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Wall-crossing, Rogers dilogarithm, and the QK/HK correspondence

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Abstract: When formulated in twistor space, the D-instanton corrected hypermultiplet moduli space in $\mathcal{N} = 2$ string vacua and the Coulomb branch of rigid $\mathcal{N} = 2$ gauge theories on $\mathbb{R}^3 \times S^1$ are strikingly similar and, to a large extent, dictated by consistency with wall-crossing. We elucidate this similarity by showing that these two spaces are related under a general duality between, on one hand, quaternion-Kähler manifolds with a quaternionic isometry and, on the other hand, hyperkähler manifolds with a rotational isometry, equipped with a canonical hyperholomorphic circle bundle and a connection. We show that the transition functions of the hyperholomorphic circle bundle relevant for the hypermultiplet moduli space are given by the Rogers dilogarithm function, and that consistency across walls of marginal stability is ensured by the motivic wall-crossing formula of Kontsevich and Soibelman. We illustrate the construction on some simple examples of wall-crossing related to cluster algebras for rank 2 Dynkin quivers. In an appendix we also provide a detailed discussion on the general relation between wall-crossing and cluster algebras.

Keywords: Supersymmetric gauge theory, Solitons Monopoles and Instantons, D-branes, Superstring Vacua

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Contents

1 Introduction

In four-dimensional string vacua and gauge theories with $\mathcal{N} = 2$ supersymmetry, BPS states, more often than not, tend to decay across certain real codimension-one walls in the space of couplings or moduli z^a [1–4]. A useful way to keep track of the resulting dependence of the BPS index $\Omega(\gamma, z)$ on the moduli is to consider the same theory on $\mathbb{R}^3 \times S^1$: for large but finite radius of S^1, the low-energy effective action in three dimensions receives instanton corrections from four-dimensional BPS states whose Euclidean worldline winds around the circle [5–8]. Single-instanton corrections are weighted by the BPS index, and are therefore discontinuous across walls of marginal stability. Multi-instanton corrections ensure that the low energy effective action is nonetheless regular in the vicinity of the wall, both in the context of $\mathcal{N} = 2$ gauge theories [9] and $\mathcal{N} = 2$ string vacua [10].

In this work we elucidate the geometric origin of the striking similarity between the twistorial constructions of the moduli space in $\mathcal{N} = 2$ string and gauge theories on $\mathbb{R}^3 \times S^1$, and clarify some aspects of our earlier computation of D-instanton corrections to the hypermultiplet metric in $\mathcal{N} = 2$ string vacua [10]. In particular, we show that both metrics are related by an instance of the QK/HK correspondence, a general duality between quaternion-Kähler (QK) spaces with a quaternionic isometry, and hyperkähler (HK) spaces with a rotational isometry, equipped with a canonical hyperholomorphic circle bundle. We show that the BPS invariants determine a canonical hyperholomorphic circle bundle $\mathcal{P}$ on the HK space $\mathcal{M}'$ dual to the hypermultiplet moduli space $\mathcal{M}$, which in turn determines the D-instanton corrected QK metric on $\mathcal{M}$. The functional equations obeyed by the Rogers dilogarithm play an essential role in ensuring the consistency of the construction. To explain our results in more detail, we start with a brief recap of [9, 10].

$\mathcal{N} = 2$ field theory and symplectic geometry. For gauge theories with rigid $\mathcal{N} = 2$ supersymmetry in four dimensions, the discontinuity of the BPS index across walls of marginal stability is captured by the Kontsevich-Soibelman (KS) wall-crossing formula [11] (see e.g. [12] for a review of the KS formula and equivalent versions thereof). This claim can be justified by examining instanton corrections to the non-linear sigma model which describes the effective low-energy dynamics on $\mathbb{R}^3 \times S^1$ [9]. Due to supersymmetry, the target space $\mathcal{M}'$ of this sigma model must carry a hyperkähler (HK) metric of real dimension $4n$, where n is the rank of the gauge group. Instanton corrections to this metric are most conveniently formulated in terms of the twistor space $\mathcal{Z}' = \mathbb{C}P^1 \times \mathcal{M}'$ [13–15] (see section 2.1 for a summary of this approach). Viewing $\mathcal{Z}'$ as a fibration over $\mathbb{C}P^1$ with stereographic coordinate ζ, the fiber $\mathcal{M}'(\zeta)$ is isomorphic to a (twisted) complex symplectic torus $(\mathbb{C}^\times)^{2n}$ in complex structure $J(\zeta)$. Consequently, there exist canonical complex Darboux coordinates $\Xi' = (\eta^\Lambda, \mu_\Lambda)$ (which are functions of $(\zeta, x'^\mu) \in \mathbb{C}P^1 \times \mathcal{M}'$), such that the symplectic form $\omega'(\zeta)$ on $\mathcal{M}'(\zeta)$ is proportional to

$$\langle \mathrm{d}\Xi', \mathrm{d}\Xi' \rangle = 2\, \mathrm{d}\mu_\Lambda \wedge \mathrm{d}\eta^\Lambda \,, \tag{1.1}$$

and such that the torus action corresponds to integer translations of $(\eta^\Lambda, \mu_\Lambda)$ [16]. As argued in [9], due to instanton corrections the Darboux coordinates Ξ' are discontinuous across certain meridian lines on $\mathbb{C}P^1$, known as BPS rays, whose azimuthal angle is equal to the phase of the central charge $Z_\gamma(z)$ of the corresponding BPS state. It is natural to identify the symplectomorphism relating the Darboux coordinate systems Ξ' across the BPS ray ℓ_γ with the abstract operator U_γ appearing in the KS formula. Upon doing so, the KS formula ensures the consistency of the construction across walls of marginal stability, and hence the smoothness of the hyperkähler metric.

$\mathcal{N} = 2$ supergravity and contact geometry. For string vacua with local $\mathcal{N} = 2$ supersymmetry in four dimensions, a similar relation holds between the BPS spectrum in $D = 4$ and the target space metric of the non-linear sigma model $\mathcal{M}$ describing the vector multiplet moduli space in $D = 3$ (or, by T-duality, the hypermultiplet moduli space in $D = 4$), but with some important wrinkles [10]. First, unlike in field theory, the index $\Omega(\gamma)$ of BPS states in string vacua tends to grow exponentially, and the resulting instantonic series is divergent. This divergence is expected to be resolved by gravitational instantons (or, in the T-dual set-up, NS5-brane instantons), which have no counterpart in field theory [17]. These additional corrections are characterized by a non-trivial dependence on the NUT potential (or Neveu-Schwarz axion) σ, which parametrizes a certain compact direction in $\mathcal{M}$. However, for small string coupling they are exponentially suppressed compared to the BPS-instantons. In this paper we restrict to this weak coupling limit where gravitational (or NS5-) instantons are negligible, so that the metric on $\mathcal{M}$ has a Killing vector field ∂_σ, and we ignore the divergence of the instantonic series caused by the exponential growth of $\Omega(\gamma)$.

Second, unlike the rigid case, the target space $\mathcal{M}$ is no longer hyperkähler but rather quaternion-Kähler (see section 2.1.2 for a reminder of basis properties of QK manifolds). As a result, and in contrast to the hyperkähler situation, the twistor space $\mathcal{Z}$ of a quaternion-Kähler manifold $\mathcal{M}$ is a non-trivial fibration $\mathbb{C}P^1 \to \mathcal{Z} \to \mathcal{M}$, and the fiber $\mathcal{M}(t)$ of the opposite local fibration $\mathcal{M} \to \mathcal{Z} \to \mathbb{C}P^1$ is not a complex manifold (we denote by t the stereographic coordinate on $\mathbb{C}P^1$, at this point unrelated to the coordinate ζ in the HK construction). Nevertheless, the full twistor space $\mathcal{Z}$ does admit a canonical complex structure, in fact a complex *contact structure*, which serves as a replacement for the complex symplectic structure in the hyperkähler case [18]. In particular, there still exist canonical 'contact-Darboux' coordinates $(\Xi, \alpha) = (\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$ (which are locally functions of $(t, x^\mu) \in \mathbb{C}P^1 \times \mathcal{M}$), such that the complex contact structure is given by the kernel of the holomorphic one-form [19]

$$\mathcal{X} = \mathrm{d}\alpha + \xi^\Lambda \mathrm{d}\tilde{\xi}_\Lambda. \tag{1.2}$$

The Killing vector field ∂_σ on $\mathcal{M}$ lifts to a holomorphic vector field on $\mathcal{Z}$ which preserves the contact structure. The Darboux coordinates $(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$ can be chosen such that this vector field is the Reeb vector[1] ∂_α of the contact structure [20]. For this property to hold globally, the complex contact transformations $V^{[ij]}$ relating the Darboux coordinate systems on the overlap $\mathcal{U}_i \cap \mathcal{U}_j$ of two patches on $\mathcal{Z}$, must descend to complex symplectomorphisms on the *reduced twistor space* $\mathcal{Z}/\partial_\alpha$. Put differently, the complex contact transformations must decompose into a symplectomorphism $U^{[ij]}$ acting on the Darboux coordinates $\Xi = (\xi^\Lambda, \tilde{\xi}_\Lambda)$, and a shift of the Darboux coordinate α,

$$V^{[ij]}: \quad \Xi^{[i]} = U^{[ij]} \cdot \Xi^{[j]}, \qquad \alpha^{[i]} = \alpha^{[j]} - \frac{1}{4\pi^2} S^{[ij]}(\Xi^{[j]}), \tag{1.3}$$

such that the combined transformation preserves the contact one-form (1.2). The function $S^{[ij]}(\Xi^{[j]})$ is determined by the generating function of the symplectomorphism $U^{[ij]}$ (see [15, 19]).

As argued in [10] on the basis of duality, mirror symmetry and wall-crossing, it is again natural to identify the symplectomorphism $U^{[ij]}$ with the KS operator U_γ, such that the KS wall-crossing formula ensures the consistency of the construction of the reduced twistor space $\mathcal{Z}/\partial_\alpha$ (if not of the full twistor space $\mathcal{Z}$) across walls of marginal stability. In fact, upon trading the central charge function (or stability data) $Z_\gamma(z)$ and the BPS invariants $\Omega(\gamma, z)$ of the $\mathcal{N} = 2$ gauge theory with those relevant for the string compactification, identifying the canonical Darboux coordinates $\Xi'(\zeta)$ and $\Xi(t)$ and ignoring the contact Darboux coordinate $\alpha(t)$, the construction of the twistor space $\mathcal{Z}$ in [10] is formally isomorphic to the construction given in [9] of the HK metric on the Coulomb branch $\mathcal{M}'$ of $\mathcal{N} = 2$ gauge theories. As a result, the reduced twistor space $\mathcal{Z}/\partial_\alpha$ carries a HK metric, similar to the HK metric on $\mathcal{M}'$. There are however two notable differences: for prepotentials arising in Calabi-Yau compactifications, the HK metric on $\mathcal{Z}/\partial_\alpha$ has indefinite signature $(4, 4n-4)$; in addition, since the prepotential is homogeneous of degree 2, the HK metric always admits a Killing vector field $\partial_{\theta'}$ acting by R-symmetry rotations. Keeping these differences in mind, we henceforth denote by $\mathcal{M}'$ the space $\mathcal{Z}/\partial_\alpha$ equipped with the above Lorentzian-hyperkähler metric.

QK/HK correspondence, hyperholomorphic bundles and dilogarithm identities. The fact that the D-instanton corrected QK metric on $\mathcal{M}$ is captured by a (Lorentzian) hyperkähler metric on a different manifold $\mathcal{M}'$ is a consequence of a general geometric construction, which we dub the *QK/HK correspondence*. We study this correspondence for its own sake in section 2, and briefly outline it here (see figure 1 for orientation).

[1] Recall that the Reeb vector of a contact structure is the generator R of the kernel of $\mathrm{d}\mathcal{X}$, normalized such that $\mathcal{X}(R) = 1$.

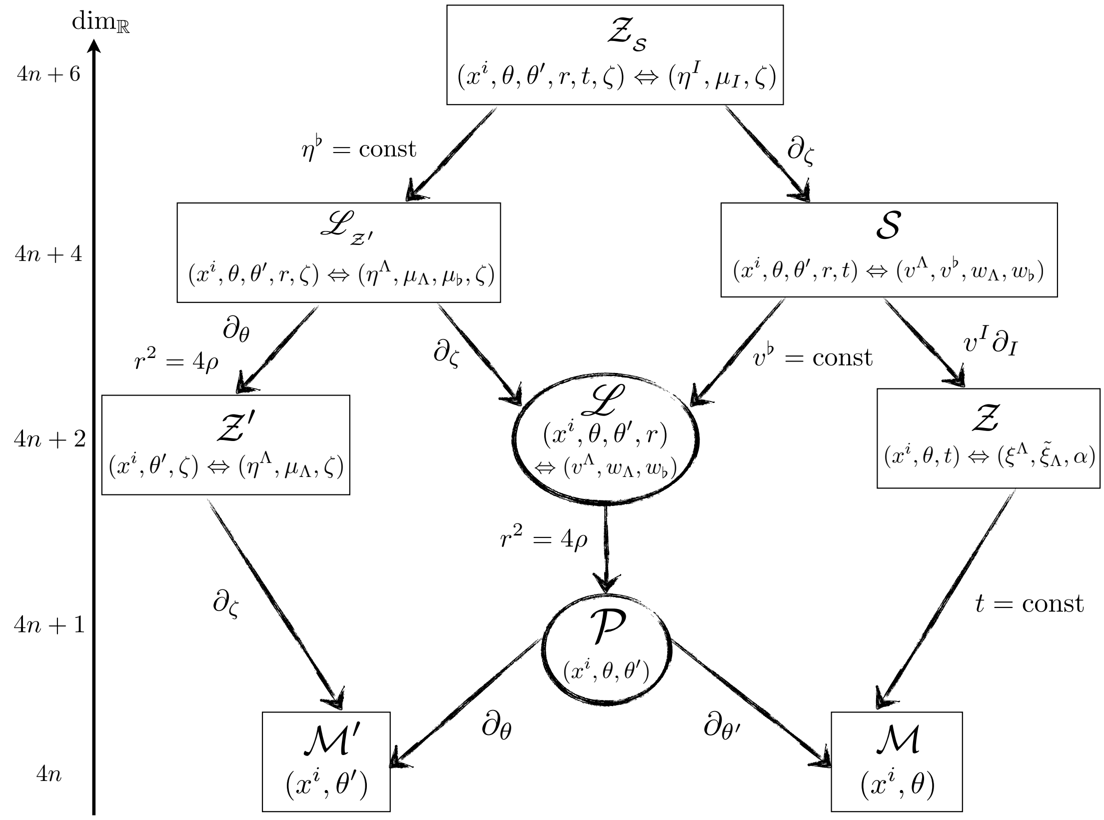

Figure 1. Overview of the QK/HK correspondence.

To any QK manifold $\mathcal{M}$ with a quaternionic circle action $\mathrm{U}(1)_A$, generated by a Killing vector field ∂_θ, the correspondence associates a HK manifold $\mathcal{M}'$ of the same dimension with an isometric circle action $\mathrm{U}(1)_R$, generated by a rotational Killing vector field $\partial_{\theta'}$, equipped with a canonical hyperholomorphic circle bundle $\mathcal{P}$ and a connection λ. This relation proceeds by lifting the circle action $\mathrm{U}(1)_A$ on $\mathcal{M}$ to a tri-holomorphic action on the Swann bundle $\mathcal{S}$ (or hyperkähler cone) of $\mathcal{M}$, and then constructing the hyperkähler quotient

$$\mathcal{M}' \equiv \mathcal{S}///\mathrm{U}(1)_A = \mathcal{S} \cap \{\vec{\mu} = \vec{m}\}/\mathrm{U}(1)_A\,, \tag{1.4}$$

where $\vec{\mu}$ is the moment map of $\mathrm{U}(1)_A$ action on $\mathcal{S}$, and $\vec{m}$ is a fixed unit norm vector (the direction of $\vec{m}$ is immaterial, since it rotates under the $\mathrm{SU}(2)_R$ isometric action on $\mathcal{S}$). The result is a hyperkähler manifold $\mathcal{M}'$ with positive signature if $\mathcal{M}$ has positive scalar curvature, or Lorentzian signature if $\mathcal{M}$ has negative scalar curvature. Moreover, $\mathcal{M}'$ admits an isometric $\mathrm{U}(1)_R$ action which rotates the complex structures (here $\mathrm{U}(1)_R$ is the subgroup of $\mathrm{SU}(2)_R$ which preserves the vector $\vec{m}$). The level set $\mathcal{P} = \mathcal{S} \cap \{\vec{\mu} = \vec{m}\}$ is generically a circle bundle $\mathcal{P}$ over $\mathcal{M}$, equipped with a connection λ given by the restriction of the Levi-Civita connection on $\mathcal{S}$. It is well known that this connection is hyperholomorphic, in the sense that its curvature $\mathscr{F} = \mathrm{d}\lambda$ is of type (1,1) in all complex structures. In a fixed complex structure, the circle bundle $\mathcal{P}$ can be extended to a complex line bundle $\mathscr{L}$ with first Chern class $c_1(\mathscr{L}) = \mathrm{d}\lambda/(2\pi)$. By the standard twistor construction, it can be further lifted to a holomorphic line bundle $\mathscr{L}_{\mathcal{Z}'}$ on the twistor space $\mathcal{Z}'$, with the property of being trivial along the $\mathbb{C}P^1$-fiber.

The QK/HK correspondence becomes particularly transparent at the level of the twistor spaces $\mathcal{Z}$ and $\mathcal{Z}'$. Each of them is determined by an open covering $\{\mathcal{U}_i\}$ (respectively $\{\mathcal{U}'_i\}$) and a set of transition functions $H^{[ij]}_{\mathrm{QK}}(\Xi, \alpha)$ (respectively $H^{[ij]}_{\mathrm{HK}}(\Xi', \zeta)$) describing contact (respectively symplectic) transformations between Darboux coordinates in different patches (see section 2.1). It turns out that the twistor spaces of a dual pair $(\mathcal{M}, \mathcal{M}')$ can be described by isomorphic coverings and identical transition functions

$$H^{[ij]}_{\mathrm{HK}}(\Xi) = H^{[ij]}_{\mathrm{QK}}(\Xi)\,. \tag{1.5}$$

This identification in (1.5) is meaningful since the $\mathrm{U}(1)_A$ isometry on $\mathcal{M}$ requires the transition functions $H^{[ij]}_{\mathrm{QK}}$ to be independent of α, while the $\mathrm{U}(1)_R$ isometry on $\mathcal{M}'$ requires $H^{[ij]}_{\mathrm{HK}}$ to be independent of the $\mathbb{C}P^1$ coordinate ζ. In such a scheme, the Darboux coordinates (Ξ, α) on $\mathcal{M}$ and (Ξ', Υ) on $\mathscr{L}_{\mathcal{Z}'} \to \mathcal{Z}'$ (where Υ is a holomorphic section of $\mathscr{L}_{\mathcal{Z}'}$) are simply related by[2]

$$\eta^{\Lambda}_{[i]}(\zeta) = \xi^{\Lambda}_{[i]}(t)\,, \qquad \mu^{[i]}_{\Lambda}(\zeta) = \tilde{\xi}^{[i]}_{\Lambda}(t)\,, \qquad \Upsilon^{[i]}(\zeta) = e^{-2\mathrm{i}\pi\alpha^{[i]}(t)}\,, \tag{1.6}$$

while the fiber coordinates t on $\mathcal{Z}$ and ζ on $\mathcal{Z}'$ are related by a phase rotation

$$t = \zeta\, e^{-\mathrm{i}\theta'}. \tag{1.7}$$

The transition functions for the line bundle $\mathscr{L}_{\mathcal{Z}'}$ are furthermore determined by the holomorphic section Υ via

$$f_{ij} = \frac{\Upsilon^{[i]}}{\Upsilon^{[j]}} = \exp\left(\frac{\mathrm{i}}{2\pi}\, S^{[ij]}(\Xi)\right), \tag{1.8}$$

where $S^{[ij]}$ is the same function which governs the shift of α under the contact transformation (1.3) (the consistency conditions on $V^{[ij]}$ spelled out in [19] ensure that f_{ij} is indeed a Cech cocycle). Thus, the geometry of the twistor space $\mathcal{Z}$ of the QK manifold $\mathcal{M}$ is completely encoded in a suitable line bundle over the twistor space $\mathcal{Z}'$ of the dual HK manifold $\mathcal{M}'$. As we shall see, one advantage of this dual point of view is that it provides a rigorous definition of the Darboux coordinates $\Xi(t, x^\mu), \alpha(t, x^\mu)$ on $\mathcal{Z}$, which is difficult on the QK side due to the non-trivial fibration of the twistor sphere $\mathbb{C}P^1$ over $\mathcal{M}$.

On the basis of this correspondence, specifying the D-instanton corrections to the QK metric on $\mathcal{M}$ is therefore equivalent to constructing a specific hyperholomorphic line bundle $\mathscr{L} \to \mathcal{M}'$, defined by transition functions of the form (1.8) such that the transformation (1.3) leaves the contact one-form (1.2) invariant. Although not phrased in this way, this problem was already addressed in [10, 21], where the generating function $S^{[ij]}$ of the symplectomorphism $U^{[ij]}$ was computed in terms of the Spence dilogarithm function $\mathrm{Li}_2(z)$ and the BPS invariants $\Omega(\gamma, z)$. While the formula obtained in [21] (reproduced

[2]The last relation in (1.6) must be altered in the presence of an anomalous dimension, see (2.50).

below in (3.27)) was rather unilluminating, one of the main new insights in the present work is to recognize that the shifted, symplectic invariant coordinate

$$\tilde{\alpha} = -2\alpha - \xi^\Lambda \tilde{\xi}_\Lambda \tag{1.9}$$

transforms in a much simpler fashion, namely $\tilde{\alpha} \mapsto \tilde{\alpha} + \Delta_\gamma \tilde{\alpha}$ with[3]

$$\Delta_\gamma \tilde{\alpha} = \frac{1}{2\pi^2}\, \Omega(\gamma)\, L\left(e^{-2\pi \mathrm{i} \langle \gamma, \Xi \rangle}\right), \tag{1.10}$$

where $L(z)$ is known as the Rogers dilogarithm,

$$L(z) = \mathrm{Li}_2(z) + \frac{1}{2} \log z \log(1-z)\,. \tag{1.11}$$

As we shall see, the functional identities obeyed by the Rogers dilogarithm, such as the five-term relation

$$L(x) - L\left(\frac{x(1-y)}{1-xy}\right) - L\left(\frac{y(1-x)}{1-xy}\right) + L(y) - L(xy) = 0\,, \quad 0 < x, y < 1, \tag{1.12}$$

are instrumental for ensuring the consistency of the gluing conditions (1.3) across walls of marginal stability. In section 3.4 we show that the semi-classical limit of the motivic wall-crossing formula of Kontsevich and Soibelman [11] (a much more powerful statement than the usual numerical KS formula) implies a set of functional identities for the Rogers dilogarithm which guarantee the existence of the line bundle $\mathscr{L}_{\mathcal{Z}'}$. We illustrate the construction on some simple examples of wall-crossing involving a finite number of BPS states on either side, where consistency is ensured by the famous pentagon identity and perhaps less known hexagon and octagon identities for the Rogers dilogarithm.

Relation with cluster algebras. Although this point of view is not essential for attaining our results, it should be mentioned that the special wall-crossing identities mentioned above occur naturally in the context of cluster algebras (specifically, cluster algebras associated to rank 2 Dynkin quivers). Indeed, cluster algebras provide a powerful method to generate wall-crossing identities and dilogarithm identities[4] [27–31]. In fact, our derivation in section 3.4 is a simple generalization of the analysis of [32] (in turn generalizing [33]), where it was pointed out that these functional identities can be obtained as semi-classical limits of quantum dilogarithm product identities for simply laced quivers established in [34–36]. Our derivation shows that the conjectural formulas of Nakanishi [29] for cluster algebras associated to non-simply laced quivers follow in the same manner from the motivic wall-crossing formula of Kontsevich and Soibelman [11]. In appendix B, we summarize some basic facts about cluster algebras, cluster transformations and their relation to wall-crossing, and derive the pentagon, hexagon, and octagon identities from periods of the cluster algebras associated to the A_2, B_2 and G_2 Dynkin quivers.

[3]This relation holds when the quadratic refinement $\sigma(\gamma)$ is equal to $+1$, see (3.29) for the general statement.

[4]Such dilogarithm identities appeared first in studies of two-dimensional integrable models [22–25], see e.g. [26] for a recent review. Cluster algebras can be regarded as an abstraction of the Y-systems appearing in these models.

Historical remarks. We close this introduction with some historical remarks and pointers to related literature. The basic tenets of the QK/HK correspondence were noticed by A. Neitzke and the third-named author in 2008 [16], in trying to understand the geometric meaning of the 'freezing procedure' used to extract the 'rigid limit' of local c-map spaces [37, 38]. After the main results in this article were obtained, we learned from A. Neitzke that the QK/HK correspondence had been independently discovered by A. Haydys [39] in 2007 (cf. [40, 41] for further accounts of Haydys' construction). For completeness, we shall incorporate some further insights gleaned from [39]. Moreover, the fact that the local and rigid c-map metrics are related by the QK/HK-correspondence (see section 2.4) appears to have been noticed independently by O. Macia and A. Swann [41]. Our construction of the hyperholomorphic line bundle $\mathscr{L}$ also seems closely related to work in progress by V. Fock and A. Goncharov in the context of quantization of cluster varieties [42]. Finally, A. Neitzke has independently constructed a hyperholomorphic line bundle over the Coulomb branch of $\mathcal{N} = 2$ gauge theories on $\mathbb{R}^3 \times S^1$ [43].[5] His construction appears to match ours in superconformal cases where the prepotential is homogeneous of degree 2.

Outline. This paper is organized as follows. In section 2 we present the generalities of the QK/HK correspondence, and illustrate it in the case of four-dimensional HK and QK metrics with a rotational isometry (which can all be described in terms of solutions to Toda equation) and in the case of c-map metrics. In particular, we show that the 'local' and rigid' c-map metrics, for the same choice of prepotential, are related by the QK/HK correspondence. In section 3, we specialize to the hypermultiplet moduli space of $\mathcal{N} = 2$ string vacua, and express the contact transformations (1.3) across BPS rays in terms of the Rogers dilogarithm. We show that the consistency of the construction across walls of marginal stability is ensured by the classical limit of the motivic wall-crossing formula. We illustrate this mechanism on simple examples of wall-crossing, related to cluster algebras of type A_2, B_2 and G_2. Appendix A recalls the definition and main properties of the Rogers dilogarithm and some of its variants. Finally, for completeness we review in appendix B some basic aspects of cluster algebras and their relation to wall crossing.

2 The QK/HK correspondence

In this section we present a general geometric correspondence between QK and HK manifolds with a rotational Killing vector field. More precisely, to any real $4n$-dimensional QK manifold $\mathcal{M}$ with a quaternionic isometry, we associate a HK manifold $\mathcal{M}'$ of the same dimension, equipped with a rotational Killing vector field and a canonical hyperholomorphic circle bundle $\mathcal{P} \to \mathcal{M}'$ with connection λ. In section 2.1 we start with a brief reminder on the twistorial description of HK and QK manifolds. In section 2.2 we construct $(\mathcal{M}', \mathcal{P})$ by lifting the quaternionic isometry on $\mathcal{M}$ to a tri-holomorphic action on the Swann bundle $\mathbb{R}^4/\mathbb{Z}_2 \to \mathcal{S} \to \mathcal{M}$, and then taking the hyperkähler quotient. In section 2.3, we study the correspondence in detail for QK/HK manifolds of real dimension 4, where both sides can be

[5] We are grareful to A. Neitzke for drawing our attention to [39, 40] and sharing with us an advance draft of [43].

described in terms of the same solution to Toda equation. Finally, in section 2.4 we work out the QK/HK correspondence for c-map spaces, and show that the rigid and the local c-map (including the one-loop deformation parameter) are related by the QK/HK correspondence.

2.1 Twistorial descriptions of HK and QK manifolds: a reminder

In this section we briefly recall some basic features and relevant formulae for the twistorial description of HK and QK manifolds (see [15, 19] for more details).

2.1.1 Hyperkähler manifolds and symplectic geometry

A dimension $4n$ Riemannian manifold $\mathcal{M}'$ is hyperkähler (HK) if it has restricted holonomy group $\mathrm{USp}(n) \subset \mathrm{SO}(4n)$. We shall also allow for Lorentzian-HK manifolds, whose holonomy group lies in $\mathrm{USp}(1, n-1) \subset \mathrm{SO}(4, 4n-4)$. In either case, $\mathcal{M}'$ can be described analytically in terms of its twistor space $\mathcal{Z}' = \mathcal{M}' \times \mathbb{C}P^1$. $\mathcal{Z}'$ admits a complex symplectic structure, more precisely a closed two-form

$$\omega' = \omega'_+ - \mathrm{i}\zeta\omega'_3 + \zeta^2\omega'_- \,, \qquad \omega'_\pm = -\frac{1}{2}\left(\omega'_1 \mp \mathrm{i}\omega'_2\right) \tag{2.1}$$

valued in the $\mathcal{O}(2)$ line bundle over $\mathbb{C}P^1$, and non-degenerate along the fibers of the projection $\mathcal{Z}' \to \mathcal{M}'$. Here ω'_i are the symplectic forms on $\mathcal{M}'$ associated to the complex structures J_i, $i = 1, 2, 3$, satisfying the quaternion algebra and ζ is a complex coordinate on $\mathbb{C}P^1 \cong \mathrm{SU}(2)/\mathrm{U}(1)$ parametrizing the complex structure

$$J(\zeta) = \mathrm{i}\,\frac{\bar\zeta - \zeta}{1 + |\zeta|^2}\, J_1 + \frac{\zeta + \bar\zeta}{1 + |\zeta|^2}\, J_2 + \frac{1 - |\zeta|^2}{1 + |\zeta|^2}\, J_3 \,. \tag{2.2}$$

In particular, the pull-back of ω' to $\mathcal{M}'$ is the Kähler form with respect to $J(\zeta)$.

Locally, on a patch $\mathcal{U}'_i$ of an open covering $\cup\mathcal{U}'_i$ of $\mathcal{Z}'$, there exist complex Darboux coordinates $(\eta^\Lambda_{[i]}, \mu^{[i]}_\Lambda)$ such that[6]

$$\omega' = \kappa_i \, \mathrm{d}\eta^\Lambda_{[i]} \wedge \mathrm{d}\mu^{[i]}_\Lambda \,. \tag{2.3}$$

The complex symplectic structure is conveniently encoded in a set of holomorphic functions $H^{[ij]}_{\mathrm{HK}}(\eta_{[i]}, \mu^{[j]}, \zeta)$ which generate symplectomorphisms between the Darboux coordinates $(\eta^\Lambda_{[i]}, \mu^{[i]}_\Lambda)$ on the overlap $\mathcal{U}'_i \cap \mathcal{U}'_j$ [15]. The functions $H^{[ij]}_{\mathrm{HK}}$ are subject to certain cocycle and reality conditions. Moreover, since ω' is defined only up to closed two-forms which vanish on the fiber of the projection $\mathcal{Z}' \to \mathcal{M}'$, $H^{[ij]}_{\mathrm{HK}}$ may in general depend explicitly on ζ. To obtain the metric, one needs to 'parametrize the twistor lines', i.e. to find the Darboux coordinates $(\eta^\Lambda, \mu_\Lambda)$ as functions of ζ and of the coordinates on $\mathcal{M}'$. The Darboux coordinates are determined by the following integral equations [21]

$$\begin{aligned} \eta^\Lambda_{[i]}(\zeta) &= x^\Lambda + \zeta^{-1} v^\Lambda - \zeta \bar v^\Lambda - \frac{1}{2} \sum_j \oint_{C_j} \frac{\mathrm{d}\zeta'}{2\pi\mathrm{i}\zeta'}\, \frac{\zeta' + \zeta}{\zeta' - \zeta}\, \partial_{\mu^{[j]}_\Lambda} H^{[ij]}_{\mathrm{HK}}(\zeta'), \\ \mu^{[i]}_\Lambda(\zeta) &= \varrho_\Lambda + \frac{1}{2} \sum_j \oint_{C_j} \frac{\mathrm{d}\zeta'}{2\pi\mathrm{i}\zeta'}\, \frac{\zeta' + \zeta}{\zeta' - \zeta}\, \partial_{\eta^\Lambda_{[i]}} H^{[ij]}_{\mathrm{HK}}(\zeta'), \end{aligned} \tag{2.4}$$

[6] We use a patch-dependent normalization, such that in the patch $\mathcal{U}'_+$ around the north pole of $\mathbb{C}P^1$ ($\zeta = 0$) it is given by $\kappa_+ = \mathrm{i}\zeta/4$.

where C_j is the contour surrounding the projection of $\mathcal{U}'_j$ on $\mathbb{C}P^1$ in the counterclockwise direction, while the complex variables v^Λ and real variables $x^\Lambda, \varrho_\Lambda$ serve as coordinates on $\mathcal{M}'$. The sums in (2.4) run over all patches including those which do not intersect with $\mathcal{U}'_i$ — the corresponding transition functions are obtained by analytic continuation and by applying the cocycle condition. Once the Darboux coordinates are known, the Kähler potential in complex structure $J_3 \equiv J(\zeta = 0)$ is given by the contour integral [21, 44]

$$K_{\mathcal{M}'} = \frac{1}{8\pi} \sum_j \oint_{C_j} \frac{\mathrm{d}\zeta}{\zeta} \left[H_{\rm HK}^{[ij]} - \eta_{[i]}^\Lambda \partial_{\eta_{[i]}^\Lambda} H_{\rm HK}^{[ij]} + \left(\zeta^{-1} v^\Lambda - \zeta \bar{v}^\Lambda \right) \partial_{\eta_{[i]}^\Lambda} H_{\rm HK}^{[ij]} \right]. \tag{2.5}$$

A set of complex coordinates on $\mathcal{M}'$ in this complex structure is given by the leading Laurent coefficients in the expansion of $\eta_{[+]}^\Lambda$ and $\mu_\Lambda^{[+]}$ at small ζ, namely v_Λ and

$$w_\Lambda \equiv \frac{\mathrm{i}}{2} \mu_\Lambda^{[+]}|_{\zeta=0} = \frac{\mathrm{i}}{2} \varrho_\Lambda + \frac{1}{8\pi} \sum_j \oint_{C_j} \frac{\mathrm{d}\zeta}{\zeta} \partial_{\eta_{[i]}^\Lambda} H_{\rm HK}^{[+j]}. \tag{2.6}$$

For HK manifolds with a rotational isometry, the transition functions $H_{\rm HK}^{[ij]}$ must have no explicit dependence on the fiber coordinate ζ.

2.1.2 Quaternion-Kähler manifolds and contact geometry

A $4n$-dimensional Riemannian manifold $\mathcal{M}$ is quaternion-Kähler (QK) if it has restricted holonomy group $\mathrm{USp}(n) \times \mathrm{SU}(2) \subset \mathrm{SO}(4n)$. The Ricci scalar R is then constant, and the curvature of the SU(2) part of the Levi-Civita connection, rescaled by $1/R$, provides a triplet of quaternionic 2-forms $\vec{\omega}$. While R can take either sign, hypermultiplet moduli spaces in $\mathcal{N} = 2$ supergravity or string theory models have $R < 0$. The degenerate limit $R \to 0$ recovers the case of HK manifolds discussed in section 2.1.1.

A QK manifold $\mathcal{M}$ can be described analytically in (at least) two equivalent ways, either in terms of its twistor space $\mathcal{Z}$, or in terms of the Swann bundle $\mathcal{S}$ (and its twistor space $\mathcal{Z}_\mathcal{S}$). The Swann bundle, or HK cone, is the total space $\mathcal{S}$ of the $\mathbb{R}^4/\mathbb{Z}_2$ bundle over $\mathcal{M}$ twisted with the SU(2) part of the Levi-Civita connection on $\mathcal{M}$ [45, 46]. It is a HK manifold[7] with a homothetic action of $\mathbb{R}^+$ and an isometric action of $\mathrm{SU}(2)_R$ (here R stands for R-symmetry). The twistor space $\mathcal{Z}_\mathcal{S}$ of the Swann bundle provides an analytic description of the QK space $\mathcal{M}$ in terms of a homogeneous complex symplectic structure. On the other hand, the twistor space $\mathcal{Z}$ is the total space of the $\mathbb{C}P^1$ bundle over $\mathcal{M}$ twisted with the *projectivized* $\mathrm{SU}(2)_R$ connection on $\mathcal{M}$. $\mathcal{Z}$ is a Kähler-Einstein space equipped with a canonical complex contact structure, given by the kernel of the one-form

$$Dt = \mathrm{d}t + p_+ - \mathrm{i}p_3 t + p_- t^2, \tag{2.7}$$

where t is a complex coordinate on $\mathbb{C}P^1$, and $p_\pm, p_3$ is the SU(2) part of the Levi-Civita connection on $\mathcal{M}$. The curvature of the latter is related to the triplet of covariantly constant two-forms by (we set the cosmological constant $\Lambda = -6$)

$$\omega_+ = -\frac{1}{2} \left(\mathrm{d}p_+ + \mathrm{i}p_+ \wedge p_3 \right), \qquad \omega_3 = -\frac{1}{2} \left(\mathrm{d}p_3 - 2\mathrm{i}p_+ \wedge p_- \right). \tag{2.8}$$

[7]More precisely, $\mathcal{S}$ carries a hyperkähler metric if $\mathcal{M}$ has positive scalar curvature, or a pseudo-hyperkähler metric with signature $(4, 4n)$ if $\mathcal{M}$ has negative scalar curvature.

Note that Dt is defined only projectively, as it rescales under SU(2) rotations. More precisely, it is valued in the $\mathcal{O}(2)$ line bundle on $\mathbb{C}P^1$ [18].

The two descriptions of $\mathcal{M}$ outlined above are closely related, since the twistor space $\mathcal{Z}$ is the quotient of the Swann bundle $\mathcal{S}$ by the $\mathbb{C}^\times$ action which combines the dilation and $U(1)_R \subset SU(2)_R$ rotation. The complex contact structure on $\mathcal{Z}$ is then simply the projectivization of the homogeneous complex symplectic structure on $\mathcal{S}$. The HKC metric on $\mathcal{S}$ and the Kähler-Einstein metric on $\mathcal{Z}$ are related to the quaternion-Kähler metric on $\mathcal{M}$ by [19]

$$\mathrm{d}s^2_{\mathcal{S}} = \mathrm{d}r^2 + r^2 \left[\frac{1}{4} \left(D\theta'\right)^2 + \mathrm{d}s^2_{\mathcal{Z}}\right], \qquad \mathrm{d}s^2_{\mathcal{Z}} = \frac{|Dt|^2}{(1+t\bar{t})^2} - \frac{1}{2}\,\mathrm{d}s^2_{\mathcal{M}}\,, \tag{2.9}$$

where (r, θ') parametrizes the $\mathbb{C}^\times$ fiber and

$$D\theta' = \mathrm{d}\theta' + \frac{\mathrm{i}}{1+t\bar{t}}\left[t\mathrm{d}\bar{t} - \bar{t}\mathrm{d}t - \mathrm{i}(1-t\bar{t})p_3 + 2tp_- - 2\bar{t}p_+\right]. \tag{2.10}$$

Moreover, the Kähler forms on $\mathcal{S}$ and $\mathcal{Z}$ are given by

$$\omega^3_{\mathcal{S}} = \frac{1}{2}\, r\,\mathrm{d}r \wedge D\theta' + r^2\,\omega_{\mathcal{Z}}\,, \qquad \omega_{\mathcal{Z}} = \frac{\mathrm{i}}{2}\,\frac{Dt\wedge D\bar{t}}{(1+t\bar{t})^2} - \frac{(1-t\bar{t})\omega_3 + 2\mathrm{i}t\omega_- - 2\mathrm{i}\bar{t}\omega_+}{2(1+t\bar{t})} \tag{2.11}$$

while the complex symplectic form on $\mathcal{S}$ is

$$\omega^+_{\mathcal{S}} = -\frac{r^2 e^{\mathrm{i}\theta'}}{2(1+t\bar{t})}\left[\omega_+ - \mathrm{i}t\omega_3 + t^2\omega_- - \left(\frac{\mathrm{i}}{2}\,D\theta' + \frac{\mathrm{d}r}{r}\right)\wedge Dt\right]. \tag{2.12}$$

Locally, on a patch of an open covering $\{\mathcal{U}_i\}$ of $\mathcal{Z}$, one can always find complex Darboux coordinates $(\xi^\Lambda_{[i]}, \tilde{\xi}^{[i]}_\Lambda, \alpha^{[i]})$ such that the contact one-form (2.7), suitably rescaled by a function $e^{\Phi^{[i]}}$, takes the form

$$\mathcal{X}^{[i]} \equiv 4\, e^{\Phi^{[i]}}\,\frac{Dt}{\mathrm{i}t} = \mathrm{d}\alpha^{[i]} + \xi^\Lambda_{[i]}\mathrm{d}\tilde{\xi}^{[i]}_\Lambda\,. \tag{2.13}$$

The function $\Phi^{[i]}$, which we refer to as the 'contact potential', is holomorphic along the $\mathbb{C}P^1$ fiber, and defined up to an additive holomorphic function on $\mathcal{U}_i$. It provides, among other things, a Kähler potential for the Kähler-Einstein metric on $\mathcal{Z}$ [19]:

$$K^{[i]}_{\mathcal{Z}} = \log\frac{4(1+t\bar{t})}{|t|} + \Phi^{[i]}\,. \tag{2.14}$$

Globally, the complex contact structure on $\mathcal{Z}$ can be specified by a set of generating functions $H^{[ij]}_{\rm QK}(\xi^\Lambda_{[i]}, \tilde{\xi}^{[j]}_\Lambda, \alpha^{[j]})$ for complex contact transformations between Darboux coordinates on overlaps $\mathcal{U}_i \cap \mathcal{U}_j$, subject to cocycle and reality conditions [19]. Unlike the HK case, the transition functions $H^{[ij]}_{\rm QK}$ are independent of the coordinate t on the $\mathbb{C}P^1$ fiber. In the case when the QK-manifold $\mathcal{M}$ has a quaternionic isometry ∂_θ, one may choose the Darboux coordinates such that the Killing vector lifts to the holomorphic action ∂_α. As a result, the transition functions $H^{[ij]}_{\rm QK}$ become independent of the coordinate $\alpha^{[j]}$, and the contact potential $\Phi^{[i]}$ becomes constant on $\mathbb{C}P^1$ [10].[8] In this case the Darboux coordinates are determined by the following system of integral equations:

$$\begin{aligned}
\xi^\Lambda_{[i]}(t) &= A^\Lambda + t^{-1}Y^\Lambda - t\,\bar Y^\Lambda - \frac{1}{2}\sum_j \oint_{C_j} \frac{\mathrm{d}t'}{2\pi\mathrm{i}t'}\,\frac{t'+t}{t'-t}\,\partial_{\tilde\xi^{[j]}_\Lambda} H^{[ij]}_{\rm QK},\\
\tilde\xi^{[i]}_\Lambda(t) &= B_\Lambda + \frac{1}{2}\sum_j \oint_{C_j} \frac{\mathrm{d}t'}{2\pi\mathrm{i}t'}\,\frac{t'+t}{t'-t}\,\partial_{\xi^\Lambda_{[i]}} H^{[ij]}_{\rm QK},\\
\alpha^{[i]}(t) &= B_\alpha + \frac{1}{2}\sum_j \oint_{C_j} \frac{\mathrm{d}t'}{2\pi\mathrm{i}t'}\,\frac{t'+t}{t'-t}\left(H^{[ij]}_{\rm QK} - \xi^\Lambda_{[i]}\partial_{\xi^\Lambda_{[i]}} H^{[ij]}_{\rm QK}\right) + 4\mathrm{i}c\log t.
\end{aligned} \tag{2.15}$$

Here the complex variables Y^Λ, up to an overall phase rotation which can be absorbed into a phase rotation of t, and the real variables $A^\Lambda, B_\Lambda, B_\alpha$ serve as coordinates on $\mathcal{M}$. It is convenient to fix the phase freedom in Y^Λ by requiring $Y^0 \equiv \mathcal{R}$ to be real. Moreover, B_α is related to the coordinate θ along the isometric direction by $\partial_{B_\alpha} = \frac{1}{4}\partial_\theta$. Finally, c is a real constant known as an anomalous dimension [19], which characterizes the singular behavior of the Darboux coordinate α at the north and south poles of $\mathbb{C}P^1$. It plays an important physical role in describing the one-loop correction to the hypermultiplet moduli space metric in type II string compactifications.[9]

The procedure to extract the metric from the solutions of (2.15) was outlined in [10, 19]. Similarly to $K_{\mathcal{M}'}$, the contact potential can be computed from the transition functions $H^{[ij]}_{\rm QK}$ and the solutions of (2.15) using

$$e^\Phi = \frac{1}{16\pi}\sum_j \oint_{C_j} \frac{\mathrm{d}t}{t}\left(t^{-1}Y^\Lambda - t\,\bar Y^\Lambda\right)\partial_{\xi^\Lambda_{[i]}} H^{[ij]}_{\rm QK} - c. \tag{2.16}$$

2.2 The QK/HK correspondence

We start from a QK manifold $\mathcal{M}$ of real dimension $4n$, with a quaternionic Killing vector field[10] which we denote by ∂_θ. We assume that the action of ∂_θ exponentiates to a circle action of a group which we denote by $\mathrm{U}(1)_A$ (A stands for axion). By the moment map construction [47, 48], the vector field ∂_θ lifts to a tri-holomorphic Killing vector field on the Swann bundle $\mathcal{S}$. We abuse notation and denote by the same symbol ∂_θ the vector field on $\mathcal{M}$ and its tri-holomorphic lift to $\mathcal{S}$, and by $\vec\mu$ its moment map.

Let us now perform the hyperkähler quotient of $\mathcal{S}$ by $\mathrm{U}(1)_A$. This proceeds by first restricting to a fixed level set of the moment map,

$$\mathcal{P}(\vec m) = \mathcal{S} \cap \{\vec\mu = \vec m\} \tag{2.17}$$

[8] As we shall see, e^Φ is related to the norm of the moment map of ∂_θ on $\mathcal{M}$, or to the moment map of $\partial_{\theta'}$ on the dual HK manifold $\mathcal{M}'$.

[9] In fact, there are other anomalous dimensions, called c_Λ, which introduce logarithmic singularities in $\tilde\xi_\Lambda$ and further affect α [19]. In this work we restrict ourselves to the case of vanishing c_Λ because they do not seem to play any role in physical applications. From the point of view of the QK/HK correspondence, it appears that their inclusion does not affect the dual HK metric, but does affect the hyperholomorphic connection.

[10] Recall that a quaternionic vector field is a vector field which preserves the canonical closed 4-form $\vec\omega\wedge\vec\omega$ on the QK manifold $\mathcal{M}$.

and then performing the usual Riemannian quotient by $\mathrm{U}(1)_A$. For $\vec{m} \neq 0$, the action of $\mathrm{U}(1)_A$ on $\mathcal{P}(\vec{m})$ has at most a finite stabilizer, and the quotient

$$\mathcal{M}' = \mathcal{P}(\vec{m})/\mathrm{U}(1)_A \tag{2.18}$$

is a hyperkähler orbifold. Due to the $\mathrm{SU}(2)_R$ and dilation symmetries on $\mathcal{S}$, the spaces $\mathcal{P}(\vec{m})$ (respectively, $\mathcal{M}'(\vec{m})$) for varying $\vec{m} \neq 0$ are canonically isomorphic, and the induced (respectively, quotient) metric depends only on the norm of $\vec{m}$, by an overall factor. We shall set $|\vec{m}| = 1$ in the following and omit the dependence on the vector $\vec{m}$.

As usual, the hyperkähler quotient may be decomposed in three steps: (i) impose $\mu_+ = 0$, where μ_+ is the complex valued projection of the moment map $\vec{\mu}$ on the plane orthogonal to $\vec{m}$, (ii) impose $\mu_3 = 1$, where $\mu_3 = \vec{\mu} \cdot \vec{m}$, and (iii) mod out by $\mathrm{U}(1)_A$. Step (i) defines a complex submanifold of $\mathcal{S}$ in complex structure $\vec{m} \cdot \vec{J}$. Steps (ii) and (iii) are equivalent to modding out by the complexification $\mathbb{C}^\times_A$ of $\mathrm{U}(1)_A$, which equips $\mathcal{M}'$ with a complex structure which we continue to denote by $\vec{m} \cdot \vec{J}$. Since $\mathcal{M}'$ is independent of the direction $\vec{m}$, it admits an S^2 worth of complex structures, and is indeed hyperkähler, with positive signature if $\mathcal{M}$ has positive scalar curvature, or Lorentzian signature is $\mathcal{M}$ has negative scalar curvature. Moreover, since $\mathcal{P}$ is invariant under the $\mathrm{U}(1)_R$ subgroup of $\mathrm{SU}(2)_R$ which leaves invariant the direction of the vector $\vec{m}$, and since $\mathrm{U}(1)_A$ commutes with $\mathrm{SU}(2)_R$, the action of $\mathrm{U}(1)_R$ descends to an isometric action on the quotient $\mathcal{M}'$. We denote by $\partial_{\theta'}$ the corresponding vector field. Denoting by J_3 the projection of $\vec{J}$ along $\vec{m}$ and by J_+ the orthogonal projection, the $\mathrm{U}(1)_R$ action rotates the complex structure according to

$$J_3 \mapsto J_3, \qquad J_+ \mapsto e^{\mathrm{i}\theta'} J_+ \,. \tag{2.19}$$

By construction, the level set $\mathcal{P}$ is the total space of a circle bundle over $\mathcal{M}'$. It is equipped with a canonical connection one-form λ, namely the restriction of the Levi-Civita connection on $\mathcal{S}$. It is a well-known fact that this connection is hyperholomorphic, i.e. that the curvature $\mathscr{F} = \mathrm{d}\lambda$ is of type (1,1) in all complex structures on $\mathcal{M}'$ [49–51]. The complex line bundle $\mathscr{L}$ associated to the circle bundle $\mathcal{P}$ in complex structure J_3 is isomorphic to the locus $\{\mu_+ = 0\} \cap \mathcal{S}$ referred to above. The connection λ on $\mathcal{P}$ defines a unitary connection on $\mathscr{L}$ which we continue to denote by the same symbol.

2.2.1 From QK to HK

Let us now perform the quotient procedure discussed above using the explicit formula for the metric on the Swann bundle (2.9). To this end, note that any QK metric with a quaternionic Killing vector ∂_θ can be written as

$$\mathrm{d}s^2_{\mathcal{M}} = \tau \left(\mathrm{d}\theta + \Theta\right)^2 + \mathrm{d}s^2_{\mathcal{M}/\partial_\theta}, \tag{2.20}$$

where Θ is a connection one-form and τ is a function on $\mathcal{M}$ invariant under ∂_θ.[11] We choose an SU(2) frame such that the Lie derivative of $\vec{p}$ with respect to ∂_θ vanishes, and such that the QK moment map $\vec{\mu}_{\mathcal{M}}$ for ∂_θ is aligned along the third axis. Denoting by $1/(4\rho^2)$ its

[11] Note that θ and Θ depend on the choice of coordinate but τ and $D\theta \equiv \mathrm{d}\theta + \Theta$ are defined unambiguously.

squared norm, so that $\vec{\mu}_{\mathcal{M}} = (0,0,1)/(2\rho)$, from (2.8) it follows that the component p_3 of the SU(2) connection on $\mathcal{M}$ must take the form

$$p_3 = -\frac{1}{\rho}\left(\mathrm{d}\theta + \Theta\right) + \Theta' \tag{2.21}$$

for a certain one-form connection Θ' on $\mathcal{M}/\partial_\theta$. For later convenience, we further trade the function τ in (2.20) for a function ν such that

$$\tau = \frac{\nu+\rho}{2\rho^2\nu}. \tag{2.22}$$

From (2.9), one then finds the following metric on the Swann bundle

$$\mathrm{d}s^2_{\mathcal{S}} = \mathrm{d}r^2 + r^2\left[\frac{1}{4}\left(D\theta'\right)^2 + \frac{|Dt|^2}{(1+t\bar{t})^2} - \frac{\tau}{2}\left(\mathrm{d}\theta + \Theta\right)^2 - \frac{1}{2}\,\mathrm{d}s^2_{\mathcal{M}/\partial_\theta}\right]. \tag{2.23}$$

We now perform the hyperkähler quotient of $\mathcal{S}$ with respect to the tri-holomorphic action of ∂_θ. From (2.11) and (2.12), one finds that the components of the moment map $\vec{\mu}$ on the Swann bundle are given by

$$\mu_3 = -\frac{1-t\bar{t}}{1+t\bar{t}}\,\frac{r^2}{4\rho}\,, \qquad \mu_+ = \frac{\mathrm{i}t\,e^{\mathrm{i}\theta'}}{1+t\bar{t}}\,\frac{r^2}{4\rho}\,. \tag{2.24}$$

Therefore, the level set $\mathcal{P}(\vec{m})$ in (2.17), with $\vec{m}$ a fixed unit norm vector, is obtained by setting

$$r = 2\sqrt{\rho}, \qquad t = \zeta e^{-\mathrm{i}\theta'}, \tag{2.25}$$

and holding ζ constant. As we shall see momentarily, ζ parametrizes the twistor fiber of $\mathcal{M}'$, and the last equality in (2.25) establishes the relation (1.7) between the $\mathbb{C}P^1$ coordinates on $\mathcal{Z}$ and $\mathcal{Z}'$. After completing squares, the restriction to $\mathcal{P}(\vec{m})$ of the metric on $\mathcal{S}$ can be written

$$\mathrm{d}s^2_{\mathcal{P}} = \mathrm{d}s^2_{\mathcal{M}'} - \frac{1}{\nu}\left(\mathrm{d}\theta + \lambda\right)^2\,, \tag{2.26}$$

where $\mathrm{d}s^2_{\mathcal{M}'}$ is a metric which is degenerate along ∂_θ and invariant under $\partial_{\theta'}$,

$$\mathrm{d}s^2_{\mathcal{M}'} = \frac{\mathrm{d}\rho^2}{\rho} + 4\rho\,|p_+|^2 + (\nu+\rho)\left(\mathrm{d}\theta' + \Theta'\right)^2 - 2\rho\,\mathrm{d}s^2_{\mathcal{M}/\partial_\theta} \tag{2.27}$$

and λ is the one-form

$$\lambda = \nu\left(\mathrm{d}\theta' + \Theta'\right) + \Theta\,. \tag{2.28}$$

Performing the quotient with respect to $\partial_{\theta'}$, the metric (2.27) gives the metric on the HK space $\mathcal{M}'$ dual to the QK space $\mathcal{M}$, while the one-form λ on the circle bundle $\mathcal{P}(\vec{m})$ is the hyperholomorphic connection afforded by the QK/HK correspondence. It is noteworthy that both the metric (2.27) and connection (2.28) are independent of the parameter ζ.

To check that the metric (2.27) is hyperkähler, one may construct the Kähler form $\omega'_3(\zeta)$ and complex symplectic form $\omega'_+(\zeta)$ on $\mathcal{M}'$ by restricting the corresponding forms (2.11), (2.12) on $\mathcal{S}$ to $\mathcal{P}(\vec{m})$. With some further efforts, one finds that $\omega_3'(\zeta), \omega_+'(\zeta)$ can be integrated to the one-forms

$$p_3'(\zeta) = \frac{(1-\zeta\bar{\zeta})p_3' + 2\mathrm{i}\zeta p_-' - 2\mathrm{i}\bar{\zeta}p_+'}{1+\zeta\bar{\zeta}}, \qquad p_+'(\zeta) = \frac{p_+' - \mathrm{i}\zeta p_3' + \zeta^2 p_-'}{1+\zeta\bar{\zeta}}, \tag{2.29}$$

where p_3', p_+' are related to the SU(2) connection on $\mathcal{M}$ by

$$p_3' = \rho\left(\mathrm{d}\theta' + p_3\right) + \mathrm{d}\theta, \qquad p_+' = \rho\, e^{\mathrm{i}\theta'} p_+ . \tag{2.30}$$

In the first of these equations, the last term $\mathrm{d}\theta$ was chosen so as to cancel the contraction $\partial_\theta \cdot p_3'$. In particular, these formulae identify the parameter ζ as the standard stereographic coordinate on the twistor space of $\mathcal{M}'$, as anticipated below (2.25). They also show that the Killing vector $\partial_{\theta'}$ on $\mathcal{M}'$ leaves the complex structure $J_3 = J(\zeta=0)$ on $\mathcal{M}'$ invariant, and rotate $J_\pm = J_\pm(\zeta = 0)$ according to (2.19). Furthermore, they identify the coordinate ρ as the moment map of the Killing vector $\partial_{\theta'}$ with respect to $\omega_3' = \omega_3'(\zeta=0)$. By the usual argument, this implies that the coefficient of $\mathrm{d}\rho^2$ in the metric (2.27) must be inversely related to the coefficient of $(\mathrm{d}\theta' + \Theta')^2$, namely

$$\mathrm{d}s^2_{\mathcal{M}'} = \frac{\mathrm{d}\rho^2}{\nu+\rho} + (\nu+\rho)\left(\mathrm{d}\theta' + \Theta'\right)^2 + \mathrm{d}s^2_{\mathcal{M}'//\partial_{\theta'}} \tag{2.31}$$

where $\mathrm{d}s^2_{\mathcal{M}'//\partial_{\theta'}}$ is the metric on the Kähler quotient of $\mathcal{M}'$ by $\mathrm{U}(1)_R$. In particular, the complex structure J_3 maps the one-form $\mathrm{d}\rho$ to $(\nu+\rho)\,(\mathrm{d}\theta' + \Theta')$, and as a result

$$\mathrm{i}(\partial - \bar{\partial})\rho = -(\nu+\rho)\left(\mathrm{d}\theta' + \Theta'\right), \tag{2.32}$$

where ∂ is the Dolbeault derivative in complex structure J_3. Combining (2.21), (2.28), (2.30) and (2.32), we see that the hyperholomorphic one-form λ can be rewritten as

$$\lambda = -\mathrm{i}(\partial - \bar{\partial})\rho - p_3', \tag{2.33}$$

with curvature

$$\mathscr{F} = \mathrm{d}\lambda = 2\mathrm{i}\,\partial\bar{\partial}\rho - \omega_3', \tag{2.34}$$

in agreement with eq. (14) in [39]. In particular, the hyperholomorphic curvature $\mathscr{F}$ can be derived from the Kähler potential

$$K_{\mathscr{L}} = 2\rho - K_{\mathcal{M}'}, \tag{2.35}$$

where $K_{\mathcal{M}'}$ is a Kähler potential for ω_3' in complex structure J_3.

2.2.2 From HK to QK

The above construction can be inverted as follows. Let $\mathcal{M}'$ be a HK manifold with a rotational Killing vector $\partial_{\theta'}$, which lifts to a $\mathrm{U}(1)_R$ circle action, and acts on the Kähler form ω_3' and complex symplectic form ω_+' (in a fixed complex structure J_3) via

$$\mathcal{L}_{\partial_{\theta'}}\omega_3' = 0, \qquad \mathcal{L}_{\partial_{\theta'}}\omega_\pm' = \pm\mathrm{i}\,\omega_\pm'. \tag{2.36}$$

We assume that ω'_3 lies in an integer cohomology class. Let ρ be the moment map of $\partial_{\theta'}$ with respect to ω'_3. As explained in [40], the two-form $\mathscr{F} \equiv 2\mathrm{i}\, \partial\bar{\partial}\rho - \omega'_3$, where ∂ is the Darboux derivative in complex structure J_3, is of type (1,1) in all complex structures, hence it defines a hyperholomorphic circle bundle $\mathcal{P}$ on $\mathcal{M}'$ with first Chern class $c_1(\mathcal{P}) = \mathscr{F}/(2\pi)$. Let λ be a connection on $\mathcal{M}'$, such that $\mathrm{d}\lambda = \mathscr{F}$.

Since ρ is the moment map for the circle action, the hyperkähler metric on $\mathcal{M}'$ can always be written in the form (2.31) for some function ν and connection 1-form Θ' with $\partial_{\theta'} \cdot \nu = \partial_{\theta'} \cdot \Theta' = 0$. Here, ρ and ν are defined up to an additive constant, but their sum $\rho + \nu$ is unambiguous. In addition, θ' and Θ' depend on the choice of coordinate but the combination $\mathrm{d}\theta' + \Theta'$ is unambiguous. Since λ satisfies (2.34), it can be written as (2.33) for some one-form p'_3 such that $\mathrm{d}p'_3 = \omega'_3$. Using (2.32), this can be further decomposed into

$$p'_3 = \rho \left(\mathrm{d}\theta' + \Theta'\right) - \Theta, \qquad \lambda = \nu \left(\mathrm{d}\theta' + \Theta'\right) + \Theta \tag{2.37}$$

for some connection Θ with $\partial_{\theta'} \cdot \Theta = 0$.

We now equip the circle bundle $\mathcal{P}$ with the metric (2.26), invariant under the $\mathrm{U}(1)_A$ action generated by the Killing vector ∂_θ. The $\mathrm{U}(1)_R$ isometric action on $\mathcal{M}'$ lifts to a $\mathrm{U}(1)_R$ isometric action on $\mathcal{P}$ generated by the Killing vector $\partial_{\theta'}$ (indeed, any other lift of the form $\partial_{\theta'} + \nu_0 \partial_\theta$ can be brought to this form by tuning the additive constant in ν). Using the formulae (2.20) and (2.27), the metric on $\mathcal{P}$ can be rewritten as

$$\mathrm{d}s^2_{\mathcal{P}} = -2\rho\, \mathrm{d}s^2_{\mathcal{M}} + \frac{\mathrm{d}\rho^2}{\rho} + 4\rho\, |p_+|^2 + \rho\, (\mathrm{d}\theta' + p_3)^2\,, \tag{2.38}$$

where the metric element $\mathrm{d}s^2_{\mathcal{M}}$ is degenerate along the direction $\partial_{\theta'}$, and expressed in terms of the HK metric on $\mathcal{M}'$ and connection λ via

$$\mathrm{d}s^2_{\mathcal{M}} = \frac{(\mathrm{d}\rho)^2}{4\tau\rho^4} + \tau\, (\mathrm{d}\theta + \Theta)^2 + 2\frac{|p'_+|^2}{\rho^2} - \frac{1}{2\rho}\mathrm{d}s^2_{\mathcal{M}'//\partial_{\theta'}}\,, \tag{2.39}$$

where the function τ is defined in terms of ν and ρ by (2.22). The metric element (2.39) defines a non-degenerate QK metric on the quotient $\mathcal{P}/\mathrm{U}(1)_R$, whose SU(2) connection $\vec{p}$ is obtained in terms of the Kähler and complex symplectic connections p'_3, p'_+ on $\mathcal{M}'$ by inverting (2.30). Note that the metric (2.39) on $\mathcal{P}/\mathrm{U}(1)_R$ differs from the standard Riemanniann quotient metric $\mathrm{d}s^2_{\mathcal{P}} - \rho\, (\mathrm{d}\theta' + p_3)^2$ due the second and third terms in (2.38), as well as the conformal rescaling by $-1/(2\rho)$. The fact that the metric (2.39) is quaternion-Kähler follows from the Swann bundle construction in section 2.2.1.

It is also important to emphasize that the QK metric (2.39) depends on the hyperholomorphic connection λ, and not only on its curvature. In particular, shifting $\lambda \mapsto \lambda + c\, \mathrm{d}\theta'$ where c is an arbitrary constant does not affect the curvature $\mathscr{F} = \mathrm{d}\lambda$, but does in general affect the functions ν and τ and therefore the dual QK metric, leading to a one-parameter family of inequivalent QK metrics. On the other hand, a shift of λ (and, simultaneously, an opposite shift of p'_3 such that (2.33) is preserved) by a closed one-form $\mathrm{d}\phi$ with $\partial_{\theta'} \cdot \mathrm{d}\phi = 0$ can be reabsorbed by a redefinition of the coordinate θ and one-form Θ, such that the dual QK space is unaffected.

Finally, to make contact with [39] we observe that (2.39) can alternatively be written as

$$\mathrm{d}s^2_{\mathcal{M}} = -\frac{1}{2\rho}\left[\mathrm{d}s^2_{\mathcal{M}'} - \frac{1}{\nu}\left(\mathrm{d}\theta+\lambda\right)^2\right] + \frac{1}{2\rho^2}\left[\left(\mathrm{d}\theta+\lambda-K_0\right)^2+\vec{K}^2\right], \tag{2.40}$$

where K_0 and $\vec{K}=\partial_{\theta'}\cdot\vec{\omega}'$ the one-forms obtained by contracting $\partial_{\theta'}$ with the HK metric and the triplet of symplectic forms $\vec{\omega}'$, respectively:

$$K_0 = (\nu+\rho)\left(\mathrm{d}\theta'+\Theta'\right), \qquad K_3=\mathrm{d}\rho, \qquad K_+=\mathrm{i}p'_+. \tag{2.41}$$

Eq. (2.40) reproduces eq. (18) in [39] (after correcting a misprint in this reference, namely ψ^2 should appear outside the bracket).

2.2.3 Darboux coordinates and transition functions

In this subsection we shall relate the twistorial descriptions of the QK manifold $\mathcal{M}$ and of the HK manifold $\mathcal{M}'$ together with the hyperholomorphic line bundle $\mathscr{L}$. The relation is provided by the formulae (1.5) and (1.6) previewed in the introduction. Here we establish them from the quotient procedure underlying the QK/HK correspondence.

Let us choose Darboux coordinates $(\xi^\Lambda,\tilde\xi_\Lambda,\alpha)$ on $\mathcal{Z}$ such that the isometry ∂_θ lifts to the holomorphic action ∂_α on $\mathcal{Z}$. Denoting by $\sqrt{v^\flat}$ the complex coordinate on the fiber of the Swann bundle $\mathbb{C}^\times\to\mathcal{S}\to\mathcal{Z}$ (following the notations of [19, 20]), one can choose the following combinations

$$v^\flat, \qquad v^\Lambda=v^\flat\xi^\Lambda, \qquad w_\Lambda=\frac{\mathrm{i}}{2}\,\tilde\xi_\Lambda, \qquad w_\flat=\frac{\mathrm{i}}{2}\,\alpha+2c\log v^\flat \tag{2.42}$$

as complex Darboux coordinates on $\mathcal{S}$ in complex structure J_3, such that $\omega^+_{\mathcal{S}}=\mathrm{d}v^I\wedge\mathrm{d}w_I$ with $I=\flat,0,1,\dots$. The coordinate $v^\flat$ is related to the moment map $\vec{\mu}$ by $\mu_+=-\mathrm{i}v^\flat$.

We now consider the hyperkähler quotient $\mathcal{M}'=\mathcal{S}///\partial_\theta$ at level $\vec{m}$. A set of complex coordinates on $\mathcal{M}'$ in complex structure J_3 can be obtained by restricting the complex coordinates v^Λ, w_Λ to the locus $\mu_+=\text{const}$. Thus, we can identify the complex Darboux coordinates $\Xi'=(\eta^\Lambda,\mu_\Lambda)$ on $\mathcal{M}'$ with the complex Darboux coordinates $\Xi=(\xi^\Lambda,\tilde\xi_\Lambda)$ on $\mathcal{Z}$. On the other hand, it was shown in [19] that the coordinate t on the $\mathbb{C}P^1$ fiber of $\mathcal{Z}$ is related to the coordinate ζ on the $\mathbb{C}P^1$ fiber of $\mathcal{Z}_{\mathcal{S}}$ by $t=(\bar\pi_2\zeta+\pi^1)/(-\bar\pi_1\zeta+\pi^2)$, where (π_1,π_2) are coordinates on the $\mathbb{C}^2/\mathbb{Z}_2$ fiber of the bundle $\mathcal{S}\to\mathcal{M}$. Since we work in complex structure J_3 on $\mathcal{S}$, we have $\pi^1=0$ and therefore $t=(\bar\pi_2/\pi^2)\zeta$. Using (3.43) in [19], we arrive at the following identifications of the Darboux coordinates on $\mathcal{Z}'$ and $\mathcal{Z}$,

$$\eta^\Lambda_{[i]}(\zeta)=\xi^\Lambda_{[i]}(t)\,, \qquad \mu^{[i]}_\Lambda(\zeta)=\tilde\xi^{[i]}_\Lambda(t)\,, \qquad t=\zeta\,e^{-\mathrm{i}\theta'}\,. \tag{2.43}$$

This identification implies that the patches on $\mathcal{Z}$ and $\mathcal{Z}'$ are in one-to-one correspondence and allows to conclude that the same transition functions and covering which define the complex contact structure on $\mathcal{Z}$ also define the complex symplectic structure on $\mathcal{Z}'$, i.e.

$$H^{[ij]}_{\mathrm{HK}}(\Xi)=H^{[ij]}_{\mathrm{QK}}(\Xi). \tag{2.44}$$

In particular, the fact that $\mathcal{M}'$ admits a rotational isometry follows from the independence of $H^{[ij]}_{\mathrm{HK}}$ on the $\mathbb{C}P^1$ coordinate ζ. Moreover, the coordinates $(v^\Lambda,x^\Lambda,\varrho_\Lambda)$ on $\mathcal{M}'$ which appear in the integral equations (2.4) for $\mathcal{M}'$ are naturally identified with the coordinates $(Y^\Lambda, A^\Lambda, B_\Lambda)$ appearing in the integral equations (2.15) for $\mathcal{M}$ through

$$v^\Lambda = Y^\Lambda e^{\mathrm{i}\theta'}, \qquad x^\Lambda = A^\Lambda, \qquad \varrho_\Lambda = B_\Lambda. \tag{2.45}$$

Together with the identification of the $\mathbb{C}P^1$ variables in (2.43), this allows to relate the coverings $\mathcal{U}_i$ and $\mathcal{U}'_i$ of the two dual twistor spaces.

Having identified the Darboux coordinates $\xi^\Lambda = \eta^\Lambda$, $\tilde{\xi}_\Lambda = \mu_\Lambda$ on $\mathcal{M}$ and $\mathcal{M}'$ via (2.45), it is natural to apply the same identifications to the contact Darboux coordinate $\alpha^{[i]}$ in (2.15), which gives

$$\alpha^{[i]}(\zeta) = B_\alpha + \frac{1}{2}\sum_j \oint_{C_j} \frac{\mathrm{d}\zeta'}{2\pi\mathrm{i}\zeta'}\,\frac{\zeta'+\zeta}{\zeta'-\zeta}\left(H^{[ij]}_{\mathrm{HK}} - \eta^\Lambda_{[i]}\partial_{\eta^\Lambda_{[i]}} H^{[ij]}_{\mathrm{HK}}\right) + 4c\,(\theta' + \mathrm{i}\log\zeta), \tag{2.46}$$

and ask about its meaning on the HK side. We first restrict to the case with no anomalous dimension, $c=0$. By construction, $\alpha^{[i]}$ is holomorphic in complex structure $J(\zeta)$ in the patch $\mathcal{U}'_i$, and on the overlap of two patches $\mathcal{U}'_i \cap \mathcal{U}'_j$, satisfies

$$S^{[ij]} \equiv \frac{1}{(2\pi)^2}\left(\alpha^{[j]} - \alpha^{[i]}\right) = \frac{1}{(2\pi)^2}\left(H^{[ij]}_{\mathrm{HK}} - \eta^\Lambda_{[i]}\partial_{\eta^\Lambda_{[i]}} H^{[ij]}_{\mathrm{HK}}\right). \tag{2.47}$$

Thus, $\Upsilon^{[i]} \equiv e^{-2\mathrm{i}\pi\alpha^{[i]}}$ can be viewed as a holomorphic section of a line bundle $\mathscr{L}_{\mathcal{Z}'}$ over the twistor space $\mathcal{Z}'$, with transition functions given by $S^{[ij]}$. The restriction of the line bundle $\mathscr{L}_{\mathcal{Z}'}$ to the fibers of the fibration $\mathcal{Z}' \to \mathcal{M}'$ is trivial, since it admits a nowhere-vanishing section Υ. By the usual twistor correspondence (see e.g. [52]) this descends to a hyperholomorphic line bundle $\mathscr{L}$ on $\mathcal{M}'$, and therefore to a hyperholomorphic circle bundle $\mathcal{P}$, whose fiber is parametrized by B_α. The connection λ on $\mathscr{L}$ can be obtained by requiring that the covariant derivative $D\Upsilon \equiv (\mathrm{d} + 8\pi\mathrm{i}\lambda)\Upsilon$ be of type (1,0) in complex structure $J(\zeta)$ [53]. Equivalently,

$$\lambda = \frac{1}{4}\left(\bar\partial^{(\zeta)}\alpha^{[i]} + \partial^{(\zeta)}\bar\alpha^{[i]}\right), \tag{2.48}$$

where $\partial^{(\zeta)}$ is the Dolbeault derivative in complex structure $J(\zeta)$. In particular, the r.h.s. of (2.48) is independent of ζ, since the connection λ is. Moreover, a Kähler potential[12] for $\mathscr{F} = \mathrm{d}\lambda$ in complex structure $J(\zeta)$ is given by the log-norm of Υ,

$$\mathscr{F} = \mathrm{i}\partial^{(\zeta)}\bar\partial^{(\zeta)} K^{[i]}_{\mathscr{L}}(\zeta), \qquad K^{[i]}_{\mathscr{L}}(\zeta) = \frac{1}{2}\,\mathrm{Im}\,\alpha^{[i]}. \tag{2.49}$$

Let us now discuss the effect of the anomalous dimension. In this case $\alpha^{[+]}$ and $\alpha^{[-]}$ are no longer regular in their respective patches, but have a logarithmic singularity $4\mathrm{i}c\log\zeta$ at $\zeta=0$ and $\zeta=\infty$, respectively. This singularity can however be cancelled by singling out one of the Darboux coordinates, say η^0, and defining

$$\Upsilon^{[i]} \equiv (\eta^0_{[i]})^{8\pi\, c^{[i]}} e^{-2\mathrm{i}\pi\alpha^{[i]}}, \tag{2.50}$$

[12] It should be noted that $\frac{1}{2\mathrm{i}}(\partial^{(\zeta)} - \bar\partial^{(\zeta)})K^{[i]}_{\mathscr{L}}(\zeta)$ in general differs from λ by a closed, ζ-dependent one-form.

where $c^{[+]} = -c^{[-]} = c$ and $c^{[i]} = 0$ otherwise. This defines a section of a holomorphic line bundle $\mathscr{L}_{\mathcal{Z}'}$ on $\mathcal{Z}'$ which is trivial along the real twistor lines, and therefore again a hyperholomorphic curvature $\mathscr{F} = \mathrm{d}\lambda$ on $\mathcal{M}'$ with Kähler potential

$$K_{\mathscr{L}}^{[i]}(\zeta) = \frac{1}{2}\,\mathrm{Im}\left[\alpha^{[i]} + 4\mathrm{i}c^{[i]}\log\eta^{0}_{[i]}\right]. \tag{2.51}$$

On the other hand, comparing (2.21) with the expression following from (2.13), one finds that the moment map ρ of the $\mathrm{U}(1)_R$ action on $\mathcal{M}'$ coincides with the contact potential on $\mathcal{M}$, $\rho = e^{\Phi}$. Plugging these results into (2.35), one obtains, in particular, that a Kähler potential for the HK metric on $\mathcal{M}'$ in complex structure J_3 is given by

$$K_{\mathcal{M}'}(0) = 2e^{\Phi} - \frac{1}{2}\lim_{\zeta\to 0}\,\mathrm{Im}\left[\alpha - 4\mathrm{i}c\log\zeta\right] - c\log v^0\bar{v}^0\,. \tag{2.52}$$

This indeed agrees with (2.5), (2.15) and (2.16) up to a Kähler transformation given by the last term.

Thus, we see that the twistorial description of the QK manifold $\mathcal{M}$ is completely equivalent to the twistorial description of the HK manifold $\mathcal{M}'$ endowed with a hyperholomorphic circle bundle $\mathcal{P} \to \mathcal{M}'$ with connection (2.33). The advantage of the description in terms of $(\mathcal{M}', \mathcal{P})$ is that the twistor space $\mathcal{Z}'$ is trivially fibered over $\mathbb{C}P^1$, so that the Darboux coordinates (2.4) are valid globally on $\mathcal{Z}'$, whereas $\mathcal{Z}$ is a non-trivial fibration by $\mathbb{C}P^1$'s, and therefore does not admit such coordinates globally. In the rest of this paper, we shall use t and ζ interchangeably for the twistor coordinate on $\mathcal{Z}'$, and similarly Ξ and Ξ' for the Darboux coordinates on $\mathcal{M}'$, keeping in mind the identifications (2.43).

2.3 The QK/HK correspondence in one quaternionic dimension

In this subsection we consider the QK/HK correspondence for one-dimensional quaternionic manifolds. Recall that in one quaternionic dimension, HK and QK manifolds correspond to self-dual Einstein spaces with zero and non-zero cosmological constant, respectively. Moreover, the triplet of hyperkähler forms is self-dual, while hyperholomorphic connections are connections with anti-self dual curvature. In addition, self-dual Einstein metrics with one rotational Killing vector field are classified by solutions of the continual Toda equation

$$\partial_z\partial_{\bar z}T + \partial_\rho^2\,\mathrm{e}^T = 0\,. \tag{2.53}$$

We shall see that the QK/HK correspondence relates QK and HK manifolds associated to the same solution of (2.53).

2.3.1 Tod Ansatz for one-dimensional QK manifolds with one isometry

On the QK side, self-dual Einstein metrics with one Killing vector field can be cast locally into the form of Tod's Ansatz [54]

$$\mathrm{d}s^2_{\mathcal{M}} = \frac{1}{2}\left[\frac{P}{\rho^2}\left(\mathrm{d}\rho^2 + 4\mathrm{e}^T\mathrm{d}z\mathrm{d}\bar z\right) + \frac{1}{P\rho^2}\left(\mathrm{d}\theta + \Theta\right)^2\right], \tag{2.54}$$

where $(\rho, z, \bar{z}, \theta)$ are local coordinates, with ∂_θ coresponding to the Killing vector field. Here T is a solution of the Toda equation (2.53), $P \equiv 1 - \frac{1}{2}\,\rho\partial_\rho T$, and Θ is a connection one-form such that

$$\mathrm{d}\Theta = \mathrm{i}(\partial_z P \mathrm{d}z - \partial_{\bar{z}} P \mathrm{d}\bar{z}) \wedge \mathrm{d}\rho - 2\mathrm{i}\,\partial_\rho(P\mathrm{e}^T)\mathrm{d}z \wedge \mathrm{d}\bar{z}\,. \tag{2.55}$$

This condition is integrable by virtue of (2.53) and gauge transformations of the one-form Θ can be reabsorbed into redefinitions of the coordinate θ. The self-dual part of the Levi-Civita connection can be chosen as

$$p_3 = -\frac{1}{\rho}\,(\mathrm{d}\theta + \Theta) + \Theta'\,, \qquad p_+ = \frac{e^{T/2}}{\rho}\,\mathrm{d}z = (p_-)^*\,, \tag{2.56}$$

where we introduced another one-form

$$\Theta' \equiv \frac{\mathrm{i}}{2}\,(\partial_z T \mathrm{d}z - \partial_{\bar{z}} T \mathrm{d}\bar{z})\,. \tag{2.57}$$

The triplet of quaternionic two-forms (2.8) is then covariantly constant, verifying the quaternion-Kähler property of the metric.

It will be important to note that the Toda equation (2.53) is invariant under the symmetry

$$T(\rho, z, \bar{z}) \mapsto \tilde{T}(\rho, z, \bar{z}) = T(\rho + c, g(z), \bar{g}(\bar{z})) + \log|\mathrm{d}g/\mathrm{d}z|^2, \tag{2.58}$$

where $g(z)$ is any holomorphic function of z and c any real constant. The effect of the function $g(z)$ can be absorbed by a holomorphic change of coordinates, but this is not so for the constant c. Thus, QK metrics with one Killing vector come (at least locally) in one-parameter families. For later reference we note that under the symmetry (2.58), the one-forms Θ and Θ' vary by

$$\Theta \mapsto \Theta - c\,\Theta', \qquad \Theta' \mapsto \Theta' - \operatorname{Im} \mathrm{d} \log \frac{\mathrm{d}g}{\mathrm{d}z}, \tag{2.59}$$

where all quantities on the right hand side are understood as functions of $\rho + c$ and $g(z)$. As a result of (2.59), the curvature of the circle bundle generated by ∂_θ receives a contribution proportional to $\mathrm{d}\Theta'$.

2.3.2 Toda Ansatz for one-dimensional HK manifolds with one rotational isometry

On the HK side, self-dual Ricci-flat metrics with one rotational Killing vector field can be cast into the Boyer-Finley Ansatz [55–58],

$$\mathrm{d}s^2_{\mathcal{M}'} = \frac{1}{2}\left[\partial_\rho T\left(\mathrm{d}\rho^2 + 4\mathrm{e}^T \mathrm{d}z\mathrm{d}\bar{z}\right) + \frac{4}{\partial_\rho T}\left(\mathrm{d}\theta' + \Theta'\right)^2\right], \tag{2.60}$$

where $T(\rho, z, \bar{z})$ is again a solution of the Toda equation (2.53), and Θ' is the connection one-form (2.57). We choose

$$\mathbf{u} = \frac{\sqrt{\partial_\rho T}}{2}\,\mathrm{d}\rho + \frac{\mathrm{i}}{\sqrt{\partial_\rho T}}\left(\mathrm{d}\theta' + \Theta'\right), \qquad \mathbf{v} = e^{\frac{1}{2}T + \mathrm{i}\theta'}\sqrt{\partial_\rho T}\mathrm{d}z \tag{2.61}$$

as a basis of the space of $(1,0)$ forms in complex structure J_3. The self-dual two-forms

$$\omega'_3 = \mathrm{i}\,(\mathbf{u}\wedge\bar{\mathbf{u}}+\mathbf{v}\wedge\bar{\mathbf{v}}) = \mathrm{d}\rho\wedge\left(\mathrm{d}\theta'+\Theta'\right)+\mathrm{i}\,e^{T}\partial_\rho T\,\mathrm{d}z\wedge\mathrm{d}\bar{z} \tag{2.62}$$

$$\omega'_+ = \mathbf{u}\wedge\mathbf{v} = \mathrm{d}\left(e^{T/2+\mathrm{i}\theta'}\right)\wedge\mathrm{d}z\,, \qquad \omega'_- = (\omega'_+)^*\,, \tag{2.63}$$

are closed by virtue of the Toda equation, and the corresponding complex structures satisfy the quaternion algebra, verifying the hyperkähler property. They can be integrated to one-forms

$$p'_3 = \rho\left(\mathrm{d}\theta'+\Theta'\right)-\Theta, \qquad p'_+ = e^{T/2+\mathrm{i}\theta'}\,\mathrm{d}z\,, \tag{2.64}$$

where Θ is the same connection which features in the QK metric (2.54). In fact, the one-forms p'_3, p'_+ are related to the SU(2) connection (2.56) of the QK metric by exactly the same equations as (2.30).

The Kähler connection p'_3 can be further integrated to a Kähler potential [55]. For this purpose, one must first integrate the Toda potential $T(\rho,z,\bar{z})$ to a function $\mathcal{L}(\rho,z,\bar{z})$ such that

$$\partial_\rho\mathcal{L} = T\,, \qquad \partial_z\partial_{\bar{z}}\mathcal{L}+\partial_\rho\,\mathrm{e}^{T} = 0\,. \tag{2.65}$$

This determines the function $\mathcal{L}$ up to the addition of the real part of a holomorphic function of z. We fix this ambiguity by requiring that

$$\Theta = \rho\,\Theta' - \frac{\mathrm{i}}{2}\left(\partial_z\mathcal{L}\,\mathrm{d}z-\partial_{\bar{z}}\mathcal{L}\,\mathrm{d}\bar{z}\right). \tag{2.66}$$

Indeed, any solution of (2.55) can be put in this form. Then the Legendre transform of $\mathcal{L}$ with respect to ρ

$$K_{\mathcal{M}'}(z,\bar{z},u,\bar{u}) = \langle\rho\,\log(u\bar{u})-\mathcal{L}(\rho,z,\bar{z})\rangle_\rho\,, \tag{2.67}$$

provides a Kähler potential for the HK metric in the complex structure J_3 [55], with complex coordinates z,u. Using (2.65), one verifies that the Kähler potential (2.67) satisfies the Monge-Ampère equation

$$\partial^2_{z\bar{z}}K_{\mathcal{M}'}\,\partial^2_{u\bar{u}}K_{\mathcal{M}'}-\partial^2_{z\bar{u}}K_{\mathcal{M}'}\,\partial^2_{u\bar{z}}K_{\mathcal{M}'} = 1 \tag{2.68}$$

and reproduces the metric (2.60) provided one identifies $u = e^{T/2+\mathrm{i}\theta'}$. Moreover, using (2.66) one may check that it also reproduces the connection (2.64),

$$p'_3 = \frac{1}{2\mathrm{i}}(\partial-\bar{\partial})K_{\mathcal{M}'} = \rho\,\mathrm{d}\theta'+\frac{\mathrm{i}}{2}\left(\partial_z\mathcal{L}\,\mathrm{d}z-\partial_{\bar{z}}\mathcal{L}\,\mathrm{d}\bar{z}\right). \tag{2.69}$$

2.3.3 Hyperholomorphic connection and QK/HK correspondence

Using the general prescription in section 2.2.1 with

$$\tau = \frac{1}{2P\rho^2}\,, \qquad \nu = \frac{2P}{\partial_\rho T}\,, \qquad \nu+\rho = \frac{2}{\partial_\rho T} \tag{2.70}$$

it is immediate to check that the HK metric (2.60) is related to the QK metric (2.54) with the same Toda potential under the QK/HK correspondence. The hyperholomorphic connection afforded by this correspondence is given by (2.28),

$$\lambda = \frac{2P}{\partial_\rho T}\left(\mathrm{d}\theta'+\Theta'\right)+\Theta. \tag{2.71}$$

Indeed, one may check that the curvature $\mathscr{F} = \mathrm{d}\lambda$ is a linear combination of anti-self-dual forms,

$$\mathscr{F} = \frac{2\mathrm{i}}{(\partial_\rho T)^2}\Big[e^{-\frac{1}{2}T-\mathrm{i}\theta'}\partial^2_{\rho z}T\,\bar{\mathbf{u}}\wedge\mathbf{v} - e^{-\frac{1}{2}T+\mathrm{i}\theta'}\partial^2_{\rho\bar z}T\,\mathbf{u}\wedge\bar{\mathbf{v}} + \left(\frac{1}{2}(\partial_\rho T)^2 + e^{-T}\partial^2_{z\bar z}T\right)(\mathbf{u}\wedge\bar{\mathbf{u}} - \mathbf{v}\wedge\bar{\mathbf{v}})\Big], \tag{2.72}$$

and derives from the Kähler potential (2.35) where $K_{\mathcal{M}'}$ is given in (2.67).

Under the symmetry (2.58), we note that the HK metric (2.60) is invariant, up to a change of coordinates $(\rho, z, \theta') \mapsto (\tilde\rho = \rho + c, \tilde z = g(z), \tilde\theta' = \theta' - \operatorname{Im}\log g'(z))$. However, the hyperholomorphic connection, Kähler connection and Kähler potential do transform,

$$\lambda \mapsto \lambda + c\mathrm{d}\tilde\theta'\,, \qquad p'_3 \mapsto p'_3 - c\mathrm{d}\tilde\theta'\,, \qquad K_{\mathcal{M}'} \mapsto K_{\mathcal{M}'} - c\log(u\bar u)\,. \tag{2.73}$$

The parameter c determines how the $\mathrm{U}(1)_R$ action on $\mathcal{M}'$ lifts to an action on the total space of the circle bundle $\mathcal{P}$, and leads to a one-parameter family of dual QK metrics. It is also worth noting that in one quaternionic dimension, unlike in higher dimensions (cf. (2.27)), the quotients of the QK and HK manifold by their respective U(1) action are related by a conformal rescaling,

$$\mathrm{d}s^2_{\mathcal{M}/\partial_\theta} \propto \mathrm{d}s^2_{\mathcal{M}'//\partial_{\theta'}} \propto \mathrm{d}\rho^2 + 4e^T\mathrm{d}z\,\mathrm{d}\bar z\,. \tag{2.74}$$

The QK/HK correspondence also provides a relation between the above description based on the Toda equation and the twistor framework. This can be done using expressions of the Toda coordinates ρ, z and potential T in terms of the data on the twistor space $\mathcal{Z}$ found in [59] and the dictionary (2.45) between the coordinates on the dual QK and HK spaces. As a result, one finds that

$$T = \log(v\bar v/4), \qquad \rho = e^\Phi, \qquad z = \frac{\mathrm{i}}{2}\varrho + \frac{1}{8\pi}\sum_j\oint_{C_j}\frac{\mathrm{d}t}{t}\,\partial_{\xi_{[i]}}H^{[ij]}, \tag{2.75}$$

where Φ given in (2.16) is understood as a function of v and ϱ. Note that here z coincides with w defined in (2.6). Using these identifications, one can also show that the Kähler potential obtained by the Legendre transform (2.67) differs from the one given in (2.5) by a Kähler transformation proportional to the anomalous dimension $c\log(v\bar v/4)$.

2.3.4 Examples

Let us now illustrate the general formulae obtained in this subsection on three simple examples of QK manifolds, the sphere S^4, hyperbolic space H^4, and the 'perturbative universal hypermultiplet moduli space' (a deformation of a non-compact version of $\mathbb{C}P^2$). The latter is a special case of the local c-map spaces discussed in section 2.4. As it turns out, these three QK manifolds are dual to the same HK manifold, namely $\mathbb{R}^4$ with its flat metric (with negative definite signature when $\mathcal{M} = H^4$), but equipped with a different hyperholomorphic connection.

S^4 and H^4 vs. flat space. The standard round metric on S^4 (respectively, the standard metric on the four-dimensional hyperbolic space H^4) can be cast into the Tod Ansatz (2.54) by choosing

$$T = 2\log\frac{\epsilon(4\rho-1)}{4\cosh(z+\bar{z})}\,, \qquad \Theta' = 4\Theta = -\mathrm{i}\tanh(z+\bar{z})(\mathrm{d}z-\mathrm{d}\bar{z})\,, \tag{2.76}$$

where $\epsilon = 1$ for $\mathcal{M} = S^4$ and $\epsilon = -1$ for $\mathcal{M} = H^4$, and ρ lies in the range where $\epsilon(4\rho-1) > 0$ [59]. By changing coordinates to

$$\rho = \frac{1}{4}(1+\epsilon R^2), \qquad z = \frac{1}{2}\left(\log\tan\delta + \mathrm{i}(\beta-\gamma)\right), \qquad \theta' = \beta+\gamma, \tag{2.77}$$

the metric on the dual HK space (2.60) can be written as

$$\mathrm{d}s^2_{\mathcal{M}'} = \epsilon\left[\mathrm{d}R^2 + R^2\left(\mathrm{d}\delta^2 + \sin^2\delta\,\mathrm{d}\beta^2 + \cos^2\delta\,\mathrm{d}\gamma^2\right)\right], \tag{2.78}$$

which is recognized as the flat metric on $\mathbb{R}^4$ in Hopf coordinates, with positive signature for $\epsilon = 1$, or negative signature for $\epsilon = -1$. The hyperholomorphic connection (2.71) evaluates to the flat connection

$$\lambda = -\frac{1}{4}\,\mathrm{d}\theta' = -\frac{1}{4}\,\mathrm{d}(\beta+\gamma), \qquad \mathscr{F} = 0. \tag{2.79}$$

It may be checked explicitly that it derives from the Kähler potential (2.35), where

$$K_{\mathcal{M}'} = 2\epsilon|u|\cosh(z+\bar{z}) + \frac{1}{4}\log u\bar{u} = 2\rho + \frac{1}{4}\log u\bar{u} - \frac{1}{2} \tag{2.80}$$

follows by Legendre transform from the potential

$$\mathcal{L} = \frac{1}{2}(4\rho-1)\left(\log\frac{\epsilon(4\rho-1)}{4\cosh(z+\bar{z})} - 1\right). \tag{2.81}$$

The twistor space associated to $\mathcal{M}$ is $\mathcal{Z} = \mathbb{C}P^3$ for $\epsilon = 1$, or $\mathbb{C}P^{2,1}$ for $\epsilon = -1$. In either cases $\mathcal{Z}$ can be described in the language of section 2.1 by three open patches $\mathcal{U}_+, \mathcal{U}_-, \mathcal{U}_0$ where $\mathcal{U}_\pm$ covers a neighborhood of $t = \mp\epsilon e^{\pm(z+\bar{z})}$ and $\mathcal{U}_0$ covers the rest of $\mathbb{C}P^1$, with transition functions and anomalous dimension [59]

$$H^{[0\pm]} = \pm\frac{1}{2}\,\xi\log\xi\,, \qquad c = -\frac{1}{4}\,. \tag{2.82}$$

The Darboux coordinates (2.15) in the patch $\mathcal{U}_0$ are then given by

$$\begin{aligned}
\xi &= \epsilon(4\rho-1)\left(\frac{t^{-1}-t}{2\cosh(z+\bar{z})} - \epsilon\tanh(z+\bar{z})\right),\\
\tilde{\xi} &= -\mathrm{i}\left(2z + \log\frac{1+\epsilon t e^{-z-\bar{z}}}{1-\epsilon t e^{z+\bar{z}}}\right),\\
\alpha &= 4\theta - \mathrm{i}\log t.
\end{aligned} \tag{2.83}$$

After performing the replacement (2.43), the same formulae provide Darboux coordinates η, μ on $\mathcal{Z}'$ and a holomorphic section $\Upsilon = e^{-2\pi\mathrm{i}\alpha}$ of $\mathscr{L}_{\mathcal{Z}'}$. Indeed, one may check that the one-forms $\mathrm{d}\xi, \mathrm{d}\tilde{\xi}$ and $\mathrm{d}\alpha - 4\lambda$ are of type (1,0) in complex structure $J(\zeta)$, consistently with (2.48).

Perturbative universal hypermultiplet. We now consider the family of QK metrics [60]

$$\mathrm{d}s^2_{\mathcal{M}} = \frac{\rho+2c}{4\rho^2(\rho+c)}\,\mathrm{d}\rho^2 + \frac{\rho+2c}{16\rho^2}\left((\mathrm{d}\zeta^0)^2+4(\mathrm{d}\tilde\zeta_0)^2\right) + \frac{\rho+c}{64\rho^2(\rho+2c)}(\mathrm{d}\sigma+\tilde\zeta_0\,\mathrm{d}\zeta^0-\zeta^0\,\mathrm{d}\tilde\zeta_0)^2, \tag{2.84}$$

where c is a real parameter, and ρ lies in the range $\rho > \max(0,-2c)$. The metric (2.84) describes the weak coupling limit of the hypermultiplet moduli space in type IIA string theory compactified on a rigid Calabi-Yau three-fold X, where c is determined by the Euler number of X (see section 2.4 and section 3.2 for further details on this set-up). For $c=0$, the metric (2.84) reduces to the SU(2,1)-invariant metric on $\mathbb{C}P^{1,1}$ (a non-compact version of $\mathbb{C}P^2$). The metric (2.84) may be cast into Tod's Ansatz (2.54) by choosing

$$z = -\frac{1}{4}\,(\zeta^0-2\mathrm{i}\tilde\zeta_0), \qquad \theta = -\frac{1}{8}\,\sigma, \qquad T=\log(\rho+c). \tag{2.85}$$

Using

$$P=\frac{\rho+2c}{2(\rho+c)}\,, \qquad \Theta = -\frac{\mathrm{i}}{2}(z\mathrm{d}\bar z-\bar z\mathrm{d}z) = \frac{1}{8}(\zeta^0\mathrm{d}\tilde\zeta_0-\tilde\zeta_0\mathrm{d}\zeta^0)\,, \qquad \Theta'=0\,, \tag{2.86}$$

we find the dual HK metric (2.60) and hyperholomorphic connection (2.71)

$$\mathrm{d}s^2_{\mathcal{M}'} = \frac{\mathrm{d}\rho^2}{2(\rho+c)} + 2\mathrm{d}z\,\mathrm{d}\bar z + 2(\rho+c)(\mathrm{d}\theta')^2, \qquad \lambda = (\rho+2c)\mathrm{d}\theta' - \frac{\mathrm{i}}{2}\,(z\mathrm{d}\bar z-\bar z\mathrm{d}z). \tag{2.87}$$

By changing coordinates to $(\rho,z) = (\frac{1}{2}\,R^2-c, \tilde R e^{\mathrm{i}\tilde\theta'}/\sqrt{2})$, we recognize (2.87) as the flat metric on $\mathbb{R}^4$ in bi-polar coordinates, equipped with a constant anti-self dual field,[13]

$$\mathrm{d}s^2_{\mathcal{M}'} = \mathrm{d}R^2+R^2(\mathrm{d}\theta')^2+\mathrm{d}\tilde R^2+\tilde R^2(\mathrm{d}\tilde\theta')^2, \qquad \lambda = \frac{1}{2}\left(R^2\,\mathrm{d}\theta'-\tilde R^2\,\mathrm{d}\tilde\theta'\right)+c\mathrm{d}\theta'. \tag{2.88}$$

One may check that λ derives from the Kähler potential

$$K_{\mathscr{L}} = -z\bar z + u\bar u + c\log u\bar u\,, \tag{2.89}$$

related via (2.35) and (2.67) to the function $\mathcal{L}$ and Kähler potential $K_{\mathcal{M}'}$

$$\mathcal{L} = (\rho+c)\,\log(\rho+c)-\rho-z\bar z\,, \qquad K_{\mathcal{M}'} = z\bar z+u\bar u - c\log(u\bar u)\,. \tag{2.90}$$

Although the logarithmic term in $K_{\mathcal{M}'}$ and $K_{\mathscr{L}}$ can be removed by a Kähler transformation, it is needed in order to correctly reproduce the hyperholomorphic connection and, as a consequence, the dual QK metric (2.84).

The twistor space $\mathcal{Z}$ can be covered by three patches $\mathcal{U}_+$,$\mathcal{U}_-$ and $\mathcal{U}_0$, covering the north pole ($t=0$), south pole ($t=\infty$), and equator in $\mathbb{C}P^1$, respectively, with transition functions [19, 59]

$$H^{[+0]} = -\frac{\mathrm{i}}{4}\,\xi^2\,, \qquad H^{[-0]} = \frac{\mathrm{i}}{4}\,\xi^2\,, \tag{2.91}$$

[13] Since the metric (2.84) has a curvature singularity at $\rho=-2c$ when $c<0$, this example shows that the HK dual of a smooth QK manifold need not be geodesically complete.

and anomalous dimension c. The Darboux coordinates (2.15) in the patch $\mathcal{U}_0$ read

$$\begin{aligned} \xi/2 &= -(z+\bar z) + \sqrt{\rho+c}\,\left(t^{-1}-t\right) \\ \mathrm{i}\tilde\xi &= z-\bar z + \sqrt{\rho+c}\,\left(t^{-1}+t\right) \\ -2\alpha-\xi\tilde\xi &= -8\theta + 4\mathrm{i}\sqrt{\rho+c}\,\left(z\,t^{-1}+\bar z\,t\right) - 8\mathrm{i}c\log t\,. \end{aligned} \tag{2.92}$$

Upon performing the replacement (2.43), the same formulae provide Darboux coordinates η,μ on $\mathcal{Z}'$ and a holomorphic section $\Upsilon = e^{-2\pi\mathrm{i}\alpha}$ of $\mathscr{L}_{\mathcal{Z}'}$. As in the previous example, one may check that the one-forms $\mathrm{d}\xi, \mathrm{d}\tilde\xi$ and $\mathrm{d}\alpha - 4\lambda$ are of type (1,0) in complex structure $J(\zeta)$.

2.4 Local c-map vs. rigid c-map

In this subsection we demonstrate that the $4n$-dimensional QK space $\mathcal{M}$, obtained by the local c-map procedure [61] from a $(2n-2)$-dimensional projective special Kähler manifold $\mathcal{SK}$ with homogeneous prepotential $F(X^\Lambda)$, is dual via the QK/HK correspondence to the HK manifold $\mathcal{M}'$ obtained by the rigid c-map [62] from the $2n$-dimensional rigid special Kähler manifold with the same (homogeneous) prepotential $F(X^\Lambda)$. The correspondence continues to hold for the one-loop deformed local c-map [19, 63], which is dual to the same HK manifold but with a different hyperholomorphic connection. The universal hypermultiplet manifold considered above is a particular example with quadratic prepotential. The main results in this subsection were obtained in [16].

2.4.1 Local c-map

We start by describing a one-parameter family of QK metrics associated to any special Kähler space $\mathcal{SK}$. Its relevance to physics comes from the fact that this family describes the perturbative hypermultiplet moduli space in type II string theory compactified on a Calabi-Yau threefold (see section 3.2). Let $F(X^\Lambda)$ be a holomorphic function of n coordinates X^Λ, $\Lambda = 0,\dots,n-1$, homogeneous of degree 2, which encodes the geometry of $\mathcal{SK}$. The family of QK metrics, parametrized by a real constant c is given in coordinates $\rho, z^a, \zeta^\Lambda, \tilde\zeta_\Lambda, \sigma$ by [63, 64]

$$\mathrm{d}s^2_{\mathcal{M}} = \frac{\rho+2c}{4\rho^2(\rho+c)}\,\mathrm{d}\rho^2 + \frac{\rho+c}{2\rho}\,\mathrm{d}s^2_{\mathcal{SK}} + \frac{\mathrm{d}s^2_{\mathcal{T}}}{4\rho} + \frac{c}{2\rho^2}\,e^{\mathcal{K}}\,|X^\Lambda \mathrm{d}\tilde\zeta_\Lambda - F_\Lambda \mathrm{d}\zeta^\Lambda|^2 + \frac{\rho+c}{64\rho^2(\rho+2c)} D\sigma^2\,, \tag{2.93}$$

where $\mathrm{d}s^2_{\mathcal{SK}} = 2\mathcal{K}_{a\bar b}\mathrm{d}z^a\mathrm{d}\bar z^b$ is the projective special Kähler metric with Kähler potential and Kähler connection

$$\mathcal{K} = -\log[\mathrm{i}(\bar X^\Lambda F_\Lambda - X^\Lambda \bar F_\Lambda)], \qquad \mathcal{A}_K = \frac{\mathrm{i}}{2}\,(\mathcal{K}_a \mathrm{d}z^a - \mathcal{K}_{\bar a}\mathrm{d}\bar z^{\bar a}), \tag{2.94}$$

$\mathrm{d}s^2_{\mathcal{T}}$ is the Kähler metric on the torus $\mathcal{T}$

$$\mathrm{d}s^2_{\mathcal{T}} = -\frac{1}{2}\,(\mathrm{d}\tilde\zeta_\Lambda - \bar{\mathcal{N}}_{\Lambda\Lambda'}\mathrm{d}\zeta^{\Lambda'})\,\mathrm{Im}\,\mathcal{N}^{\Lambda\Sigma}(\mathrm{d}\tilde\zeta_\Sigma - \mathcal{N}_{\Sigma\Sigma'}\mathrm{d}\zeta^{\Sigma'}), \tag{2.95}$$

$\mathcal{N}_{\Lambda\Lambda'}$ is the 'Weil period matrix'[14]

$$\mathcal{N}_{\Lambda\Lambda'} = \bar{\tau}_{\Lambda\Lambda'} + 2\mathrm{i}\frac{[\mathrm{Im}\,\tau \cdot X]_\Lambda [\mathrm{Im}\,\tau \cdot X]_{\Lambda'}}{X^\Sigma\ \mathrm{Im}\,\tau_{\Sigma\Sigma'} X^{\Sigma'}}\,, \qquad \tau_{\Lambda\Sigma} = \partial_{X^\Lambda}\partial_{X^\Sigma} F\,, \tag{2.96}$$

and

$$D\sigma \equiv \mathrm{d}\sigma + \tilde{\zeta}_\Lambda \mathrm{d}\zeta^\Lambda - \zeta^\Lambda \mathrm{d}\tilde{\zeta}_\Lambda + 8c\,\mathcal{A}_K. \tag{2.97}$$

For prepotentials F arising in Calabi-Yau compactifications, the quadratic forms $\mathrm{Im}\,\mathcal{N}$ and $\mathrm{Im}\,\tau$ have signature $(0,n)$ and $(n-1,1)$, respectively. In the one-modulus case with $F = -\frac{\mathrm{i}}{4}\,(X^0)^2$, $\mathcal{SK}$ is trivial and the metric (2.93) reduces to (2.84).

The twistor space $\mathcal{Z}$ can be read off from the Legendre transform construction of (2.93) [37, 64]. It can be covered by three patches $\mathcal{U}_+, \mathcal{U}_-, \mathcal{U}_0$ around the north pole $(t=0)$, south pole $(t=\infty)$ and equator, respectively, with transition functions

$$H^{[+0]} = F(\xi^\Lambda), \qquad H^{[-0]} = \bar{F}(\xi^\Lambda)\,, \tag{2.98}$$

and anomalous dimension c. The canonical Darboux coordinates on $\mathcal{Z}$ in the patch $\mathcal{U}_0$ are given by [20]

$$\begin{aligned} \xi^\Lambda &= \zeta^\Lambda + 2e^{\mathcal{K}/2}\sqrt{\rho+c}\,\left(t^{-1}X^\Lambda - t\,\bar{X}^\Lambda\right), \\ \tilde{\xi}_\Lambda &= \tilde{\zeta}_\Lambda + 2e^{\mathcal{K}/2}\sqrt{\rho+c}\,\left(t^{-1}F_\Lambda - t\,\bar{F}_\Lambda\right), \\ \tilde{\alpha} &= \sigma + 2e^{\mathcal{K}/2}\sqrt{\rho+c}\left[t^{-1}(F_\Lambda\zeta^\Lambda - X^\Lambda\tilde{\zeta}_\Lambda) - t\,(\bar{F}_\Lambda\zeta^\Lambda - \bar{X}^\Lambda\tilde{\zeta}_\Lambda)\right] - 8\mathrm{i}c\log t\,, \end{aligned} \tag{2.99}$$

where $\tilde{\alpha}$ is related to α by (1.9). Unlike the latter, the former is invariant under simultaneous symplectic transformations of the vectors (X^Λ, F_Λ) and $(\zeta^\Lambda, \tilde{\zeta}_\Lambda)$. The metric (2.93) is invariant under the Killing vector field ∂_σ (as well as translations along $\zeta^\Lambda, \tilde{\zeta}_\Lambda$, which we ignore in this section). The vector field ∂_σ lifts to the holomorphic vector field ∂_α on $\mathcal{Z}$.

2.4.2 Rigid c-map

On the other hand, the same prepotential F, via the rigid c-map construction[15] produces the following hyperkähler metric [62]

$$\mathrm{d}s^2_{\mathcal{M}'} = \frac{1}{4}\,\mathrm{d}s^2_{\mathcal{RSK}} + \frac{1}{2}\,\mathrm{d}s^2_{\mathcal{T}_{\rm rig}}, \tag{2.100}$$

where

$$\mathrm{d}s^2_{\mathcal{RSK}} = -4\,\mathrm{Im}\,\tau_{\Lambda\Sigma}\mathrm{d}Z^\Lambda \mathrm{d}\bar{Z}^\Sigma \tag{2.101}$$

is the metric on the rigid special Kähler manifold $\mathcal{RSK}$ with prepotential $F(Z^\Lambda)$, and

$$\mathrm{d}s^2_{\mathcal{T}_{\rm rig}} = -\frac{1}{2}\left(\mathrm{d}\tilde{\zeta}_\Lambda - \bar{\tau}_{\Lambda\Sigma}\mathrm{d}\zeta^\Sigma\right)\mathrm{Im}\,\tau^{\Lambda\Lambda'}\left(\mathrm{d}\tilde{\zeta}_{\Lambda'} - \tau_{\Lambda'\Sigma'}\mathrm{d}\zeta^{\Sigma'}\right) \tag{2.102}$$

with $\tau_{\Lambda\Sigma} = \partial_{Z^\Lambda}\partial_{Z^\Sigma}F(Z)$ is the flat metric on its cotangent space.

[14] This terminology is borrowed from the context of Type IIA string theory compactified on a Calabi-Yau, where $\mathcal{N}_{\Lambda\Lambda'}$ corresponds to the period matrix of the Weil intermediate Jacobian, while $\tau_{\Lambda\Lambda'}$ is the period matrix of the Griffiths intermediate Jacobian.

[15] The rigid c-map construction does not require that F be homogeneous, but we restrict to this case as it is the one relevant for the QK/HK correspondence.

For prepotentials F arising in Calabi-Yau compactifications, the metric (2.100) has signature $(4, 4n-4)$. In complex structure J_3 where Z^Λ and

$$W_\Lambda = \frac{\mathrm{i}}{2}(\tilde{\zeta}_\Lambda - \tau_{\Lambda\Sigma}\zeta^\Sigma) \tag{2.103}$$

are complex coordinates, the Kähler form and holomorphic symplectic form are

$$\omega'_3 = \frac{1}{2\mathrm{i}}\,\mathrm{Im}\,\tau_{\Lambda\Sigma}\mathrm{d}Z^\Lambda\wedge\mathrm{d}\bar{Z}^\Sigma + \frac{1}{2\mathrm{i}}\,[\mathrm{Im}\,\tau^{-1}]^{\Lambda\Sigma}\,DW_\Lambda\wedge D\bar{W}_\Sigma, \quad \omega'_+ = \frac{1}{2}\,\mathrm{d}Z^\Lambda\wedge\mathrm{d}W_\Lambda, \tag{2.104}$$

where DW_Λ are the (1,0)-forms

$$DW_\Lambda \equiv \mathrm{d}W_\Lambda - \frac{1}{2\mathrm{i}}\,\partial_{Z^\Lambda}\tau_{KL}[\,\mathrm{Im}\,\tau^{-1}]^{KK'}(W_{K'}+\bar{W}_{K'})\mathrm{d}Z^L\,. \tag{2.105}$$

It is straightforward to check that $\omega'_3, \omega'_\pm$ are closed and that the associated complex structures $J_3 = \omega'_3\, g^{-1}, J_\pm = \omega'_\pm\, g^{-1}$ satisfy the quaternion algebra. The Kähler form $\omega'_3 = \mathrm{i}\partial\bar{\partial}K_{\mathcal{M}'}$ derives from the Kähler potential

$$K_{\mathcal{M}'} = \frac{1}{4\mathrm{i}}\left(Z^\Lambda\bar{G}_\Lambda - \bar{Z}^\Lambda G_\Lambda\right) - \frac{1}{4}\,(W_\Lambda+\bar{W}_\Lambda)[\,\mathrm{Im}\,\tau^{-1}]^{\Lambda\Sigma}(W_\Sigma+\bar{W}_\Sigma), \tag{2.106}$$

where $G_\Lambda \equiv \partial F(Z^\Lambda)/\partial Z^\Lambda$. The metric (2.100) is invariant under $(Z^\Lambda, W_\Lambda) \mapsto (e^{\mathrm{i}\theta'}\, Z^\Lambda, W_\Lambda)$, while the complex structures transform as in (2.19).

The twistor space $\mathcal{Z}'$ can again be read off from the Legendre transform construction of (2.100) [65]. It involves three patches $\mathcal{U}'_+, \mathcal{U}'_-, \mathcal{U}'_0$ around $\zeta = 0$, $\zeta = \infty$ and around the equator, and the same transition functions as in (2.98),

$$H^{[+0]} = F(\eta^\Lambda), \qquad H^{[-0]} = \bar{F}(\eta^\Lambda)\,. \tag{2.107}$$

In fact, the Legendre construction of the HK metric (2.100) is closely similar to that of the Swann bundle $\mathcal{S}$ of the QK metric (2.93). To wit, the Legendre construction of $\mathcal{S}$ involves one additional $\mathcal{O}(2)$ multiplet $\eta^\flat$, known as the superconformal compensator, and the generalized prepotential for $\mathcal{S}$ is just obtained by rescaling the generalized prepotential for $\mathcal{M}'$ by a factor of $1/\eta^\flat$, ensuring the proper homogeneity degree for superconformal invariance. Thus, in this Legendre construction procedure, the rigid c-map $\mathcal{M}$ is obtained from $\mathcal{S}$ by freezing the $\mathcal{O}(2)$ multiplet to a fixed value, which determines the complex structure in which $\mathcal{M}'$ is obtained. Mathematically, this freezing corresponds to performing the HK quotient with respect to the $\mathrm{U}(1)_A$ isometry. As a result, the Darboux coordinates (2.4) obtained from (2.107) are closely similar to (2.99),

$$\begin{aligned} \eta^\Lambda &= \zeta^\Lambda + \zeta^{-1}Z^\Lambda - \zeta\bar{Z}^\Lambda\,, \\ \mu_\Lambda &= \tilde{\zeta}_\Lambda + \zeta^{-1}G_\Lambda - \zeta\bar{G}_\Lambda\,. \end{aligned} \tag{2.108}$$

The use of the same variables $\zeta^\Lambda, \tilde{\zeta}_\Lambda$ as in the local c-map metric (2.93) will be justified in (2.109) below.

2.4.3 QK/HK-correspondence for c-map spaces

The fact that the local c-map $\mathcal{M}$ and rigid c-map $\mathcal{M}'$ are described by identical transition functions (2.98), (2.107) shows, by itself, that $\mathcal{M}$ and $\mathcal{M}'$ are dual under the QK/HK correspondence. To confirm this, we note that the Darboux coordinate systems (2.99) and (2.108) are related under the general identification (2.43), provided the coordinates Z^Λ, W_Λ on $\mathcal{M}'$ are related to $\rho, z^a, \zeta^\Lambda, \tilde\zeta_\Lambda$ on $\mathcal{M}/\partial_\sigma$ via

$$Z^\Lambda = \sqrt{2}\, R\, e^{\mathrm{i}\theta' + \mathcal{K}/2} X^\Lambda(z^a), \qquad W_\Lambda = \frac{\mathrm{i}}{2}\left(\tilde\zeta_\Lambda - \tau_{\Lambda\Sigma}\zeta^\Lambda\right), \tag{2.109}$$

where, as in (2.88), we define $R = \sqrt{2(\rho + c)}$.

Moreover, by applying the procedure of section 2.2.1 to the QK metric (2.93) with

$$\tau = \frac{\rho + c}{\rho^2(\rho + 2c)}, \qquad \nu = \rho + 2c, \qquad \mathrm{d}\theta + \Theta = -\frac{1}{8} D\sigma, \qquad \Theta' = \mathcal{A}_K, \tag{2.110}$$

and using (2.27), (2.109) one finds that the HK metric dual to (2.93) is given by

$$\mathrm{d}s^2_{\mathcal{M}'} = \mathrm{d}R^2 - \frac{1}{2}R^2\, \mathrm{d}s^2_{\mathcal{SK}} + R^2(\mathrm{d}\theta' + \mathcal{A}_K)^2 - \frac{1}{2}\mathrm{d}s^2_{\mathcal{T}} + \frac{1}{2R^2}|Z^\Lambda \mathrm{d}\tilde\zeta_\Lambda - G_\Lambda \mathrm{d}\zeta^\Lambda|^2 . \tag{2.111}$$

It is straightforward to check that this agrees with the rigid c-map metric (2.100). The hyperholomorphic connection afforded by the QK/HK correspondence is easily obtained from (2.28),

$$\lambda = (\rho + c)(\mathrm{d}\theta' + \mathcal{A}_K) + \frac{1}{8}\left(\zeta^\Lambda \mathrm{d}\tilde\zeta_\Lambda - \tilde\zeta_\Lambda \mathrm{d}\zeta^\Lambda\right) + c\,\mathrm{d}\theta' . \tag{2.112}$$

Its curvature can be expressed as

$$\mathscr{F} = \frac{1}{2\mathrm{i}}\, \mathrm{Im}\,\tau_{\Lambda\Sigma} \mathrm{d}Z^\Lambda \wedge \mathrm{d}\bar Z^\Sigma - \frac{1}{2\mathrm{i}}\, [\mathrm{Im}\,\tau^{-1}]^{\Lambda\Sigma}\, DW_\Lambda \wedge D\bar W_\Sigma , \tag{2.113}$$

and is indeed of type (1,1) in all complex structures. It is worthwhile to note that it differs from the Kähler form in (2.104) by a flip of sign (and a rescaling). Indeed, it can be derived from the Kähler potential

$$K_{\mathscr{L}} = \frac{1}{4\mathrm{i}}\left(Z^\Lambda \bar G_\Lambda - \bar Z^\Lambda G_\Lambda\right) + \frac{1}{4}(W_\Lambda + \bar W_\Lambda)[\mathrm{Im}\,\tau^{-1}]^{\Lambda\Sigma}(W_\Sigma + \bar W_\Sigma) \tag{2.114}$$

in complex structure J_3, which differs also from (2.106) by a flip of sign. Furthermore, it can be checked that the Darboux coordinate

$$\tilde\alpha = \sigma + 2\mathrm{i}\left(\zeta^{-1} Z^\Lambda W_\Lambda + \zeta \bar Z^\Lambda \bar W_\Lambda\right) - 8c\left(\theta' + \mathrm{i}\log\zeta\right) , \tag{2.115}$$

obtained by replacing t by $\zeta e^{-\mathrm{i}\theta'}$ in the third equation of (2.99), satisfies

$$\lambda = -\frac{1}{2}\left(\bar\partial^{(\zeta)}\tilde\alpha + \partial^{(\zeta)}\bar{\tilde\alpha}\right) \tag{2.116}$$

in any complex structure. This of course agrees with (2.48), since α and $-\frac{1}{2}\tilde\alpha$ differ by a holomorphic function. Thus, $\tilde\Upsilon \equiv e^{\mathrm{i}\pi\tilde\alpha}$ provides a holomorphic section of the bundle $\mathscr{L}_{\mathcal{Z}'}$, related to $\Upsilon = e^{-2\mathrm{i}\pi\alpha}$ by a (complexified) gauge transformation.

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3 D-instantons, wall-crossing and contact geometry

In this section we apply the general correspondence discussed in section 2 to the geometry of the hypermultiplet moduli space $\mathcal{M}$ in $\mathcal{N}=2$ string vacua. As indicated in the introduction, the twistorial construction of the QK manifold $\mathcal{M}$ presented in [10] is closely similar to the twistorial construction of the HK moduli space $\mathcal{M}'$ of the Coulomb branch of $\mathcal{N}=2$ rigid field theories in [9]. In this section, we shall show that $\mathcal{M}$ arises by applying the QK/HK correspondence to the HK manifold $\mathcal{M}'$ constructed in [9] (using the central charge function and BPS invariants relevant for the string vacuum at hand), equipped with a suitable hyperholomorphic line bundle $\mathscr{L}$. After reviewing the construction of $\mathcal{M}', \mathcal{M}$ in section 3.1 and section 3.2, we show in section 3.3 that the requisite line bundle $\mathscr{L}$ can be constructed by lifting the KS-symplectomorphisms U_γ across BPS rays on $\mathcal{Z}'$ to contact transformations V_γ on $\mathcal{Z}$ (or to gauge transformations of the hyperholomorphic line bundle $\mathscr{L}$ on $\mathcal{M}'$), using the Rogers dilogarithm function. Generalizing the techniques of Kashaev and Nakanishi [32], building on earlier work by Faddeev and Kashaev [33], we show in section 3.4 that the classical limit of the motivic KS wall crossing formula implies a set of functional identities for the Rogers dilogarithm which ensure the consistency of the construction. In section 3.5 we give a detailed illustration of the general contact wall crossing formula for the so called pentagon, hexagon and octagon relations for the Rogers dilogarithm.

3.1 Wall-crossing in $\mathcal{N}=2$ gauge theories and symplectic geometry

In this section we briefly review the construction of the Coulomb branch of $\mathcal{N}=2$ gauge theories on $\mathbb{R}^3\times S^1$ with emphasis on the phenomenon of wall-crossing [9].

3.1.1 The Coulomb branch of four-dimensional $\mathcal{N}=2$ gauge theories

Let us first focus on the Coulomb branch of an $\mathcal{N}=2$ gauge theory on $\mathbb{R}^4$ with rank n gauge group G. For simplicity we restrict to the case where the flavor symmetry is trivial and, as in section 2.4, assume that the theory is superconformal, as it is the case relevant for the QK/HK correspondence. The moduli space $\mathcal{B}$ is an n-dimensional rigid special Kähler manifold parametrized by n complex valued scalar fields $z^i, i=1,\dots,n$, specifying the vevs of the vector multiplet scalars. At a generic point on $\mathcal{B}$ the gauge group G is broken to $\mathrm{U}(1)^n$ and there are correspondingly n massless gauge fields. The electric and magnetic charges $\gamma=(p^\Lambda,q_\Lambda)$ are sections of a local system $\Gamma\to\mathcal{B}$ of rank $2n$ lattices $\Gamma_z\cong\mathbb{Z}^{2n}$ fibered over each point $z^i\in\mathcal{B}$. The lattice Γ_z is even and self-dual with respect to the symplectic inner product $2\langle\cdot,\cdot\rangle$,

$$\langle\gamma,\gamma'\rangle = q_\Lambda p'^\Lambda - q'_\Lambda p^\Lambda \in\mathbb{Z}\,. \tag{3.1}$$

In what follows we identify Γ with its dual $\Gamma^\star$.

The geometry of $\mathcal{B}$ can be encoded in a holomorphic Lagrangian section

$$\Omega(z) = (Z^\Lambda(z), G_\Lambda(z)), \tag{3.2}$$

$\Lambda=0,1,\dots,n-1$, of the symplectic vector bundle $\Gamma\otimes\mathbb{C}$ over $\mathcal{B}$. The Lagrangian property $\langle\mathrm{d}\Omega,\mathrm{d}\Omega\rangle=0$ implies that G_Λ is locally given by $\partial_{Z^\Lambda}F(Z)$ for some holomorphic

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function $F(Z^\Lambda)$ known as the prepotential. Superconformal invariance implies that F is homogeneous of degree 2. The Kähler metric on $\mathcal{B}$ derives from the Kähler potential

$$K_{\mathcal{B}} = \mathrm{i}\langle \Omega, \bar{\Omega}\rangle = \mathrm{i}(\bar{Z}^\Lambda G_\Lambda - Z^\Lambda \bar{G}_\Lambda), \tag{3.3}$$

while the complexified gauge coupling (or period matrix) is given by the second derivative of the prepotential, $\tau_{\Lambda\Sigma}$. A key object in the study of $\mathcal{N}=2$ theories is the central charge function (or stability data) $Z' : \Gamma \to \mathbb{C}$, defined as the inner product:

$$Z'_\gamma(z) = \langle \gamma, \Omega(z)\rangle = q_\Lambda Z^\Lambda(z) - p^\Lambda G_\Lambda(z). \tag{3.4}$$

3.1.2 Compactification to three dimensions and the semi-flat metric

Upon compactification on a circle, the low-energy dynamics is described by an $\mathcal{N}=4$ supersymmetric sigma model on $\mathbb{R}^3$, with complex $2n$-dimensional hyperkähler target space $\mathcal{M}'$. Topologically, $\mathcal{M}'$ is a twisted torus bundle $\mathcal{T}_z \to \mathcal{M}' \to \mathcal{B}$ over the four-dimensional Coulomb branch $\mathcal{B}$. The torus fiber $\mathcal{T}_z = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}$ over each point $z^i \in \mathcal{B}$ parametrizes the holonomies $C = (\zeta^\Lambda, \tilde{\zeta}_\Lambda)$ of the electric and magnetic Abelian gauge fields around the circle. Invariance under large gauge transformations requires that the holonomies are valued in $\mathbb{R}/\mathbb{Z}$, i.e. are periodic under integer translations

$$\zeta^\Lambda \mapsto \zeta^\Lambda + n^\Lambda, \qquad \tilde{\zeta}_\Lambda \mapsto \tilde{\zeta}_\Lambda + m_\Lambda, \qquad H = (n^\Lambda, m_\Lambda) \in \mathbb{Z}^{2n}. \tag{3.5}$$

In the infinite radius limit,[16] the HK metric on $\mathcal{M}'$ is given by the rigid c-map metric (2.100), with $\mathcal{B}$ playing the role of the rigid special Kähler manifold. In this context, the rigid c-map metric (2.100) is also known as the 'semi-flat' metric on $\mathcal{M}'$.

The corresponding twistor space $\mathcal{Z}'$ admits a canonical set of complex Darboux coordinates $\Xi' = (\eta^\Lambda, \mu_\Lambda)$ given in (2.108). Defining, for any $\gamma \in \Gamma$,

$$\Xi'_\gamma \equiv \langle \gamma, \Xi'\rangle, \qquad \Theta_\gamma = \langle \gamma, C\rangle, \tag{3.6}$$

these Darboux coordinates can then be written as

$$\Xi'^{\mathrm{sf}}_\gamma \equiv \Theta_\gamma + \zeta^{-1} Z'_\gamma - \zeta \bar{Z}'_\gamma \,. \tag{3.7}$$

The translations (3.5) of $(\zeta^\Lambda, \tilde{\zeta}_\Lambda)$ then lift to a holomorphic action on $\mathcal{Z}'$:

$$\eta^\Lambda \mapsto \eta^\Lambda + n^\Lambda, \qquad \mu_\Lambda \mapsto \mu_\Lambda + m_\Lambda. \tag{3.8}$$

This twistorial description of the semi-flat metric will play an important role in what follows.

While the metric (2.100) is correct in the strict $R=\infty$ limit, at finite radius it fails to take into account instanton effects arising from $D=4$ BPS states whose Euclidean worldline winds around the circle. These effects are particularly important near singularities in $\mathcal{B}$ where these BPS states become massless, and are expected to resolve the singularity of the semi-flat metric (2.100). As shown in [9], the corresponding quantum corrections to the HK metric on $\mathcal{M}'$ are largely dictated by consistency with wall-crossing, to which we now turn.

[16] We set the radius of the circle to $r=2$ using superconformal invariance. The infinite radius limit then corresponds to the boundary of $\mathcal{B}$ where Z^Λ is scaled to infinity.

3.1.3 BPS-instantons and wall-crossing

The Hilbert space $\mathcal{H}(z)$ of single-particle states in a four-dimensional gauge theory depends on the values of the scalar fields z^i and is graded by the charge lattice Γ,

$$\mathcal{H}(z) = \bigoplus_{\gamma\in\Gamma} \mathcal{H}_\gamma(z). \tag{3.9}$$

The index (or second helicity supertrace)

$$\Omega(\gamma, z) = -\frac{1}{2}\mathrm{Tr}_{\mathcal{H}_\gamma(z)}(2J_3)^2(-1)^{2J_3} \in \mathbb{Z}\,, \tag{3.10}$$

where J_3 is generator of the little group in $D = 4$, is sensitive only to BPS states, i.e. to single-particle states whose mass M saturates the bound $M \geq |Z'_\gamma(z)|$, which is determined by the central charge function (3.4). The index $\Omega(\gamma, z)$ is a locally constant function of $z \in \mathcal{B}$ but may jump at co-dimension one subspaces corresponding to walls of marginal stability:

$$W(\gamma_1, \gamma_2) = \{z \in \mathcal{B} \,:\, \arg[Z'_{\gamma_1}(z)] = \arg[Z'_{\gamma_2}(z)]\}\,, \tag{3.11}$$

where (γ_1, γ_2) are two primitive charge vectors. Across the wall $W(\gamma_1, \gamma_2)$, BPS bound states of particles with charges $\gamma^{(m)}$ lying in the two-dimensional sublattice spanned by γ_1 and γ_2 become unstable, leading to a jump in the index $\Omega(\gamma, z)$ for $\gamma = \sum_m \gamma^{(m)}$. The jump of $\Omega(\gamma, z)$ across the wall is determined by the Kontsevich-Soibelman (KS) wall-crossing formula [11].

The KS wall-crossing formula holds the key to the construction of the twistor space $\mathcal{Z}'$ for the instanton-corrected metric on $\mathcal{M}'$ as follows [9, 11]. At a fixed point z^i on the 4D Coulomb branch, the instanton correction from a BPS state with total charge γ induces a discontinuity in the canonical Darboux coordinates Ξ' across a meridian line ℓ_γ on the twistor fiber, also known as a BPS ray, which extends from the north ($\zeta = 0$) to the south ($\zeta = \infty$) pole at a longitude determined by the phase of the central charge $Z'_\gamma(z)$:

$$\ell_\gamma = \{\zeta \in \mathbb{C}^\times \,:\; Z'_\gamma(z)/\zeta \in \mathrm{i}\mathbb{R}^-\}\,. \tag{3.12}$$

The discontinuity in Ξ' is given by the action of a (twisted) complex symplectomorphism U_γ which is most conveniently represented in terms of its action on (twisted) holomorphic Fourier modes $\mathcal{X}'_\gamma$ (with respect to the abelian translation group (3.8)), defined by

$$\mathcal{X}'_\gamma \equiv \sigma(\gamma)\, e^{-2\pi \mathrm{i}\langle\gamma,\Xi'\rangle}\,. \tag{3.13}$$

Here $\sigma(\gamma)$ is a quadratic refinement of the intersection form on Γ, i.e. a homomorphism $\sigma : \Gamma \to \mathrm{U}(1)$ satisfying the cocycle relation

$$\sigma(H + H') = (-1)^{\langle H,H'\rangle}\, \sigma(H)\, \sigma(H')\,. \tag{3.14}$$

In the basis where $H = (n^\Lambda, m_\Lambda)$ the quadratic refinement can be parametrized by characteristics $\Theta = (\theta^\Lambda, \phi_\Lambda) \in (\Gamma \otimes \mathbb{R})/\Gamma$ such that [66]

$$\sigma_\Theta(H) = e^{-\pi \mathrm{i} m_\Lambda n^\Lambda + 2\pi\mathrm{i}\left(m_\Lambda\theta^\Lambda - n^\Lambda\phi_\Lambda\right)}\,. \tag{3.15}$$

We shall restrict ourselves to the case where $\sigma_\Theta(H)$ is real, i.e. where the characteristics are half integer. The action of the symplectomorphism U_γ on $\mathcal{X}'_\gamma$ is then

$$U_\gamma(z): \quad \mathcal{X}'_{\gamma'} \longmapsto \mathcal{X}'_{\gamma'}(1-\mathcal{X}'_\gamma)^{\Omega(\gamma,z)\langle\gamma,\gamma'\rangle}. \tag{3.16}$$

The symplectomorphisms U_γ are naturally identified with the abstract operators featuring in the KS wall-crossing formula, which we are now ready to state: as $z \in \mathcal{B}$ crosses the wall $W(\gamma_1,\gamma_2)$, the jump in the index $\Omega(\gamma,z)$ for γ lying in the two-dimensional sublattice spanned by γ_1,γ_2 should be such that the following product of symplectomorphisms stays constant:

$$A(\gamma_1,\gamma_2;z) = \prod_{\substack{\gamma=m_1\gamma_1+m_2\gamma_2\\ m_1,m_2\geq 0}} U_\gamma(z), \tag{3.17}$$

where the factors are ordered so that $\arg(Z'_\gamma)$ decreases from left to right (corresponding to a clockwise ordering of the BPS rays ℓ_γ). Equivalently, this may be rewritten as

$$\prod_{\substack{m_1\geq 0, m_2\geq 0,\\ m_1/m_2\downarrow}} U_{m_1\gamma_1+m_2\gamma_2}(z_+) = \prod_{\substack{m_1\geq 0, m_2\geq 0,\\ m_1/m_2\uparrow}} U_{m_1\gamma_1+m_2\gamma_2}(z_-), \tag{3.18}$$

where $z_\pm$ denote points infinitesimally close on opposite sides of the wall. By applying the Baker-Campbell-Hausdorff formula repeatedly, one may rewrite the product of factors appearing on the l.h.s. in the opposite order and express the BPS index $\Omega(\gamma)$ on one side of the wall in terms of its value on the other side (see e.g. [12] for more details). With this twistorial interpretation of the operators U_γ, it is now clear that the KS formula ensures that the complex symplectic structure on $\mathcal{Z}'$ defined by the collection of symplectomorphisms $\{U_\gamma(z), \gamma \in \Gamma\}$ is unchanged as $z \in \mathcal{B}$ crosses the wall.

According to a standard procedure, the HK metric can be obtained by 'parametrizing the twistor lines', i.e. determining the Darboux coordinates Ξ'_γ in terms of the coordinates $X^\Lambda, \zeta^\Lambda, \tilde\zeta_\Lambda$ on $\mathcal{M}'$ and of the coordinate ζ on the twistor fiber, and plugging them into the complex symplectic two-form (1.1). The gluing conditions (3.16) for the Darboux coordinates across the BPS rays, as well as the boundary conditions at $\zeta = 0$ and $\zeta = \infty$, can be summarized in the following system of integral equations [9, 21]:

$$\mathcal{X}'_\gamma = \mathcal{X}'^{\rm sf}_\gamma \exp\left[\frac{1}{4\pi \mathrm{i}}\sum_{\gamma'}\Omega(\gamma')\,\langle\gamma,\gamma'\rangle\int_{\ell_{\gamma'}}\frac{\mathrm{d}\zeta'}{\zeta'}\,\frac{\zeta+\zeta'}{\zeta-\zeta'}\,\log\left(1-\mathcal{X}'_{\gamma'}(\zeta')\right)\right]. \tag{3.19}$$

Solving (3.19) iteratively with respect to increasing number of instantons (by first plugging in $\mathcal{X}'^{\rm sf}_\gamma$ on the r.h.s. and successively repeating this procedure) generates an infinite series of multi-instanton corrections to the semi-flat metric. The one-instanton corrections correspond to the first correction in this iterative scheme and are weighted by a factor $\Omega(\gamma,z)e^{-4\pi|Z'_\gamma(z)|-2\pi \mathrm{i}\langle\gamma,C\rangle}$, which is discontinuous across walls of marginal stability. Nevertheless, multi-instanton corrections conspire so as to produce a smooth metric across the walls. In addition, these instanton corrections resolve the codimension 2 singularities arising from BPS states becoming massless, at least when the BPS state has primitive charge [9].

3.2 Wall-crossing in $\mathcal{N} = 2$ supergravity and contact geometry

We now turn to wall-crossing in the context of four-dimensional $\mathcal{N} = 2$ supergravities arising as low-energy limits of type II string compactifications on compact Calabi-Yau threefolds.

3.2.1 The hypermultiplet sector of $\mathcal{N} = 2$ string vacua

In string theory the analogue of the hyperkähler Coulomb branch $\mathcal{M}'$ is the vector multiplet moduli space $\mathcal{M}$ in type IIA/B on $X \times \mathbb{R}^3 \times S^1$, where X is a compact Calabi-Yau threefold. By T-duality along the compactification circle, the same space $\mathcal{M}$ also describes the hypermultiplet moduli space in type IIB/A on $X \times \mathbb{R}^4$. In either case local $\mathcal{N} = 2$ supersymmetry requires that $\mathcal{M}$ be quaternion-Kähler [67]. For definiteness we shall use a terminology adapted to the hypermultiplet sector of type IIA string theory. The dictionary for translating to other set-ups can be found in [10, 68].

In this language, $\mathcal{M}$ is parametrized by the expectation value ρ of the dilaton, the NS-axion σ (the 4D dual of the B-field), the periods $(\zeta^\Lambda, \tilde\zeta_\Lambda)$ of the RR 3-form C along a symplectic basis $(\mathcal{A}^\Lambda, \mathcal{B}_\Lambda)$ of the (D-brane) charge lattice $\Gamma \equiv H_3(X, \mathbb{Z})$, together with $n - 1$ complex scalars z^a, $a = 1, \ldots, n - 1$, corresponding to coordinates on the complex structure moduli space $\mathcal{M}_X$ of the Calabi-Yau. In the weak coupling limit, to all orders in $1/\rho$, the metric on $\mathcal{M}$ is given by the c-map metric (2.93) after identifying the projective special Kähler manifold $\mathcal{SK}$ with the complex structure moduli space $\mathcal{M}_X$, and fixing the parameter c to

$$c = -\chi(X)/(192\pi)\,, \tag{3.20}$$

where $\chi(X)$ is the Euler number of X. The twisted torus $\mathcal{T}$ is then identified with the intermediate Jacobian $H^3(X, \mathbb{R})/H^3(X, \mathbb{Z})$ in the Weil complex structure, with metric given in (2.95). This torus is in turn fibered over $\mathcal{M}_X$ with total space $\mathcal{J}_X \to \mathcal{M}_X$ known as the relative intermediate Jacobian. Similarly as in field theory the torus coordinates $C \equiv (\zeta^\Lambda, \tilde\zeta_\Lambda)$ are periodic with integer periods, due to large gauge transformations of the RR 3-form. However now the large gauge transformations involve an additional shift of the NS-axion σ:

$$\zeta^\Lambda \mapsto \zeta^\Lambda + n^\Lambda, \qquad \tilde\zeta_\Lambda \mapsto \tilde\zeta_\Lambda + m_\Lambda, \qquad \sigma \mapsto \sigma + 2\kappa + \langle C - 2\Theta, H\rangle - n^\Lambda m_\Lambda, \tag{3.21}$$

where $(n^\Lambda, m_\Lambda, \kappa) \in \mathbb{Z}^{2n} \times \mathbb{Z}$. The characteristics $\Theta = (\theta^\Lambda, \phi_\Lambda) \in H^3(X, \mathbb{R})/H^3(X, \mathbb{Z})$ appearing in (3.21) are conjecturally identified with those appearing in the quadratic refinement (3.15) [69]. Due to the periodicity under $\sigma \to \sigma + 2$, the NS-axion σ parametrizes the fiber of a circle bundle over $\mathcal{J}_X$. Hence, at fixed large value of ρ (weak-coupling limit) the manifold $\mathcal{M}_\rho$ is the total space of the fibration [68, 69]:

$$S^1_\sigma \longrightarrow \mathcal{M}_\rho \longrightarrow \mathcal{J}_X\,, \tag{3.22}$$

equipped with a connection $D\sigma$ given by (2.97). The fibration (3.22) is the stringy generalization of the twisted torus bundle $\mathcal{T}_z \to \mathcal{M}' \to \mathcal{B}$ in $\mathcal{N} = 2$ field theories on $\mathbb{R}^3 \times S^1$.

As explained in section 2.4.1, the perturbative metric (2.93) on $\mathcal{M}$ is most conveniently described in terms of its twistor space, with Darboux coordinates $\Xi = (\xi^\Lambda, \tilde\xi_\Lambda)$ and $\tilde\alpha$ given in (2.99). In particular, the large gauge transformations (3.21) lift to the holomorphic action on $\mathcal{Z}$,

$$\xi^\Lambda \mapsto \xi + n^\Lambda, \qquad \tilde\xi_\Lambda \mapsto \tilde\xi_\Lambda + m_\Lambda, \qquad \tilde\alpha \mapsto \tilde\alpha + 2\kappa + \langle \Xi - 2\Theta, H\rangle - n^\Lambda m_\Lambda, \tag{3.23}$$

leaving invariant the holomorphic Fourier modes

$$\mathcal{X}_\gamma = \sigma(\gamma)\, e^{-2\pi \mathrm{i} \langle\gamma,\Xi\rangle}\,, \tag{3.24}$$

which are the direct analogues of the holomorphic Fourier modes (3.13) in the field theory case.

3.2.2 D-brane instantons and wall-crossing

Similarly to the semi-flat metric on $\mathcal{M}'$, the perturbative metric on $\mathcal{M}$ is only physically valid in the strict weak-coupling limit where $\rho = \infty$. At finite values of the coupling there are additional effects arising from D-brane instantons, i.e. Euclidean D-branes wrapping supersymmetric cycles in the internal manifold. For type IIA compactified on a Calabi-Yau X these correspond to D2-branes wrapping special Lagrangian 3-cycles (sLags) in X, with homology class $\gamma = [q_\Lambda \mathcal{A}^\Lambda - p^\Lambda \mathcal{B}_\Lambda] \in H_3(X,\mathbb{Z})$. In the weak coupling limit $\rho \to \infty$ and in the one-instanton approximation, corrections to the metric on $\mathcal{M}$ are of the form $\Omega(\gamma, z) e^{-8\pi |Z_\gamma|/g_s - 2\pi \mathrm{i}\langle\gamma, C\rangle}$, where $g_s \equiv \rho^{-1/2}$ is the string coupling, $Z_\gamma(z)$ is the central charge function given by a period integral of the holomorphic 3-form $\Omega^{3,0} \in H^{3,0}(X,\mathbb{C})$:

$$Z_\gamma(z) = e^{\mathcal{K}/2} \int_\gamma \Omega^{3,0} = e^{\mathcal{K}/2} \left(q_\Lambda X^\Lambda(z) - p^\Lambda F_\Lambda(z) \right) \in \mathbb{C}\,, \tag{3.25}$$

and $\Omega(\gamma, z)$ is the generalized Donaldson-Thomas (DT) invariant, counting the number of stable sLags in homology class γ. Just like the BPS indices in rigid $\mathcal{N} = 2$ field theories, the DT invariants $\Omega(\gamma)$ are locally constant functions of $z^a \in \mathcal{M}_X$ but may jump on codimension 2 subspaces $W(\gamma_1, \gamma_2)$ defined as in (3.11) (with $\mathcal{B}$ replaced by $\mathcal{M}_X$), with a jump determined by the KS wall-crossing formula (3.18). Importantly, D-brane instanton corrections (unlike NS5-instanton corrections) are independent of the NS-axion σ, and therefore preserve the Killing vector ∂_σ.

The D-instanton corrected metric on $\mathcal{M}$, or rather its twistor space $\mathcal{Z}$, was constructed in [10, 21], based on consistency with S-duality and mirror symmetry. The construction is formally identical to the construction of the twistor space $\mathcal{Z}'$ of instanton corrected Coulomb branch $\mathcal{M}'$ in $\mathcal{N} = 2$ gauge theories, in particular the holomorphic Fourier modes $\mathcal{X}_\gamma$ satisfy the same discontinuities (3.16) across BPS rays and integral equations (3.19) as in the field theory case.[17] In addition, the discontinuity of the contact coordinate α was specified in [10, 21] in terms of the Spence dilogarithm function, ensuring that the combined transformation of $(\xi^\Lambda, \tilde\xi_\Lambda, \tilde\alpha)$ preserves the contact one-form. However, this construction was unsatisfactory on two counts: i) requiring that the change of Darboux coordinates is a contact transformation determined the shift of $\tilde\alpha$ only up to an additive constant, and there could have been a global obstruction in choosing these constants, and ii) the notion of

[17] The quadratic refinement $\sigma(\gamma)$ was ignored in [10, 21], but has been included later in [68].

Darboux coordinates being discontinuous across BPS rays ignored the fact that, unlike the field theory twistor space $\mathcal{Z}'$, the stringy twistor space $\mathcal{Z}$ is *not* trivially fibered over $\mathbb{C}P^1$, but rather is a non-trivial $\mathbb{C}P^1$ bundle over $\mathcal{M}$. In the rest of this section, we shall use the QK/HK correspondence to put the construction of [10, 21] on a more rigorous basis.

3.3 D-brane instantons, QK/HK correspondence and Rogers dilogarithm

As outlined in section 1, our approach is based on the fact that D-instanton corrections preserve the continuous isometry ∂_σ corresponding to shifts along the NS-axion σ. Therefore, the D-instanton corrected quaternion-Kähler metric on $\mathcal{M}$ can be equivalently described in terms of the dual HK metric on $\mathcal{M}'$ and hyperholomorphic line bundle $\mathscr{L}$. At the level of twistor spaces, the complex contact geometry on the twistor space $\mathcal{Z}$ over $\mathcal{M}$ is then equivalently described by the complex symplectic geometry of $\mathcal{Z}'$, equipped with the holomorphic line bundle $\mathscr{L}_{\mathcal{Z}'}$.

The results of section 2.2.3 imply that the Darboux coordinates $\Xi' = (\eta^\Lambda, \mu_\Lambda)$ on $\mathcal{Z}'$ are identified with the Darboux coordinates $\Xi = (\xi^\Lambda, \tilde\xi_\Lambda)$ on the reduced twistor space $\mathcal{Z}/\partial_\alpha$, while the additional contact coordinate $\Upsilon = e^{-2\mathrm{i}\pi\alpha}$ on $\mathcal{Z}$ parametrizes the $\mathbb{C}^\times$-fiber of $\mathscr{L}_{\mathcal{Z}'} \to \mathcal{Z}'$. The construction of the twistor space $\mathcal{Z}$ obtained in [10] can now be rephrased as follows:

i) The dual twistor space $\mathcal{Z}'$ is given by the same construction as in $\mathcal{N}=2$ field theory, with the appropriate central charge function (3.25) and BPS invariants $\Omega(\gamma, z)$. In other words, $\mathcal{Z}'$ is described by complex symplectomorphisms U_γ (3.16) relating the Darboux coordinates Ξ' across the BPS rays (3.12).

ii) The holomorphic line bundle $\mathscr{L}_{\mathcal{Z}'}$ over $\mathcal{Z}'$ is defined by transition functions (1.8) evaluated on the dual coordinates $\Xi' = (\eta^\Lambda, \mu_\Lambda)$:

$$f_\gamma = \frac{\Upsilon_+}{\Upsilon_-} = \exp\left(\frac{\mathrm{i}}{2\pi}\, S_\gamma(\Xi')\right), \tag{3.26}$$

where $S_\gamma(\Xi')$ can be computed from eq. (3.32) in [21] and reads

$$S_\gamma(\Xi') = \Omega(\gamma)\left[\mathrm{Li}_2\left(\mathcal{X}'_\gamma\right) - 2\pi\mathrm{i}\, q_\Lambda \eta^\Lambda \log\left(1-\mathcal{X}'_\gamma\right) + \frac{1}{2}\Omega(\gamma) p^\Lambda q_\Lambda \left[\log\left(1-\mathcal{X}'_\gamma\right)\right]^2\right]. \tag{3.27}$$

To elucidate the transition function (3.27), it is useful to change coordinates and use the symplectic invariant coordinate $\tilde\alpha$ (1.9) in place of α. Eq. (3.27) can then be rewritten as

$$\Delta_\gamma \tilde\alpha = \tilde\alpha_+ - \tilde\alpha_- = \frac{1}{2\pi^2}\, \tilde S_\gamma(\Xi'), \tag{3.28}$$

where the function $\tilde S_\gamma$ is defined by

$$\tilde S_\gamma(\Xi') = \Omega(\gamma)\, L_{\sigma(\gamma)}(\mathcal{X}'_\gamma)\,. \tag{3.29}$$

Here, $L_\epsilon(z)$ is a variant of the Rogers dilogarithm $L(z)$ (1.11) defined by

$$L_\epsilon(z) \equiv \mathrm{Li}_2(z) + \frac{1}{2}\,\log(\epsilon^{-1} z)\log(1-z)\,. \tag{3.30}$$

To check that the combination of the shift (3.28) and the symplectomorphism (3.16) preserves the contact one-form

$$\mathcal{X} = -\frac{1}{2}\Big(\mathrm{d}\tilde\alpha + \langle \Xi', \mathrm{d}\Xi'\rangle\Big), \tag{3.31}$$

we note that under the complex symplectomorphism U_γ,

$$U_\gamma \; : \; \langle \Xi', \mathrm{d}\Xi'\rangle \; \mapsto \; \langle \Xi', \mathrm{d}\Xi'\rangle - \frac{\Omega(\gamma)}{2\pi\mathrm{i}}\left[\Xi'_\gamma \,\mathrm{d}\log(1-\mathcal{X}'_\gamma) - \log(1-\mathcal{X}'_\gamma)\,\mathrm{d}\Xi'_\gamma\right], \tag{3.32}$$

and use the following properties[18] of the function $L_\epsilon(z)$:

$$L_\epsilon(z) \,=\, L(z) - \frac{1}{2}\,\log\epsilon\,\log(1-z)\,, \tag{3.33}$$

$$\mathrm{d}L_\epsilon(z) \,=\, -\frac{1}{2}\left(\frac{\log(1-z)}{z} + \frac{\log(\epsilon^{-1}z)}{1-z}\right)\mathrm{d}z\,. \tag{3.34}$$

It is also important to note that the shift (3.28) is consistent with the invariance under the large gauge transformations (3.23) thanks to the monodromy of the Rogers dilogarithm around $z=0$. Indeed, consider the action (3.23) on the Darboux coordinates $(\Xi'_-,\tilde\alpha_-)$ on one side of a BPS ray. Under this action, the Fourier mode $\mathcal{X}'_\gamma$ rotates by $e^{-2\pi\mathrm{i}\langle\gamma,H\rangle}$, where $H=(n^\Lambda,m_\Lambda)\in\mathbb{Z}^{2n}$. On the other side, the Darboux coordinates $(\Xi'_+,\tilde\alpha_+)$ determined by (3.36) will transform the same way as in (3.23) provided $\tilde S_\gamma$ transforms as

$$\tilde S_\gamma \mapsto \tilde S_\gamma - \mathrm{i}\pi\,\Omega(\gamma)\,\langle\gamma,H\rangle\,\log\left(1-\mathcal{X}'_\gamma\right). \tag{3.35}$$

This property is indeed ensured by the last term in (3.30).

Postponing the issue of consistency with wall-crossing to the next subsection, we conclude that the geometry of the D-instanton corrected hypermultiplet moduli space $\mathcal{M}$ is obtained via the QK/HK correspondence from the HK manifold $\mathcal{M}'$ and hyperholomorphic connection λ, whose twistor space $\mathcal{Z}'$ is governed by the discontinuity conditions

$$V_\gamma \quad : \quad (\Xi',\tilde\alpha) \;\longmapsto\; \left(U_\gamma\cdot\Xi',\,\tilde\alpha + \frac{1}{2\pi^2}\,\tilde S_\gamma(\Xi')\right) \tag{3.36}$$

across the BPS ray (3.12). These discontinuities can be viewed either as contact transformations on $\mathcal{Z}$, or as a combination of a complex symplectomorphism on $\mathcal{Z}'$ and a gauge transformation on $\mathscr{L}_{\mathcal{Z}'}$. These gluing conditions together with the regularity conditions at $\zeta=0$ and $\zeta=\infty$ can be summarized by the same integral equations (3.19) for the holomorphic Fourier modes $\mathcal{X}'_\gamma$, supplemented by an integral formula for the holomorphic section $\tilde\Upsilon\equiv e^{\mathrm{i}\pi\tilde\alpha}$ of $\mathscr{L}'_{\mathcal{Z}}$

$$\tilde\Upsilon = \exp\left[\mathrm{i}\pi\left(\sigma+\zeta^{-1}\mathcal{W}-\zeta\bar{\mathcal{W}}\right) - \frac{\chi_X}{24}\left(\log\zeta - \mathrm{i}\theta'\right) + \frac{1}{8\pi^2}\sum_\gamma\int_{\ell_\gamma}\frac{\mathrm{d}\zeta'}{\zeta'}\,\frac{\zeta+\zeta'}{\zeta-\zeta'}\,\tilde S_\gamma\left(\Xi\right)\right], \tag{3.37}$$

where we defined

$$\mathcal{W} = G_\Lambda\zeta^\Lambda - Z^\Lambda\tilde\zeta_\Lambda + \frac{1}{8\pi^2}\sum_\gamma\Omega(\gamma)\,Z'_\gamma(z)\int_{\ell_\gamma}\frac{\mathrm{d}\zeta'}{\zeta'}\,\log\left(1-\mathcal{X}'_\gamma\right). \tag{3.38}$$

[18] See (A.29), (A.32) for a precise statement of these properties.

This result can be easily translated into an expression for the contact coordinate $\tilde{\alpha}$ on the QK side using (1.6) and (2.109). This expression is equivalent to the one obtained in [21], eq. (3.32), but is considerably simpler thanks to the use of the Rogers dilogarithm. By plugging the solution of (3.19) and (3.37) (after implementing the identifications (2.43)) into the contact one-form (3.31) and matching to (2.7), one may extract the D-instanton corrections to the perturbative metric (2.93) in a systematic fashion.

3.4 Dilogarithm identities and wall-crossing

We now return to the issue of the consistency of the set of discontinuities (3.36) with wall-crossing. For the construction to be independent of the value of the moduli $z \in \mathcal{M}_X$, the transformations V_γ should satisfy the obvious generalization of the KS wall-crossing identity,

$$\prod_{\substack{m_1\geq 0, m_2 \geq 0,\\ m_1/m_2 \downarrow}} V_{m_1\gamma_1+m_2\gamma_2}(z_+) = \prod_{\substack{m_1\geq 0, m_2 \geq 0,\\ m_1/m_2 \uparrow}} V_{m_1\gamma_1+m_2\gamma_2}(z_-) \,. \tag{3.39}$$

By construction this formula reduces to (3.18) when projecting onto the base of the fibration $\mathscr{L}_{\mathcal{Z}'} \to \mathcal{Z}'$. This implies in particular that the left and right-hand sides of (3.39) differ at most by a translation $\tilde{\alpha} \to \tilde{\alpha} + \Delta\tilde{\alpha}$ along the $\mathbb{C}^\times$-fiber, which is given by the cumulative effect of the translations in (3.36). To present the explicit form of this shift it is useful to first rewrite the wall-crossing formula (3.39) by assembling all the operators on one side:

$$\prod_s V_{\gamma_s}^{\epsilon_s} = 1 \,, \tag{3.40}$$

where the product runs over all charge vectors appearing in (3.39), and ϵ_s is a sign which changes from $+1$ on the right of the product (corresponding to the r.h.s. of (3.39)) to -1 on the left (corresponding to the inverse of the l.h.s. of (3.39)). The total translation along the fiber of $\mathscr{L}_{\mathcal{Z}'} \to \mathcal{Z}'$ is now given by

$$\Delta\tilde{\alpha} = \frac{1}{2\pi^2} \sum_s \epsilon_s \, \Omega(\gamma_s)\, L_{\sigma(\gamma_s)}\big(\mathcal{X}_{\gamma_s}(s)\big), \tag{3.41}$$

where $\mathcal{X}_{\gamma_s}(s)$ denotes the Fourier mode $\mathcal{X}_{\gamma_s}$ successively acted upon by all preceding gauge transformations:

$$\mathcal{X}_{\gamma_s}(s) = U_{\gamma_{s-1}} \circ U_{\gamma_{s-2}} \circ \cdots \circ U_{\gamma_2} \circ U_{\gamma_1} \cdot \mathcal{X}_{\gamma_s}. \tag{3.42}$$

We thus need to show that the total shift $\Delta\tilde{\alpha}$ in (3.41) vanishes modulo the natural periodicity of the variable $\tilde{\alpha}$,

$$\Delta\tilde{\alpha} = 0 \mod 2 \,. \tag{3.43}$$

Fortunately, we shall now see that the motivic wall-crossing formula of Kontsevich and Soibelman [11] ensures that the non-trivial functional identity (3.43) for the Rogers dilogarithm indeed holds. Our strategy will be to consider the semi-classical limit of the motivic wall-crossing formula, using the techniques of [32, 33].

Recall that the motivic wall-crossing formula pertains to the 'refined index' [70] (more accurately, the protected spin character [71])

$$\Omega(\gamma, y, z^a) = \operatorname{Tr}'(-y)^{2J_3} = \sum_{n\in\mathbb{Z}} (-y)^n \Omega_n(\gamma, z^a) \tag{3.44}$$

(here Tr' denotes a trace on the space orthogonal to the bosonic and fermionic translational zero-modes). Although this quantity is not protected in string theory, it is nevertheless a useful construct, since its behavior under wall-crossing can be computed using localization methods which would break down at $y = 1$. The motivic wall-crossing formula takes a similar form as (3.18),

$$\prod_{\substack{m_1 \geq 0, m_2 \geq 0, \\ m_1/m_2 \downarrow}} \hat{U}_{m_1\gamma_1+m_2\gamma_2}(z_+) = \prod_{\substack{m_1 \geq 0, m_2 \geq 0, \\ m_1/m_2 \uparrow}} \hat{U}_{m_1\gamma_1+m_2\gamma_2}(z_-) \tag{3.45}$$

but the operators $\hat{U}_\gamma$ are now given by [11, 72][19]

$$\hat{U}_\gamma = \prod_{n\in\mathbb{Z}} \left[\Psi_{q^{1/2}}(y^n \hat{\mathcal{X}}_\gamma)\right]^{-(-1)^n \Omega_n(\gamma, z^a)}, \quad y = -q^{1/2} = e^{\mathrm{i}\pi\hbar}, \tag{3.46}$$

where $\Psi_{q^{1/2}}(x)$ is the quantum dilogarithm defined in (A.33), and $\hat{\mathcal{X}}_\gamma$ are generators of the quantum torus

$$\hat{\mathcal{X}}_\gamma\, \hat{\mathcal{X}}_{\gamma'} = (-y)^{\langle\gamma,\gamma'\rangle}\, \hat{\mathcal{X}}_{\gamma+\gamma'}\,. \tag{3.47}$$

In particular, for a hypermultiplet BPS state with $\Omega(y) = 1$, $\hat{U}_\gamma = \Psi^{-1}_{q^{1/2}}(\hat{\mathcal{X}}_\gamma)$. In the classical limit $y \to 1$, the adjoint action

$$\mathrm{Ad}\,\hat{U}_\gamma \ : \ \hat{\mathcal{X}}_{\gamma'} \mapsto \hat{U}_\gamma \hat{\mathcal{X}}_{\gamma'} (\hat{U}_\gamma)^{-1} \tag{3.48}$$

reduces to the usual twisted symplectomorphism (3.16). Thus, the motivic wall-crossing formula (3.45) implies the numerical wall-crossing formula (3.18). However, we shall see that it also implies a functional identity for the Rogers dilogarithm, which yields the stronger contact wall-crossing formula (3.39).

To see this, we proceed as in [32, 33], and realize the generators of the quantum torus (3.47) as unitary operators acting on $L^2(\mathbb{R}^{2r})$:

$$\hat{\mathcal{X}}_\gamma = \sigma(\gamma) \exp\left(Q^i(\hat{p}_i + \epsilon_{ij}\hat{u}^j)\right), \tag{3.49}$$

where $\gamma = Q^i e_i$ (so Q^i contains both the electric and magnetic charges), $\langle\gamma,\gamma'\rangle = \epsilon_{ij} Q^i Q'^j$, and

$$\left[\hat{u}^i, \hat{u}^j\right] = [\hat{p}_i, \hat{p}_j] = 0\,, \qquad \left[\hat{p}_i, \hat{u}^j\right] = -\mathrm{i}\pi\hbar\delta_{ij}\,. \tag{3.50}$$

It will be convenient to use a complete basis of wave functions $\langle u|$ and $|p\rangle$ which diagonalize the action of $\hat{u}^i$ and $\hat{p}_i$, respectively:

$$\langle u|\, \hat{u}^i = u^i\, \langle u|\,, \qquad \langle u|\, \hat{p}_i = -\mathrm{i}\pi\hbar\, \partial_{u^i}\, \langle u| \tag{3.51}$$

and similarly

$$\hat{p}_i\, |p\rangle = p_i\, |p\rangle\,, \qquad \hat{u}^i\, |p\rangle = \mathrm{i}\pi\hbar\, \partial_{p_i}\, |p\rangle\,. \tag{3.52}$$

[19] We use the conventions of [73]. Note that $\hbar$ differs by a factor of π from the one used in [32], and that y differs by a sign from the one used in [71].

The inner products and completeness relations are

$$\langle u|p\rangle = \langle p|u\rangle^{-1} = e^{\frac{\mathrm{i}}{\pi\hbar}u^i p_i}\,, \qquad \int \mathrm{d}u\,|u\rangle\langle u| = \frac{1}{(2\pi^2\hbar)^n}\int \mathrm{d}p\,|p\rangle\langle p| = 1\,. \tag{3.53}$$

We now assume that both sides of (3.45) have a finite number of factors, and rewrite it in a similar way as (3.40):

$$\hat{U} \equiv \prod_{s=1}^{N} \hat{U}_{\gamma_s}^{\epsilon_s} = 1\,, \tag{3.54}$$

where, as before, ϵ_s changes from $+1$ on the right of the product (corresponding to the r.h.s. of (3.45)) to -1 on the left (corresponding to the inverse of the l.h.s. of (3.45)). Thus, for any u, p, we have $\langle u|\hat{U}|p\rangle/\langle u|p\rangle = 1$. Inserting a complete basis of states between each of the factors in (3.54), we arrive, as in eq. (5.10) of [32], at

$$\begin{aligned}
&(2\pi^2\hbar)^{-n(N-1)} \int \mathrm{d}p(1)\mathrm{d}u(2)\mathrm{d}p(2)\ldots\mathrm{d}u(N) \\
&\langle p(0)|u(1)\rangle\,\frac{\langle u(1)|\hat{U}_{\gamma_1}^{\epsilon_1}|p(1)\rangle}{\langle u(1)|p(1)\rangle}\langle u(1)|p(1)\rangle \times \langle p(1)|u(2)\rangle\,\frac{\langle u(2)|\hat{U}_{\gamma_2}^{\epsilon_s}|p(2)\rangle}{\langle u(2)|p(2)\rangle}\langle u(2)|p(2)\rangle \\
&\cdots \times \langle p(N-1)|u(N)\rangle\,\frac{\langle u(N)|\hat{U}_{\gamma_N}^{\epsilon_N}|p(N)\rangle}{\langle u(N)|p(N)\rangle}\langle u(N)|p(N)\rangle = 1\,,
\end{aligned} \tag{3.55}$$

where $u \equiv u(1) \equiv u(N+1), p \equiv p(N) \equiv p(0)$. Now, we use the fact that

$$\frac{\langle u|\hat{U}_{\gamma}^{\epsilon}|p\rangle}{\langle u|p\rangle} = \prod_{n\in\mathbb{Z}} \left(\Psi_{q^{1/2}}\left[y^n\sigma(\gamma)\exp\left(Q^i(p_i+\epsilon_{ij}u^j)\right)\right]\right)^{(-1)^{n+1}\Omega_n(\gamma)\,\epsilon}\,. \tag{3.56}$$

In the semi-classical limit $\hbar \to 0$, the integral (3.55) can be evaluated in the saddle point approximation. Using (A.36), we arrive at

$$(2\pi^2\hbar)^{-n(N-1)} \int \mathrm{d}p(1)\mathrm{d}u(2)\mathrm{d}p(2)\ldots\mathrm{d}u(N)\,\exp\left(\frac{1}{\pi\mathrm{i}\hbar}\,S\right) \sim 1\,, \tag{3.57}$$

where

$$S = \sum_{s=1}^{N} \left[\frac{\epsilon_s}{2}\,\Omega(\gamma_s)\mathrm{Li}_2\left[\sigma(\gamma_s)\,\mathcal{Y}_s\right] - u^i(s)(p_i(s)-p_i(s-1))\right] \tag{3.58}$$

and we defined $\mathcal{Y}_s \equiv e^{Q_s^i[p_i(s)+\epsilon_{ij}u^j(s)]}$.

We thus need to extremize S with respect to $p_i(s), s = 1\ldots N-1$, and $u^i(s), s = 2\ldots N$. Using (A.3), we arrive at

$$\begin{aligned}
p_i(s) - p_i(s-1) &= \frac{1}{2}\,\epsilon_{ij}Q_s^j\,\epsilon_s\,\Omega(\gamma_s)\,\log\left[1-\sigma(\gamma_s)\,\mathcal{Y}_s\right], \qquad s = 2,\ldots,N \\
u^i(s) - u^i(s+1) &= -\,\frac{1}{2}\,Q_s^i\,\epsilon_s\,\Omega(\gamma_s)\,\log\left[1-\sigma(\gamma_s)\,\mathcal{Y}_s\right], \qquad s = 1,\ldots,N-1.
\end{aligned} \tag{3.59}$$

From this we conclude that for $s = 2,\ldots,N$ the quantity $\delta_i(s) \equiv p_i(s-1) - \epsilon_{ij}u^j(s)$ is independent of s. Furthermore, we can choose the initial and final states to satisfy the same relation, i.e. $p_i(0) - \epsilon_{ij} w^j(1) = \delta_i$. On the other hand, the same quantity can be evaluated using the saddle point equations (3.59). Equating the result to δ_i, one arrives at the following requirement

$$\sum_{s=1}^{N} \epsilon_{ij} Q^j(s)\, \epsilon_s\, \Omega(\gamma_s)\, \log\left[1 - \sigma(\gamma_s)\, \mathcal{Y}_s\right] = 0. \tag{3.60}$$

The left-hand side is recognized as the product of KS factors corresponding to the product (3.54) of quantum dilogarithms, provided that we can identify $\sigma(\gamma_s)\mathcal{Y}_s$ with $\mathcal{X}_{\gamma_s}$. To establish this identification, note that for all s one has

$$Q_s^i \epsilon_{ij} w^j(s) = Q_s^i (p_i(s) - \delta_i). \tag{3.61}$$

Therefore, for arbitrary charge γ we can define $\mathcal{Y}_\gamma(s) \equiv e^{Q^i(2p_i(s)-\delta_i)}$ such that $\mathcal{Y}_{\gamma_s}(s) = \mathcal{Y}_s$. The advantage of these new functions is that for all s they satisfy the following recurrence relation

$$\mathcal{Y}_{\gamma'}(s-1) = \mathcal{Y}_{\gamma'}(s)\, \left(1 - \sigma(\gamma_s)\mathcal{Y}_s\right)^{\epsilon_s \langle\gamma_s,\gamma'\rangle\, \Omega(\gamma_s)}. \tag{3.62}$$

This is precisely the symplectomorphism (3.16) for $\epsilon_s = 1$, or its inverse for $\epsilon_s = -1$, which allows to identify $\mathcal{X}_\gamma = \sigma(\gamma)\mathcal{Y}_\gamma$.

Given these results, in particular the constraint (3.60), it is now easy to show that the action S at the saddle point can be rewritten as

$$S = \frac{1}{2}\sum_{s=1}^{N} \epsilon_s\, \Omega(\gamma_s)\, \left[\mathrm{Li}_2\left(\sigma(\gamma_s)\mathcal{Y}_s\right) + \frac{1}{2}\, Q_s^i\left(2p_i(s) - \delta_i\right) \log\left(1 - \sigma(\gamma_s)\mathcal{Y}_s\right)\right]. \tag{3.63}$$

Thus, the vanishing of S at the saddle point leads to the dilogarithm identity

$$\sum_{s=1}^{N} \epsilon_s\, \Omega(\gamma_s)\, L_{\sigma(\gamma_s)}\left(\sigma(\gamma_s)\mathcal{Y}_s\right) = 0\,. \tag{3.64}$$

This formula generalizes the "non-simply laced" Rogers dilogarithm identities (B.23), proven and conjectured in [29–31] using techniques from the theory of cluster categories. The general identity (3.64) shows that the constant shift (3.41) vanishes identically, at least on the slice where all the Fourier modes $\mathcal{X}_\gamma$ are real. By analytic continuation, it will continue to vanish on the universal cover of the complex torus. In the next subsection, we carry out this analytic continuation in detail for some simple examples of wall-crossing, where (3.64) reduces to the known 5-term, 6-term and 8-term relations for the Rogers dilogarithm.

3.5 Analytic continuation of the pentagon, hexagon and octagon identities

In this subsection, we show the consistency of the prescription (3.29) with wall-crossing in three simple examples which involve only a finite number of BPS states on either side of the wall. Since wall-crossing involves only a two-dimensional sublattice of the total

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charge lattice, we can restrict to the rank 2 case, and parametrize the complex torus by two $\mathbb{C}^\times$-valued variables

$$x = e^{2\pi\mathrm{i}\tilde{\xi}} = e^{u}, \qquad y = e^{-2\pi\mathrm{i}\xi} = e^{\tilde{u}}. \tag{3.65}$$

The KS symplectomorphism (3.16) acts as on x, y as

$$U_{p,q}^{(\Omega)}: \quad [x, y] \mapsto [(1-\mathcal{X}_{p,q})^{q\,\Omega} x, (1-\mathcal{X}_{p,q})^{-p\,\Omega} y], \quad \mathcal{X}_{p,q} \equiv \sigma_{p,q} x^p y^q\,, \tag{3.66}$$

preserving the symplectic form

$$\mathrm{d}\xi \wedge \mathrm{d}\tilde{\xi} = \frac{1}{4\pi^2} \frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y}\,. \tag{3.67}$$

When $\Omega = 1$, we omit the superscript and denote $U_{p,q} = U_{p,q}^{(1)}$. The inverse of $U_{p,q}^{(\Omega)}$ is $U_{p,q}^{(-\Omega)}$. The contact transformation $V_\gamma \equiv V_{p,q}^{(\Omega)}$ is obtained by supplementing the action (3.66) by a translation of the contact variable $\tilde{\alpha} \equiv z/(2\pi^2)$,

$$\tilde{\alpha} \mapsto \tilde{\alpha} + \frac{\Omega}{2\pi^2} L_{\sigma_{p,q}}(\mathcal{X}_{p,q})\,. \tag{3.68}$$

More accurately, one should choose logarithms $u_{p,q}, v_{p,q}$, such that

$$e^{u_{p,q}} = \mathcal{X}_{p,q}\,, \qquad e^{v_{p,q}} = 1 - \mathcal{X}_{p,q}\,, \qquad u_{p,q} = pu + q\tilde{u} + 2\pi\mathrm{i} c_{p,q}\,, \tag{3.69}$$

where $c_{p,q}$ is an element of $\mathbb{R}/\mathbb{Z}$ such that $\sigma_{p,q} = (-1)^{2c_{p,q}}$, and express the variation of $\tilde{\alpha}$ in terms of the enhanced Rogers dilogarithm, whose definition and basic properties are recalled in appendix A:

$$\tilde{\alpha} \mapsto \tilde{\alpha} + \frac{\Omega}{2\pi^2} \left[L(u_{p,q}, v_{p,q}) - \mathrm{i}\pi c_{p,q} v_{p,q} \right]. \tag{3.70}$$

By construction, $V_{p,q}^{(\Omega)}$ preserves the contact one-form

$$-2\mathcal{X} = \mathrm{d}\tilde{\alpha} + \tilde{\xi}\mathrm{d}\xi - \xi\mathrm{d}\tilde{\xi} = \frac{1}{2\pi^2} \left(\mathrm{d}z + \frac{1}{2}(u\,\mathrm{d}\tilde{u} - \tilde{u}\,\mathrm{d}u) \right). \tag{3.71}$$

Moreover, the inverse of $V_{p,q}^{(\Omega)}$ is $V_{p,q}^{(-\Omega)}$.

The simplest example involves a single BPS state of charge $\gamma_1 + \gamma_2$ with $\langle \gamma_1, \gamma_2 \rangle = 1$ decaying into its components of charge γ_1 and γ_2. The corresponding product of KS symplectomorphisms is the usual 'pentagon identity'

$$U_{0,1}^{-1}\, U_{1,0}^{-1}\, U_{0,1}\, U_{1,1}\, U_{1,0} = 1\,, \tag{3.72}$$

which holds whenever $\sigma_{1,0}\sigma_{0,1} = -\sigma_{1,1}$, as required by the quadratic refinement condition (3.14). The successive images $(x_s, y_s)_{s=0,\dots 4}$ of $(x_0, y_0) \equiv (x, y)$ under the sequence of symplectomorphisms (3.72) (from right to left), as well as the monomials $\mathcal{X}_s \equiv \mathcal{X}_{\gamma_s}^{-\epsilon_s}$ and $1 - \mathcal{X}_s$ are displayed in table 1. Upon extending the range of s from $0, \dots 4$ to $\mathbb{Z}$ by

s	ϵ_s	x_s	y_s	$\mathcal{X}_s$	$1-\mathcal{X}_s$
0	1	x	y	$\frac{\sigma_{1,0}}{x}$	$\frac{x-\sigma_{1,0}}{x}$
1	1	x	$\frac{y}{1-\sigma_{1,0}x}$	$\frac{\sigma_{1,1}(1-\sigma_{1,0}x)}{xy}$	$-\frac{\sigma_{1,1}(1-\sigma_{1,0}x-\sigma_{1,1}xy)}{xy}$
2	1	$\frac{x(1-\sigma_{1,0}x-\sigma_{1,1}xy)}{1-\sigma_{1,0}x}$	$\frac{y}{1-\sigma_{1,0}x-\sigma_{1,1}xy}$	$\frac{\sigma_{0,1}(1-\sigma_{1,0}x-\sigma_{1,1}xy)}{y}$	$\frac{(\sigma_{1,0}x-1)(\sigma_{0,1}-y)}{y}$
3	-1	$x(1-\sigma_{0,1}y)$	$\frac{y}{1-\sigma_{1,0}x-\sigma_{1,1}xy}$	$\sigma_{1,0}x(1-\sigma_{0,1}y)$	$1-\sigma_{1,0}x-\sigma_{1,1}xy$
4	-1	$x(1-\sigma_{0,1}y)$	y	$\sigma_{0,1}y$	$1-\sigma_{0,1}y$
5	1	x	y	$\frac{\sigma_{1,0}}{x}$	$\frac{x-\sigma_{1,0}}{x}$

Table 1. The sequence of symplectomorphisms U_γ corresponding to the pentagon identity.

requiring periodicity modulo 5, one easily checks that $\mathcal{X}_s$ satisfies the recursion relation

$$\mathcal{X}_{s-1}\mathcal{X}_{s+1} = 1 - \mathcal{X}_s\,, \tag{3.73}$$

with periodicity 5. As we discuss in appendix B, this recursion relation finds its origin in the periodicity of mutations of the cluster algebra associated to the Dynkin quiver A_2. The cluster algebras associated to B_2 and G_2 lead to two other simple examples of wall-crossing described by the 'hexagon formula'

$$U_{0,1}^{(-2)}U_{1,0}^{(-1)}U_{0,1}^{(2)}U_{1,2}^{(1)}U_{1,1}^{(2)}U_{1,0}^{(1)} = 1 \tag{3.74}$$

and the 'octagon formula'

$$U_{0,1}^{(-3)}U_{1,0}^{(-1)}U_{0,1}^{(3)}U_{1,3}^{(1)}U_{1,2}^{(3)}U_{2,3}^{(1)}U_{1,1}^{(3)}U_{1,0}^{(1)} = 1, \tag{3.75}$$

respectively. Using (3.66), it is straightforward to check that the products of symplectomorphisms (3.74) and (3.75) are indeed equal to the identity. In fact, the monomials $\mathcal{X}_s \equiv \mathcal{X}_{\gamma_s}^{-\epsilon_s}$ in all three cases satisfy the recursion relation

$$\mathcal{X}_{s-1}\mathcal{X}_{s+1} = (1-\mathcal{X}_s)^{\Omega_s}, \tag{3.76}$$

with periodicity $N = 5, 6, 8$ in the A_2, B_2, G_2 cases, respectively. Here $\Omega_s = 1$ if s is even and $\Omega_s = 1, 2, 3$ if s is odd, respectively. In the rest of this section, we shall show that the corresponding product of contact transformations V_γ is indeed the identity for these three cases.

For this purpose, let us denote by u_s, v_s, u'_s, v'_s the logarithms of $(\mathcal{X}_s)^{\pm 1}$ and $1-(\mathcal{X}_s)^{\pm 1}$:

$$e^{u_s} = \mathcal{X}_s\,, \qquad e^{v_s} = 1-\mathcal{X}_s\,, \qquad e^{u'_s} = 1/\mathcal{X}_s\,, \qquad e^{v'_s} = 1-1/\mathcal{X}_s\,. \tag{3.77}$$

The logarithms (u_s, v_s) and (u'_s, v'_s) are related by

$$u'_s = -u_s + 2\pi\mathrm{i}\eta_s, \qquad v'_s = v_s - u_s + \mathrm{i}\pi\eta_s, \tag{3.78}$$

where η_s are odd integers. We choose the logarithms (3.77) such that the recursion relation (3.76) lifts to

$$\Omega_s\, v_s = u_{s-1} + u_{s+1} - \mathrm{i}\pi(\eta_{s-1}+\eta_{s+1})\,. \tag{3.79}$$

Now, according to (3.70), the total variation of z under the composition of the contact transformations V_γ is given by

$$\Delta z = \sum_{s:\epsilon_s=1} \Omega_s \left(L(u'_s, v'_s) - \mathrm{i}\pi c_s\, v'_s\right) - \sum_{s:\epsilon_s=-1} \Omega_s \left(L(u_s, v_s) - \mathrm{i}\pi c_s\, v_s\right), \tag{3.80}$$

where we recall that c_s is a half integer chosen such that $\sigma(\gamma_s) = (-1)^{2c_s}$. Since the symplectomorphisms U_γ compose to the identity, Δz is constant. We shall now show that this constant vanishes modulo $4\pi^2$, provided the odd integers η_s are suitably chosen. Since z is related to the contact coordinate $\tilde{\alpha}$ by $z = 2\pi^2\tilde{\alpha}$, this $4\pi^2$-ambiguity is consistent with the mod 2 periodicity of $\tilde{\alpha}$.

To show that (3.80) vanishes modulo $4\pi^2$, we combine (A.24) together with (A.18) to obtain

$$L(u_s, v_s) + L(u'_s, v'_s) = \frac{\mathrm{i}\pi\eta_s}{2}(2v_s - u_s) - \pi^2\eta_s^2 + 2L(1)\,. \tag{3.81}$$

Now, we use the key property

$$\sum_{s=0\ldots N-1} \Omega_s L(u_s, v_s) = \frac{\mathrm{i}\pi}{2} \sum_{s=0\ldots N-1} \Omega_s \eta_s v_s - N_+\, L(1)\,, \tag{3.82}$$

where N_+ is the total BPS index in the "positive chamber",

$$N_+ = \sum_{\substack{s=0\ldots N-1\\ \epsilon_s=1}} \Omega_s\,. \tag{3.83}$$

In the A_2 case, the relation (3.82) agrees with (A.22) upon using (A.18). More generally, (3.82) can be justified as follows. First we note that the differential of the same sum as on the l.h.s. of (3.82) for the variant $L_{-1}(u,v)$ of Rogers dilogarithm vanishes, i.e. $\mathrm{d}\big(\sum_s \Omega_s L_{-1}(u_s, v_s)\big) = 0$, and hence this sum must be constant:

$$\sum_{s=0\ldots N-1} \Omega_s L_{-1}(u_s, v_s) = -N_+ L(1). \tag{3.84}$$

The precise value of the constant can be easily verified for the A_2, B_2, G_2-examples analyzed in section B.5, and in fact for finite Dynkin quivers the formula (3.84) follows directly from the dilogarithm identities proven in [29–31]. We do not know how to establish the identity (3.84) in complete generality, but we note that it is consistent with the conjectural eqs. 6.35 and 6.36 in [29] (reproduced in (B.23)). The desired formula (3.82) now follows from (3.84) by using eq. (A.32).

Combining (3.80), (3.81) and (3.82), we can rewrite the total variation of z as

$$\Delta z = \frac{\mathrm{i}\pi}{2} \sum_{s:\epsilon_s=1} \Omega_s\,(\eta_s - 2c_s)v'_s - \frac{\mathrm{i}\pi}{2} \sum_{s:\epsilon_s=-1} \Omega_s\,(\eta_s - 2c_s)v_s + \frac{\pi^2}{2} \sum_{s:\epsilon_s=1} \Omega_s\,(1-\eta_s^2)\,. \tag{3.85}$$

Since the η_s are odd, the last term vanishes modulo $4\pi^2$. Using the recursion relation (3.79), (3.85) can be rewritten as

$$\Delta z = \frac{\mathrm{i}\pi}{2} \sum_{s=0}^{N-1} \epsilon_s(\eta_s - 2c_s)\big(u_{s-1} + u_{s+1} - \mathrm{i}\pi(\eta_{s-1} + \eta_{s+1})\big) - \frac{\mathrm{i}\pi}{2} \sum_{s:\epsilon_s=1} \Omega_s(\eta_s - 2c_s)(u_s - \mathrm{i}\pi\eta_s)\,, \tag{3.86}$$

where the equality holds modulo $4\pi^2$. In the special case where $2c_s$ is odd for all s (corresponding to $\sigma(\gamma) = -1$ for all BPS states), one may simply choose $\eta_s = 2c_s$ for all s, so that Δz indeed vanishes. More generally, however, we find that the condition that Δz should vanish (modulo $4\pi^2$) for all u_s, v_s subject to (3.79) selects a two-dimensional linear subspace[20] in the space of the η_s. E.g. in the pentagonal case one has

$$\eta_0 - \eta_3 = 2(c_0 - c_3), \qquad \eta_2 - \eta_4 = 2(c_2 - c_4), \qquad \eta_1 - \eta_3 - \eta_4 = 2(c_1 - c_3 - c_4) \tag{3.87}$$

while in the hexagonal case we find

$$\begin{aligned} \eta_0 - \eta_4 &= 2(c_0 - c_4)\,, & \eta_1 - \eta_4 - \eta_5 &= 2(c_1 - c_4 - c_5), \\ \eta_2 - \eta_4 - 2\eta_5 &= 2(c_2 - c_4 - 2c_5), & \eta_3 - \eta_5 &= 2(c_3 - c_5), \end{aligned} \tag{3.88}$$

and, finally, in the octagonal case one has

$$\begin{aligned} \eta_0 - \eta_6 &= 2(c_0 - c_6)\,, & \eta_1 - \eta_6 - \eta_7 &= 2(c_1 - c_6 - c_7), \\ \eta_3 - \eta_6 - 2\eta_7 &= 2(\eta_3 - c_6 - 2c_7), & \eta_4 - \eta_6 - 3\eta_7 &= 2(\eta_4 - c_6 - 3c_7), \\ \eta_2 - 2\eta_6 - 3\eta_7 &= 2(\eta_2 - 2c_6 - 3c_7), & \eta_5 - \eta_7 &= 2(c_5 - c_7). \end{aligned} \tag{3.89}$$

In all these cases, a solution with odd η_s's can be shown to exist for any half integer c_s obeying the quadratic refinement condition (3.14). Thus, at least in these cases, we have shown that the product of KS symplectomorphisms can be lifted to a product of contact transformations consistent with wall-crossing.

4 Discussion

In the first part of this work we have presented a general duality between quaternion-Kähler and hyperkähler manifolds with isometric circle actions. More precisely, this QK/HK correspondence associates, to a real $4n$-dimensional QK manifold $\mathcal{M}$ with a quaternionic S^1-isometry, a HK manifold $\mathcal{M}'$ of the same dimension with a rotational S^1-isometry, equipped with a hyperholomorphic circle bundle $\mathcal{P}$ and a connection λ. The construction proceeds by lifting the S^1-isometry of $\mathcal{M}$ to a triholomorphic isometry of the associated Swann bundle $\mathcal{S} \to \mathcal{M}$, and then performing the standard hyperkähler quotient at non-zero level $\vec{r}$. The circle bundle $\mathcal{P}$ is the level set $\vec{\mu} = \vec{r}$, and λ is induced from the Levi-Cevita connection on $\mathcal{S}$. $\mathcal{P}$ arises as the unit circle bundle in a holomorphic line bundle $\mathscr{L}$ over $\mathcal{M}'$ with unitary connection λ in complex structure determined by $\vec{r}$. By the usual twistor correspondence, $\mathscr{L}$ can be lifted to a holomorphic line bundle $\mathscr{L}_{\mathcal{Z}'}$ on the twistor space $\mathcal{Z}'$ over $\mathcal{M}'$. Unlike the twistor space $\mathcal{Z}$ over $\mathcal{M}$, the former is a trivial product $\mathbb{C}P^1 \times \mathcal{M}'$. Thus, the QK/HK correspondence gives a way to bypass the non-trivial topology of the twistor space $\mathcal{Z}$, at least for QK spaces with a quaternionic circle action.

In the second part of the paper, we have applied this correspondence to the hypermultiplet moduli space $\mathcal{M}$ in type II string theory on a Calabi-Yau threefold. In the absence of NS5-brane or Kaluza-Klein monopole corrections (i.e. for weak coupling, or

[20] These conditions are essentially equivalent to the flattening conditions in [74, 75].

large radius), the latter has a quaternionic circle action corresponding to shifts of the NS-axion (respectively, NUT scalar). We have shown that the twistorial construction of the D-instanton corrected metric on $\mathcal{M}$ given in [10] can be reformulated as the construction of a certain hyperholomorphic circle bundle $\mathcal{P}$ over the dual hyperkähler manifold $\mathcal{M}'$ (or equivalently, a holomorphic line bundle $\mathscr{L}_{\mathcal{Z}'}$ on the twistor space $\mathcal{Z}'$), whose transition functions are expressed in terms of the BPS degeneracies $\Omega(\gamma)$ by means of the Rogers dilogarithm function. The existence of $\mathscr{L}_{\mathcal{Z}'}$ is ensured by the semi-classical limit of the motivic Kontsevich-Soibelman wall-crossing formula. This reformulation clarifies the geometric origin of the similarity with the construction of the HK metric on the Coulomb branch of $\mathcal{N}=2$ gauge theories in 3 dimensions [9]. In particular, it provides a rigorous basis for the notion of 'complex contact transformations across BPS rays' used in [10], which should be interpreted as transition functions for the holomorphic line bundle $\mathscr{L}_{\mathcal{Z}'}$ over the twistor space $\mathcal{Z}'$ of the HK space $\mathcal{M}'$ which is dual to the QK-space $\mathcal{M}$.

Our work also reveals new aspects of the intriguing links between wall-crossing in $\mathcal{N}=2$ theories, dilogarithm identities and cluster algebras, which have emerged in recent years (see [11, 36, 71, 76]). The generalized wall-crossing formula (3.39) for gauge transformations V_γ acting on $\mathscr{L}_{\mathcal{Z}'}$ generates a wealth of new functional identities (3.64) for the Rogers dilogarithm $L(x)$, which generalize the identities established (or conjectured) in [29–31] using cluster algebra techniques. Moreover, as mentioned in section 1, our construction of the line bundle $\mathscr{L}_{\mathcal{Z}'}$ is very reminiscent of recent work of Fock and Goncharov [42], pertaining to the geometric quantization of cluster $\mathcal{A}$-varieties (see section B), where the Rogers dilogarithm also plays the central role. This suggests that the complex torus $\mathcal{M}'(\zeta)\cong(\mathbb{C}^\times)^{2n}$, constructed from $\mathcal{M}$ via the QK/HK correspondence, should be identified with a cluster seed torus whose associated cluster variety $\mathcal{A}$ is equipped with a hyperkähler metric. In this picture the holomorphic fibration $\mathscr{L}_{\mathcal{Z}'}\to\mathcal{Z}'$ arises as the prequantum line bundle over the $\mathcal{A}$-cluster variety. Further support for this relation is found in the fact that the contact one-form $\mathcal{X}$ in (3.31) defines a holomorphic connection on the line bundle $\mathscr{L}_{\mathcal{Z}'}$, whose curvature $\mathrm{d}\mathcal{X}$ is proportional to the holomorphic symplectic form $\omega'(\zeta)$ on the torus $\mathcal{M}'(\zeta)$, as is characteristic for geometric quantization. The fact that $\mathscr{L}_{\mathcal{Z}'}$ is equipped with a connection goes beyond the standard relation between hyperholomorphic connections on $\mathcal{M}'$ and holomorphic line bundles on $\mathcal{Z}'$, which usually do not carry a natural connection [52].

It is natural to speculate that other semi-classical limits of the motivic KS formula, where the quantization parameter $q=e^{2\pi\mathrm{i}\hbar}$ approaches other roots of unity, may ensure the existence of higher rank hyperholomorphic bundles on $\mathcal{M}'$ (which would be Morita-equivalent to the rank 1 bundle constructed in this work). Indeed, on the cluster algebra side, the limit $\hbar\to s/k\in\mathbb{Q}$ produces a holomorphic vector bundle $\mathscr{V}_\hbar\to\mathcal{A}$ of rank $k^{\frac{1}{2}\mathrm{rk}\mathcal{B}}$, where $\mathcal{B}$ is the exchange matrix of $\mathcal{A}$ (see section B.1), and first Chern class proportional to s. Specializing for simplicity to $s=1$ and $\mathrm{rk}\mathcal{B}=2$, holomorphic sections of $\mathscr{V}_\hbar$ have a 'non-Abelian' Fourier expansion with respect to translations along the cluster seed torus and along the $\mathbb{C}^\times$-fiber, which is equivalent to the Fourier expansion

$$H_k(\xi,\tilde\xi)\Upsilon^k=\sum_{\ell\in\mathbb{Z}/(|k|\mathbb{Z})}\ \sum_{m\in\mathbb{Z}+\ell/|k|}\tilde\Psi_{k,\ell}(\tilde\xi-m)e^{-2\pi\mathrm{i}km\tilde\xi}\Upsilon^k \tag{4.1}$$

of the holomorphic sections of $H^1(\mathcal{Z},\mathcal{O}(2))$ which parametrize deformations of $\mathcal{M}$ consistent with invariance under the large gauge transformations (3.23) [68, 77–79] (see [80] for a recent survey). Thus, the hyperholomorphic vector bundle on $\mathcal{M}'$ arising from a variant of our construction at $\hbar = 1/k$ appears to be the right framework to discuss instanton corrections from k NS5-branes consistently with wall-crossing, at least perturbatively away from the D-instanton corrected geometry.

Finally, we note that our construction of the hyperholomorphic line bundle $\mathscr{L}$ makes sense also in the context of the Hitchin moduli space of Higgs bundles, and more generally in the context of $\mathcal{N}=2$ gauge theories in 3 dimensions. It would be very interesting to understand their physical significance.

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A Properties of the Rogers and quantum dilogarithms

In this appendix we recall the definition and main properties of the Rogers dilogarithm and its variants. More details can be found in [81, 82] and references therein. We also include a brief summary of the most important properties of the (non-compact) quantum dilogarithm.

A.1 The Rogers dilogarithm and its analytic continuations

The Spence dilogarithm $\mathrm{Li}_2(z)$ is defined for $|z|<1$ by the absolutely convergent series

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}\,. \tag{A.1}$$

By analytic continuation, it defines a multi-valued function on $\mathbb{C}$, with a logarithmic branch cut from $z=1$ to $z=+\infty$ (more precisely, a univalued function on the universal cover of $\mathbb{C}\backslash\{1\}$) . For a given contour γ extending from 0 to z, $\mathrm{Li}_2(z)$ is given by

$$\mathrm{Li}_2(z) = -\int_\gamma \frac{\log(1-y)}{y}\,\mathrm{d}y, \tag{A.2}$$

where $-\log(1-y)$ is the analytic continuation of the series $\sum_{n=1}^{\infty} y^n/n$ along the path γ. In particular,

$$\mathrm{d}\mathrm{Li}_2(z) = -\frac{\log(1-z)}{z}\mathrm{d}z\,, \qquad \mathrm{Li}_2(0)=0\,, \qquad \mathrm{Li}_2(1)=\frac{\pi^2}{6}\,. \tag{A.3}$$

The Spence dilogarithm satisfies many functional relations, which however take a more pleasant form when expressed in terms of the Rogers dilogarithm.

For $|z| < 1$ and $|1 - z| < 1$, the Rogers dilogarithm is defined by

$$L(z) = \mathrm{Li}_2(z) + \frac{1}{2} \log z \log(1 - z)\,. \tag{A.4}$$

In particular, $L(z)$ takes the special values

$$L(0) = 0\,, \qquad L(1/2) = \frac{\pi^2}{12}\,, \qquad L(1) = \frac{\pi^2}{6}\,. \tag{A.5}$$

By checking that the derivative of the left-hand side vanishes and evaluating $L(z)$ at one of these special values, one easily shows that for x, y, z close to the real interval $[0, 1]$, such that all arguments of L below satisfy $|z|, |1 - z| < 1$, the following functional relations are obeyed:

$$L(z) + L(1 - z) \;=\; L(1), \tag{A.6}$$

$$L(x) - L\left(\frac{x(1-y)}{1-xy}\right) - L\left(\frac{y(1-x)}{1-xy}\right) + L(y) - L(xy) \;=\; 0\,. \tag{A.7}$$

Using (A.6), one may rewrite this last relation as

$$L(x) + L\left(\frac{1-x}{1-xy}\right) + L\left(\frac{1-y}{1-xy}\right) + L(y) + L(1-xy) = 3L(1)\,, \tag{A.8}$$

which has the advantage of making all terms appear with the same sign. Moreover, the arguments z_s of L appearing from left to right in (A.8) satisfy the period 5 recursion relation

$$1 - z_s = z_{s-1} z_{s+1}\,. \tag{A.9}$$

It is possible to extend $L(z)$ from the interval $[0, 1]$ to the real axis such that the above relations, together with the additional identity $L(z)+L(1/z) = 2L(1)$, are satisfied modulo $3L(1)$, but we shall not make use of this extension in this work, as we instead need an analytic extension of $L(z)$ modulo $24L(1)$ into the full complex plane. Let us also record the nine-term relation, which follows by applying the five-term relation three times [83]:

$$\begin{aligned} L(abc) + L\left(\frac{a(b-1)c}{1-ac}\right) + L\left(\frac{c(b-a)}{1-ac}\right) + L\left(\frac{\left(1-\frac{1}{a}\right)(1-ac)}{\left(1-\frac{1}{b}\right)(1-bc)}\right) \\ + L\left(\frac{a(bc-1)}{1-a}\right) + L\left(\frac{b}{a}\right) + L\left(\frac{b-a}{1-a}\right) + L\left(-\frac{b}{1-b}\right) = 0\,. \end{aligned} \tag{A.10}$$

This reduces to the five-term relation upon setting $a = b = x, c = y/x$.

By analytic continuation, the Rogers dilogarithm extends to a multi-valued function on $\mathbb{C}$, with two logarithmic branch cuts, from $z = 1$ to $z = +\infty$ and from $z = 0$ to $z = -\infty$ (more precisely, a univalued function on the universal cover of $\mathbb{C}\backslash\{0, 1\}$). For a given contour γ extending from $1/2$ to z, $L(z)$ is given by

$$L(z) = \frac{\pi^2}{12} - \frac{1}{2}\int_\gamma \left[\frac{\log(1-y)}{y} + \frac{\log(y)}{1-y}\right] \mathrm{d}y, \tag{A.11}$$

where $\log(1-y)$, $\log(y)$ are analytically continued away from $y = 1/2$. In particular, the derivative of $L(z)$ is given by

$$\mathrm{d}L(z) = -\frac{1}{2}\left[\frac{\log(1-z)}{z} + \frac{\log(z)}{1-z}\right] \mathrm{d}z. \tag{A.12}$$

If one is interested only in the value of $L(z)$ modulo $\mathbb{Z}(2) \equiv (2\pi\mathrm{i})^2\mathbb{Z} = 24L(1)\mathbb{Z}$ (which suffices for the purpose of this work), one may trade the universal cover of $\mathbb{C}\backslash\{0, 1, \infty\}$ for the Abelian cover $\hat{Y}_+$ of $Y_+ = \mathbb{C}P^1\backslash\{0, 1, \infty\}$, defined as

$$\hat{Y}_+ = \{(u, v) \in \mathbb{C}^2 \mid e^u + e^v = 1\}, \tag{A.13}$$

with covering map $\hat{Y}_+ \to Y_+$ given by

$$(u, v) \mapsto e^u = z = 1 - e^v\,. \tag{A.14}$$

In other words, u and v run over all possible choices of logarithms of z and $1-z$. The analytically continued Rogers dilogarithm (sometimes referred to as 'enhanced') is then the univalued function

$$L: \quad \hat{Y}_+ \to \mathbb{C}/\mathbb{Z}(2), \tag{A.15}$$

defined by

$$L(u, v) = \mathrm{Li}_2(e^u) + \frac{1}{2}\,uv\,. \tag{A.16}$$

The enhanced Rogers dilogarithm is ambiguous modulo $\mathbb{Z}(2)$, due to the fact that the derivative

$$\mathrm{d}L = \frac{1}{2}\,(u\mathrm{d}v - v\mathrm{d}u) \tag{A.17}$$

(subject to the relation (A.13)) has simple poles at $v \in 2\pi\mathrm{i}\mathbb{Z}$ with residues belonging to $2\pi\mathrm{i}\mathbb{Z}$. Under covering transformations $\hat{Y}_+ \to Y_+$ (i.e. upon changing the choice of logarithms of z and $1-z$) one has

$$L(u + 2\pi\mathrm{i}r, v + 2\pi\mathrm{i}s) = L(u, v) + \mathrm{i}\pi(rv - su) + 2\pi^2 rs\,. \tag{A.18}$$

This construction of the enhanced Rogers dilogarithm $L(u, v)$ is essentially identical to the one presented in [74, 75] in the context of Chern-Simons invariants of hyperbolic three-manifolds. We recall that the Bloch-Wigner dilogarithm $D(u, v)$, defined by

$$D(u, v) = \mathrm{Im}\left[L(u, v) + \frac{1}{2}\,\bar{u}v\right], \tag{A.19}$$

is invariant under the covering transformations (A.18), so descends to a univalued function $D: Y_+ \to \mathbb{R}$, which computes the volume of an ideal hyperbolic tetrahedron with vertices at $0, 1, \infty, z$. In contrast, the real part of $L/(2\pi)$ is inherently ambiguous modulo 2π, as it computes the Chern-Simons invariant of the same tetrahedron (see e.g. [74] and references therein).

At the special points $(0,\infty)$ and $(\infty,0)$, corresponding to $z=1$ and $z=0$, the enhanced Rogers dilogarithm is continuous and takes the values

$$L(0,\infty)=L(1)=\frac{\pi^2}{6}\,,\qquad L(\infty,0)=L(0)=0\,. \tag{A.20}$$

The functional relations (A.6), (A.8), transcribed as

$$\begin{aligned} & L(u,v)+L(v,u) = L(1) \quad \mod \mathbb{Z}(2), && \text{(A.21)}\\ v_s=u_{s-1}+u_{s+1} \quad \Rightarrow \quad & \sum_{s \bmod 5} L(u_s,v_s) = 3L(1) \quad \mod \mathbb{Z}(2)\,, && \text{(A.22)} \end{aligned}$$

now hold throughout the Abelian cover $\hat{Y}_+$, as one can check by differentiation and evaluating the l.h.s at

$$(u_1,v_1)=(u_4,v_4)=(\infty,0),\quad (u_2,v_2)=(u_3,v_3)=(u_5,v_5)=(0,\infty). \tag{A.23}$$

In addition, the analogue of the functional relation $L(z)+L(1/z)=2L(1)$ alluded to below (A.9) becomes

$$L(u,v)+L(-u,v-u+\mathrm{i}\pi\eta)=2L(1)+\frac{\mathrm{i}\pi\eta}{2}u \quad \mod \mathbb{Z}(2)\,, \tag{A.24}$$

where η is any *odd* integer (as is necessary for the argument to belong to $\hat{Y}_+$).

As explained in section 3.2, it is also advantageous to introduce a variant of the Rogers dilogarithm defined for $|z|, 1-|z|<1$ by

$$L_{-1}(-z)=\mathrm{Li}_2(-z)+\frac{1}{2}\log(z)\log(1+z)\,. \tag{A.25}$$

As before, one may analytically continue this to a function $L_{-1}\,:\,\hat{Y}_-\to\mathbb{C}/\mathbb{Z}(2)$, defined by

$$L_{-1}(\hat{u},\hat{v})=\mathrm{Li}_2\left(e^{\hat{u}}\right)+\frac{1}{2}\hat{u}\hat{v} \tag{A.26}$$

where

$$\hat{Y}_-=\{(u,v)\in\mathbb{C}^2\mid e^{\hat{u}}-e^{\hat{v}}=-1\}, \tag{A.27}$$

is the Abelian cover of $Y_-=\mathbb{C}P^1\backslash\{0,-1,\infty\}$, with covering map

$$z=-e^{\hat{u}}=e^{\hat{v}}-1\,. \tag{A.28}$$

The function $L_{-1}(\hat{u},\hat{v})$ satisfies properties analogous to (A.17), (A.24), (A.22):

$$\begin{aligned} \mathrm{d}L_{-1}(\hat{u},\hat{v}) &= -\frac{1}{2}(\hat{u}\mathrm{d}\hat{v}-\hat{v}\mathrm{d}\hat{u}) \quad \mod \mathbb{Z}(2)\,, && \text{(A.29)}\\ L_{-1}(\hat{u},\hat{v})+L_{-1}(-\hat{u},\hat{v}-\hat{u}) &= -L(1) \quad \mod \mathbb{Z}(2)\,, && \text{(A.30)}\\ \hat{v}_s=\hat{u}_{s-1}+\hat{u}_{s+1} \quad \Rightarrow \quad \sum_{s \bmod 5} L(\hat{u}_s,\hat{v}_s) &= -3L(1) \quad \mod \mathbb{Z}(2)\,. && \text{(A.31)} \end{aligned}$$

The relation between $L(u,v)$ and $L(\hat{u},\hat{v})$ is given by

$$\hat{u}=u-\mathrm{i}\pi\eta\,,\quad \hat{v}=v \quad \Rightarrow \quad L_{-1}(\hat{u},\hat{v})=L(u,v)-\frac{\mathrm{i}\pi\eta}{2}v \quad \mod \mathbb{Z}(2) \tag{A.32}$$

whenever η_s is an *odd* integer.

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A.2 The quantum dilogarithm

We now turn to the quantum dilogarithm, defined by [33]

$$\Psi_{q^{1/2}}(x) = \prod_{n=0}^{\infty}(1+q^{n+\frac{1}{2}}x)^{-1} = \frac{1}{(-xq^{1/2};q)_{\infty}} \tag{A.33}$$

where $(x;q)_\infty \equiv \prod_{n=0}^{\infty}(1-q^n x)$. Alternatively,

$$\Psi_{q^{1/2}}(x) = \sum_{n=0}^{\infty}\frac{(-q^{1/2}x)^n}{(1-q)\cdots(1-q^n)} = \exp\left[\sum_{n=1}^{\infty}\frac{(-q^{1/2}x)^n}{n(1-q^n)}\right]. \tag{A.34}$$

The main property of the quantum dilogarithm is the pentagon identity

$$\Psi_{q^{1/2}}(x_1)\Psi_{q^{1/2}}(x_2) = \Psi_{q^{1/2}}(x_2)\Psi_{q^{1/2}}(x_{12})\Psi_{q^{1/2}}(x_1), \tag{A.35}$$

where $x_1x_2 = qx_2x_1$ and $x_{12} = q^{-1/2}x_1x_2$. In the classical limit $\hbar \to 0$, $q^{1/2} = -e^{\mathrm{i}\pi\hbar}$, the quantum dilogarithm reduces to the ordinary dilogarithm,

$$\Psi_{q^{1/2}}(x) = \exp\left(-\frac{1}{2\mathrm{i}\pi\hbar}\,\mathrm{Li}_2(x) + \frac{\mathrm{i}\pi\hbar\, x}{12(1-x)} + \mathcal{O}(\hbar^3)\right). \tag{A.36}$$

In this limit, the pentagon identity (A.35) reduces to the five-term relation (A.7) [33].

B Cluster varieties and dilogarithm identities

In this appendix we will introduce and apply some technology from the theory of cluster algebras and cluster varieties, as developed by Fomin-Zelevinsky [84, 85] and Fock-Goncharov [27, 34]. This formalism gives powerful algorithmic methods of finding dilogarithm identities, and consequently wall crossing formulas. We begin by reviewing some basic properties of cluster varieties and cluster mutations. In section B.3 we discuss cluster transformations over a tropical semi-field, a point of view which elucidates the periodicity properties of sequences of cluster transformations. In section B.4 we introduce the notion of a framed quiver, which is useful for extracting a particular class of (quasi-)periodic mutation sequences, called ν-periods. This allows us to give an explicit expression for the Kontsevich-Soibelman symplectomorphisms U_γ in terms of certain birational cluster automorphisms, conjugated by products of simple monomial transformations. In section B.5 we discuss some explicit examples corresponding to the cluster algebras of type A_2, B_2 and G_2, which are associated with the pentagon, hexagon and octagon dilogarithm identities studied in section 3.5.

B.1 Cluster varieties

The defining data for a cluster variety (or, more generally, a cluster ensemble) consists of a finite set I of cardinality n, a subset $I_0 \subset I$ of cardinality n_0, a $\mathbb{Q}$-valued function $\mathcal{B}_{ij}$ on $I \times I$, such that $\mathcal{B}_{ij} \in \mathbb{Z}$ unless $(i,j) \in I_0 \times I_0$, and a set of coprime integers d_i such that the function $\hat{\mathcal{B}}_{ij} = \mathcal{B}_{ij}/d_j$ is antisymmetric [27]. The function $\mathcal{B}_{ij}$ is often called the exchange matrix. It is customary to represent this data by a quiver diagram Q with n nodes, $|\mathcal{B}_{ij}/d_j|$ arrows going from node i to node j if $\mathcal{B}_{ij} > 0$, or from j to i if $\mathcal{B}_{ij} < 0$, and with each node decorated by the integer d_i. The nodes associated to I_0 are called frozen nodes. In addition, we introduce a set of distinguished $\mathbb{C}^\times$-coordinates $\{x_i\}_{i\in I}$ and $\{a_i\}_{i\in I}$ on two complex n-dimensional tori $\mathcal{X}$ and $\mathcal{A}$, which respectively carry a Poisson structure

$$P = d_j^{-1}\mathcal{B}_{ij}x_i x_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}, \tag{B.1}$$

and a pre-symplectic structure (i.e. possibly degenerate) given by the closed 2-form

$$\omega = d_i\mathcal{B}_{ij}\frac{\mathrm{d}a_i}{a_i}\wedge\frac{\mathrm{d}a_j}{a_j}\,. \tag{B.2}$$

The data $(I, I_0, \mathcal{B}_{ij}, d_i, \mathcal{X}, \mathcal{A})$ together with the distinguished coordinates (x_i, a_i) are sometimes called the initial seed.

For any initial seed $(I, I_0, \mathcal{B}_{ij}, d_i, \mathcal{X}, \mathcal{A}; x_i, a_i)$ and any choice of $k \in I\backslash I_0$, one may construct a new seed $(I, I_0, \mathcal{B}'_{ij}, d_i, \mathcal{X}', \mathcal{A}'; x'_i, a'_i)$ with the same set of nodes and multipliers, but with a new exchange matrix given by the explicit formula

$$\mathcal{B}'_{ij} = \begin{cases} -\mathcal{B}_{ij} & i = k \text{ or } j = k \\ \mathcal{B}_{ij} & \mathcal{B}_{ik}\mathcal{B}_{kj} < 0 \\ \mathcal{B}_{ij} + \mathcal{B}_{ik}|\mathcal{B}_{kj}| & \mathcal{B}_{ik}\mathcal{B}_{kj} \geq 0 \end{cases}, \qquad d'_i = d_i\,, \tag{B.3}$$

and new distinguished $\mathbb{C}^\times$-coordinates related by the birational transformation

$$x'_i = \begin{cases} x_i\big(1 + x_k^{-\mathrm{sgn}(\mathcal{B}_{ik})}\big)^{-\mathcal{B}_{ik}} & i \neq k \\ x_k^{-1} & i = k \end{cases}, \qquad a'_i = \begin{cases} a_i & i \neq k \\ a_k^{-1}\big(\mathbb{A}_k^+ + \mathbb{A}_k^-\big) & i = k \end{cases}, \tag{B.4}$$

where we defined

$$\mathbb{A}_k^+ \equiv \prod_{j:\mathcal{B}_{kj}>0} a_j^{\mathcal{B}_{kj}}, \qquad \mathbb{A}_k^- \equiv \prod_{j:\mathcal{B}_{kj}<0} a_j^{-\mathcal{B}_{kj}}. \tag{B.5}$$

The combined transformation (B.3), (B.4) is involutive and preserves the Poisson structure (B.1) and closed 2-form (B.2). Such transformations were first introduced by Fomin and Zelevinsky [84, 85][21] and are called cluster transformations. We shall denote the combination of (B.3) and (B.4) by μ_k, and refer to it as the mutation along the node k. Since I, I_0, d_i are invariant under mutation, we shall henceforth omit them.

Starting from an initial seed $(\mathcal{B}_{ij}(0), \mathcal{X}(0), \mathcal{A}(0); x_i(0), a_i(0))$ and applying sequences of mutations, we arrive at a collection of seeds $(\mathcal{B}_{ij}(s), \mathcal{X}(s), \mathcal{A}(s); x_i(s), a_i(s))$ attached to the vertices of an 'exchange graph', whose edges correspond to mutations. The seed tori $(\mathcal{X}(s), \mathcal{A}(s))$ together with the mutations μ_k between neighboring vertices of the exchange graph form an atlas for the cluster varieties $(\mathcal{X}, \mathcal{A})$ introduced by Fock and Goncharov.

[21] In [84, 85],the coordinates a_i and x_i are called cluster variables and principal coefficients and are denoted by x_i and y_i, respectively. The action on the variables a_i given in (B.4) corresponds to the case where the coefficients are set to one; moreover, the exchange matrix in [84, 85] is the transpose of the one in [27], whose conventions we follow.

The former carries a Poisson structure given locally by (B.1), while the latter carries a (possibly degenerate) symplectic structure given locally by (B.2). In addition, there exists a homomorphism from $\mathcal{A}$ to $\mathcal{X}$, which maps the local coordinates on seed tori as

$$p\,:\,(a_i)_{i\in I}\;\mapsto\;(x_i)_{i\in I},\qquad x_i=\prod_{j\in I}a_j^{\mathcal{B}_{ij}}=\mathbb{A}_i^+/\mathbb{A}_i^-\,. \tag{B.6}$$

The fibers of the map p are the leaves of the null-foliation of the 2-form ω, while the subtorus $p(\mathcal{A})$ is a symplectic leaf of the Poisson structure P. The spaces $\mathcal{A}$ and $\mathcal{X}$ are in some sense "Langlands-dual" to each other [27]. It is worth noting that the cluster variables $a_i(s)$ satisfy the Laurent phenomenon, in that they always turn out to be finite Laurent polynomials in the initial cluster variables $a_i(0)$ [86].

B.2 Monomial transformations and birational automorphisms

In order to relate the cluster transformations (B.3) with the KS symplectomorphisms U_γ of section 3.1, the first step is to decompose the birational transformation (B.3) into a 'birational automorphism', which preserves the symplectic form (B.2) for fixed exchange matrix $\mathcal{B}_{ij}$, and a 'monomial map', which acts by a simple change of basis on the seed tori. For this purpose, it is useful to rewrite x_i' $(i\neq k)$ and a_k' in (B.4) as

$$x_i' = x_i\,x_k^{[\mathcal{B}_{ik}]_+}\left(1+x_k\right)^{-\mathcal{B}_{ik}} = x_i\,x_k^{[-\mathcal{B}_{ik}]_+}\left(1+1/x_k\right)^{-\mathcal{B}_{ik}} \tag{B.7}$$

$$a_k' = a_k^{-1}\,\mathbb{A}_k^-\left(1+x_k\right) = a_k^{-1}\,\mathbb{A}_k^+\left(1+1/x_k\right), \tag{B.8}$$

where we defined $[z]_+=\max(0,z)$ and identified x_k with $p(a_k)=\mathbb{A}_k^+/\mathbb{A}_k^-$ in the second line. It is then apparent that μ_k can be decomposed in two different ways,

$$\mu_k=\tau_+\circ\mu_{k,+}=\tau_-\circ\mu_{k,-}\,, \tag{B.9}$$

where $\mu_{k,\epsilon},\epsilon=\pm1$, acts via the birational map

$$\mu_{k,\epsilon}\,:\;x_i\mapsto\begin{cases}x_i\left(1+x_k^\epsilon\right)^{-\mathcal{B}_{ik}} & i\neq k\\ x_k & i=k\end{cases},\qquad a_i\mapsto\begin{cases}a_i & i\neq k\\ a_k\left(1+x_k^\epsilon\right)^{-1} & i=k\end{cases}, \tag{B.10}$$

keeping $\mathcal{B}_{ij}$ unchanged, while τ_ϵ acts by the monomial map

$$\tau_{k,\epsilon}\,:\;x_i\mapsto\begin{cases}x_i\,x_k^{[\epsilon\,\mathcal{B}_{ik}]_+} & i\neq k\\ 1/x_k & i=k\end{cases},\qquad a_i\mapsto\begin{cases}a_i & i\neq k\\ \mathbb{A}_k^{-\epsilon}/a_k & i=k\end{cases} \tag{B.11}$$

accompanied by the transformation $\mathcal{B}_{ij}\mapsto\mathcal{B}'_{ij}$ in (B.3). Importantly, $\mu_{k,\epsilon}$ preserves the symplectic form (B.2), while $\tau_{k,\epsilon}$ is still involutive.

B.3 Tropicalization and framed quivers

It is important that the action of μ_k on x_i does not involve any subtraction, implying that the variables $x_i(s)$ effectively lie in the universal semi-field spanned by subtraction-free rational expressions in the initial variables $x_i(0)$. Another natural semi-field is the tropical semi-field Trop($\{x_i(0)\}$), i.e. the free Abelian multiplicative group generated by elements $x_i(0)$ with the usual addition rule $+$ replaced by the tropical addition $\oplus$, defined by

$$\prod_j x_j^{a_j} \oplus \prod_j x_j^{b_j} := \prod_j x_j^{\min(a_j,b_j)}. \tag{B.12}$$

There is a canonical homomorphism $x \mapsto x^T$ from the universal semi-field to the tropical semi-field, obtained by replacing $x_i(s)$ by its leading Laurent monomial in the limit where all the initial variables $x_i(0)$ are scaled to zero at the same rate. This homomorphism, sometimes known as the tropicalization, commutes with the mutations, and one may therefore ask how mutations act at the tropical level. It turns out that at each step, the tropicalisation of the variable $x_i(s)$ is a monomial of either positive degree in all initial variables $x_i(0)$, or of negative degree in all $x_i(0)$ (Conj. 5.4 and Prop. 5.6 in [85])). Equivalently, the 'c-vectors' $\mathbf{c}(s)$ with components

$$c_i(s) = \left(\frac{\partial \log x_i^T(s)}{\partial \log x_j^T(0)} \right)_{j\in I} \tag{B.13}$$

satisfy the 'sign-coherence' property, i.e. all non-zero entries in the vector $\mathbf{c}(s)$ have the same sign [87]. Following the terminology of [32], we refer to this sign as the tropical sign of $x_i(s)$, and denote it by $\epsilon(x_i(s))$. The sign-coherence property of the c-vectors will play a crucial role below for establishing dilogarithm identities.

Upon replacing $+$ by $\oplus$ in the birational automorphism (B.10) and choosing $\epsilon = \epsilon(x_k)$, it is then clear that $\mu_{k,\epsilon(x_k)}$ acts trivially at the tropical level, and therefore the tropicalization of μ_k reduces to the action of the monomial map $\tau_{k,\epsilon(x_k)}$:

$$\mu_k^T \equiv \tau_{k,\epsilon(x_k)} :\ x_i \mapsto \begin{cases} x_i\, x_k^{[\epsilon(x_k)\,\mathcal{B}_{ik}]_+} & i\neq k, \\ 1/x_k & i=k \end{cases}, \qquad a_i \mapsto \begin{cases} a_i & i\neq k \\ \mathbb{A}_k^{-\epsilon(x_k)}/a_k & i=k. \end{cases} \tag{B.14}$$

Correspondingly, the action on the c-vectors is obtained by

$$\log \mu_k^T :\ c_i \mapsto \begin{cases} c_i + [\epsilon(x_k)\,\mathcal{B}_{ik}]_+ c_k & i\neq k \\ -c_k & i=k \end{cases}, \tag{B.15}$$

which provides the gluing conditions for the tropical variety in [27]. As we shall see momentarily, the birational automorphism $\mu_{k,\epsilon(x_k)}$, suitably conjugated by a product of monomial transformations, is the one to be identified with the KS symplectomorphism U_γ.

A useful way to compute the tropical variables $x_i^T(s)$ is to extend the set of nodes $I_u = I\backslash I_0$ by a copy I'_u, and extend the exchange matrix $\mathcal{B}$ into $\tilde{\mathcal{B}}$ such that $\tilde{\mathcal{B}}_{ij} = \mathcal{B}_{ij}$ when $i,j\in I$, $\mathcal{B}_{ii'} = \mathcal{B}_{i'i} = 1$ between a node in $i\in I_u$ and its copy in $i'\in I'_u$, and $\mathcal{B}_{i'i'} = 0$ for $i',j'\in I'_u$. The full set of nodes is then $\tilde{I} = I_u \cup I'_u \cup I_0$, where $\tilde{I}_0 = I'_u \cup I_0$ are frozen (this construction is in fact the main reason for introducing the notion of frozen node). We refer to $\tilde{\mathcal{B}}$ as the framed exchange matrix, and to the corresponding quiver $\tilde{Q}$ as the framed quiver. One can then show that the c-vectors are given by the off-diagonal part of the framed exchange matrix, namely $c_i(s) = \tilde{\mathcal{B}}_{ii'} e_{i'}$ where $e_{i'}$ is the unit vector in the i'-th direction [85]. In particular, the sign $\epsilon(x_i(s))$ can be read off straightforwardly from the sign of the i-th row in the upper-right hand block of the framed exchange matrix (which is independent of the column by the sign-coherence property).

B.4 Wall crossing and dilogarithm identities from ν-periods

For a sequence of N mutations $\mu_{\mathbf{k}} \equiv \mu_{k_N} \circ \cdots \circ \mu_{k_2} \circ \mu_{k_1}$ with $\mathbf{k} = (k_1, k_2, \dots k_n) \in (I \backslash I_0)^N$, we shall denote by

$$(\mathcal{B}_{ij}(s), x_i(s), a_i(s)) = \mu_{k_s}(\mathcal{B}_{ij}(s-1), x_i(s-1), a_i(s-1)) \tag{B.16}$$

the associated sequence of seeds, with $(\mathcal{B}_{ij}(0), x_i(0), a_i(0))$ corresponding as before to the initial seed. More generally, for $s \le N$ we denote by $\mathbf{k}_s = (k_1, \dots, k_s)$ the subsequence formed by the first s mutations. It is sometimes the case that a sequence of N mutations $\mu_{\mathbf{k}}$ composes to the identity transformation, $\mu_{\mathbf{k}}(\mathcal{B}_{ij}, x_i, a_i) = (\mathcal{B}_{ij}, x_i, a_i)$. More generally, it may happen that

$$\mu_{\mathbf{k}}(\mathcal{B}_{ij}, x_i, a_i) = \nu(\mathcal{B}_{ij}, x_i, a_i), \tag{B.17}$$

where ν is an automorphism of the seed, i.e. a permutation of the nodes $i \in I$ (with the corresponding action on $\mathcal{B}_{ij}, x_i, a_i$) which fixes the frozen nodes $i \in I_0$. Such sequences of mutations are called ν-periods of length N. An important theorem asserts that $\mu_{\mathbf{k}}$ is ν-periodic if and only if its tropicalization $\mu^T_{\mathbf{k}}$ is ν-periodic [30, 88].

The ν-periods provide a powerful source of wall-crossing identities, as we will now demonstrate. As proposed in [11] and further elaborated upon in [71, 89], we can identify the s:th birational automorphism[22] inside the sequence $\mu_{\mathbf{k}}$ with a KS symplectomorphism U_{γ_s} for a suitable charge vector γ_s and index $\Omega(s)$, up to conjugation by a product of monomial maps. More precisely, let $\{e_i\}$ be an integer basis of vectors of the "charge lattice" Γ equipped with the antisymmetric product[23]

$$\langle e_i, e_j \rangle = \mathcal{B}_{ij}/d_j, \tag{B.18}$$

let $\mu^T_{\mathbf{k}_s}$ be the following product of monomial transformations

$$\mu^T_{\mathbf{k}_s} = \mu^T_{k_{s-1}} \circ \cdots \circ \mu^T_{k_1}\,, \tag{B.19}$$

and let γ_s be the c-vector $\mathbf{c}_{\mathbf{k}_s}(s)$. Now denote an arbitrary monomial by $\mathcal{Y}_\gamma = \prod x_i^{c_i}$ where c_i are the components of the c-vector $\mathbf{c} \equiv \gamma$. One may then show that the birational automorphism $\mu_{k_s,\epsilon(x_k(s))}$, conjugated by $\mu^T_{\mathbf{k}_s}$, acts on any monomial $\mathcal{Y}_{\gamma'}$ according to

$$(\mu^T_{\mathbf{k}_s})^{-1} \circ \mu_{k_s,\epsilon(x_k(s))} \circ \mu^T_{\mathbf{k}_s} : \quad \mathcal{Y}_{\gamma'} \longmapsto \mathcal{Y}_{\gamma'}\,(1 + \mathcal{Y}_{\gamma_s})^{d_{k_s}\epsilon_s \langle \gamma_s, \gamma' \rangle}\,, \tag{B.20}$$

where $\epsilon_s = \epsilon(x_{k_s}(s))$ and $\mathcal{Y}_{\gamma_s} = (x_{k_s}(s))^{\epsilon_s}$. If we now identify $\mathcal{Y}_\gamma \equiv \sigma(\gamma)^{-1}\mathcal{X}_\gamma$ as in section 3.4, this agrees precisely with the KS symplectomorphism U_{γ_s} defined in (3.16) when $\epsilon_s = 1$, or with its inverse $(U_{\gamma_s})^{-1}$ when $\epsilon_s = -1$, in the special case where the BPS index and quadratic refinement are taken to be

$$\Omega(\gamma_s) = d_{k_s}\,, \qquad \sigma(\gamma_s) = -1\,. \tag{B.21}$$

[22]We consider only the action of $\mu_{k_s,\epsilon(x_{k_s})}$ on x_i, the variables a_i seem to play no role at this stage.

[23]Although the integrality of this antisymmetric product does not seem to be guaranteed by the axioms, it appears to hold in all cases of interest.

Under this identification, the periodicity ν of the mutation sequence $\mu_{\mathbf{k}}$ and of its tropicalization $\mu_{\mathbf{k}}^T$ imply that the following product is the identity:

$$\prod_{s=0}^{N-1} U_{\gamma_s}^{\epsilon_s} = 1, \tag{B.22}$$

where the product is ordered from right to left. Moreover, it was shown[24] in [29] that the following dilogarithm identity holds:

$$\sum_{s=0}^{N-1} \epsilon_s \, d_{k_s} \, L\left(\frac{(x_{k_s}(s))^{\epsilon_s}}{1+(x_{k_s}(s))^{\epsilon_s}}\right) = 0 \ . \tag{B.23}$$

This is recognized as a special case of the general formula (3.64).

B.5 Wall-crossing and dilogarithm identities for rank 2 Dynkin quivers

To illustrate the general construction, we now consider the rank 2 cases with seed

$$I = \{1,2\}\,, \quad I_0 = \emptyset\,, \quad \mathcal{B}^{\pm}_{ij} = \pm \begin{pmatrix} 0 & c \\ -1 & 0 \end{pmatrix}, \quad (d_1, d_2) = (1, c)\,, \tag{B.24}$$

where $c = 1, 2$ or 3. The associated quiver then corresponds to the Dynkin diagram of A_2, B_2 or G_2, respectively. A mutation with respect to either of the nodes maps $\mathcal{B}^{\pm}_{ij}$ to $\mathcal{B}^{\mp}_{ij}$. The framed exchange matrix corresponding to the choice of lower sign in (B.24), which we shall take as the initial seed, is then

$$\tilde{I} = \{1,2,3,4\}, \quad \tilde{I}_0 = \{3,4\}, \quad \tilde{\mathcal{B}}^{-}_{ij} = \begin{pmatrix} 0 & -c & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad (d_i) = (1, c, 1, 1). \tag{B.25}$$

The upper-right 2×2 block of $\mathcal{B}(s)$ then gives the tropicalization of the variables $x_i(s)$ as row vectors. We shall denote by $\mu_i^{\pm}$ the mutations with respect to the node i for the exchange matrix $\pm\mathcal{B}_{ij}$. Moreover, for convenience we define

$$\mu_i^{\pm\sharp} = \mu_{i,+}^{\pm}, \quad \mu_i^{\pm\flat} = \mu_{i,-}^{\pm}, \quad \mu_i'^{\pm} = \tau_{i,+}, \quad \mu_i''^{\pm} = \tau_{i,-}. \tag{B.26}$$

Clearly $\mu_i''^{\pm} = \mu_i'^{\mp}$ when acting on the x-variables, while $\mu_i^{+} = \mu_i^{-}$ when acting on the a-variables. The action of these transformations is summarized in table 2.

B.5.1 Example: A_2

We now consider the A_2 case, corresponding to $c = 1$ in (B.24). The sequence of five mutations $\mu_{\mathbf{k}} = \mu_1^- \mu_2^+ \mu_1^- \mu_2^+ \mu_1^-$ is a ν-period of length 5, where ν exchanges the nodes 1 and 2, as is evident from table 3. According to a general result of [30, 88] quoted above, the

[24] This statement was proven for antisymmetric exchange matrix $\mathcal{B}_{ij}$, and conjectured to hold in the antisymmetrizable case.

	a_1	a_2	x_1	x_2
μ_1^+	$\frac{1+a_2^c}{a_1}$	a_2	$\frac{1}{x_1}$	$x_2(1+x_1)$
$\mu_1^{\sharp+}$	$\frac{a_1}{1+a_2^c}$	a_2	x_1	$x_2(1+x_1)$
$\mu_1'^{+}$	$1/a_1$	a_2	$1/x_1$	x_2
$\mu_1^{\flat+}$	$\frac{a_1}{1+a_2^{-c}}$	a_2	x_1	$x_2(1+1/x_1)$
$\mu_1''^{+}$	a_1/a_2^c	a_2	$1/x_1$	x_1x_2
μ_2^+	a_1	$\frac{1+a_1}{a_2}$	$\frac{x_1}{(1+1/x_2)^c}$	$\frac{1}{x_2}$
$\mu_2^{\sharp+}$	a_1	$\frac{a_1a_2}{1+a_1}$	$x_1/(1+x_2)^c$	x_2
$\mu_2'^{+}$	a_1	a_1/a_2	$x_1x_2^c$	$1/x_2$
$\mu_2^{\flat+}$	a_1	$\frac{a_2}{1+a_1}$	$\frac{x_1}{(1+1/x_2)^c}$	x_2
$\mu_2''^{+}$	a_1	a_1a_2	x_1	$1/x_2$
μ_1^-	$\frac{1+a_2^c}{a_1}$	a_2	$\frac{1}{x_1}$	$\frac{x_1x_2}{1+x_1}$
$\mu_1^{\sharp-}$	$\frac{a_1}{1+a_2^{-c}}$	a_2	x_1	$x_2/(1+x_1)$
$\mu_1'^{-}$	a_2^c/a_1	a_2	$1/x_1$	x_1x_2
$\mu_1^{\flat-}$	$\frac{a_1}{1+a_2^c}$	a_2	x_1	$\frac{x_1x_2}{1+x_1}$
$\mu_1''^{-}$	$1/a_1$	a_2	$1/x_1$	x_2
μ_2^-	a_1	$\frac{1+a_1}{a_2}$	$x_1(1+x_2)^c$	$\frac{1}{x_2}$
$\mu_2^{\sharp-}$	a_1	$\frac{a_2}{1+a_1}$	$x_1(1+x_2)^c$	x_2
$\mu_2'^{-}$	a_1	$1/a_2$	x_1	$1/x_2$
$\mu_2^{\flat-}$	a_1	$\frac{a_1a_2}{1+a_1}$	$x_1(1+1/x_2)^c$	x_2
$\mu_2''^{-}$	a_1	a_2/a_1	$x_1x_2^c$	$1/x_2$

Table 2. Mutation sequences for rank 2 quivers of type A_2, B_2, G_2, corresponding to $c = 1, 2, 3$.

periodicity of $\mu_{\mathbf{k}}$ follows from the periodicity of its tropicalization, i.e. from the identity

$$(1+)(2-)(1-)(2+)(1-) = \nu, \tag{B.27}$$

where we abbreviated the monomial transformation $\mu_i'^{\pm} = \mu_i''^{\mp}$ acting on the x-variables by $(i\pm)$. The tropical sequence is

$$x^T_{\mathbf{k}_s}(s) = (x_1, x_1x_2, x_2, 1/x_1, 1/x_2), \tag{B.28}$$

so the tropical sign sequence is $\epsilon_s = (+,+,+,-,-)$. Thus, one may construct the sequence of symplectomorphisms

$$\begin{aligned} U_{1,0} &= \mu_1^{\sharp-}, \\ U_{1,1} &= (1-)\,\mu_2^{\sharp+}\,(1-), \\ U_{0,1} &= (1-)(2+)\,\mu_1^{\sharp-}(2+)(1-), \\ U_{1,0}^{-1} &= (1-)(2+)(1-)\,\mu_2^{\flat+}(1-)(2+)(1-), \\ U_{0,1}^{-1} &= (1-)(2+)(1-)(2-)\,\mu_1^{\flat-}\,(2-)(1-)(2+)(1-) \end{aligned} \tag{B.29}$$

so that the ν-periodicity of $\mu_{\mathbf{k}}$ translates exactly into the pentagonal identity (3.72).

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s	$\hat{B}_{ij}(s)$	$x_1(s)$	$x_2(s)$	$x_1^T(s)$	$x_2^T(s)$	$a_1(s)$	$a_2(s)$
0	$\begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$	x_1	x_2	x_1	x_2	a_1	a_2
1	$\begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$	$\frac{1}{x_1}$	$\frac{x_1x_2}{1+x_1}$	$1/x_1$	x_1x_2	$\frac{1+a_2}{a_1}$	a_2
2	$\begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}$	$\frac{x_2}{1+x_1x_2+x_1}$	$\frac{1+x_1}{x_1x_2}$	x_2	$1/x_1x_2$	$\frac{1+a_2}{a_1}$	$\frac{1+a_1+a_2}{a_1a_2}$
3	$\begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	$\frac{1+x_1x_2+x_1}{x_2}$	$\frac{1}{x_1(1+x_2)}$	$1/x_2$	$1/x_1$	$\frac{1+a_1}{a_2}$	$\frac{1+a_1+a_2}{a_1a_2}$
4	$\begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	$\frac{1}{x_2}$	$x_1(1+x_2)$	$1/x_2$	x_1	$\frac{1+a_1}{a_2}$	a_1
5	$\begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$	x_2	x_1	x_2	x_1	a_2	a_1

Table 3. The ν-period $\mu_{\mathbf{k}} = \mu_1^- \mu_2^+ \mu_1^- \mu_2^+ \mu_1^-$ of length 5 for A_2.

Upon identifying the coordinates x, y with the initial variables x_1, x_2, one may check that the generators U_γ take the standard form given in (3.66) with quadratic refinement $\sigma_{p,q} = (-1)^{pq+p+q}$:

$$\begin{aligned} &U_{1,0} : [x, y] \mapsto [x, \frac{y}{1+x}], \qquad U_{1,1} : [x, y] \mapsto [(1+xy)x, \frac{y}{1+xy}], \\ &U_{0,1} : [x, y] \mapsto [(1+y)x, y], \quad U_{1,0}^{-1} : [x, y] \mapsto [x, y(1+x)], \quad U_{0,1}^{-1} : [x, y] \mapsto [\frac{x}{1+y}, y]. \end{aligned} \tag{B.30}$$

The dilogarithm identity (B.23) specializes to

$$L\left(\frac{x}{x+1}\right) + L\left(\frac{xy}{xy+x+1}\right) + L\left(\frac{y}{(x+1)(y+1)}\right) - L\left(\frac{x(y+1)}{xy+x+1}\right) - L\left(\frac{y}{y+1}\right) = 0\,. \tag{B.31}$$

The validity of this formula can be most easily verified by taking the limit $y \to 0$ and using the fact that $L(0) = 0$. Setting $x' = x(1+y)/(1+x+xy), y' = y/(1+y)$, one recovers (A.7).

As a side remark, we note that the tropical sequence $x^T_{\mathbf{k}_s}$, or equivalently the charge vector γ_s, is in one-to-one correspondence with the positive roots $\alpha_1, \alpha_1+\alpha_2, \alpha_2$ and the negative simple roots $-\alpha_1, -\alpha_2$ of the finite Lie algebra A_2.

s	$x_1(s)$	$x_2(s)$	$x_1^T(s)$	$x_2^T(s)$	$a_1(s)$	$a_2(s)$
0	x_1	x_2	x_1	x_2	a_1	a_2
1	$\frac{1}{x_1}$	$\frac{x_1x_2}{1+x_1}$	$1/x_1$	x_1x_2	$\frac{1+a_2^2}{a_1}$	a_2
2	$\frac{x_1x_2^2}{(1+x_1+x_1x_2)^2}$	$\frac{1+x_1}{x_1x_2}$	$x_1x_2^2$	$1/(x_1x_2)$	$\frac{1+a_2^2}{a_1}$	$\frac{1+a_1+a_2^2}{a_1a_2}$
3	$\frac{(1+x_1+x_1x_2)^2}{x_1x_2^2}$	$\frac{x_2}{1+x_1(1+x_2)^2}$	$1/(x_1x_2^2)$	x_2	$\frac{(1+a_1)^2+a_2^2}{a_1a_2^2}$	$\frac{1+a_1+a_2^2}{a_1a_2}$
4	$\frac{1}{x_1(1+x_2)^2}$	$\frac{1+x_1(1+x_2)^2}{x_2}$	$1/x_1$	$1/x_2$	$\frac{(1+a_1)^2+a_2^2}{a_1a_2^2}$	$\frac{1+a_1}{a_2}$
5	$x_1(1+x_2)^2$	$\frac{1}{x_2}$	x_1	$1/x_2$	a_1	$\frac{1+a_1}{a_2}$
6	x_1	x_2	x_1	x_2	a_1	a_2

Table 4. The length 6 mutation sequence $(\mu_2^+\mu_1^-)^3$ for B_2.

B.5.2 Example: B_2

We now turn to the B_2 case, corresponding to $c = 2$ in (B.24). The sequence of mutations $(\mu_2^+\mu_1^-)^3$ is a period of length 6, as displayed in table 4. The tropical sequence is

$$x^T_{k_s}(s) = (x_1, x_1x_2, x_1x_2^2, x_2, 1/x_1, 1/x_2) \tag{B.32}$$

so the tropical sign sequence is $\epsilon_s = (+,+,+,+,-,-)$. Setting $x = x_1, y = x_2$, the dilogarithm identity (B.23) reads

$$\begin{aligned} &L\left(\frac{x}{x+1}\right) + 2L\left(\frac{xy}{xy+x+1}\right) + L\left(\frac{xy^2}{(x+1)\,(x(y+1)^2+1)}\right) \\ &+2L\left(\frac{y}{(y+1)(xy+x+1)}\right) - L\left(\frac{x(y+1)^2}{x(y+1)^2+1}\right) - 2L\left(\frac{y}{y+1}\right) = 0\,. \end{aligned} \tag{B.33}$$

This is in fact a consequence of the nine-term relation (A.10) with $a = -x-y-xy$, $b = -x$, $c = 1$. The associated wall-crossing identity (consistent with eq. C.1 in [73]) reads

$$U^{(-2)}_{0,1}U^{(-1)}_{1,0}U^{(2)}_{0,1}U^{(1)}_{1,2}U^{(2)}_{1,1}U^{(1)}_{1,0} = 1 \tag{B.34}$$

where we denote the BPS index in superscript. This arises from the mutations by identifying

$$\begin{aligned} U^{(1)}_{1,0} &= \mu_1^{\sharp-}, \\ U^{(2)}_{1,1} &= (1-)\,\mu_2^{\sharp+}\,(1-), \\ U^{(1)}_{1,2} &= (1-)(2+)\,\mu_1^{\sharp-}\,(2+)(1-), \\ U^{(2)}_{0,1} &= (1-)(2+)(1-)\,\mu_2^{\sharp+}\,(1-)(2+)(1-), \\ U^{(-1)}_{1,0} &= (1-)(2+)(1-)(2+)\,\mu_1^{\flat-}\,(2+)(1-)(2+)(1-), \\ U^{(-2)}_{0,1} &= (1-)(2+)(1-)(2+)(1+)\,\mu_2^{\flat+}\,(1+)(2+)(1-)(2+)(1-), \end{aligned} \tag{B.35}$$

s	$x_1(s)$	$x_2(s)$	$x_1^T(s)$	$x_2^T(s)$	$a_1(s)$	$a_2(s)$
0	x_1	x_2	x_1	x_2	a_1	a_2
1	$\frac{1}{x_1}$	$\frac{x_1x_2}{1+x_1}$	$1/x_1$	x_1x_2	$\frac{1+a_2^3}{a_1}$	a_2
2	$\frac{x_1^2x_2^3}{(1+x_1+x_1x_2)^3}$	$\frac{1+x_1}{x_1x_2}$	$x_1^2x_2^3$	$1/x_1x_2$	$\frac{1+a_2^3}{a_1}$	$\frac{1+a_1+a_2^3}{a_1a_2}$
3	$\frac{(1+x_1+x_1x_2)^3}{x_1^2x_2^3}$	$\frac{x_1x_2^2}{1+x_1^2(1+x_2)^3+x_1(2+3x_2)}$	$1/(x_1^2x_2^3)$	$x_1x_2^2$	$\frac{(1+a_1)^3+(2+3a_1)a_2^3+a_2^6}{a_1^2a_2^3}$	$\frac{1+a_1+a_2^3}{a_1a_2}$
4	$\frac{x_1x_2^3}{(1+x_1(1+x_2)^2)^3}$	$\frac{1+x_1^2(1+x_2)^3+x_1(2+3x_2)}{x_1x_2^2}$	$x_1x_2^3$	$1/(x_1x_2^2)$	$\frac{(1+a_1)^3+(2+3a_1)a_2^3+a_2^6}{a_1^2a_2^3}$	$\frac{(1+a_1)^2+a_2^3}{a_1a_2^2}$
5	$\frac{(1+x_1(1+x_2)^2)^3}{x_1x_2^3}$	$\frac{x_2}{(1+x_1(1+x_2)^3}$	$1/(x_1x_2^3)$	x_2	$\frac{(1+a_1)^3+a_2^2}{a_1a_2^3}$	$\frac{(1+a_1)^2+a_2^3}{a_1a_2^2}$
6	$\frac{1}{x_1(1+x_2)^3}$	$\frac{(1+x_1(1+x_2)^3}{x_2}$	$1/x_1$	$1/x_2$	$\frac{(1+a_1)^3+a_2^2}{a_1a_2^3}$	$\frac{1+a_1}{a_2}$
7	$x_1(1+x_2)^3$	$\frac{1}{x_2}$	x_1	$1/x_2$	a_1	$\frac{1+a_1}{a_2}$
8	x_1	x_2	x_1	x_2	a_1	a_2

Table 5. The mutation sequence $(\mu_2^+\mu_1^-)^4$ for G_2.

and the identity (B.34) then follows from the tropical identity

$$(2-)(1+)(2+)(1-)(2+)(1-) = 1\,. \tag{B.36}$$

Finally, we note that the factors in (B.34) are in one-to-one correspondence with the positive roots $\alpha_1, \alpha_1{+}\alpha_2, \alpha_1{+}2\alpha_2, \alpha_2$ and negative simple roots $-\alpha_1, -\alpha_2$ of the Lie algebra B_2. We also note that the cluster algebra for the quiver B_2 can be obtained by folding the cluster algebra for the quiver A_3, i.e. by specializing to the locus $x_1 = x_3$, $a_1 = a_3$.

B.5.3 Example: G_2

Finally, we turn to the G_2 case, corresponding to $c = 3$ in (B.24). The sequence of mutations $(\mu_2^+\mu_1^-)^4$ is now a period of length 8, as displayed in table 5. The tropical sequence is now

$$x^T_{k_s}(s) = (x_1, x_1x_2, x_1^2x_2^3, x_1x_2^2, x_1x_2^3, x_2, 1/x_1, 1/x_2) \tag{B.37}$$

so the tropical sign sequence is $\epsilon_s = (+,+,+,+,+,+,-,-)$. Setting $x = x_1, y = x_1$, the dilogarithm identity (B.23) reads

$$\begin{aligned} &L\left(\frac{x}{x+1}\right) + 3L\left(\frac{xy}{xy+x+1}\right) + L\left(\frac{x^2y^3}{1+x+xy)^3+x^2y^3}\right) \\ +3L&\left(\frac{xy^2}{(xy+x+1)\,(x(y+1)^2+1)}\right) + L\left(\frac{xy^3}{(x(y+1)^2+1)^3+xy^3}\right) \\ +3L&\left(\frac{y}{(y+1)\,(x(y+1)^2+1)}\right) - L\left(\frac{x(y+1)^3}{x(y+1)^3+1}\right) - 3L\left(\frac{y}{y+1}\right) = 0\,. \end{aligned} \tag{B.38}$$

As in the previous case, we expect that this identity can be obtained by specializing a 16-term identity in 4 variables arising from periods of mutations of the D_4 quiver, and presumably accessible by repeated use of the five-term relation. The associated wall-crossing identity is

$$U^{(-3)}_{0,1}U^{(-1)}_{1,0}U^{(3)}_{0,1}U^{(1)}_{1,3}U^{(3)}_{1,2}U^{(1)}_{2,3}U^{(3)}_{1,1}U^{(1)}_{1,0}=1, \tag{B.39}$$

which arises from the mutations by identifying

$$\begin{aligned}
U^{(1)}_{1,0}&=\mu_1^{\sharp-},\\
U^{(3)}_{1,1}&=(1-)\,\mu_2^{\sharp+}\,(1-),\\
U^{(1)}_{2,3}&=(1-)(2+)\,\mu_1^{\sharp-}\,(2+)(1-),\\
U^{(3)}_{1,2}&=(1-)(2+)(1-)\,\mu_2^{\sharp+}\,(1-)(2+)(1-),\\
U^{(1)}_{1,3}&=(1-)(2+)(1-)(2+)\,\mu_1^{\sharp-}\,(2+)(1-)(2+)(1-),\\
U^{(3)}_{0,1}&=(1-)(2+)(1-)(2+)(1-)\,\mu_2^{\sharp+}\,(1-)(2+)(1-)(2+)(1-),\\
U^{(-1)}_{1,0}&=(1-)(2+)(1-)(2+)(1-)(2+)\,\mu_1^{\flat-}\,(2+)(1-)(2+)(1-)(2+)(1-),\\
U^{(-3)}_{0,1}&=(1-)(2+)(1-)(2+)(1-)(2+)(1+)\,\mu_2^{\flat+}\,(1+)(2+)(1-)(2+)(1-)(2+)(1-).
\end{aligned} \tag{B.40}$$

The identity (B.39) then follows from the simpler tropical identity

$$(2-)(1+)(2+)(1-)(2+)(1-)(2+)(1-)=1. \tag{B.41}$$

As before, the factors in (B.39) are in one-to-one correspondence with the positive roots $\alpha_1, \alpha_1+\alpha_2, \alpha_1+2\alpha_2, \alpha_1+3\alpha_2, 2\alpha_1+3\alpha_2$ and negative simple roots $-\alpha_1, -\alpha_2$ of the Lie algebra G_2.

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