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Covariant chiral kinetic equation in non-Abelian gauge field from “covariant gradient expansion”

Xiao-Li Luo and Jian-Hua Gao

*Shandong Provincial Key Laboratory of Optical Astronomy and Solar-Terrestrial Environment,
Institute of Space Sciences, Shandong University,
No.180 Wenhua Xilu, Weihai, Shandong 264209, China*

E-mail: xiaoli_luo@mail.sdu.edu.cn, gaojh@sdu.edu.cn

ABSTRACT: We derive the chiral kinetic equation in 8 dimensional phase space in non-Abelian $SU(N)$ gauge field within the Wigner function formalism. By using the “covariant gradient expansion”, we disentangle the Wigner equations in four-vector space up to the first order and find that only the time-like component of the chiral Wigner function is independent while other components can be explicit derivative. After further decomposing the Wigner function or equations in color space, we present the non-Abelian covariant chiral kinetic equation for the color singlet and multiplet phase-space distribution functions. These phase-space distribution functions have non-trivial Lorentz transformation rules when we define them in different reference frames. The chiral anomaly from non-Abelian gauge field arises naturally from the Berry monopole in Euclidian momentum space in the vacuum or Dirac sea contribution. The anomalous currents as non-Abelian counterparts of chiral magnetic effect and chiral vortical effect have also been derived from the non-Abelian chiral kinetic equation.

KEYWORDS: Quark-Gluon Plasma, Anomalies in Field and String Theories

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1 Introduction

In recent years, there has been a considerable amount of theoretical work on the chiral kinetic theory (CKT) in relativistic heavy ion collisions. The CKT aims to incorporate the chiral anomaly into kinetic theory and provide a consistent formalism to describe various novel chiral effects, e.g., chiral magnetic effect [1–3], chiral vortical effect [4–7], chiral separation effect [8, 9] and so on, which are all associated with the chiral anomaly. Recent progress on chiral effects and chiral kinetic theory in relativistic heavy ion collisions can be found in the reviews such as [10–14]. The chiral kinetic equation has been derived from various methods, such as semiclassical approach [15–24], Wigner function formalism [25–30], effective field theory [31–34] and world-line approach [35–37]. The numerical simulation based on chiral kinetic equation can be found in refs. [38–45].

Despite all these development, so far most of the literature focuses on the CKT in Abelian gauge field. Only very restricted work [19, 21, 24, 37] had discussed the CKT in non-Abelian gauge field. However, as we all know, the dynamics of the produced quark-gluon plasma in relativistic heavy ion collisions are mainly determined by quantum chromodynamics — non-Abelian SU(3) gauge field. Especially, in the small x physics, the initial state in relativistic nucleus-nucleus collisions can be described as a classical coherent non-Abelian gauge field configuration called the color glass condensate [46–50]. It still remains an open question how the decoherence from the classical color field to the quark-gluon plasma takes place. In order to address these problems, we need generalize the CKT in Abelian gauge field to the one in non-Abelian gauge field.

In this paper, we will be dedicated to deriving the chiral kinetic equation in $SU(N)$ gauge field from the quantum transport theory [51–54, 58] based on the Wigner functions from quantum gauge field theory. In section 2, we review the Wigner function formalism given in refs. [51–54] and present some results in ref. [58] that would be useful for our present work. In section 3, we apply the “covariant gradient expansion” given in [52–54, 58] to expanding the Wigner equations for massless fermions up to the first order and disentangle the Wigner equations by the method developed in the Abelian case in ref. [29]. We find that only the timelike component of the Wigner functions is independent and all other spacelike components can be derivative from timelike component directly. Such result is very similar to the Abelian case and reduces the Wigner equations greatly. We present the covariant chiral kinetic equation for this independent Wigner function in 8-dimensional form, i.e., 4-dimensional momentum space and 4-dimensional coordinate space. In comparison with the Abelian case, the extra constraint equation appears in non-Abelian case. In section 4, we decompose the results further in the color space and find that the color singlet phase-space distribution function and multiplet ones are totally coupled with each other. In section 5, we discuss the modified Lorentz transformation of the distribution function in phase space when we define it in different reference frames. With the results in previous sections, we calculate the vector and axial currents induced by color field and vorticity in section 6. It turns out that the non-Abelian chiral anomaly can be derived directly from the 4-dimensional Berry curvature in the vacuum contribution of the color singlet Wigner function. With specific distribution near global equilibrium, we can obtain the non-Abelian counterparts of chiral magnetic effect and chiral vortical effect. Finally, we summarize the paper in section 7.

In this work, we use the convention for the metric $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$, Levi-Civita tensor $\epsilon^{0123} = 1$. We choose natural units such that $\hbar = c = 1$ except for the cases when we want to display \hbar dependence to clarify the perturbative expansion.

2 Quantum transport theory

In quantum transport theory, the gauge invariant density matrix for spin-1/2 quarks is defined as [51–53]

$$\rho\left(x + \frac{y}{2}, x - \frac{y}{2}\right) = \bar{\psi}\left(x + \frac{y}{2}\right) U\left(x + \frac{y}{2}, x\right) \otimes U\left(x, x - \frac{y}{2}\right) \psi\left(x - \frac{y}{2}\right). \quad (2.1)$$

where the direct product is over both spinor and color indices. The element of density matrix with specific color and spinor indices is given by

$$\rho_{\alpha\beta}^{ij}\left(x + \frac{y}{2}, x - \frac{y}{2}\right) = \bar{\psi}_{\beta}^{j'}\left(x + \frac{y}{2}\right) U^{j'j}\left(x + \frac{y}{2}, x\right) U^{ii'}\left(x, x - \frac{y}{2}\right) \psi_{\alpha}^{i'}\left(x - \frac{y}{2}\right). \quad (2.2)$$

where α, β denote spinor indices, i, i', j, j' mean color indices in fundamental representation and $U^{j'j}$ or $U^{i'i}$ is the Wilson line or gauge link

$$U^{ij}(x, y) = \left[P \exp\left(\frac{ig}{\hbar} \int_y^x dz^{\mu} A_{\mu}(z)\right) \right]^{ij} \quad (2.3)$$

which is necessary to keep the operator gauge invariant. In the definition of Wilson line P denotes path ordering of the operator and the integral in the exponent is taken along the straight path from x to y . The gauge field potential is defined by $A_\mu = A_\mu^a t^a$, with the $N^2 - 1$ hermitian generators of $SU(N)$ in the fundamental representation satisfying

$$\text{Tr } t^a = 0, \quad [t^a, t^b] = if^{abc}t^c, \quad \{t^a, t^b\} = \frac{1}{N}\delta^{ab}\mathbf{1} + d^{abc}t^c. \quad (2.4)$$

For non-Abelian gauge field, the covariant derivative in the fundamental representation is defined as,

$$D_\mu(x) = \partial_\mu - \frac{ig}{\hbar}A_\mu(x), \quad (2.5)$$

and the field strength tensor follows as

$$F_{\mu\nu}(x) \equiv F_{\mu\nu}^a t^a = -\frac{\hbar}{ig} [D_\mu, D_\nu] = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - \frac{ig}{\hbar} [A_\mu(x), A_\nu(x)]. \quad (2.6)$$

The Wigner operator $\hat{W}(x, p)$ is related to the gauge invariant density matrix by Fourier transformation

$$\hat{W}(x, p) = \int \frac{d^4 y}{(2\pi)^4} e^{-ip \cdot y} \rho \left(x + \frac{y}{2}, x - \frac{y}{2} \right), \quad (2.7)$$

and the Wigner function is defined as ensemble averaging of the Wigner operator

$$W(x, p) = \langle \hat{W}(x, p) \rangle. \quad (2.8)$$

In our present work, we will concentrate on the quark matter under a purely classical external non-Abelian gauge field, which is very relevant to the earlier stage of relativistic heavy-ion collisions. When the quark-gluon plasma is produced from the decoherence of the classical color field, the gluon dynamics will play an important role and the kinetic equation of the gluons must be included. The gluon's kinetic equation can be derived from the gluon's Wigner function by generalizing the methods to deal with photon's kinetic description given in ref. [59] or [60]. We will postpone such generalization in the future. Under background field approximation, the ordinary matrix multiplication rules in spinor space or color space suffice and the Wigner equations will not generate the so-called BBGKY-hierarchy [55] and can be closed by itself

$$\begin{aligned} \left[m - \gamma^\mu \left(p_\mu + \frac{1}{2} i \mathcal{D}_\mu(x) \right) \right] W(x, p) = \frac{ig}{2} \gamma^\mu \partial_p^\nu \left\{ \int_0^1 ds \frac{1+s}{2} \left[e^{-\frac{1}{2} is \Delta} F_{\mu\nu}(x) \right] W(x, p) \right. \\ \left. + W(x, p) \int_0^1 ds \frac{1-s}{2} \left[e^{\frac{1}{2} is \Delta} F_{\mu\nu}(x) \right] \right\}, \end{aligned} \quad (2.9)$$

together with the hermitian adjoint equation

$$\begin{aligned} W(x, p) \left[m - \gamma^\mu \left(p_\mu - \frac{1}{2} i \mathcal{D}_\mu^\dagger(x) \right) \right] = -\frac{ig}{2} \partial_p^\nu \left\{ \int_0^1 ds \frac{1-s}{2} \left[e^{-\frac{1}{2} is \Delta} F_{\mu\nu}(x) \right] W(x, p) \right. \\ \left. + W(x, p) \int_0^1 ds \frac{1+s}{2} \left[e^{\frac{1}{2} is \Delta} F_{\mu\nu}(x) \right] \right\} \gamma^\mu, \end{aligned} \quad (2.10)$$

where we have introduced the definition of covariant derivative in the adjoint representation for a second-rank tensor $\mathcal{T}(x)$ in color space by

$$\mathcal{D}_\mu(x)\mathcal{T}(x) \equiv [D_\mu(x), \mathcal{T}(x)] = \partial_\mu^x \mathcal{T}(x) - \frac{ig}{\hbar} [A_\mu(x), \mathcal{T}(x)], \quad (2.11)$$

and $\Delta \equiv \partial_p \cdot \mathcal{D}(x)$ with $\mathcal{D}(x)$ only acting on $F_{\mu\nu}$ and ∂_p always on W after or in front of it. It should be noted that in the definition of the Wigner function given by eq. (2.8) and the Wigner equations (2.9) and (2.10) there is no normal ordering in the Wigner matrix because we did not make any manipulation on the order of the quark field. It has been demonstrated in [56, 57] that this plays a central role to give rise to the chiral anomaly in the quantum kinetic theory.

If we take the convention in [58], momentum derivatives standing to the right of the Wigner function are defined in the sense of partial integration as

$$W(x, p) \partial_p^{\nu_1} \dots \partial_p^{\nu_k} \equiv (-1)^k \partial_p^{\nu_k} \dots \partial_p^{\nu_1} W(x, p), \quad (2.12)$$

and define generalized non-local momentum and derivative operators Π_μ and G_μ as

$$\begin{aligned} \Pi_\mu &= p_\mu + \frac{g}{2} \int_0^1 ds \left(e^{-\frac{1}{2}is\Delta} F_{\mu\nu}(x) \right) is \partial_p^\nu, \\ G_\mu &= D_\mu + \frac{g}{2} \int_0^1 ds \left(e^{-\frac{1}{2}is\Delta} F_{\mu\nu}(x) \right) \partial_p^\nu, \end{aligned} \quad (2.13)$$

the Wigner equations can be cast into a more compact form [58],

$$2mW(x, p) = \gamma^\mu (\{\Pi_\mu, W(x, p)\} + i [G_\mu, W(x, p)]), \quad (2.14)$$

$$2mW(x, p) = (\{\Pi_\mu, W(x, p)\} - i [G_\mu, W(x, p)]) \gamma^\mu. \quad (2.15)$$

Adding or subtracting the two equations above gives

$$4mW(x, p) = \{\gamma^\mu, \{\Pi_\mu, W(x, p)\}\} + i [\gamma^\mu, [G_\mu, W(x, p)]], \quad (2.16)$$

$$0 = [\gamma^\mu, \{\Pi_\mu, W(x, p)\}] + i \{\gamma^\mu, [G_\mu, W(x, p)]\}. \quad (2.17)$$

In spinor space, we can decompose the Wigner function into

$$W = \frac{1}{4} \left[\mathcal{F} + i\gamma^5 \mathcal{P} + \gamma^\mu \mathcal{V}_\mu + \gamma^\mu \gamma^5 \mathcal{A}_\mu + \frac{1}{2} \sigma^{\mu\nu} \mathcal{J}_{\mu\nu} \right]. \quad (2.18)$$

In this work, we will restrict ourselves to the massless or chiral fermions. In consequence, if we introduce a chirality basis via

$$\mathcal{J}_s^\mu = \frac{1}{2} (\mathcal{V}^\mu + s \mathcal{A}^\mu), \quad (2.19)$$

where $s = +1/-1$ denotes right-handed/left-handed component, the equations for the chiral Wigner function \mathcal{J}_s^μ will decouple from all the other components of the Wigner function and each other as well, which leads to

$$0 = \{\Pi_\mu, \mathcal{J}_s^\mu\}, \quad (2.20)$$

$$0 = [G_\mu, \mathcal{J}_s^\mu], \quad (2.21)$$

$$0 = \{\Pi^\mu, \mathcal{J}_s^\nu\} - \{\Pi^\nu, \mathcal{J}_s^\mu\} + s \hbar \epsilon^{\mu\nu\alpha\beta} [G_\alpha, \mathcal{J}_{s\beta}], \quad (2.22)$$

where we have recovered the \hbar dependence before the generalized derivative operators in the last equation in order to make perturbative expansion in the following section. These Wigner equations will be the starting point of our present work in the following. For brevity, we will suppress the subscript s of the left-hand or right-hand Wigner function \mathcal{J}_s^μ in the subsequent sections and reinstate it when it is necessary.

3 Disentangling Wigner equations in four-vector space

In the Abelian plasma, the disentanglement theorem of Wigner functions has been demonstrated in ref. [29], which tell us that up to any order of \hbar among four components of Wigner functions \mathcal{J}^μ only the timelike component is independent and satisfies only one independent Wigner equation, the other spatial components can be totally fixed from this independent Wigner function and the Wigner equations for them are all satisfied automatically. Now let us try to generalize this disentanglement formalism from Abelian gauge field to non-Abelian gauge field. In order to achieve this goal, we will resort to the ‘‘covariant gradient expansion’’ proposed in refs. [53, 54, 58]. In this expansion scheme, when we have one extra covariant derivative D_μ or \mathcal{D}_μ , we will have one extra higher order contribution. The ‘‘covariant gradient expansion’’ preserves gauge invariance order by order automatically. Actually we can trace such expansion in powers of \hbar , e.g., in the Wigner equations (2.22) and the generalized non-local momentum and derivative operators

$$\Pi_\mu = \sum_{k=0}^{\infty} \hbar^k \Pi_\mu^{(k)} = p_\mu - \hbar \frac{ig}{2} \sum_{k=0}^{\infty} \left(-\frac{i\hbar}{2} \right)^k \frac{k+1}{(k+2)!} [(\partial_p \cdot \mathcal{D})^k F_{\nu\mu}] \partial_p^\nu \quad (3.1)$$

$$G_\mu = \sum_{k=0}^{\infty} \hbar^k G_\mu^{(k)} = D_\mu - \frac{g}{2} \sum_{k=0}^{\infty} \left(-\frac{i\hbar}{2} \right)^k \frac{1}{(k+1)!} [(\partial_p \cdot \mathcal{D})^k F_{\nu\mu}] \partial_p^\nu. \quad (3.2)$$

Up to the second order of \hbar , the non-local operators Π_μ and G_μ are given by

$$\Pi_\mu^{(0)} = p_\mu, \quad \Pi_\mu^{(1)} = \frac{ig}{4} F_{\mu\nu} \partial_p^\nu, \quad \Pi_\mu^{(2)} = \frac{g}{12} [(\partial_p \cdot \mathcal{D}) F_{\mu\nu}] \partial_p^\nu, \quad (3.3)$$

$$G_\mu^{(0)} = D_\mu + \frac{g}{2} F_{\mu\nu} \partial_p^\nu, \quad G_\mu^{(1)} = -\frac{ig}{8} [(\partial_p \cdot \mathcal{D}) F_{\mu\nu}] \partial_p^\nu. \quad (3.4)$$

We can also expand the Wigner operator as

$$W(x, p) = \sum_{k=0}^{\infty} \hbar^k W^{(k)}(x, p). \quad (3.5)$$

However it should be noted that the ‘‘covariant gradient expansion’’ is not completely identical to an expansion in powers of \hbar for non-Abelian gauge field which had been pointed out in [53, 54, 58] though it is identical for Abelian gauge field. In non-Abelian case, there is an extra gauge potential A_μ with ig/\hbar in the covariant derivative D_μ or \mathcal{D}_μ in eqs. (3.1) and (3.2) while there only exist ordinary derivative ∂_μ^x in the Abelian case.

In order to disentangle the Wigner equations further, it is convenient to introduce time-like 4-vector n^μ with normalization $n^2 = 1$. For simplicity we assume n^μ is a constant

vector. With the auxiliary vector n^μ , we can decompose any vector X^μ into the component parallel to n^μ and the other components perpendicular to n^μ ,

$$X^\mu = X_n n^\mu + \bar{X}^\mu, \quad (3.6)$$

where $X_n = X \cdot n$ and $\bar{X}^\mu = \Delta^{\mu\nu} X_\nu$ with $\Delta^{\mu\nu} = g^{\mu\nu} - n^\mu n^\nu$. The gauge field tensor $F^{\mu\nu}$ can be also decomposed into

$$F^{\mu\nu} = E^\mu n^\nu - E^\nu n^\mu - \bar{\epsilon}^{\mu\nu\sigma} B_\sigma \quad (3.7)$$

with

$$E^\mu = F^{\mu\nu} n_\nu, \quad B^\mu = \frac{1}{2} \bar{\epsilon}^{\mu\rho\sigma} F_{\rho\sigma}, \quad (3.8)$$

where for notational convenience we have defined $\bar{\epsilon}^{\mu\alpha\beta} = \epsilon^{\mu\nu\alpha\beta} n_\nu$.

Now we can decompose the Wigner functions and Wigner equations along the direction n^μ order by order. The leading order or the zeroth order result is very simple

$$0 = p_n \mathcal{J}_n^{(0)} + \bar{p}_\mu \bar{\mathcal{J}}^{(0)\mu}, \quad (3.9)$$

$$0 = [G_n^{(0)}, \mathcal{J}_n^{(0)}] + [\bar{G}_\mu^{(0)}, \bar{\mathcal{J}}^{(0)\mu}], \quad (3.10)$$

$$0 = \bar{p}^\mu \mathcal{J}_n^{(0)} - p_n \bar{\mathcal{J}}^{(0)\mu}, \quad (3.11)$$

$$0 = \bar{p}^\mu \bar{\mathcal{J}}^{(0)\nu} - \bar{p}^\nu \bar{\mathcal{J}}^{(0)\mu}. \quad (3.12)$$

From eq. (3.11), we can express the space-like component $\bar{\mathcal{J}}^{(0)\mu}$ in terms of $\mathcal{J}_n^{(0)}$

$$\bar{\mathcal{J}}^{(0)\mu} = \bar{p}^\mu \frac{\mathcal{J}_n^{(0)}}{p_n}. \quad (3.13)$$

Substituting this relation into eq. (3.9) gives rise to the on-shell condition

$$p^2 \frac{\mathcal{J}_n^{(0)}}{p_n} = 0, \quad (3.14)$$

which means $\mathcal{J}_n^{(0)}/p_n$ must be proportional to the Dirac delta function $\delta(p^2)$

$$\frac{\mathcal{J}_n^{(0)}}{p_n} = f^{(0)} \delta(p^2), \quad (3.15)$$

where $f^{(0)}$ can be regarded as the usual particle distribution function in four-dimensional momentum space and four-dimensional coordinate space. It must be non-singular function at $p^2 = 0$. Putting eqs. (3.15) and (3.13) together, we get the full Wigner function of the zeroth order

$$\mathcal{J}^{(0)\mu} = p^\mu f^{(0)} \delta(p^2). \quad (3.16)$$

The transport equation satisfied by $f^{(0)}$ can be obtained from eq. (3.10)

$$0 = [G_\mu^{(0)}, p^\mu f^{(0)} \delta(p^2)]. \quad (3.17)$$

It is obvious that eq. (3.12) is automatically satisfied with the expression (3.13).

The next-to-leading order or the first order equations are given by

$$0 = 2p_n \mathcal{J}_n^{(1)} + 2\bar{p}_\mu \bar{\mathcal{J}}^{(1)\mu} + \left\{ \Pi_n^{(1)}, \mathcal{J}_n^{(0)} \right\} + \left\{ \bar{\Pi}_\mu^{(1)}, \bar{\mathcal{J}}^{(0)\mu} \right\}, \quad (3.18)$$

$$0 = \left[G_n^{(0)}, \mathcal{J}_n^{(1)} \right] + \left[\bar{G}_\mu^{(0)}, \bar{\mathcal{J}}^{(1)\mu} \right] + \left[G_n^{(1)}, \mathcal{J}_n^{(0)} \right] + \left[\bar{G}_\mu^{(1)}, \bar{\mathcal{J}}^{(0)\mu} \right], \quad (3.19)$$

$$0 = 2\bar{p}^\mu \mathcal{J}_n^{(1)} - 2p_n \bar{\mathcal{J}}^{(1)\mu} + \left\{ \bar{\Pi}^{(1)\mu}, \mathcal{J}_n^{(0)} \right\} - \left\{ \Pi_n^{(1)}, \bar{\mathcal{J}}^{(0)\mu} \right\} + s\bar{\epsilon}^{\mu\alpha\beta} \left[G_\alpha^{(0)}, \mathcal{J}_\beta^{(0)} \right], \quad (3.20)$$

$$0 = 2\bar{p}^\mu \bar{\mathcal{J}}^{(1)\nu} - 2\bar{p}^\nu \bar{\mathcal{J}}^{(1)\mu} + \left\{ \bar{\Pi}^{(1)\mu}, \bar{\mathcal{J}}^{(0)\nu} \right\} - \left\{ \bar{\Pi}^{(1)\nu}, \bar{\mathcal{J}}^{(0)\mu} \right\} + s\bar{\epsilon}^{\mu\nu\alpha} \left(\left[G_\alpha^{(0)}, \mathcal{J}_n^{(0)} \right] - \left[G_n^{(0)}, \mathcal{J}_\alpha^{(0)} \right] \right). \quad (3.21)$$

From eq. (3.20), we can express $\bar{\mathcal{J}}^{(1)\mu}$ in terms of $\mathcal{J}_n^{(1)}$ and $\mathcal{J}_n^{(0)}$

$$\begin{aligned} \bar{\mathcal{J}}^{(1)\mu} &= \bar{p}^\mu \frac{\mathcal{J}_n^{(1)}}{p_n} + \frac{s}{2p_n} \bar{\epsilon}^{\mu\alpha\beta} \left[G_\alpha^{(0)}, \bar{p}_\beta \frac{\mathcal{J}_n^{(0)}}{p_n} \right] \\ &+ \frac{1}{2p_n} \left(\left\{ \bar{\Pi}^{(1)\mu}, p_n \frac{\mathcal{J}_n^{(0)}}{p_n} \right\} - \left\{ \Pi_n^{(1)}, \bar{p}^\mu \frac{\mathcal{J}_n^{(0)}}{p_n} \right\} \right). \end{aligned} \quad (3.22)$$

Substituting it into eqs. (3.18) and (3.19) gives rise to the modified on-shell condition and transport equation for $\mathcal{J}_n^{(1)}$, respectively,

$$\begin{aligned} p^2 \frac{\mathcal{J}_n^{(1)}}{p_n} &= -\frac{s}{2p_n} \bar{\epsilon}^{\mu\alpha\beta} \bar{p}_\mu \left[G_\alpha^{(0)}, \bar{p}_\beta \frac{\mathcal{J}_n^{(0)}}{p_n} \right] - \frac{1}{2} \left\{ \Pi_\mu^{(1)}, p^\mu \frac{\mathcal{J}_n^{(0)}}{p_n} \right\} \\ &- \frac{\bar{p}_\mu}{2p_n} \left(\left\{ \bar{\Pi}^{(1)\mu}, p_n \frac{\mathcal{J}_n^{(0)}}{p_n} \right\} - \left\{ \Pi_n^{(1)}, \bar{p}^\mu \frac{\mathcal{J}_n^{(0)}}{p_n} \right\} \right), \end{aligned} \quad (3.23)$$

$$\begin{aligned} \left[G_\mu^{(0)}, p^\mu \frac{\mathcal{J}_n^{(1)}}{p_n} \right] &= -\frac{s}{2} \bar{\epsilon}^{\mu\alpha\beta} \left[\bar{G}_\mu^{(0)}, \frac{1}{p_n} \left[G_\alpha^{(0)}, \bar{p}_\beta \frac{\mathcal{J}_n^{(0)}}{p_n} \right] \right] - \left[G_\mu^{(1)}, p^\mu \frac{\mathcal{J}_n^{(0)}}{p_n} \right] \\ &- \frac{1}{2} \left[\bar{G}_\mu^{(0)}, \frac{1}{p_n} \left(\left\{ \bar{\Pi}^{(1)\mu}, p_n \frac{\mathcal{J}_n^{(0)}}{p_n} \right\} - \left\{ \Pi_n^{(1)}, \bar{p}^\mu \frac{\mathcal{J}_n^{(0)}}{p_n} \right\} \right) \right]. \end{aligned} \quad (3.24)$$

It is easy to verify that the general expression of the constraint equation (3.23) is given by

$$\frac{\mathcal{J}_n^{(1)}}{p_n} = f^{(1)} \delta(p^2) + \frac{s}{2p_n} \bar{\epsilon}^{\mu\alpha\beta} p_\mu \left\{ \frac{g}{2} F_{\alpha\beta}, f^{(0)} \right\} \delta'(p^2) + \left\{ \Pi_\mu^{(1)}, p^\mu f^{(0)} \right\} \delta'(p^2). \quad (3.25)$$

Just like $f^{(0)}$, the function $f^{(1)}$ is also a non-singular distribution function at $p^2 = 0$ in four-dimensional momentum space and four-dimensional coordinate space and can be regarded as the first order correction to $f^{(0)}$. The transport equation for $f^{(1)}$ can be directly obtained by inserting eq. (3.25) into eq. (3.24) and will not be presented explicitly here to avoid too lengthy equations. Putting eqs. (3.22) and (3.25) together, we get the full Wigner function of the first order

$$\begin{aligned} \mathcal{J}^{(1)\mu} &= p^\mu \left[f^{(1)} \delta(p^2) + \frac{s}{2p_n} \bar{\epsilon}^{\nu\alpha\beta} p_\nu \left\{ \frac{g}{2} F_{\alpha\beta}, f^{(0)} \right\} \delta'(p^2) + \left\{ \Pi_\nu^{(1)}, p^\nu f^{(0)} \right\} \delta'(p^2) \right] \\ &+ \frac{1}{2p_n} \left(\left\{ \bar{\Pi}^{(1)\mu}, p_n f^{(0)} \delta(p^2) \right\} - \left\{ \Pi_n^{(1)}, \bar{p}^\mu f^{(0)} \delta(p^2) \right\} \right) \\ &+ \frac{s}{2p_n} \bar{\epsilon}^{\mu\alpha\beta} \left[G_\alpha^{(0)}, \bar{p}_\beta f^{(0)} \delta(p^2) \right]. \end{aligned} \quad (3.26)$$

As we note in the zeroth order case, the equation (3.12) is automatically satisfied once we have the expression (3.13). Now we can check if the first order equation (3.21) also holds automatically by using the first order expression (3.22) together with eqs. (3.10), (3.14) and (3.16). In consequence, after direct calculation we find that the first order equation (3.21) is not satisfied automatically but lead to the constraint equation for $\mathcal{J}_\mu^{(0)}$ or $f^{(0)}$

$$0 = n_\alpha \left([F^{\nu\alpha}, \mathcal{J}^{(0)\mu}] + [F^{\alpha\mu}, \mathcal{J}^{(0)\nu}] + [F^{\mu\nu}, \mathcal{J}^{(0)\alpha}] \right). \quad (3.27)$$

Because n^α is an arbitrary auxiliary vector with normalization $n^2 = 1$, the constraint equation should not depend on n^α or this equation should hold for any n^α . This leads to the Lorentz covariant constraint equation

$$[F^{\nu\alpha}, \mathcal{J}^{(0)\mu}] + [F^{\alpha\mu}, \mathcal{J}^{(0)\nu}] + [F^{\mu\nu}, \mathcal{J}^{(0)\alpha}] = 0, \quad (3.28)$$

which is equivalent to

$$[\tilde{F}^{\alpha\beta}, \mathcal{J}_\alpha^{(0)}] = 0 \quad \text{with} \quad \tilde{F}^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} F_{\mu\nu}. \quad (3.29)$$

In ref. [58], similar constraints for \mathcal{F} and $\mathcal{S}_{\mu\nu}$ in eq. (2.18) had already been obtained. Such constraints only arise in the quantum transport theory with non-Abelian gauge field. The disentanglement theorem of Wigner functions in Abelian gauge field given in ref. [29] show that all these constraint equations in Abelian cases are satisfied automatically and holds up to any order of \hbar . We also notice that the first order equation (3.21) gives the constraint for the zeroth order Wigner function $\mathcal{J}^{(0)\mu}$ because the first order Wigner functions are totally canceled due to the antisymmetry of the equation. Hence in order to get the constraint for the first order Wigner function $\mathcal{J}^{(1)\mu}$, we need the second order Wigner functions and equations. The second order expression of eq. (2.22) is given by

$$\begin{aligned} 0 = & 2\bar{p}^\mu \mathcal{J}_n^{(2)} - 2p_n \bar{\mathcal{J}}^{(2)\mu} \\ & + \left\{ \bar{\Pi}^{(1)\mu}, \mathcal{J}_n^{(1)} \right\} - \left\{ \Pi_n^{(1)}, \bar{\mathcal{J}}^{(1)\mu} \right\} + \left\{ \bar{\Pi}^{(2)\mu}, \mathcal{J}_n^{(0)} \right\} - \left\{ \Pi_n^{(2)}, \bar{\mathcal{J}}^{(0)\mu} \right\} \\ & + s\bar{\epsilon}^{\mu\alpha\beta} \left([G_\alpha^{(0)}, \mathcal{J}_\beta^{(1)}] + [G_\alpha^{(1)}, \mathcal{J}_\beta^{(0)}] \right), \end{aligned} \quad (3.30)$$

$$\begin{aligned} 0 = & 2\bar{p}^\mu \bar{\mathcal{J}}^{(2)\nu} - 2\bar{p}^\nu \bar{\mathcal{J}}^{(2)\mu} \\ & + \left\{ \bar{\Pi}^{(1)\mu}, \bar{\mathcal{J}}^{(1)\nu} \right\} - \left\{ \bar{\Pi}^{(1)\nu}, \bar{\mathcal{J}}^{(1)\mu} \right\} + \left\{ \bar{\Pi}^{(2)\mu}, \bar{\mathcal{J}}^{(0)\nu} \right\} - \left\{ \bar{\Pi}^{(2)\nu}, \bar{\mathcal{J}}^{(0)\mu} \right\} \\ & + s\bar{\epsilon}^{\mu\nu\alpha} \left([G_\alpha^{(0)}, \mathcal{J}_n^{(1)}] - [G_n^{(0)}, \mathcal{J}_\alpha^{(1)}] + [G_\alpha^{(1)}, \mathcal{J}_n^{(0)}] - [G_n^{(1)}, \mathcal{J}_\alpha^{(0)}] \right). \end{aligned} \quad (3.31)$$

From the first equation above, we can express $\bar{\mathcal{J}}^{(2)\mu}$ in terms of $\mathcal{J}_n^{(2)}$, $\mathcal{J}_n^{(1)}$ and $\mathcal{J}_n^{(0)}$ as

$$\begin{aligned} \bar{\mathcal{J}}^{(2)\mu} = & \bar{p}^\mu \frac{\mathcal{J}_n^{(2)}}{p_n} + \frac{s}{2p_n} \bar{\epsilon}^{\mu\alpha\beta} \left([G_\alpha^{(0)}, \mathcal{J}_\beta^{(1)}] + [G_\alpha^{(1)}, \mathcal{J}_\beta^{(0)}] \right) \\ & + \frac{1}{2p_n} \left(\left\{ \bar{\Pi}^{(1)\mu}, \mathcal{J}_n^{(1)} \right\} - \left\{ \Pi_n^{(1)}, \bar{\mathcal{J}}^{(1)\mu} \right\} \right) \\ & + \frac{1}{2p_n} \left(\left\{ \bar{\Pi}^{(2)\mu}, \mathcal{J}_n^{(0)} \right\} - \left\{ \Pi_n^{(2)}, \bar{\mathcal{J}}^{(0)\mu} \right\} \right). \end{aligned} \quad (3.32)$$

Similar to the first order, substituting it into eq. (3.31) and using eqs. (3.22), (3.23) and (3.24) leads to the constraint for $\mathcal{J}^{(1)\mu}$

$$\left[\tilde{F}_{\alpha\beta}, \mathcal{J}^{(1)\alpha} \right] = -\frac{3}{32} \left[\left[\tilde{F}_{\nu\alpha} \partial_\beta^p - \tilde{F}_{\nu\beta} \partial_\alpha^p, F^{\nu\kappa} \partial_\kappa^p \right], \mathcal{J}^{(0)\alpha} \right]. \quad (3.33)$$

As we just mentioned above, these constraints are unique for non-Abelian gauge field and absent for Abelian field. Such constraints actually originate from the fact that the ‘‘covariant gradient expansion’’ is not completely identical to an expansion in powers of \hbar for non-Abelian gauge field. One difference between non-Abelian and Abelian is the operator $G_\mu^{(0)}$. In the non-Abelian case, the derivative in $G_\mu^{(0)}$ is covariant derivative D_μ , while in Abelian case, it is ordinary space-time derivative ∂_μ^x . When we calculate high order contribution through iterative process, we will meet the commutator $[D_\mu, D_\nu] = igF_{\mu\nu}/\hbar$ in non-Abelian gauge field and this term will contribute to the lower power order, but for the ordinary derivative such issue will never happen in Abelian gauge field. Actually, during our calculation of (3.33), we find that if we do not use the constraints for $\mathcal{J}^{(0)\alpha}$ in eq. (3.28) or (3.29) beforehand, we will have the same term as the right side of eq. (3.27) but with minus sign. This term from the second order equation will eventually cancel the one from the first order. Although we can not give the general proof, we expect that the third order equation of (2.22) will cancel the second order result (3.33) and so on. Adding all the contributions up to any high order, the constraint equation (2.22) should also be satisfied automatically.

4 Decomposing covariant chiral kinetic equations in color space

Up to now, the Wigner function \mathcal{J}^μ is still an $N \times N$ matrix in color space. Hence it is necessary to decompose the Wigner function into color singlet and multiplet components:

$$\mathcal{J}_\mu(x, p) = \mathcal{J}_\mu^I(x, p) \mathbf{1} + \mathcal{J}_\mu^a(x, p) t^a, \quad (4.1)$$

with

$$\mathcal{J}_\mu^I(x, p) = \frac{1}{N} \text{tr} \mathcal{J}_\mu(x, p), \quad \mathcal{J}_\mu^a(x, p) = 2 \text{tr} t^a \mathcal{J}_\mu(x, p). \quad (4.2)$$

It should be noted that we use upper index ‘‘I’’ to denote singlet component. Similarly, we can decompose the operators into the color singlet and multiplet contributions:

$$G_\mu^{(0)} = D_\mu + G_\mu^{(0)a} t^a, \quad \Pi_\mu^{(1)} = \Pi_\mu^{(1)a} t^a, \quad G_\mu^{(1)} = G_\mu^{(1)a} t^a, \quad (4.3)$$

where

$$G_\mu^{(0)a} = \frac{g}{2} F_{\mu\nu}^a \partial_p^\nu, \quad \Pi_\mu^{(1)a} = \frac{ig}{4} F_{\mu\nu}^a \partial_p^\nu, \quad G_\mu^{(1)a} = -\frac{ig}{8} \left(\mathcal{D}_\lambda^{ac} F_{\mu\nu}^c \right) \partial_p^\lambda \partial_p^\nu, \quad (4.4)$$

with $\mathcal{D}_\lambda^{ac} = \delta^{ca} \partial_\lambda^x + g f^{bca} A_\lambda^b / \hbar$. With such decomposition, the singlet and multiplet components of Wigner functions at the zeroth order can be derived from eq. (3.16)

$$\mathcal{J}^{(0)I\mu} = p^\mu f^{(0)I} \delta(p^2), \quad \mathcal{J}^{(0)a\mu} = p^\mu f^{(0)a} \delta(p^2), \quad (4.5)$$

which satisfy the coupled transport equations

$$0 = \partial_\mu^x \mathcal{J}^{(0)I\mu} + \frac{1}{N} G_\mu^{(0)a} \mathcal{J}^{(0)a\mu}, \quad (4.6)$$

$$0 = \mathcal{D}_\mu^{ac} \mathcal{J}^{(0)c\mu} + 2G_\mu^{(0)a} \mathcal{J}^{(0)I\mu} + d^{bca} G_\mu^{(0)b} \mathcal{J}^{(0)c\mu}. \quad (4.7)$$

Similarly but more complicatedly, the color decomposition of first order Wigner functions can be derived from eq. (3.26)

$$\begin{aligned} \mathcal{J}^{(1)I\mu} &= p^\mu f^{(1)I} \delta(p^2) - \frac{s}{2} \epsilon^{\mu\nu\alpha\beta} p_\nu \frac{g}{2N} F_{\alpha\beta}^a f^{(0)a} \delta'(p^2) \\ &\quad + \frac{s}{2p_n} \bar{c}^{\mu\alpha\beta} p_\beta \left(\partial_\alpha^x f^{(0)I} + \frac{1}{N} G_\alpha^{(0)a} f^{(0)a} \right) \delta(p^2), \end{aligned} \quad (4.8)$$

$$\begin{aligned} \mathcal{J}^{(1)a\mu} &= p^\mu f^{(1)a} \delta(p^2) - s \epsilon^{\mu\nu\alpha\beta} p_\nu \left(\frac{g}{2} F_{\alpha\beta}^a f^{(0)I} + \frac{1}{2} d^{bca} \frac{g}{2} F_{\alpha\beta}^b f^{(0)c} \right) \delta'(p^2) \\ &\quad + \frac{s}{2p_n} \bar{c}^{\mu\alpha\beta} p_\beta \left(\mathcal{D}_\alpha^{ac} f^{(0)c} + 2G_\alpha^{(0)a} f^{(0)I} + d^{bca} G_\alpha^{(0)b} f^{(0)c} \right) \delta(p^2) \\ &\quad + \frac{1}{2p_n} i f^{bca} \left(\bar{\Pi}^{(1)b\mu} [p_n f^{(0)c} \delta(p^2)] - \Pi_n^{(1)b} [\bar{p}^\mu f^{(0)c} \delta(p^2)] \right) \\ &\quad + i f^{bca} p^\mu \left[\Pi_\nu^{(1)b} (p^\nu f^{(0)c}) \right] \delta'(p^2), \end{aligned} \quad (4.9)$$

which satisfy the corresponding transport equations

$$0 = \partial_\mu^x \mathcal{J}^{(1)I\mu} + \frac{1}{N} G_\mu^{(0)a} \mathcal{J}^{(1)a\mu}, \quad (4.10)$$

$$0 = \mathcal{D}_\mu^{ac} \mathcal{J}^{(1)c\mu} + 2G_\mu^{(0)a} \mathcal{J}^{(1)I\mu} + d^{bca} G_\mu^{(0)b} \mathcal{J}^{(1)c\mu} + i f^{bca} G_\mu^{(1)b} \mathcal{J}^{(0)c\mu}. \quad (4.11)$$

In order to attain all the results above, we have used the eq. (2.4) repeatedly. We note that the singlet distribution $f^{(0)I}$ and multiplet distribution $f^{(0)a}$ are totally coupled with each other even in the zeroth order transport equation, which displays the much complexity for non-Abelian chiral kinetic equation, in comparison with chiral kinetic equation in Abelian gauge field.

5 Frame dependence of distribution function

We can regard $f(x, p)$ as the particle distribution function in 8-dimensional phase space and $f^{(0)}(x, p)$ in eq. (3.15) and $f^{(1)}(x, p)$ in eq. (3.25) are the zeroth order and first order corrections to $f(x, p)$, respectively. However this distribution function defined in this way depends on the auxiliary vector n^μ we choose. Since we can identify this time-like vector n^μ as the velocity of the observer in a reference frame, the distribution function depends on the reference frame in which we define it. In general, the distribution function in phase space can not be Lorentz scalar when we change the reference frame from one to another. In this section, we will derive how these distribution functions transform in different reference frames. In order to do that, we rewrite the zeroth and first order results for Wigner functions with explicit dependence on the frame velocity n^μ as the following:

$$\mathcal{J}^{(0)\mu} = p^\mu \frac{n \cdot \mathcal{J}^{(0)}}{n \cdot p}, \quad (5.1)$$

$$\begin{aligned} \mathcal{J}^{(1)\mu} &= p^\mu \frac{n \cdot \mathcal{J}^{(1)}}{n \cdot p} + \frac{s}{2n \cdot p} \epsilon^{\mu\nu\alpha\beta} n_\nu \left[G_\alpha^{(0)}, \mathcal{J}_\beta^{(0)} \right] \\ &\quad + \frac{1}{2n \cdot p} \left(\left\{ \Pi^{(1)\mu}, n \cdot \mathcal{J}^{(0)} \right\} - \left\{ n \cdot \Pi^{(1)}, \mathcal{J}^{(0)\mu} \right\} \right). \end{aligned} \quad (5.2)$$

Of course, we can also define the particle distribution function in another reference frame with velocity n' ,

$$\mathcal{J}^{(0)\mu} = p^\mu \frac{n' \cdot \mathcal{J}^{(0)}}{n' \cdot p}, \quad (5.3)$$

$$\begin{aligned} \mathcal{J}^{(1)\mu} = & p^\mu \frac{n' \cdot \mathcal{J}^{(1)}}{n' \cdot p} + \frac{s}{2n' \cdot p} \epsilon^{\mu\nu\alpha\beta} n'_\nu [G_\alpha^{(0)}, \mathcal{J}_\beta^{(0)}] \\ & + \frac{1}{2n' \cdot p} \left(\{ \Pi^{(1)\mu}, n' \cdot \mathcal{J}^{(0)} \} - \{ n' \cdot \Pi^{(1)}, \mathcal{J}^{(0)\mu} \} \right). \end{aligned} \quad (5.4)$$

Since $\mathcal{J}^{(0)\mu}$ and $\mathcal{J}^{(1)\mu}$ should not depend on the auxiliary vector, we will get the modified Lorentz transformation for $\mathcal{J}_n^{(0)}/p_n$ and $\mathcal{J}_n^{(1)}/p_n$

$$\delta \left(\frac{n \cdot \mathcal{J}^{(0)}}{n \cdot p} \right) = \frac{n' \cdot \mathcal{J}^{(0)}}{n' \cdot p} - \frac{n \cdot \mathcal{J}^{(0)}}{n \cdot p} = 0, \quad (5.5)$$

$$\begin{aligned} \delta \left(\frac{n \cdot \mathcal{J}^{(1)}}{n \cdot p} \right) &= \frac{n' \cdot \mathcal{J}^{(1)}}{n' \cdot p} - \frac{n \cdot \mathcal{J}^{(1)}}{n \cdot p} \\ &= -\frac{s\epsilon^{\mu\nu\alpha\beta} n_\mu n'_\nu}{2(n \cdot p)(n' \cdot p)} [G_\alpha^{(0)}, \mathcal{J}_\beta^{(0)}] - \frac{(n_\mu n'_\nu - n_\nu n'_\mu)}{2(n \cdot p)(n' \cdot p)} \{ \Pi^{(1)\mu}, \mathcal{J}^{(0)\nu} \}. \end{aligned} \quad (5.6)$$

We note that the zeroth order $\mathcal{J}_n^{(0)}/p_n$ does not depend on the reference frame and is Lorentz scalar while the first order $\mathcal{J}_n^{(1)}/p_n$ does have non-trivial transformation and is not Lorentz scalar when we change from reference frame n_μ to n'_μ . The first term of the last line in eq. (5.6) is just the so-called side-jump term and the second term is unique for non-Abelian gauge field and absent for Abelian gauge field. We can decompose the modified Lorentz transformation into color singlet and multiplet components:

$$\delta \left(\frac{n \cdot \mathcal{J}^{(0)I}}{n \cdot p} \right) = 0, \quad \delta \left(\frac{n \cdot \mathcal{J}^{(0)a}}{n \cdot p} \right) = 0, \quad (5.7)$$

$$\delta \left(\frac{n \cdot \mathcal{J}^{(1)I}}{n \cdot p} \right) = -\frac{s\epsilon^{\mu\nu\alpha\beta} n_\mu n'_\nu}{2(n \cdot p)(n' \cdot p)} \partial_\alpha^x \mathcal{J}_\beta^{(0)I} - \frac{s\epsilon^{\mu\nu\alpha\beta} n_\mu n'_\nu}{2(n \cdot p)(n' \cdot p)N} G_\alpha^{(0)a} \mathcal{J}_\beta^{(0)a}, \quad (5.8)$$

$$\begin{aligned} \delta \left(\frac{n \cdot \mathcal{J}^{(1)a}}{n \cdot p} \right) &= -\frac{s\epsilon^{\mu\nu\alpha\beta} n_\mu n'_\nu}{2(n \cdot p)(n' \cdot p)} \left[\mathcal{D}_\alpha^{ac} \mathcal{J}_\beta^{(0)c} + 2G_\alpha^{a(0)} \mathcal{J}_\beta^{(0)I} + d^{bca} G_\alpha^{(0)b} \mathcal{J}_\beta^{(0)c} \right] \\ &\quad - \frac{n_\mu n'_\nu - n_\nu n'_\mu}{2(n \cdot p)(n' \cdot p)} i f^{bca} \Pi^{(1)b\mu} \mathcal{J}^{(0)c\nu}. \end{aligned} \quad (5.9)$$

Using eqs. (4.5), (4.8) and (4.9), we obtain the transformation of the singlet and multiplet distribution function of $f^{(0)}$ and $f^{(1)}$ when we define them in different frames, respectively,

$$\begin{aligned} \delta(p^2)\delta f^{(0)I} &= 0, \\ \delta(p^2)\delta f^{(0)a} &= 0, \end{aligned} \tag{5.10}$$

$$\delta(p^2)\delta f^{(1)I} = -\delta(p^2)\frac{s\epsilon^{\mu\nu\alpha\beta}n_\mu n'_\nu p_\beta}{2(n\cdot p)(n'\cdot p)}\left[\partial_\alpha^x f^{(0)I} + \frac{\hbar^2}{N}G_\alpha^{(0)a}f^{(0)a}\right], \tag{5.11}$$

$$\begin{aligned} \delta(p^2)\delta f^{(1)a} &= -\delta(p^2)\frac{s\epsilon^{\mu\nu\alpha\beta}n_\mu n'_\nu p_\beta}{2(n\cdot p)(n'\cdot p)}\left[\mathcal{D}_\alpha^{ac}f^{(0)c} + 2G_\alpha^{(0)a}f^{(0)I} + d^{bca}G_\alpha^{(0)b}f^{(0)c}\right] \\ &\quad - \frac{n_\mu n'_\nu - n_\nu n'_\mu}{2(n\cdot p)(n'\cdot p)}{}_i f^{bca}\Pi^{(1)b\mu}\left(p^\nu f^{(0)c}\delta(p^2)\right). \end{aligned} \tag{5.12}$$

These non-trivial transformation play very important role to choose some specific solutions. They will be used to derive chiral effects in the next section.

6 Chiral effects in non-Abelian gauge field

As we all know, chiral kinetic theory tries to incorporate chiral anomaly, a novel and prominent quantum effect, into kinetic approach in a consistent way. It can describe various chiral effects originating from chiral anomaly, such as chiral magnetic effect and chiral vortical effect. However, as far as we know, most of work in the literature on chiral kinetic theory focused on the chiral anomaly or chiral effects induced by Abelian gauge field. In this section, we will demonstrate how the non-Abelian chiral effects can arise naturally in the formalism discussed in the preceding sections.

6.1 Non-Abelian chiral anomaly

First of all, let us consider the non-Abelian chiral anomaly. In general, we can write the zeroth order Wigner function in free Dirac field as the following,

$$\mathcal{J}_{s\mu}^{(0)ij} = \frac{\delta_{ij}}{4\pi^3}\left[\theta(p_0)n_s^i + \theta(-p_0)\left(\bar{n}_s^i - 1\right)\right]p_\mu\delta(p^2) \tag{6.1}$$

where we have recovered the lower chirality index s , the upper scripts i and j indicate the color index in fundamental representation corresponding to eq. (2.2) and the repeated indices here do not denote summation. The function n_s^i/\bar{n}_s^i represent the quark/antiquark number density with color i and chirality s in phase space. They are defined as the ensemble average of the normal-ordered number density operator and are expected to vanish at infinity in phase space. The -1 term in antiparticle distribution is vacuum or Dirac sea contribution and originate from the anticommutator of the antiparticle field in the definition of Wigner function without normal ordering. This term plays a central role to generate the chiral anomaly as pointed out in [56, 57]. Decomposing it in color space gives rise to

$$\mathcal{J}_{s\mu}^{(0)ij} = \delta_{ij}\mathcal{J}_{s\mu}^{(0)I} + t_{ij}^a\mathcal{J}_{s\mu}^{(0)a} \tag{6.2}$$

where the singlet and multiplet components are given by, respectively,

$$\mathcal{J}_{s\mu}^{(0)I} = p_\mu f_s^{(0)I}\delta(p^2), \quad \mathcal{J}_{s\mu}^{(0)a} = p_\mu f_s^{(0)a}\delta(p^2), \tag{6.3}$$

with

$$f_s^{(0)I} = \frac{1}{4\pi^3 N} \sum_i \left[\theta(p_0) n_s^i + \theta(-p_0) \bar{n}_s^i \right] - \frac{1}{4\pi^3} \theta(-p_0), \quad (6.4)$$

$$f_s^{(0)a} = \frac{1}{2\pi^3} \sum_i t_{ii}^a \left[\theta(p_0) n_s^i + \theta(-p_0) \bar{n}_s^i \right]. \quad (6.5)$$

We note that only the singlet component $f_s^{(0)I}$ includes the vacuum contribution. In order to consider the chiral anomaly, we need the transport equation for the axial Wigner functions $\mathcal{A}^{I\mu}$ and $\mathcal{A}^{a\mu}$

$$\mathcal{A}^{I\mu} = \sum_{s=\pm 1} s \mathcal{J}_s^{I\mu}, \quad \mathcal{A}^{a\mu} = \sum_{s=\pm 1} s \mathcal{J}_s^{a\mu}, \quad (6.6)$$

from which we can obtain the chiral currents

$$j_5^{I\mu} = \int d^4 p \mathcal{A}^{I\mu}, \quad j_5^{a\mu} = \int d^4 p \mathcal{A}^{a\mu}. \quad (6.7)$$

The zeroth order equations can be derived trivially from eqs. (4.6), (4.7)

$$\partial_\mu^x \mathcal{A}^{(0)I\mu} = -\frac{1}{N} G_\mu^{(0)a} \mathcal{A}^{(0)a\mu}, \quad (6.8)$$

$$\mathcal{D}_\mu^{ac} \mathcal{A}^{(0)c\mu} = -2G_\mu^{(0)a} \mathcal{A}^{(0)I\mu} - d^{bca} G_\mu^{(0)b} \mathcal{A}^{(0)c\mu}. \quad (6.9)$$

From the expression (6.3), we note that the vacuum contributions in $\mathcal{A}^{(0)I\mu}$ and $\mathcal{A}^{(0)a\mu}$ are all cancelled between $s = +1$ and $s = -1$. Since the right hand sides of the equations above are all total derivatives on momentum and only normal particle distributions are involved, integrating over the 4-momentum leads to the conservation of chiral current at the zeroth order.

$$\partial_\mu^x j_5^{(0)I\mu} = 0, \quad \mathcal{D}_\mu^{ac} j_5^{(0)c\mu} = 0. \quad (6.10)$$

The first order equations can be given from eqs. (4.10), (4.11)

$$0 = \partial_\mu^x \mathcal{A}^{(1)I\mu} + \frac{1}{N} G_\mu^{(0)a} \mathcal{A}^{(1)a\mu}, \quad (6.11)$$

$$0 = \mathcal{D}_\mu^{ac} \mathcal{A}^{(1)c\mu} + 2G_\mu^{(0)a} \mathcal{A}^{(1)I\mu} + d^{bca} G_\mu^{(0)b} \mathcal{A}^{(1)c\mu} + i f^{bca} G_\mu^{(1)b} \mathcal{A}^{(0)c\mu}. \quad (6.12)$$

The right hand sides of these first order equations are still all total derivatives, after integrating over momentum, the only possible nonvanishing contribution is from the singular vacuum term,

$$\partial_\mu^x j_5^{(1)I\mu} = \frac{g^2}{2N} F_{\mu\lambda}^a \tilde{F}^{a,\mu\nu} \int d^4 p \partial_p^\lambda \left[p_\nu f_v^{(0)} \delta'(p^2) \right], \quad (6.13)$$

$$\mathcal{D}_\mu^{ac} j_5^{(1)c\mu} = \frac{g^2}{2} d^{bca} F_{\mu\lambda}^b \tilde{F}^{c,\mu\nu} \int d^4 p \partial_p^\lambda \left[p_\nu f_v^{(0)} \delta'(p^2) \right], \quad (6.14)$$

where $f_v^{(0)}$ represents the vacuum contribution

$$f_v^{(0)} = -\frac{1}{2\pi^3} \theta(-p_0). \quad (6.15)$$

Using the identity

$$F_{\mu\lambda}^a \tilde{F}^{a,\mu\nu} = \frac{1}{4} g_\lambda^\nu \tilde{F}^{\alpha\beta} F_{\alpha\beta} = g_\lambda^\nu E^a \cdot B^a, \quad (6.16)$$

$$d^{bca} F_{\mu\lambda}^b \tilde{F}^{c,\mu\nu} = \frac{1}{4} g_\lambda^\nu d^{bca} \tilde{F}^{b,\alpha\beta} F_{\alpha\beta}^c = g_\lambda^\nu d^{bca} E^b \cdot B^c, \quad (6.17)$$

we have

$$\partial_\mu^x j_5^{(1)I\mu} = \frac{g^2}{2N} E^a \cdot B^a \int d^4 p \partial_p^\lambda \left[p_\lambda f_v^{(0)} \delta'(p^2) \right], \quad (6.18)$$

$$\mathcal{D}_\mu^{ac} j_5^{(1)c,\mu} = \frac{g^2}{2} d^{bca} E^b \cdot B^c \int d^4 p \partial_p^\lambda \left[p_\lambda f_v^{(0)} \delta'(p^2) \right]. \quad (6.19)$$

As in the Abelian case [56, 57], we can finish integrating the momentum

$$C_v = \int d^4 p \partial_p^\lambda \left[p_\lambda f_v^{(0)} \delta'(p^2) \right] \quad (6.20)$$

in 4 dimensional Euclidean momentum space $p_E^\mu = (ip_0, \mathbf{p})$ by Wick rotation

$$C_v = -\frac{1}{2\pi^2} \int \frac{d^4 p_E}{2\pi^2} \partial_\mu \left(\frac{p_E^\mu}{p_E^4} \right) = -\frac{1}{2\pi^2}, \quad (6.21)$$

or 3 dimensional Euclidean momentum space \mathbf{p} after integrating over p_0

$$C_v = -\frac{1}{2\pi^2} \int \frac{d^3 \mathbf{p}}{2\pi} \partial_{\mathbf{p}} \cdot \left(\frac{\hat{\mathbf{p}}}{2\mathbf{p}^2} \right) = -\frac{1}{2\pi^2}, \quad (6.22)$$

where p_E^μ/p_E^4 and $\hat{\mathbf{p}}/2\mathbf{p}^2$ are just the Berry curvature of a 4-dimensional and 3-dimensional monopoles in Euclidean momentum space, respectively. It follows that

$$\partial_\mu^x j_5^{(1)I\mu} = -\frac{g^2}{4\pi^2 N} E^a \cdot B^a, \quad \mathcal{D}_\mu^{ac} j_5^{(1)c\mu} = -\frac{g^2}{4\pi^2} d^{bca} E^b \cdot B^c. \quad (6.23)$$

It is obvious that the non-Abelian chiral anomaly originates from the Berry curvature of the vacuum contribution.

6.2 Non-Abelian anomalous currents

As we all know that the vorticity and magnetic field imposed on a chiral system could induce some novel chiral effects such as chiral magnetic effect, chiral vortical effect and chiral separate effect. In this section, we will derive the chiral effects induced by non-Abelian gauge field. For the zeroth order distribution function in eqs. (6.4), (6.5), we assume the quark and antiquark number density is the global equilibrium Fermi-Dirac distribution

$$n_s^i = \frac{1}{1 + e^{(u \cdot p - \mu_s^i)/T}}, \quad \bar{n}_s^i = \frac{1}{1 + e^{(-u \cdot p + \mu_s^i)/T}}. \quad (6.24)$$

where μ_s^i denotes the chemical potential of the quark with chirality s and color i . The chirality chemical potential μ_s^i is related to the vector chemical potential μ^i and axial chemical

potential μ_5^i by $\mu_s^i = \mu^i + s\mu_5^i$. Now let us impose the covariant-constant field in this chiral system

$$F_{\mu\nu}^a = F_{\mu\nu}\xi^a \quad (6.25)$$

with the color index a only running in the $N - 1$ commuting Cartan generators and ξ^a being $(N - 1)$ -dimensional constant color vector. Since the field tensor $F_{\mu\nu}$ is independent of space and time, the external gauge potential A_μ^a can be chosen as

$$A_\mu^a = -\frac{1}{2}F_{\mu\nu}x^\nu\xi^a. \quad (6.26)$$

It is easy to verify that when the following constraint conditions are satisfied

$$\partial_\mu^x \frac{u_\nu}{T} + \partial_\nu^x \frac{u_\mu}{T} = 0, \quad \partial_\mu^x \frac{\mu_s^i}{T} = g\xi^a t_{ii}^a \frac{E_\mu}{T}, \quad (6.27)$$

the zeroth order Wigner function in (6.4), (6.5) with Fermi-Dirac distribution is indeed the solution of the zeroth order Wigner equations (4.6), (4.7). Once we have a special zeroth order solution, most of the terms in the first order solution are totally fixed by eqs. (4.8) and (4.9) except for the first terms with $f_s^{(1)I} = 0$ or $f_s^{(1)a} = 0$. As shown in ref. [62], we can not causally set $f_s^{(1)I} = 0$ and $f_s^{(1)a} = 0$ because they must be consistent with the transformations (5.11) and (5.12). Substituting these specific solution (6.4), (6.5), (6.24) and conditions (6.27) into the transformations of the first order, we can have

$$\delta(p^2)\delta f_s^{(1)I} = -\delta(p^2)\frac{sn'_\nu\tilde{\Omega}^{\nu\sigma}p_\sigma}{2(n'\cdot p)}\frac{df_s^{(0)I}}{dy} + \delta(p^2)\frac{sn_\nu\tilde{\Omega}^{\nu\sigma}p_\sigma}{2(n\cdot p)}\frac{df_s^{(0)I}}{dy}, \quad (6.28)$$

$$\delta(p^2)\delta f_s^{(1)a} = -\delta(p^2)\frac{sn'_\nu\tilde{\Omega}^{\nu\sigma}p_\sigma}{2(n'\cdot p)}\frac{df_s^{(0)a}}{dy} + \delta(p^2)\frac{sn_\nu\tilde{\Omega}^{\nu\sigma}p_\sigma}{2(n\cdot p)}\frac{df_s^{(0)a}}{dy}, \quad (6.29)$$

where we have defined

$$\Omega_{\mu\nu} = \frac{1}{2}\left(\partial_\mu^x \frac{u_\nu}{T} - \partial_\nu^x \frac{u_\mu}{T}\right), \quad \tilde{\Omega}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\Omega^{\alpha\beta}, \quad y = u\cdot p/T. \quad (6.30)$$

This indicates that we can choose the specific solution which is consistent with the transformations (5.11) and (5.12),

$$f_s^{(1)I} = -\frac{sn_\nu\tilde{\Omega}^{\nu\sigma}p_\sigma}{2(n\cdot p)}\frac{df_s^{(0)I}}{dy}, \quad f_s^{(1)a} = -\frac{sn_\nu\tilde{\Omega}^{\nu\sigma}p_\sigma}{2(n\cdot p)}\frac{df_s^{(0)a}}{dy}. \quad (6.31)$$

Inserting these results into eqs. (4.8) and (4.9) gives rise to

$$\mathcal{J}_s^{(1)I\mu} = -\frac{s}{2}\tilde{\Omega}^{\mu\nu}p_\nu\frac{df_s^{(0)I}}{dy}\delta(p^2) - \frac{sg}{2N}\tilde{F}^{a,\mu\nu}p_\nu f_s^{(0)a}\delta'(p^2), \quad (6.32)$$

$$\mathcal{J}_s^{(1)a,\mu} = -\frac{s}{2}\tilde{\Omega}^{\mu\nu}p_\nu\frac{df_s^{(0)a}}{dy}\delta(p^2) - sg\tilde{F}^{b,\mu\nu}p_\nu\left(\delta^{ab}f_s^{(0)I} + \frac{1}{2}d^{bca}f_s^{(0)c}\right)\delta'(p^2), \quad (6.33)$$

where we have dropped all the terms which vanish when color index runs only in the $N - 1$ commuting Cartan generators. It is obvious that the final expressions do not depend on the auxiliary vector n^μ any more and are explicitly Lorentz covariant.

Now it is straightforward to obtain the right-handed/left-handed currents by integrating the 4-dimension momentum p^μ .

$$j_s^{(1)I\mu} = -\frac{s}{2}\omega^\mu \int d^4p y \frac{df_s^{(0)I}}{dy} \delta(p^2) - \frac{sg}{2N} B^{a\mu} T \int d^4p y f_s^{(0)a} \delta'(p^2), \quad (6.34)$$

$$j_s^{(1)a\mu} = -\frac{s}{2}\omega^\mu \int d^4p y \frac{df_s^{(0)a}}{dy} \delta(p^2) - sg B^{b\mu} T \int d^4p y (\delta^{ab} f_s^{(0)I} + \frac{d^{bca}}{2} f_s^{(0)c}) \delta'(p^2), \quad (6.35)$$

where $\omega^\mu = T\tilde{\Omega}^{\mu\nu}u_\nu = \epsilon^{\mu\nu\alpha\beta}u_\nu\partial_\alpha^x u_\beta/2$. From eqs. (6.4) and (6.5) together with eq. (6.24), we can finish the integrals analytically

$$\int d^4p y \frac{df_s^{(0)I}}{dy} \delta(p^2) = -\frac{T^2}{6} - \frac{\sum_i \mu_s^{i2}}{2\pi^2 N}, \quad T \int d^4p y f_s^{(0)I} \delta'(p^2) = \frac{\sum_i \mu_s^i}{4\pi^2 N}, \quad (6.36)$$

$$\int d^4p y \frac{df_s^{(0)a}}{dy} \delta(p^2) = -\frac{\sum_i t_{ii}^a \mu_s^{i2}}{\pi^2}, \quad T \int d^4p y f_s^{(0)a} \delta'(p^2) = \frac{\sum_i t_{ii}^a \mu_s^i}{2\pi^2}, \quad (6.37)$$

It follows that

$$j_s^{(1)I\mu} = \xi_s^I \omega^\mu + \xi_{B_s}^{Ia} B^{a\mu}, \quad j_s^{(1)a\mu} = \xi_s^a \omega^\mu + \xi_{B_s}^{ab} B^{b\mu} \quad (6.38)$$

where

$$\xi_s^I = s \left(\frac{T^2}{12} + \frac{1}{4\pi^2 N} \sum_i \mu_s^{i2} \right), \quad \xi_{B_s}^{Ia} = -\frac{sg}{4\pi^2 N} \sum_i t_{ii}^a \mu_s^i, \quad (6.39)$$

$$\xi_s^a = \frac{s}{2\pi^2} \sum_i t_{ii}^a \mu_s^{i2}, \quad \xi_{B_s}^{ab} = -\frac{sg}{4\pi^2} \left(\frac{\delta^{ab}}{N} \sum_i \mu_s^i + d^{bca} \sum_i t_{ii}^c \mu_s^i \right). \quad (6.40)$$

The vector current and axial current can be obtained from right-hand and left-hand currents directly,

$$j^{(1)I\mu} = j_{+1}^{(1)I\mu} + j_{-1}^{(1)I\mu} = \xi^I \omega^\mu + \xi_B^{Ia} B^{a\mu}, \quad (6.41)$$

$$j^{(1)a\mu} = j_{+1}^{(1)a\mu} + j_{-1}^{(1)a\mu} = \xi^a \omega^\mu + \xi_B^{ab} B^{b\mu}, \quad (6.42)$$

$$j_5^{(1)I\mu} = j_{+1}^{(1)I\mu} - j_{-1}^{(1)I\mu} = \xi_5^I \omega^\mu + \xi_{B_5}^{Ia} B^{a\mu}, \quad (6.43)$$

$$j_5^{(1)a\mu} = j_{+1}^{(1)a\mu} - j_{-1}^{(1)a\mu} = \xi_5^a \omega^\mu + \xi_{B_5}^{ab} B^{b\mu}, \quad (6.44)$$

where the anomalous transport coefficients for the vector currents are given by

$$\xi^I = \frac{1}{\pi^2 N} \sum_i \mu^i \mu_5^i, \quad \xi_B^{Ia} = -\frac{g}{2\pi^2 N} \sum_i t_{ii}^a \mu_5^i, \quad (6.45)$$

$$\xi^a = \frac{2}{\pi^2} \sum_i t_{ii}^a \mu^i \mu_5^i, \quad \xi_B^{ab} = -\frac{g}{2\pi^2} \left(\frac{\delta^{ab}}{N} \sum_i \mu_5^i + d^{bca} \sum_i t_{ii}^c \mu_5^i \right) \quad (6.46)$$

and the coefficients for the axial currents are given by

$$\xi_5^I = \frac{T^2}{6} + \frac{1}{2\pi^2 N} \sum_i (\mu^{i2} + \mu_5^{i2}), \quad \xi_{B_5}^{Ia} = -\frac{g}{2\pi^2 N} \sum_i t_{ii}^a \mu^i, \quad (6.47)$$

$$\xi_5^a = \frac{1}{\pi^2} \sum_i t_{ii}^a (\mu^{i2} + \mu_5^{i2}), \quad \xi_{B_5}^{ab} = -\frac{g}{2\pi^2} \left(\frac{\delta^{ab}}{N} \sum_i \mu^i + d^{bca} \sum_i t_{ii}^c \mu^i \right). \quad (6.48)$$

These are just the non-Abelian counterparts of the chiral magnetic effect, chiral vortical effect and chiral separation effect. We note that the coefficients ξ^I , ξ_B^{Ia} , ξ_5^I and ξ_{B5}^{Ia} for the singlet current are very similar to the coefficients in the Abelian case. They can be regarded as the average value of the coefficient in Abelian currents over different colors. These results will reduce into the usual Abelian chiral effects if we set $N = 1$, $t_{ii}^a = 1$ and $g = -1$. The coefficients ξ^a , ξ_B^{ab} , ξ_5^a and ξ_{B5}^{ab} are unique for the non-Abelian currents and agree with the results obtained in different approaches in refs. [63–65].

7 Summary

In this paper, we generalize the chiral kinetic theory in Abelian gauge field to non-Abelian gauge field. Starting from the gauge invariant and Lorentz invariant quantum transport theory set up in [51–54, 58], we decompose the Wigner functions and Wigner equations completely both in spinor space and in color space. With the help of the “covariant gradient expansion”, we find that the right-handed and left-handed Wigner function are totally decoupled with all the other Wigner functions. Among the four components of right-handed or left-handed Wigner functions, we can define the time-like component as the independent Wigner function and regard it as the phase space particle distribution function in some reference frame with velocity n^μ . In consequence, all the space-like components can be totally determined by this chosen independent distribution function. Such disentangling process simplifies the Wigner equations greatly. The difference between Abelian and non-Abelian gauge field is that in Abelian gauge field the disentanglement theorem demonstrated in [29] show that the transport equation for space-like components are automatically satisfied while in non-Abelian gauge field these equations are not satisfied automatically order by order and we obtain extra constraint conditions. We present the chiral kinetic equations up to the first order in non-Abelian gauge field in 8-dimension phase space. Since the kinetic equations of the singlet component and multiplet components are totally coupled with each other, the non-Abelian chiral kinetic equation is much more complicated than Abelian chiral kinetic equation. We also give the modified Lorentz transformation of the non-Abelian phase space distribution function when we define them in different frames. Finally, we utilize it to calculate the non-Abelian chiral anomaly and the vector and axial currents induced by color field and vorticity and find that it is consistent and successful in describing the chiral effects in non-Abelian gauge field.

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References

- [1] A. Vilenkin, *Equilibrium parity violating current in a magnetic field*, *Phys. Rev. D* **22** (1980) 3080 [[INSPIRE](#)].
- [2] D.E. Kharzeev, L.D. McLerran and H.J. Warringa, *The effects of topological charge change in heavy ion collisions: ‘event by event P and CP-violation’*, *Nucl. Phys. A* **803** (2008) 227 [[arXiv:0711.0950](#)] [[INSPIRE](#)].
- [3] K. Fukushima, D.E. Kharzeev and H.J. Warringa, *The chiral magnetic effect*, *Phys. Rev. D* **78** (2008) 074033 [[arXiv:0808.3382](#)] [[INSPIRE](#)].
- [4] A. Vilenkin, *Parity violating currents in thermal radiation*, *Phys. Lett. B* **80** (1978) 150 [[INSPIRE](#)].
- [5] D. Kharzeev and A. Zhitnitsky, *Charge separation induced by P-odd bubbles in QCD matter*, *Nucl. Phys. A* **797** (2007) 67 [[arXiv:0706.1026](#)] [[INSPIRE](#)].
- [6] J. Erdmenger, M. Haack, M. Kaminski and A. Yarom, *Fluid dynamics of R-charged black holes*, *JHEP* **01** (2009) 055 [[arXiv:0809.2488](#)] [[INSPIRE](#)].
- [7] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam and P. Surowka, *Hydrodynamics from charged black branes*, *JHEP* **01** (2011) 094 [[arXiv:0809.2596](#)] [[INSPIRE](#)].
- [8] D.T. Son and A.R. Zhitnitsky, *Quantum anomalies in dense matter*, *Phys. Rev. D* **70** (2004) 074018 [[hep-ph/0405216](#)] [[INSPIRE](#)].
- [9] M.A. Metlitski and A.R. Zhitnitsky, *Anomalous axion interactions and topological currents in dense matter*, *Phys. Rev. D* **72** (2005) 045011 [[hep-ph/0505072](#)] [[INSPIRE](#)].
- [10] D.E. Kharzeev, *The chiral magnetic effect and anomaly-induced transport*, *Prog. Part. Nucl. Phys.* **75** (2014) 133 [[arXiv:1312.3348](#)] [[INSPIRE](#)].
- [11] D.E. Kharzeev, J. Liao, S.A. Voloshin and G. Wang, *Chiral magnetic and vortical effects in high-energy nuclear collisions — a status report*, *Prog. Part. Nucl. Phys.* **88** (2016) 1 [[arXiv:1511.04050](#)] [[INSPIRE](#)].
- [12] Y.-C. Liu and X.-G. Huang, *Anomalous chiral transports and spin polarization in heavy-ion collisions*, *Nucl. Sci. Tech.* **31** (2020) 56 [[arXiv:2003.12482](#)] [[INSPIRE](#)].
- [13] J.-H. Gao, G.-L. Ma, S. Pu and Q. Wang, *Recent developments in chiral and spin polarization effects in heavy-ion collisions*, *Nucl. Sci. Tech.* **31** (2020) 90 [[arXiv:2005.10432](#)] [[INSPIRE](#)].
- [14] J.-H. Gao, Z.-T. Liang and Q. Wang, *Quantum kinetic theory for spin-1/2 fermions in Wigner function formalism*, *Int. J. Mod. Phys. A* **36** (2021) 2130001 [[arXiv:2011.02629](#)] [[INSPIRE](#)].
- [15] C. Duval, Z. Horvath, P.A. Horvathy, L. Martina and P. Stichel, *Berry phase correction to electron density in solids and ‘exotic’ dynamics*, *Mod. Phys. Lett. B* **20** (2006) 373 [[cond-mat/0506051](#)] [[INSPIRE](#)].
- [16] C.H. Wong and Y. Tserkovnyak, *Quantum kinetic equation in phase-space textured multiband systems*, *Phys. Rev. B* **84** (2011) 115209 [[arXiv:1102.1121](#)] [[INSPIRE](#)].
- [17] D.T. Son and N. Yamamoto, *Berry curvature, triangle anomalies, and the chiral magnetic effect in Fermi liquids*, *Phys. Rev. Lett.* **109** (2012) 181602 [[arXiv:1203.2697](#)] [[INSPIRE](#)].

- [18] M.A. Stephanov and Y. Yin, *Chiral kinetic theory*, *Phys. Rev. Lett.* **109** (2012) 162001 [[arXiv:1207.0747](#)] [[INSPIRE](#)].
- [19] M. Stone and V. Dwivedi, *Classical version of the non-Abelian gauge anomaly*, *Phys. Rev. D* **88** (2013) 045012 [[arXiv:1305.1955](#)] [[INSPIRE](#)].
- [20] V. Dwivedi and M. Stone, *Classical chiral kinetic theory and anomalies in even space-time dimensions*, *J. Phys. A* **47** (2013) 025401 [[arXiv:1308.4576](#)] [[INSPIRE](#)].
- [21] Y. Akamatsu and N. Yamamoto, *Chiral Langevin theory for non-Abelian plasmas*, *Phys. Rev. D* **90** (2014) 125031 [[arXiv:1402.4174](#)] [[INSPIRE](#)].
- [22] J.-Y. Chen, D.T. Son, M.A. Stephanov, H.-U. Yee and Y. Yin, *Lorentz invariance in chiral kinetic theory*, *Phys. Rev. Lett.* **113** (2014) 182302 [[arXiv:1404.5963](#)] [[INSPIRE](#)].
- [23] C. Manuel and J.M. Torres-Rincon, *Chiral transport equation from the quantum Dirac Hamiltonian and the on-shell effective field theory*, *Phys. Rev. D* **90** (2014) 076007 [[arXiv:1404.6409](#)] [[INSPIRE](#)].
- [24] T. Hayata and Y. Hidaka, *Kinetic theory and anomalous transports in the presence of non-Abelian phase-space Berry curvatures*, *PTEP* **2017** (2017) 073I01 [[arXiv:1701.04012](#)] [[INSPIRE](#)].
- [25] J.-H. Gao, Z.-T. Liang, S. Pu, Q. Wang and X.-N. Wang, *Chiral anomaly and local polarization effect from quantum kinetic approach*, *Phys. Rev. Lett.* **109** (2012) 232301 [[arXiv:1203.0725](#)] [[INSPIRE](#)].
- [26] J.-W. Chen, S. Pu, Q. Wang and X.-N. Wang, *Berry curvature and four-dimensional monopoles in the relativistic chiral kinetic equation*, *Phys. Rev. Lett.* **110** (2013) 262301 [[arXiv:1210.8312](#)] [[INSPIRE](#)].
- [27] Y. Hidaka, S. Pu and D.-L. Yang, *Relativistic chiral kinetic theory from quantum field theories*, *Phys. Rev. D* **95** (2017) 091901 [[arXiv:1612.04630](#)] [[INSPIRE](#)].
- [28] A. Huang, S. Shi, Y. Jiang, J. Liao and P. Zhuang, *Complete and consistent chiral transport from Wigner function formalism*, *Phys. Rev. D* **98** (2018) 036010 [[arXiv:1801.03640](#)] [[INSPIRE](#)].
- [29] J.-H. Gao, Z.-T. Liang, Q. Wang and X.-N. Wang, *Disentangling covariant Wigner functions for chiral fermions*, *Phys. Rev. D* **98** (2018) 036019 [[arXiv:1802.06216](#)] [[INSPIRE](#)].
- [30] Y.-C. Liu, L.-L. Gao, K. Mameda and X.-G. Huang, *Chiral kinetic theory in curved spacetime*, *Phys. Rev. D* **99** (2019) 085014 [[arXiv:1812.10127](#)] [[INSPIRE](#)].
- [31] D.T. Son and N. Yamamoto, *Kinetic theory with Berry curvature from quantum field theories*, *Phys. Rev. D* **87** (2013) 085016 [[arXiv:1210.8158](#)] [[INSPIRE](#)].
- [32] S. Carignano, C. Manuel and J.M. Torres-Rincon, *Consistent relativistic chiral kinetic theory: a derivation from on-shell effective field theory*, *Phys. Rev. D* **98** (2018) 076005 [[arXiv:1806.01684](#)] [[INSPIRE](#)].
- [33] S. Lin and A. Shukla, *Chiral kinetic theory from effective field theory revisited*, *JHEP* **06** (2019) 060 [[arXiv:1901.01528](#)] [[INSPIRE](#)].
- [34] S. Carignano, C. Manuel and J.M. Torres-Rincon, *Chiral kinetic theory from the on-shell effective field theory: derivation of collision terms*, *Phys. Rev. D* **102** (2020) 016003 [[arXiv:1908.00561](#)] [[INSPIRE](#)].

- [35] N. Mueller and R. Venugopalan, *The chiral anomaly, Berry's phase and chiral kinetic theory, from world-lines in quantum field theory*, *Phys. Rev. D* **97** (2018) 051901 [[arXiv:1701.03331](#)] [[INSPIRE](#)].
- [36] N. Mueller and R. Venugopalan, *Worldline construction of a covariant chiral kinetic theory*, *Phys. Rev. D* **96** (2017) 016023 [[arXiv:1702.01233](#)] [[INSPIRE](#)].
- [37] N. Mueller and R. Venugopalan, *Constructing phase space distributions with internal symmetries*, *Phys. Rev. D* **99** (2019) 056003 [[arXiv:1901.10492](#)] [[INSPIRE](#)].
- [38] Y. Sun, C.M. Ko and F. Li, *Anomalous transport model study of chiral magnetic effects in heavy ion collisions*, *Phys. Rev. C* **94** (2016) 045204 [[arXiv:1606.05627](#)] [[INSPIRE](#)].
- [39] Y. Sun and C.M. Ko, *Chiral vortical and magnetic effects in the anomalous transport model*, *Phys. Rev. C* **95** (2017) 034909 [[arXiv:1612.02408](#)] [[INSPIRE](#)].
- [40] Y. Sun and C.M. Ko, *Λ hyperon polarization in relativistic heavy ion collisions from a chiral kinetic approach*, *Phys. Rev. C* **96** (2017) 024906 [[arXiv:1706.09467](#)] [[INSPIRE](#)].
- [41] Y. Sun and C.M. Ko, *Chiral kinetic approach to the chiral magnetic effect in isobaric collisions*, *Phys. Rev. C* **98** (2018) 014911 [[arXiv:1803.06043](#)] [[INSPIRE](#)].
- [42] Y. Sun and C.M. Ko, *Azimuthal angle dependence of the longitudinal spin polarization in relativistic heavy ion collisions*, *Phys. Rev. C* **99** (2019) 011903 [[arXiv:1810.10359](#)] [[INSPIRE](#)].
- [43] W.-H. Zhou and J. Xu, *Simulating the chiral magnetic wave in a box system*, *Phys. Rev. C* **98** (2018) 044904 [[arXiv:1810.01030](#)] [[INSPIRE](#)].
- [44] W.-H. Zhou and J. Xu, *Simulating chiral anomalies with spin dynamics*, *Phys. Lett. B* **798** (2019) 134932 [[arXiv:1904.01834](#)] [[INSPIRE](#)].
- [45] S.Y.F. Liu, Y. Sun and C.M. Ko, *Spin polarizations in a covariant angular-momentum-conserved chiral transport model*, *Phys. Rev. Lett.* **125** (2020) 062301 [[arXiv:1910.06774](#)] [[INSPIRE](#)].
- [46] L.V. Gribov, E.M. Levin and M.G. Ryskin, *Semihard processes in QCD*, *Phys. Rept.* **100** (1983) 1 [[INSPIRE](#)].
- [47] A.H. Mueller and J.-W. Qiu, *Gluon recombination and shadowing at small values of x* , *Nucl. Phys. B* **268** (1986) 427 [[INSPIRE](#)].
- [48] L.D. McLerran and R. Venugopalan, *Gluon distribution functions for very large nuclei at small transverse momentum*, *Phys. Rev. D* **49** (1994) 3352 [[hep-ph/9311205](#)] [[INSPIRE](#)].
- [49] L.D. McLerran and R. Venugopalan, *Green's functions in the color field of a large nucleus*, *Phys. Rev. D* **50** (1994) 2225 [[hep-ph/9402335](#)] [[INSPIRE](#)].
- [50] E. Iancu and R. Venugopalan, *The color glass condensate and high-energy scattering in QCD*, in *Quark-gluon plasma 4*, R.C. Hwa and X.-N. Wang eds., **World Scientific**, Singapore (2003), pg. 249 [[hep-ph/0303204](#)] [[INSPIRE](#)].
- [51] U.W. Heinz, *Kinetic theory for non-Abelian plasmas*, *Phys. Rev. Lett.* **51** (1983) 351 [[INSPIRE](#)].
- [52] H.T. Elze, M. Gyulassy and D. Vasak, *Transport equations for the QCD gluon Wigner operator*, *Phys. Lett. B* **177** (1986) 402 [[INSPIRE](#)].
- [53] H.T. Elze, M. Gyulassy and D. Vasak, *Transport equations for the QCD quark Wigner operator*, *Nucl. Phys. B* **276** (1986) 706 [[INSPIRE](#)].

- [54] H.-T. Elze and U.W. Heinz, *Quark-gluon transport theory*, *Phys. Rept.* **183** (1989) 81 [[INSPIRE](#)].
- [55] S.R. De Groot, W.A. Van Leeuwen and C.G. Van Weert, *Relativistic kinetic theory*, North-Holland, Amsterdam, The Netherlands (1980).
- [56] J.-H. Gao, Z.-T. Liang and Q. Wang, *Dirac sea and chiral anomaly in the quantum kinetic theory*, *Phys. Rev. D* **101** (2020) 096015 [[arXiv:1910.11060](#)] [[INSPIRE](#)].
- [57] R.-H. Fang and J.-H. Gao, *Chiral anomaly, Dirac sea and Berry monopole in Wigner function approach*, *Nucl. Phys. A* **1005** (2021) 121851 [[arXiv:2002.04800](#)] [[INSPIRE](#)].
- [58] S. Ochs and U.W. Heinz, *Wigner functions in covariant and single time formulations*, *Annals Phys.* **266** (1998) 351 [[hep-th/9806118](#)] [[INSPIRE](#)].
- [59] X.-G. Huang, P. Mitkin, A.V. Sadofyev and E. Speranza, *Zilch vortical effect, Berry phase, and kinetic theory*, *JHEP* **10** (2020) 117 [[arXiv:2006.03591](#)] [[INSPIRE](#)].
- [60] K. Hattori, Y. Hidaka, N. Yamamoto and D.-L. Yang, *Wigner functions and quantum kinetic theory of polarized photons*, *JHEP* **02** (2021) 001 [[arXiv:2010.13368](#)] [[INSPIRE](#)].
- [61] S. Ochs and U.W. Heinz, *Wigner functions in covariant and single time formulations*, *Annals Phys.* **266** (1998) 351 [[hep-th/9806118](#)] [[INSPIRE](#)].
- [62] J.-H. Gao, J.-Y. Pang and Q. Wang, *Chiral vortical effect in Wigner function approach*, *Phys. Rev. D* **100** (2019) 016008 [[arXiv:1810.02028](#)] [[INSPIRE](#)].
- [63] D.T. Son and P. Surowka, *Hydrodynamics with triangle anomalies*, *Phys. Rev. Lett.* **103** (2009) 191601 [[arXiv:0906.5044](#)] [[INSPIRE](#)].
- [64] Y. Neiman and Y. Oz, *Relativistic hydrodynamics with general anomalous charges*, *JHEP* **03** (2011) 023 [[arXiv:1011.5107](#)] [[INSPIRE](#)].
- [65] K. Landsteiner, E. Megias and F. Pena-Benitez, *Gravitational anomaly and transport*, *Phys. Rev. Lett.* **107** (2011) 021601 [[arXiv:1103.5006](#)] [[INSPIRE](#)].