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Magnetic charges and Wald entropy

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Abstract: Using Wald's formalism, we study the thermodynamics (first laws and Smarr formulae) of asymptotically-flat black holes, rings etc. in a higher-dimensional higher-rank generalization of the Einstein-Maxwell theory. We show how to deal with the electric and magnetic charges of the objects and how the electric-magnetic duality properties of the theory are realized in their thermodynamics.

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1 Introduction

Over most of the past century, electric-magnetic duality has been one of the main research topics in Theoretical Physics. While it arises as a symmetry of the source-free equations of motion of electromagnetism in 4 dimensions, it has been extended and generalized in many directions. Of particular interest for us is the generalization to higher-rank fields in higher dimensions and the inclusion of localized sources. The former is quite natural in the context of brane physics and the second, which goes back to Dirac's discovery of the magnetic monopole [1], arises naturally when one considers classical point-like or brane-like sources: the rotations between electric and magnetic fields must be in one-to-one correspondence with rotations of electric and magnetic sources and their charges.[1]

In d dimensions, black holes and black branes can carry some of the electric and magnetic charges associated to $(p+1)$-form fields: as a general rule, black p-branes (where $p=0$ corresponds to black holes) can carry the electric charges of $(p+1)$-form potentials and the magnetic charges of $(\tilde{p}+1)$-form potentials, with $\tilde{p}=d-p-4$ and $\tilde{\tilde{p}}=p$. When $\tilde{p}=p$, they can carry electric and magnetic charges of the same $(p+1)$-form field.

The dynamics of black p-branes should exhibit the same electric-magnetic duality properties as the theory whose equations of motion they solve. In particular, the laws of thermodynamics [4] should exhibit those properties. Thus, in $d=4$, where black holes can

[1] See, for instance, refs. [2, 3] and references therein.

carry electric and magnetic charges of the same electromagnetic field, the first law of black-hole mechanics must include work terms for the variations of both kinds of charges. The conjugate thermodynamical potentials are the electrostatic and magnetostatic potentials evaluated on the black-hole horizon. If the theory has electric-magnetic duality, then we expect the first law of black-hole mechanics in 4 dimensions and the Smarr formulae [5] to be invariant under the simultaneous rotations of the (variations of the) electric and magnetic charges and of the conjugate thermodynamical potentials. The invariance of the Smarr formula for axion-dilaton black holes was recently proven in ref. [6] using Wald's formalism [7–9] and the generalized Komar formula [10] constructed in refs. [11, 12]. The proof can be easily generalized to other theories with scalars and vectors minimally coupled to gravity with electric-magnetic dualities [13] like most ungauged supergravity theories. However, a similar proof for the first law using Wald's formalism is not yet available.

Actually, the presence of magnetic work terms and scalar work terms in the first law of 4-dimensional black hole mechanics was found by other methods and is well known [14]. In the notation of that reference, it takes this form:

$$\delta M = \frac{\kappa \delta A}{8\pi G_N^{(4)}} + \Omega \delta J + \psi^\Lambda \delta q_\Lambda + \chi_\Lambda \delta p^\Lambda - G_{ab}(\phi_\infty)\Sigma^a \delta \phi^b_\infty \,, \tag{1.1}$$

where the q_Λs and p_Λs are, respectively, electric and magnetic charges with respect to the vector field A^Λ and ψ^Λ and χ_Λ are the electrostatic and magnetostatic potentials evaluated on the horizon, $G_{ab}(\phi_\infty)$ is the metric of the scalar manifold evaluated at infinity, Σ^a the scalar charges and ϕ^a_∞ the vales of the scalars at infinity (moduli).

The invariance of this formula under electric-magnetic duality transformations presents several problems because, on general grounds [13], the terms involving electric and magnetic charges should be combined in a manifestly symplectic-invariant expression, which is not the case. The term involving the scalar charges is manifestly invariant. However, there is no good definition of the scalar charge as a conserved charge and, while this does not invalidate the result, it obscures its meaning.

In this paper we want to study the electric-magnetic duality properties of the first law of black hole mechanics for asymptotically-flat black p-branes coupled to higher-rank form potentials in d dimensions[2] using Wald's formalism, leaving the problem of understanding the term with scalar charges for later work. In previous work [16–18] we showed how to prove the first law in presence of matter fields by correctly taking into account the interplay between diffeomorphisms and gauge transformations. However, since there is only one gauge transformation per gauge field, we were unable to recover the work terms proportional to the variations of the magnetic charges or the moduli, even though the same methods correctly give the terms proportional to magnetic charges in the Smarr formula refs. [6, 11, 12]. In this paper we are going to show how those terms arise in a more careful

[2]For $p = 0$ these are black holes, for $p = 1$ black rings etc. Standard black p-branes are generically not asymptotically flat because they either extend to infinity or they are wrapped around compact dimensions and, in both cases, they would only be asymptotically flat in the transverse, non-compact dimensions. They have to be studied separately. Work in this direction is already in progress [15].

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calculation and how they do it with the appropriate sign to have electric-magnetic duality invariance.

The theory we are going to consider is the straightforward generalization of Einstein-Maxwell to higher dimensions and higher-rank form potentials. Thus, it differs from the one considered in ref. [19][3] by the absence of scalar fields, which we plan to study in future work. In ref. [19] only electric charges and electrically-charged black branes were considered and, in this work, we are going to show how to include the magnetic ones. We will not constrain the dimension of the horizon and, therefore, we consider, simultaneously, black p-branes which are electrically charged with respect to the $(p+1)$-form field and black $\tilde{p}$-branes which are magnetically charged with respect to it or electrically with respect to the dual $(\tilde{p}+1)$-form field, although we are always going to use the formulation in which the $(p+1)$-form potential appears.[4]

This paper is organized as follows: in section 2 we introduce the theories we are going to consider and we are going to explain how electric-magnetic duality is realized in them. In section 3 we are going to define the conserved charges of the theory: those associated to the gauge symmetries and the magnetic ones, whose nature is topological. In section 4 we are going to prove the restricted form of the generalized zeroth laws which we will use in section 5 to find the Smarr formulae and in section 6 to prove the first law. We end by discussing the results obtained and proposing new directions of research in section 7.

2 Electric-magnetic duality and $(p+1)$-forms

We are going to consider the generalization of the d-dimensional Einstein-Maxwell theory in which the Maxwell 1-form field is replaced by a $(p+1)$-form A[5]

$$A = \frac{1}{(p+1)!} A_{\mu_1 \cdots \mu_{p+1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}} \,, \tag{2.1}$$

whose $(p+2)$-form field strength F[6]

$$F \equiv dA = \frac{1}{(p+1)!} \partial_{\mu_1} A_{\mu_2 \cdots \mu_{p+2}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+2}} \,, \tag{2.4}$$

[3] See also ref. [20].

[4] In ref. [21] we have considered an example in which both the fundamental and the dual potential occur in the action. There are two gauge symmetries and the inclusion of magnetic charges is straightforward. These democratic formulations are often very complicated and here we are not going to use them even though in this case they would be much simpler to find.

[5] In our notation p is the dimension of the objects that couple to these potentials, namely p-branes with $(p+1)$-dimensional worldvolumes. This notation differs from the one used in ref. [19], but it is the most natural one in brane physics.

[6] With our normalization, the components of F are defined by

$$F = \frac{1}{(p+2)!} F_{\mu_1 \cdots \mu_{p+2}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+2}} \,, \tag{2.2}$$

so

$$F_{\mu_1 \cdots \mu_{p+2}} = (p+2) \partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+2}]} \,. \tag{2.3}$$

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is invariant under gauge transformations

$$\delta_\chi A = d\chi\,, \tag{2.5}$$

where χ is an arbitrary p-form.

The components of the dual $(\tilde{p}+2)$-form $\star F$ ($\tilde{p} \equiv d-p-4$) are given by

$$(\star F)_{\mu_1\cdots\mu_{\tilde{p}+2}} \equiv \frac{1}{(p+2)!\sqrt{|g|}}\varepsilon_{\mu_1\cdots\mu_{\tilde{p}+2}\nu_1\cdots\nu_{p+2}}F^{\nu_1\cdots\nu_{p+2}}\,, \tag{2.6}$$

and

$$\star^2 F = \sigma^2 F\,, \quad \text{with} \quad \sigma^2 = (-1)^{(d+1)(p+1)}\,. \tag{2.7}$$

We will call the dual form G

$$G \equiv \star F\,. \tag{2.8}$$

G and F are forms of the same rank when $p = \tilde{p}$, which happens when $d = 2(p+2)$. Real (anti-) self-duality requires $\sigma^2 = +1$ and only $(p+1)$-form fields with p odd can have this property.

We choose the Vielbein

$$e^a = e^a{}_\mu dx^\mu\,, \tag{2.9}$$

as the gravitational field. The Levi-Civita spin connection $\omega^{ab} = -\omega^{ba}$ is defined through the first Cartan structure equation

$$\mathcal{D}e^a = de^a - \omega^a{}_b \wedge e^b = 0\,, \tag{2.10}$$

and the curvature 2-form is

$$R^{ab} = d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb}\,. \tag{2.11}$$

We will also use the total covariant derivative ∇. It satisfies the Vielbein postulate

$$\nabla_\mu e^a{}_\nu - \omega_\mu{}^a{}_b e^b{}_\nu - \Gamma_{\mu\nu}{}^\rho e^a{}_\rho = 0\,, \tag{2.12}$$

that relates the components of the spin connection $\omega_\mu{}^{ab}$ to those of the affine connection $\Gamma_{\mu\nu}{}^\rho$ which are given by the Christoffel symbols

$$\Gamma_{\mu\nu}{}^\rho = \tfrac{1}{2}g^{\rho\sigma}\left\{\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right\}\,. \tag{2.13}$$

In terms of these variables and objects, the action we want to consider is

$$\begin{aligned} S[e^a, A] &= \frac{1}{16\pi G_N^{(d)}}\int\left\{(-1)^{d-1}\star(e^a\wedge e^b)\wedge R_{ab} + \frac{(-1)^{d(p-1)}}{2}F\wedge\star F\right\} \\ &\equiv \int \mathbf{L}\,. \end{aligned} \tag{2.14}$$

Since we are mainly interested in the electric-magnetic duality properties of the first law and Smarr formulae, we are not including the dilaton field that usually couples to these forms in supergravity/superstring theories. A more general study including general couplings to scalars will be made elsewhere.

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Under a general variation of the fields

$$\delta S = \int \{\mathbf{E}_a \wedge \delta e^a + \mathbf{E} \wedge \delta A + d\mathbf{\Theta}(e^a, A, \delta e^a, \delta A)\} \,, \tag{2.15}$$

where the equations of motion $\mathbf{E}_a$ (Einstein) and $\mathbf{E}$ (Maxwell) and the symplectic potential $(d-1)$-form $\mathbf{\Theta}(e^a, A, \delta e^a, \delta A)$ are given by[7]

$$\mathbf{E}_a = \imath_a \star (e^c \wedge e^d) \wedge R_{cd} + \tfrac{(-1)^{dp}}{2} \left(\imath_a F \wedge G + (-1)^{p+1} F \wedge \imath_a G\right) \,, \tag{2.16a}$$

$$\mathbf{E} = -dG \,, \tag{2.16b}$$

$$\mathbf{\Theta}(e^a, A, \delta e^a, \delta A) = - \star (e^a \wedge e^b) \wedge \delta\omega_{ab} + G \wedge \delta A \,, \tag{2.16c}$$

where $\imath_c$ stands for $\imath_{e_c}$, i.e. the interior product with the vector field $e_c = e_c{}^\mu \partial_\mu$.

This set of equations can be enlarged with the Bianchi identity

$$\mathbf{B} \equiv -dF \,. \tag{2.17}$$

In order to explore the invariance of the enlarged set of equations of motion under electric-magnetic duality transformations, it is convenient to define the vector of field strengths

$$\mathcal{F} \equiv \begin{pmatrix} F \\ G \end{pmatrix} \,, \tag{2.18}$$

in terms of which the equations take the form

$$\mathbf{E}_a = \imath_a \star (e^c \wedge e^d) \wedge R_{cd} - \tfrac{(-1)^{p(d+1)}}{2} \mathcal{F}^T \Omega \, \imath_a \mathcal{F} \,, \tag{2.19a}$$

$$\begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix} = -d\mathcal{F} \,, \tag{2.19b}$$

where we have defined the 2×2 matrix

$$\Omega \equiv \begin{pmatrix} 0 & 1 \\ \sigma^2 & 0 \end{pmatrix} \,. \tag{2.20}$$

For $\sigma^2 = +1$, this matrix is the non-diagonal metric of $\mathrm{O}(1,1)$ and for $\sigma^2 = -1$ it is the "metric" of $\mathrm{Sp}(2,\mathbb{R})$. It is, then, evident, that the above system of equations is invariant under linear transformations of $\mathcal{F}$ and that the groups of invariance are $\mathrm{O}(1,1)$ and $\mathrm{Sp}(2,\mathbb{R})$. Of course, these linear transformations only make sense for $p = \tilde{p}$. However, when this is not the case, the equations are still formally invariant under the discrete subgroups of $\mathrm{O}(1,1)$ and $\mathrm{Sp}(2,\mathbb{R}) \sim \mathrm{SL}(2,\mathbb{R})$ that simply interchange F and G (up to signs).

However, these are not the duality groups of the theory because the transformations must respect the self-duality constraint

$$\star\, \mathcal{F} = \Omega \mathcal{F} \,. \tag{2.21}$$

[7] In order to simplify the expressions, we suppress the global factors of $\left(16\pi G_N^{(d)}\right)^{-1}$. We will restore them in the final results.

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For $\sigma^2 = +1$, only the interchange of F and G (up to a global sign) survives, while, for $\sigma^2 = -1$, it is the whole continuous subgroup $\mathrm{SO}(2) \subset \mathrm{SL}(2,\mathbb{R})$ that survives.

Observe that the symplectic potential $\boldsymbol{\Theta}(e^a, A, \delta e^a, \delta A)$ is not invariant under any electric-magnetic duality transformations. This is not surprising because the action is not invariant, either.

3 Conserved charges

3.1 Lorentz charge

The action eq. (2.14) is exactly invariant under local Lorentz transformations of the Vielbein

$$\delta_\sigma e^a = \sigma^a{}_b e^b \,, \tag{3.1}$$

which induce the following transformation of the spin connection and curvature

$$\begin{aligned} \delta_\sigma \omega^{ab} &= \mathcal{D}\sigma^{ab} \,, \\ \delta_\sigma R^{ab} &= 2\sigma^{[a|}{}_c R^{c|b]} \,. \end{aligned} \tag{3.2}$$

For these particular transformations and upon use of the Noether identity (the symmetry of the Einstein equations)

$$\mathbf{E}^{[a} \wedge e^{b]} = 0 \,, \tag{3.3}$$

we find

$$\delta_\sigma S = \int d\mathbf{J}[\sigma] \,, \qquad \mathbf{J}[\sigma] = - \star (e^a \wedge e^b) \wedge \mathcal{D}\sigma_{ab} \,. \tag{3.4}$$

The off-shell invariance of the action for arbitrary parameters σ^{ab} and arbitrary integration region imply the closedness of $\mathbf{J}[\sigma]$ and its local exactness

$$\mathbf{J}[\sigma] = d\mathbf{Q}[\sigma] \,, \qquad \mathbf{Q}[\sigma] = \frac{(-1)^{d-1}}{16\pi G_N^{(d)}} \star (e^a \wedge e^b) \wedge \sigma_{ab} \,. \tag{3.5}$$

Using this $(d-2)$-form one can construct for on-shell field configurations a conserved charge for each independent parameter σ^{ab} that leaves that field configuration invariant [22]. However, there are no non-trivial parameters σ^{ab} that leave invariant a regular Vielbein $\sigma^a{}_b e^b = 0$ and, therefore, there seems to be no conserved charges associated to this symmetry. In spite of this, this $(d-2)$-form plays an important role in black-hole thermodynamics for a particular σ^{ab}, as we are going to see and we have already pointed out in refs. [17, 18].

3.2 Electric charge

The action eq. (2.14) is exactly invariant under the gauge transformations of the $(p+1)$-form field A eq. (2.5). For those particular transformations, and using the Noether identity

$$d\mathbf{E} = 0 \,, \tag{3.6}$$

the general variation of the action eq. (2.15) can be written in the form

$$\delta_\chi S = \int d\mathbf{J}[\chi] \,, \qquad \mathbf{J}[\chi] = (-1)^{d-p+1} \mathbf{E} \wedge \chi + G \wedge d\chi \,. \tag{3.7}$$

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The off-shell invariance of the action for arbitrary p-forms χ and integration region imply the closedness of $\mathbf{J}[\chi]$ and its local exactness

$$\mathbf{J}[\chi] = d\mathbf{Q}[\chi]\,, \qquad \mathbf{Q}[\chi] = \frac{(-1)^{d(p-1)}}{16\pi G_N^{(d)}}\chi \wedge G\,. \tag{3.8}$$

The $(d-2)$-form $\mathbf{Q}[\chi]$ can be used to define a charge which is conserved on-shell ($dG = 0$) for each independent gauge parameter χ leaving invariant the field configuration [22, 23]. The p-form gauge parameters that leave invariant the potential A are the closed ones $d\chi = 0$. In a compact manifold with no boundary, these can be decomposed in a linear combination of harmonic p-forms h_i plus an exact p-form de. Only the harmonic ones give non-trivial conserved charges when integrated over closed codimension-2 surfaces Σ^{d-2}

$$Q_i \equiv \frac{(-1)^{d(p+1)}}{16\pi G_N^{(d)}}\int_{\Sigma^{d-2}} h_i \wedge G\,, \tag{3.9}$$

and the addition of exact p-forms to h_i does not change their values [19, 20]. Observe that the sign we have chosen in this definition is purely conventional.

3.3 Magnetic charge

Even though there are no more gauge symmetries in our theory, we can define magnetic charges which are conserved in exactly the same sense as the electric ones:

$$P^m \equiv \frac{(-1)^{d(p+1)}}{16\pi G_N^{(d)}}\int_{\Sigma^{d-2}} \tilde{h}^m \wedge F\,, \tag{3.10}$$

where $\tilde{h}^m$ is a harmonic $\tilde{p}$-form. We are using a different set of indices for the magnetic charges since, in general, the number of harmonic p- and $\tilde{p}$-forms need not be the same. It is unclear how duality rotations or Dirac-like quantization conditions for these charges can be defined, except for the special case in which $p = \tilde{p}$ and $\tilde{h}^i = h_i$. In this case, the charges can also be arranged in vectors

$$\mathcal{Q}_i \equiv \begin{pmatrix} P_i \\ Q_i \end{pmatrix} = \frac{1}{16\pi G_N^{(d)}}\int_{\Sigma^{d-2}} h_i \wedge \mathcal{F}\,, \tag{3.11}$$

transforming in the same way as $\mathcal{F}$.

Observe that the definition of magnetic charge eq. (3.10) becomes trivial and gives zero whenever $F = dA$ *globally*. Thus, as expected, non-vanishing magnetic charges are an exclusive property of certain non-trivial gauge field configurations.

3.4 Noether-Wald charge

The action eq. (2.14) is exactly invariant under infinitesimal diffeomorphisms

$$\delta x^\mu = \xi^\mu(x)\,. \tag{3.12}$$

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Then, if we consider only the infinitesimal transformations of the fields $\delta\varphi \equiv \varphi'(x) - \varphi(x)$ the action is invariant up to a total derivative

$$\delta_\xi S = -\int d\imath_\xi \mathbf{L}\,. \tag{3.13}$$

The associated $(d-2)$-form $\mathbf{Q}[\xi]$ is called the Noether-Wald charge [8].

As discussed in refs. [6, 16–18, 21],[8] the transformation of fields with some gauge freedom under infinitesimal diffeomorphisms is, generically, of the form

$$\delta_\xi = -\pounds_\xi + \delta_{\Lambda_\xi}\,, \tag{3.14}$$

where $\pounds_\xi$ is the standard Lie derivative with respect to the vector field ξ and δ_{Λ_ξ} is a ("compensating" or "induced") gauge transformation whose parameter Λ_ξ depends on ξ and on the fields on which the transformation acts.[9] We should write, then, $\Lambda(\xi, e^a, A)$, but we will use Λ_ξ for simplicity, keeping in mind the dependence on the fields. In general the value of this parameter is only fully determined when the diffeomorphism is a symmetry of the whole field configuration. In that case we will denote the vector field that generates it by k since, in particular, it must be a Killing vector and one can define

$$\delta_k = -\pounds_k + \delta_{\Lambda_k} \equiv -\mathbb{L}_k\,, \tag{3.15}$$

where $\mathbb{L}_k$ transforms covariantly under gauge transformations, hence the name *covariant Lie derivative*. This property (which is not shared by the standard Lie derivative) has to be checked case by case. It ensures that the annihilation of all the fields by the transformation δ_k (or by the operator $\mathbb{L}_k$) is a gauge-independent condition.

In the case of the $(p+1)$-form field A, the compensating p-form gauge parameter is given by [16–18]

$$\chi_\xi = \imath_\xi A - P_\xi\,, \tag{3.16}$$

where the *momentum map* p-form P_ξ satisfies, for $\xi = k$, the *momentum map equation*

$$dP_k + \imath_k F = 0\,. \tag{3.17}$$

Then

$$\delta_\xi A = -\pounds_\xi A + \delta_{\chi_\xi} A = -(d\imath_\xi + \imath_\xi d)A + d\chi_\xi = -(dP_\xi + \imath_\xi F)\,, \tag{3.18}$$

which is guaranteed to vanish when $\xi = k$ by virtue of the momentum map equation (3.17).

The $(p+2)$-form field strength F is gauge invariant and, upon use of the Bianchi identity

$$\delta_k F = -\pounds_k F = -(d\imath_k + \imath_k d)F = -d\imath_k F\,, \tag{3.19}$$

which, yet again, vanishes identically by virtue of the momentum map equation (3.17).

[8] A different, more mathematically rigorous approach based on the theory of principal bundles was followed in ref. [24], but it cannot be applied to the $p > 0$ gauge transformations considered here or in refs. [17, 18, 21].

[9] A slightly different point of view is that of "invariance up to gauge transformations", taken in refs. [25–30].

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In the case of the Vielbein field, the compensating gauge (Lorentz) parameter is given by [16–18, 31, 32][10]

$$\sigma_\xi{}^{ab} = \imath_\xi \omega^{ab} - P_\xi{}^{ab}\,, \tag{3.20}$$

where $P_\xi{}^{ab}$ is the *Lorentz momentum map* which, for $\xi = k$, is defined to satisfy the *Lorentz momentum map equation*

$$\mathcal{D} P_k{}^{ab} + \imath_k R^{ab} = 0\,. \tag{3.21}$$

This equation is solved by the *Killing bivector*

$$P_k{}^{ab} = \nabla^a k^b = \nabla^{[a} k^{b]}\,. \tag{3.22}$$

As a matter of fact, for this value of the momentum map, the Lorentz momentum map equation (3.21) becomes the integrability condition of the Killing vector equation. Then, on the Vielbein

$$\delta_\xi e^a = -(d\imath_\xi + \imath_\xi d)e^a + \sigma_\xi{}^a{}_b e^b = \mathcal{D}\xi^a + P_\xi{}^a{}_b e^b = -\tfrac{1}{2}\left(\nabla_\mu \xi^a + \nabla^a \xi_\mu\right) dx^\mu\,, \tag{3.23}$$

which vanishes when $\xi = k$ by virtue of the Killing vector equation.

For the spin connection we have

$$\delta_\xi \omega^{ab} = -(d\imath_\xi + \imath_\xi d)\omega^{ab} + \mathcal{D}\sigma_\xi{}^{ab} = -\left(\mathcal{D} P_\xi{}^{ab} + \imath_\xi R^{ab}\right)\,, \tag{3.24}$$

that vanishes when $\xi = k$ by virtue of the Lorentz momentum map equation (3.21).

Finally, as a simple exercise, we can consider the transformation of the curvature. According to the general rule and using the explicit form of the compensating Lorentz parameter, we get

$$\begin{aligned}
\delta_\xi R^{ab} &= -\pounds_\xi R^{ab} + 2\sigma_\xi{}^{[a|}{}_c R^{c|b]} \\
&= -\imath_\xi\left(\mathcal{D} R^{ab} + 2\omega^{[a|}{}_c R^{c|b]}\right) - \left(\mathcal{D}\imath_\xi R^{ab} + 2\omega^{[a|}{}_c \imath_\xi R^{c|b]}\right) \\
&\quad + 2\imath_k \omega^{[a|}{}_c R^{c|b]} - 2P_\xi{}^{[a|}{}_c R^{c|b]} \\
&= -\mathcal{D}\imath_\xi R^{ab} - 2P_\xi{}^{[a|}{}_c R^{c|b]}\,.
\end{aligned} \tag{3.25}$$

When $\xi = k$ we can use the Lorentz momentum map equation and

$$\delta_k R^{ab} = \mathcal{D}\mathcal{D} P_k{}^{ab} - 2P_k{}^{[a|}{}_c R^{c|b]}\,, \tag{3.26}$$

which vanishes identically (Ricci identity).

In order to find the Noether-Wald charge we just have to plug the above transformations into the general variation of the action eq. (2.15)

$$\delta_\xi S = \int \left\{-\mathbf{E}_a \wedge \left(\mathcal{D}\xi^a + P_\xi{}^a{}_b e^b\right) - \mathbf{E} \wedge (dP_\xi + \imath_\xi F) + d\boldsymbol{\Theta}(e^a, A, \delta_\xi e^a, \delta_\xi A)\right\}\,, \tag{3.27}$$

with

$$\boldsymbol{\Theta}(e^a, A, \delta_\xi e^a, \delta_\xi A) = \star(e^a \wedge e^b) \wedge \left(\mathcal{D} P_\xi{}^{ab} + \imath_\xi R^{ab}\right) + G \wedge (dP_\xi + \imath_\xi F)\,. \tag{3.28}$$

[10]The resulting covariant derivative, known as *Lie-Lorentz covariant derivative* is a generalization of the spinorial derivative of refs. [31, 33–36].

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The term involving $P_\xi{}^{ab}$ in eq. (3.27) vanishes by virtue of the Noether identity associated to local Lorentz invariance eq. (3.3). Integrating by parts the two terms of eq. (3.27) that involve derivatives, we get

$$\delta_\xi S = \int \left\{(-1)^{d-1}\mathcal{D}\mathbf{E}_a\xi^a - \mathbf{E}\wedge\imath_\xi F + (-1)^{d-p-1}d\mathbf{E}\wedge P_\xi + d\boldsymbol{\Theta}'(e^a, A, \delta_\xi e^a, \delta_\xi A)\right\}, \tag{3.29}$$

with

$$\boldsymbol{\Theta}'(e^a, A, \delta_\xi e^a, \delta_\xi A) = \boldsymbol{\Theta}(e^a, A, \delta_\xi e^a, \delta_\xi A) + (-1)^d\mathbf{E}_a\xi^a + (-1)^{d-p}\mathbf{E}\wedge P_\xi\,. \tag{3.30}$$

Using the Noether identities associated to gauge transformations eq. (3.6) and diffeomorphisms[11]

$$\mathcal{D}\mathbf{E}_a\xi^a + (-1)^d\mathbf{E}\wedge\imath_\xi F = 0\,, \tag{3.31}$$

we arrive at

$$\delta_\xi S = \int d\boldsymbol{\Theta}'(e^a, A, \delta_\xi e^a, \delta_\xi A)\,, \tag{3.32}$$

which, combined with eq. (3.13), leads to

$$d\mathbf{J}[\xi] = 0\,, \qquad \mathbf{J}[\xi] \equiv \boldsymbol{\Theta}'(e^a, A, \delta_\xi e^a, \delta_\xi A) + \imath_\xi\mathbf{L}\,. \tag{3.33}$$

As usual, this implies the local existence of the $(d-2)$-form $\mathbf{Q}[\xi]$

$$\mathbf{J}[\xi] = d\mathbf{Q}[\xi]\,, \qquad \mathbf{Q}[\xi] = \frac{1}{16\pi G_N^{(d)}}\left\{(-1)^d\star(e^a\wedge e^b)P_{\xi\,ab} - (-1)^{d(p-1)}P_\xi\wedge G\right\}, \tag{3.34}$$

which is a straightforward generalization of the Noether-Wald $(d-2)$-form obtained in the Einstein-Maxwell case in ref. [16]. It is manifestly not invariant under any electric-magnetic duality transformations.

4 Restricted, generalized, zeroth laws

Before we derive the Smarr formula and the first law of black hole thermodynamics we must derive the generalized zeroth laws: the constancy of the potentials associated to the charges over the event horizon. Our techniques only allow us to prove them restricted to the bifurcation surface (hence the name *restricted, generalized zeroth laws*), but this is sufficient for our purposes. The statements may, in some cases, be extended to the rest of the horizon using the ideas proposed in ref. [37]

These laws apply to the bifurcation surfaces ($\mathcal{BH}$) of Killing horizons ($\mathcal{H}$) associated to the Killing vector k, which is also assumed to generate a diffeomorphism that leaves invariant all the fields of the theory. Thus, $k^2 \overset{\mathcal{H}}{=} 0$, $k \overset{\mathcal{BH}}{=} 0$. In stationary black-hole spacetimes, the Killing vector whose Killing horizon coincides with the black-hole event horizon, k is an asymptotically timelike linear combination of the one generating time translations $t = t^\mu\partial_\mu$ and those generating rotations in orthogonal planes $\phi_n = \phi_n^\mu\partial_\mu$,

$$k = t + \Omega^n\phi_n\,, \tag{4.1}$$

where the constants Ω_n are the associated angular velocities of the horizon.

[11] The proof of this identity is a trivial generalization of the proof given in ref. [16] for the case $p = 0$.

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If the $(p+2)$-form field F is invariant under the diffeomorphism generated by k, then we can define the momentum map p-form P_k satisfying the momentum map equation (3.17) and, assuming that F is regular on the horizon,

$$dP_k = -\imath_k F \overset{\mathcal{BH}}{=} 0\,. \tag{4.2}$$

Then, using the Hodge decomposition theorem

$$P_k \overset{\mathcal{BH}}{=} \Phi^i h_i + de\,, \tag{4.3}$$

where the h_i are harmonic p-forms on the bifurcation surface and the constants Φ^i are going to play the role of potentials associated to the charges Q_i defined in eq. (3.9) now computed by integration over the bifurcation surface.[12]

We can also define potentials associated to the magnetic charges. The invariance of the metric and gauge field under the diffeomorphism generated by k, plus the equations of motion $dG = 0$, lead to the existence of a magnetic momentum map $\tilde{P}_k$[13]

$$\delta_k G \doteq -d\imath_k G = 0\,, \quad \Rightarrow \quad \exists\ \tilde{P}_k \quad | \quad d\tilde{P}_k + \imath_k G \doteq 0\,. \tag{4.4}$$

In an analogous fashion, the regularity of G over the horizon leads to

$$\tilde{P}_k \overset{\mathcal{BH}}{=} \Phi_m \tilde{h}^m + de\,, \tag{4.5}$$

where the $\tilde{h}^m$ are harmonic $\tilde{p}$-forms on the bifurcation surface and the constants Φ_m are going to play the role of potentials associated to the magnetic charges P^m defined in eq. (3.10), now computed by integration over the bifurcation surface.

Observe that the same reasoning can be applied to the Lorentz momentum map equation, obtaining

$$\mathcal{D}P_k{}^{ab} \overset{\mathcal{BH}}{=} 0\,, \tag{4.6}$$

which implies that $P_k{}^{ab}$ can be expanded as a linear combination with constant coefficients of covariantly constant antisymmetric Lorentz tensors. It is a well-known result that

$$P_k{}^{ab} \overset{\mathcal{BH}}{=} \kappa n^{ab}\,, \tag{4.7}$$

where κ is the surface gravity (constant over the whole event horizon, according to the standard zeroth law) and n^{ab} is the binormal to the horizon with the normalization $n^{ab}n_{ab} = -2$. Clearly, n^{ab} is covariantly constant over the bifurcation surface and κ can be interpreted as the "potential" associated to the Lorentz charge, which is, essentially, the area of the (spatial sections of the) horizon.

[12] The fact that the charges and conjugate potentials that occur in the first law of black-hole mechanics and in the Smarr formulae are defined and computed over the horizon cannot be overemphasized. Its implications in situations in which the topology of the horizon and the topology of spatial infinity are different are dramatic [15].

[13] We are going to use the symbol $\doteq$ for identities that only hold on-shell.

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5 Smarr formulae

Smarr formulae for stationary black-hole solutions [5] can be systematically obtained using Komar integrals [10, 38, 39]. Wald's formalism, in its turn, can be used to construct the $(d-2)$-form integrands of Komar integrals, that we are going to call *Komar charges* as explained in refs. [6, 11, 12, 21] (see also [40]).

The main observation is that, on-shell and for a Killing vector k that generates a symmetry of the whole field configuration, the only non-vanishing contribution to $\mathbf{J}[k]$ is $\imath_k\mathbf{L}$

$$\mathbf{J}[k] \doteq \imath_k\mathbf{L}\,. \tag{5.1}$$

Furthermore, under the same conditions,

$$0 = -\delta_k\mathbf{L} \doteq \pounds_k\mathbf{L} = d\imath_k\mathbf{L}\,, \tag{5.2}$$

which implies the local existence of the $(d-2)$-form ω_k

$$d\omega_k \doteq \imath_k\mathbf{L}\,. \tag{5.3}$$

Since we have proven that $\mathbf{J}[\xi] = d\mathbf{Q}[\xi]$, eq. (5.1) leads to the identity

$$d\mathbf{K}[k] \doteq 0\,, \tag{5.4}$$

for the *Komar charge* $(d-2)$-form $\mathbf{K}[k]$ defined by

$$\mathbf{K}[k] \equiv -\left(\mathbf{Q}[k] - \omega_k\right)\,. \tag{5.5}$$

Smarr formulae for stationary black holes are obtained by integrating eq. (5.4) on hypersurfaces Σ with boundaries at a spatial section of the event horizon $\partial\Sigma_h$ (usually, the bifurcation surface $\mathcal{BH}$) and at spatial infinity $\partial\Sigma_\infty$. Applying Stokes' theorem to that integral one gets

$$\int_{\partial\Sigma_\infty}\mathbf{K}[k] = \int_{\mathcal{BH}}\mathbf{K}[k]\,, \tag{5.6}$$

and performing the integrals one arrives at the Smarr formula.

In order to apply this algorithm we must first construct the Komar charge $\mathbf{K}[k]$ finding ω_k. This can be done for general configurations using the techniques of ref. [6]. The trace of the Einstein equation (2.16a) can be written in terms of the Lagrangian as follows:

$$e^a\wedge\mathbf{E}_a = (-1)^{d-1}(d-2)\left\{\mathbf{L} - (-1)^{d(p+1)}\frac{(p+1)}{(d-2)}F\wedge G\right\}\,, \tag{5.7}$$

which implies that the on-shell Lagrangian takes the value

$$\mathbf{L} \doteq (-1)^{d(p+1)}\frac{(p+1)}{(d-2)}F\wedge G\,. \tag{5.8}$$

Next, using the momentum map equations (3.17) and (4.4)

$$\imath_k\mathbf{L} \doteq -(-1)^{d(p+1)}\frac{(p+1)}{(d-2)}\left[dP_k\wedge G + (-1)^{p(d-1)}d\tilde{P}_k\wedge F\right]\,. \tag{5.9}$$

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and integrating by parts and using the equation of motion and Bianchi identity, we arrive at

$$\omega_k = -(-1)^{d(p+1)} \frac{(p+1)}{(d-2)} \left[P_k \wedge G + (-1)^{p(d-1)} \tilde{P}_k \wedge F \right] . \tag{5.10}$$

The Komar charge $(d-2)$-form is, then, given by

$$\begin{aligned} \mathbf{K}[k] = \frac{1}{16\pi G_N^{(d)}} \Big\{ & (-1)^{d-1} \star (e^a \wedge e^b) P_{k\,ab} \\ & + \frac{(-1)^{d(p+1)}}{d-2} \left[(\tilde{p}+1) P_k \wedge G - (-1)^d \sigma^2 (p+1) \tilde{P}_k \wedge F \right] \Big\} . \end{aligned} \tag{5.11}$$

When $p = \tilde{p}$ (so d is even), defining the vector of momentum maps

$$\mathcal{P}_k \equiv \begin{pmatrix} P_k \\ \tilde{P}_k \end{pmatrix} , \tag{5.12}$$

which transforms as $\mathcal{F}$ under electric-magnetic duality because it satisfies the equation

$$d\mathcal{P}_k + \imath_k \mathcal{F} = 0 \, , \tag{5.13}$$

$\mathbf{K}[k]$ can be rewritten in the manifestly duality-symmetric form

$$\mathbf{K}[k] = \frac{1}{16\pi G_N^{(d)}} \left\{ (-1)^{d-1} \star (e^a \wedge e^b) P_{k\,ab} + \frac{(p+1)}{(d-2)} \mathcal{P}_k^T \wedge \Omega \mathcal{F} \right\} . \tag{5.14}$$

We now plug the Komar charge eq. (5.11) in the integrals of eq. (5.6). For asymptotically-flat black holes, only the gravitational term in the first line contributes to the integral over spatial infinity since the products of potentials and gauge fields fall off too fast approaching infinity if we impose adequate boundary conditions. Using also the restricted generalized zeroth laws for the momentum maps eqs. (4.3) and (4.5), we get

$$\begin{aligned} \frac{1}{16\pi G_N^{(d)}} \int_{\partial\Sigma_\infty} (-1)^{d-1} \star (e^a \wedge e^b) P_{k\,ab} = & \frac{1}{16\pi G_N^{(d)}} \int_{\mathcal{BH}} (-1)^{d-1} \star (e^a \wedge e^b) P_{k\,ab} \\ & + \frac{1}{(d-2)} \left[(\tilde{p}+1) \Phi^i Q_i + (-1)^d (p+1) \sigma^2 \tilde{\Phi}_m P^m \right] . \end{aligned} \tag{5.15}$$

For the Killing vector eq. (4.1), the integral in the left-hand side of this equation gives

$$\frac{1}{16\pi G_N^{(d)}} \int_{\partial\Sigma_\infty} (-1)^{d-1} \star (e^a \wedge e^b) P_{k\,ab} = \frac{(d-3)}{(d-2)} \left(M - \Omega^n J_n \right) , \tag{5.16}$$

where M is the mass and J_n are the components of the angular momentum. Furthermore, using eq. (4.7), the integral in the right-hand side gives

$$\frac{1}{16\pi G_N^{(d)}} \int_{\mathcal{BH}} (-1)^{d-1} \star (e^a \wedge e^b) P_{k\,ab} = -\frac{1}{16\pi G_N^{(d)}} \int_{\mathcal{BH}} dA n^{ab} P_{k\,ab} = \frac{\kappa A}{8\pi G_N^{(d)}} = TS \, , \tag{5.17}$$

where T is the Hawking temperature and S is the Bekenstein-Hawking entropy.

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Thus, we get the Smarr equation

$$M = \frac{(d-2)}{(d-3)} TS + \Omega^n J_n + \frac{(\tilde{p}+1)}{(d-3)} \Phi^i Q_i + (-1)^d \sigma^2 \frac{(p+1)}{(d-3)} \tilde{\Phi}_m P^m \,. \tag{5.18}$$

For $p = \tilde{p}$ this formula takes the manifestly electric-magnetic duality invariant form

$$M = \frac{(d-2)}{(d-3)} TS + \Omega^n J_n + \frac{(p+1)}{(d-3)} \hat{\Phi}^{T\,i} \wedge \Omega \mathcal{Q}_i \,, \tag{5.19}$$

where $\mathcal{Q}_i$ is the charge vector defined in eq. (3.11) and $\hat{\Phi}^i$ is the vector of potentials

$$\hat{\Phi}^i \equiv \begin{pmatrix} \tilde{\Phi}^i \\ \Phi^i \end{pmatrix} \,, \tag{5.20}$$

so that

$$\hat{\Phi}^{T\,i} \wedge \Omega \mathcal{Q}_i = \Phi^i Q_i + \sigma^2 \tilde{\Phi}_i P^i \,. \tag{5.21}$$

6 First law

We are going to review the derivation of the first law in full detail, improving the derivations made in refs. [6, 16–18, 21] and showing where and how the variation of the magnetic charges, missed in those works, arise.

Following refs. [7–9], and denoting by φ all the fields of the theory, we define the symplectic $(d-1)$-form

$$\omega(\varphi, \delta_1\varphi, \delta_2\varphi) \equiv \delta_1 \mathbf{\Theta}(\varphi, \delta_2\varphi) - \delta_2 \mathbf{\Theta}(\varphi, \delta_1\varphi) \,, \tag{6.1}$$

and we choose $\delta_1\varphi = \delta\varphi$, variations which satisfy the linearized equations of motion but which are, otherwise, arbitrary, and $\delta_2\varphi = \delta_\xi\varphi$, the transformations under diffeomorphisms that we have defined in section 3.4. On-shell $\mathbf{\Theta} = \mathbf{\Theta}'$ and using the definitions of $\mathbf{J}[\xi]$ eq. (3.33) and δ_ξ eq. (3.14)

$$\begin{aligned}
\omega(\varphi, \delta\varphi, \delta_\xi\varphi) &\doteq \delta\mathbf{\Theta}'(\varphi, \delta_\xi\varphi) - \delta_\xi\mathbf{\Theta}'(\varphi, \delta\varphi) \\
&= \delta\left(\mathbf{J}[\xi] - \imath_\xi \mathbf{L}\right) - \left(-\pounds_\xi + \delta_{\Lambda_\xi}\right)\mathbf{\Theta}'(\varphi, \delta\varphi) \\
&= \delta\mathbf{J}[\xi] - \imath_\xi \delta\mathbf{L} + \left(\imath_\xi d + d\imath_\xi - \delta_{\Lambda_\xi}\right)\mathbf{\Theta}'(\varphi, \delta\varphi) \\
&= \delta d\mathbf{Q}[\xi] - \imath_\xi\left(\mathbf{E}_\varphi \wedge \delta\varphi + d\mathbf{\Theta}(\varphi, \delta\varphi)\right) + \left(\imath_\xi d + d\imath_\xi - \delta_{\Lambda_\xi}\right)\mathbf{\Theta}'(\varphi, \delta\varphi) \\
&\doteq d\left[\delta\mathbf{Q}[\xi] + \imath_\xi \mathbf{\Theta}'(\varphi, \delta\varphi)\right] - \delta_{\Lambda_\xi}\mathbf{\Theta}'(\varphi, \delta\varphi) \,.
\end{aligned} \tag{6.2}$$

This result differs from the standard one by the last term, which does not look like a total derivative. Let us study it in more detail in the theory at hand:

$$\begin{aligned}
\delta_{\Lambda_\xi}\mathbf{\Theta}'(\varphi, \delta\varphi) &\doteq \delta_{\Lambda_\xi}\left\{-\star\left(e^a \wedge e^b\right) \wedge \delta\omega_{ab} + G \wedge \delta A\right\} \\
&= -\delta_{\sigma_\xi}\left\{\star(e^a \wedge e^b)\right\} \wedge \delta\omega_{ab} - \star(e^a \wedge e^b) \wedge \delta_{\sigma_\xi}\delta\omega_{ab} + G \wedge \delta_{\chi_\xi}\delta A \,,
\end{aligned} \tag{6.3}$$

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since G is gauge invariant. Now, let us consider the last term. By definition, and taking into account that the parameter of the compensating gauge transformation depends on the field on which the transformation acts

$$\delta_{\chi_\xi}\delta A = \delta_{\chi_\xi}(A' - A) = d\chi(\xi, A') - d\chi(\xi, A) = d\delta\chi_\xi\,. \tag{6.4}$$

By the same token

$$\delta_{\sigma_\xi}\delta\omega_{ab} = \mathcal{D}\delta\sigma_\xi - 2\delta\omega_{[a|}{}^c\sigma_{\xi\,c|b]}\,, \tag{6.5}$$

while the first term transforms in the standard fashion

$$\delta_{\sigma_\xi}\left\{\star(e^a\wedge e^b)\right\} = 2\sigma_\xi{}^{[a|}{}_c\star(e^c\wedge e^{|b]})\,. \tag{6.6}$$

Combining these results, integrating by parts and using the equation of motion $dG = 0$ we arrive at another total derivative

$$\begin{aligned}\delta_{\Lambda_\xi}\boldsymbol{\Theta}'(\varphi,\delta\varphi) &\doteq -\left\{\star(e^a\wedge e^b)\right\}\wedge\mathcal{D}\delta\sigma_{\xi\,ab} + G\wedge d\delta\chi_\xi\\ &\doteq d\left\{(-1)^{d-1}\star(e^a\wedge e^b)\delta\sigma_{\xi\,ab} + (-1)^{\tilde{p}}G\wedge\delta\chi_\xi\right\}\,,\end{aligned} \tag{6.7}$$

which allows us to rewrite the complete symplectic $(d-1)$-form as the total derivative of a $(d-2)$-form that we will denote by $\boldsymbol{\Omega}(\varphi,\delta\varphi,\delta_\xi\varphi)$, which is defined up to total derivatives

$$\omega(\varphi,\delta\varphi,\delta_\xi\varphi) \doteq -d\boldsymbol{\Omega}(\varphi,\delta\varphi,\delta_\xi\varphi)\,, \tag{6.8a}$$

$$\begin{aligned}\boldsymbol{\Omega}(\varphi,\delta\varphi,\delta_\xi\varphi) \equiv{}& -\delta\mathbf{Q}[\xi] - \imath_\xi\boldsymbol{\Theta}'(\varphi,\delta\varphi)\\ &+\frac{1}{16\pi G_N^{(d)}}\left\{(-1)^{d-1}\star(e^a\wedge e^b)\delta\sigma_{\xi\,ab} + (-1)^{\tilde{p}}G\wedge\delta\chi_\xi\right\}\,.\end{aligned} \tag{6.8b}$$

Plugging into $\boldsymbol{\Omega}(\varphi,\delta\varphi,\delta_\xi\varphi)$ the expressions we have obtained for $\mathbf{Q}[\xi]$ and $\boldsymbol{\Theta}'(\varphi,\delta\varphi)$ and operating, we can put $\boldsymbol{\Omega}(\varphi,\delta\varphi,\delta_\xi\varphi)$ in this form:

$$\begin{aligned}\boldsymbol{\Omega}(\varphi,\delta\varphi,\delta_\xi\varphi) = (-1)^{d-1}\delta\star(e^a\wedge e^b)\wedge P_{\xi\,ab} + \imath_\xi\star(e^a\wedge e^b)\wedge\delta\omega_{ab}\\ +(-1)^{d(p+1)}P_\xi\wedge\delta G - \imath_\xi G\wedge\delta A\,,\end{aligned} \tag{6.9}$$

again, up to total derivatives. We are going to profit from this freedom to rewrite this charge as follows:

$$\begin{aligned}\boldsymbol{\Omega}(\varphi,\delta\varphi,\delta_\xi\varphi) = \delta\left[(-1)^{d-1}\star(e^a\wedge e^b)\wedge P_{\xi\,ab}\right] - (-1)^{d-1}\star(e^a\wedge e^b)\wedge\delta P_{\xi\,ab}\\ +\imath_\xi\star(e^a\wedge e^b)\wedge\delta\omega_{ab} + (-1)^{d(p+1)}P_\xi\wedge\delta G - \left(\imath_\xi G + d\tilde{P}_\xi\right)\wedge\delta A\\ +(-1)^{\tilde{p}+1}\tilde{P}_\xi\wedge\delta F\,.\end{aligned} \tag{6.10}$$

Now, when $\xi = k$, since $\delta_k\varphi = 0$ implies $\omega(\varphi,\delta\varphi,\delta_k\varphi) = 0$, we have the identity

$$d\boldsymbol{\Omega}(\varphi,\delta\varphi,\delta_k\varphi) \doteq 0\,, \tag{6.11}$$

where, upon use of the definition of the dual momentum map eq. (4.4) $\boldsymbol{\Omega}(\varphi,\delta\varphi,\delta_k\varphi)$ takes the final form

$$\begin{aligned}\boldsymbol{\Omega}(\varphi,\delta\varphi,\delta_k\varphi) = \delta\left[(-1)^{d-1}\star(e^a\wedge e^b)\wedge P_{k\,ab}\right] - (-1)^{d-1}\star(e^a\wedge e^b)\wedge\delta P_{k\,ab}\\ +\imath_k\star(e^a\wedge e^b)\wedge\delta\omega_{ab} + (-1)^{d(p+1)}\left[P_k\wedge\delta G + (-1)^d\sigma^2\tilde{P}_k\wedge\delta F\right]\,.\end{aligned} \tag{6.12}$$

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To proceed, we integrate the identity eq. (6.11) over the same hypersurface over which we integrated the analogous identity involving the Komar charge $\mathbf{K}[k]$ in the previous section. Using Stokes' theorem

$$\int_{\partial\Sigma_\infty} \mathbf{\Omega}(\varphi, \delta\varphi, \delta_k\varphi) = \int_{\mathcal{BH}} \mathbf{\Omega}(\varphi, \delta\varphi, \delta_k\varphi)\,. \tag{6.13}$$

For the Killing vector eq. (4.1) the integral at spatial infinity can be shown to give [9, 41][14]

$$\int_{\partial\Sigma_\infty} \mathbf{\Omega}(\varphi, \delta\varphi, \delta_k\varphi) = \delta M - \Omega_n \delta J^n\,. \tag{6.14}$$

When evaluating the integral over the bifurcation surface, we can use the reasoning in ref. [9] to show that the second term in eq. (6.12) does not contribute and that the first gives, simply $\kappa\delta A/(8\pi G_N^{(d)})$. The third simply vanishes on the bifurcation surface. Using these results, the restricted, generalized, zeroth laws eqs. (4.3) and (4.5) and the definitions of electric and magnetic charges eqs. (3.9) and (3.10), the integral over the bifurcation surface gives

$$\int_{\mathcal{BH}} \mathbf{\Omega}(\varphi, \delta\varphi, \delta_k\varphi) = \frac{\kappa\delta A}{8\pi G_N^{(d)}} + (-1)^{d(p-1)} \left[\Phi^i \delta Q_i + (-1)^d \sigma^2 \tilde{\Phi}_m \delta P^m\right]\,, \tag{6.15}$$

and we arrive at the first law

$$\delta M = \frac{\kappa\delta A}{8\pi G_N^{(d)}} + \Omega_n \delta J^n + (-1)^{d(p-1)} \left[\Phi^i \delta Q_i + (-1)^d \sigma^2 \tilde{\Phi}_m \delta P^m\right]\,, \tag{6.16}$$

which, for the $\tilde{p} = p$ case takes the manifestly electric-magnetic duality-invariant form

$$\delta M = \frac{\kappa\delta A}{8\pi G_N^{(d)}} + \Omega_n \delta J^n + \Phi^i \Omega \delta \mathcal{Q}_i\,. \tag{6.17}$$

7 Discussion

In this paper we have studied how to deal with magnetic charges in a d-dimensional generalization of the Einstein-Maxwell theory with $(p+1)$-form potentials. Our main results are

1. The Komar charge eqs. (5.11), which, for $p = \tilde{p}$, takes the manifestly electric-magnetic duality-invariant form eq. (5.14).

2. The Smarr formula eq. (5.18), which, again, takes the manifestly electric-magnetic duality-invariant form eq. (5.19) in the $p = \tilde{p}$ case.

3. The first law eq. (6.16) and the manifestly electric-magnetic duality-invariant form eq. (6.17) that it takes when $p = \tilde{p}$.

[14] As in the calculation of the Smarr formula, the additional terms that we have found will not contribute at infinity if we impose suitable boundary conditions to the fields and their variations.

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We have assumed in the derivation of these results the asymptotic flatness of the solutions. Thus, they are valid for black holes, black rings and their generalizations, but, in order to apply them to infinite, planar, p-branes, a few, simple, modifications would be necessary to replace mass by tension and charges by charge densities removing the infinite volume factors. Wrapping these branes on compact dimensions would introduce additional effects (KK and winding modes) that need to be studied separately.[15]

Furthermore, observe that the Smarr formulae and first laws obtained are generic: a particular solution may not be able to carry the electric, the magnetic or either charge. For instance, a black hole in 6 dimensions in a theory with a 2-form will not be able to carry electric nor magnetic charge with respect to the 2-form. In 5 dimensions, a black hole can carry the electric charge of a 1-form potential but not the magnetic charge (electric with respect to a 2-form potential), while a black ring can, in principle, carry the opposite.

In the $\tilde{p} = p$ cases, black p-branes can carry electric and magnetic charges of the same $(p+1)$ potential and, as it is well known, since electric-magnetic duality leaves invariant the metric, all their geometric properties including their surface gravity and area are also duality invariant. Thus, the first law of their dynamics should also be invariant. Our results show that this is, indeed, the case.

As mentioned in the Introduction, the first law also has a term proportional to the variation of the moduli where the proportionality constants are the scalar charges, for which no good definition as conserved charges has ever been given [14]. Here we have avoided this problem by studying a theory with no scalar fields, but this is a problem that has to be confronted and understood and we plan to do so in future work.

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References

[1] P.A.M. Dirac, *Quantised singularities in the electromagnetic field,*, *Proc. Roy. Soc. Lond. A* **133** (1931) 60 [INSPIRE].

[2] M.J. Duff, R.R. Khuri and J.X. Lu, *String solitons*, *Phys. Rept.* **259** (1995) 213 [hep-th/9412184] [INSPIRE].

[15] Work in this direction is in progress [15].

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[3] T. Ortín, *Gravity and Strings*, Cambridge Monographs on Mathematical Physics, Cambridge University Press (2015), 2nd ed., 10.1017/CBO9781139019750 [INSPIRE].

[4] J.M. Bardeen, B. Carter and S.W. Hawking, *The Four laws of black hole mechanics*, *Commun. Math. Phys.* **31** (1973) 161 [INSPIRE].

[5] L. Smarr, *Mass formula for Kerr black holes*, *Phys. Rev. Lett.* **30** (1973) 71 [*Erratum ibid.* **30** (1973) 521] [INSPIRE].

[6] D. Mitsios, T. Ortín and D. Pereñíguez, *Komar integral and Smarr formula for axion-dilaton black holes versus S duality*, *JHEP* **08** (2021) 019 [arXiv:2106.07495] [INSPIRE].

[7] J. Lee and R.M. Wald, *Local symmetries and constraints*, *J. Math. Phys.* **31** (1990) 725 [INSPIRE].

[8] R.M. Wald, *Black hole entropy is the Noether charge*, *Phys. Rev. D* **48** (1993) R3427 [gr-qc/9307038] [INSPIRE].

[9] V. Iyer and R.M. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, *Phys. Rev. D* **50** (1994) 846 [gr-qc/9403028] [INSPIRE].

[10] A. Komar, *Covariant conservation laws in general relativity*, *Phys. Rev.* **113** (1959) 934 [INSPIRE].

[11] S. Liberati and C. Pacilio, *Smarr Formula for Lovelock Black Holes: a Lagrangian approach*, *Phys. Rev. D* **93** (2016) 084044 [arXiv:1511.05446] [INSPIRE].

[12] T. Ortín, *Komar integrals for theories of higher order in the Riemann curvature and black-hole chemistry*, *JHEP* **08** (2021) 023 [arXiv:2104.10717] [INSPIRE].

[13] M.K. Gaillard and B. Zumino, *Duality Rotations for Interacting Fields*, *Nucl. Phys. B* **193** (1981) 221 [INSPIRE].

[14] G.W. Gibbons, R. Kallosh and B. Kol, *Moduli, scalar charges, and the first law of black hole thermodynamics*, *Phys. Rev. Lett.* **77** (1996) 4992 [hep-th/9607108] [INSPIRE].

[15] P. Meessen, T. Ortín, D. Pereñíguez and M. Zatti, work in progress.

[16] Z. Elgood, P. Meessen and T. Ortín, *The first law of black hole mechanics in the Einstein-Maxwell theory revisited*, *JHEP* **09** (2020) 026 [arXiv:2006.02792] [INSPIRE].

[17] Z. Elgood, D. Mitsios, T. Ortín and D. Pereñíguez, *The first law of heterotic stringy black hole mechanics at zeroth order in α'*, *JHEP* **07** (2021) 007 [arXiv:2012.13323] [INSPIRE].

[18] Z. Elgood, T. Ortín and D. Pereñíguez, *The first law and Wald entropy formula of heterotic stringy black holes at first order in α'*, *JHEP* **05** (2021) 110 [arXiv:2012.14892] [INSPIRE].

[19] G. Compere, *Note on the First Law with p-form potentials*, *Phys. Rev. D* **75** (2007) 124020 [hep-th/0703004] [INSPIRE].

[20] K. Copsey and G.T. Horowitz, *The Role of dipole charges in black hole thermodynamics*, *Phys. Rev. D* **73** (2006) 024015 [hep-th/0505278] [INSPIRE].

[21] P. Meessen, D. Mitsios and T. Ortín, *Black hole chemistry, the cosmological constant and the embedding tensor*, arXiv:2203.13588 [INSPIRE].

[22] G. Barnich and F. Brandt, *Covariant theory of asymptotic symmetries, conservation laws and central charges*, *Nucl. Phys. B* **633** (2002) 3 [hep-th/0111246] [INSPIRE].

[23] G. Barnich, *Boundary charges in gauge theories: Using Stokes theorem in the bulk*, *Class. Quant. Grav.* **20** (2003) 3685 [hep-th/0301039] [INSPIRE].

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[24] K. Prabhu, *The First Law of Black Hole Mechanics for Fields with Internal Gauge Freedom*, *Class. Quant. Grav.* **34** (2017) 035011 [arXiv:1511.00388] [INSPIRE].

[25] D. Sudarsky and R.M. Wald, *Extrema of mass, stationarity, and staticity, and solutions to the Einstein Yang-Mills equations*, *Phys. Rev. D* **46** (1992) 1453 [INSPIRE].

[26] D. Sudarsky and R.M. Wald, *Mass formulas for stationary Einstein Yang-Mills black holes and a simple proof of two staticity theorems*, *Phys. Rev. D* **47** (1993) R5209 [gr-qc/9305023] [INSPIRE].

[27] G. Barnich and G. Compere, *Surface charge algebra in gauge theories and thermodynamic integrability*, *J. Math. Phys.* **49** (2008) 042901 [arXiv:0708.2378] [INSPIRE].

[28] S. McCormick, *The Phase Space for the Einstein-Yang-Mills Equations and the First Law of Black Hole Thermodynamics*, *Adv. Theor. Math. Phys.* **18** (2014) 799 [arXiv:1302.1237] [INSPIRE].

[29] K. Hajian, A. Seraj and M.M. Sheikh-Jabbari, *NHEG Mechanics: Laws of Near Horizon Extremal Geometry (Thermo)Dynamics*, *JHEP* **03** (2014) 014 [arXiv:1310.3727] [INSPIRE].

[30] K. Hajian and M.M. Sheikh-Jabbari, *Solution Phase Space and Conserved Charges: A General Formulation for Charges Associated with Exact Symmetries*, *Phys. Rev. D* **93** (2016) 044074 [arXiv:1512.05584] [INSPIRE].

[31] T. Ortín, *A Note on Lie-Lorentz derivatives*, *Class. Quant. Grav.* **19** (2002) L143 [hep-th/0206159] [INSPIRE].

[32] T. Jacobson and A. Mohd, *Black hole entropy and Lorentz-diffeomorphism Noether charge*, *Phys. Rev. D* **92** (2015) 124010 [arXiv:1507.01054] [INSPIRE].

[33] A. Lichnerowicz, *Spineurs harmoniques*, *C. R. Acad. Sci. Paris* **257** (1963) 7.

[34] Y. Kosmann, *Dérivées de Lie des spineurs*, *C. R. Acad. Sci. Paris A* **262** (1966) A289.

[35] Y. Kosmann, *Dérivées de Lie des spineurs*, *Annali Mat. Pura Appl.* **91** (1971) 317.

[36] D.J. Hurley and M.A. Vandyck, *On the concepts of Lie and covariant derivatives of spinors. Part 1*, *J. Phys. A* **27** (1994) 4569 [INSPIRE].

[37] T. Jacobson, G. Kang and R.C. Myers, *On black hole entropy*, *Phys. Rev. D* **49** (1994) 6587 [gr-qc/9312023] [INSPIRE].

[38] D. Kastor, *Komar Integrals in Higher (and Lower) Derivative Gravity*, *Class. Quant. Grav.* **25** (2008) 175007 [arXiv:0804.1832] [INSPIRE].

[39] D. Kastor, S. Ray and J. Traschen, *Smarr Formula and an Extended First Law for Lovelock Gravity*, *Class. Quant. Grav.* **27** (2010) 235014 [arXiv:1005.5053] [INSPIRE].

[40] T. Jacobson and M. Visser, *Gravitational Thermodynamics of Causal Diamonds in (A)dS*, *SciPost Phys.* **7** (2019) 079 [arXiv:1812.01596] [INSPIRE].

[41] G. Barnich and G. Compere, *Conserved charges and thermodynamics of the spinning Godel black hole*, *Phys. Rev. Lett.* **95** (2005) 031302 [hep-th/0501102] [INSPIRE].

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