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Soft gravitational radiation from multi-body collisions

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ABSTRACT: We derive a universal expression for the gravitational radiation energy spectrum $dE^{GW}/d\omega$ at sub-leading order emitted from a generic gravitational hard scattering of multi-particles or multi-bodies. Our result includes all $\mathcal{O}(\omega)$ corrections to the gravitational radiation flux from a generic $2 \rightarrow N$ collision, in both the cases of massless and massive particles/bodies. We also show the dependence of the radiation energy flux by the quantum spin in case of particle collisions. Then, we consider the specific case of a gravitational elastic scattering of two massive bodies, i.e. $m + M \rightarrow m + M$ with m, M the masses of the two bodies respectively. We demonstrate that in this case all $\mathcal{O}(\omega)$ contributions to the energy flux exactly cancel each others. Nevertheless, we also show that, for a $2 \rightarrow 2$ inelastic scattering, the inclusion of sub-leading soft gravitons leads to a not zero radiation flux, having a simple expression in certain asymptotic regimes. Our results can be applied to the case of Black Hole collisions with possible testable implications in gravitational waves physics.

KEYWORDS: Black Holes, Classical Theories of Gravity, Models of Quantum Gravity

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1 Introduction

As it is well known, the gravitational bremsstrahlung radiation, emitted from a generic gravitational scattering, has a simple general expression in the leading-order (LO) zero-frequency limit (ZFL) derived many decades ago [1–8]. Nevertheless in the recent years the interest to soft theorems has come back with several new results about sub- and sub-sub-leading terms beyond the soft leading expression [9–16]. Indeed, many recent works were dedicated on several possible preservation and violation cases of universality (of soft theorems) [17–26] as well as to the connection of soft theorems with the Bondi-Metzner-Van der Burg-Sachs (BMS) symmetry group of asymptotically-flat space-time metric [27–32].

Certainly, nowadays these theoretical issues, that may appear in Weinberg’s paper as academic, are now revitalized by the direct observations of Gravitational Waves from Black Hole (BH) and Neutron star mergings [33]. Indeed recently it was suggested that the *gravitational memory effect* related to soft theorems and BMS may be tested from future gravitational waves experiments [34]. On the other hand, the analysis of gravitational radiation in soft regime can also be important for the detection of quantum gravity effects

such possible α' corrections on radiation energy flux predicted by string theory in BH mergings dubbed *string memory effect* [35–38].

For these reasons, it is interesting to explore the possibility of using new tools developed from soft theorems and scattering amplitudes to extend the $dE^{GW}/d\omega$ results beyond the leading ZFL order, i.e. to the Next-to-leading order (NLO). Recent progresses of this program have been shown with different approaches, but substantial agreements of results [39–43].

On the other hand, ZFL radiation analysis can be compared with other results from the perspective of bremsstrahlung emission in high energy gravitational scatterings. In particular, a old standing program still in progress explores aspects of gravitational scattering at the transplanckian energy limit [44–54] (e.g. see also [55–57] for recent updates in the subject). This can lead to important insights on the information paradox in energy regimes where the BH should be generated [58–61] and should evaporate in form of Hawking radiation [62]. On the other hand, this can also scrutinize string theory predictions from finite size of scattering vertices and tidal excitations [44, 47, 63] as well as possible modifications of gravity at short-distance and generalized uncertainty principle (GUP) [64, 65].

Let us consider a generic process from **IN** \rightarrow **OUT**, including virtual and real soft graviton effects. Let us denote the S-matrix of such a process as¹

$$\mathcal{S}^{(0)} \rightarrow \mathcal{S} = \exp\left(\frac{d^3q}{\sqrt{2\omega}}\left(\lambda_q^* a_q^\dagger - \lambda_q a_q\right)\right) \mathcal{S}^{(0)}, \quad (1.1)$$

where a_q, a_q^\dagger are destruction and creation operators for soft gravitons of momenta q and maximal cutoff energy $\Lambda \ll E$ — with E the characteristic energy scale of the collision process — and λ_q is a process-dependent function of q . Such an exponential operator applied on the bare S-matrix \mathcal{S}^0 corresponds to a coherent state operator and soft gravitons are in a coherent vacuum state.

The energy spectrum of soft graviton emitted by a generic **IN** \rightarrow **OUT** process is related to the eq. (1.1) as follows

$$\frac{dE^{GW}}{d^3q} = \frac{\hbar}{2} |\lambda_q|^2, \quad (1.2)$$

where λ_q depends from the specific **IN** \rightarrow **OUT** scattering.

In general such an expression can be expanded in powers of q and, integrating on the momenta solid angle, one can obtain a $dE^{GW}/d\omega$ with a leading term going in the ZFL $\omega \rightarrow 0$ as $dE_0^{GW}/d\omega \sim \omega^{-2}$.

Nevertheless, a universal factorized expression for the sub-leading graviton emission is not known and it takes the form of a differential operator acting on the bare amplitudes. Indeed the subleading soft current depends by the total angular momentum operators acting on the bare S-matrix.

The main purpose of this paper is to compute the sub-leading order of the $dE^{GW}/d\omega$ spectrum. A first step towards it was done by *Bianchi, Veneziano* and one of the author of

¹Let us also remark that in our paper we will focus on the 3 + 1 space-time dimensional case.

this paper [41], considering high energy ultra-relativistic scattering of spin-less particles. Our aim is to extend the previous Addazi-Bianchi-Veneziano (ABV) analysis to the case of generic scatterings of massive particles and bodies, including non-relativistic regimes and having in mind the possible application of it for soft GW emission from BH mergings.

As a warm up let us consider the first three leading orders of soft graviton emissions with universal behavior

$$\begin{aligned} \mathcal{M}_{N+1}(p_i; q) &= \sqrt{8\pi G} \sum_{i=1}^N \left[\frac{p_i h p_i}{q p_i} + \frac{p_i h J_i q}{q p_i} - \frac{q J_i h J_i q}{2 q p_i} \right] \mathcal{M}_N(p_i) + \mathcal{O}(q^2) \\ &\equiv (S_0 + S_1 + S_2) \mathcal{M}_N(p_i) \end{aligned} \quad (1.3)$$

where $\mathcal{M}_{N+1} \equiv \mathcal{M}_{N+1}(p_i; q)$ denotes a generic $(N+1)$ -particle on-shell scattering amplitude including an external graviton with momentum q^μ and polarization $h_{\mu\nu}$.

The q^μ and $h_{\mu\nu}$, defined above, satisfy

$$q^2 = 0, \quad h_{\mu\nu} = h_{\nu\mu}, \quad q^\mu h_{\mu\nu} = 0. \quad (1.4)$$

The leading, subleading and sub-subleading soft factors are given by

$$S_0 \equiv \sqrt{8\pi G} \sum_{i=1}^N \frac{p_i h p_i}{q p_i}, \quad S_1 = \sqrt{8\pi G} \sum_{i=1}^N \frac{p_i h J_i q}{q p_i}, \quad S_2 = -\sqrt{8\pi G} \sum_{i=1}^N \frac{q J_i h J_i q}{2 q p_i} \quad (1.5)$$

and $J_i^{\mu\nu} \equiv p_i^\mu \partial_i^\nu - p_i^\nu \partial_i^\mu + S_i^{\mu\nu}$ denotes the total angular momentum of the i -th ‘hard’ particle.

The expression eq. (1.3) is gauge invariant under the rank-2 tensor shift

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + q_\mu \zeta_\nu + q_\nu \zeta_\mu, \quad (1.6)$$

following from the conservation of momentum and angular momentum.

The three soft terms can be related by the respective first three terms of the q -expansion of λ_q . This allows to compute the gravitational energy spectrum $dE^{GW}/d\omega$ after the integration over the emission direction. As we will see, such an operation involves non-trivial vector and tensor integrals that we perform in full generality.

The plan of the paper is the following. In section 2 we will briefly review Weinberg’s derivation of the leading B -factor and $dE^{GW}/d\omega$. In section 3, we will show a complete computation of the sub-leading emission of soft radiation, in the case of massive particle collisions, after a review of the previous ABV result. In section 4 & 5, we will show a new surprising result: in case of $2 \rightarrow 2$ elastic scattering all sub-leading effects exactly vanish for generic particle/body collision. In section 6, we analyze the case of $2 \rightarrow 2$ inelastic collision: we find an exact non-zero result which is, in full generality, complicated, but it has simpler expressions in certain kinematic regimes.

2 Soft gravitational radiation from leading order: short review

The soft theorem up to leading order is universal and the dominant behavior reads

$$|\mathcal{M}_{N+1}(p_i; q)|^2 = 8\pi G \sum_{s=\pm 2} \left| \sum_{i=1}^N \frac{p_i h^s p_i}{q p_i} \right|^2 |\mathcal{M}_N(p_i)|^2, \quad (2.1)$$

where notation is defined in the introduction above. The polarization tensor h^s satisfies $h_{\mu\nu}^{-s} = (h_{\mu\nu}^s)^*$ and after tedious but straightforward manipulations one can obtain that the sum on the spins of the hh 4-tensor corresponds to

$$\sum_{s=\pm 2} h_{\mu\nu}^s h_{\rho\sigma}^{-s} = \Pi_{\mu\nu,\rho\sigma} = \frac{1}{2}(\pi_{\mu\rho}\pi_{\nu\sigma} + \pi_{\mu\sigma}\pi_{\nu\rho} - \pi_{\mu\nu}\pi_{\rho\sigma}), \quad (2.2)$$

where

$$\pi_{\mu\nu} = \eta_{\mu\nu} - q_\mu \bar{q}_\nu - q_\nu \bar{q}_\mu, \quad \bar{q}^2 = 0 \quad \text{and} \quad \bar{q}q = 1. \quad (2.3)$$

Weinberg's B-factor can be obtained after the integration over the momentum of the soft graviton in the final state $q = |q|(-1, \vec{n})$:

$$B_0 = \int \frac{d^3q}{2|q|(2\pi)^3} |\mathcal{M}_{N+1}|^2$$

which corresponds to the number density as follows

$$\frac{dN_0}{d\omega} = \frac{dB_0}{d\omega} = \int \frac{d^3q \delta(|q| - \omega)}{2|q|(2\pi)^3} |\mathcal{M}_{N+1}|^2 \quad (2.4)$$

and the energy spectrum of the gravitational wave is then

$$\frac{dE_0^{\text{GW}}}{d\omega} = \hbar\omega \frac{dN_0}{d\omega}. \quad (2.5)$$

To evaluate the soft factor in eq. (2.1), one first needs to compute the polarization sum and the result reads as

$$\sum_{i,j} \frac{p_i^\mu p_i^\nu}{qp_i} \Pi_{\mu\nu,\rho\sigma} \frac{p_j^\rho p_j^\sigma}{qp_j} = \sum_{i,j} \frac{(p_i p_j)^2 - \frac{1}{2} p_i^2 p_j^2}{qp_i qp_j} = \sum_{i,j} \frac{(p_i p_j)^2 - \frac{1}{2} m_i^2 m_j^2}{qp_i qp_j}. \quad (2.6)$$

The evaluation of the integral in the B-factor

$$I = \int \frac{d^3q}{|q| qp_i qp_j} \quad (2.7)$$

leads to the well know result

$$I = -\ln \frac{\Lambda}{\lambda} \left(\frac{2\pi}{\beta_{ij} p_i p_j} \right) \ln \frac{1 + \beta_{ij}}{1 - \beta_{ij}}, \quad (2.8)$$

where we have used λ and Λ to denote the IR cutoff scale and the upper limit for the validity of the leading soft behavior respectively. Finally, one obtains the well-known Weinberg B-factor [2, 41, 66]:

$$\begin{aligned} B_0 &= -\frac{8\pi G}{2(2\pi)^3} \sum_{i,j} \frac{(p_i p_j)^2 - \frac{1}{2} m_i^2 m_j^2}{E_i E_j} \ln \frac{\Lambda}{\lambda} \frac{2\pi E_i E_j}{\beta_{ij} p_i p_j} \ln \frac{1 + \beta_{ij}}{1 - \beta_{ij}} \\ &= -\frac{G}{2\pi} \ln \frac{\Lambda}{\lambda} \sum_{i,j} \frac{(p_i p_j)(1 + \beta_{ij}^2)}{\beta_{ij}} \ln \frac{1 + \beta_{ij}}{1 - \beta_{ij}}. \end{aligned} \quad (2.9)$$

3 Soft gravitational radiation from sub-leading order

The sub-leading contribution to the GW spectrum comes from the interference between S_0 and S_1 of eq. (1.3)

$$B_1 = 8\pi G \int \frac{d^3q}{2|q|(2\pi)^3} \sum_{i,j} \sum_{s=\pm 2} \left[\frac{(p_i h^s p_i) (p_j h^{(-s)} J_j q)}{qp_i qp_j} + (i \leftrightarrow j) \right]. \quad (3.1)$$

Now, contrary to Winberg's B_0 , the sub-leading factor B_1 involves the presence of the angular momentum operator though as acting on the S -matrix.

3.1 Massless case: a review of the ABV result

In this section, we will review the previous ABV result as an introduction to the main results of our paper.

Let us consider the case in which the 'hard' particles are massless: the sum over helicities of the emitted soft gravitons is

$$\begin{aligned} \sum_{i,j} \frac{p_i^\mu p_i^\nu p_j^\rho (J_j q)^\sigma}{qp_i qp_j} \Pi_{\mu\nu,\rho\sigma} + (i \leftrightarrow j) &= \sum_{i,j} \frac{(p_i p_j) (p_i J_j q)}{qp_i qp_j} + (i \leftrightarrow j) \\ &= \sum_{i,j} \frac{p_i p_j}{qp_i qp_j} [p_i J_j q + p_j J_i q]. \end{aligned} \quad (3.2)$$

Thus, B_1 factor can be rewritten in a simplified form as

$$B_1 = 8\pi G \int \frac{d^3q}{2|q|(2\pi)^3} \sum_{i,j} \frac{p_i p_j}{qp_i qp_j} [p_i \vec{J}_j + p_j \overleftarrow{J}_i] q, \quad (3.3)$$

where \overleftarrow{J}_i and \vec{J}_j act on \mathcal{M}_N and its complex conjugate \mathcal{M}_N^* respectively.

The four-vector integral that we need to evaluate for our purposes is

$$I_{ij}^\mu = \int \frac{d^3q}{2|q|(2\pi)^3} \frac{p_i p_j q^\mu}{qp_i qp_j} = \int \frac{d^4q}{(2\pi)^3} \delta_+(q^2) \frac{p_i p_j q^\mu}{qp_i qp_j}, \quad (3.4)$$

where we have defined $\delta_+(q^2) \equiv \delta(q^2)\Theta(-q_0)$ and we used

$$\int_{-\infty}^{\infty} dq_0 \delta(q^2)\Theta(-q_0) = \frac{1}{2|q|}, \quad (3.5)$$

where the Heaviside step function constrains the energy to be positive.

When one considers the angle between \vec{q} and \vec{p}_i going to zero, apparent collinear divergences appear in the integral eq. (3.4). This can be explicitly seen as $qp_i = |q|E_i(-1 + \cos\theta_{qi}) \sim |q|E_i\theta_{qi}^2$ as $\theta_{qi} \rightarrow 0$ while $\sin\theta_{qi}d\theta_{qi}$ inside the phase space integral d^3q goes like $\theta_{qi}d\theta_{qi}$, thus the θ_{qi} -integral becomes $d\ln\theta_{qi}$ which is divergent for $\theta_{qi} \rightarrow 0$ as well as for θ_{qj} . However, we would expect that gravity is not an interaction plagued by collinear divergences. Thus, in order to avoid these apparent divergences, ABV [41] introduced the following ij -sum zero-equivalent shift terms as follows

$$I_{ij}^\mu \rightarrow \tilde{I}_{ij}^\mu = \int \frac{d^4q}{(2\pi)^3} \delta_+(q^2) \frac{[(p_i p_j)q^\mu - (qp_j)p_i^\mu - (qp_i)p_j^\mu]}{(p_i q)(p_j q)}, \quad (3.6)$$

the shift terms in eq. (3.6) vanish after summing over i and j thanks to momentum and angular momentum conservations.

The four-vector integral eq. (3.6) can be rewritten in a Lorentz contravariant form as follows

$$K_{ij}^\mu(P, \Lambda) = \int \frac{d^4q}{(2\pi)^3} \frac{\delta_+(q^2)\delta\left(\frac{qP}{\Lambda^2} + 1\right)}{(qp_i)(qp_j)} [(p_i p_j)q - (qp_j)p_i - (qp_i)p_j]^\mu, \quad (3.7)$$

where P is an arbitrary four vector and Λ is a constant with the dimension of energy.

In a generic $n \rightarrow m$ process, P can be identified as the total momentum of n incoming (or m outgoing) particles and thus we can choose it as $P = (E_{\text{CM}}, 0)$. With this choice, $\Lambda^2 = \sqrt{s}\hbar\omega_0$ where $s = -P^2 = E_{\text{CM}}^2$ is the Mandelstam variable of the corresponding channel and ω_0 is the center-of-mass frequency at which we wish to compute the energy spectrum. Then the Dirac-Delta can be rewritten as

$$\delta\left(\frac{qP}{\Lambda^2} + 1\right) = \delta\left(1 - \frac{\omega}{\omega_0}\right).$$

The Lorentz-invariant (graviton number) spectrum $dB_1/d\omega_0$ is then given by

$$\frac{dB_1}{d\omega_0} = 8\pi G \sum_{i,j} \frac{K_{ij}^\mu(P, \omega_0)}{\omega_0} [p_i \vec{J}_j + p_j \overleftarrow{J}_i]_\mu. \quad (3.8)$$

With Lorentz covariance, K_{ij}^μ can be expanded as

$$K_{ij}^\mu(p_i p_j) = K_P(s)P^\mu + K_i(Pp_i)p_i^\mu + K_j(Pp_j)p_j^\mu, \quad (3.9)$$

since K_{ij}^μ is orthogonal to both p_i and p_j , K_{ij}^μ can be rewritten in the form

$$K_{ij}^\mu = K[(p_i p_j)P^\mu - (Pp_j)p_i^\mu - (Pp_i)p_j^\mu] \equiv K(p_i p_j)Q_{ij}^\mu, \quad (3.10)$$

where we have defined $K \equiv \frac{K_P}{p_i p_j}$ and a new vector

$$Q_{ij}^\mu \equiv P^\mu - \frac{Pp_j}{p_i p_j} p_i^\mu - \frac{Pp_i}{p_i p_j} p_j^\mu. \quad (3.11)$$

The main integrals involved in eq. (3.8) are

$$\int \frac{d^3q}{|q|} \delta(\omega - \omega_0) \frac{(p_i p_j)(qP)}{qp_i qp_j} = -\frac{(p_i p_j)\sqrt{s}}{E_i E_j} J = \frac{2\pi\sqrt{s}}{\beta_{ij}} \ln \frac{1 + \beta_{ij}}{1 - \beta_{ij}}, \quad (3.12)$$

and

$$\begin{aligned} \int \frac{d^3q}{|q|} \delta(\omega - \omega_0) \left(\frac{Pp_i}{qp_i} + \frac{Pp_j}{qp_j} \right) &= \sqrt{s} (L_i + L_j) \\ &= -2\pi\sqrt{s} \left(\frac{1}{|\vec{v}_i|} \ln \frac{1 - |\vec{v}_i|}{1 + |\vec{v}_i|} + \frac{1}{|\vec{v}_j|} \ln \frac{1 - |\vec{v}_j|}{1 + |\vec{v}_j|} \right). \end{aligned} \quad (3.13)$$

In the massless limit, the above expressions has the form

$$\int \frac{d^3q}{|q|} \delta(\omega - \omega_0) \frac{(p_i p_j)(qP)}{qp_i qp_j} = 4\pi\sqrt{s} \ln\left(-\frac{2p_i p_j}{m_i m_j}\right), \quad (3.14)$$

and

$$\int \frac{d^3q}{|q|} \delta(\omega - \omega_0) \left(\frac{Pp_i}{qp_i} + \frac{Pp_j}{qp_j} \right) = -4\pi\sqrt{s} \ln \frac{m_i m_j}{4E_i E_j}. \quad (3.15)$$

Contracting K_{ij}^μ with P^μ , we get from eq. (3.7):

$$\begin{aligned} K_{ij}P &= \int \frac{d^3q}{2|q|(2\pi)^3} \delta\left(1 - \frac{\omega}{\omega_0}\right) \left[\frac{(p_i p_j)(Pq)}{(qp_i)(qp_j)} - \left(\frac{Pp_i}{qp_i} + \frac{Pp_j}{qp_j} \right) \right] \\ &= -\frac{4\pi\omega_0\sqrt{s}}{2(2\pi)^3} \left[\ln\left(-\frac{2p_i p_j}{m_i m_j}\right) + \ln\left(\frac{m_i m_j}{4E_i E_j}\right) \right] = -\frac{\omega_0\sqrt{s}}{4\pi^2} \ln\left(-\frac{p_i p_j}{2E_i E_j}\right) \end{aligned} \quad (3.16)$$

and from eq. (3.10):

$$K_{ij}P = K[(p_i p_j)P^2 - 2(Pp_j)(Pp_i)]. \quad (3.17)$$

K is then determined by equating eq. (3.16) and eq. (3.17)

$$K(p_i p_j) = \frac{\omega_0\sqrt{s}}{4\pi^2 \tilde{s}_{ij}} \ln\left(-\frac{p_i p_j}{2E_i E_j}\right), \quad (3.18)$$

where we have introduced the quantity

$$\tilde{s}_{ij} = -Q_{ij}^2 = s + \frac{2(Pp_i)(Pp_j)}{p_i p_j}. \quad (3.19)$$

Substituting the above result into eq. (3.7) and renaming ω_0 as ω , we have a sub-leading factor as follows:

$$\begin{aligned} \frac{dB_1}{d\omega} &= -2\frac{G\sqrt{s}}{\pi} \sum_{i,j} \frac{1}{\tilde{s}_{ij}} \ln\left(-\frac{p_i p_j}{2E_i E_j}\right) Q_{ij}^\mu [p_i \vec{J}_j + p_j \overleftarrow{J}_i]_\mu \\ &= -2\frac{G\sqrt{s}}{\pi} \sum_{i,j} \frac{(p_i p_j)}{\tilde{s}_{ij}} \ln\left(-\frac{p_i p_j}{2E_i E_j}\right) Q_{ij}^\mu [\vec{\partial}_j + \overleftarrow{\partial}_i]_\mu. \end{aligned} \quad (3.20)$$

In [41], ABV showed that the sub-leading correction $dB_1/d\omega$ vanishes for a $2 \rightarrow 2$ process by direct computation in the Briet frame (see figure 1). We will show in the following sections that the zero result is also re-obtained in the massive case. In next section, we will derive a new B -factor for the massive case.

3.2 Massive case: a generalization of ABV result

In this section, we will generalize the above discussion to the massive case.

First, we notice that the polarization sum in the massive case has an additional term and it reads as

$$\begin{aligned} &\sum_{i,j} \frac{p_i^\mu p_i^\nu p_j^\rho (J_j q)^\sigma}{qp_i qp_j} \Pi_{\mu\nu, \rho\sigma} + (i \leftrightarrow j) \\ &= \sum_{i,j} \frac{(p_i p_j)(p_i J_j q) - \frac{1}{2} p_i^2 (p_j J_j q)}{qp_i qp_j} + (i \leftrightarrow j) \\ &= \sum_{i,j} \frac{p_i p_j}{qp_i qp_j} [p_i J_j q + p_j J_i q] - \frac{1}{2} \frac{1}{qp_i qp_j} [p_i^2 (p_j J_j q) + p_j^2 (p_i J_i q)]. \end{aligned} \quad (3.21)$$

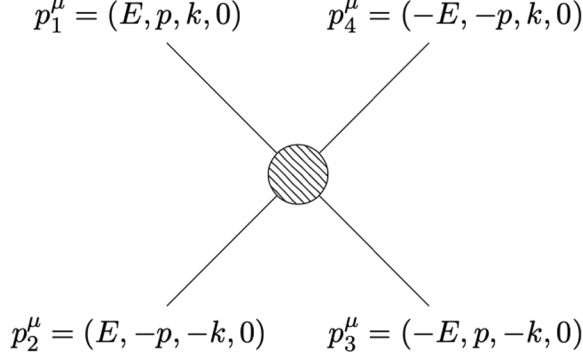


Figure 1. $2 \rightarrow 2$ scattering in Breit frame.

Considering the polarization sum in eq. (3.1) with eq. (3.21), the sub-leading B -factor becomes

$$B_1 = 8\pi G \int \frac{d^3 q}{2|q|(2\pi)^3} \sum_{i,j} \left\{ \frac{p_i p_j q^\mu}{q p_i q p_j} \left(p_i \vec{J}_j + p_j \overleftarrow{J}_i \right)_\mu - \frac{1}{2} \frac{q^\mu}{q p_i q p_j} \left[p_i^2 \left(p_j \vec{J}_j \right)_\mu + p_j^2 \left(p_i \overleftarrow{J}_i \right)_\mu \right] \right\}. \quad (3.22)$$

Then B_1 can be rewritten in terms of the integral eq. (3.4) as follows

$$B_1 = 8\pi G \sum_{i,j} I_{ij}^\mu \left\{ \left(p_i \vec{J}_j + p_j \overleftarrow{J}_i \right)_\mu - \frac{1}{2} \left[\frac{p_i^2}{p_i p_j} \left(p_j \vec{J}_j \right)_\mu + \frac{p_j^2}{p_i p_j} \left(p_i \overleftarrow{J}_i \right)_\mu \right] \right\}. \quad (3.23)$$

Following the same procedure in section 3.1, one gets the number density spectrum

$$\frac{dB_1}{d\omega_0} = -8\pi G \sum_{i,j} \frac{K_{ij}^\mu(P, \omega_0)}{\omega_0} \left[p_i \vec{J}_j - \frac{1}{2} \frac{p_i^2}{p_i p_j} \left(p_j \vec{J}_j \right)_\mu + (i \leftrightarrow j) \right] \quad (3.24)$$

where K_{ij}^μ is given by

$$K_{ij}^\mu(P, \omega_0) = \int \frac{d^4 q}{(2\pi)^3} \frac{\delta_+(q^2) \delta\left(1 - \frac{\omega}{\omega_0}\right)}{(q p_i)(q p_j)} \left[(p_i p_j) q - (q p_j) p_i - (q p_i) p_j \right]^\mu, \quad (3.25)$$

and it can be written in the form

$$K_{ij}^\mu = K \left[(p_i p_j) P^\mu - (P p_j) p_i^\mu - (P p_i) p_j^\mu \right] \equiv K (p_i p_j) Q_{ij}^\mu. \quad (3.26)$$

Contracting K_{ij}^μ with P_μ , with eq. (3.12) and eq. (3.14) and from eq. (3.25), one obtains

$$\begin{aligned} K_{ij} P &= \int \frac{d^3 q}{2|q|(2\pi)^3} \delta\left(1 - \frac{\omega}{\omega_0}\right) \left[\frac{(p_i p_j)(q P)}{q p_i q p_j} - \left(\frac{P p_i}{q p_i} + \frac{P p_j}{q p_j} \right) \right] \\ &= -\frac{\omega_0}{2(2\pi)^3} \left[\frac{2\pi\sqrt{s}}{\beta_{ij}} \ln \frac{1 + \beta_{ij}}{1 - \beta_{ij}} + 2\pi\sqrt{s} \left(\frac{1}{|\vec{v}_i|} \ln \frac{1 - |\vec{v}_i|}{1 + |\vec{v}_i|} + \frac{1}{|\vec{v}_j|} \ln \frac{1 - |\vec{v}_j|}{1 + |\vec{v}_j|} \right) \right], \end{aligned}$$

and from eq. (3.26)

$$K_{ij} P = K \left[(p_i p_j) P^2 - 2(P p_j)(P p_i) \right].$$

Thus K is determined by

$$K(p_i p_j) = \frac{\omega_0 \sqrt{s}}{8\pi^2 \tilde{s}_{ij}} \left[\frac{1}{\beta_{ij}} \ln \frac{1 + \beta_{ij}}{1 - \beta_{ij}} + \left(\frac{1}{|\vec{v}_i|} \ln \frac{1 - |\vec{v}_i|}{1 + |\vec{v}_i|} + \frac{1}{|\vec{v}_j|} \ln \frac{1 - |\vec{v}_j|}{1 + |\vec{v}_j|} \right) \right]. \quad (3.27)$$

Substituting the above K -factor into eq. (3.24) gives

$$\begin{aligned} \frac{dB_1}{d\omega} = & -\frac{G\sqrt{s}}{\pi} \sum_{i,j} \frac{1}{\tilde{s}_{ij}} \left[\frac{1}{\beta_{ij}} \ln \frac{1 + \beta_{ij}}{1 - \beta_{ij}} + \left(\frac{1}{|\vec{v}_i|} \ln \frac{1 - |\vec{v}_i|}{1 + |\vec{v}_i|} + \frac{1}{|\vec{v}_j|} \ln \frac{1 - |\vec{v}_j|}{1 + |\vec{v}_j|} \right) \right] \\ & \times Q_{ij}^\mu \left[p_i \vec{J}_j + p_j \overleftarrow{J}_i \right]_\mu. \end{aligned} \quad (3.28)$$

Considering that

$$Q_{ij}^\mu \left[p_i \vec{J}_j + p_j \overleftarrow{J}_i \right]_\mu = (p_i p_j) Q_{ij}^\mu \left(\overleftarrow{\partial}_i + \overrightarrow{\partial}_j \right)_\mu + \frac{P p_i}{p_i p_j} p_j^2 \left(\overrightarrow{\partial}_j p_i \right) + \frac{P p_j}{p_i p_j} p_i^2 \left(\overleftarrow{\partial}_i p_j \right),$$

we finally obtain the new general subleading B -factor for the massive case

$$\begin{aligned} \frac{dB_1}{d\omega} = & -\frac{G\sqrt{s}}{\pi} \sum_{i,j} \frac{1}{\tilde{s}_{ij}} \left[\frac{1}{\beta_{ij}} \ln \frac{1 + \beta_{ij}}{1 - \beta_{ij}} + \left(\frac{1}{|\vec{v}_i|} \ln \frac{1 - |\vec{v}_i|}{1 + |\vec{v}_i|} + \frac{1}{|\vec{v}_j|} \ln \frac{1 - |\vec{v}_j|}{1 + |\vec{v}_j|} \right) \right] \\ & \times \left[(p_i p_j) Q_{ij}^\mu \left(\overleftarrow{\partial}_i + \overrightarrow{\partial}_j \right)_\mu + \frac{P p_i}{p_i p_j} p_j^2 \left(\overrightarrow{\partial}_j p_i \right) + \frac{P p_j}{p_i p_j} p_i^2 \left(\overleftarrow{\partial}_i p_j \right) \right]. \end{aligned} \quad (3.29)$$

4 Test in two-body elastic scattering

In this section, we use the result (3.29) to study a two-body elastic collision. The kinematics for this process is similar to the one shown in figure 1 with the difference that here we consider all the ‘hard’ particles to be massive and with the same mass m . Thus the energy-momentum conservation condition becomes $E^2 = p^2 + k^2 + m^2$ and the Mandelstam variables satisfy the relation: $s + t + u = 4m^2$. The contribution from the (i, j) -pair takes the form

$$\begin{aligned} \frac{dB_1^{(i,j)}}{d\omega} = & \eta_i \eta_j \frac{G}{\pi} \frac{1 - \vec{v}_i \cdot \vec{v}_j}{1 + \vec{v}_i \cdot \vec{v}_j} \left[\frac{1}{\beta_{ij}} \ln \frac{1 + \beta_{ij}}{1 - \beta_{ij}} + \frac{2E}{\sqrt{p^2 + k^2}} \ln \frac{E - \sqrt{p^2 + k^2}}{E + \sqrt{p^2 + k^2}} \right] \\ & \times \left\{ E^2 [\delta_0^\mu (1 + \vec{v}_i \cdot \vec{v}_j) + \delta_r^\mu (\vec{v}_i + \vec{v}_j)^r] \left(\overleftarrow{\partial}_i + \overrightarrow{\partial}_j \right)_\mu \right. \\ & \left. - \frac{m^2}{1 - \vec{v}_i \cdot \vec{v}_j} \left[(\delta_0^\mu + \delta_r^\mu (\vec{v}_i)^r) \left(\overrightarrow{\partial}_j \right)_\mu + (\delta_0^\mu + \delta_r^\mu (\vec{v}_j)^r) \left(\overleftarrow{\partial}_i \right)_\mu \right] \right\}, \end{aligned} \quad (4.1)$$

where β_{ij} and $|\vec{v}_i|$ are given by

$$|\vec{v}_i| = \sqrt{\frac{p^2 + k^2}{E^2}} \quad \text{and} \quad \beta_{ij} \equiv \sqrt{1 - \frac{m^4}{(p_i \cdot p_j)^2}}.$$

First considering the case $i, j = 1, 2$ and $i, j = 3, 4$, a direct computation gives

$$\begin{aligned} & \frac{dB_1^{(1,2)}}{d\omega} + \frac{dB_1^{(2,1)}}{d\omega} \\ &= \frac{2G}{\pi} \left[\frac{E^2 + p^2 + k^2}{2E\sqrt{p^2 + k^2}} \ln \frac{E^2 + p^2 + k^2 + 2E\sqrt{p^2 + k^2}}{E^2 + p^2 + k^2 - 2E\sqrt{p^2 + k^2}} + \frac{2E}{\sqrt{p^2 + k^2}} \ln \frac{E - \sqrt{p^2 + k^2}}{E + \sqrt{p^2 + k^2}} \right] \\ & \times \left\{ (p^2 + k^2) \overleftrightarrow{\partial}_E + E \left(p \overleftrightarrow{\partial}_p + k \overleftrightarrow{\partial}_k \right) \right\}, \end{aligned}$$

and

$$\begin{aligned} & \frac{dB_1^{(3,4)}}{d\omega} + \frac{dB_1^{(4,3)}}{d\omega} \\ &= -\frac{2G}{\pi} \left[\frac{E^2 + p^2 + k^2}{2E\sqrt{p^2 + k^2}} \ln \frac{E^2 + p^2 + k^2 + 2E\sqrt{p^2 + k^2}}{E^2 + p^2 + k^2 - 2E\sqrt{p^2 + k^2}} + \frac{2E}{\sqrt{p^2 + k^2}} \ln \frac{E - \sqrt{p^2 + k^2}}{E + \sqrt{p^2 + k^2}} \right] \\ & \times \left\{ (p^2 + k^2) \overleftrightarrow{\partial}_E + E \left(p \overleftrightarrow{\partial}_p + k \overleftrightarrow{\partial}_k \right) \right\}. \end{aligned}$$

Thus, we obtain

$$\frac{dB_1^{(1,2)}}{d\omega} + \frac{dB_1^{(2,1)}}{d\omega} + \frac{dB_1^{(3,4)}}{d\omega} + \frac{dB_1^{(4,3)}}{d\omega} = 0 \quad (4.2)$$

which means that the contributions from (1,2) and (3,4) cancel with each other.

For $i, j = 1, 3$, after a long but straightforward computation, one finds

$$\begin{aligned} \frac{dB_1^{(1,3)}}{d\omega} &= -\frac{G}{\pi} \frac{E^2 + p^2 - k^2}{E^2 - p^2 + k^2} \left[\frac{E^2 + p^2 - k^2}{2p\sqrt{E^2 - k^2}} \ln \frac{E^2 + p^2 - k^2 + 2p\sqrt{E^2 - k^2}}{E^2 + p^2 - k^2 - 2p\sqrt{E^2 - k^2}} \right. \\ & \left. + \frac{2E}{\sqrt{p^2 + k^2}} \ln \frac{E - \sqrt{p^2 + k^2}}{E + \sqrt{p^2 + k^2}} \right] \left\{ E^2 \left[\frac{E^2 - p^2 + k^2}{E^2} \left(\overleftrightarrow{\partial}_E - \overleftarrow{\partial}_E \right) + \frac{2k}{E} \left(\overleftarrow{\partial}_k - \overrightarrow{\partial}_k \right) \right] \right. \\ & \left. - \frac{E^2 m^2}{E^2 + p^2 - k^2} \left[\left(-\overrightarrow{\partial}_E + \overleftarrow{\partial}_E \right) + \frac{p}{E} \left(\overrightarrow{\partial}_p - \overleftarrow{\partial}_p \right) + \frac{k}{E} \left(-\overrightarrow{\partial}_k + \overleftarrow{\partial}_k \right) \right] \right\} \\ &= -\frac{dB_1^{(3,1)}}{d\omega}. \end{aligned}$$

Once again, the total contribution from (1,3)-pair vanishes. We have also checked that the same happens for the other pairs: (1,4), (2,3), (2,4), proving that $dB_1/d\omega = 0$ is also valid in the case of massive two-body collisions with all particles having the same mass.

5 Gravitational elastic scattering

In this section, we apply the new sub-leading differential B -factor obtained in previous sections to the case of a gravitational elastic scattering.

Indeed, from a classical prospective this can correspond to the case of a test light body elastically deflected but not captured by a Black Hole of mass M .

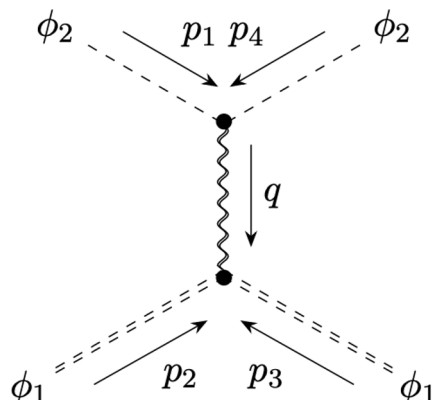


Figure 2. Feynman's diagram of gravitational elastic scattering $\phi_1(p_1)\phi_2(p_2) \rightarrow \phi_1(p_3)\phi_2(p_4)$, where $\phi_{1,2}$ are two scalar fields (dashed lines), the curly line represent the propagator of the graviton field and $p_{1,2,3,4}$ are the four-momenta of in and out states respectively.

For the case in which the test body has a mass $m \ll M$, the differential B -factor has an asymptotic expression in the limit of $m \rightarrow 0$ (see appendix A for detail derivation) as follows

$$\frac{dB_1}{d\omega} \sim \frac{GM}{\pi} \left(6 + \frac{2}{3} \frac{P^2}{M^2} \right) \left(\vec{p}_f \overleftrightarrow{\partial}_{\vec{p}_f} - \vec{p}_i \overleftrightarrow{\partial}_{\vec{p}_i} \right), \quad (5.1)$$

where $P = |\vec{p}|$, M is the mass of the black hole, \vec{p}_i, \vec{p}_f are initial and final momenta in the CM frame. Then the soft sub-leading radiation in ZFL can be obtained applying the differential B-factor in eq. (5.1) on the amplitude of gravitational $2 \rightarrow 2$ elastic scattering

$$\mathcal{M}^* \frac{dB_1}{d\omega} \mathcal{M}, \quad (5.2)$$

where the partial derivatives with left (right) arrow act on \mathcal{M}^* (\mathcal{M}).

From field theory prospective, let us start from the Einstein-Hilbert action coupled to two scalar fields ϕ_1 and ϕ_2

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G} R - \frac{1}{2} \partial^\mu \phi_1 \partial_\mu \phi_1 - \frac{1}{2} M^2 \phi_1^2 - \frac{1}{2} \partial^\nu \phi_2 \partial_\nu \phi_2 - \frac{1}{2} m^2 \phi_2^2 \right]. \quad (5.3)$$

Indeed, for the moment, as an illustrative example, we consider spin-less field gravitational scatterings (see figure 2). The amplitude for this classical tree-level diagram is [67]

$$\mathcal{M} = \frac{16\pi G}{q^2} \left(M^2 m^2 - 2(p_1 \cdot p_2)^2 - (p_1 \cdot p_2) q^2 \right), \quad (5.4)$$

where p_1, p_2 (p_3, p_4) are incoming (outgoing) momenta, $p_1^2 = p_4^2 = -m^2$, $p_2^2 = p_3^2 = -M^2$ and the momentum transfer is $q = p_1 + p_4 = -(p_2 + p_3)$. With the kinematics discussed in appendix A, the amplitude in the center of mass frame becomes

$$\begin{aligned} \mathcal{M} &= \frac{8\pi G}{(P^2 + kk' - pp')} \left[M^2 m^2 - 2P \left(M + P + \frac{P^2}{2M} \right) \left(P \left(M + \frac{P^2}{2M} \right) - kk' + pp' \right) \right] \\ &\sim \frac{8\pi G}{(P^2 + kk' - pp')} \left[M^2 m^2 - 2P(M + P)(MP - kk' + pp') \right]. \end{aligned} \quad (5.5)$$

The derivatives of this amplitude give

$$\begin{aligned}
 \vec{\partial}_p \mathcal{M} &= \frac{8\pi G}{(P^2 + kk' - pp')^2} p' \left(M^2 m^2 - 2P^2 (M + P)^2 \right) = \mathcal{M}^* \overleftarrow{\partial}_p, \\
 \vec{\partial}_k \mathcal{M} &= \frac{8\pi G}{(P^2 + kk' - pp')^2} k' \left(2P^2 (M + P)^2 - M^2 m^2 \right) = \mathcal{M}^* \overleftarrow{\partial}_k, \\
 \vec{\partial}_{p'} \mathcal{M} &= \frac{8\pi G}{(P^2 + kk' - pp')^2} p \left(M^2 m^2 - 2P^2 (M + P)^2 \right) = \mathcal{M}^* \overleftarrow{\partial}_{p'}, \\
 \vec{\partial}_{k'} \mathcal{M} &= \frac{8\pi G}{(P^2 + kk' - pp')^2} k \left(2P^2 (M + P)^2 - M^2 m^2 \right) = \mathcal{M}^* \overleftarrow{\partial}_{k'}.
 \end{aligned} \tag{5.6}$$

Then it is easy to check that

$$\begin{aligned}
 p \left[(\mathcal{M}^* \overleftarrow{\partial}_p) \mathcal{M} + \mathcal{M}^* (\vec{\partial}_p \mathcal{M}) \right] - p' \left[(\mathcal{M}^* \overleftarrow{\partial}_{p'}) \mathcal{M} + \mathcal{M}^* (\vec{\partial}_{p'} \mathcal{M}) \right] &= 0, \\
 k \left[(\mathcal{M}^* \overleftarrow{\partial}_k) \mathcal{M} + \mathcal{M}^* (\vec{\partial}_k \mathcal{M}) \right] - k' \left[(\mathcal{M}^* \overleftarrow{\partial}_{k'}) \mathcal{M} + \mathcal{M}^* (\vec{\partial}_{k'} \mathcal{M}) \right] &= 0,
 \end{aligned}$$

which implies

$$\mathcal{M} \left(p' \overleftarrow{\partial}_{p'} + k' \overleftarrow{\partial}_{k'} - p \overleftarrow{\partial}_p - k \overleftarrow{\partial}_k \right) \mathcal{M}^* = 0. \tag{5.7}$$

Thus once again, we get a vanishing result

$$\mathcal{M}^* \frac{dB_1}{d\omega} \mathcal{M} = 0, \tag{5.8}$$

even if the differential B -factor is non-zero. Furthermore, in appendix B, we find that eq. (5.8) is still re-obtained in the case of massive m test particle.

6 Inelastic scattering

Here, we will show that $dB_1/d\omega$ gives a finite non-zero result in the $2 \rightarrow 2$ inelastic scattering case. Indeed, in the case of test particle deflected by a BH, it is reasonable to consider the elastic scattering regime only for small deflection angles. In case of harder deflections we would expect that inelastic channels would be opened.

Following similar discussion as appendix A, but this time take

$$p^2 + k^2 = P^2 \neq P'^2 = p'^2 + k'^2 \tag{6.1}$$

with eq. (3.29), we can get the contributions from different particle pairs. As shown in appendix C, however, the result is a long expression.

Nevertheless, in the non-relativistic regime, performing a Taylor series expansion up to the second order of $O(P/M)$ and $O(P'/M')$, the six contributions simplify to

$$\begin{aligned}
 \frac{dB_1^{(1,2)}}{d\omega} + \frac{dB_1^{(2,1)}}{d\omega} &\sim \frac{GM}{\pi} \left(-2 + \frac{1}{3} \frac{P^2}{M^2} \right) \left(P \overleftarrow{\partial}_{P+p} + p \overleftarrow{\partial}_p + k \overleftarrow{\partial}_k \right), \\
 \frac{dB_1^{(3,4)}}{d\omega} + \frac{dB_1^{(4,3)}}{d\omega} &\sim \frac{GM'}{\pi} \left(2 - \frac{1}{3} \frac{P'^2}{M'^2} \right) \left(P' \overleftarrow{\partial}_{P'+p'} + p' \overleftarrow{\partial}_{p'} + k' \overleftarrow{\partial}_{k'} \right),
 \end{aligned}$$

$$\begin{aligned}
 \frac{dB_1^{(2,4)}}{d\omega} + \frac{dB_1^{(4,2)}}{d\omega} &\sim \frac{GM'}{\pi} \left(-2 + \frac{1}{3} \frac{P'^2}{M'^2} \right) \left(P \overleftrightarrow{\partial}_P + p \overleftrightarrow{\partial}_p + k \overleftrightarrow{\partial}_k \right), \\
 \frac{dB_1^{(1,3)}}{d\omega} + \frac{dB_1^{(3,1)}}{d\omega} &\sim \frac{GM}{\pi} \left(-2 + \frac{1}{3} \frac{P^2}{M^2} \right) \left(-P' \overleftrightarrow{\partial}_{P'} - p' \overleftrightarrow{\partial}_{p'} - k' \overleftrightarrow{\partial}_{k'} \right), \\
 \frac{dB_1^{(1,4)}}{d\omega} + \frac{dB_1^{(4,1)}}{d\omega} &\sim \frac{G}{\pi} \left[-2 - \frac{2}{3} \left(\frac{P^2}{M^2} + \frac{P'^2}{M'^2} \right) \right] \left(pM' \overleftrightarrow{\partial}_p + kM' \overleftrightarrow{\partial}_k - p'M \overleftrightarrow{\partial}_{p'} - k'M \overleftrightarrow{\partial}_{k'} \right), \\
 \frac{dB_1^{(2,3)}}{d\omega} + \frac{dB_1^{(3,2)}}{d\omega} &= \frac{2G}{\pi} \frac{PP' - (pp' - kk')}{PP' + (pp' - kk')} \ln \left[\frac{PP' - (pp' - kk')}{2PP'} \right] \left\{ [PP' + (pp' - kk')] \right. \\
 &\quad \left. \times \left(\overleftrightarrow{\partial}_P - \overleftrightarrow{\partial}_{P'} \right) + (pP' + p'P) \left(\overleftrightarrow{\partial}_p - \overleftrightarrow{\partial}_{p'} \right) + (kP' - k'P) \left(\overleftrightarrow{\partial}_k + \overleftrightarrow{\partial}_{k'} \right) \right\}.
 \end{aligned}$$

Summing the single contributions, we find that the total sub-leading B -factor for the inelastic scattering is dominated by the following leading term:

$$\begin{aligned}
 \frac{dB_1}{d\omega} &= \left[\frac{GM}{\pi} \left(2 - \frac{1}{3} \frac{P^2}{M^2} \right) + \frac{GM'}{\pi} \left(2 - \frac{1}{3} \frac{P'^2}{M'^2} \right) \right] \left(P' \overleftrightarrow{\partial}_{P'} - P \overleftrightarrow{\partial}_P + p' \overleftrightarrow{\partial}_{p'} \right. \\
 &\quad \left. - p \overleftrightarrow{\partial}_p + k' \overleftrightarrow{\partial}_{k'} - k \overleftrightarrow{\partial}_k \right) + \frac{G}{\pi} \left[-2 - \frac{2}{3} \left(\frac{P^2}{M^2} + \frac{P'^2}{M'^2} \right) \right] \\
 &\quad \times \left[M' \left(p \overleftrightarrow{\partial}_p + k \overleftrightarrow{\partial}_k \right) - M \left(p' \overleftrightarrow{\partial}_{p'} - k' \overleftrightarrow{\partial}_{k'} \right) \right].
 \end{aligned} \tag{6.2}$$

7 Conclusions and remarks

In this paper, we computed the sub-leading order expansion of the gravitational radiation energy spectrum $dE^{GW}/d\omega$ in the ZFL for a generic multi-body or multi-particle collision. Our result can be applied on both massless and massive fundamental/composite particle scatterings with every spin as well as to bodies such as Black Holes.

As an application, we considered the case of sub-leading soft emission from a gravitational elastic scattering. From the point of view of BH physics, it can be the case of a test body deflected, but not captured, by a BH.

Surprisingly, we obtain that all sub-leading terms of the energy radiation spectrum, emitted from an elastic collision, exactly cancel each others, in both massless and massive particle collisions. On the other hand, for inelastic $2 \rightarrow 2$ scatterings, we obtain a general and complicated non-zero analytic result which may be applied in the case of GW soft emission from BH gravitational inelastic scatterings. It may certainly be an attractive possibility to relate our result on searches of BMS gravitational memory effects from GW physics. In this sense, our results may be applied in BH mergings seen as a gravitational BH capture inelastic scattering process. On the other hand, violations of universality of our results from loop radiative corrections would be expected from previous analysis as mentioned in the introduction above.

Another intriguing possibility is to consider *footprints* of string theory effects on leading and sub-leading gravitational radiation from α' corrections and Regge poles that may survive with a polynomial decay in radius, dubbed string memory effect [35].

Finally, the very next step for ZFL program would be to compute sub-sub-leading order on the $dE^{GW}/d\omega$ expansion in ω . In ref. [41] the case of massless particle high energy collisions was considered — having in mind an application to transplanckian scattering regimes. However a more general result is missing yet and it is beyond the purpose of this paper.

A Gravitational elastic scattering I

In this appendix, we give a detail computation of the differential B -factor for the gravitational elastic scattering process with the test particle being massless as discussed in section 4. In the center of mass frame, the kinematical variables of this process are

$$\begin{aligned}
 p_1 &= \left(M + \frac{P^2}{2M}, p, k, 0 \right), \\
 p_2 &= (P, -p, -k, 0), \\
 p_3 &= (-P, p', -k', 0), \\
 p_4 &= \left(-M - \frac{P^2}{2M}, -p', k', 0 \right),
 \end{aligned} \tag{A.1}$$

where p, p', k, k' satisfy the relation

$$p^2 + k^2 = P^2 = p'^2 + k'^2.$$

Some properties relating eq. (3.29) are the following

$$\begin{aligned}
 P = p_1 + p_2 &= \left(M + P + \frac{P^2}{2M}, \vec{0} \right), \\
 s = -(p_1 + p_2)^2 &= \left(M + P + \frac{P^2}{2M} \right)^2, \\
 \tilde{s}_{ij} &= - \left(M + P + \frac{P^2}{2M} \right)^2 \frac{1 + \vec{v}_i \cdot \vec{v}_j}{1 - \vec{v}_i \cdot \vec{v}_j}, \\
 Q_{ij}^\mu &= - \left(M + P + \frac{P^2}{2M} \right) \left(\frac{1 + \vec{v}_i \cdot \vec{v}_j}{1 - \vec{v}_i \cdot \vec{v}_j}, \frac{\vec{v}_i + \vec{v}_j}{1 - \vec{v}_i \cdot \vec{v}_j} \right),
 \end{aligned}$$

and the velocities are given by

$$\begin{aligned}
 \vec{v}_1 &= \frac{1}{M}(p, k, 0), & \vec{v}_2 &= \frac{1}{P}(-p, -k, 0), & |\vec{v}_1| &= \frac{P}{M}, & |\vec{v}_2| &= \frac{P}{\sqrt{m^2 + P^2}} \rightarrow 1, \\
 \vec{v}_3 &= \frac{1}{P}(-p', k', 0), & \vec{v}_4 &= \frac{1}{M}(p', -k', 0), & |\vec{v}_3| &= \frac{P}{\sqrt{m^2 + P^2}} \rightarrow 1, & |\vec{v}_4| &= \frac{P}{M}.
 \end{aligned}$$

Their are six (i, j) -pairs we need to take into consideration, namely: $(1, 2)$, $(1, 3)$, $(1, 4)$, $(2, 3)$, $(2, 4)$, $(3, 4)$. After some tedious but straightforward computations, we obtain the contributions from $(1, 2) + (3, 4)$:

$$\begin{aligned} & \frac{dB_1^{(1,2)}}{d\omega} + \frac{dB_1^{(2,1)}}{d\omega} + \frac{dB_1^{(3,4)}}{d\omega} + \frac{dB_1^{(4,3)}}{d\omega} \\ &= \frac{G}{\pi} \left[\ln \left(\frac{M+P+\frac{P^2}{2M}}{M} \right)^2 + \frac{M}{P} \ln \frac{M-P}{M+P} \right] \frac{M+P}{M-P} \left(P-M-\frac{P^2}{2M} \right) \\ & \quad \times \left(p' \overleftrightarrow{\partial}_{p'} + k' \overleftrightarrow{\partial}_{k'} - p \overleftrightarrow{\partial}_p - k \overleftrightarrow{\partial}_k \right), \end{aligned}$$

and from $(1, 3) + (2, 4)$:

$$\begin{aligned} & \frac{dB_1^{(1,3)}}{d\omega} + \frac{dB_1^{(3,1)}}{d\omega} + \frac{dB_1^{(2,4)}}{d\omega} + \frac{dB_1^{(4,2)}}{d\omega} \\ &= \frac{G}{\pi} \frac{PM + (pp' - kk')}{PM - (pp' - kk')} \left[\ln \left(\frac{P \left(M + \frac{P^2}{2M} \right) + pp' - kk'}{MP} \right)^2 + \frac{M}{P} \ln \frac{M-P}{M+P} \right] \\ & \quad \times \left\{ \left(P \left(M + \frac{P^2}{2M} \right) + (pp' - kk') \right) \left[\frac{(p-p')(P+M)}{PM + (pp' - kk')} \left(\overleftrightarrow{\partial}_p + \overleftrightarrow{\partial}_{p'} \right) \right. \right. \\ & \quad \left. \left. + \frac{(k+k')(P+M)}{PM + (pp' - kk')} \left(\overleftrightarrow{\partial}_k - \overleftrightarrow{\partial}_{k'} \right) \right] + \frac{P \left[P^2 - \left(M + \frac{P^2}{2M} \right)^2 \right]}{P \left(M + \frac{P^2}{2M} \right) + pp' - kk'} \right. \\ & \quad \left. \times \left(p \overleftrightarrow{\partial}_{p'} - k \overleftrightarrow{\partial}_{k'} - p' \overleftrightarrow{\partial}_p + k' \overleftrightarrow{\partial}_k \right) \right\}, \end{aligned}$$

from $(1, 4)$:

$$\begin{aligned} & \frac{dB_1^{(1,4)}}{d\omega} + \frac{dB_1^{(4,1)}}{d\omega} \\ &= \frac{G}{\pi} \left[\frac{M}{\sqrt{2} \sqrt{P^2 - (pp' - kk')}} \ln \frac{1 + \sqrt{2} \sqrt{\frac{P^2 - (pp' - kk')}{M}}}{1 - \sqrt{2} \sqrt{\frac{P^2 - (pp' - kk')}{M}}} + 2 \frac{M}{P} \ln \frac{M-P}{M+P} \right] M \\ & \quad \times \left(p \overleftrightarrow{\partial}_p - p' \overleftrightarrow{\partial}_{p'} + k \overleftrightarrow{\partial}_k - k' \overleftrightarrow{\partial}_{k'} \right), \end{aligned}$$

and finally from $(2, 3)$:

$$\begin{aligned} & \frac{dB_1^{(2,3)}}{d\omega} + \frac{dB_1^{(3,2)}}{d\omega} \\ &= \frac{G}{\pi} \frac{P^2 - (pp' - kk')}{P^2 + (pp' - kk')} \ln \left[\frac{P^2 - (pp' - kk')}{2P^2} \right]^2 \left[P(p+p') \left(\overleftrightarrow{\partial}_p - \overleftrightarrow{\partial}_{p'} \right) - P(k'-k) \left(\overleftrightarrow{\partial}_k + \overleftrightarrow{\partial}_{k'} \right) \right]. \end{aligned}$$

To get a compact and meaningful differential B -factor, we do the Taylor series expansion up to the second order at P/M . Then the above results are simplified to

$$\frac{dB_1^{(1,2)}}{d\omega} + \frac{dB_1^{(2,1)}}{d\omega} + \frac{dB_1^{(3,4)}}{d\omega} + \frac{dB_1^{(4,3)}}{d\omega} \sim \frac{GM}{\pi} \left(2 - \frac{1}{3} \frac{P^2}{M^2} \right) \left(p' \overleftrightarrow{\partial}_{p'} + k' \overleftrightarrow{\partial}_{k'} - p \overleftrightarrow{\partial}_p - k \overleftrightarrow{\partial}_k \right),$$

$$\frac{dB_1^{(1,3)}}{d\omega} + \frac{dB_1^{(3,1)}}{d\omega} + \frac{dB_1^{(2,4)}}{d\omega} + \frac{dB_1^{(4,2)}}{d\omega} \sim \frac{GM}{\pi} \left(-2 + \frac{1}{3} \frac{P^2}{M^2} \right) \left(p \overleftrightarrow{\partial}_p - p' \overleftrightarrow{\partial}_{p'} + k \overleftrightarrow{\partial}_k - k' \overleftrightarrow{\partial}_{k'} \right),$$

and

$$\begin{aligned} \frac{dB_1^{(1,4)}}{d\omega} + \frac{dB_1^{(4,1)}}{d\omega} &= \frac{2G}{\pi} \left[1 + \frac{M}{P} \ln \frac{1 - \frac{P}{M}}{1 + \frac{P}{M}} \right] M \left(p \overleftrightarrow{\partial}_p - p' \overleftrightarrow{\partial}_{p'} + k \overleftrightarrow{\partial}_k - k' \overleftrightarrow{\partial}_{k'} \right) \\ &\sim \frac{2G}{\pi} \left[-1 - \frac{2}{3} \frac{P^2}{M^2} \right] M \left(p \overleftrightarrow{\partial}_p - p' \overleftrightarrow{\partial}_{p'} + k \overleftrightarrow{\partial}_k - k' \overleftrightarrow{\partial}_{k'} \right). \end{aligned}$$

Summing up the above results, we find that the total differential B -factor: $dB_1/d\omega = \sum_{i,j}^{i \neq j} dB_1^{(i,j)}/d\omega$ has a leading expression as

$$\begin{aligned} \frac{dB_1}{d\omega} &\sim \frac{GM}{\pi} \left(6 + \frac{2}{3} \frac{P^2}{M^2} \right) \left(p' \overleftrightarrow{\partial}_{p'} + k' \overleftrightarrow{\partial}_{k'} - p \overleftrightarrow{\partial}_p - k \overleftrightarrow{\partial}_k \right) \\ &= \frac{GM}{\pi} \left(6 + \frac{2}{3} \frac{P^2}{M^2} \right) \left(\vec{p}_f \overleftrightarrow{\partial}_{\vec{p}_f} - \vec{p}_i \overleftrightarrow{\partial}_{\vec{p}_i} \right), \end{aligned}$$

where \vec{p}_i (\vec{p}_f) are momenta of initial (final) states and $\overleftrightarrow{\partial} \equiv \overleftarrow{\partial} + \overrightarrow{\partial}$.

B Gravitational elastic scattering II

Our goal in this appendix is to generalize the discussion in appendix A to the case in which the test particle has mass m . The kinematics is similar to the previous case and reads

$$\begin{aligned} p_1 &= \left(M + \frac{P^2}{2M}, p, k, 0 \right), \\ p_2 &= \left(\sqrt{P^2 + m^2}, -p, -k, 0 \right), \\ p_3 &= \left(-\sqrt{P^2 + m^2}, p', -k', 0 \right), \\ p_4 &= \left(-M - \frac{P^2}{2M}, -p', k', 0 \right), \end{aligned} \tag{B.1}$$

with $p^2 + k^2 = P^2 = p'^2 + k'^2$.

Again, let us consider the contributions from six pairs separately: first for $(1, 2) + (3, 4)$, considering that $\tilde{s}_{12} = \tilde{s}_{34}$ and $p_1 \cdot p_2 = p_3 \cdot p_4$, we obtain

$$\begin{aligned} \frac{dB_1^{(1,2)}}{d\omega} + \frac{dB_1^{(2,1)}}{d\omega} &= -\frac{G\sqrt{s}}{\pi} \frac{1}{\tilde{s}_{12}} \left[\frac{1}{\beta_{12}} \ln \frac{1 + \beta_{12}}{1 - \beta_{12}} + \left(\frac{1}{|\vec{v}_1|} \ln \frac{1 - |\vec{v}_1|}{1 + |\vec{v}_1|} + \frac{1}{|\vec{v}_2|} \ln \frac{1 - |\vec{v}_2|}{1 + |\vec{v}_2|} \right) \right] \\ &\times \left[(p_1 p_2) Q_{12}^\mu \left(\overleftrightarrow{\partial}_1 + \overleftrightarrow{\partial}_2 \right)_\mu + \frac{P p_1}{p_1 p_2} p_2^2 \overleftrightarrow{\partial}_2 p_1 + \frac{P p_2}{p_1 p_2} p_1^2 \overleftrightarrow{\partial}_1 p_2 \right], \end{aligned}$$

and

$$\begin{aligned} \frac{dB_1^{(3,4)}}{d\omega} + \frac{dB_1^{(4,3)}}{d\omega} &= -\frac{G\sqrt{s}}{\pi} \frac{1}{\tilde{s}_{34}} \left[\frac{1}{\beta_{34}} \ln \frac{1+\beta_{34}}{1-\beta_{34}} + \left(\frac{1}{|\vec{v}_3|} \ln \frac{1-|\vec{v}_3|}{1+|\vec{v}_3|} + \frac{1}{|\vec{v}_4|} \ln \frac{1-|\vec{v}_4|}{1+|\vec{v}_4|} \right) \right] \\ &\times \left[(p_3 p_4) Q_{34}^\mu \left(\overleftrightarrow{\partial}_3 + \overleftrightarrow{\partial}_4 \right)_\mu + \frac{P p_3}{p_3 p_4} p_4^2 \overleftrightarrow{\partial}_{4p_3} + \frac{P p_4}{p_3 p_4} p_3^2 \overleftrightarrow{\partial}_{3p_4} \right], \end{aligned}$$

where the Q^μ vectors are given by

$$\begin{aligned} Q_{12}^\mu &= -\frac{M + \frac{P^2}{2M} + \sqrt{P^2 + m^2}}{M\sqrt{P^2 + m^2} + P^2} \left(M\sqrt{P^2 + m^2} - P^2, (\sqrt{P^2 + m^2} - M)(p, k, 0) \right), \\ Q_{34}^\mu &= -\frac{M + \frac{P^2}{2M} + \sqrt{P^2 + m^2}}{M\sqrt{P^2 + m^2} + P^2} \left(M\sqrt{P^2 + m^2} - P^2, (\sqrt{P^2 + m^2} - M)(p', -k', 0) \right). \end{aligned}$$

Thus we have

$$\begin{aligned} &\frac{dB_1^{(1,2)}}{d\omega} + \frac{dB_1^{(2,1)}}{d\omega} + \frac{dB_1^{(3,4)}}{d\omega} + \frac{dB_1^{(4,3)}}{d\omega} \\ &= -\frac{G\sqrt{s}}{\pi} \frac{1}{\tilde{s}_{12}} \left[\frac{1}{\beta_{12}} \ln \frac{1+\beta_{12}}{1-\beta_{12}} + \left(\frac{1}{|\vec{v}_1|} \ln \frac{1-|\vec{v}_1|}{1+|\vec{v}_1|} + \frac{1}{|\vec{v}_2|} \ln \frac{1-|\vec{v}_2|}{1+|\vec{v}_2|} \right) \right] \\ &\times \left\{ \frac{P p_1}{p_1 p_2} m^2 \left(p \overleftrightarrow{\partial}_p + k \overleftrightarrow{\partial}_k - p' \overleftrightarrow{\partial}_{p'} - k' \overleftrightarrow{\partial}_{k'} \right) + \frac{P p_2}{p_1 p_2} \left[P^2 - \left(M + \frac{P^2}{2M} \right)^2 \right] \right. \\ &\left. \times \left(p' \overleftrightarrow{\partial}_{p'} + k' \overleftrightarrow{\partial}_{k'} - p \overleftrightarrow{\partial}_p - k \overleftrightarrow{\partial}_k \right) \right\}. \end{aligned}$$

The combination: $p' \overleftrightarrow{\partial}_{p'} + k' \overleftrightarrow{\partial}_{k'} - p \overleftrightarrow{\partial}_p - k \overleftrightarrow{\partial}_k$ factorized implies that $\mathcal{M}^*[(1,2)+(3,4)]\mathcal{M} = 0$ (see eq. (5.7)).

Then for (1,3) + (2,4), similarly, considering that $\tilde{s}_{13} = \tilde{s}_{24}, p_1 \cdot p_3 = p_2 \cdot p_4$, we obtain

$$\begin{aligned} &\frac{dB_1^{(1,3)}}{d\omega} + \frac{dB_1^{(3,1)}}{d\omega} + \frac{dB_1^{(2,4)}}{d\omega} + \frac{dB_1^{(4,2)}}{d\omega} \\ &= \frac{G\sqrt{s}}{\pi} \frac{1}{\tilde{s}_{13}} \left[\frac{1}{\beta_{13}} \ln \frac{1+\beta_{13}}{1-\beta_{13}} + \left(\frac{1}{|\vec{v}_1|} \ln \frac{1-|\vec{v}_1|}{1+|\vec{v}_1|} + \frac{1}{|\vec{v}_3|} \ln \frac{1-|\vec{v}_3|}{1+|\vec{v}_3|} \right) \right] \\ &\times \left\{ -(p_1 p_3) \frac{\left(M + \frac{P^2}{2M} + \sqrt{P^2 + m^2} \right) \left(\sqrt{P^2 + m^2} + M \right)}{M\sqrt{P^2 + m^2} - (kk' - pp')} \left(p \overleftrightarrow{\partial}_p - p' \overleftrightarrow{\partial}_{p'} + k \overleftrightarrow{\partial}_k - k' \overleftrightarrow{\partial}_{k'} \right. \right. \\ &\left. \left. - p' \overleftrightarrow{\partial}_p + p \overleftrightarrow{\partial}_{p'} + k' \overleftrightarrow{\partial}_k - k \overleftrightarrow{\partial}_{k'} \right) + \frac{P p_1}{p_1 p_3} m^2 \left[p' \overleftrightarrow{\partial}_p - p \overleftrightarrow{\partial}_{p'} + k \overleftrightarrow{\partial}_{k'} - k' \overleftrightarrow{\partial}_k \right] \right. \\ &\left. + \frac{P p_2}{p_1 p_3} \left[P^2 - \left(M + \frac{P^2}{2M} \right)^2 \right] \left(p \overleftrightarrow{\partial}_{p'} - p' \overleftrightarrow{\partial}_p + k' \overleftrightarrow{\partial}_k - k \overleftrightarrow{\partial}_{k'} \right) \right\}. \end{aligned}$$

Let us now note that all the three terms in the above expression contain the combination $p \overleftrightarrow{\partial}_p - p' \overleftrightarrow{\partial}_{p'} + k' \overleftrightarrow{\partial}_k - k \overleftrightarrow{\partial}_{k'}$, which guarantees the cancellation after the application of

the expression on the amplitudes

$$\begin{aligned} & \mathcal{M}^* \left[-p' \overleftrightarrow{\partial}_p + p \overleftrightarrow{\partial}_{p'} + k' \overleftrightarrow{\partial}_k - k \overleftrightarrow{\partial}_{k'} \right] \mathcal{M} \\ &= \frac{16\pi G (2P^2(M+P)^2 - M^2 m^2)}{(P^2 + kk' - pp')^2} (p'^2 - p^2 + k'^2 - k^2) \mathcal{M} = 0, \end{aligned}$$

thus $\mathcal{M}^* [(1, 3) + (2, 4)] \mathcal{M} = 0$.

Similarly, the final expression for the contribution from (2, 3)-pair is

$$\begin{aligned} & \frac{dB_1^{(2,3)}}{d\omega} + \frac{dB_1^{(3,2)}}{d\omega} \\ &= \frac{G\sqrt{s}}{\pi} \frac{1}{\tilde{s}_{23}} \left[\frac{1}{\beta_{23}} \ln \frac{1 + \beta_{23}}{1 - \beta_{23}} + \frac{2}{|\vec{v}_2|} \ln \frac{1 - |\vec{v}_2|}{1 + |\vec{v}_2|} \right] \left\{ - \left(M + \frac{P^2}{2M} + \sqrt{P^2 + m^2} \right) \right. \\ & \quad \times \sqrt{P^2 + m^2} \left(p \overleftrightarrow{\partial}_p - p' \overleftrightarrow{\partial}_{p'} + k \overleftrightarrow{\partial}_k - k' \overleftrightarrow{\partial}_{k'} + p' \overleftrightarrow{\partial}_p - p \overleftrightarrow{\partial}_{p'} - k' \overleftrightarrow{\partial}_k + k \overleftrightarrow{\partial}_{k'} \right) \\ & \quad \left. - \frac{(\sqrt{P^2 + m^2}) \left(M + \frac{P^2}{2M} + \sqrt{P^2 + m^2} \right)}{(P^2 + m^2) - (pp' - kk')} m^2 \left(p \overleftrightarrow{\partial}_{p'} - p' \overleftrightarrow{\partial}_p + k' \overleftrightarrow{\partial}_k - k \overleftrightarrow{\partial}_{k'} \right) \right\} \end{aligned}$$

and from (1, 4)-pair

$$\begin{aligned} & \frac{dB_1^{(1,4)}}{d\omega} + \frac{dB_1^{(4,1)}}{d\omega} \\ &= \frac{G}{\pi} \left[\frac{\left(M + \frac{P^2}{2M} \right)^2 - (pp' - kk')}{\sqrt{(2M^2 + P^2 + \frac{P^4}{4M^2}) - (pp' - kk')} \sqrt{(P^2 + \frac{P^4}{4M^2}) - (pp' - kk')}} \right. \\ & \quad \times \ln \frac{\left(M + \frac{P^2}{2M} \right)^2 - (pp' - kk') + \sqrt{(2M^2 + P^2 + \frac{P^4}{4M^2}) - (pp' - kk')}}{\left(M + \frac{P^2}{2M} \right)^2 - (pp' - kk') - \sqrt{(2M^2 + P^2 + \frac{P^4}{4M^2}) - (pp' - kk')}} \sqrt{(P^2 + \frac{P^4}{4M^2}) - (pp' - kk')}} \\ & \quad + 2 \frac{M}{P} \ln \frac{M-P}{M+P} \left. \left\{ \left[\left(M + \frac{P^2}{2M} \right)^2 - (pp' - kk') \right] \left(\frac{M}{M^2 - (pp' - kk')} \right) \right. \right. \\ & \quad \times \left(p \overleftrightarrow{\partial}_p + k \overleftrightarrow{\partial}_k - p' \overleftrightarrow{\partial}_{p'} - k' \overleftrightarrow{\partial}_{k'} + k \overleftrightarrow{\partial}_{k'} - p \overleftrightarrow{\partial}_{p'} + p' \overleftrightarrow{\partial}_p - k' \overleftrightarrow{\partial}_k \right) \\ & \quad \left. \left. + \frac{\left(M + \frac{P^2}{2M} \right) \left[P^2 - \left(M + \frac{P^2}{2M} \right)^2 \right]}{\left(M + \frac{P^2}{2M} \right)^2 - (pp' - kk')} \left(k \overleftrightarrow{\partial}_{k'} - p \overleftrightarrow{\partial}_{p'} + p' \overleftrightarrow{\partial}_p - k' \overleftrightarrow{\partial}_k \right) \right\} \right]. \end{aligned}$$

Once again, these terms will give us a zero result after applying them to the amplitude.

In conclusion, we have generalized $\mathcal{M}^* \frac{dB_1}{d\omega} \mathcal{M} = 0$ to all gravitational elastic scatterings.

C $\frac{dB_1^{(i,j)}}{d\omega}$ for the inelastic case

In this appendix we show the general results (before expansion) of the $dB_1^{(i,j)}/d\omega$ factor for the inelastic scattering discussed in section 6. Adopting the kinematics (6.1), one gets from

eq. (3.29) contributions from six pairs respectively and the results read²

$$\begin{aligned}
 \frac{dB_1^{(1,2)}}{d\omega} + \frac{dB_1^{(2,1)}}{d\omega} &= \frac{G}{\pi} \left[\ln \left(\frac{M + \frac{P^2}{2M} + P}{M} \right)^2 + \frac{M}{P} \ln \frac{M-P}{M+P} \right] \left[\left(M + \frac{P^2}{2M} + P \right) \right. \\
 &\quad \times \left(P \overleftrightarrow{\partial}_{M+\frac{P^2}{2M}} + P \overleftrightarrow{\partial}_P \right) + \left(P - M - \frac{P^2}{2M} \right) \frac{M+P}{M-P} \\
 &\quad \left. \times \left(P \overleftrightarrow{\partial}_{M+\frac{P^2}{2M}} - p \overleftrightarrow{\partial}_p - k \overleftrightarrow{\partial}_k \right) \right], \\
 \frac{dB_1^{(3,4)}}{d\omega} + \frac{dB_1^{(4,3)}}{d\omega} &= \frac{G}{\pi} \left[\ln \left(\frac{M' + P' + \frac{P'^2}{2M'}}{M'} \right)^2 + \frac{M'}{P'} \ln \frac{M'-P'}{M'+P'} \right] \left[- \left(M' + P' + \frac{P'^2}{2M'} \right) \right. \\
 &\quad \times \left(P' \overleftrightarrow{\partial}_{P'} + P' \overleftrightarrow{\partial}_{M'+\frac{P'^2}{2M'}} \right) - \left(P' - M' - \frac{P'^2}{2M'} \right) \frac{M'+P'}{M'-P'} \\
 &\quad \left. \times \left(P' \overleftrightarrow{\partial}_{M'+\frac{P'^2}{2M'}} - p' \overleftrightarrow{\partial}_{p'} - k' \overleftrightarrow{\partial}_{k'} \right) \right], \\
 \frac{dB_1^{(1,3)}}{d\omega} + \frac{dB_1^{(3,1)}}{d\omega} &= \frac{G}{\pi} \frac{P'M + (pp' - kk')}{P'M - (pp' - kk')} \left[\ln \left(\frac{P' \left(M + \frac{P^2}{2M} \right) + pp' - kk'}{MP'} \right)^2 \right. \\
 &\quad + \frac{M}{P} \ln \frac{M-P}{M+P} \left\{ \left[P' \left(M + \frac{P^2}{2M} \right) + (pp' - kk') \left(\frac{P'M - (pp' - kk')}{P'M + (pp' - kk')} \right) \right. \right. \\
 &\quad \times \left(\overleftrightarrow{\partial}_{M+\frac{P^2}{2M}} - \overleftrightarrow{\partial}_{P'} \right) + \frac{pP' - p'M}{P'M + (pp' - kk')} \left(\overleftrightarrow{\partial}_p + \overleftrightarrow{\partial}_{p'} \right) \\
 &\quad + \frac{k'M + kP'}{P'M + (pp' - kk')} \left(\overleftrightarrow{\partial}_k - \overleftrightarrow{\partial}_{k'} \right) \left. \right] - \frac{P' \left[P^2 - \left(M + \frac{P^2}{2M} \right)^2 \right]}{P' \left(M + \frac{P^2}{2M} \right) + pp' - kk'} \\
 &\quad \left. \times \left(-P' \overleftrightarrow{\partial}_{M+\frac{P^2}{2M}} + p' \overleftrightarrow{\partial}_p - k' \overleftrightarrow{\partial}_k \right) \right\}, \\
 \frac{dB_1^{(2,4)}}{d\omega} + \frac{dB_1^{(4,2)}}{d\omega} &= \frac{G}{\pi} \frac{PM' + (pp' - kk')}{PM' - (pp' - kk')} \left[\ln \left(\frac{P \left(M' + \frac{P'^2}{2M'} \right) + pp' - kk'}{M'P} \right)^2 \right. \\
 &\quad + \frac{M'}{P'} \ln \frac{M'-P'}{M'+P'} \left\{ \left[P \left(M' + \frac{P'^2}{2M'} \right) + (pp' - kk') \right] \left[\frac{PM' - (pp' - kk')}{PM' + (pp' - kk')} \right. \right. \\
 &\quad \times \left(\overleftrightarrow{\partial}_{P'} - \overleftrightarrow{\partial}_{M'+\frac{P'^2}{2M'}} \right) - \frac{p'P - pM'}{PM' + (pp' - kk')} \left(\overleftrightarrow{\partial}_p + \overleftrightarrow{\partial}_{p'} \right) \\
 &\quad + \frac{(kM' + k'P)}{PM' + (pp' - kk')} \left(\overleftrightarrow{\partial}_k - \overleftrightarrow{\partial}_{k'} \right) \left. \right] + \frac{P \left[P'^2 - \left(M' + \frac{P'^2}{2M'} \right)^2 \right]}{P \left(M' + \frac{P'^2}{2M'} \right) + pp' - kk'} \\
 &\quad \left. \times \left(-P \overleftrightarrow{\partial}_{M'+\frac{P'^2}{2M'}} + p \overleftrightarrow{\partial}_{p'} - k \overleftrightarrow{\partial}_{k'} \right) \right\},
 \end{aligned}$$

²The computation for the inelastic case is similar to those in appendix A and B (although much more complicated), thus to avoid repeatability we only show the results in this appendix.

$$\begin{aligned}
 & \frac{dB_1^{(1,4)}}{d\omega} + \frac{dB_1^{(4,1)}}{d\omega} \\
 = & \frac{G}{\pi} \frac{MM' - (pp' - kk')}{MM' + (pp' - kk')} \left[\frac{\left(M + \frac{P^2}{2M}\right) \left(M' + \frac{P'^2}{2M'}\right) - (pp' - kk')}{\sqrt{\left[\left(M + \frac{P^2}{2M}\right) \left(M' + \frac{P'^2}{2M'}\right) - (pp' - kk')\right]^2 - M^2 M'^2}} \right. \\
 & \times \ln \frac{\left(M + \frac{P^2}{2M}\right) \left(M' + \frac{P'^2}{2M'}\right) - (pp' - kk') + \sqrt{\left[\left(M + \frac{P^2}{2M}\right) \left(M' + \frac{P'^2}{2M'}\right) - (pp' - kk')\right]^2 - M^2 M'^2}}{\left(M + \frac{P^2}{2M}\right) \left(M' + \frac{P'^2}{2M'}\right) - (pp' - kk') - \sqrt{\left[\left(M + \frac{P^2}{2M}\right) \left(M' + \frac{P'^2}{2M'}\right) - (pp' - kk')\right]^2 - M^2 M'^2}} \\
 & + \frac{M}{P} \ln \frac{M-P}{M+P} + \frac{M'}{P'} \ln \frac{M'-P'}{M'+P'} \left. \left\{ \left[\left(M + \frac{P^2}{2M}\right) \left(M' + \frac{P'^2}{2M'}\right) - (pp' - kk') \right] \right. \right. \\
 & \times \left[\frac{MM' + (pp' - kk')}{MM' - (pp' - kk')} \left(\overleftrightarrow{\partial}_{M + \frac{P^2}{2M}} - \overleftrightarrow{\partial}_{M' + \frac{P'^2}{2M'}} \right) + \frac{pM' + p'M}{MM' - (pp' - kk')} \left(\overleftrightarrow{\partial}_p - \overleftrightarrow{\partial}_{p'} \right) \right. \\
 & + \left. \left. \frac{kM' - k'M}{MM' - (pp' - kk')} \left(\overleftrightarrow{\partial}_k + \overleftrightarrow{\partial}_{k'} \right) \right] - \frac{\left(M + \frac{P^2}{2M}\right) \left[P'^2 - \left(M' + \frac{P'^2}{2M'}\right)^2 \right]}{\left(M + \frac{P^2}{2M}\right) \left(M' + \frac{P'^2}{2M'}\right) - (pp' - kk')} \right. \\
 & \times \left[\left(M + \frac{P^2}{2M}\right) \overleftrightarrow{\partial}_{M' + \frac{P'^2}{2M'}} + p \overleftrightarrow{\partial}_{p'} - k \overleftrightarrow{\partial}_{k'} \right] + \frac{\left(M' + \frac{P'^2}{2M'}\right) \left[P^2 - \left(M + \frac{P^2}{2M}\right)^2 \right]}{\left(M + \frac{P^2}{2M}\right) \left(M' + \frac{P'^2}{2M'}\right) - (pp' - kk')} \\
 & \times \left. \left. \left[\left(M' + \frac{P'^2}{2M'}\right) \overleftrightarrow{\partial}_{M + \frac{P^2}{2M}} + p' \overleftrightarrow{\partial}_p - k' \overleftrightarrow{\partial}_k \right] \right\}, \right. \\
 & \frac{dB_1^{(2,3)}}{d\omega} + \frac{dB_1^{(3,2)}}{d\omega} \\
 = & \frac{2G}{\pi} \frac{PP' - (pp' - kk')}{PP' + (pp' - kk')} \ln \left[\frac{PP' - (pp' - kk')}{2PP'} \right] \left\{ [PP' + (pp' - kk')] \left(\overleftrightarrow{\partial}_p - \overleftrightarrow{\partial}_{p'} \right) \right. \\
 & \left. + (pP' + p'P) \left(\overleftrightarrow{\partial}_p - \overleftrightarrow{\partial}_{p'} \right) + (kP' - k'P) \left(\overleftrightarrow{\partial}_k + \overleftrightarrow{\partial}_{k'} \right) \right\}.
 \end{aligned}$$

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