

Pauli-Lubanski limit and stress-energy tensor for infinite-spin fields

Karl-Henning Rehren

*Institute for Theoretical Physics, Georg-August-University Göttingen,
Friedrich-Hund-Platz 1, 37077 Göttingen, Germany*

E-mail: rehren@theorie.physik.uni-goettingen.de

ABSTRACT: String-localized quantum fields transforming in Wigner's infinite-spin representations were originally introduced in [18, 19]. We construct these fields as limits of fields of finite mass $m \rightarrow 0$ and finite spin $s \rightarrow \infty$. We determine a string-localized infinite-spin quantum stress-energy tensor with a novel prescription that does not refer to a classical Lagrangean.

KEYWORDS: Higher Spin Symmetry, Space-Time Symmetries

ARXIV EPRINT: [1709.04858](https://arxiv.org/abs/1709.04858)

Dedicated to Klaus Fredenhagen on the occasion of his 70th birthday

Contents

1	Introduction	1
1.1	Quantum fields in the infinite-spin representations	1
1.2	Contents and plan of the paper	5
2	String-localized infinite-spin fields	6
2.1	The scalar standard field	8
2.2	Tensor fields I	10
2.3	Tensor fields II	11
3	Preparations for the Pauli-Lubanski limit	13
3.1	Two-point functions and commutators: basics	13
3.2	Two-point functions of string-localized spin 1 fields	14
3.3	Two-point functions of infinite-spin fields	15
3.4	Massive fields of finite spin: definitions and properties	16
4	The Pauli-Lubanski limit	18
5	Stress-energy tensors	21
5.1	Quantum stress-energy tensors	22
5.2	Proof of Proposition 5.1	25
5.3	Properties of the infinite-spin stress-energy tensor	27
5.3.1	Matrix elements	27
5.3.2	Two-point functions and vacuum fluctuations	28
5.3.3	Commutators	29
5.3.4	Thermal states: equation of state and equipartition	30
6	Conclusion	31
A	Pauli-Lubanski limit of Wigner representations	32

1 Introduction**1.1 Quantum fields in the infinite-spin representations**

Starting from the posit that one-particles states look like one-particle states in every inertial frame, Wigner concluded that particles should be identified with unitary positive-energy representations of the (proper orthochronous) Poincaré group or its twofold covering. His famous classification [32] contains massive spin representations, massless helicity representations, and two one-parameter families of true resp. projective massless representations

called “infinite spin” or “continuous spin”. The parameter $\kappa^2 > 0$ is the eigenvalue of the Pauli-Lubanski operator $W^2 = (P \wedge M)^2$ (P_ρ is the momentum and $M_{\sigma\tau}$ are the Lorentz generators).

Weinberg [28–30] showed how one can associate local (or anti-local, in the projective case according to the spin-statistics theorem) quantum fields to all these representations except infinite spin. Let us from now on consider only the true (bosonic) representations.

In the massless case, only the field strengths can be constructed as local fields on the Fock space over the sum of Wigner representations with helicities $\pm h$ (which is irreducible when the parity is included). Their potentials necessarily violate either locality and covariance (e.g., in the Coulomb gauge) [28–30], or they must be constructed on an indefinite-metric Krein space of which the Fock space over the Wigner representation is a quotient. The latter option underlies all gauge-theoretic approaches to modern quantum field theory, with its introduction of more unphysical ghost fields to “compensate” states of negative probability.

For the irreducible infinite-spin representations, Yngvason [34] has shown a no-go theorem, that covariant local Wightman fields cannot exist. This was taken for a long time as a serious reason to consider these representations as “unphysical”.

This conclusion is, however, a misunderstanding. From work of Buchholz and Fredenhagen [5] we know that quantum operators connecting scattering states with the vacuum may not in general be assumed to be localized in bounded spacetime regions. Instead, the best localization that can be proven is (a narrow “spacelike cone” arising by smearing) a “string” $S_e(x) = \{x + se : s \geq 0\}$ ($e^2 = -1$). So string-localization of interpolating fields may be a necessary feature of charged states in interacting theories, and only observables need to be point-localized. While the result in [5] applies to massive theories, it has a massless counterpart in the violation of Lorentz covariance of charged sectors of QED [9], due to the presence of “photon clouds” attached to charged fields that can — because of Gauß’ law — at best be localized in a narrow cone.

String-localization surprisingly emerged in the context of infinite-spin representations, when Mund, Schroer and Yngvason [18], based on work by Brunetti, Guido and Longo [4], discovered a construction of fields $\varphi(e, x)$ transforming in the infinite-spin representation, that are localized along strings. Two such fields commute with each other whenever their strings $S_e(x)$ and $S_{e'}(x')$ are spacelike separated.

The same authors also noticed that string-localized free fields may also be useful in the case of finite spin: e.g., one can construct a potential $A_\mu(e, x)$ of the Maxwell field strength $F_{\mu\nu}(x)$ directly on the ghost-free Fock space of the latter. This potential is manifestly presented as a string-integral over the field strength:

$$A_\mu(e, x) := \int_0^\infty ds F_{\mu\nu}(x + se)e^\nu. \tag{1.1}$$

Similar constructions are possible for any mass and finite spin ([20] and [16] for the massless case, eq. (3.11) below and [16] for the massive case). Such potentials have the benefit [22, 23] that

- (i) in the massive case: they have an improved short-distance behaviour compared to the point-localized Proca potentials, which is a promising feature when the free fields are used perturbatively to set up an interacting theory; and unlike the latter, they admit a massless limit;
- (ii) in the massless case: they are directly defined on the physical Hilbert space without the need to introduce Gupta-Bleuler conditions or compensating ghost fields.

These potentials were systematically studied for every integer spin s in [20], and in [16, 17] with the focus on their massless limit at fixed s , and on the discrepancy between the massless string-localized potentials and the massless limit of the massive string-localized potentials.

In contrast to the string-localized fields of finite spin like eq. (1.1), the infinite-spin fields of [18, 19] are “intrinsically string-localized”: they cannot be expressed as string-integrals over point-localized fields. This makes them quite non-trivial objects to study.

Also from different points of view, there is a lot of renewed interest in the infinite-spin representations. Schuster and Toro [25, 26] and Rivelles [21] study quantum wave equations in a one-particle setting. Also their wave functions depend on an auxiliary four-vector e which has, however, no direct geometric interpretation. “Localization” in the sense of causal commutators has no meaning in a one-particle (quantum-mechanical) approach. In [27], Schuster and Toro write down canonical commutation relations which are local both in x and e , and it is not clear to the author how they realize such commutation relations in a Hilbert space — which would be at variance with Yngvason’s no-go theorem.

Bekaert et al. [1–3] are pursuing a “Fierz-Pauli program” attempting to identify a classical action principle leading to wave equations compatible with the infinite-spin representation. With quantization beyond the scope of this program, the constraints due to Hilbert space positivity and causal commutation relations (addressed in, e.g., [16, 24]) play no role in their work.

In contrast, our work is placed in the setting of “Wigner quantization”, where free fields are associated with a unitary Wigner representation by

$$\phi_M(x) = \int d\mu_0(p) \sum_n [u_{Mn}(p)(\vec{k}) a_n^*(p) e^{ipx} + v_{Mn}(p) a_n(p, \vec{k}) e^{-ipx}], \quad (1.2)$$

and the matrices u and v are “intertwiners” (see eq. (5.7)) between the unitary Wigner representation of the Lorentz group and the matrix representation (typically a tensor product of Lorentz matrices) under which the field transforms. They are needed to absorb the “Wigner rotations” that would otherwise spoil the covariant transformation law. For hermitean Bose fields, $v(p) = \overline{u(p)}$.

In this setting, Hilbert space positivity is manifest from the outset because eq. (1.2) is defined on the Fock space over the Wigner representation. Field equations, two-point functions and commutation relations follow without the need of a variational principle and “canonical” equal time commutation relations, i.e., they follow intrinsically (except for the choice of the localizing intertwiner functions) from Wigner representation theory. In this

setting, the non-existence of a covariant quantum Maxwell potential on the physical Hilbert space is just the fact that the intertwining relation has no solution.

This approach offers also an important new flexibility in perturbation theory [24]. While the Wigner representation fixes all properties of the *particles*, the choice of intertwiners determines (among other things) the short-distance behaviour of free *fields* that create these particles from the vacuum. Since UV singularities are a major problem of perturbation theory, one can benefit from the fact that string-localized fields have better UV properties and therefore admit renormalizable couplings that do not exist with point-localized fields (if the latter exist at all on a ghost-free Hilbert space). The case of massive vector bosons may be taken as an example: in order to control the UV problems of point-localized massive vector fields, the prevalent prescription treats them as massless fields in an indefinite-metric Hilbert space which “behave as if they were massive” thanks to the Higgs mechanism. In the string-localized approach, one may instead start with massive vector bosons and interpolating fields in their physical Hilbert space from the outset. (The Higgs field is still needed, but for a different reason, see below.)

In all cases of interactions mediated by string-localized fields, one has to observe that the obvious hazard of violating causality through the use of string-localized Lagrangeans in perturbation theory, can be controlled in terms of a certain cohomological “pair condition” [24] on the interaction terms: it secures the string-independence of the classical action, and is the first order condition for the string-independence of the perturbative quantum causal S-matrix. Higher order conditions may require additional “induced” interaction terms.

E.g., in the presence of selfinteracting massive vectormesons one needs an additional coupling of vectormesons to an Hermitian (Higgs) field in order to uphold second order renormalizability. (Such compensations between fields with different spins have hitherto been expected to take place in the presence of supersymmetry; but whereas there are serious problems to maintain supersymmetry in second order, in the case of selfinteracting massive vector mesons this compensation is the very *raison d’être* for the Higgs particle — without invoking a mechanism of spontaneous symmetry breaking, and without the need of unphysical ghost degrees of freedom.)

In the case of infinite-spin particles, the use of string-localized fields is not a choice (to improve the UV behaviour) but an intrinsic necessity [4]. Whether a pair condition can be fulfilled for any interaction with ordinary particles, is presently unknown. Schroer [24] discusses indications why this might not be the case (for infinite spin or already for some finite spin beyond a maximal value). As a consequence, these particles would be invisible in detectors (“inert”); the identification of a stress-energy tensor in the present work may be a starting point in order to investigate whether they might at least cause semiclassical gravitational back reactions.

The reader only interested in the infinite-spin stress-energy tensor, may jump directly to section 5, retaining from the preceding sections only the properties Proposition 2.9 of the string-localized fields. These properties, although derived through the Pauli-Lubanski limit of finite mass and finite spin, refer directly to the Fock space over the infinite-spin representation, so that the construction of the stress-energy tensor is intrinsic.

1.2 Contents and plan of the paper

Pauli-Lubanski limit. We study in section 4, how the string-localized infinite-spin fields of [18] are approximated by massive string-localized fields of finite spin, in the “Pauli-Lubanski limit” $s \rightarrow \infty$ at fixed Pauli-Lubanski parameter $\kappa^2 = m^2 \cdot s(s+1)$. This limit is suggested by the fact that the Pauli-Lubanski operator $W^2 = (P \wedge M)^2$ is a Casimir operator of the Lie algebra of the Poincaré group with eigenvalue $m^2 \cdot s(s+1)$ in the representation (m, s) , and with eigenvalue κ^2 in the massless infinite-spin representation U_κ^1 .

The Pauli-Lubanski limit is well known for the Wigner representations themselves [14], basically because the massless little group $E(2)$ [32] is a contraction of the massive little group $SO(3)$, see appendix A. In the infinite-spin representations, the pseudo-translations (the subgroup $\mathbb{R}^2 \subset E(2)$ embedded into the proper orthochronous Lorentz group $SO(1, 3)_+^\uparrow$) are non-trivially represented with spectrum of their generators lying on a circle of radius κ . The corresponding basis of eigenfunctions $e^{im\varphi}$ of the rotations of the little group ($m \in \mathbb{Z}$ is the magnetic quantum number) can be approximated by the eigenfunctions Y_{lm} ($l = s \rightarrow \infty, -l \leq m \leq l$) of the finite-spin representations of $SO(3)$, see appendix A.

But a “lift” of the limit of representations to the associated quantum fields is not known so far. An obvious obstruction seems to be that the (conserved and traceless) Proca potentials have a number of indices increasing with s , so that they are not even candidates for a “converging family of fields”. Another obvious obstruction is that a limit of local commutator functions, if it exists, should be a local, and not a string-local commutator function that we know to be the best possible thing for infinite spin. But the most important obstruction is the singularity of the Proca potentials at $m = 0$ which become stronger with increasing spin. It is related to the non-existence of point-localized currents and stress-energy tensors for the massless representations of finite helicity [31].

With string-localized massive potentials, these obstructions are absent. The potentials are manifestly string-localized from the beginning, they are regular at $m = 0$, and they are neither traceless nor conserved, so that one may consider their divergences (called “escort fields”) of fixed rank as natural candidates for converging families. Indeed this will turn out to be true, see section 4.

This finding is a bit surprising. In [16], we have found that the “scalar escort field” converges in the massless limit at fixed s to a true massless scalar field, while we are now claiming that in the Pauli-Lubanski limit, it converges to an infinite-spin field! Indeed, there is no contradiction. At finite mass, the scalar escort is coupled to the other escorts of any rank $r \leq s$ by field equations (eq. (3.12), eq. (3.15)), and each escort carries the entire spin s representation. This coupling goes to zero in the massless limit at fixed s . But as our results implicitly show, it “remains stable” when s increases at the same time, so that the limit field carries the entire infinite-spin representation.

Stress-energy tensor. In [16], we have constructed currents and stress-energy tensors for the massive finite spin representations that have a regular massless limit at fixed s . In the second part of our paper (section 5), we present a general construction that produces string-localized such densities also for the infinite-spin representations, and elucidate whether these exist as Wightman fields. In fact, this is expected not to be the case: their

vacuum two-point functions are expected to diverge due to the infinitely many inner degrees of freedom that are summed over, and we give indications that this is indeed the case. While the vacuum two-point function is tedious to compute exactly, it is very easy to compute the thermal one-point function in KMS states. Here, the expected divergence [33] proportional to $2s + 1$ (in accord with the thermodynamical equipartition theorem) can be explicitly seen.

On the other hand, the commutator of the densities with the fields is a derivation that integrates to the infinitesimal gauge or Poincaré transformations. Because the latter are meaningful also at infinite spin, we expect the limit of the densities to exist at least “as derivations” on the algebra of fields.

Studying the existence and properties of currents and stress-energy tensors for the infinite-spin representations is of great interest, because even a classical Lagrangean from which these could possibly be derived, is not known (see the “Fierz-Pauli program” of [3]). The intricacies of the quantum field theory, due to the conflict between Hilbert space positivity and causal point-localization, can only be overcome with string-localized fields.

In section 2, we review the essential features of string-localized infinite-spin fields and introduce the special fields that will appear in the stress-energy tensor. Section 3 prepares the ground for the Pauli-Lubanski limit. After these preparations, the initial main result of section 4, Proposition 4.3, which entails everything else, is very quickly obtained.

2 String-localized infinite-spin fields

The authors of [18, 19] constructed string-localized fields $\phi(e, x)$ on the Fock space over the infinite-spin representation U_κ^1 that transform like

$$U_\kappa(a, \Lambda)\phi(e, x)U_\kappa(a, \Lambda)^* = \phi(\Lambda e, \Lambda x + a). \tag{2.1}$$

The action of $U_\kappa^1(a, \Lambda)$ on the one-particle space is constructed, as in Wigner’s original approach [32], by induction from a representation d_κ of the stabilizer group $E(2)$ of the reference four-momentum $p_0 = (1, 0, 0, 1)^t$, and a family of Lorentz transformations B_p for every $p \in H_0 = \{p \in \mathbb{R}^4 : p^2 = 0, p^0 > 0\}$, such that $B_p p_0 = p$. The representation space $\mathcal{H}_\kappa = L^2(\kappa S^1)$ of d_κ are the square-integrable functions of a two-dimensional vector \vec{k} , $\vec{k}^2 = \kappa^2$. On such functions, the rotations and pseudo-translations act like

$$(d_\kappa(R_\alpha)f)(\vec{k}) = f(R_{-\alpha}\vec{k}), \quad (d_\kappa(T_{\vec{a}})f)(\vec{k}) = e^{i\vec{a}\cdot\vec{k}}f(\vec{k}). \tag{2.2}$$

The irreducible one-particle representation U_κ^1 induced from d_κ is defined on square-integrable functions on the zero mass shell H_0 with values in \mathcal{H}_κ , see appendix A. It immediately lifts to the representation U_κ on the Fock space. The construction of hermitean fields out of creation and annihilation operators $a(p, \vec{k})$, $a^*(p, \vec{k})$ ($p \in H_0$, $\vec{k} \in \kappa S^1$) then proceeds in terms of “intertwiners” $u(e, p)$:

$$\phi_u(e, x) = \int d\mu_0(p) \int d\mu_\kappa(\vec{k}) [u(e, p)(\vec{k})a^*(p, \vec{k})e^{ipx} + \overline{u(e, p)(\vec{k})}a(p, \vec{k})e^{-ipx}] \tag{2.3}$$

The intertwiners are distributions in p and e with values in \mathcal{H}_κ , that satisfy

$$u(\Lambda e, \Lambda p) = d_\kappa(W_{\Lambda,p})u(e, p) \quad (\Lambda \in \text{SO}(1, 3)_+^\uparrow), \quad (2.4)$$

where $W_{\Lambda,p} = B_{\Lambda p}^{-1}\Lambda B_p \in \text{E}(2)$ is the Wigner “rotation”.¹ This property ensures the transformation law eq. (2.1). In order to ensure that the commutator function vanishes for spacelike separated strings, it is crucial ([18], Theorem 3.3) that $u(e, p)$ is analytic in the e variable in the complex tube $\mathcal{T}_+ = \{e \in \mathbb{C}^4 : e^2 = -1, \text{Im } e \in V_+\}$, and satisfies certain local bounds in the tube as specified in ([18], Definition 3.1). The analyticity is necessary to ensure locality by a contour deformation argument. The bounds ensure that the boundary value of the analytic function u at $\text{Im } e \rightarrow 0$ defines an operator-valued distribution.

The intertwiner condition eq. (2.4) is equivalent to the pair of relations

$$u(e, p) = u(B_p^{-1}e, p_0) \quad (p \in H_0) \quad (2.5)$$

(which determines $u(e, p)$ for all $p \in H_0$ as soon as $u_0(e) = u(e, p_0)$ is given), and

$$u_0(We) = d_\kappa(W)u_0(e) \quad (W \in \text{E}(2)), \quad (2.6)$$

which determines $u_0(e)$ along each orbit of e under the little group. The argument is standard: let $\xi_0 = \frac{1}{2}(1, 0, 0, -1)^t$ and $\xi(\vec{a}) := T_{\vec{a}}\xi_0 = \frac{1}{2}(\vec{a}^2 + 1, 2a^1, 2a^2, \vec{a}^2 - 1)^t$. The orbits of $\text{E}(2)$ are parametrized by the invariants e^2 and (ep_0) , and e along each orbit with $(ep_0) \neq 0$ is parametrized by $e(\vec{a}) = \frac{e^2}{2(ep_0)}p_0 + (ep_0)\xi(\vec{a})$, $\vec{a} \in \mathbb{R}^2$. Because $e_0 = e(\vec{0})$ is fixed by the rotations of $\text{E}(2)$, $u_0(e_0)(\vec{k})$ is a constant function of $\vec{k} \in \kappa S^1$ by the first of eq. (2.2). Its value $f(e^2, (ep_0))$ is an arbitrary function of the orbit. The translations of $\text{E}(2)$ then determine

$$u_0(e)(\vec{k}) = f(e^2, (ep_0)) \cdot e^{-i\frac{(eE(\vec{k}))}{(ep_0)}}$$

everywhere along the orbit, by the second of eq. (2.2). Here, $E : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ is the standard embedding $E(\vec{k}) := (0, k^1, k^2, 0)^t$ into Minkowski space. The function $f(x, y)$ is not determined by eq. (2.6). By eq. (2.5), one gets

$$u_f(e, p)(\vec{k}) = f(e^2, (ep)) \cdot e^{-i\frac{(eE_p(\vec{k}))}{(ep)}} \quad (2.7)$$

where $E_p(\vec{k}) := B_p E(\vec{k})$.

These are identical with the “smooth” solutions to the three differential equations (3.6)–(3.8) in [25], that are the infinitesimal version of eq. (2.6), lifted to $p \in H_0$ by eq. (2.5). The “singular” solutions supported on the orbits with $(ep) = 0$ are not admissible as intertwiners.

¹Of course, it is not a rotation, nor is B_p . We sloppily adopt the terminology Wigner “rotation” and standard “boost” for $B_p : p_0 \mapsto p$ from the massive case.

2.1 The scalar standard field

Analyticity in the forward tube \mathcal{T}_+ of e requires to take $1/(ep)$ in eq. (2.7) as the distribution $1/(ep)_+$, because $(ep) \in \mathbb{C}_+$ if $e \in \mathcal{T}_+$. The simplest solution

$$\tilde{u}^\kappa(e, p)(\vec{k}) := e^{-i \frac{(eE_p(\vec{k}))}{(ep)_+}} \quad (2.8)$$

solves the wave equations eq. (3.17)–(3.18) in [25] along with their subsidiary conditions eq. (3.19)–(3.20) with $n = 0$. But it does not satisfy the bounds required to prove the vanishing of commutators when the strings are spacelike separated ([16], Theorem 3.3). E.g., for $p = (\mu, 0, 0, \mu)^t$ and $e = (ix, \sqrt{1-x^2}, 0, 0)^t \in \mathcal{T}_+$, the estimate $\|\tilde{u}^\kappa(e, p)\| \geq \exp\left(\frac{\sqrt{1-x^2}}{2x} \frac{\kappa}{\mu}\right)$ violates every power law bound as $x \rightarrow 0$ and $\mu \rightarrow 0$. Instead:

Proposition 2.1 (see also [12], pp. 42ff and Remark after [19], Definition 3.1) *In the tube \mathcal{T}_+ , it holds $|e^{-i \frac{(eE_p(\vec{k}))}{(ep)_+}}| \leq |e^{\frac{i\kappa}{(ep)}}|$. Therefore, the solution*

$$u^{\kappa(0)}(e, p)(\vec{k}) = e^{-i\kappa \frac{\sqrt{-e^2}}{(ep)_+}} \cdot e^{-i \frac{(eE_p(\vec{k}))}{(ep)_+}} \quad (2.9)$$

at real e exists as a weakly continuous $L^2(\kappa S^1)$ -valued function. It is the intertwiner of a string-localized field $\phi^{\kappa(0)}(e, x)$.

Remark 2.2 *Although we always put $e^2 = -1$, we write $\sqrt{-e^2}$ because we are going to take derivatives w.r.t. e by defining intertwiners as homogeneous functions $u(\lambda e) := u(e)$ ($\lambda > 0$) for all spacelike e . In the sequel, we shall refer to eq. (2.9) as the “standard intertwiner”, and the associated field $\phi^{\kappa(0)}$ as the “standard (string-localized infinite-spin) field”. We call $\omega^\kappa(e, p) = e^{-i\kappa \frac{\sqrt{-e^2}}{(ep)_+}}$ the “Köhler factor” [12]. It appears only in the combination eq. (2.9) where it cancels the essential singularity of eq. (2.8).*

Proof of Proposition 2.1. If $e = e' + ie''$ is in the tube, then $e'' \in V_+$ and $(e'e'') = 0$, hence e' is spacelike (or $= 0$). $e^2 = -1$ implies $-1 < e'^2 \leq 0$ and $0 < e''^2 \leq 1$.

For $e \in \mathcal{T}_+$, the real part of the exponent $-i \frac{(eE_p(\vec{k}))}{(ep)}$ is

$$\frac{(e''E_p(\vec{k}))(e'p) - (e'E_p(\vec{k}))(e''p)}{|(ep)|^2} = \frac{(f''E(\vec{k}))(f'p_0) - (f'E(\vec{k}))(f''p_0)}{|(fp_0)|^2}$$

where $f := B_p^{-1}e = f' + if''$, $(f'f'') = 0$, $f'' \in V_+$, $f'^2 - f''^2 = -1$. Parametrize $f' = \alpha'p_0 + \beta'\xi_0 + E(\vec{f}')$, $f'' = \alpha''p_0 + \beta''\xi_0 + E(\vec{f}'')$, such that $(\vec{f}'\vec{f}'') = \alpha'\beta'' + \alpha''\beta'$, and $\beta'' > 0$, $\vec{f}'^2 < 2\alpha'\beta''$, and $2\alpha'\beta' - \vec{f}'^2 - 2\alpha''\beta'' + \vec{f}''^2 = -1$, hence $\vec{f}'^2 < 2\alpha'\beta' + 1$.

Then, the numerator is (cf. [10])

$$\begin{aligned} (\beta''\vec{f}' - \beta'\vec{f}'') \cdot \vec{k} &\leq \kappa|\beta''\vec{f}' - \beta'\vec{f}''| = \kappa\sqrt{\beta''^2\vec{f}'^2 + \beta'^2\vec{f}''^2 - 2\beta'\beta''(\vec{f}'\vec{f}'')} \leq \\ &\leq \kappa\sqrt{\beta''^2(2\alpha'\beta' + 1) + \beta'^2 2\alpha''\beta'' - 2\beta'\beta''(\alpha'\beta'' + \alpha''\beta')} = \kappa\beta''. \end{aligned}$$

Thus $|e^{-i \frac{(eE_p(\vec{k}))}{(ep)}}| \leq e^{\kappa \frac{\beta''}{|(ep_0)|^2}} = |\omega^\kappa(e, p)|^{-1}$. It follows that $\omega^\kappa(e, p) \cdot e^{-i \frac{(eE_p(\vec{k}))}{(ep)}}$ is an analytic function in \mathcal{T}_+ bounded by 1. Thus it satisfies the bounds specified in ([16], Definition 3.1)

and the remark following it, hence its boundary value eq. (2.9) is well-defined as a function, and defines a string-localized field by ([16], Theorem 3.3). \square

Mund, Schroer and Yngvason in [19] also gave solutions to the intertwiner relation eq. (2.4) in a different form (for $e^2 = -1$):

$$U_F(e, p)(\vec{k}) = \int d^2 a e^{i\vec{k}\cdot\vec{a}} F((eB_p\xi(\vec{a}))). \quad (2.10)$$

The function $F(z)$ must be analytic and polynomially bounded in the upper half-plane, hence its Fourier transform \widehat{F} is supported on \mathbb{R}_+ . One can bring this form into the form eq. (2.7): with the Fourier representation of $F(z)$, the \vec{a} -integration becomes Gaussian and can be performed when $e \in \mathcal{T}_+$, with the result (see [10])

$$U_F(e, p)(\vec{k}) = u^{\kappa(0)}(e, p)(\vec{k}) \cdot \left[\frac{2\pi i}{(ep)_+} \int_0^\infty \frac{dt}{t} \widehat{F}\left(\frac{\kappa}{\sqrt{-e^2}} t\right) e^{-i\frac{\kappa\sqrt{-e^2}}{(ep)_+} \frac{(t-1)^2}{2t}} \right]. \quad (2.11)$$

The choice $2\pi\widehat{F}\left(\frac{\kappa}{\sqrt{-e^2}} t\right) = \delta(t-1)$ gives the intertwiner $U_F = \frac{i}{(ep)_+} u^{\kappa(0)}$, i.e., the standard field is $\phi^{\kappa(0)}(e, x) = -(e\partial_x)\phi_F(e, x)$.

We introduce the string-integration operator

$$(I_e X)(x) := \int_0^\infty ds X(x + se), \quad (2.12)$$

already occurring in eq. (1.1). If $X(e, x)$ is localized along the string e , then so is $I_e X(e, x)$. One has

$$(e\partial_x)I_e X = I_e(e\partial_x)X = -X. \quad (2.13)$$

In momentum space, acting on e^{ipx} , this is the multiplication operator by

$$I_e(p) = \frac{i}{(ep)_+} \equiv \lim_{\varepsilon \searrow 0} \frac{i}{(ep) + i\varepsilon} \quad (2.14)$$

as a distribution. We also introduce

$$(J_e)_\mu{}^\nu := I_e[e^\nu\partial_\mu - \delta_\mu^\nu(e\partial_\nu)] = \delta_\mu^\nu + I_e e^\nu\partial_\mu, \quad (2.15)$$

because of eq. (2.13). In momentum space, this is the multiplication operator by

$$J_e(p)_\mu{}^\nu = (ep)\partial_{e^\mu} \frac{e^\nu}{(ep)_+} = \delta_\mu^\nu - p_\mu \frac{e^\nu}{(ep)_+}, \quad (2.16)$$

so that

$$(ep)\partial_{e^\mu} \widetilde{u}^\kappa(e, p)(\vec{k}) = -i(J_e(p)E_p(\vec{k}))_\mu \cdot \widetilde{u}^\kappa(e, p)(\vec{k}). \quad (2.17)$$

Varying w.r.t. e , one has

$$(ep)\partial_{e^\lambda} J_e(p)_\mu{}^\nu = -p_\mu J_e(p)_\lambda{}^\nu. \quad (2.18)$$

Because we are going to take derivatives w.r.t. e of intertwiners multiplied with the Kähler factor, it is convenient to introduce

$$D_e(p) := \partial_e - \frac{i\kappa}{\sqrt{-e^2}} \frac{J_e(p)e}{(ep)_+}, \quad (2.19)$$

so that $D_e(p)(\omega^\kappa \tilde{u}(e, p)) = \omega^\kappa \partial_e \tilde{u}(e, p)$. Acting on the corresponding fields, this is the operator

$$D_e = \partial_e - \frac{\kappa}{\sqrt{-e^2}} I_e J_e e. \quad (2.20)$$

Proposition 2.3 *The standard string-localized infinite-spin field $\phi^{\kappa(0)}(e, x)$ satisfies the equations of motion*

$$\square_x \phi^{\kappa(0)} = (e\partial_e)\phi^{\kappa(0)} = (\partial_x D_e)\phi^{\kappa(0)} = ((e\partial_x)^2 D_e^2 + \kappa^2)\phi^{\kappa(0)} = 0. \quad (2.21)$$

The last two equations in eq. (2.21) are equivalent to

$$\sqrt{-e^2} (\partial_x \partial_e)\phi^{\kappa(0)} = \sqrt{-e^2} (e\partial_x)\square_e \phi^{\kappa(0)} = -\kappa \phi^{\kappa(0)}. \quad (2.22)$$

The Pauli-Lubanski equation follows:

$$W^2 \phi^{\kappa(0)} = \kappa^2 \phi^{\kappa(0)}, \quad (2.23)$$

where $W^2 = \square_x (e^2 \square_e - (e\partial_e)^2 - (e\partial_e)) + 2(e\partial_x)(\partial_x \partial_e)(e\partial_e) - e^2(\partial_x \partial_e)^2 - (e\partial_x)^2 \square_e$ is the Pauli-Lubanski operator.

Proof. The Klein-Gordon equation is fulfilled by construction, and the homogeneity in e is manifest from eq. (2.9). Using eq. (2.17) and $p^2 = (pE_p(\vec{k})) = 0$, hence $(pJ_e E_p(\vec{k})) = 0$ and $(J_e E_p(\vec{k}))^2 = (E_p(\vec{k}))^2 = -\kappa^2$, one computes

$$(p\partial_e)\tilde{u}^\kappa(e, p) = 0, \quad ((ep)^2 \square_e - \kappa^2)\tilde{u}^\kappa(e, p) = 0,$$

hence

$$(pD_e)u^{\kappa(0)}(e, p) = 0, \quad ((ep)^2 D_e^2 - \kappa^2)u^{\kappa(0)}(e, p) = 0.$$

These relations for the intertwiner are equivalent to the last two equations in eq. (2.21). The first of eq. (2.22) is equivalent to the third in eq. (2.21) by eq. (2.13), and the second of eq. (2.22) follows from the last of eq. (2.21) by a lengthy calculation using also the second and third of eq. (2.21). Eq. (2.23) is then a consequence of the previous. \square

2.2 Tensor fields I

We shall later also need tensor fields that transform like

$$U_\kappa(a, \Lambda)\phi_{\mu_1 \dots \mu_r}(e, x)U_\kappa(a, \Lambda)^* = \prod \Lambda_{\mu_i}^{\nu_i} \phi_{\nu_1 \dots \nu_r}(\Lambda e, \Lambda x + a). \quad (2.24)$$

They are formed with intertwiners that, regarded as functions with values in $(\mathbb{R}^4)^{\otimes r} \otimes \mathcal{H}_\kappa$, satisfy

$$u(\Lambda e, \Lambda p) = ((\Lambda^{\otimes r}) \otimes d_\kappa(W_{\Lambda,p})) u(e, p). \quad (2.25)$$

To simplify notation, we write contractions as $a^\mu b_\mu \equiv (ab) \equiv a^t b$, and for symmetric tensors of rank r , we write

$$X(v) \equiv v^{\mu_1} \dots v^{\mu_r} X_{\mu_1 \dots \mu_r} \equiv v^{t \otimes r} X \quad (v \in \mathbb{R}^4).$$

The tensor components are recovered by differentiation w.r.t. v^μ .

Definition 2.4 We introduce the symmetric rank r tensor intertwiners

$$u^{\kappa(r)}(e, p)(\vec{k}) := (-\kappa)^{-r} (J_e(p) E_p(\vec{k}))^{\otimes r} \cdot u^{\kappa(0)}(e, p)(\vec{k}), \quad (2.26)$$

where $E_p : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ is the anti-isometric embedding ($E_p(\vec{k})^2 = -\vec{k}^2$) introduced in eq. (2.7). The associated tensor fields are called $\phi^{\kappa(r)}(e)$.

Proposition 2.5 The symmetric tensor fields $\phi^{\kappa(r)}$ are recursively related to $\phi^{\kappa(0)}$ by

$$-\kappa \phi^{\kappa(r+1)}(e, x, v) = (r \cdot (v \partial_x) + (e \partial_x)(v D_e)) \phi^{\kappa(r)}(e, x, v). \quad (2.27)$$

Proof. Eq. (2.17) and $(r \cdot p_\mu + (ep) \partial_{e^\mu}) \circ J_e^{\otimes r} = J_e^{\otimes r}(ep) \circ \partial_{e^\mu}$ imply

$$-i(v^t J_e(p) E_p(\vec{k}))^{r+1} \cdot \tilde{u}^\kappa(e, p)(\vec{k}) = (r \cdot (vp) + (ep)(v \partial_e))(v^t J_e(p) E_p(\vec{k}))^r \cdot \tilde{u}^\kappa(e, p)(\vec{k}).$$

Restoring the Kähler factor, we have to replace ∂_e by $D_e(p)$ defined in eq. (2.19):

$$i\kappa u^{\kappa(r+1)}(e, p, v)(\vec{k}) = (r \cdot (vp) + (ep)(v D_e(p))) u^{\kappa(r)}(e, p, v)(\vec{k}). \quad (2.28)$$

This is the momentum space version of eq. (2.27). \square

We may take eq. (2.27) as recursive definitions. The covariance of eq. (2.27) and eq. (2.28) ensures that $\phi^{\kappa(r)}$ and their intertwiners $u^{\kappa(r)}$ transform according to eq. (2.24) and eq. (2.25), respectively, and the bounds of ([16], Definition 3.1) are inherited from the bound of $u^{\kappa(0)}$ because $(J_e(p) E_p(\vec{k}))^2 = -\kappa^2$.

Corollary 2.6 $\phi_{\mu_1 \dots \mu_r}^{\kappa(r)}(e, x)$ are string-localized tensor fields.

2.3 Tensor fields II

Definition 2.7 We introduce a second family of symmetric rank r tensor fields

$$\Phi^{\kappa(r)}(e, x) = 2^{r/2} \cdot \left(S_r \sum_{2k \leq r} \gamma_k^r \cdot (J_e^{\otimes 2} \eta \circ \eta^t)^{\otimes k} \right) \phi^{\kappa(r)}(e, x) \quad (2.29)$$

with the coefficients $\gamma_k^r = \frac{1}{4^k k!} \frac{r!}{(r-2k)!} \frac{1}{(1-r)_k}$, and S_r the projection onto the symmetric tensors in $(\mathbb{R}^4)^{\otimes r}$. The associated string-localized tensor fields are called $\Phi^{\kappa(r)}(e)$. J_e is the operator given in eq. (2.15). (Trivially, $\Phi^{\kappa(0)} = \phi^{\kappa(0)}$ and $\Phi^{\kappa(1)} = \sqrt{2} \cdot \phi^{\kappa(1)}$.)

Remark 2.8 The operations on $\phi^{\kappa(r)}$ that define $\Phi^{\kappa(r)}$ preserve the localization and covariance eq. (2.24). Thus, the latter are again string-localized tensor fields.

The following proposition exhibits the advantage of the new fields $\Phi^{\kappa(r)}$, that will become important in section 5.

Proposition 2.9 (i) The intertwiner of $\Phi^{\kappa(r)}$ equals

$$U^{\kappa(r)}(e, p)(\vec{k}) = \frac{2^{r/2}}{(-\kappa)^r} \cdot (J_e(p)E_p)^{\otimes r} (\Pi_2^{(r)}(\vec{k}^{\otimes r})) \cdot u^{\kappa(0)}(e, p)(\vec{k}). \quad (2.30)$$

Here, $\Pi_2^{(r)}$ is the projection (of two-dimensional range) onto the symmetric traceless tensors in $(\mathbb{R}^2)^{\otimes r}$.

(ii) For $\vec{k} = \kappa \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}$ and $\varepsilon_{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$, one has the decomposition into helicity eigenstates

$$\frac{2^{r/2}}{\kappa^r} \Pi_2^{(r)}(\vec{k}^{\otimes r}) = \begin{cases} 1 & (r = 0) \\ e^{-ir\varphi} \varepsilon_+^{\otimes r} + e^{ir\varphi} \varepsilon_-^{\otimes r} & (r > 0), \end{cases} \quad (2.31)$$

hence $U^{\kappa(0)} = u^{\kappa(0)}$, and for $r > 0$

$$U^{\kappa(r)}(e, p)(\vec{k}) = (-1)^r \left(e^{-ir\varphi} \mathcal{E}_+(e, p)^{\otimes r} + e^{ir\varphi} \mathcal{E}_-(e, p)^{\otimes r} \right) \cdot u^{\kappa(0)}(e, p)(\vec{k}), \quad (2.32)$$

where $\mathcal{E}_{\pm}(e, p) := J_e(p)E_p \varepsilon_{\pm}$.

Remark 2.10 The presence of the factor $u^{\kappa(0)}(\vec{k})$ prevents the interpretation of $\Phi^{\kappa(r)}(e)$ as “fields of sharp helicity”, in accord with the irreducibility of the infinite-spin representation.

Proof of Proposition 2.9. By eq. (2.29), the intertwiner of $\Phi^{\kappa(r)}$ is

$$U^{\kappa(r)}(e, p)(\vec{k}) = 2^{r/2} \cdot \left(S_r \sum_{2k \leq r} \gamma_k^r \cdot (J_e(p)^{\otimes 2} \eta \circ \eta^t)^{\otimes k} \right) u^{\kappa(r)}(e, p)(\vec{k}) \quad (2.33)$$

with $J_e(p)$ given in eq. (2.16). By eq. (2.26), the intertwiner $u^{\kappa(r)}$ is in the range of the operator $(J_e(p)E_p)^{\otimes r}$. Therefore, we may consider the operators

$$J_e(p)^{\otimes 2} \eta \circ \eta^t (J_e(p)E_p)^{\otimes 2} : (\mathbb{R}^2)^{\otimes 2} \rightarrow (\mathbb{R}^4)^{\otimes 2}$$

appearing in eq. (2.33) when eq. (2.26) is inserted. Using the standard vectors $p_0 = (1, 0, 0, 1)^t$ and $\xi_0 = (\frac{1}{2}, 0, 0, -\frac{1}{2})^t$, we have $\eta = -E^{\otimes 2} \delta_2 + p_0 \otimes \xi_0 + \xi_0 \otimes p_0$ as tensors in $(\mathbb{R}^4)^{\otimes 2}$, where $\delta_2 = \vec{e}_1 \otimes \vec{e}_1 + \vec{e}_2 \otimes \vec{e}_2$ as a tensor in $(\mathbb{R}^4)^{\otimes 2}$. Applying the standard “boost” $B_p : p_0 \mapsto p$, we get

$$\eta = -E^{\otimes 2} \delta_2 + p \otimes B_p \xi_0 + B_p \xi_0 \otimes p.$$

Because $p^2 = 0$, we have $\eta^t J_e(p)^{\otimes 2} = \eta^t - (B_p \xi_0)^t \otimes p^t - p^t \otimes (B_p \xi_0)^t$, hence

$$J_e(p)^{\otimes 2} \eta = (J_e(p)E_p)^{\otimes 2} \delta_2, \quad \eta^t (J_e(p)E_p)^{\otimes 2} = \eta^t E_p^{\otimes 2} = \delta_2^t E_p^{t \otimes 2} E_p^{\otimes 2} = \delta_2^t,$$

because p is orthogonal to the range of E_p . Thus, eq. (2.33) can be rewritten as

$$U^{\kappa(r)}(e, p)(\vec{k}) = \frac{2^{r/2}}{\kappa^r} \cdot (J_e(p)E_p)^{\otimes r} \left(S_r \sum_{2k \leq r} \gamma_k^r \cdot (\delta_2 \delta_2^t)^{\otimes k} \otimes \mathbb{I}_2^{\otimes r-2k} \right) \vec{k}^{\otimes r} \cdot u^{\kappa(0)}(e, p)(\vec{k}).$$

The operator in brackets is the projection $\Pi_2^{(r)}$ onto the symmetric traceless tensors in $(\mathbb{R}^2)^{\otimes r}$ (this is in fact the defining property of the coefficients γ_k^r [11]). This proves the claim (i). Now, write $\vec{k} = \frac{\kappa}{\sqrt{2}}(e^{-i\varphi}\varepsilon_+ + e^{+i\varphi}\varepsilon_-)$. Then eq. (2.31) is a well-known identity (that may be proven by induction in r), and eq. (2.32) follows. \square

Proposition 2.11 *Besides the massless Klein-Gordon equation, the infinite-spin symmetric tensor fields $\Phi_{\mu_1 \dots \mu_r}^{\kappa(r)}(e, x)$ satisfy the equations of motion and constraints*

$$\eta^{\mu\nu} \Phi_{\mu\nu\mu_3 \dots \mu_r}^{\kappa(r)} = 0, \quad \partial^\mu \Phi_{\mu\mu_2 \dots \mu_r}^{\kappa(r)} = 0, \quad e^\mu \Phi_{\mu\mu_2 \dots \mu_r}^{\kappa(r)} = 0, \quad (2.34)$$

$$(e\partial_e)\Phi_{\mu_1 \dots \mu_r}^{\kappa(r)} = 0, \quad (\partial_x D_e)\Phi_{\mu_1 \dots \mu_r}^{\kappa(r)} = 0, \quad (2.35)$$

as well as the coupling relations for $r \geq 2$

$$\begin{aligned} & \left((e\partial_x)D_{e^{\mu_1}}\Phi_{\mu_2 \dots \mu_{r+1}}^{\kappa(r)} + \sum_{i=2}^{r+1} \partial_{\mu_i} \Phi_{\mu_1 \dots \mu_{i-1} \mu_{i+1} \dots \mu_{r+1}}^{\kappa(r)} \right) + (\mu_1 \leftrightarrow \mu_2) = \\ & = -\sqrt{2}\kappa \left(\Phi_{\mu_1 \dots \mu_{r+1}}^{\kappa(r+1)} - \frac{1}{2}(E_e)_{\mu_1 \mu_2} \Phi_{\mu_3 \dots \mu_{r+1}}^{\kappa(r-1)} \right). \end{aligned} \quad (2.36)$$

Here, $(E_e)_{\mu\nu}$ is the integral- and differential operator $((J_e \otimes J_e)\eta)_{\mu\nu}$. For $r = 0$, the second term in the bracket on the r.h.s. is absent, and for $r = 1$ is replaced by $-(E_e)_{\mu_1 \mu_2} \Phi^{\kappa(0)}(v)$.

Proof. We proceed in momentum space, where $\partial_x = ip$ on the intertwiners eq. (2.32). Eq. (2.34) and eq. (2.35) follow by a direct computation, using $(\mathcal{E}_\pm \mathcal{E}_\pm) = 0$ and $(p\mathcal{E}_\pm) = (e\mathcal{E}_\pm) = (p\partial_e)\mathcal{E}_\pm = 0$, as well as $D_e u^{\kappa(0)} = \omega^\kappa \partial_e \tilde{u}^\kappa$. For the coupling relations, apply $i(ep)D_{e^{\mu_1}}$ to $U_{\mu_2 \dots \mu_{r+1}}^{\kappa(r)}$. Use $\partial_{e^\mu} \mathcal{E}_{\pm\nu} = -p_\nu \mathcal{E}_{\pm\mu}$ by eq. (2.18), which cancels the spatial derivatives on the l.h.s. of eq. (2.36), and $i(ep)D_{e^\mu} u^{\kappa(0)} = (J_e(p)E_p(\vec{k}))_\mu u^{\kappa(0)} = \frac{\kappa}{\sqrt{2}}(e^{-i\varphi}\mathcal{E}_+ + e^{i\varphi}\mathcal{E}_-)u^{\kappa(0)}$ by eq. (2.17). This produces the term $(\text{id} + \pi_{12})(\tilde{\mathcal{E}}_+ + \tilde{\mathcal{E}}_-) \otimes (\tilde{\mathcal{E}}_+^{\otimes r} + \tilde{\mathcal{E}}_-^{\otimes r})$ where $\tilde{\mathcal{E}}_\pm \equiv e^{\mp i\varphi} \mathcal{E}_\pm$ and π_{12} is the permutation of the first two tensor factors. This tensor trivially equals

$$= 2 \left(\tilde{\mathcal{E}}_+^{\otimes r+1} + \tilde{\mathcal{E}}_-^{\otimes r+1} \right) + (\mathcal{E}_+ \otimes \mathcal{E}_- + \mathcal{E}_- \otimes \mathcal{E}_+) \otimes \left(\tilde{\mathcal{E}}_+^{\otimes r-1} + \tilde{\mathcal{E}}_-^{\otimes r-1} \right).$$

Finally, $\mathcal{E}_+ \otimes \mathcal{E}_- + \mathcal{E}_- \otimes \mathcal{E}_+ = (J_e(p)E_p)^{\otimes 2}(\delta_2) = -E_e(p)$, as in the proof of Proposition 2.9, implies the claim. \square

3 Preparations for the Pauli-Lubanski limit

3.1 Two-point functions and commutators: basics

Our fields are free fields, so that the entire information resides in their two-point functions, which in particular determine the commutator functions. It is convenient to express the

two-point functions as integral or differential operators acting on the canonical scalar two-point function $(\Omega, \varphi(x)\varphi(y)\Omega) = \Delta_m(x-y)$. This amounts to the insertion of a “two-point kernel” into the Fourier representation:

$$(\Omega, X(x)Y(y)\Omega) = \int d\mu_m(p) e^{-ip(x-y)} \cdot M_m^{X,Y}(p), \quad (3.1)$$

where $d\mu_m(p) = (2\pi)^{-3} d^4p \delta(p^2 - m^2) \theta(p^0)$ is the Lorentz invariant measure on the positive energy mass shell H_m . E.g., the canonical scalar field and the Proca field have the two-point kernels

$$M_m^{\varphi,\varphi} = 1 \quad \text{resp.} \quad M_m^{A_\mu^P, A_\nu^P}(p) = -\pi_{\mu\nu} \equiv -(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}). \quad (3.2)$$

3.2 Two-point functions of string-localized spin 1 fields

For the massive Proca field we introduce the string-localized potential by the same formula as for the Maxwell potential eq. (1.1); but in this case it can also be expressed in terms of the point-localized potential $A_\nu^P = -\frac{1}{m^2} \partial^\mu F_{\mu\nu}^P$ that exists on the Hilbert space:

$$A_\mu(e, x) := (I_e F_{\mu\nu}^P)(x) e^\nu = A_\mu^P(x) + \partial_\mu (I_e A_\nu^P)(x) e^\nu \equiv (J_e)_\mu{}^\nu A_\nu^P \quad (3.3)$$

with I_e and J_e defined in eq. (2.12), eq. (2.15). Its two-point function arises by integration over the Proca two-point function, that results in a multiplication of the Proca two-point kernel eq. (3.2) with the matrices $J_e(p)$ defined in eq. (2.16):

$$M_m^{A_\mu^{(-e)}, A_{\mu'}^{(e')}}(p) = -E(e, e')(p)_{\mu\mu'} \equiv -J_e(p)_\mu{}^\nu J_{e'}(p)_{\mu'}{}^{\nu'} \eta_{\nu\nu'}. \quad (3.4)$$

Explicitly,

$$E(e, e')(p)_{\mu\mu'} = \left[\eta_{\mu\mu'} - \frac{e'_\mu p_{\mu'}}{(pe')_+} - \frac{p_\mu e_{\mu'}}{(pe)_+} + \frac{(ee') p_\mu p_{\mu'}}{(pe)_+(pe')_+} \right]. \quad (3.5)$$

We have chosen the string $-e$ in the left argument in eq. (3.4) in order to avoid denominators $1/(-pe)_+ = -1/(pe)_-$ if the argument were $+e$. In particular, $E(e, e)(p)$ is the momentum space version of the operator E_e in eq. (2.36).

It follows from eq. (3.4)

$$M_m^{(\partial A)^{(-e)}, (\partial A)^{(e')}}(p) = -p^\mu p^\nu E(e, e')(p)_{\mu\nu} = m^2 \left(1 - m^2 \frac{(ee')}{(ep)_+(e'p)_+} \right). \quad (3.6)$$

Therefore, the field $a(e, x) := -m^{-1} \partial_\mu A^\mu(e, x)$ is regular at $m = 0$ and converges to the canonical scalar field φ .

For the string-localized Maxwell potential eq. (1.1), which has no covariant point-localized potential on the Hilbert space, one has to compute the two-point function by integration over the field strength whose two-point kernel is $M_0^{F_{\mu\nu}, F_{\kappa\lambda}} = -p_\mu p_\kappa \eta_{\nu\lambda} + p_\nu p_\kappa \eta_{\mu\lambda} + p_\mu p_\lambda \eta_{\nu\kappa} - p_\nu p_\lambda \eta_{\mu\kappa}$. This gives the same formula eq. (3.4) except that the mass is zero and $p^2 = 0$.

This continuity property does not persist at $s > 1$, see [16, 17], where the decoupling of the lower helicities is more subtle than at $s = 1$.

3.3 Two-point functions of infinite-spin fields

The two-point kernels of infinite spin fields eq. (2.3) given in terms of their intertwiners are

$$M_0^{\phi_u(-e), \phi_u(e')}(p) = \int d\mu_\kappa(\vec{k}) \overline{u(-e, p)(\vec{k})} u(e', p)(\vec{k}). \quad (3.7)$$

For the standard field $\phi^{\kappa(0)} = \Phi^{\kappa(0)}$, we get

Proposition 3.1 (see also [12], chapter 4.1) *The two-point kernel of $\phi^{\kappa(0)}(e, x)$ is*

$$M_0^{\phi^{\kappa(0)}(-e), \phi^{\kappa(0)}(e')} = \overline{\omega^\kappa(-e, p)} \omega^\kappa(e', p) \cdot J_0(\kappa R_{e, e', p}), \quad (3.8)$$

where $q_e(p) := \frac{e}{(ep)_+}$ and $R_{e, e', p}^2 = -(q_e(p) - q_{e'}(p))^2$.

Notice that $q_e - q_{e'}$ is orthogonal to p because $(pq_e) = (pq_{e'}) = 1$, hence $q_e - q_{e'}$ is spacelike and the argument of the Bessel function is real. The distributional nature of $R_{e, e', p}$ is absorbed by the Köhler factors, see Proposition 2.1.

Proof. Recall the definition of the Bessel functions

$$\frac{1}{2\pi} \int d\varphi e^{i\nu\varphi} e^{iz \cos \varphi} = i^\nu J_\nu(z) = i^{-\nu} J_{-\nu}(z). \quad (3.9)$$

Thus, with \tilde{u}^κ given by eq. (2.8), the integral in eq. (3.7) can be performed:

$$\int d\mu_\kappa(\vec{k}) e^{i \frac{(e E_p(\vec{k}))}{(ep)_+}} e^{-i \frac{(e' E_p(\vec{k}))}{(e'p)_+}} = \int d\mu_\kappa(\vec{k}) e^{-i \vec{k} \cdot \vec{f}} = J_0(\kappa |\vec{f}|) \quad (3.10)$$

where \vec{f} is the transverse (1-2-)part of the four-vector

$$f = B_p^{-1} \left(\frac{e}{(ep)_+} - \frac{e'}{(e'p)_+} \right) = B_p^{-1}(q_e - q_{e'}).$$

Because $(fp_0) = 0$, one has $f^2 = (fp_0)(f\xi_0) - |\vec{f}|^2 = -|\vec{f}|^2$. The claim follows by multiplying with the Köhler factors. \square

Proposition 3.2 *The two-point functions among $\Phi^{\kappa(r)}$ are given by their two-point kernels (no respective sums if $r = 0$ or $r' = 0$)*

$$M_0^{\Phi^{\kappa(r)}(-e, v), \Phi^{\kappa(r')}(e', v')} = \overline{\omega^\kappa(-e, p)} \omega^\kappa(e', p) \cdot \sum_{\nu=\pm r} (v\mathcal{E}_\pm)^\nu \sum_{\nu'=\pm r'} (v'\mathcal{E}'_\pm)^{\nu'} \cdot e^{-i(\nu+\nu')\alpha_{e, e', p}} \cdot i^{\nu+\nu'} J_{\nu+\nu'}(\kappa R_{e, e', p}).$$

Here, $R_{e, e', p} \begin{pmatrix} \cos \alpha_{e, e', p} \\ \sin \alpha_{e, e', p} \end{pmatrix}$ parametrizes the 1-2-part of $B_p^{-1}(q_e(p) - q_{e'}(p))$.

Proof. By a direct computation, using eq. (2.32) and eq. (3.9). \square

Remark 3.3 *The fields $\Phi^{\kappa(r)}$ mutually decouple in the limit $\kappa \rightarrow 0$, because $J_\nu(0) = \delta_{\nu, 0}$. Their limits as $\kappa \rightarrow 0$ coincide with the massless potentials $A^{(r, r)}$, i.e., the infinite-spin field “decays” into a direct sum of massless fields of every helicity ([12], chapter 4).*

3.4 Massive fields of finite spin: definitions and properties

Our aim is to approximate eq. (3.8) by higher-spin generalizations of eq. (3.4) and eq. (3.6) in the Pauli-Lubanski limit. We take stock of the relevant results in [16, 17], and supplement it by the crucial recursive formula Proposition 3.4.

We are going to work with the string-localized fields² $a_{\mu_1 \dots \mu_r}^{(s,r)}(e, x)$ defined in eq. (3.11) and eq. (3.12), and the string-localized fields $A^{(s,r)}(e, x)$ defined in eq. (3.24). All these fields are derived from the total field strengths $F_{[\mu_1 \nu_1] \dots [\mu_s \nu_s]}^{P(s)}(x)$ of the point-localized massive “Proca” fields $A_{\mu_1 \dots \mu_s}^{P(s)}(x)$ of spin s [8]. We start with the definition

$$a_{\mu_1 \dots \mu_s}^{(s,s)}(e, x) := \left((I_e)^s F_{[\mu_1 \nu_1] \dots [\mu_s \nu_s]}^{P(s)} \right) (x) \cdot e^{\nu_1} \dots e^{\nu_s} \quad (3.11)$$

and the descending recursion

$$-m \cdot a_{\mu_1 \dots \mu_r}^{(s,r)}(e, x) := \partial^\mu a_{\mu \mu_1 \dots \mu_r}^{(s,r+1)}(e, x). \quad (3.12)$$

$a^{(s,s)}$ differs from $A^{P(s)}$ by derivatives of $a^{(s,r)}$ ($r < s$) ([16], Proposition 3.5), so that it is also a potential for the field strength $F^{P(s)}$. Unlike the Proca potential, the field strength and hence also the string-localized potential $a^{(s,s)}$ and its escort fields $a^{(s,r)}$ are regular in the limit $m \rightarrow 0$ at fixed spin s . The discrepancy between these limits and potentials $A^{(r)}$ for the massless field strengths $F^{(r)}$ was studied in [16].

Let us identify some general patterns. The composition of the operations

$$(\text{contraction with } e) \circ (\text{string integration}) \circ (\text{curl}),$$

applied to each index of $A^{P(s)}$ in the definition eq. (3.11), is the matrix operator $(J_e)_\mu^\nu = \delta_\mu^\nu + I_e e^\nu \partial_\mu$ already seen in eq. (2.15) and eq. (3.3). Thus, eq. (3.11) and eq. (3.12) become

$$\begin{aligned} a_{\mu_1 \dots \mu_s}^{(s,s)}(e, x) &= (J_e)_{\mu_1}^{\nu_1} \dots (J_e)_{\mu_s}^{\nu_s} A_{\nu_1 \dots \nu_s}^{P(s)}(x), \\ a_{\mu_1 \dots \mu_r}^{(s,r)}(e, x) &= (J_e)_{\mu_1}^{\nu_1} \dots (J_e)_{\mu_r}^{\nu_r} (m e^{\nu_{r+1}} I_e) \dots (m e^{\nu_s} I_e) A_{\nu_1 \dots \nu_s}^{P(s)}(x), \end{aligned} \quad (3.13)$$

Because $J_{\lambda e} = J_e$ ($\lambda > 0$), all $a^{(s,r)}(e, x)$ are homogeneous distributions in e .

The following formula for the e -dependence of $a^{(s,s)}(e, x)$ was displayed in ([16], Corollary 3.3) with an erroneous factor m^{-1} :

$$\partial_{e^\mu} a_{\mu_1 \dots \mu_s}^{(s,s)}(e, x) = \sum_{i=1}^s \partial_{\mu_i} I_e a_{\dots \mu_{i-1} \mu_{i+1} \dots}^{(s,s)}(e, x). \quad (3.14)$$

It follows from the definition of $a^{(s,s)}$ and the identities eq. (2.18) and eq. (2.13).

Proposition 3.4 *The descending defining recursion eq. (3.12) is inverted by the ascending recursion, involving the variation of the direction e ,*

$$-m(s-r) \cdot a_{\mu \mu_1 \dots \mu_r}^{(s,r+1)}(e, x) = \sum_{i=1}^r \partial_{\mu_i} a_{\dots \mu_{i-1} \mu_{i+1} \dots}^{(s,r)} + (e \partial_x) \partial_{e^\mu} a_{\mu_1 \dots \mu_r}^{(s,r)}(e, x). \quad (3.15)$$

In particular, the r.h.s. is completely symmetric in μ, μ_1, \dots, μ_r .

This result conversely exhibits also the e -dependence of $a_{\mu_1 \dots \mu_r}^{(s,r)}(e, x)$ via eq. (3.16).

²The first superscript s was suppressed in [16, 17], where we worked at fixed s .

*Proof.*³ Apply $\partial^{\mu_s} \dots \partial^{\mu_{r+1}}$ to eq. (3.14) and use the defining recursion eq. (3.12). This gives

$$\partial_{e^\mu} a_{\mu_1 \dots \mu_r}^{(s,r)}(e, x) = m(s-r) \cdot I_e a_{\mu_1 \dots \mu_r}^{(s,r+1)}(e, x) + \sum_{i=1}^r \partial_{\mu_i} I_e a_{\dots \mu_{i-1} \mu_{i+1} \dots}^{(s,r)}, \quad (3.16)$$

which can be solved for $a_{\mu_1 \dots \mu_r}^{(s,r+1)}(e, x)$ using eq. (2.13). \square

We also report the identities

Proposition 3.5 (see [16], Proposition 3.6) *For $r \geq 2$, resp. $r \geq 0$*

$$\eta^{\kappa\lambda} a_{\kappa\lambda\mu_3 \dots \mu_r}^{(s,r)} = -a_{\mu_3 \dots \mu_r}^{(s,r-2)}, \quad e^\kappa a_{\kappa\mu_2 \dots \mu_r}^{(s,r)} = 0. \quad (3.17)$$

Proof. By inspection of eq. (3.13), using that $A^{P(s)}$ is traceless and conserved. \square

We now turn to two-point functions.

The intertwiner for the (m, s) Proca potential is given by the s -fold tensor product of the standard intertwiner for spin 1 [28–30], preceded by the projection onto the traceless symmetric subrepresentation (= spin s representation) of the tensor product of spin 1 representations of the little group $\text{SO}(3)$:

$$u_n^{P(s)}(p) = (B_p E_3)^{\otimes s} T_n, \quad (3.18)$$

where T_n , $n = 1, \dots, 2s+1$, is an orthonormal basis of traceless symmetric tensors in $(\mathbb{R}^3)^{\otimes s}$, and $E_3 : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ the standard embedding into Minkowski space.

The resulting two-point kernel of the Proca field is (in the notations introduced in section 2.2 and section 3.1)

Proposition 3.6 (see [11] and [16], section 2.1)

$$M_m^{A^{P(s)}(v), A^{P(s)}(v')} = (-1)^s \sum_{2n \leq s} \beta_n^s \cdot [(v\pi v)(v'\pi v')]^n \cdot (v\pi v')^{s-2n}. \quad (3.19)$$

Here, $\pi_{\mu\nu} = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}$, and $(v\pi v)$ etc. are the contractions with v resp. v' . The coefficients $\beta_n^s = \frac{1}{4^n n!} \frac{s!}{(s-2n)!} \frac{1}{(\frac{1}{2}-s)_n}$ (ensuring the tracelessness) are the coefficients of the hypergeometric function

$$F_s(z) \equiv \sum_{2n \leq s} \beta_n^s z^n = {}_2F_1\left(\frac{-s}{2}, \frac{1-s}{2}; \frac{1}{2} - s; z\right). \quad (3.20)$$

This function is in fact a polynomial of order $\lfloor \frac{s}{2} \rfloor$, because either $\frac{-s}{2}$ or $\frac{1-s}{2}$ is a non-positive integer.

The two-point kernel of the massive spin s string-localized potential $a^{(s,s)}$ follows from the definition eq. (3.11):

Proposition 3.7 (see [16], section 3)

$$M_m^{a^{(s,s)}(-e,v), a^{(s,s)}(e',v')} = (-1)^s \sum_{2n \leq s} \beta_n^s \cdot [(vE v)(v'E'' v')]^n \cdot (vE' v')^{s-2n}. \quad (3.21)$$

Here, $E \equiv E(e, e)(p)$, $E' \equiv E(e, e')(p)$, $E'' \equiv E(e', e')(p)$ as in eq. (3.4).

³C. Gaß has suggested this simpler proof of a stronger statement than in a previous version.

From eq. (3.21), one gets the correlations of all escort fields $a^{(s,r)}$ by descending in r with the defining recursion eq. (3.12). It turns out to be more convenient to descend directly to $r = 0$: $a^{(s,0)}(e, x) = \frac{1}{s!}(-m)^{-s}(\partial_x \partial_v) a^{(s,s)}(e, x, v)$, and then use the ascending recursion eq. (3.15). This strategy will allow to study limits of $a_{\mu_1 \dots \mu_r}^{(s,r)}$ for fixed r while s increases.

Taking the divergence in all indices via $(\partial_x \partial_v) = -i(p \partial_v)$ and $(\partial_{x'} \partial_{v'}) = +i(p \partial_{v'})$, just amounts to putting $v = v' = p/m$ in the two-point kernel eq. (3.21). One gets the two-point kernel of the string-localized “scalar” escort field $a^{(s,0)}$:

Proposition 3.8 (see [16], section 3)

$$M_m^{a^{(s,0)}(-e), a^{(s,0)}(e')} = (-1)^s m^{-2s} \sum \beta_n^s \cdot [(pEp)(pE''p)]^n \cdot (pE'p)^{s-2n}, \quad (3.22)$$

where $(pEp) = E(e, e)(p) = -m^2(1 - m^2 q_e^2)$, and similar for $(pE'p)$, $(pE''p)$.

We rewrite eq. (3.22) as

$$M_m^{a^{(s,0)}(-e), a^{(s,0)}(e')} = (1 - m^2(q_e q_{e'}))^s \cdot F_s(z_m(q_e, q_{e'})) \equiv P_m^s(q_e, q_{e'}) \quad (3.23)$$

where $z_m(q, q') = \frac{(1-m^2 q^2)(1-m^2 q'^2)}{(1-m^2(qq'))^2}$. Notice that $P_m^s(q_e, q_{e'})$ is a polynomial in $q_e, q_{e'}$, hence immediately well-defined as a distribution.

From this, one may obtain the two-point kernels for $a^{(s,r)}$ by ascending with eq. (3.15), using eq. (2.16) and eq. (2.18) in momentum space.

We have also defined string-localized fields $A^{(s,r)}$ that decouple in the massless limit at fixed s , and become potentials for the massless field strengths $F^{(r)}$ of helicity $\pm r$:

Definition 3.9 (see [16], Proposition 3.8)

$$A^{(s,r)}(e, x, v) := N_0^{(s,r)} \cdot \sum_{2k \leq r} \gamma_k^r \cdot (-E_e(v))^k a^{(s,r-2k)}(e, x, v) \quad (3.24)$$

with E_e as in eq. (2.36). The coefficients are $\gamma_k^r = \frac{1}{4^k k!} \frac{r!}{(r-2k)!} \frac{1}{(1-r)_k}$, and $N_0^{(s,r)} := \left[\binom{s}{r} \frac{\Gamma(\frac{1}{2}+s)\Gamma(1+r)}{\Gamma(\frac{1}{2}+\frac{r+s}{2})\Gamma(1+\frac{r+s}{2})} \right]^{\frac{1}{2}}$.

Proposition 3.10 (see [16], Corollary 3.10) *In the limit $m \rightarrow 0$ at fixed s and r , the string-localized fields $A^{(s,r)}(e, x)$ mutually decouple, and are potentials for the point-localized massless field strengths $F^{(r)}(x)$ associated with the Wigner representations of helicities $\pm r$. In particular, at $m = 0$ they no longer depend on s , they are traceless, conserved, and satisfy $e^\mu A_{\mu\mu_2 \dots \mu_r}^{(s,r)} = 0$.*

4 The Pauli-Lubanski limit

After these preparations, we turn to the Pauli-Lubanski limit. The limit of the “scalar” escort fields $a^{(s,0)}(e, x)$ is rather easy. It relies on two lemmas.

Lemma 4.1 For $s \in \mathbb{N}$, the identity of polynomials of degree $\lfloor \frac{s}{2} \rfloor$

$${}_2F_1\left(\frac{-s}{2}, \frac{1-s}{2}; \frac{1}{2} - s; z\right) = \frac{\Gamma\left(1 + \frac{s}{2}\right)\Gamma\left(\frac{1}{2} + \frac{s}{2}\right)}{\Gamma\left(\frac{1}{2} + s\right)} \cdot {}_2F_1\left(\frac{-s}{2}, \frac{1-s}{2}; 1; 1-z\right)$$

holds. We henceforth abbreviate this identity as

$$F_s(z) = F_s(1) \cdot G_s(1-z), \quad (4.1)$$

in accord with the previous notation eq. (3.20). In particular, $F_s(1) = \frac{\Gamma(1+\frac{s}{2})\Gamma(\frac{1}{2}+\frac{s}{2})}{\Gamma(\frac{1}{2}+s)} \equiv (N_0^{(s,0)})^{-2}$. For large s , this decays asymptotically like $F_s(1) \approx 2^{-s} \sqrt{\pi s}$.

Proof. The identity is the formula 2.9(43) in [7] with parameters $a = \frac{-s}{2}$, $b = \frac{1-s}{2}$, $c = \frac{1}{2} - s$. The value $F_s(1)$ follows because $G_s(0) = 1$. The asymptotic form follows by Stirling's approximation of the Γ function. \square

Lemma 4.2 In the limit $s \rightarrow \infty$, the pointwise limit holds

$$\lim_{s \rightarrow \infty} G_s\left(-\frac{u^2}{s^2}\right) \equiv \lim_{s \rightarrow \infty} {}_2F_1\left(\frac{-s}{2}, \frac{1-s}{2}; 1; -\frac{u^2}{s^2}\right) = J_0(u). \quad (4.2)$$

Proof. The power series expansion reads

$$G_s\left(-\frac{u^2}{s^2}\right) = \sum_{k \geq 0} \frac{(-s)_{2k}}{4^k k!^2} \left(-\frac{u^2}{s^2}\right)^k \rightarrow \sum_{k \geq 0} \frac{1}{k!^2} \left(-\frac{u^2}{4}\right)^k = J_0(u),$$

where the limit of the coefficients is taken separately for each k . The pointwise convergence in u follows by absolute convergence of the sums. \square

Now we turn to the fields.

Proposition 4.3 In the Pauli-Lubanski limit, the two-point kernel of the rescaled scalar escort fields $N_0^{(s,0)} \cdot a^{(s,0)}(e)$ converges to $J_0(\kappa \sqrt{-(q_e(p) - q_{e'}(p))^2})$. Since $J_0(z)$ is a power series in z^2 , the two-point kernel is a power series in q_e and $q_{e'}$. The convergence is pointwise, i.e., it holds formally for fixed values of q_e and $q_{e'}$, and more precisely for fixed test functions in e , e' and p , on whose support $(q_e(p) - q_{e'}(p))^2$ is bounded.

Proof. Using Lemma 4.1, we rewrite eq. (3.23) as

$$M_m^{a^{(s,0)}(-e), a^{(s,0)}(e')} = (1 - m^2(q_e q_{e'}))^s \cdot F_s(1) G_s(1 - z_m(q_e, q_{e'})). \quad (4.3)$$

The prefactor $\left(1 - \frac{\kappa^2(qq')}{s(s+1)}\right)^s$ converges separately to 1 by Euler's formula. The claim then follows by Lemma 4.2. \square

Comparing the limit obtained in Proposition 4.3 with the two-point kernel eq. (3.8) of the standard string-localized field $\phi^{\kappa(0)}$, one notes that the Kähler factors are missing. But they can be produced by applying the operators $(1 - m\sqrt{-e^2}I_e)^s$ before the limit is taken. Namely, by Euler's formula, $\omega^\kappa(e, p) = e^{-i\kappa \frac{\sqrt{-e^2}}{(ep)_+}}$ is the limit of $\left(1 - im \frac{\sqrt{-e^2}}{(ep)_+}\right)^s$, and $\frac{i}{(ep)_+}$ is the momentum space version of the string-integration I_e .

Corollary 4.4 *The standard string-localized field $\phi^{\kappa(0)}(e, x)$ is, up to unitary equivalence, the Pauli-Lubanski limit of $N_0^{(s,0)} \cdot (1 - m\sqrt{-e^2}I_e)^s a^{(s,0)}(e, x)$.*

Remark 4.5 *The convergence of the two-point kernels is much easier to see than that of the intertwiners, because the former is basis independent. The reason is that “convergence” of vectors on different Hilbert spaces makes only sense with a suitable inductive limit (a sequence of embeddings of the Hilbert spaces). For the case at hand, this inductive limit of the representation spaces of the massive little group $SO(3)$ is described in appendix A, in such a way that the matrix elements converge to a representation of the massless little group $E(2)$, and this extends to the induced Wigner representation [14].*

With the given inductive identification of bases, one should be able to prove the convergence of intertwiners up to unitary equivalence. We refrain from doing this because the two-point function uniquely specifies a free field, and hence we may conclude the convergence of the intertwiners and of the fields up to unitary equivalence. In this sense, we may say

$$\lim_{\kappa} N_0^{(s,0)} \cdot (1 - m\sqrt{-e^2}I_e)^s a^{(s,0)}(e, x) = \phi^{\kappa(0)}(e, x). \quad (4.4)$$

Proposition 4.6 *The fields $N_0^{(s,0)} \cdot (1 - m\sqrt{-e^2}I_e)^s a^{(s,r)}(e, x)$ converge in the Pauli-Lubanski limit (in the same sense as specified in Remark 4.5) to the infinite-spin fields $\phi^{\kappa(r)}(e, x)$ defined in Definition 2.4.*

Proof. (i) The recursion eq. (3.15) for $a^{(s,r)}$ implies the recursion

$$-\kappa \tilde{\phi}^{\kappa(r+1)}(e, x, v) = (r \cdot (v\partial_x) + (e\partial_x)(v\partial_e)) \tilde{\phi}^{\kappa(r)}(e, x, v)$$

for the limits of $N_s^{(s,0)} \cdot a^{(s,r)}$, because for each fixed r , $m(s - r)$ can be replaced by κ . Restoring the Kähler factors of the intertwiners by means of the operators $(1 - m\sqrt{-e^2}I_e)^s$ as in Proposition 4.3, implies the recursion eq. (2.27) for the limits of $N_0^{(s,0)} \cdot (1 - m\sqrt{-e^2}I_e)^s a^{(s,r)}(e, x)$. The claim follows, because eq. (2.27) defines $\phi^{\kappa(r)}$. \square

We now study the Pauli-Lubanski limit of the fields $A^{(s,r)}$, defined in Definition 3.9. Using the trace identity in eq. (3.17), we rewrite eq. (3.24) as

$$A^{(s,r)}(e, x) = \frac{N_0^{(s,r)}}{N_0^{(s,0)}} \cdot \left(S_r \sum_{2k \leq r} \gamma_k^r \cdot ((J_e \otimes J_e)\eta \circ \eta^t)^{\otimes k} \right) \cdot N_0^{(s,0)} a^{(s,r)}(e, x), \quad (4.5)$$

regarding $a^{(s,r)}$ as a tensor in $(\mathbb{R}^4)^{\otimes r}$ of which pairs of indices are contracted with η and restored by $E_e = (J_e \otimes J_e)\eta$, and the result in $(\mathbb{R}^4)^{\otimes r}$ is symmetrized.

Corollary 4.7 *In the Pauli-Lubanski limit, the fields $(1 - m\sqrt{-e^2}I_e)^s A^{(s,r)}(e, x)$ converge (in the same sense as specified in Remark 4.5) to the infinite-spin fields $\Phi^{\kappa(r)}(e, x)$ defined in Definition 2.7.*

Proof. One easily sees from the formula given in Definition 3.9 that the prefactor $\frac{N_0^{(s,r)}}{N_0^{(s,0)}}$ converges to $2^{r/2}$ as $s \rightarrow \infty$. The operator does not depend on s and m and commutes with $(1 - m\sqrt{-e^2}I_e)^s$, and $N_0^{(s,0)} \cdot (1 - m\sqrt{-e^2}I_e)^s a^{(s,r)}(e, x)$ converges to $\phi^{\kappa(r)}$ by Proposition 4.6. Thus, the limit of eq. (4.5) is equivalent to eq. (2.33). \square

5 Stress-energy tensors

In [16], we have introduced several stress-energy tensors for the (m, s) fields that all yield the correct infinitesimal generators

$$P_\sigma = \int_{x_0=t} d^3\vec{x} T_{0\sigma}, \quad M_{\sigma\tau} = \int_{x_0=t} d^3\vec{x} (x_\sigma T_{0\tau} - x_\tau T_{0\sigma}) \quad (5.1)$$

of the Poincaré group. They differ by derivative terms that vanish upon the integrations eq. (5.1). We display here: the point-localized “reduced” stress-energy tensor

$$T_{\rho\sigma}^{(s)\text{red}}(x) = -\frac{1}{4}(-1)^s :A^{\text{P}(s)\mu_1\dots\mu_s} \overset{\leftrightarrow}{\partial}_\rho \overset{\leftrightarrow}{\partial}_\sigma A_{\mu_1\dots\mu_s}^{\text{P}(s)}:(x) + \partial^\mu \Delta T_{\rho\sigma;\mu}^{(s)\text{red}} \quad (5.2)$$

and the string-localized “regular” stress-energy tensor

$$\begin{aligned} t_{\rho\sigma}^{(s)\text{reg}}(e_1, e_2, x) &= \sum_{r \leq s} \binom{s}{r} t_{\rho\sigma}^{(s,r)}(e_1, e_2, x), \\ t_{\rho\sigma}^{(s,r)}(e_1, e_2, x) &= -\frac{1}{4}(-1)^r :a^{(s,r)\mu_1\dots\mu_r}(e_1) \overset{\leftrightarrow}{\partial}_\rho \overset{\leftrightarrow}{\partial}_\sigma a_{\mu_1\dots\mu_r}^{(s,r)}(e_2):(x) + \partial^\mu \Delta t_{\rho\sigma;\mu}^{(r)\text{reg}}. \end{aligned} \quad (5.3)$$

The derivative terms $\partial^\mu \Delta T_{\rho\sigma;\mu}$ do not affect the momentum generators, but they have to be added to get the correct infinitesimal Lorentz transformations. Explicit expressions can be found in [16]. We do not need them at this point, see however Example 5.5. The tensors $t^{(s,r)}$ are separately conserved, but only their sum eq. (5.3) generates the correct Poincaré transformations.

We have also given massless stress-energy tensors

$$\begin{aligned} T_{\rho\sigma}^{(s),m=0}(e_1, e_2, x) &= \sum_{r \leq s} \binom{s}{r} T_{\rho\sigma}^{(s,r),m=0}(e_1, e_2, x), \\ T_{\rho\sigma}^{(s,r),m=0}(e_1, e_2, x) &= -\frac{1}{4}(-1)^r :A^{(s,r)\mu_1\dots\mu_r}(e_1) \overset{\leftrightarrow}{\partial}_\rho \overset{\leftrightarrow}{\partial}_\sigma A_{\mu_1\dots\mu_r}^{(s,r)}(e_2):(x) + \partial^\mu \Delta T_{\rho\sigma;\mu}^{(s,r),m=0}, \end{aligned} \quad (5.4)$$

that, in spite of the definition of the string-localized fields $A^{(s,r)}$ as massless limits of fields on the massive Fock space of spin s (in terms of $a^{(s,r' \leq r)}$, see Definition 3.9), do not depend on s . At $m = 0$, eq. (5.4) describes the decoupling of the massive spin s representation into a direct sum of helicity representations [16, 17].

Let us discuss the possible role of these tensors in the Pauli-Lubanski limit. The point-localized reduced stress-energy tensor $T^{(s)\text{red}}$ does not admit a massless limit because of inverse powers m^{-4s} in the two-point function. The string-localized regular stress-energy tensor $t^{(s)\text{reg}}$ becomes in the Pauli-Lubanski limit an infinite sum over $r \leq s \rightarrow \infty$ of terms $t^{(s,r)}$, that each converge to zero (due to the explicit factor $F_s(1) \approx 2^{-s}$ present in every two-point function $(\Omega, a^{(s,r)} a^{(s,r')} \Omega)$). Yet, the sum is not zero and is still a valid stress-energy tensor, but it cannot be expressed as a sum of limits of $t^{(s,r)}$. (These and other interesting features are nicely illustrated by the expectation values of the energy density and the pressure in thermal states at inverse temperature β , cf. section 5.3.4. E.g., while the contribution of each r to the thermal energy goes to zero, the sum over r diverges like $2s + 1$.)

Eq. (5.4) seems to be better suited for the Pauli-Lubanski limit, because each term $T^{(s,r)}$ has a limit. At fixed s , the massless stress-energy tensor $T^{(s),m=0}$ is the limit of massive conserved tensors $T^{(s),m}$ [16]. The latter differ from $t^{(s)\text{reg}}$, apart from irrelevant terms that do not affect the Poincaré generators, by further terms that do disturb the generators, and that decay like $O(m)$ at fixed s ([16], Propositions 4.5 and 4.6). But such terms may grow with s , so that it is difficult to keep control, whether the Pauli-Lubanski limit produces the correct generators.

Our main result in this section computes the stress-energy tensor directly in the infinite-spin representation:

Proposition 5.1 *Let $\Phi^{\kappa(r)}(e, x)$ be the string-localized fields defined in Definition 2.7 (that may be obtained as Pauli-Lubanski limits by Corollary 4.7). The conserved symmetric tensor*

$$T_{\rho\sigma}^{\kappa}(e, x) = \sum_{r=0}^{\infty} T_{\rho\sigma}^{\kappa(r)}(e, x), \tag{5.5}$$

$$T_{\rho\sigma}^{\kappa(r)}(e, x) = -\frac{1}{4}(-1)^r : \Phi^{\kappa(r)\mu_1 \dots \mu_r}(e) \overset{\leftrightarrow}{\partial}_\rho \overset{\leftrightarrow}{\partial}_\sigma \Phi_{\mu_1 \dots \mu_r}^{\kappa(r)}(e) : (x) + \partial^\mu \Delta T_{\rho\sigma; \mu}^{\kappa(r)}$$

is a string-localized stress-energy tensor for the infinite-spin representation. An expression for $\Delta T_{\rho\sigma; \mu}^{\kappa(r)}$ will be given in eq. (5.13).

The proof in section 5.2 exhibits $\Delta T^{\kappa(r)}$ as a sum of two pieces. The “second piece” $\Delta_2 T^{\kappa(r)}$ is absent at finite s and must be identified with the accumulation of the above-mentioned uncontrolled errors.

Remark 5.2 *In order to get the correct generators, the two string-localized fields in the Wick product have to be taken with $e_1 = -e_2$, see Remark 5.7. Mund has recently shown that the Wick product with parallel strings is well-defined as a distribution in x and e .*

The problematic issue with eq. (5.5) is instead the infinite sum over r . Because the fields $\Phi^{\kappa(r)}$ do not decouple (see eq. (2.36)), correlation functions and matrix elements involving T^κ may be divergent sums. Recall from Remark 2.10 that each $T^{\kappa(r)}$ will have a non-vanishing expectation value in a state with sharp magnetic quantum number $n \in \mathbb{Z}$. The consequences of this feature for two-point functions and commutators of T^κ with the fields $\Phi^{\kappa(r)}$ will be sketched in section 5.3.

5.1 Quantum stress-energy tensors

We obtained Proposition 5.1 with a new systematic strategy to find stress-energy tensors for free quantum fields, that does not refer to a classical action principle. Instead, it is intrinsically based on the Wigner representation theory, along with a choice of intertwiners that allow to “decompose” the (global) generators into integrals over localized densities.

We first outline the general strategy, that is flexible enough to include also point- and string-localized finite-spin tensor fields and Dirac fields. The stress-energy tensors eq. (5.2), eq. (5.3) and eq. (5.4) could have been found by this strategy.

Applied to the infinite-spin case (section 5.2), it does not use the Pauli-Lubanski approximation, i.e., it proceeds directly in the limit, using just the results of Proposition 2.9.

To keep the argument transparent, we present only the case of bosonic hermitean fields, where the u - and v -intertwiners multiplying creation and annihilation operators are complex conjugates of each other.

We start with the familiar global “second quantization” formula for the momentum operator

$$P_\sigma = \int d\mu_m(p) \sum_n p_\sigma a_n^*(p) a_n(p), \quad (5.6)$$

where the sum extends over an orthonormal basis of the representation space \mathcal{H}_d of the unitary representation d of the little group. We write this as

$$P_\sigma = \int d\mu_m(p_1) d\mu_m(p_2) \sum_{n_1 n_2} p_{1\sigma} a_{n_1}^*(p_1) \delta_{n_1 n_2} (2\pi)^3 \delta(\vec{p}_1 - \vec{p}_2) (p_{10} + p_{20}) a_{n_2}(p_2),$$

and insert $(2\pi)^3 \delta(\vec{p}_1 - \vec{p}_2) = \int d^3 \vec{x} e^{-i(\vec{p}_1 - \vec{p}_2)\vec{x}} = \int d^3 \vec{x} e^{i(p_1 - p_2)x}$, which is independent of x^0 . Separating the factors that depend on p_1 and on p_2 , respectively, and interchanging the x - and p -integrations, one gets an \vec{x} -integral over the product of (derivatives of) two expressions $\int d\mu_m(p_1) e^{ip_1 x} a_{n_1}^*(p_1)$ and $\int d\mu_m(p_2) e^{-ip_2 x} a_{n_2}(p_2)$. These are of course *not* the creation and annihilation parts of a local and covariant quantum field, by the well-known problem of the nonlocal Wigner “rotations”, that is the reason why one has to use intertwiners in Wigner quantization [28–30].

So, let there be a (possibly reducible) representation D of the Lorentz group and intertwiners $u_{Mn}(p)$ satisfying

$$u(\Lambda p) = (D(\Lambda) \otimes d(W_{\Lambda,p})) u(p) \quad (\Lambda \in \text{SO}(1,3)_+^\uparrow) \quad (5.7)$$

(generalizing eq. (2.4) and eq. (2.25); $D(\Lambda)$ may include the action on a string variable), and that fulfill the completeness relation

$$g^{MN} u_{Mn_1}(p) \overline{u_{Nn_2}(p)} = \delta_{n_1 n_2} \quad (5.8)$$

with a suitable “metric” g^{MN} . In order for eq. (5.8) to be fulfilled, $D(\Lambda)$ may have to be a direct sum of tensor representations (in the bosonic case), and hence g a corresponding direct sum of tensor products of η ; we present examples for such u_{Mn} below (Example 5.5 and section 5.2).

Inserting the partition of unity eq. (5.8) into the previous expression, we get

$$P_\sigma = -\frac{1}{2} g^{MN} \int d^3 \vec{x} \phi_M^+(x) \overset{\leftrightarrow}{\partial}_0 \overset{\leftrightarrow}{\partial}_\sigma \phi_N^-(x) = -\frac{1}{4} g^{MN} \int d^3 \vec{x} : \phi_M \overset{\leftrightarrow}{\partial}_0 \overset{\leftrightarrow}{\partial}_\sigma \phi_N : (x),$$

where $\phi_M^+(x) = \int d\mu_m(p) e^{ipx} u_{Mn}(p) a_n^*(p)$, and $\phi_M^-(x)$ its hermitean conjugate. $\phi_M(x) = \phi_M^+(x) + \phi_M^-(x)$ is a covariant field, point-localized or string-localized according to the choice of the intertwiners. The second equality holds by symmetry of the Wick product and because the operator $\overset{\leftrightarrow}{\partial}_0$ vanishes on the creation-creation and annihilation-annihilation parts of the Wick product thanks to $p_{10} = p_{20}$.

The last expression is the desired local representation of eq. (5.6). The integrand

$$\tilde{T}_{\rho\sigma}(x) = -\frac{1}{4}g^{MN}:\phi_M \overset{\leftrightarrow}{\partial}_\rho \overset{\leftrightarrow}{\partial}_\sigma \phi_N:(x) \quad (5.9)$$

is a first candidate for a stress-energy tensor, that by construction produces the correct generators P_σ of translations. Of course, this construction is only unique up to terms that vanish upon the \vec{x} -integration, and we shall see presently that we need to add such terms in order to produce also the correct Lorentz generators.

We obtain the global form of the Lorentz generators from the transformation law of the creation operators:

$$U(\Lambda)a_n^*(p)U(\Lambda)^* = a_m^*(\Lambda p)d_{mn}(W_{\Lambda,p}) \quad (\Lambda \in \text{SO}(1,3)_+^\uparrow).$$

Infinitesimally:

$$i[M_{\sigma\tau}, a_n^*(p)] = (\delta_{nn'}(p_\sigma \partial_{p^\tau} - p_\tau \partial_{p^\sigma}) + d(\omega_{\sigma\tau})_{nn'}^t)a_{n'}^*(p),$$

where $\omega_{\sigma\tau}$ is the infinitesimal Wigner “rotation”. The latter depends on the choice of the standard “boosts”; we do not display it because it is going to cancel anyway. What matters is that $d(\omega_{\sigma\tau})$ is anti-hermitean on \mathcal{H}_d because the representation d is unitary, hence $-i(\mathbb{I}_d p \wedge \partial_p + d(\omega)^t)_{\sigma\tau} \equiv -i(\mathbb{I}_d(p_\sigma \partial_{p^\tau} - p_\tau \partial_{p^\sigma}) + d(\omega_{\sigma\tau})^t)$ is selfadjoint on the one-particle space. It follows:

Lemma 5.3 *The selfadjoint infinitesimal generators $M_{\sigma\tau}$ of the Lorentz transformations are the second quantization*

$$M_{\sigma\tau} = -i \int d\mu_m(p) ((\delta_{nn'} p \wedge \partial_p + d(\omega)_{nn'}^t)_{\sigma\tau} a_{n'}^*(p)) a_n(p)$$

of the operators $-i(\mathbb{I}_d p \wedge \partial_p + d(\omega)^t)_{\sigma\tau}$.

We now proceed as before with the momentum generators, inserting the partition of unity for the momenta via an \vec{x} -integration, and the partition of unity for the spin components via the sum eq. (5.8) over intertwiners. By partial integration in p_1 , the operators $-i(\mathbb{I}_d p_1 \wedge \partial_{p_1} + d(\omega)^t)$ acting on the creation operators $a^*(p_1)$ are shifted to the wave function $u(p_1)e^{ip_1x}$ where they act like

$$-i(-p_1 \wedge \partial_{p_1} + d(\omega))(u(p_1)e^{ip_1x}) = ((i p_1 \wedge \partial_{p_1} - i d(\omega) + x \wedge p_1)u(p_1))e^{ip_1x}.$$

The term $x \wedge p_1$ is treated exactly as before, and gives the expected contribution $\int d^3\vec{x}(x_\sigma \tilde{T}_{0\tau} - x_\tau \tilde{T}_{0\sigma})$ to $M_{\sigma\tau}$. We are now going to compute the remaining term.

The infinitesimal version of the intertwining property eq. (5.7) is

$$(p \wedge \partial_p)u(p) = (D(\Omega) + d(\omega))u(p),$$

therefore $i(p \wedge \partial_p - d(\omega))_{\sigma\tau}u = iD(\Omega_{\sigma\tau})u$ involves the infinitesimal Lorentz transformation $D(\Omega_{\sigma\tau})$ of the intertwiner (which is the same as that of the field). Thus,

$$M_{\sigma\tau} = \int d^3\vec{x}(x_\sigma \tilde{T}_{0\tau} - x_\tau \tilde{T}_{0\sigma}) + \Delta M_{\sigma\tau}$$

where

$$\Delta M_{\sigma\tau} = -\frac{1}{2}g^{MN} \int d^3\vec{x} : (D(\Omega_{\sigma\tau})\phi)_M \overset{\leftrightarrow}{\partial}_0 \phi_N :. \quad (5.10)$$

We now use Lemma B.1 in [16]: for a symmetric and conserved tensor $\Theta_{\rho\sigma} = \partial^\mu [X_{\rho\mu} \overset{\leftrightarrow}{\partial}_\sigma Y + X_{\sigma\mu} \overset{\leftrightarrow}{\partial}_\rho Y]$, where $X_{\rho\mu}$ is anti-symmetric and both $X(x)$ and $Y(x)$ satisfy the Klein-Gordon equation, one has

$$\int d^3\vec{x} (x_\sigma \Theta_{0\tau} - x_\tau \Theta_{0\sigma}) = 2 \int d^3\vec{x} X_{\sigma\tau} \overset{\leftrightarrow}{\partial}_0 Y.$$

Corollary 5.4 *Under the assumption eq. (5.8), the total stress-energy tensor $\tilde{T}_{\rho\sigma} + \partial^\mu \Delta T_{\rho\sigma;\mu}$ produces the correct generators of the Poincaré group eq. (5.1), where $\tilde{T}_{\rho\sigma}$ is given by eq. (5.9) and*

$$\Delta T_{\rho\sigma;\mu} = -\frac{1}{4}g^{MN} \left[D(\Omega_{\rho\mu})_M{}^{M'} : \phi_{M'} \overset{\leftrightarrow}{\partial}_\sigma \phi_N : (x) + (\rho \leftrightarrow \sigma) \right]. \quad (5.11)$$

Notice that $\tilde{T}_{\rho\sigma}$ and $\partial^\mu \Delta T_{\rho\sigma;\mu}$ are separately manifestly symmetric and conserved.

Example 5.5 *For the massive Proca field of spin s , the standard intertwiner $u_n^{\text{P}(s)}(p) = (B_p E_3)^{\otimes s} T_n$ as in the proof of Proposition 3.4 fulfills eq. (5.8) with the metric $g = (-1)^r \eta^{\otimes r}$. The sign is due to the anti-isometric property of the embedding $E_3 : \mathbb{R}^3 \rightarrow \mathbb{R}^4$. The field $A^{\text{P}(s)}$ transforms in the representation $D(\Lambda) = \Lambda^{\otimes s}$ of the Lorentz group. With $(\Omega_{\sigma\tau})_\mu{}^{\mu'} = \eta_{\tau\mu} \delta_\sigma^{\mu'} - \eta_{\sigma\mu} \delta_\tau^{\mu'}$, one gets*

$$\Delta M_{\sigma\tau} = -(-1)^r r \int d^3x : A_{\sigma\mu_2 \dots \mu_s}^{\text{P}(s)} \overset{\leftrightarrow}{\partial}_0 A_\tau^{\text{P}(s)\mu_2 \dots \mu_s} : (x)$$

and

$$\Delta T_{\rho\sigma;\mu}^{\text{P}(s)} = -(-1)^r \frac{r}{2} \left[: A_{\rho\mu_2 \dots \mu_s}^{\text{P}(s)} \overset{\leftrightarrow}{\partial}_\sigma A_\mu^{\text{P}(s)\mu_2 \dots \mu_s} : (x) + (\rho \leftrightarrow \sigma) \right].$$

This gives the reduced stress-energy tensor eq. (5.2), found in [16] by a less systematic approach. Its main part $\tilde{T}_{\rho\sigma}^{\text{P}(s)}$ (without the derivative term) appeared already in Fierz' paper [8]. For the same Wigner representation (m, s) , none of the string-localized intertwiners $u^{(s,r)}$ fulfills eq. (5.8) separately. They must be combined, in a manner similar to the infinite-spin case below. In this case, Corollary 5.4 gives the regular stress-energy tensor eq. (5.3).

5.2 Proof of Proposition 5.1

In order to prove Proposition 5.1, we apply the prescription of the preceding subsection. By Corollary 5.4, we need to fulfill eq. (5.8) with intertwiners of string-localized fields. For the infinite-spin representations, the representation space \mathcal{H}_d is $L^2(\kappa S^1)$ with $d\mu_\kappa(\vec{k}) = \frac{d\varphi}{2\pi}$, hence $\delta_{n_1 n_2}$ is replaced by $\delta_\kappa(\vec{k}_1, \vec{k}_2) = 2\pi \cdot \delta_{2\pi}(\varphi_1 - \varphi_2)$.

Lemma 5.6 *The completeness relation eq. (5.8) is fulfilled by*

$$\sum_{r \geq 0} (-1)^r \eta^{\mu_1 \nu_1} \dots \eta^{\mu_r \nu_r} U_{\mu_1 \dots \mu_r}^{\kappa(r)}(e, p)(\vec{k}_1) \overline{U_{\nu_1 \dots \nu_r}^{\kappa(r)}(e, p)(\vec{k}_2)} = \delta_\kappa(\vec{k}_1, \vec{k}_2). \quad (5.12)$$

Proof. By Proposition 2.9(ii), the left-hand side equals $u^{\kappa(0)}(e, p)(\vec{k}_1) \overline{u^{\kappa(0)}(e, p)(\vec{k}_2)}$ times

$$\left(\sum_{r \in \mathbb{Z}} e^{-ir(\varphi_1 - \varphi_2)} \right) = 2\pi \delta_{2\pi}(\varphi_1 - \varphi_2) = \delta_\kappa(\vec{k}_1, \vec{k}_2),$$

and at $\vec{k}_1 = \vec{k}_2$ the factor is = 1 because $u^{\kappa(0)}(e, p)(\vec{k})$ is a complex phase. \square

Remark 5.7 *One sees why it is crucial that the two strings in eq. (5.12) are equal: otherwise the phase factors $u^{\kappa(0)} \overline{u^{\kappa(0)}}$ would fail to cancel. It is also essential that $u^{\kappa(0)}$ is a function (Proposition 2.1), since otherwise the product of distributions involving $\frac{e}{(ep)_+}$ and $\overline{\frac{e}{(ep)_+}} = \frac{e}{(ep)_-}$ were ill-defined. Because the factors $u^{\kappa(0)}$ only appear through the inductive limit appendix A, cf. Remark 4.5, they are absent in the case of finite spin, and the analogous requirement of coinciding strings may be dropped, as in eq. (5.3).*

We can thus apply Corollary 5.4 *mutatis mutandis*. Apart from the specific partition of unity eq. (5.12), the only change is the dependence of the intertwiners on e , which are also transformed along with the Lorentz tensors by $D(\Lambda)$, specifying eq. (5.7) as

$$U^{\kappa(r)}(e, \Lambda p) = (\Lambda^{\otimes r} \otimes d_\kappa(W_{\Lambda, p})) U^{\kappa(r)}(\Lambda^{-1} e, p) \quad (\Lambda \in \text{SO}(1, 3)_+^\uparrow).$$

Therefore, $D(\Omega_{\sigma\tau})$ contains, besides the infinitesimal Lorentz matrices, the additional term $-(e \wedge \partial_e)_{\sigma\tau}$, and ΔT is a sum of two terms:

$$\Delta T_{\rho\sigma; \mu}^\kappa(e, x) = \Delta_1 T_{\rho\sigma; \mu}^\kappa(e, x) + \Delta_2 T_{\rho\sigma; \mu}^\kappa(e, x) \quad (5.13)$$

where

$$\Delta_1 T_{\rho\sigma; \mu}^\kappa(e, x) = - \sum_{r \geq 0} (-1)^r \frac{r}{2} \left[: \Phi_{\rho\mu_2 \dots \mu_r}^{\kappa(r)}(e) \overset{\leftrightarrow}{\partial}_\sigma \Phi_\mu^{\kappa(r)\mu_2 \dots \mu_r}(e) : (x) + (\rho \leftrightarrow \sigma) \right]$$

similar as in Example 5.5, and

$$\Delta_2 T_{\rho\sigma; \mu}^\kappa(e, x) = - \sum_{r \geq 0} \frac{(-1)^r}{8} \left[: \Phi_{\mu_1 \dots \mu_r}^{\kappa(r)}(e) (e \wedge \partial_e)_{\rho\mu} \overset{\leftrightarrow}{\partial}_\sigma \Phi^{\kappa(r)\mu_1 \dots \mu_r}(e) : (x) + (\rho \leftrightarrow \sigma) \right].$$

With the computation of \tilde{T} , coinciding with the expression displayed in eq. (5.5), and the specification of $\Delta T = \Delta_1 T + \Delta_2 T$, the proof of Proposition 5.1 is complete. \square

Remark 5.8 *Derivatives w.r.t. e seem to mix $\Phi^{\kappa(r)}$ with $\Phi^{\kappa(r \pm 1)}$ by eq. (2.36), but because eq. (2.36) involves a symmetrization, we cannot simply write $(e \wedge \partial_e) \Phi^{\kappa(r)}$ as a combination of $\Phi^{\kappa(r \pm 1)}$. Recall, however, that the fields $\Phi^{\kappa(r)}(e)$ are simultaneously defined for all e on the same Hilbert space, and the derivative w.r.t. e does not change the localization. Therefore, each $T^{\kappa(r)}$ in eq. (5.5) is well-defined.*

The potential problems due to the infinite summation over r in eq. (5.5) will be discussed in the next section.

The same general strategy outlined in section 5.1 applies to conserved currents of complex fields. In the case of infinite spin, the partition of unity eq. (5.12) inserted into the charge operator

$$Q = \int d\mu_0(p) d\mu_\kappa(\vec{k}) (a^*(p, \vec{k}) a(p, \vec{k}) - b^*(p, \vec{k}) b(p, \vec{k}))$$

gives rise to the current

$$J_\rho^\kappa(e, x) = i \sum_{r \geq 0} (-1)^r : \Phi_{\mu_1 \dots \mu_r}^{\kappa(r)*}(e) \overleftrightarrow{\partial}_\rho \Phi^{\kappa(r)\mu_1 \dots \mu_r}(e) : (x). \quad (5.14)$$

Remark 5.9 *Infinite-spin fields admit no subalgebra of compactly localized observables (“field strengths” or currents) whose charged sectors they would generate from the vacuum [12, 13]. Therefore, “neutral” operators like the current densities or stress-energy tensor cannot be point-localized as in the massive case; the localization on a pair of opposite strings seems to be the best that is possible.*

5.3 Properties of the infinite-spin stress-energy tensor

We present here some qualitative material that helps to assess the mathematical nature of fields like the infinite-spin stress-energy tensor eq. (5.5). The rigorous analytical treatment is beyond the scope of this article.

5.3.1 Matrix elements

For a generic one-particle state $\Psi = \int d\mu_0(p) d\mu_\kappa(\vec{k}) \psi(p, \vec{k}) a^*(p, \vec{k}) \Omega$, we compute matrix elements

$$(\Psi, \Phi^{\kappa(r)}(e, x) \Omega) = \int d\mu_0(p) e^{ipx} (\psi(p), U^{\kappa(r)}(e, p))_\kappa$$

where $(\cdot, \cdot)_\kappa$ is the scalar product of $L^2(\kappa S^1)$. Let $R_{e,p} \begin{pmatrix} \cos \alpha_{e,p} \\ \sin \alpha_{e,p} \end{pmatrix}$ parametrize the 1-2-part of $B_p^{-1} q_e(p)$, so that $u^{\kappa(0)}(e, p)(\vec{k}) = e^{-i(q_e E_p(\vec{k}))} = e^{i\kappa R_{e,p} \cos(\varphi - \alpha_{e,p})}$.

For $\psi^n(p, \vec{k}) = \psi(p) e^{in\varphi}$, the \vec{k} -integration produces Bessel functions eq. (3.9). Thus, with $\mathcal{E}_\pm(e, p) = J_e(p) E_p \varepsilon_\pm$ as in Proposition 2.9

$$(\psi^n(p), U^{\kappa(r)}(e, p))_\kappa = (-1)^r \overline{\psi(p)} \omega^\kappa(e, p) \sum_{\pm} J_{n \pm r}(\kappa R_{e,p}) e^{-i(n \pm r)(\alpha_{e,p} - \frac{\pi}{2})} \cdot \mathcal{E}_\pm(e, p)^{\otimes r}.$$

For $r = 0$, the sum of two terms is replaced by $J_n(\kappa R_{e,p}) e^{-in(\alpha_{e,p} - \frac{\pi}{2})}$.

This formula is not particularly useful, but it shows that there is no correlation between r and n (cf. Remark 2.10), and that infinite sums over r , as in the stress-energy tensor or the current, are potentially dangerous, as already pointed out in Remark 5.2.

Let us exemplarily investigate this issue in various situations: matrix elements, two-point function, and commutators of the stress-energy tensor or the current. In order to simplify the presentation, we consider the scalar Wick square

$$W^\kappa(e, x) = \sum_{r=0}^{\infty} (-1)^r \cdot \Phi^{\kappa(r)\mu_1 \dots \mu_r}(e) \Phi_{\mu_1 \dots \mu_r}^{\kappa(r)}(e)(x), \quad (5.15)$$

in which the characteristic features of the infinite sum can be seen as well. For the actual stress-energy tensor, one basically has to insert polynomial factors of p corresponding to the derivatives $\overset{\leftrightarrow}{\partial}_\rho \overset{\leftrightarrow}{\partial}_\sigma$, and add a similar contribution from ΔT .

We compute matrix elements of the Wick square, for simplicity in one-particle states Ψ_i with wave functions $\psi_i^0(p)$ (i.e., $n = 0$):

$$\begin{aligned} (\Psi_1^0, W^\kappa(e, x) \Psi_2^0) &= 2 \int d\mu_0(p_1) d\mu_0(p_2) e^{i(p_1 - p_2)x} \overline{\psi_1(p_1)} \psi_2(p_2) \cdot \omega^\kappa(e, p_1) \overline{\omega^\kappa(e, p_2)} \cdot \\ &\quad \cdot \sum_{\nu \in \mathbb{Z}} e^{-i\nu(\alpha_{e,p_1} - \alpha_{e,p_2})} J_\nu(\kappa R_{e,p_1}) J_\nu(\kappa R_{e,p_2}). \end{aligned}$$

Similar expressions with $J_{\nu+n_1} J_{\nu+n_2}$ hold for matrix elements in states with $n_i \neq 0$, or for matrix elements between the vacuum and two-particle states.

The point is that the sum over $r \in \mathbb{Z}$ is absolutely convergent thanks to the Cauchy-Schwartz inequality applied to the square-summability of the Bessel functions:

$$\sum_{\nu \in \mathbb{Z}} J_\nu(x)^2 = 1. \quad (5.16)$$

Together with Wick's theorem for matrix elements between multi-particle states, this observation supports our

Conjecture 5.10 *The Wick square eq. (5.15) and likewise the stress-energy tensor eq. (5.5) and the current eq. (5.14) have finite matrix elements in states of finite particle number and finite energy. Because such states are dense in the Fock space, these fields exist as quadratic forms with a dense domain.*

5.3.2 Two-point functions and vacuum fluctuations

The two-point function of $W^{(\kappa)}$ is a double sum over r and r' of the fully contracted squares of two-point functions $(\Omega, \Phi_\mu^{\kappa(r)}(x) \Phi_\nu^{\kappa(r')}(y) \Omega)$ given in Proposition 3.2.

If $R_{e,e',p} \begin{pmatrix} \cos \alpha_{e,e',p} \\ \sin \alpha_{e,e',p} \end{pmatrix}$ parametrizes the 1-2-part of $B_p^{-1}(q_e(p) - q_{e'}(p))$, then

$$\begin{aligned} (\Omega, W^\kappa(e, x) W^\kappa(e', x') \Omega) &= 2 \int d\mu_0(p_1) d\mu_0(p_2) e^{-i(p_1 + p_2)(x - x')} \prod_{i=1,2} \overline{\omega^\kappa(e, p_i)} \omega^\kappa(e', p_i) \cdot \\ &\quad \cdot \sum_{\nu, \nu' \in \mathbb{Z}} (-1)^{\nu - \nu'} e^{i(\nu + \nu')(\alpha_{e,e',p_1} - \alpha_{e,e',p_2})} J_{\nu + \nu'}(\kappa R_{e,e',p_1}) J_{\nu + \nu'}(\kappa R_{e,e',p_2}). \end{aligned}$$

The problem is that the double sum may not exist, because the square-summability and Cauchy-Schwartz argument (as for the matrix elements) does not apply: the convolution product of square-summable sequences need not be square-summable.

Of course, smearing with test functions does not help. This supports our

Conjecture 5.11 *The two-point functions of W^κ , $T_{\rho\sigma}^\kappa$, and J_ρ^κ do not exist.*

Mathematically, this means that the stress-energy tensor does not exist as an operator-valued distribution with a stable domain containing the vacuum vector, as required by the Wightman axioms. In physical terms, the divergence of the two-point function signals infinitely strong vacuum fluctuations. Stress-energy tensors that exist as quadratic forms (Conjecture 5.10), but not as Wightman fields (Conjecture 5.11), occur also for generalized free fields [6]. Here, the vacuum fluctuations are also divergent, but not because of the infinitely degenerate spin component, but because a continuous mass distribution cannot be “square-summable”.

5.3.3 Commutators

We have seen that the decisive difference between the “good” behaviour of matrix elements and the “bad” behaviour of two-point functions is due to the summation structure. Let us therefore study the commutator of the Wick square with a field just under this aspect.

The summation structure of the commutator is the same as that of a matrix element $(\Psi, W^\kappa \Phi^{\kappa(r)} \Omega)$ with a one-particle state. Choose for simplicity $r = 0$, and Ψ^n of helicity n as in section 5.3.1. Then, one computes

$$\begin{aligned} (\Psi^n, W^\kappa(e, x) \Phi^{\kappa(0)}(e', x') \Omega) &= \\ &= 2 \int d\mu_0(p_1) d\mu_0(p_2) e^{ip_1 x} e^{-ip_2(x-x')} \cdot \overline{\psi^n(p_1)} \cdot \omega^\kappa(e, p_1) \overline{\omega^\kappa(e, p_2)} \omega^\kappa(e', p_2) \cdot \\ &\quad \cdot \sum_{\nu \in \mathbb{Z}} i^n e^{-i(n+\nu)\alpha_{e,p_1}} e^{i\nu\alpha_{e',p_2}} J_{\nu+n}(\kappa R_{e,p_1}) J_\nu(\kappa R_{e',p_2}). \end{aligned}$$

This sum is absolutely convergent, as for the matrix elements above. The same expression with a different $i\varepsilon$ prescription (hidden in the argument R_{e,e',p_2} of the Bessel function) holds for the matrix element $(\Psi, \Phi^{\kappa(r)} W^\kappa \Omega)$, and hence the sum also converges for the commutator. This sketch of an argument supports our

Conjecture 5.12 *The commutators of W^κ , $T_{\rho\sigma}^\kappa$, and J_ρ^κ with the linear fields $\Phi_{\mu_1 \dots \mu_r}^{\kappa(r)}$ exist and can be defined as derivations on the algebra generated by smeared fields.*

In view of Conjecture 5.11, this property would rescue the stress-energy tensor as a “good” physical quantity. Namely, the prime role of the stress-energy tensor in quantum field theory is to generate infinitesimal Poincaré transformations via commutators. Of course, other technical issues remain concerning the convergence of the commutator with a smeared stress-energy tensor when the smearing functions becomes constant in space and sharp in time.

More interestingly, Conjecture 5.12 could also secure the existence of the perturbative expansion of a coupling of infinite-spin matter to linearized gravity via its stress-energy tensor, because this expansion is a series in retarded commutators.

More detailed investigations of these issues are beyond the scope of this paper.

5.3.4 Thermal states: equation of state and equipartition

Further interesting quantities to study are the energy density and the pressure in thermal equilibrium.

The computation of thermal expectation values of quadratic fields $:XY:(x)$ is most easily done by first considering $\omega_\beta(X(x)Y(x'))$ at $x \neq x'$ and using the KMS condition in momentum space (e.g., [19], eq. (16)). It determines the thermal two-point kernel on the negative mass shell by “detailed balance”:

$$M_{m,\beta}^{X,Y}(-p) = e^{-\beta p^0} \cdot M_{m,\beta}^{Y,X}(p).$$

Then one exploits the fact that the commutator is the same in the vacuum and in the thermal state, hence $M_{m,\text{vac}}^{X,Y}(p) = M_{m,\beta}^{X,Y}(p) - M_{m,\beta}^{Y,X}(-p)$. This implies

$$M_{m,\beta}^{X,Y}(p) = \frac{1}{1 - e^{-\beta p^0}} \cdot M_{m,\text{vac}}^{X,Y}(p)$$

on the positive mass shell. Subtracting the vacuum expectation value, one gets

$$\omega_\beta(:X(x)Y(x'):)= \int d\mu_m(p) \frac{1}{e^{\beta p^0} - 1} \left[M_{m,\text{vac}}^{X,Y}(p) e^{-ip(x-y)} + M_{m,\text{vac}}^{Y,X}(p) e^{ip(x-y)} \right].$$

Here, one can put $x = x'$, and thus obtains the thermal expectation value of $:XY:(x)$ from the vacuum two-point kernels.

This very efficient method reduces the computations of thermal expectation values to the inspection of the vacuum kernels, without any computation of partition functions in finite volume. It immediately gives the thermal energy density $\varepsilon = \omega_\beta(T_{00}^{\text{red}(s)})$ and the pressure $p = \omega_\beta(T_{ii}^{\text{red}(s)})$ of massive matter of finite spin

$$\varepsilon = \frac{(2s + 1)}{2\pi^2 \beta^4} \cdot I_+(\beta m), \quad 3p = \frac{(2s + 1)}{2\pi^2 \beta^4} \cdot I_-(\beta m), \quad (5.17)$$

where $I_\pm(x) = \int_0^\infty \frac{u^3 du}{e^{\sqrt{u^2+x^2}} - 1} \left(\frac{\sqrt{u^2+x^2}}{u} \right)^{\pm 1}$. The manifest factor $2s + 1$ reflects the law of equipartition. The result is independent of the choice of the stress-energy tensor, because KMS states are translation invariant, hence the derivative terms by which various stress-energy tensors differ, do not contribute. Interestingly, the individual contributions from $t^{(s,r)}$ in eq. (5.3) depend on e_1 and e_2 , while only their sum is independent of the strings. E.g., for $s = 1$, the contributions are $1 - m^2(q_{e_1}q_{e_2})$ from $r = 0$ and $2 + m^2(q_{e_1}q_{e_2})$ from $r = 1$. In the Pauli-Lubanski limit, each contribution from $t^{(s,r)}$ converges to zero (because of the factor $F_s(1)$ in eq. (4.3)), but their sum diverges as $2s + 1$ (because of eq. (5.17)). The total energy density per degree of freedom and the pressure per degree of freedom remain finite, and obey the usual massless equation of state.

The string-localized stress-energy tensor $T^{(s)m=0}$ of massless fields of finite helicity $|h| > 0$ gives the factor 2, as expected. At $m = 0$, the finite values $I_+(0) = I_-(0) = \frac{\pi^4}{15}$ reproduce the Stefan-Boltzmann law and the massless equation of state $p(\varepsilon) = \frac{1}{3}\varepsilon$. (Interestingly, while the trace of the reduced stress-energy tensor is non-zero and not even defined at $m = 0$, its thermal expectation value vanishes in the limit $m \rightarrow 0$).

For the infinite-spin stress-energy tensor, the contribution from each $T^{\kappa(r)}$ is 2 (resp. 1 for $r = 0$). Thus, the sum over r diverges as $2r + 1$, confirming the heuristic expectation. Wigner argued in [33] that this need not imply that infinite-spin matter must be unphysical, because it might never reach thermal equilibrium. Of course, this question cannot be physically addressed without a dynamical model for the coupling to ordinary matter. E.g., Schroer [24] argues that infinite-spin matter cannot couple to ordinary matter because there is no interaction Lagrangean that yields a string-independent action, as is needed to preserve causality in the quantum perturbation theory [16]. Thus infinite-spin matter is “inert”, and has no mechanism to approach thermal equilibrium at all.

6 Conclusion

We have “liberated quantum field theory from its classical crutches” (in the words of P. Jordan) by finding a construction scheme for covariant quantum stress-energy tensors that does not refer to a classical action. The method is applicable to arbitrary (in this paper: integer or infinite) spin. Auxiliary fields implementing higher-spin constraints, negative probability states, and compensating ghosts never appear.

Instead, the prescription is based on Wigner’s unitary representation theory of the Poincaré group and Weinberg’s construction of covariant quantum fields with the help of intertwiners whose analytic properties entail the localization properties of the fields.

The achieved stress-energy tensors are not unique, depending on a choice of intertwiners fulfilling the localizing completeness relation eq. (5.8). However, their densities all differ by “irrelevant derivatives” in the sense that they all produce the same Poincaré generators when integrated over space at a fixed time. Even for low spin, our “reduced” stress-energy tensors (Example 5.5) differ from the canonical or Hilbert stress-energy tensors by irrelevant derivative terms.

We applied this method in the case of the infinite-spin representations, where the best possible localization is on strings of the form $S_e(x) = x + \mathbb{R}_+ \cdot e$. In this case, the completeness relation eq. (5.8) requires an infinite direct sum of representations of the Lorentz group, which causes the stress-energy tensor to be an infinite sum of quadratic expressions in the corresponding string-localized fields. We sketched in section 5.3 the ensuing analytical implications (problems of convergence) with indications for “one bad and two good” features.

The good features are that matrix elements and commutators of the stress-energy tensor are well-behaved, while its correlations functions suffer from infinite vacuum fluctuations (the price of infinite spin).

The involved string-localized fields are defined on the Fock space over Wigner’s infinite-spin representation. We constructed these fields as Pauli-Lubanski limits of tensor fields of increasing spin and decreasing mass with fixed Pauli-Lubanski parameter $\kappa^2 = m^2 s(s + 1)$. Although it is not needed for the determination of the stress-energy tensor, this approximation is of some interest of its own. E.g., it exhibits how the dynamical coupling between escort fields $A^{(s,r)}$ of different r , that goes to zero with the mass at fixed s , remains stable (proportional to κ) when the spin increases. This may also play a role in higher spin theories.

Acknowledgments

I am grateful for invitations to the Universidade de Juiz de Fora, where this project has started, and to the University of York, where parts of it have been done. I thank Bert Schroer and Jens Mund for helpful discussions and Jakob Yngvason for his encouraging interest. This work had been impossible without their groundbreaking work on the infinite-spin representations. I thank the referee for pointing out ref. [14].

A Pauli-Lubanski limit of Wigner representations

We give a non-technical presentation of the Pauli-Lubanski limit of the (one-particle) Wigner representations. For a more rigorous treatment, see [14].

The standard reference vector of the massive Wigner representation (m, s) is $p_m = (m, 0, 0, 0)$. Its stabilizer group is $\text{Stab}(p_m) = \text{SO}(3) \subset \text{SO}(1, 3)$. We denote its generators L_i as usual, and K_i the generators of the boosts. The reference vector for a massless Wigner representation is $p_0 = (1, 0, 0, 1)^t$. We approximate it by massive vectors $p_\tau = m(\cosh \tau, 0, 0, \sinh \tau)^t$ with $e^\tau = \frac{2}{m}$ in the limit $m \rightarrow 0$. Let B^τ be the Lorentz 3-boost such that $B^\tau p_m = p_\tau$. The stabilizer group of p_τ is $\text{Stab}(p_\tau) = B^\tau \text{SO}(3) B^{\tau-1}$.

For the generators of $\text{Stab}(p_\tau)$ one computes $L_3^\tau := B^\tau L_3 B^{\tau-1} = L_3$ and $L_i^\tau := B^\tau L_i B^{\tau-1} = \cosh \tau L_i + \sinh \tau K_i$ for $i = 1, 2$. Thus, in $\text{so}(1, 3)$

$$\lim_{\tau \rightarrow \infty} 2e^{-\tau} L_i^\tau = Q_i$$

are the ‘‘translation’’ generators of $\text{Stab}(p_0) = \text{E}(2)$, while L_3 is the generator of the rotations in $\text{E}(2)$.

Now, we choose the unitary representation of $\text{SO}(3)$ with standard orthonormal basis $|s, n\rangle \in \mathcal{H}_s$ for each spin, for which $L_3 |s, n\rangle = n \cdot |s, n\rangle$ and $L_\pm |s, n\rangle = \sqrt{s(s+1) - n(n \pm 1)} \cdot |s, n \pm 1\rangle$. For increasing s , we isometrically embed $\mathcal{H}_s \rightarrow \mathcal{H}_{s+1}$ by $|s, n\rangle \mapsto |s+1, n\rangle$. The Pauli-Lubanski limit is the inductive limit of the representations of the generators $2e^{-\tau} L_i^\tau$ and L_3^τ in this sequence of representations as $s \rightarrow \infty$, while $m^2 s(s+1) = \kappa^2$ is constant. Thus, $2e^{-\tau} = m = \kappa(s(s+1))^{-\frac{1}{2}}$, hence

$$2e^{-\tau} L_\pm^\tau |s, n\rangle = \kappa \left(1 - \frac{n(n \pm 1)}{s(s+1)} \right)^{\frac{1}{2}} \cdot |s, n \pm 1\rangle, \quad L_3^\tau |s, n\rangle = n \cdot |s, n\rangle.$$

In the inductive limit, this becomes the representation D_κ of $\text{E}(2)$:

$$Q_\pm |n\rangle = \kappa \cdot |n \pm 1\rangle, \quad L_3 |n\rangle = n \cdot |n\rangle.$$

(Representing $|n\rangle \in \mathcal{H}_\kappa = L^2(\kappa S^1)$ by the wave function $\psi_n(\varphi) = e^{in\varphi}$, Q_\pm act by multiplication with $\kappa e^{\pm i\varphi}$, hence \vec{Q} act by multiplication with $\vec{k} = \kappa(\cos \varphi, \sin \varphi)$.)

Once the representation d of the respective stabilizer group is specified, the corresponding induced Wigner representation of the Poincaré group is defined on $L^2(H_m, \mathcal{H}_d)$. The translations act on wavefunctions $\psi(p)$ with values in \mathcal{H}_d by multiplication with e^{ipx} , and the Lorentz transformations act by

$$(U(\Lambda)\psi)(p) = d(W_{\Lambda, \Lambda^{-1}p})\psi(\Lambda^{-1}p),$$

where $W_{\Lambda,p} = B_{\Lambda p}^{-1} \Lambda B_p$ is the Wigner “rotation” in the stabilizer group of the respective reference vector p_0 . It depends on the choice of the standard “boosts” B_p that take p_0 to p , but the dependence is a unitary equivalence of U . This unitary equivalence acts on $L^2(H_m, \mathcal{H}_d)$ as a multiplication with a function $H_m \rightarrow U(\mathcal{H})$, and is of course irrelevant for abstract properties.

The inductive limit of representations of the stabilizer groups, outlined before, naturally extends to the induced representations of the Poincaré group.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] X. Bekaert and J. Mourad, *The Continuous spin limit of higher spin field equations*, *JHEP* **01** (2006) 115 [[hep-th/0509092](https://arxiv.org/abs/hep-th/0509092)] [[INSPIRE](#)].
- [2] X. Bekaert, J. Mourad and M. Najafizadeh, *Continuous-spin field propagator and interaction with matter*, [arXiv:1710.05788](https://arxiv.org/abs/1710.05788) [[INSPIRE](#)].
- [3] X. Bekaert and E.D. Skvortsov, *Elementary particles with continuous spin*, *Int. J. Mod. Phys. A* **32** (2017) 1730019 [[arXiv:1708.01030](https://arxiv.org/abs/1708.01030)] [[INSPIRE](#)].
- [4] R. Brunetti, D. Guido and R. Longo, *Modular localization and Wigner particles*, *Rev. Math. Phys.* **14** (2002) 759 [[math-ph/0203021](https://arxiv.org/abs/math-ph/0203021)] [[INSPIRE](#)].
- [5] D. Buchholz and K. Fredenhagen, *Locality and the Structure of Particle States*, *Commun. Math. Phys.* **84** (1982) 1 [[INSPIRE](#)].
- [6] M. Dütsch and K.-H. Rehren, *Generalized free fields and the AdS-CFT correspondence*, *Ann. Inst. H. Poincaré* **4** (2003) 613 [[math-ph/0209035](https://arxiv.org/abs/math-ph/0209035)].
- [7] A. Erdélyi, *Higher Transcendental Functions. Vol. I (Bateman Manuscript Project)*, McGraw-Hill, New York U.S.A. (1953).
- [8] M. Fierz, *Über die relativistische Theorie kräftefreier Teilchen mit beliebigem Spin*, *Helv. Phys. Acta* **12** (1939) 3 [[INSPIRE](#)].
- [9] J. Fröhlich, G. Morchio and F. Strocchi, *Infrared problem and spontaneous breaking of the Lorentz group in QED*, *Phys. Lett.* **89B** (1979) 61 [[INSPIRE](#)].
- [10] R. Gonzo, *The infinite-spin representations of the Poincaré group*, MSc Thesis, University of Padova, Padova Italy (2017).
- [11] G. Grensing, *Symmetric and Traceless Tensors on Minkowski Space*, *Rept. Math. Phys.* **14** (1978) 19 [[INSPIRE](#)].
- [12] C. Köhler, *On the localization properties of quantum fields with zero mass and infinite spin*, Ph.D. Thesis, University of Vienna, Vienna Austria (2015).
- [13] R. Longo, V. Morinelli and K.-H. Rehren, *Where Infinite Spin Particles Are Localizable*, *Commun. Math. Phys.* **345** (2016) 587 [[arXiv:1505.01759](https://arxiv.org/abs/1505.01759)] [[INSPIRE](#)].
- [14] A. McKerrell, *Canonical representations for massless particles and zero-mass limits of the helicity representation*, *Proc. Roy. Soc. A* **285** (1965) 287.

- [15] J. Mund, *String-localized vector bosons without ghosts and indefinite metric: the example of massive QED*, work in progress.
- [16] J. Mund, K.-H. Rehren and B. Schroer, *Helicity decoupling in the massless limit of massive tensor fields*, *Nucl. Phys. B* **924** (2017) 699 [[arXiv:1703.04407](#)] [[INSPIRE](#)].
- [17] J. Mund, K.-H. Rehren and B. Schroer, *Relations between positivity, localization and degrees of freedom: The Weinberg-Witten theorem and the van Dam-Veltman-Zakharov discontinuity*, *Phys. Lett. B* **773** (2017) 625 [[arXiv:1703.04408](#)] [[INSPIRE](#)].
- [18] J. Mund, B. Schroer and J. Yngvason, *String-localized quantum fields and modular localization*, *Commun. Math. Phys.* **268** (2006) 621 [[math-ph/0511042](#)] [[INSPIRE](#)].
- [19] J. Mund, B. Schroer and J. Yngvason, *String localized quantum fields from Wigner representations*, *Phys. Lett. B* **596** (2004) 156 [[math-ph/0402043](#)] [[INSPIRE](#)].
- [20] M. Plaschke and J. Yngvason, *Massless, String Localized Quantum Fields for Any Helicity*, *J. Math. Phys.* **53** (2012) 042301 [[arXiv:1111.5164](#)] [[INSPIRE](#)].
- [21] V.O. Rivelles, *Remarks on a Gauge Theory for Continuous Spin Particles*, *Eur. Phys. J. C* **77** (2017) 433 [[arXiv:1607.01316](#)] [[INSPIRE](#)].
- [22] B. Schroer, *A Hilbert space setting for interacting higher spin fields and the Higgs issue*, *Found. Phys.* **45** (2015) 219 [[arXiv:1407.0360](#)] [[INSPIRE](#)].
- [23] B. Schroer, *Beyond gauge theory: positivity and causal localization in the presence of vector mesons*, *Eur. Phys. J. C* **76** (2016) 378 [[arXiv:1601.04528](#)] [[INSPIRE](#)].
- [24] B. Schroer, *Wigner's infinite spin representations and inert matter*, *Eur. Phys. J. C* **77** (2017) 362 [[arXiv:1601.02477](#)] [[INSPIRE](#)].
- [25] P. Schuster and N. Toro, *On the Theory of Continuous-Spin Particles: Wavefunctions and Soft-Factor Scattering Amplitudes*, *JHEP* **09** (2013) 104 [[arXiv:1302.1198](#)] [[INSPIRE](#)].
- [26] P. Schuster and N. Toro, *On the Theory of Continuous-Spin Particles: Helicity Correspondence in Radiation and Forces*, *JHEP* **09** (2013) 105 [[arXiv:1302.1577](#)] [[INSPIRE](#)].
- [27] P. Schuster and N. Toro, *A Gauge Field Theory of Continuous-Spin Particles*, *JHEP* **10** (2013) 061 [[arXiv:1302.3225](#)] [[INSPIRE](#)].
- [28] S. Weinberg, *The Quantum Theory of Fields, Vol. I*, Cambridge University Press, Cambridge U.K. (1995).
- [29] S. Weinberg, *Feynman Rules for Any Spin*, *Phys. Rev. B* **133** (1964) 1318.
- [30] S. Weinberg, *Feynman Rules for Any Spin. II. Massless Particles*, *Phys. Rev. B* **134** (1964) 882.
- [31] S. Weinberg and E. Witten, *Limits on massless particles*, *Phys. Lett. B* **96** (1980) 59 [[INSPIRE](#)].
- [32] E.P. Wigner, *On Unitary Representations of the Inhomogeneous Lorentz Group*, *Annals Math.* **40** (1939) 149 [[INSPIRE](#)].
- [33] E.P. Wigner, *Relativistische Wellengleichungen*, *Z. Phys.* **124** (1948) 665.
- [34] J. Yngvason, *Zero-mass infinite spin representations of the Poincaré group and quantum field theory*, *Commun. Math. Phys.* **18** (1970) 195 [[INSPIRE](#)].