AGT on the S-duality wall

Kazuo Hosomichi,[a] **Sungjay Lee,**[b,c] **and Jaemo Park**[d]

[a] *Yukawa Institute for Theoretical Physics, Kyoto University*
  Kitashirakawa-Oiwakecho, Sakyo, Kyoto 606-8502, Japan

[b] *Korea Institute for Advanced Study*
  Hoegiro 87(207-43 Cheongnyangni-dong), Dongdaemun-gu, Seoul 130-722, Korea

[c] *DAMTP, Centre for Mathematical Sciences, University of Cambridge,*
  Wilberforce Road, Cambridge, CB3 0WA, U.K.

[d] *Department of Physics & Postech Center for Theoretical Physics, POSTECH*
  Pohang 790-784, Korea

  E-mail: hosomiti@yukawa.kyoto-u.ac.jp, s.lee@damtp.cam.ac.uk,
  jaemo@postech.ac.kr

Abstract: Three-dimensional gauge theory $T[G]$ arises on a domain wall between four-dimensional $\mathcal{N} = 4$ SYM theories with the gauge groups G and its S-dual G^{L}. We argue that the $\mathcal{N} = 2^*$ mass deformation of the bulk theory induces a mass-deformation of the theory $T[G]$ on the wall. The partition functions of the theory $T[\mathrm{SU}(2)]$ and its mass-deformation on the three-sphere are shown to coincide with the transformation coefficient of Liouville one-point conformal block on torus under the S-duality.

Contents

1 Introduction

Gauge theories with eight supersymmetry in various dimensions are interesting playgrounds in which one can test by rigorous computations our intuitive understandings of physics at strong gauge coupling. One remarkable achievement in recent years was the exact computation of partition function of 4D $\mathcal{N} = 2$ gauge theories on Omega-background [1] which made use of the localization techniques. More recently it was found that the same idea of localization applies to the computation of partition function of gauge theories on S^4 [2] or S^3 [3, 4]. These exact results have also led to a discovery of a surprising relationship between 4D gauge theories and 2D CFTs, which we call AGT relation. A correspondence between SU(2) quiver gauge theories and Liouville theory has been found by Alday, Gaiotto and Tachikawa (AGT) [5], and it has been generalized by Wyllard [6] to higher rank gauge theories and Toda theories.

After the discovery of this remarkable relation, various generalization have been considered to include external charged objects or topological defects. As an example, a correspondence was proposed in [7, 8] between Wilson or 't Hooft loops in gauge theory and Verlinde's loops or the loops of topological defect in 2D theories, and it was studied further in [9, 10]. It was also argued in [7] that the effect of some external vortex strings or surface operators in gauge theories can be identified with that of degenerate field operators in 2D theories (see also [11]), and the correspondence has been verified explicitly in [12]. There are other surface operators which, according to [13, 14], changes the 2D theories from Liouville or Toda theories to those with affine Lie algebra symmetry.

More recently, several suggestions have been made in [15] on the possible correspondences between external domain walls in 4D gauge theories and certain topological defect operators in 2D theories. Of particular interest are the configurations with a Janus domain wall where the gauge couplings on the two sides of the wall are related by a generalized S-duality group action. By taking the S-duality on one side of the wall, we obtain the *S-duality wall* separating two gauge theories which are S-dual to each other. By studying such domain walls one is led to expect that the partition function of the 3D field theory on the wall should coincide with the S-duality transformation coefficients of conformal blocks, which we call "S-duality kernel" for short.

In this paper we focus on a 3D gauge theory called $T[\mathrm{SU}(N)]$ which arises on the S-duality wall in $\mathcal{N} = 4$ $\mathrm{SU}(N)$ SYM theory. We start with the simplest theory $T[\mathrm{SU}(2)]$ and identify its mass-deformation corresponding to the bulk $\mathcal{N} = 2^*$ deformation. Then we show the agreement up to some normalization factors between the partition function of the mass-deformed theory on S^3 and the S-duality kernel of Liouville conformal blocks on one-punctured torus under the S-duality action which maps the complex structure of the torus from τ to $-1/\tau$. By a simple generalization of our argument, one can find the mass-deformation of $T[\mathrm{SU}(N)]$ theories with $N \geq 3$ corresponding to the bulk $\mathcal{N} = 2^*$ deformation. We conjecture that the partition function of the mass-deformed $T[\mathrm{SU}(N)]$ theory on S^3 corresponds to the S-duality kernel of Toda theory on one-punctured torus, with a certain degenerate operator inserted at the puncture.

2 Reviews

Here we begin by reviewing recent works on 1/2 BPS boundaries and domain walls of $\mathcal{N} = 4$ SYM theory and the 3D superconformal theory $T[G]$. Then we move on to the AGT relation and explain how it is generalized in the presence of Janus domain walls. These are necessary to formulate the conjecture on the S-duality wall which will be tested later on.

2.1 3D theory $T[G]$ on boundaries and domain walls in $\mathcal{N} = 4$ SYM

Boundary conditions on 4D $\mathcal{N} = 4$ SYM have been classified in a recent paper [16], and their property under S-duality transformation has been extensively discussed in [17]. It was shown in [17] that the S-dual of Dirichlet boundary condition on the $\mathcal{N} = 4$ SYM with gauge group G (more precisely, Dirichlet condition on $\mathcal{N} = 2$ vectormultiplet fields and Neumann condition on hypermultiplet fields) is described by a boundary field theory $T[G]$ coupled to the bulk gauge field.

$T[G]$ is a 3D $\mathcal{N} = 4$ gauge theory with global symmetry $G \times G^{\mathrm{L}}$ (Langlands dual of G) at IR superconformal fixed point. It has two distinct vacuum moduli spaces, namely Higgs and Coulomb branches, which admit the action of G and G^{L} respectively. Under the $\mathcal{N} = 4$ mirror transformation, $T[G]$ is mapped to $T[G^{\mathrm{L}}]$.

For $G = \mathrm{SU}(N)$, there is a simple type IIB brane construction for $T[\mathrm{SU}(N)]$. We realize the $\mathrm{SU}(N)$ SYM on N coincident D3-branes stretching along the directions 0126 with $x_6 \geq 0$. In order to implement ordinary Dirichlet boundary condition without any
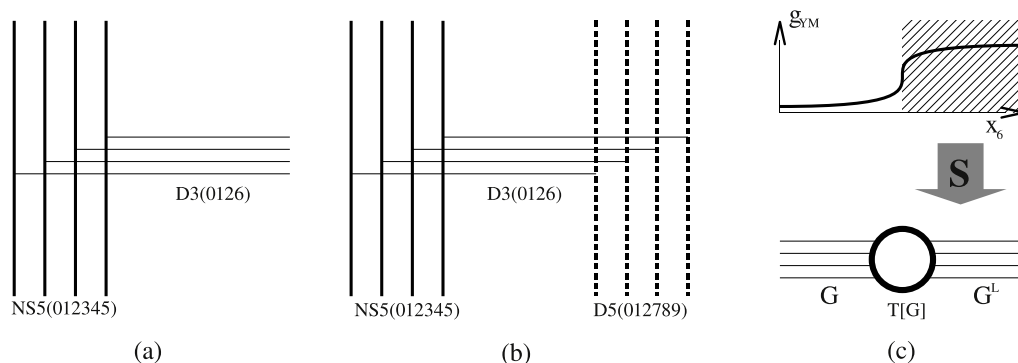
Figure 1. (a) Type IIB brane construction of the 3D gauge theory $T[\mathrm{SU}(N)]$ coupled to 4D $\mathcal{N} = 4$ SYM theory. (b) Type IIB brane construction for $T[\mathrm{SU}(N)]$. (c) Starting from Janus domain wall of G SYM theory and S-dualizing only the right of the wall, one obtains $T[G]$ connecting G and G^{L} SYM theories.

Nahm pole structure at $x_6 = 0$, we need N D5-branes extending along the directions 012345 at $x_6 = 0$ so that each D3-brane ends on different D5-branes.

We take the S-dual of the SYM theory which maps the D5-branes into NS5-branes. See figure 1(a). The resulting brane configuration gives rise to a 3D $\mathcal{N} = 4$ gauge theory on the boundary which is characterized by the following quiver diagram.
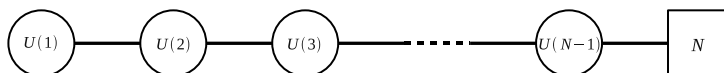
The 3D gauge group $\otimes_{n=1}^{N-1} \mathrm{U}(n)$ and global symmetry $\mathrm{SU}(N)$ correspond to the round and square nodes respectively, and there is a hypermultiplet corresponding to each link. This theory couples to the bulk fields through the gauging of $\mathrm{SU}(N)$ global symmetry, and defines the S-dual of Dirichlet boundary condition of $\mathcal{N} = 4$ SYM.

In this S-dual picture, the four-dimensional fields on D3-branes can be frozen by introducing another N D5-branes stretching along the directions 012789 at positive x_6 and terminating the D3-branes on them. See figure 1(b). The resulting brane configuration defines the 3D theory $T[\mathrm{SU}(N)]$. It is manifestly invariant under S-duality transformation if the spatial directions 345 and 789 are exchanged at the same time. It follows that the Coulomb and Higgs branch moduli spaces of $T[\mathrm{SU}(N)]$ are isomorphic and both admit the action of $\mathrm{SU}(N)$.

It was also argued in [17] that the theory $T[G]$ arise in relation to Janus domain walls of $\mathcal{N} = 4$ SYM connecting the same theory at different values of coupling. In the presence of such domain walls, the holomorphic gauge coupling τ varies as a function of one of the coordinates, say x_6. Suppose the gauge coupling is τ on the left of the wall ($x_6 < 0$) and $-1/\tau$ on the right ($x_6 > 0$). If the theory on the left is weakly coupled, it is strongly coupled on the right. By taking the S-dual only on the right half space $x_6 > 0$, one obtains a wall connecting the G SYM on the left and G^{L} SYM on the right, with the same coupling τ being small on both sides. This is called the S-duality wall. See figure 1(c).

Note that, if we introduce the Dirichlet boundary on G^L SYM at some positive x_6 and take the S-duality of the region $x_6 > 0$ again, the system flows down to the G SYM on left half-space with the S-dual of Dirichlet boundary for G^L SYM. It follows that the 3D theory on the S-duality wall is nothing but $T[G]$ coupled to the bulk fields through the gauging of its $G \times G^L$ global symmetry.

As the simplest example, $T[SU(2)]$ is a $\mathcal{N} = 4$ SQED with two electron hypermultiplets which is long known to be self-dual [18, 19]. In this letter we will mainly focus on this theory and its mass-deformation to $\mathcal{N} = 2$.

2.2 AGT relation

In a recent work [5] a remarkable correspondence was found between partition functions of certain $\mathcal{N} = 2$ superconformal gauge theories on S^4 [2] and correlation functions of Liouville theory with central charge $c = 25$, Liouville coupling $b = 1$. Both can be written in the following form,

$$Z = \int d\nu(\alpha) \, \overline{\mathcal{F}}_{\alpha,E}^{(\sigma)}(\tau) \mathcal{F}_{\alpha,E}^{(\sigma)}(\tau). \tag{2.1}$$

Let us explain the ingredients of the formula for each side of the correspondence.

In Liouville theory, Z is the correlation function of primary operators on a Riemann surface Σ. We denote its complex structure by τ, and choose an arbitrary pants decomposition σ. We also think of the corresponding Moore-Seiberg graph drawn on Σ, which consists of trivalent vertices and legs. The external legs originate from the punctures and are assigned the Liouville momenta $E = \{E_i\}$, and similarly $\alpha = \{\alpha_a\}$ is the collection of Liouville momenta assigned to the internal edges. $\mathcal{F}$ and $\overline{\mathcal{F}}$ are the Virasoro conformal blocks for the holomorphic and anti-holomorphic sectors. To construct correlation functions, one takes the product of conformal blocks and integrate over α with a measure $d\nu(\alpha)$ which is uniquely determined from single-valuedness and other consistency conditions. Among those conditions is the requirement that, although the conformal blocks are defined with respect to a specific pants decomposition σ, the correlation functions do not depend on the choice of σ. This is a strong requirement since the Liouville conformal blocks are subject to nontrivial integral transformations of the form

$$\mathcal{F}_{\alpha,E}^{(\sigma)}(\tau) = \int d\nu(\alpha') \, g_{(\alpha,\alpha',E)}^{(\sigma,\sigma')} \mathcal{F}_{\alpha',E}^{(\sigma')}(\tau), \tag{2.2}$$

under changes of pants decomposition from σ to σ'. The coefficient $g_{(\alpha,\alpha',E)}^{(\sigma,\sigma')}$ is called the S-duality kernel in what follows.

In the gauge theory side, we take a generalized superconformal quiver theory [20] with an SU(2) gauge group at each node. Some of such theories have type IIA brane construction, and at low energy they are described by two M5-branes wrapping a Riemann surface Σ with punctures. In [20] it was shown that there is a family of ganeralized quiver theories for each punctured Riemann surface Σ. Each member of the family has a definite generalized quiver structure and corresponds to a specific pants decomposition of Σ, and different members are related to one another by S-dualities. The gauge couplings τ determine the complex structure of Σ, the parameter E determines the masses and α

corresponds to the expectation value of scalars in vectormultiplet of the theory on $\mathbb{R}^4$. $\mathcal{F}, \overline{\mathcal{F}}$ are each identified with the Nekrasov partition function of the theory on $\mathbb{R}^4$ with the Omega-deformation parameter $\epsilon_1 = \epsilon_2 = 1$ in unit of $\hbar$. The integration measure $d\nu(\alpha)$ now represents the perturbative contribution to the partition function. By taking the bilinear product of $\mathcal{F}$ and integrating over the Coulomb branch moduli space one obtains the partition function of the gauge theory on S^4. It is noteworthy here that (2.2) can be understood as the generalized S-duality action on the Nekrasov partition function of $\mathcal{N} = 2$ gauge theories.

2.3 Partition function with domain walls: a conjecture

Among various generalizations of the AGT relation that have been proposed so far, we are interested in the inclusion of Janus domain walls.

Based on the origin of $\mathcal{F}, \overline{\mathcal{F}}$ in the computation of [2], it has been proposed in [15] that the gauge theory partition function in the presence of Janus domain wall wrapping on S^3 can be expressed by simply choosing different complex structure moduli for $\mathcal{F}$ and $\overline{\mathcal{F}}$,

$$ Z = \int d\nu(\alpha) \, \overline{\mathcal{F}}^{(\sigma)}_{\alpha,E}(\tau) \mathcal{F}^{(\sigma)}_{\alpha,E}(\tau'). \qquad (2.3) $$

As a special case, if the two moduli are not the same but are related by an element g of the S-duality group as $\tau' = g \cdot \tau$, then one has

$$ \mathcal{F}^{(\sigma)}_{\alpha,E}(g \cdot \tau) = \mathcal{F}^{(g \cdot \sigma)}_{\alpha,E}(\tau) = \int d\nu(\alpha') \, g^{(\sigma)}_{(\alpha,\alpha',E)} \mathcal{F}^{(\sigma)}_{\alpha',E}(\tau), \qquad (2.4) $$

with an S-duality kernel $g^{(\sigma)}_{(\alpha,\alpha',E)}$.

The partition function in the presence of the S-duality wall, obtained from the Janus domain wall by applying the S-duality only on one side of the wall, thus becomes

$$ Z = \int d\nu(\alpha)d\nu(\alpha') \, \overline{\mathcal{F}}^{(\sigma)}_{\alpha,E}(\tau) g^{(\sigma)}_{(\alpha,\alpha',E)} \mathcal{F}^{(\sigma)}_{\alpha',E}(\tau). \qquad (2.5) $$

Here $\mathcal{F}, \bar{\mathcal{F}}$ arise from the two hemispheres separated by the wall which are labelled by the Coulomb branch parameters α, α'. One is then naturally led to propose that the S-duality kernel for g coincides with the partition fucntion of the 3D theory on the S-duality wall coupled to given $\mathcal{N} = 2$ gauge theory

$$ g^{(\sigma)}_{(\alpha,\alpha',E)} = Z_{3D}[\alpha, \alpha', E]. \qquad (2.6) $$

For the specific example of $\mathcal{N} = 4$ SYM with gauge group G and g the modular S-duality of the torus, the 3D theory on the wall is known to be $T[G]$. The bulk Coulomb branch parameters α, α' on two sides of the wall become the FI and mass parameters of the 3D theory and break the global symmetries G, G^{L} to their Cartan subgroups. It has been conjectured in [15] that the partition function of $T[G]$ should agree with the modular S-matrix element of the Toda characters (zero-point blocks on the torus). As we will see below, the actual correspondence is a little more involved than that.
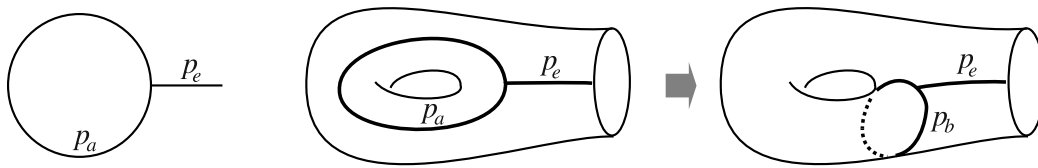
Figure 2. (left) Moore-Seiberg graph for one-point block on the torus. (right) The action of S-duality on Moore-Seiberg graph on one-punctured torus.

3   Tests on $\mathcal{N} = 4/\mathcal{N} = 2^*\, \mathrm{SU}(2)$ SYM

Here we test the proposal (2.6) concretely in the example of $\mathrm{SU}(2)$ $\mathcal{N} = 4$ SYM as well as its mass-deformation called $\mathcal{N} = 2^*$ theory. These theories describe the dynamics of two M5-branes wrapping T^2 with a puncture. The Nekrasov partition function for the theory on $\mathbb{R}^4$ is a function of the gauge coupling τ, the mass m for adjoint hypermultiplet and the expectation value a of the vectormultiplet scalar. (It also depends on the parameters of Omega-deformation, $\epsilon_1 = \epsilon_2 = \hbar$.) The corresponding Virasoro conformal block is the one-point block on a torus of modulus τ, and is described by the Moore-Seiberg graph of figure 2. The representation of Virasoro algebra labelled by p_i corresponds to the Liouville vertex operator $\exp(Q + 2ip_i)\phi$, and is of conformal weight $p_i^2 + Q^2/4$. The labels p_a and p_e are associated to the loop and the external leg of the Moore-Seiberg graph. They are related to a, m in a certain way, as we will find out below.

We introduce a Janus domain wall so that the gauge couplings on the left and right of the wall, τ and $-1/\tau$, are related by the S-duality. The partition function in the presence of this wall is a product of conformal blocks $\overline{\mathcal{F}}_{p_a, p_e}(\tau)$, $\mathcal{F}_{p_b, p_e}(\tau)$ and the S-duality kernel $S_{(p_a, p_b, p_e)}$ integrated over p_a, p_b. The conjecture (2.6) is that this kernel agrees with the partition function of a suitable mass deformation of the 3D theory $T[\mathrm{SU}(2)]$.

The relevant S-duality kernel has been obtained in [21] by making use of the relationship between Liouville conformal blocks and wave functions in quantum Teichmüller theory [21–24].

$$S_{(p_a, p_b, p_e)} = \frac{2^{\frac{3}{2}}}{s_b(p_e)} \int_{\mathbb{R}} dr \frac{s_b\left(p_b + r + \frac{1}{2}p_e + \frac{iQ}{4}\right) s_b\left(p_b - r + \frac{1}{2}p_e + \frac{iQ}{4}\right)}{s_b\left(p_b + r - \frac{1}{2}p_e - \frac{iQ}{4}\right) s_b\left(p_b - r - \frac{1}{2}p_e - \frac{iQ}{4}\right)} e^{4\pi i p_a r} . \qquad (3.1)$$

The special function $s_b(x)$ appearing here is characterized by the normalization condition $s_b(0) = 1$, the unitarity condition $s_b(x)s_b(-x) = 1$ and the poles at $x = \frac{iQ}{2} + imb + inb^{-1}$ $(m, n \in \mathbb{Z}_{\geq 0})$. It has an infinite product representation

$$s_b(x) = \prod_{m,n \in \mathbb{Z}_{\geq 0}} \frac{mb + nb^{-1} + \frac{Q}{2} - ix}{mb + nb^{-1} + \frac{Q}{2} + ix} . \qquad (3.2)$$

For more detailed explanation on this function, we refer to [25, 26].

When $\frac{Q}{2} + ip_e \equiv 2\delta \to 0$, the external momentum vanishes and the conformal block $\mathcal{F}_{p_a, p_e}(\tau)$ reduces to the Virasoro character. The S-duality kernel also simplifies in this

limit to

$$S_{(p_a,p_b,p_e)} \;=\; \frac{\sqrt{2}\cos(4\pi p_a p_b)}{\sinh(2\pi b p_b)\sinh(2\pi p_b/b)} \,. \tag{3.3}$$

To see how this happens, notice first that the factor $1/s_b(p_e)$ turns to vanish in this limit. It is cancelled by a divergence arising from the r-integral over the regions $p_b = \pm r$, where two poles pinch the integration contour. The integral near $r \simeq p_b$ takes the form

$$S_{(p_a,p_b,p_e)} \;=\; 2^{\frac{3}{2}} \cdot 4\pi\delta \int dr \frac{e^{4\pi i p_a r}}{4\sinh(2\pi b p_b)\sinh(2\pi p_b/b)} \frac{1}{4\pi^2} \frac{1}{(r-p_b)^2+\delta^2} + \cdots . \tag{3.4}$$

The integrand becomes proportional to delta function in the limit $\delta \to 0$. There is a similar contribution from the region $r \simeq -p_b$, and by adding the two contributions we obtain (3.3).

One can determine the measure of integration over p_a, p_b from the requirement that S-duality operation squares to identity.

$$d\nu(p) \;\equiv\; dp\sinh(2\pi pb)\sinh(2\pi p/b) \,. \tag{3.5}$$

The factors of sinh functions cancel the denominator of (3.3), and one is left with the modular S-matrix element for non-degenerate Virasoro representations.

3.1 $\mathcal{N} = 4$ SYM and $T[\mathrm{SU}(2)]$

The 3D theory $T[\mathrm{SU}(2)]$ is an $\mathcal{N} = 4$ SQED with two electron hypermultiplets. In terms of $\mathcal{N} = 2$ superfields, this theory has one abelian vector multiplet V, a neutral chiral multiplet ϕ and four chiral superfields $q_1, q_2, \tilde{q}^1, \tilde{q}^2$ with charge $+1, +1, -1, -1$. The Lagrangian takes the following form

$$\mathcal{L} \;=\; \int d^4\theta \, \frac{1}{g^2}\Big[-\Sigma^2 + |\phi|^2 \Big] + \Big[q^{i\dagger}e^{-2V}q_i + \tilde{q}^\dagger_i e^{+2V}\tilde{q}^i \Big] + \Big[\int d^2\theta \, \sqrt{2}\tilde{q}^i\phi q_i + \mathrm{c.c.} \Big] \tag{3.6}$$

The theory has an $\mathrm{SU}(2)_f$ flavor symmetry which rotates q_i as fundamental and $\tilde{q}^i$ as anti-fundamental doublet. Under its $\mathrm{U}(1)$ subgroup the four chiral matters carry charges $+1, -1, -1, +1$. The expectation value of the $\mathcal{N} = 2$ vector multiplet scalar is denoted by σ. It becomes the only parameter for the saddle points in the localization computation of partition function on S^3.

One can introduce real masses μ to the matter fields and lift the Higgs branch by weakly gauging the flavor symmetry $\mathrm{U}(1) \subset \mathrm{SU}(2)_f$ by a background vector superfield $V_{\mathrm{mass}} = -i\theta\bar{\theta}\mu$,

$$\mathcal{L}_{\mathrm{mass}} \;=\; \int d^4\theta \, \Big[q^{1\dagger}e^{-2V_{\mathrm{mass}}}q_1 + q^{2\dagger}e^{+2V_{\mathrm{mass}}}q_2 + \tilde{q}^\dagger_1 e^{+2V_{\mathrm{mass}}}\tilde{q}^1 + \tilde{q}^\dagger_2 e^{-2V_{\mathrm{mass}}}\tilde{q}^2 \Big] \,. \tag{3.7}$$

One can also lift the Coulomb branch by introducing the Fayet-Iliopoulos (FI) parameter ζ, or in other words by weakly gauging the shift symmetry of dual photon by another background vector superfield $V_{\mathrm{FI}} = -i\theta\bar{\theta}\zeta$,

$$\mathcal{L}_{\mathrm{FI}} \;=\; -\frac{4}{\pi} \int d^4\theta \, V_{\mathrm{FI}}\Sigma \;=\; -\frac{2}{\pi}\zeta D \,. \tag{3.8}$$

Here the normalization $\frac{4}{\pi}$ is chosen for later convenience. If this theory appears on the S-duality wall of $\mathcal{N} = 4$ SYM theory with the gauge group SU(2), these background vector multiplets $(V_{\mathrm{mass}}, \phi_{\mathrm{mass}})$ and $(V_{\mathrm{FI}}, \phi_{\mathrm{FI}})$ can be identified with bulk vector multiplets on the two sides of the wall. In particular, Coulomb branch parameters on the two sides of the wall are identified with μ and ζ after a suitable $SU(2)_N \times SU(2)_R$ R-symmetry rotation.

The partition functions of general 3D $\mathcal{N} = 2$ gauge theory on S^3 have been analyzed in [3, 4], and their result immediately applies to our problem. The partition function is an integral over the Coulomb branch moduli σ, and the integrand consists of one-loop determinants of gauge and matter multiplets on the saddle point labeled by σ. In our problem, the one-loop determinant of vector multiplet is trivial but the four matter chiral multiplets yield

$$Z(\sigma + \mu)Z(\sigma - \mu)Z(-\sigma - \mu)Z(-\sigma + \mu) \,, \tag{3.9}$$

where [3]

$$Z(\sigma) \equiv \prod_{n=1}^{\infty} \left(\frac{n + \frac{1}{2} + i\sigma}{n - \frac{1}{2} - i\sigma} \right)^n = s_{b=1}\left(\frac{i}{2} - \sigma \right). \tag{3.10}$$

The FI coupling gives rise to another factor $e^{4\pi i \sigma \zeta}$ in the integrand. The partition function of the defect theory $T[SU(2)]$ coupled to $\mathcal{N} = 4$ SYM is thus given by

$$Z_{\mathrm{3D}}^{\mathcal{N}=4} = \int d\sigma \, \frac{s_{b=1}\left(\mu + \sigma + \frac{i}{2} \right) s_{b=1}\left(\mu - \sigma + \frac{i}{2} \right)}{s_{b=1}\left(\mu + \sigma - \frac{i}{2} \right) s_{b=1}\left(\mu - \sigma - \frac{i}{2} \right)} e^{4\pi i \zeta \sigma} \,, \tag{3.11}$$

up to a normalization constant.

By a straightforward comparison, one finds that the partition function agrees with the S-duality kernel (3.1) under the identification of the parameters

$$b = 1 \,, \quad r = \sigma \,, \quad p_a = \zeta \,, \quad p_b = \mu \,, \quad p_e = 0 \,. \tag{3.12}$$

The last relation implies that, against our intuition, the Liouville conformal block correspoding to $\mathcal{N} = 4$ SU(2) SYM should be a torus one-point block with a non-vanishing external momentum, $\frac{Q}{2} + ip_e \neq 0$. One would naturally ask if this value has some special meaning. In [3] it has been shown that the one-loop determinant of a general charged $\mathcal{N} = 4$ hypermultiplet can be expressed by the function

$$Z(\sigma)Z(-\sigma) = \frac{1}{2\cosh \pi\sigma} \,. \tag{3.13}$$

We expect that for $b \neq 1$ it will generalize to an identity of the function s_b,

$$s_b\left(\frac{ib}{2} - \sigma \right) s_b\left(\frac{ib}{2} + \sigma \right) = \frac{1}{2\cosh \pi b \sigma} \,, \tag{3.14}$$

or similar identity with b replaced by $1/b$. If this is the case, then the value of the external momentum for general b should be that of the Liouville interaction operators

$$\frac{Q}{2} + ip_e = b^{-1} \quad \mathrm{or} \quad b \,.$$

This is in accordance with the observation of [27].

3.2 $\mathcal{N} = 2^*$ SYM and mass-deformed $T[\mathrm{SU}(2)]$

The above result implies that the $\mathcal{N} = 4$ SYM corresponds to the Liouville theory on a one-punctured torus with external momentum $p_e = 0$. Other values of p_e should correspond to the mass-deformation to $\mathcal{N} = 2^*$ theory. What 3D theory should arise on the S-duality wall of the $\mathcal{N} = 2^*$ theory?

The mass deformation of the bulk 4D $\mathcal{N} = 4$ SYM theory will induce a mass deformation of $T[\mathrm{SU}(2)]$ on the wall which preserves the $\mathcal{N} = 2$ supersymmetry as well as $\mathrm{SU}(2) \times \mathrm{SU}(2)$ global symmetries. Due to these symmetry constraints, the deformation of the theory on the wall is easily identified as the real mass deformation by weakly gauging a U(1) symmetry under which $q_1, q_2, \tilde{q}^1, \tilde{q}^2$ all carry the same charge $+1$ and ϕ carries the charge -2. The deformation to the Lagrangian is

$$\mathcal{L}_{\mathrm{def}} = \int d^4\theta \; \phi^\dagger e^{4V_{\mathrm{def}}} \phi + \sum_{i=1}^{2} \left[q^{i\dagger} e^{-2V_{\mathrm{def}}} q_i + \tilde{q}_i^\dagger e^{-2V_{\mathrm{def}}} \tilde{q}^i \right], \tag{3.15}$$

where $V_{\mathrm{def}} = im\theta\bar{\theta}/2$.

As was explained in [28], this U(1) is an anti-diagonal sum of the R-symmetries $\mathrm{U}(1)_N$ and $\mathrm{U}(1)_R$ which are subgroups of the $\mathrm{SU}(2)_N \times \mathrm{SU}(2)_R$ R-symmetry group of the undeformed $\mathcal{N} = 4$ theory. Since $\mathrm{SU}(2)_N$ and $\mathrm{SU}(2)_R$ are interchanged under the mirror symmetry (or S-duality in the bulk), the mass-parameter m is mapped to minus itself under the S-duality.

Applying the localization technique again, one can show that the one-loop determinant of four chiral multiplets is now modified to

$$Z\left(\sigma + \mu - \frac{m}{2}\right) Z\left(\sigma - \mu - \frac{m}{2}\right) Z\left(-\sigma - \mu - \frac{m}{2}\right) Z\left(-\sigma + \mu - \frac{m}{2}\right). \tag{3.16}$$

If the contribution from the fields in the $\mathcal{N} = 4$ vector multiplet (V, ϕ) remains trivial, the partition function of the mass-deformed $T[\mathrm{SU}(2)]$ is given by

$$Z_{3\mathrm{D}}^{\mathcal{N}=2^*} = \int d\sigma \; \frac{s_{b=1}\left(\mu + \sigma + \frac{m}{2} + \frac{i}{2}\right) s_{b=1}\left(\mu - \sigma + \frac{m}{2} + \frac{i}{2}\right)}{s_{b=1}\left(\mu + \sigma - \frac{m}{2} - \frac{i}{2}\right) s_{b=1}\left(\mu - \sigma - \frac{m}{2} - \frac{i}{2}\right)} e^{4\pi i \zeta \sigma}. \tag{3.17}$$

By carefully relating the external Liouville momentum p_e and $\mathcal{N} = 2^*$ mass parameter m, one finds that when $b = 1$ and

$$m = p_e, \tag{3.18}$$

the partition function (3.17) agrees with the S-duality kernel (3.1) up to a factor $s_b(-m)$ which is independent of p_a, p_b.

In gauge theory, an additional factor of $s_b(-m)$ is naively expected to arise from a neutral chiral matter of mass $-2m$. We claim that it is precisely the contribution of the neutral chiral field ϕ in $\mathcal{N} = 4$ vectormultiplet after the mass deformation. The field ϕ actually has non-canonical R-charge 1, and the one-loop determinant for such chiral matters is not known yet. It would be interesting to explicitly work it out, but we leave it as a future problem. We give one supporting argument for our claim in the next subsection.

3.3 Self-mirror property

An interesting observation made in [4] is that the function $1/\cosh \pi x$ is invariant under Fourier transform, and it was used in the proof of dualities in 3D $\mathcal{N} = 4$ gauge theories. Here we consider how this can be generalized to mass-deformed $\mathcal{N} = 2$ theories.

We begin by recalling that the function s_b is characterized by the (3.14) as well as $s_b = s_{1/b}$ and $s_b(x)s_b(-x) = 1$. Let us define

$$F_{m,b}(x) \equiv \frac{s_b\left(x + \frac{m}{2} + \frac{iQ}{4}\right)}{s_b\left(x - \frac{m}{2} - \frac{iQ}{4}\right)} \qquad (3.19)$$

which is an even function of x. One can show that it obeys a difference equation

$$\left[\cosh \pi b \left(x + \frac{m}{2} + \frac{iQ}{4}\right) e^{-\pi bp} - \cosh \pi b \left(x - \frac{m}{2} - \frac{iQ}{4}\right) e^{\pi bp}\right] F_{m,b}(x) = 0, \qquad (3.20)$$

where $p \equiv -\frac{i}{2\pi}\partial_x$, and a similar equation with b replaced by $1/b$. Remarkably, they are invariant if x, p, m are mapped to $p, x, -m$. This implies the invariance of $F_{m,b}$ under Fourier transform [26]

$$\int dx e^{-2\pi i p x} F_{m,b}(x) = s_b(m) F_{-m,b}(p). \qquad (3.21)$$

It is now easy to show that our partition function for mass-deformed $T[SU(2)]$ theory is invariant under the exchange of μ and ζ if m is sign-flipped at the same time.

$$\begin{aligned}
Z_{\text{3D}}^{\mathcal{N}=2^*} &= \frac{1}{s_b(m)} \int d\sigma F_{m,b=1}(\sigma + \mu) F_{m,b=1}(\sigma - \mu) e^{4\pi i \sigma \zeta} \\
&= \frac{1}{s_b(-m)} \int d\tilde{\sigma} F_{-m,b=1}(\tilde{\sigma} + \zeta) F_{-m,b=1}(\tilde{\sigma} - \zeta) e^{4\pi i \tilde{\sigma} \mu}.
\end{aligned} \qquad (3.22)$$

Recall that the overall factor $1/s_b(p_e)$ in (3.1) played an important role when we saw that the S-duality kernel reduces to modular S-matrix element as the external momentum is turned off. Here this factor is necessary for the 3D partition function to be precisely invariant under the mirror transformation. Therefore, this prefactor should arise from the path integral of some fields except the four charged chiral matters. Again, we claim it is the determinant of the neutral chiral field ϕ.

4 Generalization to SU(N)

Our result can be immediately generalized to the $\mathcal{N} = 2^*$ theories with other gauge groups G. The theory on the S-duality wall should be given by $T[G]$ with a real mass deformation. The mass should be turned on by gauging a U(1) symmetry which commutes with the global symmetry $G \times G^L$ and flips sign under the mirror transformation. The anti-diagonal combination of U(1)$_N$ and U(1)$_R$ is the only candidate which meets all these requirements.

It is straightforward to apply the result of [3, 4] to compute the partition function of mass-deformed $T[SU(N)]$ theory on the S-duality wall S^3. The saddle points of path

integral are labelled by the Coulomb branch parameters $\vec{\sigma}_n = (\sigma_{n,i})$, where $n = 1, \cdots, N-1$ refers to the gauge group $U(n)$ and $i = 1, \cdots, n$. The Coulomb branch parameters of the bulk $\mathcal{N} = 2^*$ theories on the two sides of the wall appear as the FI parameter $\vec{\zeta}$ and the masses $\vec{\mu}$ for N fundamental hypermultiplets. In the brane picture of figure 1, ζ_i are identified with the positions of NS5-branes in the x_3-direction and μ_i with the position of D5-branes in the x_7-direction. We also require

$$\sum_i \zeta_i = \sum_i \mu_i = 0.$$

The partition function reads

$$Z_{3D}^{\mathcal{N}=2^*} = \int \prod_{n=1}^{N-1} d^n \vec{\sigma}_n \prod_{n=1}^{N-1} Z_{V_n}(\vec{\sigma}_n) Z_{\phi_n}(\vec{\sigma}_n) e^{2\pi i \sum_i (\zeta_n - \zeta_{n+1}) \sigma_{n,i}}$$
$$\cdot \prod_{n=1}^{N-2} Z_{q_n}(\vec{\sigma}_n, \vec{\sigma}_{n+1}) \cdot Z_{q_{N-1}}(\vec{\sigma}_{N-1}, \vec{\mu}), \tag{4.1}$$

where Z_{V_n} and Z_{ϕ_n} are the one-loop determinants of the $\mathcal{N} = 4$ $U(n)$ vectormultiplet,

$$Z_{V_n}(\vec{\sigma}_n) = \prod_{i<j}^n \sinh^2 \pi(\sigma_{n,i} - \sigma_{n,j}),$$

$$Z_{\phi_n}(\vec{\sigma}_n) = \prod_{i,j=1}^n s_{b=1}(\sigma_{n,i} - \sigma_{n,j} - m), \tag{4.2}$$

and Z_{q_n} is that of (bi-)fundamental matters,

$$Z_{q_n}(\vec{\sigma}_n, \vec{\sigma}_{n+1}) = \prod_{i=1}^n \prod_{j=1}^{n+1} F_{m,b=1}(\sigma_{n,i} - \sigma_{n+1,j}). \tag{4.3}$$

We expect this partition function to coincide with the S-duality kernel of Toda conformal blocks on one-punctured torus with external momentum m.

Here we recall that general non-degenerate representations of $\mathcal{W}_N$ algebra are labeled by the Toda momenta which have $N-1$ components. Our Toda conformal blocks, on the other hand, have a puncture labelled by only one mass parameter m. The punctures in generalized quiver gauge theories have been classified and their relation to Toda vertex operators have been explained in [6, 20, 29]. Each puncture in $SU(N)$ generalized quiver theory is labelled by a Young diagram with N boxes, which in turn determines the global symmetry and mass parameters of the theory associated to that puncture. In this classification, $\mathcal{N} = 2^*$ theories correspond to a torus with one *simple puncture* carrying $U(1)$ global symmetry.

To obtain a fully general formula for the S-duality kernel of Toda conformal blocks on one-punctured torus, one needs a generalized quiver gauge theory for a torus with one full puncture. Such theories are not connected with $\mathcal{N} = 2^*$ theories in a simple manner. No Lagrangian description is available, although one can construct it from the theory T_N, corresponding to the sphere with three full punctures, by coupling the two $SU(N)$ symmetries to a single vectormultiplet.

5 Conclusion

In this letter we have shown the agreement between the partition function of mass-deformed $T[\mathrm{SU}(2)]$ theory and the S-duality kernel for Liouville torus one-point conformal blocks. Our result indicates that the AGT relation can be extended to include the S-duality domain walls. As suggested in [15], this agreement may be understood in terms of M5-branes wrapped on $S^3 \times \mathcal{M}_3$, where $\mathcal{M}_3$ is a three-sphere with a web of defect lines made by connecting the external legs of two Moore-Seiberg graphs related to each other by S-duality.

We notice that the evaluation of partition functions in [3, 4] have been performed on a round S^3. It would be a reasonable guess that the one-loop determinants on a squashed S^3 are given by a function s_b with $b \neq 1$. Also, to show the precise matching between the 3D partition function and the S-duality kernel, we need a full understanding of the contribution of chiral matters with non-canonical R-charge assignments. These are interesting future problems.

Very little is known about the three-dimensional theories on the boundary or domain walls of four-dimensional $N = 2$ gauge theories. It would be an interesting problem to classify such theories and study their properties under S-duality maps.

Note added in proof. Our conjecture about the one-loop determinant for chiral matters with R-charge 1 was confirmed in the paper by Jafferis [30] and also independently in [31].

References

[1] N.A. Nekrasov, *Seiberg-Witten Prepotential From Instanton Counting*, *Adv. Theor. Math. Phys.* **7** (2004) 831 [`hep-th/0206161`] [SPIRES].

[2] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, `arXiv:0712.2824` [SPIRES].

[3] A. Kapustin, B. Willett and I. Yaakov, *Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter*, *JHEP* **03** (2010) 089 [`arXiv:0909.4559`] [SPIRES].

[4] A. Kapustin, B. Willett and I. Yaakov, *Nonperturbative Tests of Three-Dimensional Dualities*, *JHEP* **10** (2010) 013 [`arXiv:1003.5694`] [SPIRES].

[5] L.F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville Correlation Functions from Four-dimensional Gauge Theories*, *Lett. Math. Phys.* **91** (2010) 167 [`arXiv:0906.3219`] [SPIRES].

[6] N. Wyllard, A_{N-1} *conformal Toda field theory correlation functions from conformal* $N = 2$ SU(N) *quiver gauge theories*, *JHEP* **11** (2009) 002 [arXiv:0907.2189] [SPIRES].

[7] L.F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa and H. Verlinde, *Loop and surface operators in* $N = 2$ *gauge theory and Liouville modular geometry*, *JHEP* **01** (2010) 113 [arXiv:0909.0945] [SPIRES].

[8] N. Drukker, J. Gomis, T. Okuda and J. Teschner, *Gauge Theory Loop Operators and Liouville Theory*, *JHEP* **02** (2010) 057 [arXiv:0909.1105] [SPIRES].

[9] F. Passerini, *Gauge Theory Wilson Loops and Conformal Toda Field Theory*, *JHEP* **03** (2010) 125 [arXiv:1003.1151] [SPIRES].

[10] J. Gomis and B. Le Floch, *'t Hooft Operators in Gauge Theory from Toda CFT*, arXiv:1008.4139 [SPIRES].

[11] K. Maruyoshi and M. Taki, *Deformed Prepotential, Quantum Integrable System and Liouville Field Theory*, *Nucl. Phys.* **B 841** (2010) 388 [arXiv:1006.4505] [SPIRES].

[12] H. Awata, H. Fuji, H. Kanno, M. Manabe and Y. Yamada, *Localization with a Surface Operator, Irregular Conformal Blocks and Open Topological String*, arXiv:1008.0574 [SPIRES].

[13] L.F. Alday and Y. Tachikawa, *Affine* SL(2) *conformal blocks from 4d gauge theories*, *Lett. Math. Phys.* **94** (2010) 87 [arXiv:1005.4469] [SPIRES].

[14] C. Kozcaz, S. Pasquetti, F. Passerini and N. Wyllard, *Affine sl(N) conformal blocks from* $N = 2$ SU(N) *gauge theories*, arXiv:1008.1412 [SPIRES].

[15] N. Drukker, D. Gaiotto and J. Gomis, *The Virtue of Defects in* 4D *Gauge Theories and* 2D *CFTs*, arXiv:1003.1112 [SPIRES].

[16] D. Gaiotto and E. Witten, *Supersymmetric Boundary Conditions in* $N = 4$ *Super Yang-Mills Theory*, arXiv:0804.2902 [SPIRES].

[17] D. Gaiotto and E. Witten, *S-duality of Boundary Conditions In* $N = 4$ *Super Yang-Mills Theory*, arXiv:0807.3720 [SPIRES].

[18] K.A. Intriligator and N. Seiberg, *Mirror symmetry in three dimensional gauge theories*, *Phys. Lett.* **B 387** (1996) 513 [hep-th/9607207] [SPIRES].

[19] A. Kapustin and M.J. Strassler, *On Mirror Symmetry in Three Dimensional Abelian Gauge Theories*, *JHEP* **04** (1999) 021 [hep-th/9902033] [SPIRES].

[20] D. Gaiotto, $N = 2$ *dualities*, arXiv:0904.2715 [SPIRES].

[21] J. Teschner, *From Liouville theory to the quantum geometry of Riemann surfaces*, hep-th/0308031 [SPIRES].

[22] J. Teschner, *Quantum Liouville theory versus quantized Teichmueller spaces*, *Fortsch. Phys.* **51** (2003) 865 [hep-th/0212243] [SPIRES].

[23] J. Teschner, *On the relation between quantum Liouville theory and the quantized Teichmueller spaces*, *Int. J. Mod. Phys.* **A 19S2** (2004) 459 [hep-th/0303149] [SPIRES].

[24] J. Teschner, *An analog of a modular functor from quantized Teichmüller theory*, math/0510174.

[25] S. Kharchev, D. Lebedev and M. Semenov-Tian-Shansky, *Unitary representations of* $U_q(\mathfrak{sl}(2,\mathbb{R}))$, *the modular double and the multiparticle q-deformed Toda chains*, *Commun. Math. Phys.* **225** (2002) 573 [hep-th/0102180] [SPIRES].

[26] A.G. Bytsko and J. Teschner, *Quantization of models with non-compact quantum group symmetry: Modular XXZ magnet and lattice sinh-Gordon model*, *J. Phys.* **A 39** (2006) 12927 [hep-th/0602093] [SPIRES].

[27] T. Okuda and V. Pestun, *On the instantons and the hypermultiplet mass of* $N = 2^*$ *super Yang-Mills on* S^4, arXiv:1004.1222 [SPIRES].

[28] D. Tong, *Dynamics of* $N = 2$ *supersymmetric Chern-Simons theories*, *JHEP* **07** (2000) 019 [hep-th/0005186] [SPIRES].

[29] S. Kanno, Y. Matsuo, S. Shiba and Y. Tachikawa, *N=2 gauge theories and degenerate fields of Toda theory*, *Phys. Rev.* **D 81** (2010) 046004 [arXiv:0911.4787] [SPIRES].

[30] D.L. Jafferis, *The Exact Superconformal R-Symmetry Extremizes Z*, arXiv:1012.3210 [SPIRES].

[31] N. Hama, K. Hosomichi and S. Lee, *Notes on SUSY Gauge Theories on Three-Sphere*, arXiv:1012.3512 [SPIRES].