm{d}k}{k^{\alpha+\beta+j_{12}}} \\
&= -\frac{1}{\lambda^{\alpha+\beta-1}} \left(-\frac{M}{\lambda}\right)^{j_1} \left(-\frac{m}{\lambda}\right)^{j_2} \frac{(\beta)_{j_1}\,(\alpha)_{j_2}}{j_1!\,j_2!\,(\alpha+\beta-1+j_{12})} \,.
\end{aligned}
\tag{B.22}
$$

The summations of these terms over j_1 diverge, because $\Lambda/M \gg 1$ and $M/\lambda \gg 1$. So by choosing to expand the integral F (B.1) via extending its upper boundary to infinity and introducing a third region, we still cannot completely avoid the problem of diverging series expansions encountered in section B.1. But at least we have isolated the divergence problems in terms which disappear from the final result (B.21) in a well-understood way.

Note that also other series expansions which appear in intermediate steps of the derivation, in particular $\sum_{j_1} F^{(s,h)}_{j_1,j_2\,[h]}$ and $\sum_{j_1} F^{(h,uh)}_{j_2,j_1\,[h]}$, diverge because the domain D_h has boundaries touching both the soft and the ultrahard domain. So we can only sum the expansions over j_1 up to ∞ once the terms have been combined into the integrals (B.18) performed over the complete integration domain. For those the convergence of the series expansions is required by condition 4 in section 3.

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