A composite axion from a supersymmetric product group

Benjamin Lillard and Tim M.P. Tait

Department of Physics and Astronomy, University of California,
Irvine, CA 92697, U.S.A.

E-mail: blillard@uci.edu, ttait@uci.edu

ABSTRACT: A global $U(1)_{PQ}$ symmetry is protected from gravitational effects in the s-confining $SU(N)^k$ product group theory with $A + 4Q + N\bar{Q}$ matter. If the $SU(4)$ family symmetry is gauged and an appropriate tree-level superpotential is added, then the dynamically generated superpotential spontaneously breaks $SU(4) \times U(1)_{PQ} \to SU(3)_c$ and produces a QCD axion. Small values of the CP-violating θ parameter are then possible without any fine-tuning, as long as the product group is suitably large. By introducing a second copy of the s-confining $SU(N)$ product group also coupled to the gauged $SU(4)$, we find that values as small as $N = 7$ are consistent with $\bar{\theta} < 10^{-10}$, even under the pessimistic assumption that the dominant contribution to the axion quality is at tree level.

Contents

1 Introduction

Despite its success at predicting the results of particle experiments, the Standard Model remains widely unloved. Its unpopularity is due in part to a few inexplicably small parameters, including the $\mathcal{O}(10^{-16})$ ratio between the electroweak and Planck scales, the puzzling array of Yukawa couplings, and the degree to which QCD conserves the discrete charge (C) and parity (P) symmetries, $|\theta| < 10^{-10}$. In addition, the Standard Model is clearly incomplete, failing to describe gravitation, dark matter, and neutrino masses.

Prominent solutions to these theoretical shortcomings include supersymmetry (SUSY), which stabilizes the electroweak scale and can support dark matter; extra dimensions and composite models, which can generate hierarchies dynamically; and axions, which explain the smallness of the QCD CP parameter θ while supplying a dark matter candidate. In this paper we consider a hybrid of these elements, a supersymmetric composite axion model, as a solution to the strong CP problem that is free from fine-tuning.

At issue (for more complete discussion, see refs. [1, 2]) is the θ term of the QCD Lagrangian,

$$\mathcal{L} = \frac{g^2}{32\pi^2}\bar{\theta}\,\epsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G^a_{\rho\sigma} \equiv \frac{g^2}{32\pi^2}\bar{\theta}\,G^a_{\mu\nu}\widetilde{G}^{a\mu\nu}, \tag{1.1}$$

which violates both P and CP. $\bar{\theta}$ is the physical combination of the intrinsic coefficient θ and a phase in the quark mass matrix,

$$\bar{\theta} \equiv \theta + \arg\det M_Q. \tag{1.2}$$

Measurements of the neutron electric dipole moment require $\left|\bar{\theta}\right| < 10^{-10}$ [3]. Such a tiny value appears to require an extraordinary cancellation between two apparently unrelated quantities.

In a simple axion model, $\bar{\theta}$ is associated with the transformation parameter of an approximate global $\mathrm{U}(1)_{\mathrm{PQ}}$ symmetry [4–10]. $\mathrm{U}(1)_{\mathrm{PQ}}$ is spontaneously broken at some high scale f_a by the expectation value of a $\mathrm{U}(1)_{\mathrm{PQ}}$-charged scalar field or the formation of a $\mathrm{U}(1)_{\mathrm{PQ}}$-charged fermion condensate, resulting in a pseudo-Nambu-Goldstone boson (pNGB): the axion a. Due to the nonzero $\mathrm{SU}(3)_c^2$-$\mathrm{U}(1)_{\mathrm{PQ}}$ anomaly, non-perturbative QCD dynamics induce an expectation value for the axion such that CP is a symmetry of the vacuum, and the axion acquires a small mass.

At energies below f_a, the effective Lagrangian contains the term:

$$\mathcal{L} = \frac{g^2}{32\pi^2}\left(\bar{\theta} + \mathcal{A}\frac{a}{f_a}\right)G^a_{\mu\nu}\widetilde{G}^{a\mu\nu}, \tag{1.3}$$

where $\mathcal{A}$ is the $\mathrm{SU}(3)_c^2$-$\mathrm{U}(1)_{\mathrm{PQ}}$ anomaly coefficient. Nonperturbative QCD generates a periodic potential for the axion which can be heuristically described by

$$V[a] = m_\pi^2 f_\pi^2\left(1 - \cos\left[\mathcal{A}\frac{a}{f_a} + \bar{\theta}\right]\right), \tag{1.4}$$

where m_π and f_π are the pion mass and decay constant, respectively. This potential is minimized when $\langle a \rangle = -f_a\bar{\theta}/\mathcal{A}$, leading to CP conservation in the vacuum. We choose to normalize the $\mathrm{U}(1)_{\mathrm{PQ}}$ charges so that $\mathcal{A} = 1$, for which the axion mass is,[1]

$$m_a^2 = \frac{m_\pi^2 f_\pi^2}{f_a^2}. \tag{1.5}$$

Experimental observations set bounds on the value of f_a. A lower bound $f_a \gtrsim 10^9$ GeV is derived from constraints on stellar and supernova cooling [12], while the axion relic abundance suggests $f_a \lesssim 10^{12}$ GeV in the absence of cosmological fine tuning [13].

Axion quality problem. Simple axion models are plagued by the theoretical inconsistencies endemic to theories containing fundamental scalar fields. The expectation value of the new complex scalar $\langle \phi \rangle \sim f_a$ receives additive corrections from high-energy physics which, while less severe than the electroweak hierarchy [14], remains a concerning source of fine-tuning. Models of axions also suffer from a different concern which is potentially much more troubling: the *axion quality problem*. Any $\mathrm{U}(1)_{\mathrm{PQ}}$-violating effects in the scalar potential can shift the axion VEV away from $\theta = 0$, inducing the strong CP problem rather than solving it. In particular, non-perturbative quantum gravity is expected to violate global symmetries [15–20], leading to terms in the low energy effective action of the form

$$\mathcal{L}_g \sim \frac{|\phi|^p\,(\phi + \phi^\star)}{M_{\mathrm{P}}^{p-3}}, \tag{1.6}$$

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[1]More careful treatments based on the QCD chiral Lagrangian [11] result in a potential given by: $V[a] = m_\pi^2 f_\pi^2\left(2 - \sqrt{1 + \frac{2m_u m_d}{(m_u+m_d)^2}\left(\cos\left[\mathcal{A}\frac{a}{f_a} + \bar{\theta}\right]\right)}\right)$, where $m_{u,d}$ are the up- and down-quark masses, and leading to an axion mass $m_a^2 = \frac{m_u m_d}{(m_u+m_d)^2}\frac{m_\pi^2 f_\pi^2}{f_a^2}$. The distinction between these two expressions for $V[a]$ is unimportant in terms of assessing the axion quality, and we use eq. (1.4) for our analysis.

which is inconsistent with $|\theta| < 10^{-10}$ unless the $p = 4$ term has a coefficient smaller than $\mathcal{O}(10^{-55})$. Considering that the axion is introduced to explain fine-tuning of $\mathcal{O}(10^{-10})$, this calls its motivation into serious question, and any successful axion model must prevent linear shifts of the form $\langle a \rangle \to \langle a \rangle + f_a \Delta\theta$ with $\Delta\theta > 10^{-10}$.

More generally, we can analyze arbitrary U(1)$_{\mathrm{PQ}}$ violation by including it in the axion potential $V[a]$ as

$$\delta V[a] = (Q \, f_a^4) \cos\left(\kappa \left[\frac{a}{f_a} + \bar{\theta}\right] + \theta_0\right), \tag{1.7}$$

for a dimensionless "quality factor" Q, an integer κ and an angle θ_0. Experimental measurements of $\langle\theta\rangle$ set a maximum bound on Q; we derive the general expression in appendix A. For $\kappa \sin\theta_0 \sim \mathcal{O}(1)$, $|\theta| < 10^{-10}$ requires:

$$Q < 10^{-62} \left(\frac{10^{12} \text{ GeV}}{f_a}\right)^4 = 10^{-50} \left(\frac{10^9 \text{ GeV}}{f_a}\right)^4. \tag{1.8}$$

Consistent axion models. Several solutions to the axion quality problem are known, in which the U(1)$_{\mathrm{PQ}}$ is protected by associating it with new gauged symmetries. In the simplest solutions a gauged discrete $\mathbb{Z}_N$ symmetry [21] forbids U(1)$_{\mathrm{PQ}}$-violating operators of dimensions smaller than N. More sophisticated models can employ discrete groups as small as $\mathbb{Z}_4$ while forbidding the problematic operators [22, 23]. Solutions without gauged discrete symmetries also exist: for example, a composite model [24] with a gauged SU(N) × SU(m) × SU(3)$_c$ protects U(1)$_{\mathrm{PQ}}$ to arbitrarily high order. More recently [25], a qualitatively different SU(N)$_L$ × SU(N)$_R$ × SU(3)$_c$ model has been shown to suppress Planck scale corrections appropriately.

Other constructions protect U(1)$_{\mathrm{PQ}}$ by gauging a related Abelian group. In one model [26] with a compact extra dimension, a gauged U(1) symmetry is spontaneously broken by fields localized on two separated four-dimensional branes. One combination of the fields is eaten by the gauge field, while the other acts as the QCD axion and is protected from gravitational corrections. A related model [27] gauges a product group of the form U(1)^k with $k \geq 14$, which can also be interpreted as a k site deconstruction of a compact fifth dimension. In a different class of models [18, 28], the fields are assigned large and relatively prime U(1) charges, so that an accidental U(1)$_{\mathrm{PQ}}$ is protected from low-dimensional operators.

Some of these models, while successful at forbidding low-dimensional U(1)$_{\mathrm{PQ}}$-breaking operators, still suffer from a hierarchy problem. One resolution is supersymmetry (SUSY), which protects f_a from loop-level corrections, so that the theory is technically natural if the SUSY-breaking scale is not much larger than f_a. Another compelling direction is composite models, which can suppress dangerous gravitational contributions to the axion potential while allowing the scale of U(1)$_{\mathrm{PQ}}$ breaking to be determined from the confining dynamics. For asymptotically free gauge theories the confinement scale is expected to be exponentially suppressed compared to M_{P}, so the hierarchy between f_a and M_{P} can be naturally generated dynamically.

In this article, we present a qualitatively new supersymmetric composite axion model which tames both the quality and hierarchy problems. The axion is a composite formed of

large product of fundamental fields, such that the quality problem is ameliorated by a sufficiently large power of $(\Lambda/M_{\rm P})^n$, where $f_a \sim \Lambda$ is dynamically generated by the confinement of a product of non-Abelian gauge theories. Supersymmetry allows for control over the low energy physics of the non-perturbative confining dynamics, and additionally stabilizes any other mass scales (including, perhaps, the electroweak scale). Our work is laid out as follows: in section 2, we explore a minimal construction in terms of its UV degrees of freedom. In section 2.1, we analyze its low energy behavior after confinement, with section 2.2 discussing the breaking of the global symmetries, including $\mathrm{U}(1)_{\rm PQ}$. Section 2.3 estimates the size of the leading gravitational corrections, and determines parameters such that the axion quality problem is ameliorated to a sufficient degree. In section 3, we show how a simple extension of the basic model can dynamically generate superpotential terms on which the basic module relies, resulting in a theory in which all of the essential mass scales are dynamically generated. In section 4, we conclude. As we shall see, solving the quality problem can imply that a theory whose low energy limit looks like a rather standard invisible axion model may blossom at high energies into a rich interlocking structure of gauge dynamics.

2 Axion from a supersymmetric product group

The construction of our axion model begins with a gauge group $\mathrm{SU}(N)_{(1)}$, with one matter field A transforming in the antisymmetric ($\boxminus$) representation; four quarks, Q; and N antiquarks $\overline{Q}_1$. This theory is known to s-confine [29–32]: that is, a set of gauge-invariant operators provides a smooth description of the moduli space which is valid at the origin, and a dynamically generated superpotential enforces the classical constraints between operators [33, 34]. When supplemented by an appropriately chosen external superpotential, $\mathrm{U}(1)_{\rm PQ}$ is spontaneously broken when $\mathrm{SU}(N)_{(1)}$ confines.

High axion quality is enforced by expanding $\mathrm{SU}(N)_{(1)}$ into a product gauge group, $\mathrm{SU}(N)^r = \mathrm{SU}(N)_{(1)} \times \mathrm{SU}(N)_{(2)} \times \ldots \times \mathrm{SU}(N)_{(r)}$. In addition to the $\mathrm{SU}(N)_{(1)}$-charged $A + 4Q$, the matter fields include a set of bifundamentals $\overline{Q}_i$ which transform under $\mathrm{SU}(N)_{(i)} \times \mathrm{SU}(N)_{(i+1)}$, and N antiquarks $\overline{Q}_r$ charged only under $\mathrm{SU}(N)_{(r)}$. It has recently been demonstrated that this product group model s-confines [35], and that the gauge-invariant operators include "mesons" of the form $(Q\overline{Q}_1\overline{Q}_2\ldots\overline{Q}_r)$ and $(A\overline{Q}_1^2\ldots\overline{Q}_r^2)$; "baryons" $(\overline{Q}_i^N)$ for each $i = 1\ldots r$; and special baryons $(A^{\frac{N-p}{2}}Q^p)$ for $0 \leq p \leq 4$, subject to the condition that $(N-p)$ is even. An axion living in a combination of these fields enjoys the feature that increasing r and N results in increasingly suppressed gravitational corrections.

To accommodate QCD within the model, we introduce a second copy of the matter fields $\overline{A} + 4q + \overline{q}_1 + \ldots + \overline{q}_{\ell-1} + N\overline{q}_\ell$ charged under a new s-confining $\mathrm{SU}(N)^\ell$ gauge group, and we let Q and q transform in the fundamental ($\square$) and antifundamental ($\overline{\square}$) representations under a weakly gauged $\mathrm{SU}(4)$ which contains $\mathrm{SU}(3)_c$ as a subgroup. The full matter content of our theory is thus $\{A, Q, \overline{Q}_1\ldots\overline{Q}_r; \overline{A}, q, \overline{q}_1\ldots\overline{q}_\ell\}$, with the gauge group $\mathrm{SU}(N)^r \times \mathrm{SU}(4) \times \mathrm{SU}(N)^\ell$. The gauge structure and matter assignments is represented as a moose diagram in figure 1, and is vaguely reminiscent of a deconstructed extra dimension with a bulk $\mathrm{SU}(N)$ broken to $\mathrm{SU}(4)$ on a defect. As we show in section 2.3, this structure permits smaller values of N for a given axion qualty.
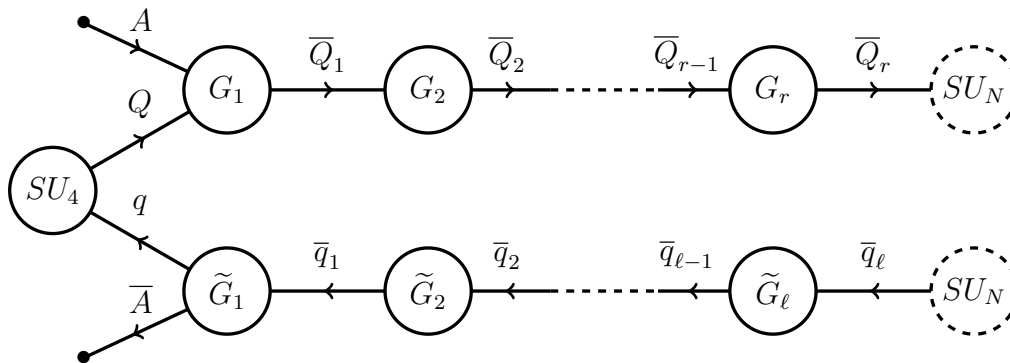
Figure 1. Moose diagram indicating the matter content and gauge interactions of the $\mathrm{SU}(N)^{\ell} \times \mathrm{SU}(4) \times \mathrm{SU}(N)^r$ composite axion model. Each G_i and $\widetilde{G}_i$ corresponds to a gauged $\mathrm{SU}(N)$, whereas $\mathrm{SU}(N)$ flavor symmetries are represented by dashed circles. The bifundamental fields Q, $\overline{Q}_i$, q, and $\overline{q}_i$ are depicted as directed line segments connecting adjacent groups, while the field A $(\overline{A})$ transforms under G_1 $(\widetilde{G}_1)$ in the antisymmetric two-tensor representation.

For convenience, we introduce the notation $\mathrm{SU}(N)^{\ell} = \widetilde{G}_1 \times \widetilde{G}_2 \times \ldots \times \widetilde{G}_{\ell}$ and $\mathrm{SU}(N)^r = G_1 \times G_2 \times \ldots \times G_r$, where $\widetilde{G}_i$ and G_i represent $\mathrm{SU}(N)$ groups that confine at scales $\widetilde{\Lambda}_i$ and Λ_i respectively. Up to a constant, the holomorphic scales $\widetilde{\Lambda}_i$ and Λ_i are defined as

$$\widetilde{\Lambda}_i^b \equiv \mu^b \exp\{-8\pi^2/\widetilde{g}_i^2 + i\widetilde{\theta}_i\}, \qquad \Lambda_i^b \equiv \mu^b \exp\{-8\pi^2/g_i^2 + i\theta_i\}, \qquad (2.1)$$

where $\widetilde{g}_i$ and g_i are the coupling constants of the gauge groups $\widetilde{G}_i$ and G_i. In the dynamically generated superpotential for each group there is an overall constant that is not determined by symmetry arguments; to simplify the notation, we absorb these constants into $\widetilde{\Lambda}_i^b$ and Λ_i^b.

In the absence of an external superpotential, there is a conserved $\mathrm{U}(1)_A \times \mathrm{U}(1)_B \times \mathrm{U}(1)_C \times \mathrm{U}(1)_R \times \mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ global symmetry, and an approximate $\mathrm{U}(1)_{\mathrm{PQ}}$ that is broken by the $\mathrm{SU}(4)^2$-$\mathrm{U}(1)$ anomaly. Charges are shown in table 1, where for convenience, we have taken the $\mathrm{U}(1)_R$ charges of Q and A to be equal to q and $\overline{A}$, respectively, with $q_Q = \frac{N-4}{N}$ and $q_A = \frac{16-2N}{N(N-2)}$. By defining $\mathrm{U}(1)_{\mathrm{PQ}}$ as in table 1, we assume that the operator $(A\overline{Q}_1^2 \ldots \overline{Q}_r^2)$ is more suppressed than $(\overline{A}\overline{q}_1^2 \ldots \overline{q}_{\ell}^2)$, so that $\mathrm{U}(1)_{\mathrm{PQ}}$ is expected to be a better symmetry than $\mathrm{U}(1)_A$. Appropriate $\mathrm{U}(1)_{\mathrm{PQ}}$ charges in the opposite limit can be recovered by performing the following outer automorphism on the moose diagram:

$$\ell \leftrightarrow r, \qquad G_i \leftrightarrow \widetilde{G}_i, \qquad \Lambda_i \leftrightarrow \widetilde{\Lambda}_i, \qquad A \leftrightarrow \overline{A}, \qquad Q \leftrightarrow q, \qquad \overline{Q}_i \leftrightarrow \overline{q}_i. \qquad (2.2)$$

At a generic point on the moduli space the full global symmetry is spontaneously broken, producing a number of Nambu-Goldstone bosons. Although the explicit symmetry breaking from gravity would supply masses for the pNGBs, a tree-level external superpotential

$$W_{\mathrm{tree}} = \frac{(\overline{A}\overline{q}_1^2\overline{q}_2^2 \ldots \overline{q}_{\ell}^2)}{M_A^{2\ell-2}} + \frac{(\overline{Q}_1^N)}{M_B^{N-3}} + \frac{(\overline{q}_1^N)}{M_C^{N-3}} + \frac{(A^m Q)(A^{m-1}Q^3)}{M_R^{N-1}} + \frac{(\overline{A}^m q)(\overline{A}^{m-1}q^3)}{M_r^{N-1}} \qquad (2.3)$$

	$\mathrm{SU}(N)_L$	$\widetilde{G}_\ell$	$\ldots$	$\widetilde{G}_1$	$\mathrm{SU}(4)$	G_1	$\ldots$	G_r	$\mathrm{SU}(N)_R$	U_A	U_B	U_C	U_R	$\mathrm{U}(1)_{\mathrm{PQ}}$
$\overline{q}_\ell$	□	□								0	0	± 1	0	0
$\overline{q}_{\ell-1}$		$\overline{□}$	□							0	0	∓ 1	0	0
$\vdots$			$\ddots$							$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\overline{q}_1$			$\overline{□}$	□						0	0	1	0	0
$\overline{A}$				$\overline{□□}$						-4	0	$\frac{-N}{N-2}$	q_A	0
q				□	$\overline{□}$					$N-2$	0	0	q_Q	0
Q					□	□				$2-N$	0	0	q_Q	$\frac{2-N}{N}$
A						□□				4	$\frac{-N}{N-2}$	0	q_A	$4/N$
$\overline{Q}_1$						$\overline{□}$	□			0	1	0	0	0
$\vdots$							$\ddots$			$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\overline{Q}_{r-1}$							$\overline{□}$	□		0	∓ 1	0	0	0
$\overline{Q}_r$								$\overline{□}$	□	0	± 1	0	0	0

Table 1. Representations of the matter fields under the gauged $\mathrm{SU}(N)^\ell \times \mathrm{SU}(4) \times \mathrm{SU}(N)^r$ symmetries, the flavor symmetries $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R \times \mathrm{U}(1)^4$, and the approximate $\mathrm{U}(1)_{\mathrm{PQ}}$ symmetry.

increases the pNGB masses by breaking the global symmetries more severely. This is essential in the case of the second (M_B) term, which as we shall see below determines the PQ symmetry breaking scale f_a after confinement. The remaining M_i could be safely taken to be M_{P} without harm. In addition, to avoid deforming the G_1 confinement, we choose them to satisfy $\Lambda_1 \lesssim M_i$.

In section 3 we discuss the possibility that some of the terms in eq. (2.3) are generated dynamically through the s-confinement of a strongly coupled $\mathrm{Sp}(2n)$ gauge group, providing a natural and completely dynamical origin for the scale f_a.

2.1 Confinement

We choose the UV gauge couplings such that $\mathrm{SU}(N)^\ell$ and $\mathrm{SU}(N)^r$ confine at an intermediate scale where $\mathrm{SU}(4)$ remains weakly coupled and supersymmetry is unbroken. For odd $N = 2m + 1$, the groups $\mathrm{SU}(N)^\ell$ and $\mathrm{SU}(N)^r$ confine separately to produce the following hadrons:

$$J_L = (\overline{q}_\ell \overline{q}_{\ell-1} \ldots \overline{q}_1 q), \quad K_L = (\overline{q}_\ell^2 \overline{q}_{\ell-1}^2 \ldots \overline{q}_1^2 \overline{A}), \quad x_1 = (\overline{A}^m q), \quad y_1 = (\overline{A}^{m-1} q^3), \quad z_i = (\overline{q}_i)^N,$$
$$(2.4)$$
$$J_R = (Q \overline{Q}_1 \overline{Q}_2 \ldots \overline{Q}_r), \quad K_R = (A \overline{Q}_1^2 \overline{Q}_2^2 \ldots \overline{Q}_r^2), \quad X_1 = (A^m Q), \quad Y_1 = (A^{m-1} Q^3), \quad Z_i = (\overline{Q}_i)^N.$$
$$(2.5)$$

Their transformation properties under the global symmetries are summarized in table 2. These operators obey quantum-modified equations of motion, for which we define the

	SU(4)	SU(N)$_L$	SU(N)$_R$	U(1)$_{\rm PQ}$
x_1	$\overline{\square}$			0
y_1	$\square$			0
z_i	1			0
J_L	$\overline{\square}$	$\square$		0
K_L	1	$\substack{\square\\\square}$		0
X_1	$\square$			1
Y_1	$\overline{\square}$			-1
Z_i	1			0
J_R	$\square$		$\square$	$\frac{2-N}{N}$
K_R	1		$\substack{\square\\\square}$	$4/N$

Table 2. Operators describing infrared degrees of freedom in the confined phase of SU(N)$^\ell \times$ SU(N)^r, and their transformation properties under the approximate SU(N)$_L \times$ SU(N)$_R \times$ U(1)$_{\rm PQ}$ flavor symmetries.

shorthand notation:

$$(\widetilde{\Pi}_1^\ell z) = \begin{cases} \text{even } \ell: & \begin{aligned} &(z_1 z_2 z_3 \ldots z_\ell) - \widetilde{\Lambda}_2^b(z_3 z_4 \ldots z_\ell) - z_1 \widetilde{\Lambda}_3^b(z_4 \ldots z_\ell) + \widetilde{\Lambda}_2^b \widetilde{\Lambda}_4^b(z_5 \ldots z_\ell) + \ldots \\ &+ (\widetilde{\Lambda}_2^b \widetilde{\Lambda}_4^b \widetilde{\Lambda}_6^b \ldots \widetilde{\Lambda}_{\ell-2}^b) z_{\ell-1} z_\ell + (\widetilde{\Lambda}_2^b \widetilde{\Lambda}_4^b \widetilde{\Lambda}_6^b \ldots \widetilde{\Lambda}_{\ell-2}^b \widetilde{\Lambda}_\ell^b), \end{aligned} \\[2ex] \text{odd } \ell: & \begin{aligned} &(z_1 z_2 z_3 \ldots z_\ell) - \widetilde{\Lambda}_2^b(z_3 z_4 \ldots z_\ell) - z_1 \widetilde{\Lambda}_3^b(z_4 \ldots z_\ell) + \widetilde{\Lambda}_2^b \widetilde{\Lambda}_4^b(z_5 \ldots z_\ell) + \ldots \\ &+ z_1 (\widetilde{\Lambda}_3^b \widetilde{\Lambda}_5^b \widetilde{\Lambda}_7^b \ldots \widetilde{\Lambda}_\ell^b) + \ldots + (\widetilde{\Lambda}_2^b \widetilde{\Lambda}_4^b \widetilde{\Lambda}_6^b \ldots \widetilde{\Lambda}_{\ell-1}^b z_\ell); \end{aligned} \end{cases} \tag{2.6}$$

$$(\widetilde{\Pi}_1^r Z) = \begin{cases} \text{even } r: & \begin{aligned} &(Z_1 Z_2 Z_3 \ldots Z_r) - \Lambda_2^b(Z_3 Z_4 \ldots Z_r) - Z_1 \Lambda_3^b(Z_4 \ldots Z_r) + \Lambda_2^b \Lambda_4^b(Z_5 \ldots Z_r) + \ldots \\ &+ (\Lambda_2^b \Lambda_4^b \Lambda_6^b \ldots \Lambda_{r-2}^b) Z_{r-1} Z_r + (\Lambda_2^b \Lambda_4^b \Lambda_6^b \ldots \Lambda_{r-2}^b \Lambda_r^b), \end{aligned} \\[2ex] \text{odd } r: & \begin{aligned} &(Z_1 Z_2 Z_3 \ldots Z_r) - \Lambda_2^b(Z_3 Z_4 \ldots Z_r) - Z_1 \Lambda_3^b(Z_4 \ldots Z_r) + \Lambda_2^b \Lambda_4^b(Z_5 \ldots Z_r) + \ldots \\ &+ Z_1 (\Lambda_3^b \Lambda_5^b \Lambda_7^b \ldots \Lambda_r^b) + \ldots + (\Lambda_2^b \Lambda_4^b \Lambda_6^b \ldots \Lambda_{r-1}^b Z_r). \end{aligned} \end{cases} \tag{2.7}$$

The constraint equations include:

$$\begin{aligned} K_L^m J_L &= x(\widetilde{\Pi}_1^\ell z) & K_L^{m-1} J_L^3 &= y(\widetilde{\Pi}_1^\ell z) & xy &= 0 \\ K_R^m J_R &= X(\widetilde{\Pi}_1^r Z) & K_R^{m-1} J_R^3 &= Y(\widetilde{\Pi}_1^r Z) & XY &= 0. \end{aligned} \tag{2.8}$$

Not shown above, X, Y, x, and y each carry an SU(4) gauge index, which is summed over in the expressions $x^\alpha y_\alpha = X_\alpha Y^\alpha = 0$. Each term in the equations above is invariant under the SU(N)$_L \times$ SU(N)$_R$ family symmetry. Combinatoric coefficients have been suppressed for clarity.

The analysis is simplified by introducing spurion superfields $X_{i>1}$, $Y_{i>1}$, $x_{i>1}$ and $y_{i>1}$, such that the constraints between operators follow directly from the dynamically generated

superpotential $W_d = W_L + W_R$, where

$$W_L = \frac{x_1 y_1 z_1 - x_1 y_2 - y_1 x_2}{\widetilde{\Lambda}_1^b} + \sum_{i=2}^{\ell-1} \frac{x_i y_i z_i - x_i y_{i+1} - y_i x_{i+1}}{\widetilde{\Lambda}_1^b \widetilde{\Lambda}_2^b \ldots \widetilde{\Lambda}_i^b}$$
$$+ \frac{x_\ell y_\ell z_\ell - x_\ell K_L^{m-1} J_L^3 - y_\ell K_L^m J_L}{\widetilde{\Lambda}_1^b \widetilde{\Lambda}_2^b \ldots \widetilde{\Lambda}_\ell^b} \tag{2.9}$$

$$W_R = \frac{X_1 Y_1 Z_1 - X_1 Y_2 - Y_1 X_2}{\Lambda_1^b} + \sum_{i=2}^{r-1} \frac{X_i Y_i Z_i - X_i Y_{i+1} - Y_i X_{i+1}}{\Lambda_1^b \Lambda_2^b \ldots \Lambda_i^b}$$
$$+ \frac{X_r Y_r Z_r - X_r K_R^{m-1} J_R^3 - Y_r K_R^m J_R}{\Lambda_1^b \Lambda_2^b \ldots \Lambda_r^b}. \tag{2.10}$$

Each of the fields $\{X_{i>1}, Y_{i>1}, x_{i>1}, y_{i>1}\}$ is a redundant operator: that is, the equations of motion determine the low-energy behavior of each superfield exactly, leaving no independent degrees of freedom. For example, the constraint $\partial W_d / \partial X_i = 0$ determines the value of Y_{i+1}:

$$Y_2 = Y_1 Z_1, \qquad Y_3 = Y_1(Z_1 Z_2 - \Lambda_2^b), \qquad Y_{i+1} = Y_i Z_i - \Lambda_i^b Y_{i-1} = Y_1(\widetilde{\Pi}_1^i Z). \tag{2.11}$$

After confinement, the tree-level superpotential eq. (2.3) leads to

$$W_{\text{tree}} \to \frac{(K_L)_{i_1 i_2}}{M_A^{2\ell-2}} + \frac{Z_1}{M_B^{N-3}} + \frac{z_1}{M_C^{N-3}} + \frac{X_1^\alpha Y_1^\alpha}{M_R^{N-1}} + \frac{x_1^\alpha y_1^\alpha}{M_r^{N-1}}, \tag{2.12}$$

where the indices i and α refer to $SU(N)_L$ and $SU(4)$, respectively. In the discussion that follows, we assume that M_B is several orders of magnitude below M_P, and that $M_B \lesssim M_{A,C,R,r} \lesssim M_P$.

2.2 Symmetry breaking

Each term in W_{tree} is introduced to break an undesired global symmetry: however, the Z_1 and z_1 tadpoles induced by W_{tree} also have a significant effect on the vacuum structure. Added to the full superpotential,

$$W = W_{\text{tree}} + W_L + W_R, \tag{2.13}$$

the Z_1 and z_1 tadpole terms in W_{tree} shift the moduli space away from the origin: specifically, their equations of motion cause $\langle X_1 Y_1 \rangle$ and $\langle x_1 y_1 \rangle$ to be nonzero. In this section we consider the case $\langle X_1 Y_1 \rangle \gg \langle x_1 y_1 \rangle$ and show that $SU(4) \times U(1)_{PQ}$ is spontaneously broken to $SU(3)_c$.

It is convenient to normalize the infrared operators by appropriate factors of Λ_i so as to give them canonical mass dimension $+1$:

$$\widetilde{J}_L \equiv \frac{J_L}{\Lambda_L^\ell}, \qquad \widetilde{K}_L \equiv \frac{K_L}{(\Lambda_L^\ell)^2}, \qquad \widetilde{x} \equiv \frac{x_1}{\widetilde{\Lambda}_1^m}, \qquad \widetilde{y} \equiv \frac{y_1}{\widetilde{\Lambda}_1^{m+1}}, \qquad \widetilde{z}_i \equiv \frac{z_i}{\widetilde{\Lambda}_i^{N-1}} \tag{2.14}$$

$$\widetilde{J}_R \equiv \frac{J_R}{\Lambda_R^r}, \qquad \widetilde{K}_R \equiv \frac{K_R}{(\Lambda_R^r)^2}, \qquad \widetilde{X} \equiv \frac{X_1}{\Lambda_1^m}, \qquad \widetilde{Y} \equiv \frac{Y_1}{\Lambda_1^{m+1}}, \qquad \widetilde{Z}_i \equiv \frac{Z_i}{\Lambda_i^{N-1}} \tag{2.15}$$

where

$$\Lambda_L^\ell \equiv (\widetilde{\Lambda}_1 \widetilde{\Lambda}_2 \dots \widetilde{\Lambda}_\ell), \qquad\qquad \Lambda_R^r \equiv (\Lambda_1 \Lambda_2 \dots \Lambda_r). \qquad (2.16)$$

In terms of these operators, the tree-level superpotential eq. (2.3) becomes

$$W_{\text{tree}} \to \Lambda_L^2 \left(\frac{\Lambda_L}{M_A}\right)^{2\ell-2} (\widetilde{K}_L)_{i_1 i_2} + \Lambda_1^2 \left(\frac{\Lambda_1}{M_B}\right)^{N-3} \widetilde{Z}_1 + \widetilde{\Lambda}_1^2 \left(\frac{\widetilde{\Lambda}_1}{M_C}\right)^{N-3} \widetilde{z}_1$$
$$+ \Lambda_1 \left(\frac{\Lambda_1}{M_R}\right)^{N-1} \widetilde{X}\widetilde{Y} + \widetilde{\Lambda}_1 \left(\frac{\widetilde{\Lambda}_1}{M_r}\right)^{N-1} \widetilde{x}\widetilde{y}, \qquad (2.17)$$

and the dynamically generated superpotential includes the leading terms

$$W_L + W_R = \widetilde{x}\widetilde{y}\widetilde{z}_1 + \widetilde{X}\widetilde{Y}\widetilde{Z}_1 - \frac{x_1 y_2 + y_1 x_2}{\widetilde{\Lambda}_1^b} - \frac{X_1 Y_2 + Y_1 X_2}{\Lambda_1^b} + \dots \qquad (2.18)$$

The equation of motion $\partial W / \partial \widetilde{Z}_1 = 0$ enforces:

$$\widetilde{X}_\alpha \widetilde{Y}^\alpha = -\frac{\Lambda_1^{N-1}}{M_B^{N-3}} \equiv \sigma^2. \qquad (2.19)$$

By performing an SU(4) gauge transformation, the nonzero expectation values can be rotated into the $\alpha = 4$ component such that

$$\langle \widetilde{X} \rangle_{(4)} = \beta\sigma, \qquad \langle \widetilde{Y} \rangle_{(4)} = \frac{1}{\beta}\sigma, \qquad \langle \widetilde{X} \rangle_{\alpha=1,2,3} = \langle \widetilde{Y} \rangle_{\alpha=1,2,3} = 0, \qquad (2.20)$$

where β parametrizes a flat direction of the degenerate vacua, which is likely to be lifted in a particular model of SUSY breaking; we treat it as a free parameter. An $SU(3)_c$ subgroup of SU(4) remains as an infrared symmetry, and the other $15 - 8 = 7$ generators of SU(4) are broken. Through the super-Higgs mechanism, 7 of the 8 would-be NGBs are eaten by the SU(4) superfields to make them massive, and a single NGB remains massless. The matter fields decompose into irreducible representations of $SU(3)_c$ as follows:

$$\begin{aligned} \square &\longrightarrow \square \oplus \mathbf{1}, & \overline{\square} &\longrightarrow \overline{\square} \oplus \mathbf{1}, & \mathbf{Adj} &\longrightarrow \mathbf{Adj} \oplus \square \oplus \overline{\square} \oplus \mathbf{1}, \\ \widetilde{X}_{\alpha'} &\longrightarrow \widetilde{X}_\alpha \oplus \widetilde{X}_{(4)}, & \widetilde{Y}_{\alpha'} &\longrightarrow \widetilde{Y}_\alpha \oplus \widetilde{Y}_{(4)}, & \lambda_a &\longrightarrow \lambda_a' \oplus \lambda^+ \oplus \lambda^- \oplus \lambda^0. \end{aligned} \qquad (2.21)$$

A combination of the superfields $\widetilde{X}_{\alpha=1,2,3}$ and $\widetilde{Y}_{\alpha=1,2,3}$ are eaten by the massive $\lambda^\pm$ vector supermultiplets. Another linear combination of $\widetilde{X}$ and $\widetilde{Y}$ is eaten by the diagonal T^{15} generator of SU(4), leaving exactly one massless superfield to play the role of the axion.

We introduce the real scalar fields ϕ_1, ϕ_2, a and η to describe the bosonic degrees of freedom:

$$\begin{aligned} \widetilde{X}_{(4)} &= \left(\frac{\phi_1}{\sqrt{2}} + \langle \widetilde{X}_{(4)} \rangle\right) \exp\left[\frac{i}{f_a}(a + \alpha\eta)\right] \\ \widetilde{Y}_{(4)} &= \left(\frac{\phi_2}{\sqrt{2}} + \langle \widetilde{Y}_{(4)} \rangle\right) \exp\left[\frac{i}{f_a}\left(-a + \frac{1}{\alpha}\eta\right)\right], \end{aligned} \qquad (2.22)$$

where f_a is the axion decay constant, and α is a constant determined by requiring canonical normalization of the scalar kinetic terms. It is convenient to define $v_{1,2}$ such that

$$v_1 = \sqrt{2}\left|\langle\widetilde{X}_{(4)}\rangle\right| = \sqrt{2}\left|\beta\sigma\right| \qquad v_2 = \sqrt{2}\left|\langle\widetilde{Y}_{(4)}\rangle\right| = \sqrt{2}\left|\frac{\sigma}{\beta}\right|, \qquad (2.23)$$

so that normalization of the scalar fields requires

$$f_a^2 = v_1^2 + v_2^2, \qquad\qquad \alpha = \frac{v_2}{v_1}. \qquad (2.24)$$

In the discussion above we assume that $\widetilde{X}$ and $\widetilde{Y}$ are the only $\mathrm{U}(1)_{\mathrm{PQ}}$-charged fields with nonzero expectation values. This is not necessarily true: for example, $\langle K_R\rangle$ may acquire an expectation value without breaking $\mathrm{SU}(3)_c$. In the limit where $\langle K_R\rangle \ll \sigma$ its contribution to the axion potential is vanishingly small, and the physics remains approximately as discussed here. For completeness, in appendix B we derive the composition of the physical axion in the more general $\langle K_R\rangle \neq 0$ case.

To preserve $\mathrm{SU}(3)_c$ in the vacuum, the QCD-charged components of the scalars $\widetilde{x}$, $\widetilde{y}$, $\widetilde{J}_L$ and $\widetilde{J}_R$ must not acquire expectation values, which places mild constraints on the unspecified nature of SUSY-breaking. Nonzero VEVs for the $i = 4$ components of the scalar fields are permitted.

2.3 Gravitational corrections

Non-perturbative gravity produces $\mathrm{U}(1)_{\mathrm{PQ}}$-violation, which at low energies are described by local gauge invariant operators in an effective superpotential. The leading (in $1/M_{\mathrm{P}}$) terms are:

$$W_g = \rho_1 \frac{(\overline{q}_\ell\overline{q}_{\ell-1}\ldots\overline{q}_1 q Q\overline{Q}_1\overline{Q}_2\ldots\overline{Q}_r)}{M_{\mathrm{P}}^{\ell+r-1}} + \rho_2 \frac{(\overline{q}_\ell\overline{q}_{\ell-1}\ldots\overline{q}_1 q)(A^m Q)}{M_{\mathrm{P}}^{\ell+m-1}} + \rho_3 \frac{(\overline{A}^m q)(A^m Q)}{M_{\mathrm{P}}^{2m-1}}$$
$$+ \rho_4 \frac{(A\overline{Q}_1^2\overline{Q}_2^2\ldots\overline{Q}_r^2)}{M_{\mathrm{P}}^{2r-2}}, \qquad (2.25)$$

with coefficients ρ_i which encode the details of the unknown quantum gravitational physics. Naive power counting would argue for $\rho_i \sim \mathcal{O}(1)$, whereas computations based on wormhole configurations or stringy realizations of quantum gravity favor $\rho_i \sim \mathcal{O}(\exp[-S_{\mathrm{wh}}])$ with $S_{\mathrm{wh}} \sim M_{\mathrm{P}}/f_a$. To capture the range of possibilities, we will consider a range of ρ_i (all taken to have roughly equal magnitudes) in our analysis below.

After confinement, W_g maps on to:

$$W_g \to \rho_1 \frac{\Lambda_L^\ell\Lambda_R^r}{M_{\mathrm{P}}^{\ell+r-1}}(\widetilde{J}_L\widetilde{J}_R) + \rho_2 \frac{\Lambda_L^\ell\Lambda_1^m}{M_{\mathrm{P}}^{\ell+m-1}}(\widetilde{J}_L\widetilde{X}) + \rho_3 \frac{\widetilde{\Lambda}_1^m\Lambda_1^m}{M_{\mathrm{P}}^{2m-1}}(\widetilde{x}\widetilde{X}) + \rho_4 \frac{(\Lambda_R^r)^2}{M_{\mathrm{P}}^{2r-2}}(\widetilde{K}_R)_{j_1 j_2}, \ (2.26)$$

where the index j refers to the $\mathrm{SU}(N)_R$ family symmetry.

There are two types of tree-level corrections to the axion potential. In the supersymmetric limit, the equations of motion from $W_{\mathrm{tree}} + W_d + W_g$ produce operators in the Lagrangian of the form

$$\mathcal{L}_g \sim \left(\prod_{i,j} \phi_i\phi_j^\star\right)(\Phi + \Phi^\star), \qquad (2.27)$$

where Φ has non-zero $U(1)_{PQ}$ charge (and thus some of its phase is part of the axion), and ϕ_i and $\phi_j^\star$ are scalar fields as determined by the equations of motion. Replacing the fields with their expectation values, $\mathcal{L}_g$ corrects the axion potential by:

$$\delta V[a] \sim \left(\prod_{i,j} \langle\phi_i\rangle\langle\phi_j^\star\rangle \right) \langle\Phi\rangle \cos\left(\frac{q_\Phi a}{f_a} + \theta_0 \right). \qquad (2.28)$$

Clearly this type of correction is only operative if all of the relevant fields $\phi_{i,j}$ have non-zero expectation values.

The second type of tree-level correction arises once SUSY is broken, and the low energy Lagrangian contains A-terms of the form

$$\mathcal{L}_g \sim m_s W_g + h.c. \qquad (2.29)$$

(where W_g should be understood to have its super-fields replaced by their scalar components, and there is a separate SUSY-breaking coefficient of $\mathcal{O}(m_s)$ for each term in W_g). In the cases where the necessary scalar fields have zero expectation values, these terms can still correct the axion potential at loop level.

As can be seen from eq. (2.8), the moduli space includes vacua with $\langle K_R \rangle = \langle J_R \rangle = 0$. These flat directions are lifted by SUSY-breaking, and thus model-dependent. Rather than getting bogged down in the details of a specific model, we make the pessimistic assumption that the resulting expectation values are large:

$$\langle \widetilde{J}_{(4)}^j \rangle, \langle \widetilde{K}^{j_1 j_2} \rangle \sim \mathcal{O}(m_s). \qquad (2.30)$$

This assumption additionally simplifies the analysis in that for such large expectation values, the tree-level corrections to the axion potential are expected to dominate over any of the loop level corrections.

Generically, the leading contributions to the axion potential are expected to arise from SUSY-breaking rather than from the equations of motion. This is because the equations of motion from W_d involve high-dimensional operators, which are only important at tree level if all of the participating fields have relatively large expectation values. For example,

$$\left| \frac{\partial W}{\partial \widetilde{J}_R} \right|^2 = \left| \frac{\Lambda_L^\ell \Lambda_R^r}{M_P^{\ell+r-1}} (\widetilde{J}_L) - \frac{(\widetilde{X}_k \widetilde{J}_R^2)\widetilde{K}_R^{m-1}}{\Lambda_r^m} - \frac{(\widetilde{Y}_k)\widetilde{K}_R^m}{\Lambda_r^{m-1}} \right|^2 \qquad (2.31)$$

reduces to

$$\mathcal{L}_g \sim \left(\frac{\Lambda_L^\ell \Lambda_R^r}{M_P^{\ell+r-1}} \frac{\langle\widetilde{K}_R^m\rangle}{\Lambda_r^{m-1}} \right) \langle\widetilde{J}_L^\star\rangle \widetilde{Y}_k + h.c. \qquad (2.32)$$

In the product $\langle\widetilde{K}_R^m\rangle$, the $SU(N)_R$ indices are contracted antisymmetrically. If some of the expectation values are close to zero, the entire product vanishes. Only in the case where $\langle\widetilde{K}\rangle$ and $\langle\widetilde{J}\rangle$ are comparable to Λ_r does eq. (2.32) contribute significantly.

Quality factors. The SUSY-breaking A-term corresponding to the ρ_1 term in W_g is

$$\mathcal{L}_g \sim m_s \rho_1 \left(\frac{\Lambda_L^\ell \Lambda_R^r}{M_{\rm P}^{\ell+r-1}} \right) (\widetilde{J}_L)_i^\alpha (\widetilde{J}_R)_j^\alpha + h.c., \tag{2.33}$$

where the indices i and j correspond to the ${\rm SU}(N)_L \times {\rm SU}(N)_R$ global symmetry. As $\widetilde{J}_R$ is charged under ${\rm U}(1)_{\rm PQ}$ $\langle \widetilde{J}_L \widetilde{J}_R \rangle \neq 0$ shifts the axion potential by

$$\delta V[a] \sim \rho_1 m_s \left(\frac{\Lambda_L^\ell \Lambda_R^r}{M_{\rm P}^{\ell+r-1}} \right) \left| \langle \widetilde{J}_L \rangle \langle \widetilde{J}_R \rangle \right| \cos \left(q_J \frac{a}{f_a} + \theta_0 \right), \tag{2.34}$$

with $q_J = \frac{2-N}{N} = \mathcal{O}(1)$. From eq. (1.7), consistency with $\left| \bar{\theta} \right| < 10^{-10}$ requires

$$\rho_1 \frac{m_s M_{\rm P} \left| \langle \widetilde{J}_L \rangle \langle \widetilde{J}_R \rangle \right|}{(10^{12}\,{\rm GeV})^4} \left(\frac{\Lambda_L^\ell \Lambda_R^r}{M_{\rm P}^{\ell+r}} \right) < 10^{-62}. \tag{2.35}$$

A limit on r is set by the ρ_4 term:

$$\delta V[a] \sim \rho_4 m_s \frac{\Lambda_R^{2r}}{M_{\rm P}^{2r-2}} \left| \langle (\widetilde{K}_R)_{j_1 j_2} \rangle \right| \cos \left(q_K \frac{a}{f_a} + \theta_0 \right), \tag{2.36}$$

where $q_K = 4/N$. Ignoring the $\mathcal{O}(1)$ number q_K,

$$\rho_4 \frac{m_s M_{\rm P}^2 \left| \langle \widetilde{K}_R \rangle \right|}{(10^{12}\,{\rm GeV})^4} \left(\frac{\Lambda_R}{M_{\rm P}} \right)^{2r} < 10^{-62}. \tag{2.37}$$

From the ρ_3 term

$$\delta V[a] \sim m_s \rho_3 \frac{\widetilde{\Lambda}_1^m \Lambda_1^m}{M_{\rm P}^{2m-1}} \left| \langle \widetilde{x}_{(4)} \rangle \langle \widetilde{X}_{(4)} \rangle \right| \cos \left(\frac{a}{f_a} + \theta_0 \right), \tag{2.38}$$

we find a constraint on $N = 2m + 1$:

$$\rho_3 \frac{m_s M_{\rm P} \langle \widetilde{x}_{(4)} \rangle \langle \widetilde{X}_{(4)} \rangle}{(10^{12}\,{\rm GeV})^4} \left(\frac{\widetilde{\Lambda}_1}{M_{\rm P}} \right)^m \left(\frac{\Lambda_1}{M_{\rm P}} \right)^m < 10^{-62}. \tag{2.39}$$

Finally, the ρ_2 term sets an additional constraint on ℓ and N:

$$\delta V[a] \sim m_s \rho_2 \frac{\Lambda_L^\ell \Lambda_1^m}{M_{\rm P}^{\ell+m-1}} \left| \langle J_L^{(4)} \rangle \langle \widetilde{X}_{(4)} \rangle \right| \cos \left(\frac{a}{f_a} + \theta_0 \right), \tag{2.40}$$

$$\rho_2 \frac{m_s M_{\rm P} \langle J_L \rangle \langle \widetilde{X}_{(4)} \rangle}{(10^{12}\,{\rm GeV})^4} \left(\frac{\Lambda_L}{M_{\rm P}} \right)^\ell \left(\frac{\Lambda_1}{M_{\rm P}} \right)^m < 10^{-62}. \tag{2.41}$$

As long as β is neither very large nor very small, eqs. (2.35), (2.37), (2.39) and (2.41) provide the most restrictive constraints on m, ℓ and r. A wide range of values is allowed for each of the parameters, as we discuss in more detail below.

B1	(GeV)		**B2**	(GeV)		**B3**	(GeV)
f_a	10^{17}		f_a	10^{12}		f_a	10^9
Λ_1	10^{17}		Λ_1	10^{12}		Λ_1	10^9
$\Lambda_{i>1}$	10^{15}		$\Lambda_{i>1}$	10^9		$\Lambda_{i>1}$	10^4
$\widetilde{\Lambda}_i$	10^{15}		$\widetilde{\Lambda}_i$	10^9		$\widetilde{\Lambda}_i$	10^4
m_s	10^6		m_s	10^4		m_s	10^4

Table 3. Three benchmark points in the parameter space of Λ_i and $\widetilde{\Lambda}_i$. With the exception of $\langle \widetilde{X} \rangle$ and $\langle \widetilde{Y} \rangle$, the expectation values of the $SU(3)_c$ singlet fields are taken to be $\mathcal{O}(m_s)$.

2.4 Benchmark models

In this section we consider the quality of the axion potential in three particular models, with $f_a = 10^{17} \, \text{GeV}$, $f_a = 10^{12} \, \text{GeV}$ and $f_a = 10^9 \, \text{GeV}$. For simplicity, we take $\Lambda_1 \sim M_B \sim f_a$ and $\Lambda_{i \neq 1} \sim \widetilde{\Lambda}_i$ for each model, and we allow all QCD singlet scalar fields to acquire $\mathcal{O}(m_s)$ expectation values. Choices for each of these scales are shown in table 3.

Model **B1** is particularly susceptible to gravitational disruptions, as the scales Λ_i and $\widetilde{\Lambda}_i$ are taken to be relatively close to the Planck scale $M_P \sim 10^{19} \, \text{GeV}$. In this model even exponential suppression of the constants $\rho_i \sim \exp(-M_P/f_a) \sim 10^{-44}$ cannot account for the high quality of the axion potential, and large values of N, ℓ and r are required. Models **B2** and **B3** have values of $f_a \lesssim 10^{12} \, \text{GeV}$ consistent with the axion dark matter hypothesis; with its smaller values of Λ_i and $\widetilde{\Lambda}_i$, model **B3** is more adept at suppressing gravitational corrections.

In figure 2 we show minimum values for $m \equiv \frac{N-1}{2}$, ℓ, and r consistent with $|\bar{\theta}| < 10^{-10}$ for the $SU(N)^\ell \times SU(4) \times SU(N)^r$ composite axion, as a function of the parameters ρ_i. A wide range is shown for ρ, to accommodate both exponentially suppressed and $\mathcal{O}(1)$ values. In the $\rho_i = \mathcal{O}(1)$ limit, the minimal gauge groups for the three benchmark models are:

$$
\begin{aligned}
\textbf{B1:} \quad & SU(23)^{11} \times SU(4) \times SU(23)^9 \\
\textbf{B2:} \quad & SU(9)^3 \times SU(4) \times SU(9)^4 \\
\textbf{B3:} \quad & SU(7)^2 \times SU(4) \times SU(7)^3.
\end{aligned}
\tag{2.42}
$$

Naturally, if after SUSY breaking the scalar fields $\widetilde{J}_{L,R}$, $\widetilde{x}$, $\widetilde{y}$, and $\widetilde{K}_R$ do not acquire expectation values, then the $U(1)_{PQ}$ violation induced by W_g affects the axion potential only at loop level, and smaller values for N, ℓ and r are permitted. In the limit where ρ is exponentially suppressed, $|\bar{\theta}| < 10^{-10}$ no longer constrains m, ℓ or r. Although eqs. (2.35), (2.37), (2.39) and (2.41) are valid only for $m \geq 2$, $r \geq 1$ and $\ell \geq 0$, smaller values for m and r are shown in figure 2 to indicate where ρ is small enough that compositeness is no longer necessary.

3 Dynamically generated W_{tree}

As described in section 2, the $SU(N)^\ell \times SU(4) \times SU(N)^r$ composite accidental axion has a high-quality scalar potential and most of the important scales are derived from the confining dynamics, with the exception of M_B in the tree-level superpotential. This is a
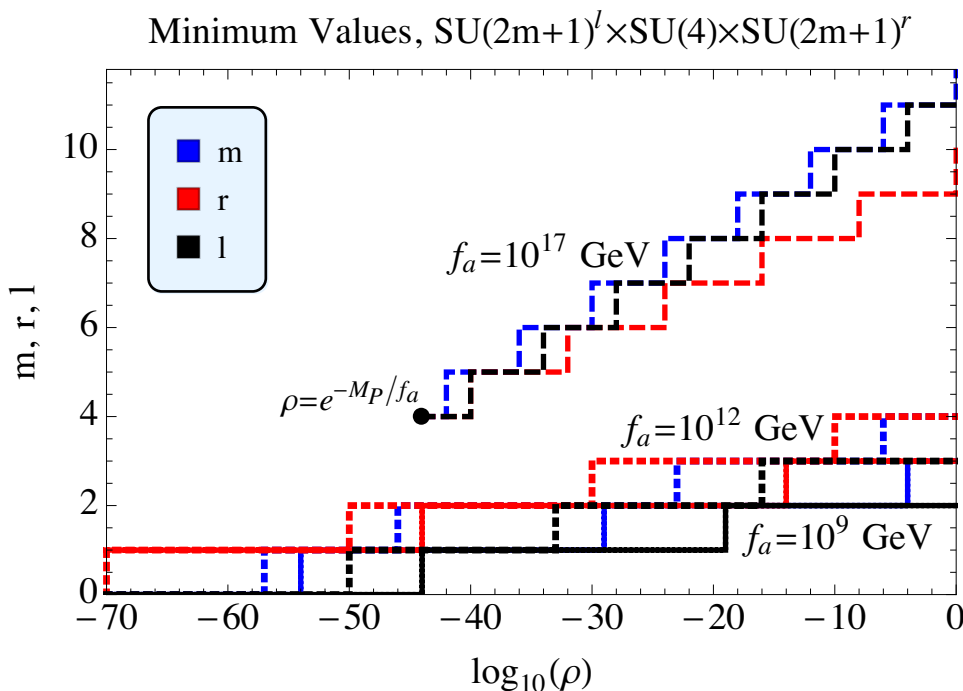
Figure 2. Minimum values for m, ℓ and r consistent with $\left|\bar{\theta}\right| < 10^{-10}$ are shown as a function of $\rho_{1\ldots 4}$. For the first benchmark model with $f_a = 10^{17}\,\text{GeV}$, we show only values of $\rho \gtrsim \exp(-M_\text{P}/f_a) \approx 10^{-43.4}$. The $f_a = 10^{12}\,\text{GeV}$ and $f_a = 10^9\,\text{GeV}$ models are depicted using dotted and solid lines, respectively.

relatively minor shortcoming: f_a is determined by the relationship between M_B, Λ_1, and $\beta^2 = \langle\widetilde{X}\rangle/\langle\widetilde{Y}\rangle$,

$$f_a^2 = 2\left|\frac{\Lambda_1^{N-1}}{M_B^{N-3}}\left(\beta^2 + \frac{1}{\beta^2}\right)\right|, \tag{3.1}$$

and the scale $M_B \ll M_\text{P}$ is added "by hand" in the tree-level superpotential. In this section we show how the M_B term in W_tree can be dynamically generated by the s-confinement of an $\text{Sp}(2N-4)$ gauge group, so that all of the important mass scales are determined by strong dynamics.

A gauge theory with $2N$ quarks ψ charged under $\text{Sp}(2N-4)$ in the fundamental representation s-confines [32] to form mesons $M_{ij} = \epsilon_{ab}\psi_i^a\psi_j^b$, with the superpotential

$$W_d = \frac{\text{Pf}\,M}{\Lambda_0^{2N-1}}. \tag{3.2}$$

We break the $\text{SU}(2N)$ flavor symmetry by gauging its $\text{SU}(N)_1 \times \text{SU}(N)_2 = G_1 \times G_2$ subgroup:

$$\square \longrightarrow (\square, \mathbf{1}) \oplus (\mathbf{1}, \square) \qquad\qquad \psi_i^a \longrightarrow (\psi_1)_\alpha^a \oplus (\psi_2)_\beta^a, \tag{3.3}$$
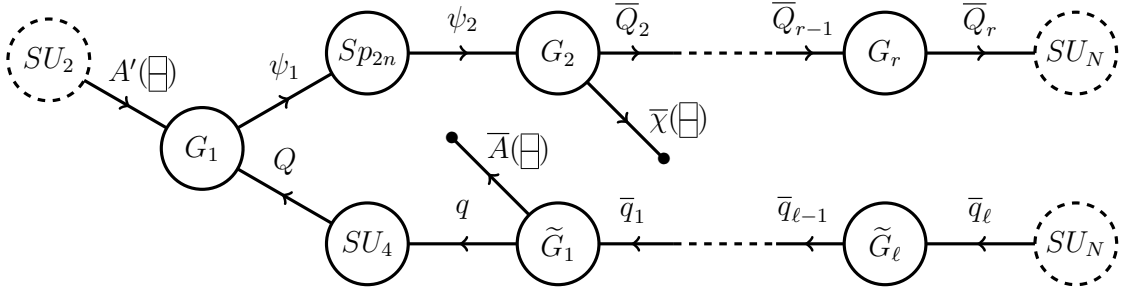
– 14 –

Figure 3. The matter content of the $\mathrm{SU}(N)^{\ell} \times \mathrm{SU}(4) \times \mathrm{Sp}(2n) \times \mathrm{SU}(N)^r$ composite axion model is depicted in the moose diagram above, with $Sp_{2n} \equiv \mathrm{Sp}(2N-4)$. The SU(2) family symmetry of the A' fields is broken explicitly by the tree-level superpotential eq. (3.7).

where α and β correspond respectively to the $\mathrm{SU}(N)_1$ and $\mathrm{SU}(N)_2$ gauge indices. The meson $M \sim \square$ decomposes into irreducible representations of $G_1 \times G_2$:

$$\widetilde{M}_1^{\alpha_1 \alpha_2} = \frac{(\psi_1)_a^{\alpha_1} (\psi_1)_b^{\alpha_2} \epsilon_{ab}}{\Lambda_0}, \quad \overline{Q}_1^{\alpha\beta} = \frac{(\psi_1)_a^{\alpha} (\psi_2)_b^{\beta} \epsilon_{ab}}{\Lambda_0}, \quad \widetilde{M}_2^{\beta_1 \beta_2} = \frac{(\psi_2)_a^{\beta_1} (\psi_2)_b^{\beta_2} \epsilon_{ab}}{\Lambda_0}, \quad (3.4)$$

where Λ_0 is the confinement scale of $\mathrm{Sp}(2N-4)$. In terms of these operators the dynamically generated superpotential is

$$W_d = \frac{\mathrm{Pf}(\psi^2)}{\Lambda_0^{2N-3}} = \frac{(\Lambda_0)^N}{\Lambda_0^{2N-3}} \left[\widetilde{M}_1^m \overline{Q}_1 \widetilde{M}_2^m + \widetilde{M}_1^{m-1} \overline{Q}_1^3 \widetilde{M}_2^{m-1} + \ldots + \widetilde{M}_1 \overline{Q}_1^{2m-1} \widetilde{M}_2 + \overline{Q}_1^{2m+1} \right], \tag{3.5}$$

in the case where $N = 2m+1$ is odd. Combinatoric factors for each term in the expansion of $\mathrm{Pf}\, M$ such as $\overline{Q}_1^N \equiv \det \overline{Q}_1$ have been suppressed.

To match this theory with the $A + 4Q + N\overline{Q}$ model, the M_1 and M_2 degrees of freedom must be removed. This is achieved by adding the following matter fields charged under $\mathrm{SU}(N)_1 \times \mathrm{SU}(N)_2$:

$$2A' + 4Q + \chi + N\overline{Q}_2 = 2(\square\hspace{-0.3em}\square, \mathbf{1}) \oplus 4(\square, \mathbf{1}) \oplus (\mathbf{1}, \overline{\square\hspace{-0.3em}\square}) \oplus N(\mathbf{1}, \overline{\square}). \tag{3.6}$$

In the $\mathrm{SU}(N)^{\ell} \times \mathrm{SU}(4) \times \mathrm{SU}(N)^r$ composite model, the SU(4) and SU(N) family symmetries of the Q and $\overline{Q}_2$ are gauged. The full matter content of the theory is shown in figure 3.

Gauge-invariant operators of the form $(A'\psi_1^2)$ and $(\chi \psi_2^2)$ can be added as marginal operators in a tree-level superpotential:

$$W_{\mathrm{tree}} = \lambda_i (A_i')^{\alpha_1 \alpha_2} (\psi_1)_{\alpha_1}^{a_1} (\psi_1)_{\alpha_2}^{a_2} \epsilon_{a_1 a_2} + \lambda_0 \chi^{\beta_1 \beta_2} (\psi_2)_{\beta_1}^{a_1} (\psi_2)_{\beta_2}^{a_2} \epsilon_{a_1 a_2}, \tag{3.7}$$

where the indices i, a, α and β correspond to SU(2), $\mathrm{Sp}(2N-4)$, $\mathrm{SU}(N)_1$ and $\mathrm{SU}(N)_2$, respectively, and λ_i and λ_0 are dimensionless coupling constants. After $\mathrm{Sp}(2N-4)$ confines, W_{tree} becomes

$$W_{\mathrm{tree}} = \lambda_i \Lambda_0 (A_i')^{\alpha_1 \alpha_2} \widetilde{M}_1^{\alpha_1 \alpha_2} + \lambda_0 \Lambda_0 \chi^{\beta_1 \beta_2} \widetilde{M}_2^{\beta_1 \beta_2}. \tag{3.8}$$

This is extremely convenient: in the limit where $\Lambda_0 \gg \Lambda_1$, the fields M_1, M_2, χ, and the linear combination "$(A_1' + A_2')$" all acquire large masses and decouple. One linear

	Sp(2N−4)	SU(N)$_1$	SU(N)$_2$	SU(N)$_3$	SU(4)	SU(2)	U(1)$_{\rm PQ}$
ψ_1	□	$\overline{\square}$					$-2/N$
ψ_2	□		□				$+2/N$
A'		□				**2**	$4/N$
χ			$\overline{\square}$			**1**	$-4/N$
Q		□			□		$\frac{2-N}{N}$
$\overline{Q}_2$			$\overline{\square}$	□			0

Table 4. A subset of the matter fields in the Sp(2N−4) model are shown with their Peccei-Quinn charges. All of the non-Abelian groups except for SU(2) are gauged.

combination of A'_1 and A'_2 remains massless, which we define as A:

$$A \equiv \frac{\lambda_2 A_1 - \lambda_1 A_2}{\mathcal{N}}, \tag{3.9}$$

with some normalization factor $\mathcal{N}$.

The dynamically generated superpotential simplifies greatly when we consider the fact that $\widetilde{M}_1$ and $\widetilde{M}_2$ have $\mathcal{O}(\Lambda_0)$ masses from $W_{\rm tree}$:

$$\frac{\partial W}{\partial A'_i} = \lambda_i \Lambda_0 \widetilde{M}_1, \qquad\qquad \frac{\partial W}{\partial \chi} = \lambda_0 \Lambda_0 \widetilde{M}_2. \tag{3.10}$$

After integrating out the heavy fields, the superpotential becomes

$$W = \frac{\overline{Q}_1^N}{\Lambda_0^{N-3}}. \tag{3.11}$$

Not only is this the desired tree-level superpotential for the composite axion model, but all of the extra matter fields A', χ, $\widetilde{M}_1$ and $\widetilde{M}_2$ have decoupled, leaving only A and $\overline{Q}_1$ as infrared degrees of freedom. In eq. (3.1) M_B is replaced by Λ_0, so that

$$f_a^2 = 2\left| \frac{\Lambda_1^{N-1}}{\Lambda_0^{N-3}} \left(\beta^2 + \frac{1}{\beta^2} \right) \right|. \tag{3.12}$$

Every important scale other than $M_{\rm P}$ is now determined solely by confining dynamics.

The nonzero Sp(2N−4)^2-U(1)$_B$ anomaly breaks U(1)$_B$ explicitly, as can be seen from the W_d of eq. (3.5). Although in principle the new fields χ and A' provide two additional anomaly-free U(1) symmetries, these are broken by the tree-level superpotential eq. (3.7), and only the SU(N)$_L \times$ SU(N)$_R \times$ U(1)$_A \times$ U(1)$_C \times$ U(1)$_R$ global symmetry remains. Introducing

$$\delta W_{\rm tree} = \frac{(\overline{A}\overline{q}_1^2\overline{q}_2^2\ldots\overline{q}_\ell^2)}{M_A^{2\ell-2}} + \frac{(\overline{q}_1^N)}{M_C^{N-3}} + \frac{(A^m Q)(A^{m-1}Q^3)}{M_R^{N-1}} + \frac{(\overline{A}^m q)(\overline{A}^{m-1}q^3)}{M_r^{N-1}} \tag{3.13}$$

with $M_A \sim M_C \sim M_R \sim M_r \sim M_{\rm P}$ is sufficient to give masses to the additional pNGBs. In table 4, the Peccei-Quinn charges of each field is shown.

Axion quality. Of the new superpotential terms which break $U(1)_{PQ}$, the leading terms are

$$W_g \sim \frac{\chi^m \overline{Q}_2 \overline{Q}_3 \ldots \overline{Q}_r}{M_P^{m+r-4}} + \sum_p \frac{(A_1^{m-p} A_2^p Q)(q\overline{q}_1 \overline{q}_2 \ldots \overline{q}_\ell)}{M_P^{m+\ell-1}} \tag{3.14}$$

As χ has a mass of $\mathcal{O}(\Lambda_0)$ and no expectation value, the χ^m interaction has no tree-level effect on the axion potential. The only effects are loop-induced and receive additional suppression.

One linear combination in the $(A_1^{m-p} A_2^p Q)$ sum corresponds to the infrared operator $(A^m Q)$, which has the expectation value $\langle X_1 \rangle$. This term is already included in the W_g of eq. (2.25). Every other term in the sum includes a power of the massive combination $(\lambda_1 A_1 + \lambda_2 A_2)$, which has no expectation value, and is therefore less disruptive to the axion potential than the effects already considered in eq. (2.25).

Aside from the replacement of M_B by Λ_0, the quality factors calculated in section 2.3 are largely unchanged. Operators involving $\overline{Q}_1$ are the exception: now that $\overline{Q}_1 = \psi_1 \psi_2 / \Lambda_0$, a suppression of Λ_0/M_P is added to the operators involving J_R and K_R, marginally improving eqs. (2.35) and (2.37):

$$\rho_1 \frac{m_s M_P \left| \langle \widetilde{J}_L \rangle \langle \widetilde{J}_R \rangle \right|}{(10^{12}\,\text{GeV})^4} \left(\frac{\Lambda_0}{M_P} \right) \left(\frac{\Lambda_L^\ell \Lambda_R^r}{M_P^{\ell+r}} \right) < 10^{-62} \tag{3.15}$$

$$\rho_4 \frac{m_s M_P^2 \left| \langle \widetilde{K}_R \rangle \right|}{(10^{12}\,\text{GeV})^4} \left(\frac{\Lambda_0}{M_P} \right)^2 \left(\frac{\Lambda_R}{M_P} \right)^{2r} < 10^{-62}. \tag{3.16}$$

For many values of ρ_i this decreases the minimum value for r by one, as can be seen from the three benchmark models at $\rho_i = \mathcal{O}(1)$:

$$
\begin{aligned}
&\textbf{B1:} \quad SU(23)^{11} \times SU(4) \times Sp(42) \times SU(23)^9 \\
&\textbf{B2:} \quad SU(9)^3 \times SU(4) \times Sp(14) \times SU(9)^3 \\
&\textbf{B3:} \quad SU(7)^2 \times SU(4) \times Sp(10) \times SU(7)^2.
\end{aligned} \tag{3.17}
$$

Alternate confinement order. Thus far, we have required that $\Lambda_0 > \Lambda_1$, simply because the dual of $SU(N) : 2A + 4Q + (2N-4)\overline{Q}$ with the tree-level superpotential $W_{\text{tree}} \sim A\overline{Q}^2$ does not appear in the literature. In principle the infrared behavior of the $2A + 4Q + (2N-4)\overline{Q}$ theory with $W_{\text{tree}} \neq 0$ can be determined using "deconfinement" techniques [29] and a sequence of dualities: a similar calculation [36] has been completed for $A + FQ + (N + F - 4)\overline{Q}$ with a superpotential of the form $W \sim A\overline{Q}^2$.

Without calculating the degrees of freedom and the superpotential in the infrared dual of $SU(N) : 2A + 4Q + (2N-4)\overline{Q}$, it is not known how the scale f_a is set in the dual theory. If in the $\Lambda_0 \ll \Lambda_1$ limit $U(1)_{PQ}$ is still broken at the scale $f_a^2 \sim \Lambda_1^{N-1}/\Lambda_0^{N-3}$, then $f_a \sim 10^{12}$ GeV can be achieved with much smaller values of Λ_0 and Λ_1, significantly improving the axion quality. We leave detailed exploration of this limit to future work.

4 Conclusions

In the composite axion model based on the gauge group $\mathrm{SU}(N)^\ell \times \mathrm{SU}(4) \times \mathrm{SU}(N)^r$, a $\mathrm{U}(1)_{\mathrm{PQ}}$ is spontaneously broken by the vacuum expectation values of the $\mathrm{SU}(4)$-charged hadrons $X_1 = (A^m Q)$ and $Y_1 = (A^{m-1} Q^3)$, simultaneously producing the QCD axion and breaking $\mathrm{SU}(4)$ to $\mathrm{SU}(3)_c$. All important scales in the axion model are generated dynamically from confinement, and are naturally small compared to the Planck scale.

By calculating the disruption to the axion potential $V[a]$ induced by Planck-scale effects, we have demonstrated that the composite model is successful at preserving the quality of the axion potential even when large expectation values are permitted for all of the $\mathrm{U}(1)_{\mathrm{PQ}}$-charged QCD-singlet scalar fields. In realistic models incorporating susy breaking with positive quadratic terms for these scalars such that no large expectation values result, the quality of the axion potential will improve significantly for any given N, ℓ and r, as the terms in W_g disrupt the axion potential to a lesser degree. It would be worthwhile to further investigate such constructions.

It is likely that the success of the $\mathrm{SU}(N)^\ell \times \mathrm{SU}(4) \times \mathrm{SU}(N)^r$ composite axion can be replicated by embedding $\mathrm{SU}(3)_c$ within the $\mathrm{SU}(N)_R$ flavor symmetry of the $A + 4Q + N\overline{Q}$ model. In this case $\mathrm{U}(1)_{\mathrm{PQ}}$ will be more closely associated with the $\mathrm{U}(1)_B$ flavor symmetry of table 1 rather than $\mathrm{U}(1)_A$, and the axion will be generated from a linear combination of $(\overline{Q}_i^N)$ baryons.

Compositeness can cure the axion quality problem, and as our models demonstrate, may provide clues to the existence of interesting dynamics in the ultraviolet.

Acknowledgments

A Axion quality

To leading order in a, the QCD-induced axion potential $V[a]$ has the form

$$V[a] = V_0 - \frac{1}{2} m_a^2 \left(a + f_a \bar{\theta} \right), \tag{A.1}$$

which is minimized when $\langle \theta \rangle \equiv (a/f_a + \bar{\theta})$ is equal to zero. It is convenient to define the shifted field $\alpha \equiv a + f_a \bar{\theta}$, so that $\langle \theta \rangle = \langle \alpha \rangle / f_a$. Explicit $\mathrm{U}(1)_{\mathrm{PQ}}$ violation elsewhere in the theory adds corrections to $V[a]$,

$$\delta V[a] = Q f_a^4 \cos \left(\kappa \left[\frac{a}{f_a} + \bar{\theta} \right] + \theta_0 \right), \tag{A.2}$$

which for small values of $\langle \theta \rangle$ is approximately

$$\delta V[a] = Q f_a^4 \left[1 - \frac{1}{2} \left(\frac{\kappa \alpha}{f_a} \right)^2 \right] \cos \theta_0 - Q f_a^4 \left(\frac{\kappa \alpha}{f_a} \right) \sin \theta_0. \tag{A.3}$$

As θ_0 is determined by the precise manner in which $U(1)_{PQ}$ is broken, we do not assume that it is smaller than $\mathcal{O}(1)$. Combining eqs. (A.1) and (A.2), $V[a]$ becomes

$$V[\alpha] = \left(V_0 + Qf_a^4 \cos\theta_0\right) - \left(Qf_a^3\kappa\sin\theta_0\right)\alpha - \frac{1}{2}\left(m_a^2 + Qf_a^2\kappa^2\cos\theta_0\right)\alpha^2, \tag{A.4}$$

so that the expectation value $\langle\alpha\rangle$ shifts away from zero:

$$\langle\alpha\rangle = -\frac{Qf_a^3\kappa\sin\theta_0}{m_a^2 + Qf_a^2\kappa^2\cos\theta_0}. \tag{A.5}$$

Experimental measurements of $\langle\theta\rangle$ set an upper bound $\langle\alpha\rangle < f_a\,|\theta_{\max}|$. Assuming $|\theta_{\max}|\,\kappa \ll \sin\theta_0$, the corresponding bound on Q is

$$Q < \frac{m_a^2}{f_a^2}\frac{|\theta_{\max}|}{\kappa\,|\sin\theta_0|}. \tag{A.6}$$

Using $m_a^2 = m_\pi^2 f_\pi^2/f_a^2$ and assuming $\kappa\sin\theta_0 = \mathcal{O}(1)$, eq. (A.6) implies

$$Q \lesssim 10^{-62}\left(\frac{10^{12}\text{ GeV}}{f_a}\right)^4. \tag{A.7}$$

B   Axion assignment in a general vacuum

Suppose there exist many fields Φ_i, each with a Peccei-Quinn charge q_i. Let us define the charge-normalized expectation value

$$v_i \equiv q_i\sqrt{2}\langle\Phi_i\rangle \tag{B.1}$$

for each field Φ_i. If there are n fields with nonzero expectation values, then let us define $n-1$ fields η_i and the axion a, with the following assignment:

$$\Phi_1 = \left(\frac{\phi_1}{\sqrt{2}} + \langle\Phi_1\rangle\right)\exp\left[\frac{iq_1}{f_a}(a + \alpha_1\eta_1)\right] \tag{B.2}$$

$$\Phi_2 = \left(\frac{\phi_2}{\sqrt{2}} + \langle\Phi_2\rangle\right)\exp\left[\frac{iq_2}{f_a}(a + \beta_1\eta_1 + \beta_2\eta_2)\right] \tag{B.3}$$

$$\Phi_3 = \left(\frac{\phi_3}{\sqrt{2}} + \langle\Phi_3\rangle\right)\exp\left[\frac{iq_3}{f_a}(a + \gamma_1\eta_1 + \gamma_2\eta_2 + \gamma_3\eta_3)\right] \tag{B.4}$$

$$\vdots$$

$$\Phi_{n-1} = \left(\frac{\phi_{n-1}}{\sqrt{2}} + \langle\Phi_{n-1}\rangle\right)\exp\left[\frac{iq_{n-1}}{f_a}(a + \alpha_1^{(n-1)}\eta_1 + \ldots + \alpha_{n-1}^{(n-1)}\eta_{n-1})\right] \tag{B.5}$$

$$\Phi_n = \left(\frac{\phi_n}{\sqrt{2}} + \langle\Phi_n\rangle\right)\exp\left[\frac{iq_n}{f_a}(a + \alpha_1^{(n)}\eta_1 + \ldots + \alpha_{n-1}^{(n)}\eta_{n-1})\right] \tag{B.6}$$

In the sequence above, the first appearance of each field η_i is in the phase of Φ_i. The field Φ_n does not introduce any new η_i fields.

Let us define the following $(n-1)$ constants:

$$x_1 = \beta_1 = \gamma_1 = \delta_1 = \ldots = \alpha_1^{(n-1)} = \alpha_1^{(n)} \tag{B.7}$$

$$x_2 = \gamma_2 = \delta_2 = \ldots = \alpha_2^{(n-1)} = \alpha_2^{(n)} \tag{B.8}$$

$$x_3 = \delta_3 = \ldots = \alpha_3^{(n-1)} = \alpha_3^{(n)} \tag{B.9}$$

$$\vdots$$

$$x_{n-2} = \alpha_{n-2}^{(n-1)} = \alpha_{n-2}^{(n)} \tag{B.10}$$

$$x_{n-1} = \alpha_{n-1}^{(n)}. \tag{B.11}$$

These equalities follow from the vanishing of the kinetic cross terms, which also give the following relationships between the x_i and $\{\alpha_1, \beta_2, \gamma_3, \ldots, \alpha_{n-1}^{(n-1)}\}$:

$$0 = 1 + x_1\alpha_1 \tag{B.12}$$

$$0 = 1 + x_1^2 + x_2\beta_2 \tag{B.13}$$

$$0 = 1 + x_1^2 + x_2^2 + x_3\gamma_3 \tag{B.14}$$

$$\vdots$$

$$0 = 1 + x_1^2 + \ldots + x_{n-2}^2 + x_{n-1}\alpha_{n-1}^{(n-1)}. \tag{B.15}$$

Finally, we require that the kinetic terms $(\partial_\mu \eta_i)^2$ and $(\partial_\mu a)^2$ are canonically normalized. This leads to the remaining n constraints:

$$\frac{f_a^2}{v_1^2} = 1 + \alpha_1^2 \tag{B.16}$$

$$\frac{f_a^2}{v_2^2} = 1 + x_1^2 + \beta_2^2 \tag{B.17}$$

$$\frac{f_a^2}{v_3^2} = 1 + x_1^2 + x_2^2 + \gamma_3^2 \tag{B.18}$$

$$\vdots$$

$$\frac{f_a^2}{v_{n-1}^2} = 1 + x_1^2 + x_2^2 + \ldots + x_{n-2}^2 + (\alpha_{n-1}^{(n-1)})^2 \tag{B.19}$$

$$\frac{f_a^2}{v_n^2} = 1 + x_1^2 + x_2^2 + \ldots + x_{n-2}^2 + x_{n-1}^2. \tag{B.20}$$

These systems of equations have the solutions:

$$\alpha_1^2 = \frac{f_a^2 - v_1^2}{v_1^2} \qquad\qquad x_1^2 = \frac{v_1^2}{f_a^2 - v_1^2} \tag{B.21}$$

$$\beta_2^2 = \frac{f_a^2(f_a^2 - v_1^2 - v_2^2)}{v_2^2(f_a^2 - v_1^2)} \qquad\qquad x_2^2 = \frac{v_2^2 f_a^2}{(f_a^2 - v_1^2 - v_2^2)(f_a^2 - v_1^2)} \tag{B.22}$$

$$\gamma_3^2 = \frac{f_a^2(f_a^2 - v_1^2 - v_2^2 - v_3^2)}{v_3^2(f_a^2 - v_1^2 - v_2^2)} \qquad\qquad x_3^2 = \frac{v_3^2 f_a^2}{(f_a^2 - v_1^2 - v_2^2 - v_3^2)(f_a^2 - v_1^2 - v_2^2)}, \tag{B.23}$$

and so on. The general solution is

$$(\alpha_i^{(i)})^2 = \frac{f_a^2(f_a^2 - v_1^2 - v_2^2 - \ldots - v_i^2)}{v_i^2(f_a^2 - v_1^2 - v_2^2 - \ldots - v_{i-1}^2)} \tag{B.24}$$

$$x_i^2 = \frac{v_i^2 f_a^2}{(f_a^2 - v_1^2 - v_2^2 - \ldots - v_i^2)(f_a^2 - v_1^2 - v_2^2 - \ldots - v_{i-1}^2)}, \tag{B.25}$$

for $i = 1 \ldots (n-1)$. Each $\alpha_i^{(i)}$ and x_i must also obey

$$\alpha_i^{(i)} x_i < 0, \tag{B.26}$$

but the signs of $\alpha^{(i)}$ and x_i are otherwise arbitrary.

Finally, the axion decay constant is:

$$f_a^2 = v_1^2 + v_2^2 + \ldots + v_{n-1}^2 + v_n^2. \tag{B.27}$$

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