$\mathcal{N} \geq 4$ supergravity amplitudes from gauge theory at two loops

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ABSTRACT: We present the full two-loop four-graviton amplitudes in $\mathcal{N} = 4, 5, 6$ supergravity. These results were obtained using the double-copy structure of gravity, which follows from the recently conjectured color-kinematics duality in gauge theory. The two-loop four-gluon scattering amplitudes in $\mathcal{N} = 0, 1, 2$ supersymmetric gauge theory are a second essential ingredient. The gravity amplitudes have the expected infrared behavior: the two-loop divergences are given in terms of the squares of the corresponding one-loop amplitudes. The finite remainders are presented in a compact form. The finite remainder for $\mathcal{N} = 8$ supergravity is also presented, in a form that utilizes a pure function with a very simple symbol.

Contents

1   Introduction

It is well known that pure Einstein gravity is ultraviolet (UV) divergent at two loops [1, 2]. This result, along with general power-counting arguments, has led to the widespread belief that a UV finite pointlike theory of gravity cannot be constructed. However, explicit calculations of scattering amplitudes in maximally supersymmetric ($\mathcal{N} = 8$) supergravity have displayed an ultraviolet behavior that is much better than prior expectations, showing that the theory in four dimensions is finite up to at least four loops. Furthermore, $\mathcal{N} = 8$ supergravity exhibits the same UV behavior, when continued to higher spacetime dimensions, as does $\mathcal{N} = 4$ super-Yang-Mills (sYM) [3–5]. Surprising cancellations are also visible at lower loop orders [6–17], and even at tree level where the amplitudes are nicely behaved at large (complex) momenta [16, 18–22].

In pure supergravity theories (where all states are related by supersymmetry to the graviton) no counterterm can be constructed below three loops. This is because the only possible two-loop counterterm, $R^3 \equiv R_{\mu\nu}^{\lambda\rho} R_{\sigma\tau}^{\mu\nu} R_{\lambda\rho}^{\sigma\tau}$, where $R_{\sigma\tau}^{\mu\nu}$ is the Riemann tensor, generates non-zero four-graviton amplitudes with helicity assignment $(\pm, +, +, +)$ [23–25]. Such amplitudes are forbidden by the Ward identities for the minimal $\mathcal{N} = 1$ supersymmetry [26, 27]. The counterterm denoted by R^4 is allowed by supersymmetry and could appear at three loops [24, 28–31]. However, as mentioned earlier, $\mathcal{N} = 8$ supergravity was found to be finite at this order [7, 8]. It was recently understood that the R^4 counterterm is forbidden [32, 33] by the nonlinear $E_{7(7)}$ symmetry realized by the 70 scalars of the theory [34–36]. In fact, $E_{7(7)}$ should delay the divergence in $\mathcal{N} = 8$ supergravity to at least seven loops, where the first $E_{7(7)}$-invariant counterterm can be constructed [37–39].

Non-maximal ($\mathcal{N} < 8$) supergravity does not have this extra $E_{7(7)}$ symmetry, and may therefore diverge at only three loops in four dimensions.

Recently, the constraints that the smaller duality symmetries of non-maximal supergravities impose on potential counterterms have also been investigated [38, 39]. In four dimensions, $\mathcal{N} = 6$ supergravity is expected to be finite at three and four loops, and $\mathcal{N} = 5$ supergravity should be finite at three loops [38]. These results still allow for a three-loop divergence in $\mathcal{N} \leq 4$ supergravities. In particular, for $\mathcal{N} = 4$ supergravity, although the volume of superspace vanishes on shell, it has been argued that the usual three-loop R^4 counterterm can appear [39]. The finiteness results for $\mathcal{N} = 5, 6$ could in principle be checked, and potential divergences for $\mathcal{N} \leq 4$ investigated, via explicit three-loop amplitude calculations in non-maximal supergravities. Because the same situation, in which the superspace volume vanishes on shell, and yet a counterterm appears to be allowed, holds for $\mathcal{N} = 8$ supergravity at seven loops, as for $\mathcal{N} = 4$ supergravity at three loops, this latter case may be of particular interest.

On the other hand, relatively few loop amplitudes have been computed for any non-maximal supergravities. At one loop, the four-point amplitudes with $\mathcal{N} \leq 8$ supersymmetries were presented in refs. [40, 41], while the $\mathcal{N} = 6$ supergravity all-point maximally-helicity-violating (MHV) and six-point non-MHV amplitudes were first obtained in ref. [42]. The $\mathcal{N} = 4$ supergravity one-loop five-point amplitude was also computed in refs. [42, 43]. In the following, we present expressions for the two-loop four-graviton amplitudes in $\mathcal{N} = 4, 5, 6$ supergravity. The calculations were performed using the gravity "squaring" relations [44, 45], or double-copy property, which follows from the color-kinematics, or Bern-Carrasco-Johansson (BCJ), duality obeyed by gauge-theory amplitudes at the loop level [46].

The BCJ relations allow us to combine the $\mathcal{N} = 4$ sYM amplitude [47] with the $\mathcal{N} = 0, 1, 2$ sYM amplitudes [48] in order to obtain the corresponding amplitudes in supergravity. Although they have been tested now in several loop-level amplitude computations [5, 17, 44, 46], the underlying mechanism or symmetry behind the general loop-level BCJ relations is still not well understood. (In the self-dual sector at tree level, a diffeomorphism Lie algebra appears to play a key role. [49].) Therefore it is important to validate results obtained using BCJ duality. We will verify the expected infrared divergences and forward-scattering behavior for the two-loop amplitudes that we compute.

This paper is organized as follows. In section 2 we review BCJ duality and the squaring relations for gravity. In section 3 we illustrate the method for $\mathcal{N} = 8$ supergravity at two loops. In section 4 we present our main formula for the two-loop amplitudes in $\mathcal{N} = 4, 5, 6$ supergravity. In section 5 we expand the (dimensionally regulated) amplitudes for $D = 4 - 2\epsilon$ around $\epsilon = 0$. We discuss the infrared (IR) pole structure, which agrees with general expectations, thus providing a cross check on the construction. We present the finite remainders in the two independent kinematic channels. In section 6 we examine the behavior of the amplitudes in the limit of forward scattering. In section 7, we present our conclusions and suggestions for future research directions. An appendix provides some one-loop results that are required for extracting the two-loop finite remainders.

2 Review of the BCJ duality and squaring relations

We now briefly review BCJ duality and the gravity squaring relations that follow from it. For a more complete treatment see, for example, the recent reviews [50, 51]. Here, we will focus solely on applications to loop amplitudes.

We can write any m-point L-loop-level gauge-theory amplitude, where all particles are in the adjoint representation, as

$$\mathcal{A}_m^{(L)} = i^L g^{m-2+2L} \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2} , \qquad (2.1)$$

where g is the gauge coupling. The sum runs over the set of distinct m-point L-loop graphs, labeled by j, with only cubic vertices, corresponding to the diagrams of a ϕ^3 theory. The product in the denominator runs over all Feynman propagators of each cubic diagram. The integrals are over $\{p_l^\mu\}$, a set of L independent D-dimensional loop momenta. The c_i are the color factors, obtained by dressing every three-vertex with a structure constant, defined by $\tilde{f}^{abc} = i\sqrt{2} f^{abc} = \text{Tr}([T^a, T^b] T^c)$. The n_j are kinematic numerator factors depending on momenta, polarizations and spinors. The S_j are the internal symmetry factors for each diagram. The form of the amplitude presented in eq. (2.1) can be obtained in various ways. For example, one can start from covariant Feynman diagrams in Feynman gauge, where the contact terms are absorbed into kinematic numerators using inverse propagators, i.e. by inserting factors of $1 = p_{\alpha_j}^2 / p_{\alpha_j}^2$.

Triplets (i, j, k) of color factors are related to each other by $c_i = c_j + c_k$ if their corresponding graphs are identical, except for a region containing (in turn for i, j, k) the three cubic four-point graphs that exist at tree level. The relation holds because the products of two $\tilde{f}^{abc}$ structure constants corresponding to the four-point tree graphs satisfy the Jacobi identity

$$\tilde{f}^{abe} \tilde{f}^{cde} = \tilde{f}^{ace} \tilde{f}^{bde} + \tilde{f}^{ade} \tilde{f}^{cbe} , \qquad (2.2)$$

and the remaining structure constant factors in the triplet of graphs are identical. The relations $c_i = c_j + c_k$ mean that the representation (2.1) is not unique; terms can be shuffled from one graph to others, in a kind of generalized gauge transformation [44].

A representation (2.1) is said to satisfy the BCJ duality if the three associated kinematic numerators are also related via Jacobi identities. Namely, we must have:

$$c_i = c_j + c_k \qquad \Rightarrow \qquad n_i = n_j + n_k , \qquad (2.3)$$

where the left-hand side follows directly from group theory, while the right-hand side is the highly non-trivial requirement of the duality. Moreover, we demand that the numerator factors have the same antisymmetry property as the color factors under the interchange of two legs attached to a cubic vertex,

$$c_i \to -c_i \qquad \Rightarrow \qquad n_i \to -n_i . \qquad (2.4)$$

The relations (2.3) were found long ago for the case of four-point tree amplitudes [52, 53]; the idea that the relations should hold for arbitrary amplitudes is more recent [44, 46].

As remarked earlier, the representation (2.1) is not unique. Work is often required in order to find a BCJ-satisfying representation of a given amplitude in a particular gauge theory. At loop level, such representations were found initially at four points through three loops for $\mathcal{N} = 4$ sYM, and through two loops for identical-helicity pure Yang-Mills amplitudes [44]. A BCJ-satisfying representation was recently obtained at five points through three loops in $\mathcal{N} = 4$ sYM [17]. Very recently, a four-point four-loop representation was found in the same theory [5].

As a remarkable consequence of the BCJ duality, one can combine two gauge-theory amplitudes in the form (2.1), in order to obtain a gravity amplitude, as long as one of the two gauge-theory representations manifestly satisfies the duality [44, 45]. We have,

$$\mathcal{M}_m^{(L)} \; = \; i^{L+1} \left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2}, \tag{2.5}$$

where either the n_j or the $\tilde{n}_j$ must satisfy eqs. (2.3) and (2.4). Here κ is the gravitational coupling constant, which is related to Newton's constant G_N and the Planck mass M_{Planck} by $\kappa^2 = 32\pi G_N = 32\pi/M_{\text{Planck}}^2$. The proof of eq. (2.5) at tree level is inductive, and uses on-shell recursion relations [54, 55] for the gauge and gravity theories, which are based on the same complex momentum shift [45]. The extrapolation to loop level is based on reconstructing loop amplitudes from tree amplitudes using (generalized) unitarity.

The relations (2.5) are similar in spirit to the KLT relations [56]. Both types of relations express gravity amplitudes as the "square" of gauge-theory amplitudes, or more generally as the product of two different types of gauge-theory amplitudes, as the n_i and $\tilde{n}_j$ numerator factors may come from two different Yang-Mills theories. However, the KLT relations only hold at tree level, which means that at loop level they can only be used on the (generalized) unitarity cuts. Although the gravity cuts can be completely determined by the KLT relations in terms of local Yang-Mills integrands, the gravity integrand found in this way is not manifestly local. That is, it does not manifestly have the form of numerator factors multiplied by scalar propagators for some set of ϕ^3 graphs. Reconstructing a local representation can be a significant task [3, 7, 8].

In contrast, eq. (2.5) is a loop-level relation, and furnishes directly a local integrand for gravity. Most of the applications of this formula to date have been to maximal $\mathcal{N} = 8$ supergravity, viewed as the tensor product of two copies of maximal $\mathcal{N} = 4$ super-Yang-Mills theory. The squaring relations were shown to reproduce the $\mathcal{N} = 8$ supergravity four-point amplitudes through four loops [5, 44] and the five-point amplitudes through two loops [17]. Quite recently, in the first loop-level applications for $\mathcal{N} < 8$, the one-loop four- and five-point $\mathcal{N} \leq 8$ supergravity amplitudes were shown to satisfy the double-copy property [43]. In this paper, we would like to extend this kind of analysis for $\mathcal{N} < 8$ supergravity to two loops. First, however, we briefly review the $\mathcal{N} = 8$ case.

3 Two-loop $\mathcal{N} = 8$ supergravity

In this section we review the construction of the two-loop four-graviton amplitude in $\mathcal{N} = 8$ supergravity based on squaring relations, as preparation for a similar construction for $\mathcal{N} = 4, 5, 6$ supergravity in the next section.
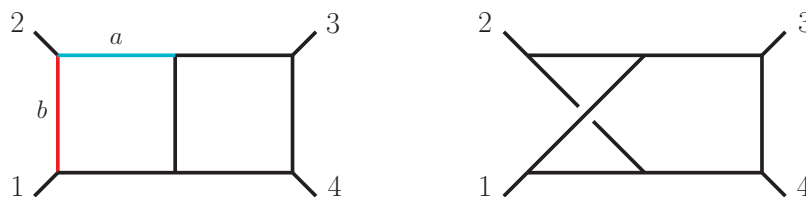
Figure 1. The planar and nonplanar cubic diagrams at two loops. The marked (colored) propagators in the planar diagram are used in the text to describe different color and kinematic Jacobi identities.

As mentioned previously, a manifestly BCJ-satisfying representation of the four-gluon $\mathcal{N} = 4$ sYM amplitude is known at two loops [44, 47],

$$\mathcal{A}_4^{(2)}(1,2,3,4) = -g^6 st \, A_4^{\text{tree}}(1,2,3,4) \Big(c_{1234}^{(\text{P})} \, s \, \mathcal{I}_4^{(\text{P})}(s,t) + c_{3421}^{(\text{P})} \, s \, \mathcal{I}_4^{(\text{P})}(s,u) \qquad (3.1)$$
$$+ c_{1234}^{(\text{NP})} \, s \, \mathcal{I}_4^{(\text{NP})}(s,t) + c_{3421}^{(\text{NP})} \, s \, \mathcal{I}_4^{(\text{NP})}(s,u) + \text{cyclic} \Big),$$

where s, t, u are the usual Mandelstam invariants ($s = (k_1 + k_2)^2$, $t = (k_2 + k_3)^2$, $u = (k_1 + k_3)^2$) and "+ cyclic" instructs one to add the two cyclic permutations of (2,3,4). The tree-level partial amplitude is

$$A_4^{\text{tree}}(1,2,3,4) = i \, \frac{\langle j \, k \rangle^4}{\langle 1 \, 2 \rangle \, \langle 2 \, 3 \rangle \, \langle 3 \, 4 \rangle \, \langle 4 \, 1 \rangle} \,, \qquad (3.2)$$

where j and k label the two negative-helicity gluons. The two-loop planar and nonplanar scalar double-box integrals are, respectively,

$$\mathcal{I}_4^{(\text{P})}(s,t) = \int \frac{d^D p}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{1}{p^2 \, (p - k_1)^2 \, (p - k_1 - k_2)^2 \, (p + q)^2 q^2 \, (q - k_4)^2 \, (q - k_3 - k_4)^2} \,,$$
$$\mathcal{I}_4^{(\text{NP})}(s,t) = \int \frac{d^D p}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{1}{p^2 \, (p - k_2)^2 \, (p + q)^2 \, (p + q + k_1)^2 \, q^2 \, (q - k_3)^2 \, (q - k_3 - k_4)^2} \,,$$

and they are depicted in figure 1. The color factors $c_{ijkl}^{(\text{P,NP})}$ are obtained by dressing each vertex of the associated diagram with a factor of $\tilde{f}^{abc}$, and each internal line with a δ_{ab}. All helicity information is encoded in the prefactor $st A_4^{\text{tree}}(1,2,3,4)$, which is invariant under all permutations, thanks to a Ward identity for $\mathcal{N} = 4$ supersymmetry.

Comparing eqs. (2.1) and (3.1) we can identify the numerators as

$$n_{1234}^{(\text{P})} = n_{3421}^{(\text{P})} = n_{1234}^{(\text{NP})} = n_{3421}^{(\text{NP})} = s \times st \, A_4^{\text{tree}}(1,2,3,4) \,,$$
$$n_{1342}^{(\text{P})} = n_{4231}^{(\text{P})} = n_{1342}^{(\text{NP})} = n_{4231}^{(\text{NP})} = u \times st \, A_4^{\text{tree}}(1,2,3,4) \,,$$
$$n_{1423}^{(\text{P})} = n_{2341}^{(\text{P})} = n_{1423}^{(\text{NP})} = n_{2341}^{(\text{NP})} = t \times st \, A_4^{\text{tree}}(1,2,3,4) \,. \qquad (3.3)$$

It is easy to see that the two-loop expression (3.1) satisfies the duality [44]. For instance, let's look at the diagrams related by a Jacobi identity applied to a four-point tree-level subdiagram of the planar double-box graph on the left-hand side of figure 1. The tree
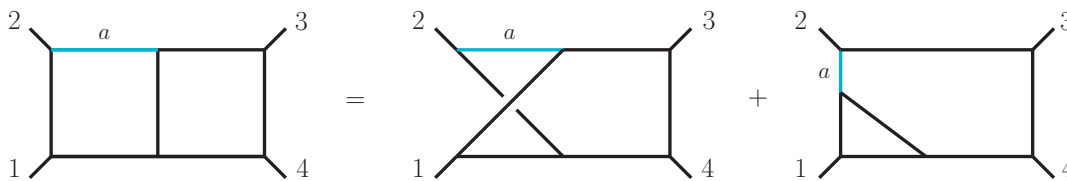
Figure 2. Two-loop diagrams related by a Jacobi identity. The Jacobi identity is applied to the four-point tree-level subdiagram that contains the (light blue) intermediate line marked a. The rest of the diagram is unchanged.

subdiagram is the one whose intermediate propagator is the light-blue line marked a in the figure. We replace the "s-channel" tree subdiagram with the corresponding t- and u-channel tree subdiagrams, by appropriately permuting the attachments of line a to the rest of the graph. This Jacobi identity is illustrated in figure 2. Because the $\mathcal{N} = 4$ sYM diagrams with triangle one-loop subdiagrams all have vanishing coefficients in eq. (3.1), the duality (2.3) requires the equality of the planar and nonplanar numerator factors, $n_{1234}^{(\mathrm{P})} = n_{1234}^{(\mathrm{NP})}$. Similarly, applying a Jacobi identity to the red propagator marked b in the planar double-box diagram in figure 1, we find two graphs, one of which again contains a vanishing triangle subgraph. Therefore the numerator of the planar box graph should be symmetric under the exchange of legs 1 and 2, or equivalently $n_{1234}^{(\mathrm{P})} = n_{3421}^{(\mathrm{P})}$. Looking at eq. (3.3), we see that these two conditions are satisfied.

Having verified that eq. (3.1) satisfies the BCJ relations, we may combine two copies of (3.1) following prescription (2.5) to obtain the two-loop four-graviton $\mathcal{N} = 8$ amplitude. We obtain

$$\mathcal{M}_4^{(2)}(1,2,3,4) = -i\left(\frac{\kappa}{2}\right)^6 [st\, A_4^{\mathrm{tree}}(1,2,3,4)]^2 \Big(s^2 \mathcal{I}_4^{(\mathrm{P})}(s,t) + s^2 \mathcal{I}_4^{(\mathrm{P})}(s,u) \\ + s^2 \mathcal{I}_4^{(\mathrm{NP})}(s,t) + s^2 \mathcal{I}_4^{(\mathrm{NP})}(s,u) + \mathrm{cyclic} \Big), \tag{3.4}$$

which is precisely the known result [6]. We also recall that the four-graviton and four-gluon tree-level partial amplitudes are related to each other by

$$stu M_4^{\mathrm{tree}} = -i\, [st\, A_4^{\mathrm{tree}}(1,2,3,4)]^2. \tag{3.5}$$

4 Two-Loop $4 \leq \mathcal{N} < 8$ supergravity

Now we move to the main subject of this paper, the construction of the two-loop four-graviton amplitudes for $\mathcal{N} = 4, 5, 6$ supergravity. As we mentioned earlier, only one of the two gauge-theory amplitudes entering the double-copy formula (2.5) needs to satisfy the BCJ duality. We will combine the duality-satisfying $\mathcal{N} = 4$ sYM amplitude (3.1) with four-gluon amplitudes for $\mathcal{N} \equiv \mathcal{N}_{\mathrm{YM}} = 0, 1, 2$ sYM, in order to obtain the corresponding two-loop four-graviton amplitudes in supergravities with $\mathcal{N} = 4 + \mathcal{N}_{\mathrm{YM}} = 4, 5, 6$. Looking at the multiplicities of states for various supergravities and super-Yang-Mills theories in

helicity	0	+1/2	+1	+3/2	+2
$\mathcal{N} = 8$ supergravity	70	56	28	8	1
$\mathcal{N} = 6$ supergravity	30	26	16	6	1
$\mathcal{N} = 5$ supergravity	10	11	10	5	1
$\mathcal{N} = 4$ supergravity	2	4	6	4	1
$\mathcal{N} = 4$ sYM	6	4	1		
$\mathcal{N} = 2$ sYM	2	2	1		
$\mathcal{N} = 1$ sYM		1	1		
$\mathcal{N} = 0$ sYM			1		

Table 1. State multiplicity as a function of helicity for relevant supersymmetric multiplets in pure supergravities and super-Yang-Mills theories. By CPT invariance, the multiplicity for helicity $-h$ is the same as that shown for h.

table 1, we can see that at the level of counting states,

$$
\begin{aligned}
\mathcal{N} = 6 \text{ supergravity}: & \quad (\mathcal{N} = 4 \text{ sYM}) \times (\mathcal{N} = 2 \text{ sYM}), \\
\mathcal{N} = 5 \text{ supergravity}: & \quad (\mathcal{N} = 4 \text{ sYM}) \times (\mathcal{N} = 1 \text{ sYM}), \\
\mathcal{N} = 4 \text{ supergravity}: & \quad (\mathcal{N} = 4 \text{ sYM}) \times (\mathcal{N} = 0 \text{ sYM}),
\end{aligned} \tag{4.1}
$$

where $\mathcal{N} = 0$ sYM refers to pure Yang-Mills theory with only gluons. Because the gauge theories with $\mathcal{N} < 4$ supersymmetry are consistent truncations of maximal $\mathcal{N} = 4$ sYM, and similarly on the gravity side, these equivalences also hold at the level of amplitudes, through either the KLT relations (at tree level) or the double-copy relations (2.5).

In ref. [57], it was shown that one could write a color decomposition of any one-loop full-color all-adjoint gauge-theory amplitude in terms of color factors called "ring diagrams". The diagrammatic representation of these color factors have all the external legs connected directly to the loop. Other conceivable color factors, in which nontrivial trees are attached to the loop, can be removed systematically by using Jacobi identities, in favor of ring graphs with different cyclic orderings of the external legs. This decomposition is independent of the (adjoint) particle content in the loop. In the same way, we can use the Jacobi identities at two loops to rewrite any full-color four-gluon amplitude in a theory with only adjoint particles, in terms of only the color factors $c_{1234}^{(\mathrm{P})}$ and $c_{1234}^{(\mathrm{NP})}$ of the diagrams of figure 1 (plus permutations).

For super-Yang-Mills theory with $\mathcal{N} = \mathcal{N}_{\mathrm{YM}}$ supersymmetries, we write

$$
\begin{aligned}
\mathcal{A}&_{\mathcal{N}_{\mathrm{YM}}}^{(2)}(1,2,3,4) \\
&= -g^6 \Big(c_{1234}^{(\mathrm{P})} A_{1234,\mathcal{N}_{\mathrm{YM}}}^{(\mathrm{P})} + c_{3421}^{(\mathrm{P})} A_{3421,\mathcal{N}_{\mathrm{YM}}}^{(\mathrm{P})} + c_{1234}^{(\mathrm{NP})} A_{1234,\mathcal{N}_{\mathrm{YM}}}^{(\mathrm{NP})} + c_{3421}^{(\mathrm{NP})} A_{3421,\mathcal{N}_{\mathrm{YM}}}^{(\mathrm{NP})} \\
&\quad + c_{1342}^{(\mathrm{P})} A_{1342,\mathcal{N}_{\mathrm{YM}}}^{(\mathrm{P})} + c_{4231}^{(\mathrm{P})} A_{4231,\mathcal{N}_{\mathrm{YM}}}^{(\mathrm{P})} + c_{1342}^{(\mathrm{NP})} A_{1342,\mathcal{N}_{\mathrm{YM}}}^{(\mathrm{NP})} + c_{4231}^{(\mathrm{NP})} A_{4231,\mathcal{N}_{\mathrm{YM}}}^{(\mathrm{NP})} \\
&\quad + c_{1423}^{(\mathrm{P})} A_{1423,\mathcal{N}_{\mathrm{YM}}}^{(\mathrm{P})} + c_{2341}^{(\mathrm{P})} A_{2341,\mathcal{N}_{\mathrm{YM}}}^{(\mathrm{P})} + c_{1423}^{(\mathrm{NP})} A_{1423,\mathcal{N}_{\mathrm{YM}}}^{(\mathrm{NP})} + c_{2341}^{(\mathrm{NP})} A_{2341,\mathcal{N}_{\mathrm{YM}}}^{(\mathrm{NP})} \Big),
\end{aligned} \tag{4.2}
$$

where $A_{1234}^{(\mathrm{P})}$ is the integrated color-ordered subamplitude associated with the color factor

$c_{1234}^{(P)}$. For example, for the $\mathcal{N} = 4$ sYM representation (3.1), we read off

$$A_{1234,\mathcal{N}_{YM}=4}^{(P)} = st\, A_4^{\text{tree}}(1,2,3,4) \times s\, \mathcal{I}_4^{(P)}(s,t)\,.$$

Normally, to implement the double-copy formula (2.5), we would need to have a representation for the *integrand* of the gauge-theory amplitudes, in particular for the $\mathcal{N} = 0, 1, 2$ sYM amplitudes we are combining with those for $\mathcal{N} = 4$ sYM. However, at two loops the numerator factors for $\mathcal{N} = 4$ sYM have no dependence on the loop momenta. The same feature holds for the one-loop four- and five-point amplitudes studied in ref. [43]. Therefore, just as in those cases, we can remove the $\mathcal{N} = 4$ sYM numerator factors from the loop integrals in eq. (2.5). Using eq. (3.3) for the $\mathcal{N} = 4$ sYM numerator factors, we obtain the remarkably simple general formula,

$$\begin{aligned}
\mathcal{M}_{\mathcal{N}_{YM}+4}^{(2)}(1,2,3,4) = &-i\Big(\frac{\kappa}{2}\Big)^6 st\, A_4^{\text{tree}}(1,2,3,4) \\
&\times \Big(s\, A_{1234,\mathcal{N}_{YM}}^{(P)} + s\, A_{3421,\mathcal{N}_{YM}}^{(P)} + s\, A_{1234,\mathcal{N}_{YM}}^{(NP)} + s\, A_{3421,\mathcal{N}_{YM}}^{(NP)} \\
&\quad + u\, A_{1342,\mathcal{N}_{YM}}^{(P)} + u\, A_{4231,\mathcal{N}_{YM}}^{(P)} + u\, A_{1342,\mathcal{N}_{YM}}^{(NP)} + u\, A_{4231,\mathcal{N}_{YM}}^{(NP)} \\
&\quad + t\, A_{1423,\mathcal{N}_{YM}}^{(P)} + t\, A_{2341,\mathcal{N}_{YM}}^{(P)} + t\, A_{1423,\mathcal{N}_{YM}}^{(NP)} + t\, A_{2341,\mathcal{N}_{YM}}^{(NP)} \Big)\,. \quad (4.3)
\end{aligned}$$

In summary, we obtain the $\mathcal{N} = 4, 5, 6$ supergravity amplitudes by first expressing the $\mathcal{N} = 0, 1, 2$ sYM helicity amplitudes from ref. [48] in terms of the color basis (4.2).[1] We then replace $g^6 \to i(\kappa/2)^6$ and perform the following additional replacements (plus their relabelings):

$$c_{1234}^{(P)} \to st A_4^{\text{tree}}(1,2,3,4) \times s, \qquad\qquad c_{1234}^{(NP)} \to st A_4^{\text{tree}}(1,2,3,4) \times s. \qquad (4.4)$$

Because $st A_4^{\text{tree}}(1,2,3,4)$ is permutation-invariant, only the single factors of s, t, u persist inside the parentheses in eq. (4.3).

In order to preserve supersymmetry, we use the four-dimensional helicity variant of dimensional regularization [58, 59] for both copies of the gauge-theory amplitudes. The results (4.3) can be expressed in terms of master integrals for the two-loop planar and non-planar double-box topologies, plus various other integrals with fewer propagators present. However, in this form the results are rather lengthy. Instead of presenting them here, we expand the dimensionally-regulated results, for $D = 4 - 2\epsilon$, around $\epsilon = 0$, as discussed in the next section.

5 Infrared poles and finite remainders

At two loops, all pure supergravity amplitudes are ultraviolet finite [23–25]. Therefore all of their divergences are infrared in nature, either soft or possibly collinear. As two massless external particles become collinear, gravitational tree amplitudes have singularities only in phase, not in magnitude. The same universal "splitting amplitude" that controls the phase behavior governs loop amplitudes as well as tree amplitudes [9]. Correspondingly, there

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[1]We thank Zvi Bern for providing us with the expressions in this format.

are no virtual divergences from purely collinear regions of integration [60]. Soft divergences were studied long ago and found to exponentiate [61]. More recent, explicit analyses can be found in refs. [60, 62, 63]. At one loop, the IR pole behavior is [40, 41, 61, 64, 65],

$$\mathcal{M}_4^{(1)} = \left(\frac{\kappa}{8\pi}\right)^2 \frac{2}{\epsilon} \left(s \ln(-s) + t \ln(-t) + u \ln(-u)\right) \mathcal{M}_4^{\text{tree}} + \mathcal{O}(\epsilon^0). \qquad (5.1)$$

At L loops, the leading divergence is at order $1/\epsilon^L$. We first checked that the leading divergence of our two-loop $\mathcal{N} = 4, 5, 6$ supergravity amplitudes is indeed at order $1/\epsilon^2$.

Moreover, the exponentiation of soft divergences implies that the full two-loop IR behavior can be expressed in terms of the one-loop amplitude as follows:

$$\frac{\mathcal{M}_4^{(2)}(\epsilon)}{\mathcal{M}_4^{\text{tree}}} = \frac{1}{2}\left[\frac{\mathcal{M}_4^{(1)}(\epsilon)}{\mathcal{M}_4^{\text{tree}}}\right]^2 + \left(\frac{\kappa}{8\pi}\right)^4 F_4^{(2)} + \mathcal{O}(\epsilon), \qquad (5.2)$$

where $F_4^{(2)}$ is the finite remainder in the limit $\epsilon \to 0$. This infrared behavior was checked explicitly for the four-point $\mathcal{N} = 8$ supergravity amplitude [64, 65], and was conjectured to hold for all supersymmetric gravity amplitudes [63]. We have checked that our expressions indeed satisfy eq. (5.2). We remark that the lack of any additional (ultraviolet) poles in ϵ confirms the absence of UV divergences for $\mathcal{N} = 4, 5, 6$ supergravity in four dimensions at two loops [23–25].

In order to verify eq. (5.2) and extract $F_4^{(2)}$, we need the $\mathcal{O}(\epsilon^0)$ and $\mathcal{O}(\epsilon^1)$ coefficients in the expansion of the corresponding one-loop amplitude $\mathcal{M}_4^{(1)}$. That is because $\mathcal{M}_4^{(1)}$ appears squared in eq. (5.2), and the $1/\epsilon$ pole in eq. (5.1) can multiply the $\mathcal{O}(\epsilon^1)$ coefficient to generate a finite term. We give the required one-loop expansions in appendix A.

Next we present the finite remainders $F_4^{(2)}$ for the different theories under consideration. It is convenient to express the remainders for $\mathcal{N} < 8$ supergravity in terms of the $\mathcal{N} = 8$ remainder plus an additional term. The result for $\mathcal{N} = 8$ supergravity was first presented in refs. [64, 65]. We always consider the helicity configuration $(1^-, 2^-, 3^+, 4^+)$. There are three separate physical kinematic regions: the s channel, with $s > 0$ and $t, u < 0$; the t channel ($t > 0$ and $s, u < 0$); and the u channel ($u > 0$ and $s, t < 0$). The s channel is singled out by the fact that it has identical-helicity incoming gravitons. For all the supergravity theories, the $(1^-, 2^-, 3^+, 4^+)$ helicity configuration chosen is symmetric under $3 \leftrightarrow 4$. Therefore we do not have to present results separately for the u channel; they can be obtained from the t-channel results by relabeling $t \leftrightarrow u$. In the case of $\mathcal{N} = 8$ supergravity, an $\mathcal{N} = 8$ supersymmetric Ward identity implies that the results in the t channel (normalized by the tree amplitude) can be obtained simply by relabeling $s \leftrightarrow t$. For $\mathcal{N} < 8$, this property no longer holds, and we will have to quote the s- and t-channel results separately.

The $\mathcal{N} = 8$ finite remainder was expressed in refs. [64, 65] partly in terms of Nielsen polylogarithms $S_{n,p}(x)$. Here we give a representation similar to ref. [64], and a second representation entirely in terms of classical polylogarithms Li_n, for consistency with the

forms we present below for $\mathcal{N} < 8$. The finite remainder is

$$F_4^{(2),\mathcal{N}=8}\Big|_{s-\text{channel}} = 8\left\{t\,u\left[f_1\left(\frac{-t}{s}\right) + f_1\left(\frac{-u}{s}\right)\right] + s\,u\left[f_2\left(\frac{-t}{s}\right) + f_3\left(\frac{-t}{s}\right)\right]\right.$$
$$\left. + s\,t\left[f_2\left(\frac{-u}{s}\right) + f_3\left(\frac{-u}{s}\right)\right]\right\}, \tag{5.3}$$

where

$$f_1(x) = S_{1,3}(1-x) + \zeta_4 + \frac{1}{24}\ln^4 x + i\pi\left[-S_{1,2}(1-x) + \zeta_3 + \frac{1}{6}\ln^3 x\right]$$
$$= -\text{Li}_4(x) + \ln x\,\text{Li}_3(x) - \frac{1}{2}\ln^2 x\,\text{Li}_2(x) + \frac{1}{24}\ln^4 x - \frac{1}{6}\ln^3 x\,\ln(1-x) + 2\,\zeta_4$$
$$+ i\pi\left[\text{Li}_3(x) - \ln x\,\text{Li}_2(x) + \frac{1}{6}\ln^3 x - \frac{1}{2}\ln^2 x\,\ln(1-x)\right], \tag{5.4}$$

$$f_2(x) = S_{1,3}\left(1 - \frac{1}{x}\right) + \zeta_4 + \frac{1}{24}\ln^4 x + i\pi\left[S_{1,2}\left(1 - \frac{1}{x}\right) - \zeta_3 + \frac{1}{6}\ln^3 x\right]$$
$$= \text{Li}_4(x) - \ln x\,\text{Li}_3(x) + \frac{1}{2}\ln^2 x\,\text{Li}_2(x) + \frac{1}{6}\ln^3 x\,\ln(1-x)$$
$$- i\pi\left[\text{Li}_3(x) - \ln x\,\text{Li}_2(x) - \frac{1}{2}\ln^2 x\,\ln(1-x)\right], \tag{5.5}$$

and

$$f_3(x) = \text{Li}_4(y) - \ln(-y)\,\text{Li}_3(y) + \frac{1}{2}\left[\ln^2(-y) + \pi^2\right]\text{Li}_2(y)$$
$$+ \frac{1}{6}\left[\ln^3(-y) + 3\,\pi^2\,\ln(-y) - 2\,i\,\pi^3\right]\ln(1-y), \tag{5.6}$$

with $y = -x/(1-x)$. The $\mathcal{N} = 8$ supergravity remainder in the t channel is given simply by relabeling the s-channel result, exchanging s and t:

$$F_4^{(2),\mathcal{N}=8}(s,t,u)\Big|_{t-\text{channel}} = F_4^{(2),\mathcal{N}=8}(t,s,u)\Big|_{s-\text{channel}}. \tag{5.7}$$

It was noted previously [64, 65] that $F_4^{(2),\mathcal{N}=8}$ has a uniform maximal transcendentality. That is, all functions appearing are degree-four combinations of polylogarithms, logarithms, and transcendental constants. A *pure function* is a function with a uniform degree of transcendentality, having only constants (rational numbers) multiplying the combinations of polylogarithms, etc. A pure function f has a well-defined *symbol*, $\mathcal{S}(f)$, which can be obtained by an iterated differentiation procedure [66, 67]. In the representation (5.3), the functions f_1, f_2 and f_3 are pure functions with very simple, one-term symbols:

$$\mathcal{S}(f_1) = x \otimes x \otimes x \otimes \frac{x}{1-x}, \tag{5.8}$$

$$\mathcal{S}(f_2) = x \otimes x \otimes x \otimes (1-x), \tag{5.9}$$

$$\mathcal{S}(f_3) = -\frac{x}{1-x} \otimes \frac{x}{1-x} \otimes \frac{x}{1-x} \otimes (1-x). \tag{5.10}$$

We have shuffled terms slightly with respect to refs. [64, 65] in order to make this property manifest. For example, our function $f_1(x)$ is very similar to the function $h(t,s,u)$ given in

eq. (2.26) of ref. [64], after multiplying it by $1/8$ and setting $-s/t \to x$. However, eq. (5.4) contains a term $\frac{1}{24} \ln^4 x$ in place of the term $\frac{1}{24} \ln^4(1-x)$ in $h/8$. Because only the sum $f_1(x) + f_1(1-x)$ appears in eq. (5.3), this swap of terms does not affect the total, but it does ensure that the branch cut origins are in the same place for all terms in f_1, and correspondingly it simplifies the symbol $\mathcal{S}(f_1)$. The functions f_2 and f_3 are related to f_1 by crossing: f_2 by the map $x \to 1/x$ ($s \leftrightarrow t$), and f_3 by the map $x \to -(1-x)/x$ ($s \to t \to u \to s$).

Curiously, the symbol of f_1 obeys a certain "final entry" condition recently observed to appear in the context of the remainder function for planar $\mathcal{N} = 4$ sYM amplitudes or Wilson loops [68, 69]. Furthermore, $f_1(x)$ obeys the generalization of this condition to functions, namely

$$\frac{df_1}{dx} = \frac{p(x)}{x(1-x)}, \tag{5.11}$$

where $p(x)$ is also a pure function, in this case

$$p(x) = \frac{1}{6} \ln^3 x + \frac{i\pi}{2} \ln^2 x. \tag{5.12}$$

When the finite remainder of the four-graviton amplitude in $\mathcal{N} = 8$ supergravity becomes available at three loops (for example by computing the integrals for one of the three available expressions for it [7, 8, 44]), it will be very interesting to see whether it can also be expressed in terms of pure functions of degree six with simple symbols. Perhaps the functions will even obey a relation like eq. (5.11).

We return now to two loops and $\mathcal{N} < 8$ supergravity. We present the finite remainder for $\mathcal{N} = 6$ supergravity, first in the s channel:

$$F_4^{(2),\mathcal{N}=6}\Big|_{s-\text{channel}} = F_4^{(2),\mathcal{N}=8}\Big|_{s-\text{channel}} + t\,u\left[f_{6,s}\left(\frac{-t}{s}\right) + f_{6,s}\left(\frac{-u}{s}\right)\right], \tag{5.13}$$

where

$$f_{6,s}(x) = f_{6,s;4}(x) + f_{6,s;3}(x) \tag{5.14}$$

gives the decomposition into a degree-four function,

$$\begin{aligned}
f_{6,s;4}(x) = {} & 20\,\text{Li}_4(x) - 4\,(1-x)\,\text{Li}_4\left(\frac{-x}{1-x}\right) - 12\ln x\,\text{Li}_3(x) + 4\ln^2 x\,\text{Li}_2(x) \\
& -4\,(1-x)\ln\left(\frac{x}{1-x}\right)\text{Li}_3(x) - \frac{1}{4}\,x\,(1-x)\left[\ln^4\left(\frac{x}{1-x}\right) + \pi^4\right] \\
& +\frac{\pi^2}{2}\left[x\ln x + (1-x)\ln(1-x)\right]^2 + \frac{2}{3}\,x\ln^4 x \\
& \qquad -\frac{2}{3}\ln x\ln(1-x)\left[(1+x)\ln^2 x - \frac{9}{4}\ln x\ln(1-x)\right] \\
& -4\,\zeta_2\left[x\,\text{Li}_2(x) + 2\ln x\ln(1-x)\right] - \frac{41}{2}\,\zeta_4 \\
& +i\,\pi\left[-12\,\text{Li}_3(x) + 8\ln x\,(\text{Li}_2(x) + \zeta_2) - \frac{2}{3}\,(1-2\,x)\ln x\,(\ln^2 x + \pi^2)\right. \\
& \qquad \left. +4\,(1-x)\ln^2 x\ln(1-x)\right], \tag{5.15}
\end{aligned}$$

and a degree-three one,

$$f_{6,s;3}(x) = -\frac{4}{3}\, x\, \ln x \left[\ln^2 x + 3\, \ln^2(1-x) + \pi^2\right]$$
$$+ 8 \left[\mathrm{Li}_3(x) - \ln x\, \mathrm{Li}_2(x) - \frac{\zeta_3}{2} + i\pi\, \zeta_2\right]. \qquad (5.16)$$

It has been observed [65] that at one loop the four-graviton amplitude in $\mathcal{N} = 6$ super-gravity has maximal transcendentality (degree two). This result extends to one-loop amplitudes with more gravitons, thanks to the absence of bubble integrals [42, 43]. However, the degree-three nature of eq. (5.16) shows that this property is broken at two loops. The breaking comes from both the two-loop amplitude $\mathcal{M}_4^{(2)}$, but also from the square of the one-loop amplitude $\mathcal{M}_4^{(1)}$, which has to be subtracted in eq. (5.2). As can be seen from eqs. (A.5) and (A.7), the one-loop $\mathcal{N} = 6$ amplitude has degree-two terms as well as degree-three terms at $\mathcal{O}(\epsilon)$; the former terms multiply the $1/\epsilon$ degree-one terms from the IR pole shown in eq. (5.1) to generate degree-three contributions to eq. (5.16). On the other hand, these contributions are purely logarithmic; the polylogarithmic terms in eq. (5.16) can be traced to $\mathcal{M}_4^{(2)}$. The complexity of the expressions (5.15) and (5.16), in terms of their power-law dependence on x, makes it unprofitable to try to separate the $\mathcal{N} < 8$ finite remainders into pure functions and to compute their symbols.

Because of the helicity assignment $(1^-, 2^-, 3^+, 4^+)$, the s-channel remainder is always symmetric under $t \leftrightarrow u$. However, in the t channel there is no such symmetry. The $\mathcal{N} = 6$ remainder in this channel is,

$$\left. F_4^{(2),\mathcal{N}=6}\right|_{t-\mathrm{channel}} = \left. F_4^{(2),\mathcal{N}=8}\right|_{t-\mathrm{channel}} + t\, u \left[f_{6,t;4}\!\left(\frac{-u}{t}\right) + f_{6,t;3}\!\left(\frac{-u}{t}\right)\right], \qquad (5.17)$$

where the degree-four part is

$$f_{6,t;4}(x) = -20\, \mathrm{Li}_4(1-x) - 20\, \mathrm{Li}_4\left(\frac{1-x}{-x}\right) - 4\, \frac{1+x}{1-x}\left(\mathrm{Li}_4(x) - \zeta_4\right) + 16\, \ln x\, \mathrm{Li}_3(1-x)$$
$$-12\, \ln(1-x)\left(\mathrm{Li}_3(x) - \zeta_3\right) + 4\, \frac{4-3\,x}{1-x}\, \ln x \left[\mathrm{Li}_3(x) - \zeta_3 + \frac{1}{2}\, \ln(1-x)\, \ln^2 x\right]$$
$$+4\, \ln x \left(\ln x - 2\, \ln(1-x)\right) \mathrm{Li}_2(1-x) + 4\, \zeta_2\, \frac{7-5\,x}{1-x}\, \mathrm{Li}_2(1-x)$$
$$-\frac{1}{6}\, \frac{5-8\,x}{(1-x)^2}\, \ln^4 x - 6\, \ln^2(1-x)\, \ln^2 x - 2\, \zeta_2\, \frac{13 - 19\,x + 12\,x^2}{(1-x)^2}\, \ln^2 x$$
$$+16\, \zeta_2\, \frac{1-2\,x}{1-x}\, \ln x\, \ln(1-x) \; + \; i\,\pi \left[16\, \mathrm{Li}_3(1-x) + \frac{4}{1-x}\left(\mathrm{Li}_3(x) - \zeta_3\right)\right.$$
$$-8\, \ln(1-x)\, \mathrm{Li}_2(1-x) + \frac{2}{3}\, \frac{1 - 2\,x + 4\,x^2}{(1-x)^2}\, \ln^3 x + 2\, \frac{2+x}{1-x}\, \ln^2 x\, \ln(1-x)$$
$$\left.-2\, \ln x\, \ln^2(1-x) - 4\, \zeta_2\, \frac{4-x}{1-x}\, \ln x\right], \qquad (5.18)$$

and the degree-three part is

$$f_{6,t;3}(x) = \frac{4}{3} \frac{x}{1-x} \ln x \left(\ln^2 x - 2\pi^2 \right) - 8 \left(\text{Li}_3(x) - \ln x \, \text{Li}_2(x) \right) \tag{5.19}$$

$$+ 4 \ln(1-x) \left(\ln^2 x - 4\zeta_2 \right) + 4 i\pi \left[\frac{x}{1-x} \ln^2 x - 2 \left(\text{Li}_2(1-x) + \zeta_2 \right) \right].$$

In the s channel, the finite remainder for $\mathcal{N} = 5$ supergravity at two loops is given by,

$$F_4^{(2),\mathcal{N}=5} \Big|_{s-\text{channel}} = F_4^{(2),\mathcal{N}=8} \Big|_{s-\text{channel}} + t \, u \left[f_{5,s}\left(\frac{-t}{s} \right) + f_{5,s}\left(\frac{-u}{s} \right) \right], \tag{5.20}$$

where

$$f_{5,s}(x) = f_{5,s;4}(x) + f_{5,s;3}(x) + f_{5,s;2}(x) \tag{5.21}$$

gives the decomposition into a degree-four function,

$$f_{5,s;4}(x) = -12 \left\{ (1-x) \left[\text{Li}_4\left(\frac{-x}{1-x} \right) - \zeta_2 \, \text{Li}_2(x) \right] - 2 \left(1 + x(1-x) \right) \text{Li}_4(x) \right.$$

$$+ \left[(2-x^2) \ln x - (1-x)^2 \ln(1-x) \right] \text{Li}_3(x) - \frac{1}{2} \ln^2 x \, \text{Li}_2(x) \right\}$$

$$- \frac{1}{16} x(1-x) \left[5 \ln^4 \left(\frac{x}{1-x} \right) + 34 \pi^2 \ln^2 \left(\frac{x}{1-x} \right) \right] + \frac{1}{2} x \ln^4 x$$

$$- (1-x) \ln^3 x \ln(1-x) + \frac{3}{4} \left(3 - 4x(1-x) \right) \ln^2 x \ln^2(1-x)$$

$$+ \frac{\pi^2}{2} \left[-(1-x)(3-2x) \ln^2 x + \frac{3}{2} \ln^2 \left(\frac{x}{1-x} \right) \right] - \frac{3}{8} \zeta_4 \left(72 + 323 \, x(1-x) \right)$$

$$+ i\pi \left\{ -12 \left[\left(1 + 2x(1-x) \right) \text{Li}_3(x) - \ln x \, \text{Li}_2(x) \right] \right.$$

$$- (1-2x)(1-x) \ln x \left(\ln^2 x + \pi^2 \right) + 3 \left(2(1-x)^2 + x \right) \ln^2 x \ln(1-x)$$

$$\left. + 2\pi^2 \ln x \right\}, \tag{5.22}$$

a degree-three function,

$$f_{5,s;3}(x) = 12 \left\{ (1+x^2) \left[\text{Li}_3(x) - \ln x \, \text{Li}_2(x) \right] - \frac{1}{2} \left(1 - x(1-x) \right) \ln^2 x \ln(1-x) \right\}$$

$$- 2x(1-x) \ln^3 x - 4\pi^2 x \ln x - 12 \zeta_3 + 12 \pi i \left[(1-x) \text{Li}_2(x) \right.$$

$$\left. + \frac{1}{4} \left(x \ln x + (1-x) \ln(1-x) \right)^2 + \frac{\zeta_2}{2} \left(2 - 3x(1-x) \right) \right], \tag{5.23}$$

and a degree-two function,

$$f_{5,s;2}(x) = -3 \left[\left(x \ln x + (1-x) \ln(1-x) \right)^2 - \pi^2 x(1-x) + 4\pi i x \ln x \right]. \tag{5.24}$$

The $\mathcal{N} = 5$ remainder function in the t channel is,

$$F_4^{(2),\mathcal{N}=5} \Big|_{t-\text{channel}} = F_4^{(2),\mathcal{N}=8} \Big|_{t-\text{channel}} + t \, u \left[f_{5,t;4}\left(\frac{-u}{t} \right) + f_{5,t;3}\left(\frac{-u}{t} \right) + f_{5,t;2}\left(\frac{-u}{t} \right) \right], \tag{5.25}$$

where the degree-four part is

$$f_{5,t;4}(x) = 12 \left\{ -\frac{1+x}{1-x}\left(\text{Li}_4(x) - \zeta_4\right) - 2\left(1 - \frac{x}{(1-x)^2}\right)\left[\text{Li}_4(1-x) + \text{Li}_4\left(\frac{1-x}{-x}\right)\right] \right.$$

$$-\left[\left(1 - \frac{2x}{(1-x)^2}\right)\ln(1-x) - \left(2 - \frac{x^2}{(1-x)^2}\right)\ln x\right]\left(\text{Li}_3(x) - \zeta_3\right)$$

$$+2\ln x \,\text{Li}_3(1-x) - \frac{1}{2}\ln x\left(\ln x - 2\ln(1-x)\right)\text{Li}_2(x)$$

$$+2\,\zeta_2 \frac{2-x}{1-x}\text{Li}_2(1-x) - \frac{8-21x}{96(1-x)^2}\ln^4 x + \frac{(1-2x)(5-x)}{12(1-x)^2}\ln^3 x \ln(1-x)$$

$$+\frac{1}{8}\left(3 + \frac{4x}{(1-x)^2}\right)\ln^2 x \ln^2(1-x) - \frac{\zeta_2}{4}\frac{10 - 12x + 11x^2}{(1-x)^2}\ln^2 x$$

$$+\zeta_2 \frac{1 - 5x + 2x^2}{(1-x)^2}\ln x \ln(1-x) \; + \; i\pi\left[\frac{\text{Li}_3(x) - \zeta_3}{(1-x)^2} + 2\,\text{Li}_3(1-x)\right.$$

$$-\ln(1-x)\,\text{Li}_2(1-x) + \frac{3}{8}\frac{x^2}{(1-x)^2}\ln^3 x$$

$$+\frac{1}{24}\frac{2+x}{1-x}\ln^2 x\left(\ln x + 6\ln(1-x)\right) - \frac{1}{4}\ln x \ln^2(1-x)$$

$$\left.\left. -\frac{\zeta_2}{2}\frac{4-x}{1-x}\ln x\right]\right\}, \tag{5.26}$$

the degree-three part is

$$f_{5,t;3}(x) = 12 \left\{ \frac{1+x}{1-x}\left[\text{Li}_3(1-x) - \ln(1-x)\,\text{Li}_2(1-x) - \frac{1}{2}\ln x \ln^2(1-x)\right] \right.$$

$$-\left(1 + \frac{x^2}{(1-x)^2}\right)\left(\text{Li}_3(x) - \ln x \,\text{Li}_2(x)\right) + \frac{x \ln^3 x}{6(1-x)^2} + \frac{1}{2}\ln^2 x \ln(1-x)$$

$$-\zeta_2\left[\frac{x(1-4x)}{(1-x)^2}\ln x + \ln(1-x)\right] - \zeta_3 \frac{1-2x}{(1-x)^2}$$

$$+i\pi\left[-\frac{1+(1-x)^2}{(1-x)^2}\text{Li}_2(1-x) - \frac{1}{(1-x)^2}\ln x \ln(1-x)\right.$$

$$\left.\left. +\frac{1}{2}\frac{x}{1-x}\left(\ln^2 x + 2\zeta_2\right)\right]\right\}, \tag{5.27}$$

and the degree-two part is

$$f_{5,t;2}(x) = -6\left[\left(\ln(1-x) + \frac{x}{1-x}\ln x\right)^2 - \pi^2 \frac{x}{1-x}\right.$$

$$\left. +\frac{2i\pi}{1-x}\left(\ln(1-x) + \frac{x}{1-x}\ln x\right)\right]. \tag{5.28}$$

The results for $\mathcal{N} = 4$ supergravity are the lengthiest of all. In the s channel, the finite remainder for $\mathcal{N} = 4$ supergravity at two loops is given by,

$$\left. F_4^{(2),\,\mathcal{N}=4}\right|_{s-\text{channel}} = \left. F_4^{(2),\,\mathcal{N}=8}\right|_{s-\text{channel}} + t\,u\left[f_{4,s}\left(\frac{-t}{s}\right) + f_{4,s}\left(\frac{-u}{s}\right)\right], \tag{5.29}$$

where

$$f_{4,s}(x) = f_{4,s;4}(x) + f_{4,s;3}(x) + f_{4,s;2}(x) + f_{4,s;1}(x) + f_{4,s;0}(x) \tag{5.30}$$

gives the decomposition into a degree-four function,

$$
\begin{aligned}
f_{4,s;4}(x) = {}& 4 \left(9 - 4\,x\,(1-x) \right) \mathrm{Li}_4(x) - 4\,(8 - 2\,x - 9\,x^2 + 8\,x^3)\,\ln x\,\mathrm{Li}_3(x) \\
& -4\,(1-x)\,(3 + 3\,x - 8\,x^2) \left[\mathrm{Li}_4\left(\frac{-x}{1-x}\right) + \zeta_2\,\mathrm{Li}_2\left(\frac{-x}{1-x}\right) - \ln(1-x)\,\mathrm{Li}_3(x) \right] \\
& +4 \left(2 - x\,(1-x) \right) \ln x\,(\ln x + 2\,i\,\pi)\,\mathrm{Li}_2(x) - 4\,i\,\pi \left(5 - 2\,x\,(1-x) \right) \mathrm{Li}_3(x) \\
& +\frac{1}{6}\,x\,(4 - 8\,x - 5\,x^2 + 21\,x^3 - 9\,x^4 + 3\,x^5) \\
& \times \left(\ln^4 x - 4\,\ln^3 x\,\ln(1-x) + 2\,\pi^2\,\ln^2 x + \frac{\pi^4}{2} \right) \\
& -\frac{2}{3}\,(2 - x\,(1-x))\,(1 - 3\,x)\left(\ln^2 x\,(\ln x - 6i\pi)\,\ln(1-x) + i\pi\,\ln x\,(\ln^2 x - \pi^2) \right) \\
& +\frac{2}{3}\,i\,\pi\,x\,\ln x \left[(2 - 13\,x + 8\,x^2)\,\ln^2 x + 3\,(4 + 10\,x - 5\,x^2)\,\ln x\,\ln(1-x) \right. \\
& \left. +(14\,(1 + x^2) - 19\,x)\,\pi^2 \right] \\
& +\frac{1}{2}\,(2 - x\,(1-x))\,(1 - x\,(1-x))^2\,\ln x\,\ln(1-x)\left(3\,\ln x\,\ln(1-x) - 2\,\pi^2 \right) \\
& -2\,\zeta_2\,x\,(8 - 16\,x + 11\,x^2)\,\ln^2 x - \frac{3}{2}\,\zeta_4\,(44 - 17\,x\,(1-x)), \tag{5.31}
\end{aligned}
$$

a degree-three function,

$$
\begin{aligned}
f_{4,s;3}(x) = {}& -\left(\frac{53}{6} + x^2 \right) \left[\mathrm{Li}_3\left(\frac{-x}{1-x}\right) - \ln\left(\frac{x}{1-x}\right) \mathrm{Li}_2\left(\frac{-x}{1-x}\right) \right] \\
& -\frac{1}{18}\,(59 - 12\,x^2 + 8\,x^3 + 54\,x^4 + 36\,x\,(1-x)^4) \\
& \times \ln x \left[\ln x \left(\ln x - 3\,\ln(1-x) \right) + \pi^2 \right] \\
& -\left(\frac{31}{3} - 12\,x + 10\,x^2 \right) \ln^2 x\,\ln(1-x) \\
& -i\,\pi \left[(1 - 2\,x)\,\ln^2 x - 9\,x\,(1-x)\left(\ln x\,\ln\left(\frac{x}{1-x}\right) + \frac{\pi^2}{2} \right) \right] \\
& +\frac{\zeta_2}{3}\,(59 - 156\,x + 132\,x^2)\,(\ln x + i\,\pi) - 33\,\zeta_3\,x\,(1-x), \tag{5.32}
\end{aligned}
$$

a degree-two function,

$$
\begin{aligned}
f_{4,s;2}(x) = {}& -\frac{6 - 7\,x + 4\,x^2}{2\,(1-x)}\,\ln x\,(\ln x + 2\,i\,\pi) \\
& +\left[3\left(1 + x^2\,(1-x)^2 \right) - \frac{13}{3}\,x\,(1-x) \right] \left[\ln x\,\ln\left(\frac{x}{1-x}\right) + \frac{\pi^2}{2} \right] \\
& +\frac{1}{3}\,\zeta_2\,x^2 \left(6\,x^2 - (1-x)\,(23 - 24\,x) \right), \tag{5.33}
\end{aligned}
$$

a degree-one function,

$$f_{4,s;1}(x) = -\frac{1}{3}\, x \left(4\,(1-x)^2 - x\,(1-2\,x) \right) (\ln x + i\,\pi)\,, \tag{5.34}$$

and a rational part,

$$f_{4,s;0}(x) = -\frac{1}{4} \left(2 + x\,(1-x) \right). \tag{5.35}$$

The $\mathcal{N} = 4$ remainder function in the t channel is,

$$F_4^{(2),\mathcal{N}=4}\bigg|_{t-\text{channel}} = F_4^{(2),\mathcal{N}=8}\bigg|_{t-\text{channel}} + t\,u \left[f_{4,t;4}\left(\frac{-u}{t}\right) + f_{4,t;3}\left(\frac{-u}{t}\right) + f_{4,t;2}\left(\frac{-u}{t}\right) \right.$$
$$\left. + f_{4,t;1}\left(\frac{-u}{t}\right) + f_{4,t;0}\left(\frac{-u}{t}\right) \right], \tag{5.36}$$

where the degree-four part is

$$f_{4,t;4}(x) = -4 \left(9 + \frac{4\,x}{(1-x)^2} \right) \left[\mathrm{Li}_4(1-x) + \mathrm{Li}_4\left(\frac{1-x}{-x}\right) \right.$$
$$\left. + \left(\ln(1-x) + \frac{i\pi}{2} \right) \left(\mathrm{Li}_3(x) - \zeta_3 \right) + \zeta_2\,\mathrm{Li}_2(1-x) \right]$$
$$-4\,\frac{1+x}{1-x} \left(3 - \frac{8\,x}{(1-x)^2} \right) \left[\mathrm{Li}_4(x) - \zeta_4 - \frac{i\pi}{2} \left(\mathrm{Li}_3(x) - \zeta_3 \right) \right]$$
$$+4\,\frac{8 - 22\,x + 11\,x^2 - 5\,x^3}{(1-x)^3}$$
$$\times \left[\ln x \left(\mathrm{Li}_3(x) - \zeta_3 \right) + \zeta_2 \left(2\,\mathrm{Li}_2(1-x) + \ln x\,\ln(1-x) \right) \right]$$
$$+4 \left(2 + \frac{x}{(1-x)^2} \right) \left\{ \left(2\,\ln(1-x) + 3\,i\,\pi \right) \left(\mathrm{Li}_3(x) - \zeta_3 \right) \right. \tag{5.37}$$
$$+ \left(\ln^2 x + 4\,\zeta_2 \right) \mathrm{Li}_2(1-x)$$
$$+2\,(\ln x + i\,\pi) \left(2\,\mathrm{Li}_3(1-x) - \ln(1-x)\,\mathrm{Li}_2(1-x) \right)$$
$$+i\,\pi \left[\left(\frac{1}{6} + \frac{x\,(1-x^2+x^3)}{2\,(1-x)^4} \right) \ln^3 x + \frac{2+x}{2\,(1-x)}\,\ln^2 x\,\ln(1-x) \right.$$
$$\left. \left. -\frac{1}{2}\,\ln x\,\ln^2(1-x) - \zeta_2\,\frac{4-x}{1-x}\,\ln x \right] \right\}$$
$$-\frac{9 - 48\,x + 104\,x^2 - 129\,x^3 + 87\,x^4 - 26\,x^5}{6\,(1-x)^6}\,\ln^2 x(\ln^2 x - 4\pi^2)$$
$$+\frac{2}{3}\,\frac{23 - 52\,x + 49\,x^2 - 17\,x^3}{(1-x)^3}\,\ln^3 x\,\ln(1-x) - \left(11 + \frac{5\,x}{(1-x)^2} \right) \ln^2 x\,\ln^2(1-x)$$
$$-4\,\zeta_2\,\frac{x\,(14 - 9\,x\,(1-x))}{(1-x)^3}\,\ln x\,\ln(1-x) - 2\,\zeta_2\,\frac{43 - 71\,x + 100\,x^2 - 25\,x^3}{(1-x)^3}\,\ln^2 x\,,$$

the degree-three part is

$$f_{4,t;3}(x) = -\left(\frac{56}{3} + \frac{2\,x}{(1-x)^2}\right)\left[\mathrm{Li}_3(x) - (\ln x + i\,\pi)\,\mathrm{Li}_2(x) - \frac{2}{3}\,i\,\pi^3 - \frac{5}{3}\,\pi^2\,\ln(1-x)\right]$$
$$+\frac{x\,(24 - 15\,x + 13\,x^2 - 32\,x^3 + 28\,x^4)}{9\,(1-x)^5}\,\ln x\,(\ln x + i\,\pi)\,(\ln x + 2\,i\,\pi)$$
$$+\left(\frac{1}{3} + 2\,\frac{5 - 4\,x\,(1-x)}{(1-x)^2}\right)\left[\ln(1-x)\left((\ln x + i\,\pi)^2 - 3\,\pi^2\right) - 2\,i\,\pi^3\right]$$
$$-\frac{2 + 10\,x - x^2}{(1-x)^2}\left[\pi^2\,(\ln x - 2\,\ln(1-x)) - i\,\pi\,(\ln^2 x + \pi^2)\right]$$
$$-\frac{1+x}{1-x}\left[(\ln x + i\,\pi)\,\ln(1-x)\,(\ln(1-x) + 2\,i\,\pi) + \frac{i\pi}{1-x}\,\ln^2 x\right]$$
$$+66\,\zeta_3\,\frac{x}{(1-x)^2}\,,\tag{5.38}$$

the degree-two part is

$$f_{4,t;2}(x) = -\frac{6 - 5\,x + 3\,x^2}{2\,(1-x)}\left(\ln^2 x - 2\,\ln(1-x)\,(\ln x + i\,\pi) + \pi^2\right)$$
$$+\left(3 + \frac{x\,(13\,(1-x)^2 + 9\,x)}{3\,(1-x)^4}\right)\left((\ln x + i\,\pi)^2 + 2\,\zeta_2\right)$$
$$+\frac{3\,(1 + x^2) - 8\,x}{2\,x}\,\ln(1-x)\,(\ln(1-x) + 2\,i\,\pi)$$
$$+\zeta_2\,\frac{14\,(1 + x^2) - 3\,x}{(1-x)^2}\,,\tag{5.39}$$

the degree-one part is

$$f_{4,t;1}(x) = -\frac{1}{3}\left[\ln\left(\frac{x}{1-x}\right) - \frac{1 + x + 4\,x^2}{(1-x)^3}\,(\ln x + i\,\pi)\right],\tag{5.40}$$

and the rational part is

$$f_{4,t;0}(x) = -\frac{(2-x)(1-2\,x)}{2\,(1-x)^2}\,.\tag{5.41}$$

6 Forward-scattering limit of the amplitudes

We now inspect the behavior of the two-loop supergravity amplitudes in the limit of small-angle, forward scattering, i.e. small momentum transfer at fixed center-of-mass energy. In particular, we want to verify the contributions from matter exchange, versus graviton exchange, in the forward-scattering limit. The results are sensitive to the helicity configuration, or for fixed helicity configuration, to which invariant is time-like and which of the two space-like invariants is becoming small.

We first consider configurations, or channels, for which the associated tree-level amplitudes have a pole at small momentum transfer. These configurations are dominated by the exchange of soft gravitons, and require helicity conservation along the forward-going graviton line. (They also require helicity conservation along the backward-going line, but

this second condition follows automatically from the first one, for the MHV amplitudes that we study.) To see the helicity conservation explicitly, we rewrite the tree amplitude as,

$$M_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+) = -is^2 \left(\frac{1}{t} + \frac{1}{u}\right) \left[\frac{\langle 1\,2\rangle}{[12]} \frac{[34]}{\langle 3\,4\rangle}\right]^2, \tag{6.1}$$

where the quantity in brackets is a pure phase. Expanding eq. (6.1) for small t at fixed s in the physical s channel ($s > 0$ kinematics), one gets a leading term of $\mathcal{O}(s^2/t)$ as $t \to 0$. Because the s-channel amplitude is symmetric under $t \leftrightarrow u$, one could also have taken the small u limit and gotten a pole-dominated behavior. However, in the physical t channel, one has to take u small in order to conserve helicity at both vertices. Then the leading tree-level behavior is $\mathcal{O}(t^2/u)$ as $u \to 0$. In contrast, the limit of small s in the t channel violates helicity conservation, and the tree amplitude is heavily power-law suppressed with respect to the dominant pole behavior, having a leading term of $\mathcal{O}(s^3/t^2)$ as $s \to 0$.

Interestingly, in the helicity-conserving channels described above, the two-loop remainders, $F_4^{(2)}$, for $\mathcal{N} = 4, 5, 6, 8$ supergravity amplitudes are all power-law suppressed. The forward-scattering leading behavior is thus fully determined by the square of the one-loop amplitude. Moreover, the dominant one-loop behavior is the same for all $4 \leq \mathcal{N} \leq 8$ supergravity amplitudes. Namely, at one loop as $t \to 0$ in the s channel, we have

$$\frac{\mathcal{M}_4^{(1)}(\epsilon)}{\mathcal{M}_4^{\text{tree}}} = \left(\frac{\kappa}{8\pi}\right)^2 (-2\pi i)\, s \left[\frac{1}{\epsilon} + \ln\left(\frac{s}{-t}\right) + \frac{\epsilon}{2}\ln^2\left(\frac{s}{-t}\right)\right] + \mathcal{O}(\epsilon^2, t), \tag{6.2}$$

and at two loops we have

$$\frac{\mathcal{M}_4^{(2)}(\epsilon)}{\mathcal{M}_4^{\text{tree}}} = \frac{1}{2}\left[\frac{\mathcal{M}_4^{(2)}(\epsilon)}{\mathcal{M}_4^{\text{tree}}}\right]^2 + \mathcal{O}(\epsilon, t). \tag{6.3}$$

Both equations hold for any number of supersymmetries. We also verified the analogous equations in the limit $u \to 0$ in the physical t channel ($t > 0$ kinematics).

As discussed in refs. [70–73], in the physical s channel only the s-channel ladder and crossed-ladder diagrams (shown in figure 1 with s flowing horizontally) contribute to the eikonal limit $t \to 0$. The limit is dominated by graviton exchanges because the coupling of a particle of spin J exchanged in the channel with small momentum transfer is proportional to E^J, where E is the center-of-mass energy. The s-channel ladder and crossed-ladder diagrams allow for the maximum number of attachments of gravitons to a hard line (one with energy of order E). This property explains why eqs. (6.2) and (6.3) are independent of the number of supersymmetries at high energy. The possible Reggeization of gravity, discussed in refs. [74, 75], remains an open question. However, this issue cannot be resolved by studying forward-scattering or eikonal limits. The t-channel ladder diagrams (obtained from figure 1 by rotating by 90° or permuting $1 \to 2 \to 3 \to 4 \to 1$), which should contribute to Reggeization, are subleading by powers of t/s because they have fewer attachments to the high-energy lines.

It is also interesting to consider the helicity-violating limit in which $s \to 0$ for $t > 0$ kinematics ($u \simeq -t$). As mentioned before, the associated tree-level amplitude is power-suppressed in this limit with respect to the dominant pole behavior; its leading behavior is $\mathcal{O}(s^3/t^2)$. In this limit, many of the finite-remainder expressions naively appear to blow up (see for instance eq. (5.18) as $x \to 1$). However, one can check in all cases that these spurious singularities cancel, and the leading behavior of the ratio of the one- and two-loop amplitudes to the tree amplitude is of $\mathcal{O}(t^L)$, $L = 1, 2$. Thus the one- and two-loop amplitudes never have a power $(1/s)$ enhancement over the tree amplitude in the helicity-violating limit, but are of the same order in s. (There is a $\ln(s)$ enhancement, but only in the pure $\mathcal{N} = 8$ supergravity terms, not in any of the matter contributions.)

7   Conclusions

In this paper, we have computed the full four-graviton two-loop amplitudes in $\mathcal{N} = 4, 5, 6$ supergravity. As expected, their IR divergences can be expressed in terms of the square of the corresponding one-loop amplitudes. The finite remainders were presented in a simple form. We also noted that the finite remainder in $\mathcal{N} = 8$ supergravity can be expressed in terms of permutations of a pure function $f_1(x)$ possessing a simple, one-term symbol.

The $\mathcal{N} = 4, 5, 6$ supergravity results were obtained using the double-copy property of gravity, which is a consequence of the recently-conjectured BCJ duality. The former property allowed us to combine the BCJ-satisfying $\mathcal{N} = 4$ sYM representation with known $\mathcal{N} = 0, 1, 2$ sYM gauge-theory amplitudes, in order to obtain the corresponding supergravity amplitudes, including all loop integrations.

Our task was vastly simplified by the fact that both sets of Yang-Mills amplitudes entering the double-copy formula were known, as well as by the lack of loop-momentum dependence for the $\mathcal{N} = 4$ sYM amplitudes in this case. As mentioned in the introduction, generic $\mathcal{N} < 8$ supergravity theories are expected to diverge at three loops (but not $\mathcal{N} = 5$ or 6 [38, 39]), because the counterterm R^4 is allowed by supersymmetry. It would thus be very interesting to compute explicit three-loop non-maximal supergravity amplitudes. If one computes in $\mathcal{N} \geq 4$ supergravity, then one can use the double-copy formula, because a BCJ-satisfying form exists for one of the two copies, namely the three-loop $\mathcal{N} = 4$ sYM amplitude [44].

However, for the other gauge-theory copy, $\mathcal{N} < 4$ sYM, the three-loop amplitudes are not known. Full-color amplitudes (including nonplanar terms) are required, and they should be known at the level of the integrand, because the BCJ form for the three-loop $\mathcal{N} = 4$ sYM amplitude contains loop-momentum dependence in its numerator factors. BCJ duality for $\mathcal{N} < 4$ sYM could help simplify these gauge-theory calculations. For instance, for the three-loop four-point $\mathcal{N} = 4$ sYM amplitude, the duality reduced the computation of the full amplitude to the evaluation of the maximal cut [76] of a single diagram [44]. Non-maximal amplitude calculations are not expected to be as simple, however. More powers of loop momentum will appear in the numerator factors, and graphs containing triangle and bubble subgraphs will also arise. It would be interesting nonetheless to investigate the simplifications that may be provided by BCJ duality in these cases.

Acknowledgments

A   One-loop expressions

In this appendix we give the $\mathcal{O}(\epsilon^0)$ and $\mathcal{O}(\epsilon^1)$ coefficients in the expansion of the one-loop four-graviton amplitude $\mathcal{M}_4^{(1)}$ in the various supergravity theories, because they enter the extraction of the two-loop finite remainder $F_4^{(2)}$ according to eq. (5.2). These amplitudes were first computed through $\mathcal{O}(\epsilon^0)$ in refs. [40, 41] for $\mathcal{N} = 4$ and $\mathcal{N} = 6$ supergravity (and the $\mathcal{N} = 5$ case is trivially related to $\mathcal{N} = 6$ at one loop). Expressions valid to all orders in ϵ, in terms of box, triangle and bubble integrals, can be found in ref. [43].

We write

$$\mathcal{M}_4^{(1)} = \left(\frac{\kappa}{8\pi}\right)^2 \left(\frac{4\pi\, e^{-\gamma}\, \mu^2}{|s|}\right)^\epsilon \mathcal{M}_4^{\text{tree}} \left[\frac{2}{\epsilon}\left(s\,\ln(-s) + t\,\ln(-t) + u\,\ln(-u)\right) + F_4^{(1)}\right], \quad \text{(A.1)}$$

where $\ln(-s) \to \ln|s| - i\pi$ in the s channel, $\ln(-t) \to \ln|t| - i\pi$ in the t channel. We will give the $\mathcal{O}(\epsilon^0)$ and $\mathcal{O}(\epsilon^1)$ coefficients for $F_4^{(1)}$ for each theory in these two channels.

For $\mathcal{N} = 8$ supergravity in the s channel we have,

$$F_4^{(1),\mathcal{N}=8}\Big|_{s-\text{channel}} = s\left[g_s\left(\frac{-t}{s}\right) + g_s\left(\frac{-u}{s}\right)\right], \quad \text{(A.2)}$$

where

$$\begin{aligned}
g_s(x) = {}& 2\,x\,(\ln x + i\pi)\,\ln(1-x) \\
& + \epsilon\left\{-2\,(2-x)\left[\text{Li}_3(x) - \frac{\zeta_3}{3} + (\ln x + i\pi)\,\text{Li}_2(1-x) + \frac{1}{2}\,\ln(1-x)\,(\ln^2 x - 4\,\zeta_2)\right]\right. \\
& \left. + \frac{1}{3}\,x\,\ln^3 x - i\pi\,(1-x)\,(\ln^2 x - 4\,\zeta_2) - \ln x\,(\ln x + i\pi)\,\ln(1-x)\right\}. \quad \text{(A.3)}
\end{aligned}$$

The t-channel result for the $\mathcal{N} = 8$ supergravity amplitude, divided by the tree, is obtained by exchanging s and t in the corresponding s-channel result. (This is not quite the case for $F_4^{(1),\mathcal{N}=8}$, due to the explicit factor of $|s|^{-\epsilon}$ extracted in eq. (A.1).)

We express the finite remainders for $\mathcal{N} < 8$ supergravities in terms of the one for $\mathcal{N} = 8$ supergravity. For $\mathcal{N} = 6$ supergravity we find, in the s channel,

$$F_4^{(1),\mathcal{N}=6}\Big|_{s-\text{channel}} = F_4^{(1),\mathcal{N}=8}\Big|_{s-\text{channel}} + s\left[g_{6,s}\left(\frac{-t}{s}\right) + g_{6,s}\left(\frac{-u}{s}\right)\right], \quad \text{(A.4)}$$

where

$$g_{6,s}(x) = \frac{1}{2}\, x\,(1-x)\left[\ln^2\left(\frac{x}{1-x}\right) + \pi^2\right] \tag{A.5}$$

$$+ \epsilon\left\{2\,x\,(1-x)\left[\mathrm{Li}_3(x) - \ln x\,\mathrm{Li}_2(x) - \frac{1}{3}\ln^3 x - \frac{\pi^2}{2}\ln x\right]\right.$$

$$\left. - \frac{1}{2}\Big[x\,(\ln x + i\pi) + (1-x)\,(\ln(1-x) + i\pi)\Big]^2 - \frac{\pi^2}{2}\,(1 - x\,(1-x))\right\}.$$

The t-channel result is

$$\left. F_4^{(1),\,\mathcal{N}=6}\right|_{t-\text{channel}} = \left. F_4^{(1),\,\mathcal{N}=8}\right|_{t-\text{channel}} + s\, g_{6,t}\!\left(\frac{-u}{t}\right), \tag{A.6}$$

where

$$g_{6,t}(x) = -\frac{x}{(1-x)^2}\ln x\,(\ln x + 2\,i\pi) + \epsilon\left\{\frac{2\,x}{(1-x)^2}\left[\mathrm{Li}_3(x) - \zeta_3 - (\ln x + i\pi)\,(\mathrm{Li}_2(x) - \zeta_2)\right.\right.$$

$$\left. + \frac{1}{3}\ln^3 x + 2\,\zeta_2\ln x + \frac{i\pi}{2}\ln^2 x - \ln x\,(\ln x + 2\,i\pi)\ln(1-x)\right]$$

$$\left. - \left[(\ln(1-x) + i\pi) + \frac{x}{1-x}\,(\ln x + i\pi)\right]^2 - \frac{1 - x\,(1-x)}{(1-x)^2}\,\pi^2\right\}. \tag{A.7}$$

The corresponding one-loop results for $\mathcal{N}=5$ supergravity are trivially related to those for $\mathcal{N}=6$, because the difference in field content from $\mathcal{N}=8$ is due to the same matter multiplet, just three copies instead of two. Therefore we have,

$$\left. F_4^{(1),\,\mathcal{N}=5}\right|_{s-\text{channel}} = \left. F_4^{(1),\,\mathcal{N}=8}\right|_{s-\text{channel}} + \frac{3}{2}\, s\left[g_{6,s}\!\left(\frac{-t}{s}\right) + g_{6,s}\!\left(\frac{-u}{s}\right)\right], \tag{A.8}$$

$$\left. F_4^{(1),\,\mathcal{N}=5}\right|_{t-\text{channel}} = \left. F_4^{(1),\,\mathcal{N}=8}\right|_{t-\text{channel}} + \frac{3}{2}\, s\, g_{6,t}\!\left(\frac{-u}{t}\right). \tag{A.9}$$

The s-channel one-loop finite remainder for $\mathcal{N}=4$ supergravity is given by

$$\left. F_4^{(1),\,\mathcal{N}=4}\right|_{s-\text{channel}} = \left. F_4^{(1),\,\mathcal{N}=8}\right|_{s-\text{channel}} + s\left[g_{4,s}\!\left(\frac{-t}{s}\right) + g_{4,s}\!\left(\frac{-u}{s}\right)\right], \tag{A.10}$$

where

$$g_{4,s}(x) = \Big[2 - x\,(1-x)\Big]\,g_{6,s}(x) + x\,(1-x)\left[(1-2\,x)\ln x + \frac{1}{2}\right]$$

$$- \frac{\epsilon}{6}\left\{\frac{x\,(3 - x^2\,(12 - 15\,x + 5\,x^2))}{1-x}\ln^2 x - 5\,x^2\,(1-x)^2\left[\ln x\,\ln(1-x) - \frac{\pi^2}{2}\right]\right.$$

$$+ i\pi\left[2\,\frac{x^2}{1-x}\,(7 - 12\,x + 6\,x^2)\ln x + 1\right]$$

$$\left. - 2\,x\,(6 - 24\,x + 17\,x^2)\ln x - 10\,x\,(1-x)\right\}. \tag{A.11}$$

The t-channel expression is

$$\left. F_4^{(1),\,\mathcal{N}=4}\right|_{t-\text{channel}} = \left. F_4^{(1),\,\mathcal{N}=8}\right|_{t-\text{channel}} + s\, g_{4,t}\!\left(\frac{-u}{t}\right), \tag{A.12}$$

where

$$g_{4,t}(x) = \left[2 + \frac{x}{(1-x)^2}\right] g_{6,t}(x) - \frac{x}{(1-x)^2}\left[\frac{1+x}{1-x}\left(\ln x + i\pi\right) + 1\right]$$
$$+ \frac{\epsilon}{6}\frac{x}{(1-x)^2}\left\{(3 - x\,(1-x))\left[\ln^2\left(\frac{x}{1-x}\right) + \pi^2\right] - \frac{5\,x}{(1-x)^2}\ln x\,(\ln x + 2i\pi)\right.$$
$$+ \frac{1 - x + 3\,x^2}{x^2}\ln(1-x)\,(\ln(1-x) + 2i\pi) + 2\frac{1 - 12\,x - 6\,x^2}{x\,(1-x)}\left(\ln x + i\pi\right)$$
$$\left. -2\frac{1 - 5\,x + x^2}{x}\ln\left(\frac{x}{1-x}\right) - 20\right\}. \tag{A.13}$$

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References

[1] M.H. Goroff and A. Sagnotti, *The Ultraviolet Behavior of Einstein Gravity*, *Nucl. Phys.* **B 266** (1986) 709 [INSPIRE].

[2] A. van de Ven, *Two loop quantum gravity*, *Nucl. Phys.* **B 378** (1992) 309 [INSPIRE].

[3] Z. Bern, J.J. Carrasco, L.J. Dixon, H. Johansson and R. Roiban, *The Ultraviolet Behavior of N = 8 Supergravity at Four Loops*, *Phys. Rev. Lett.* **103** (2009) 081301 [arXiv:0905.2326] [INSPIRE].

[4] Z. Bern, J.J.M. Carrasco, L.J. Dixon, H. Johansson and R. Roiban, *The Complete Four-Loop Four-Point Amplitude in N = 4 super-Yang-Mills Theory*, *Phys. Rev.* **D 82** (2010) 125040 [arXiv:1008.3327] [INSPIRE].

[5] Z. Bern, J.J.M. Carrasco, L.J. Dixon, H. Johansson and R. Roiban, *Simplifying Multiloop Integrands of Gauge Theory and Gravity Amplitudes*, to appear.

[6] Z. Bern, L.J. Dixon, D. Dunbar, M. Perelstein and J. Rozowsky, *On the relationship between Yang-Mills theory and gravity and its implication for ultraviolet divergences*, *Nucl. Phys.* **B 530** (1998) 401 [hep-th/9802162] [INSPIRE].

[7] Z. Bern, J.J. Carrasco, L.J. Dixon, H. Johansson, D. Kosower and R. Roiban, *Three-Loop Superfiniteness of N = 8 Supergravity*, *Phys. Rev. Lett.* **98** (2007) 161303 [hep-th/0702112] [INSPIRE].

[8] Z. Bern, J.J.M. Carrasco, L.J. Dixon, H. Johansson and R. Roiban, *Manifest Ultraviolet Behavior for the Three-Loop Four-Point Amplitude of N = 8 Supergravity*, *Phys. Rev.* **D 78** (2008) 105019 [arXiv:0808.4112] [INSPIRE].

[9] Z. Bern, L.J. Dixon, M. Perelstein and J. Rozowsky, *Multileg one loop gravity amplitudes from gauge theory*, *Nucl. Phys.* **B 546** (1999) 423 [hep-th/9811140] [INSPIRE].

[10] Z. Bern, N. Bjerrum-Bohr and D.C. Dunbar, *Inherited twistor-space structure of gravity loop amplitudes*, *JHEP* **05** (2005) 056 [hep-th/0501137] [INSPIRE].

[11] N. Bjerrum-Bohr, D.C. Dunbar, H. Ita, W.B. Perkins and K. Risager, *The No-triangle Hypothesis for N = 8 Supergravity*, *JHEP* **12** (2006) 072 [hep-th/0610043] [INSPIRE].

[12] N. Bjerrum-Bohr and P. Vanhove, *Absence of Triangles in Maximal Supergravity Amplitudes*, *JHEP* **10** (2008) 006 [arXiv:0805.3682] [InSPIRE].

[13] N. Bjerrum-Bohr and P. Vanhove, *On Cancellations of Ultraviolet Divergences in Supergravity Amplitudes*, *Fortsch. Phys.* **56** (2008) 824 [arXiv:0806.1726] [InSPIRE].

[14] S. Badger, N. Bjerrum-Bohr and P. Vanhove, *Simplicity in the Structure of QED and Gravity Amplitudes*, *JHEP* **02** (2009) 038 [arXiv:0811.3405] [InSPIRE].

[15] Z. Bern, J.J. Carrasco, D. Forde, H. Ita and H. Johansson, *Unexpected Cancellations in Gravity Theories*, *Phys. Rev.* **D 77** (2008) 025010 [arXiv:0707.1035] [InSPIRE].

[16] N. Arkani-Hamed, F. Cachazo and J. Kaplan, *What is the Simplest Quantum Field Theory?*, *JHEP* **09** (2010) 016 [arXiv:0808.1446] [InSPIRE].

[17] J.J.M. Carrasco and H. Johansson, *Five-Point Amplitudes in $N = 4$ super-Yang-Mills Theory and $N = 8$ Supergravity*, arXiv:1106.4711 [InSPIRE].

[18] P. Benincasa, C. Boucher-Veronneau and F. Cachazo, *Taming Tree Amplitudes In General Relativity*, *JHEP* **11** (2007) 057 [hep-th/0702032] [InSPIRE].

[19] H. Elvang and D.Z. Freedman, *Note on graviton MHV amplitudes*, *JHEP* **05** (2008) 096 [arXiv:0710.1270] [InSPIRE].

[20] N. Arkani-Hamed and J. Kaplan, *On Tree Amplitudes in Gauge Theory and Gravity*, *JHEP* **04** (2008) 076 [arXiv:0801.2385] [InSPIRE].

[21] A. Hall, *On-shell recursion relations for gravity*, *Phys. Rev.* **D 77** (2008) 124004 [arXiv:0803.0215] [InSPIRE].

[22] M. Bianchi, H. Elvang and D.Z. Freedman, *Generating Tree Amplitudes in $N = 4$ SYM and $N = 8$ SG*, *JHEP* **09** (2008) 063 [arXiv:0805.0757] [InSPIRE].

[23] M.T. Grisaru, *Two Loop Renormalizability of Supergravity*, *Phys. Lett.* **B 66** (1977) 75 [InSPIRE].

[24] S. Deser, J. Kay and K. Stelle, *Renormalizability Properties of Supergravity*, *Phys. Rev. Lett.* **38** (1977) 527 [InSPIRE].

[25] E. Tomboulis, *On the Two Loop Divergences of Supersymmetric Gravitation*, *Phys. Lett.* **B 67** (1977) 417 [InSPIRE].

[26] M.T. Grisaru, H. Pendleton and P. van Nieuwenhuizen, *Supergravity and the S Matrix*, *Phys. Rev.* **D 15** (1977) 996 [InSPIRE].

[27] M.T. Grisaru and H. Pendleton, *Some Properties of Scattering Amplitudes in Supersymmetric Theories*, *Nucl. Phys.* **B 124** (1977) 81 [InSPIRE].

[28] S. Ferrara and B. Zumino, *Structure of Conformal Supergravity*, *Nucl. Phys.* **B 134** (1978) 301 [InSPIRE].

[29] S. Deser and J. Kay, *Three loop counterterms for extended supergravity*, *Phys. Lett.* **B 76** (1978) 400 [InSPIRE].

[30] P.S. Howe and U. Lindström, *Higher order invariants in extended supergravity*, *Nucl. Phys.* **B 181** (1981) 487 [InSPIRE].

[31] R. Kallosh, *Counterterms in extended supergravities*, *Phys. Lett.* **B 99** (1981) 122 [InSPIRE].

[32] J. Broedel and L.J. Dixon, *R^4 counterterm and $E_{7(7)}$ symmetry in maximal supergravity*, *JHEP* **05** (2010) 003 [arXiv:0911.5704] [InSPIRE].

[33] H. Elvang and M. Kiermaier, *Stringy KLT relations, global symmetries and $E_{7(7)}$ violation*, *JHEP* **10** (2010) 108 [arXiv:1007.4813] [INSPIRE].

[34] E. Cremmer and B. Julia, *The $N = 8$ Supergravity Theory. 1. The Lagrangian*, *Phys. Lett.* **B 80** (1978) 48 [INSPIRE].

[35] E. Cremmer and B. Julia, *The SO(8) Supergravity*, *Nucl. Phys.* **B 159** (1979) 141 [INSPIRE].

[36] G. Bossard, C. Hillmann and H. Nicolai, *$E_{7(7)}$ symmetry in perturbatively quantised $N = 8$ supergravity*, *JHEP* **12** (2010) 052 [arXiv:1007.5472] [INSPIRE].

[37] N. Beisert, H. Elvang, D.Z. Freedman, M. Kiermaier, A. Morales and S. Stieberger, *$E_{7(7)}$ constraints on counterterms in $N = 8$ supergravity*, *Phys. Lett.* **B 694** (2010) 265 [arXiv:1009.1643] [INSPIRE].

[38] G. Bossard, P. Howe and K. Stelle, *On duality symmetries of supergravity invariants*, *JHEP* **01** (2011) 020 [arXiv:1009.0743] [INSPIRE].

[39] G. Bossard, P. Howe, K. Stelle and P. Vanhove, *The vanishing volume of $D = 4$ superspace*, *Class. Quant. Grav.* **28** (2011) 215005 [arXiv:1105.6087] [INSPIRE].

[40] D.C. Dunbar and P.S. Norridge, *Calculation of graviton scattering amplitudes using string based methods*, *Nucl. Phys.* **B 433** (1995) 181 [hep-th/9408014] [INSPIRE].

[41] D.C. Dunbar and P.S. Norridge, *Infinities within graviton scattering amplitudes*, *Class. Quant. Grav.* **14** (1997) 351 [hep-th/9512084] [INSPIRE].

[42] D.C. Dunbar, J.H. Ettle and W.B. Perkins, *Perturbative expansion of $\mathcal{N} < 8$ Supergravity*, *Phys. Rev.* **D 83** (2011) 065015 [arXiv:1011.5378] [INSPIRE].

[43] Z. Bern, C. Boucher-Veronneau and H. Johansson, *$\mathcal{N} \geq 4$ Supergravity Amplitudes from Gauge Theory at One Loop*, arXiv:1107.1935 [INSPIRE].

[44] Z. Bern, J.J.M. Carrasco and H. Johansson, *Perturbative Quantum Gravity as a Double Copy of Gauge Theory*, *Phys. Rev. Lett.* **105** (2010) 061602 [arXiv:1004.0476] [INSPIRE].

[45] Z. Bern, T. Dennen, Y.-t. Huang and M. Kiermaier, *Gravity as the Square of Gauge Theory*, *Phys. Rev.* **D 82** (2010) 065003 [arXiv:1004.0693] [INSPIRE].

[46] Z. Bern, J.J.M. Carrasco and H. Johansson, *New Relations for Gauge-Theory Amplitudes*, *Phys. Rev.* **D 78** (2008) 085011 [arXiv:0805.3993] [INSPIRE].

[47] Z. Bern, J. Rozowsky and B. Yan, *Two loop four gluon amplitudes in $N = 4$ super Yang-Mills*, *Phys. Lett.* **B 401** (1997) 273 [hep-ph/9702424] [INSPIRE].

[48] Z. Bern, A. De Freitas and L.J. Dixon, *Two loop helicity amplitudes for gluon-gluon scattering in QCD and supersymmetric Yang-Mills theory*, *JHEP* **03** (2002) 018 [hep-ph/0201161] [INSPIRE].

[49] R. Monteiro and D. O'Connell, *The Kinematic Algebra From the Self-Dual Sector*, *JHEP* **07** (2011) 007 [arXiv:1105.2565] [INSPIRE].

[50] J.J.M. Carrasco and H. Johansson, *Generic multiloop methods and application to $N = 4$ super-Yang-Mills*, *J. Phys.* **A 44** (2011) 454004 [arXiv:1103.3298] [INSPIRE].

[51] T. Sondergaard, *Perturbative Gravity and Gauge Theory Relations — A Review*, arXiv:1106.0033 [INSPIRE].

[52] D.-p. Zhu, *Zeros in scattering amplitudes and the structure of nonabelian gauge theories*, *Phys. Rev.* **D 22** (1980) 2266 [INSPIRE].

[53] C. Goebel, F. Halzen and J. Leveille, *Angular zeros of Brown, Mikaelian, Sahdev and Samuel and the factorization of tree amplitudes in gauge theories*, *Phys. Rev.* **D 23** (1981) 2682 [INSPIRE].

[54] R. Britto, F. Cachazo and B. Feng, *New recursion relations for tree amplitudes of gluons*, *Nucl. Phys.* **B 715** (2005) 499 [hep-th/0412308] [INSPIRE].

[55] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, *Phys. Rev. Lett.* **94** (2005) 181602 [hep-th/0501052] [INSPIRE].

[56] H. Kawai, D. Lewellen and S. Tye, *A Relation Between Tree Amplitudes of Closed and Open Strings*, *Nucl. Phys.* **B 269** (1986) 1 [INSPIRE].

[57] V. Del Duca, L.J. Dixon and F. Maltoni, *New color decompositions for gauge amplitudes at tree and loop level*, *Nucl. Phys.* **B 571** (2000) 51 [hep-ph/9910563] [INSPIRE].

[58] Z. Bern and D.A. Kosower, *The Computation of loop amplitudes in gauge theories*, *Nucl. Phys.* **B 379** (1992) 451 [INSPIRE].

[59] Z. Bern, A. De Freitas, L.J. Dixon and H. Wong, *Supersymmetric regularization, two loop QCD amplitudes and coupling shifts*, *Phys. Rev.* **D 66** (2002) 085002 [hep-ph/0202271] [INSPIRE].

[60] R. Akhoury, R. Saotome and G. Sterman, *Collinear and Soft Divergences in Perturbative Quantum Gravity*, *Phys. Rev.* **D 84** (2011) 104040 [arXiv:1109.0270] [INSPIRE].

[61] S. Weinberg, *Infrared photons and gravitons*, *Phys. Rev.* **140** (1965) B516.

[62] C.D. White, *Factorization Properties of Soft Graviton Amplitudes*, *JHEP* **05** (2011) 060 [arXiv:1103.2981] [INSPIRE].

[63] S.G. Naculich and H.J. Schnitzer, *Eikonal methods applied to gravitational scattering amplitudes*, *JHEP* **05** (2011) 087 [arXiv:1101.1524] [INSPIRE].

[64] S.G. Naculich, H. Nastase and H.J. Schnitzer, *Two-loop graviton scattering relation and IR behavior in N = 8 supergravity*, *Nucl. Phys.* **B 805** (2008) 40 [arXiv:0805.2347] [INSPIRE].

[65] A. Brandhuber, P. Heslop, A. Nasti, B. Spence and G. Travaglini, *Four-point Amplitudes in N = 8 Supergravity and Wilson Loops*, *Nucl. Phys.* **B 807** (2009) 290 [arXiv:0805.2763] [INSPIRE].

[66] A.B. Goncharov, *A simple construction of Grassmannian polylogarithms*, arXiv:0908.2238 .

[67] A.B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, *Classical Polylogarithms for Amplitudes and Wilson Loops*, *Phys. Rev. Lett.* **105** (2010) 151605 [arXiv:1006.5703] [INSPIRE].

[68] S. Caron-Huot, *Superconformal symmetry and two-loop amplitudes in planar $\mathcal{N} = 4$ super Yang-Mills*, arXiv:1105.5606 [INSPIRE].

[69] L.J. Dixon, J.M. Drummond and J.M. Henn, *Bootstrapping the three-loop hexagon*, *JHEP* **11** (2011) 023 [arXiv:1108.4461] [INSPIRE].

[70] D. Amati, M. Ciafaloni and G. Veneziano, *Higher Order Gravitational Deflection And Soft Bremsstrahlung In Planckian Energy Superstring Collisions*, *Nucl. Phys.* **B 347** (1990) 550 [INSPIRE].

[71] D. Amati, M. Ciafaloni and G. Veneziano, *Effective action and all order gravitational eikonal at Planckian energies*, *Nucl. Phys.* **B 403** (1993) 707 [INSPIRE].

[72] S.B. Giddings and R.A. Porto, *The Gravitational S-matrix*, *Phys. Rev.* **D 81** (2010) 025002 [arXiv:0908.0004] [INSPIRE].

[73] S.B. Giddings, M. Schmidt-Sommerfeld and J.R. Andersen, *High energy scattering in gravity and supergravity*, *Phys. Rev.* **D 82** (2010) 104022 [arXiv:1005.5408] [INSPIRE].

[74] H.J. Schnitzer, *Reggeization of $N = 8$ supergravity and $N = 4$ Yang-Mills theory*, hep-th/0701217 [INSPIRE].

[75] H.J. Schnitzer, *Reggeization of $N = 8$ supergravity and $N = 4$ Yang-Mills theory. II.*, arXiv:0706.0917 [INSPIRE].

[76] Z. Bern, J.J.M. Carrasco, H. Johansson and D. Kosower, *Maximally supersymmetric planar Yang-Mills amplitudes at five loops*, *Phys. Rev.* **D 76** (2007) 125020 [arXiv:0705.1864] [INSPIRE].

[77] D. Binosi and L. Theussl, *JaxoDraw: A Graphical user interface for drawing Feynman diagrams*, *Comput. Phys. Commun.* **161** (2004) 76 [hep-ph/0309015] [INSPIRE].

[78] D. Binosi, J. Collins, C. Kaufhold and L. Theussl, *JaxoDraw: A graphical user interface for drawing Feynman diagrams. Version 2.0 release notes*, *Comput. Phys. Commun.* **180** (2009) 1709 [arXiv:0811.4113] [INSPIRE].

[79] J. Vermaseren, *Axodraw*, *Comput. Phys. Commun.* **83** (1994) 45 [INSPIRE].