Isometries, gaugings and $\mathcal{N}=2$ supergravity decoupling

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ABSTRACT: We study off-shell rigid limits for the kinetic and scalar-potential terms of a single $\mathcal{N}=2$ hypermultiplet. In the kinetic term, these rigid limits establish relations between four-dimensional quaternion-Kähler and hyper-Kähler target spaces with symmetry. The scalar potential is obtained by gauging the graviphoton along an isometry of the quaternion-Kähler space. The rigid limits unveil two distinct cases. A rigid $\mathcal{N}=2$ theory on Minkowski or on AdS$_4$ spacetime, depending on whether the isometry is translational or rotational respectively. We apply these results to the quaternion-Kähler space with Heisenberg $\ltimes$ U(1) isometry, which describes the universal hypermultiplet at type-II string one-loop.

Contents

1 Introduction

The gravity decoupling is a subject of interest in supergravity and string theory. It is a precious source of information about the spectrum in models where supersymmetry is rigidly realized, controls the curvature of the background metric, and can shed light on the supersymmetry breaking mechanism.

In the framework of local $\mathcal{N} = 2$ supersymmetry, the hypermultiplet scalar dynamics is captured in σ-models with four-dimensional quaternion-Kähler target spaces [1], and interaction potentials obtained by gauging some symmetries. Similarly, global $\mathcal{N} = 2$ supersymmetry requires hyper-Kähler spaces [2]. These are Ricci-flat Kähler spaces and for one hypermultiplet Riemann-self-dual.

When supersymmetry is locally realized, the scalar curvature of the quaternion-Kähler space is directly proportional to the gravitational constant $k^2 = 8\pi M_{\text{Planck}}^{-2}$. The decoupling limit consists in taking this coupling constant to zero i.e. $M_{\text{Planck}} \to \infty$, which deforms

the quaternion-Kähler geometry into a hyper-Kähler one. Such a limiting process must smoothly interpolate between the two geometries, and its description requires care. It implies to simultaneously "zooming-in" in order to recover non-trivial hyper-Kähler geometries, as performed in ref. [3] for quaternion-Kähler spaces with Heisenberg isometry (this symmetry was discussed in ref. [4]).

Bridging hyper-Kähler and quaternion-Kähler four-dimensional spaces has been discussed in some specific cases, including the quaternionic quotient method [5, 6], or the eight-dimensional hyper-Kähler cone technique [7, 8]. Later, a general correspondence between quaternionic manifolds with an isometry and hyper-Kähler manifolds with a rotational symmetry, endowed with a hyperholomorphic connection, was found in [9] within a mathematical framework. This correspondence was further pursued in [10, 11], and developed from a more physical perspective in [12]. Finally progress has been made in the rigid limit of special quaternionic Kähler manifolds [13], constructed through the local c-map, reducing to hyper-Kähler spaces constructed by the rigid c-map [14, 15]. It it obvious that several quaternion-Kähler spaces can have the same rigid limit. For instance, $\mathrm{SO}(1,4)/\mathrm{SO}(4)$ and $\mathrm{SU}(1,2)/\mathrm{SU}(2) \times \mathrm{U}(1)$ lead to flat hyper-Kähler spaces. It is also true that one quaternion-Kähler space can have several rigid limits. A point that we discuss in this work.

More recently, a systematic pattern for connecting general quaternion-Kähler and hyper-Kähler spaces with symmetries was introduced in [16]. The aim of the present article is to recast this method in more general terms, including in particular the scalar potential and its behaviour along the decoupling, its critical points and their supersymmetry properties, as well as the value of the cosmological constant. We will also discuss alternative decoupling limits, setting the control of the symmetry at the hyper-Kähler level.

We will first discuss hyper-Kähler and quaternion-Kähler spaces with symmetries, emphasizing a simple relationship between pairs of such spaces, which translates into the coupling or decoupling of gravity. This holds for the kinetic term of hypermultiplet scalars. The behaviour of the potential will be analyzed next, when this potential is obtained by gauging (for simplicity) the graviphoton along an isometry of the quaternion-Kähler space. Two separate regimes will be studied: the case where the decoupling of gravity leaves a rigid $\mathcal{N} = 2$ theory on Minkowski background, and the alternative where the spacetime is AdS_4. For all these cases, we systematically study the mass spectrum.

We start with a short review on hyper-Kähler and quaternionic manifolds with a symmetry, section 2. In section 3, we investigate gravity decoupling limits of quaternionic manifolds with a symmetry, leading to hyper-Kähler spaces. The analysis of the scalar potential by gauging the graviphoton is performed in section 4. Finally, we study extensively in section 5 the quaternion-Kähler space with Heisenberg $\ltimes \mathrm{U}(1)$ isometry and its decoupling limits. Two appendices follow, including the discussion of the pseudo-Fubini-Study metric, which describes the universal hypermultiplet at string tree-level, section A, and an alternative exhibition of generic, Ricci-flat, scalar-flat or Einstein four-dimensional Kähler spaces with a holomorphic isometry, section B.

2 Hyper-Kähler and quaternionic manifolds with a symmetry

A hyper-Kähler space in four dimensions is a Kähler manifold with self-dual Riemann tensor:

$$R_{uvxy} - \frac{1}{2}\,\varepsilon_{uv}{}^{wz}\,R_{wzxy} = 0\,. \tag{2.1}$$

The indices $u, v, \ldots$ run from 1 to 4, and we have introduced $\varepsilon_{uvxy} = \sqrt{\det g}\,\epsilon_{uvxy}$ with $\epsilon_{0123} = 1$. This space is Ricci-flat and endowed with 3 covariantly constant anti-self-dual 2-forms J_K. These form a triplet of SU(2) complex structures normalized to satisfy

$$(J_K)_u{}^x\,(J_L)_x{}^v = -\delta_{KL}\,\delta_u^v - \varepsilon_{KL}{}^M\,(J_M)_u{}^v\,,$$
$$\sum_{K=1}^{3}\,(J_K)_u{}^v\,(J_K)_x{}^y = g_{ux}\,g^{vy} - \delta_u^y\,\delta_x^v + \varepsilon_{ux}{}^{vy}\,. \tag{2.2}$$

In the following, we will assume the existence of isometries. As a consequence of Bianchi identity, a Killing vector ξ satisfies

$$\nabla_x\nabla_v\xi_u = R_{uvxy}\xi^y\,. \tag{2.3}$$

Its (anti-)self-dual covariant derivatives[1]

$$k_{uv}^{\pm} = \frac{1}{2}\left(\nabla_u\xi_v \pm \frac{1}{2}\varepsilon_{uv}{}^{wz}\,\nabla_w\xi_z\right)\,, \tag{2.4}$$

obey remarkable identities,

$$g^{uv}\,k_{ux}^{\pm}\,k_{vy}^{\pm} = \frac{1}{4}\,g_{xy}\,k_{\pm}^2\,, \quad k_{\pm}^2 := k_{uv}^{\pm}k^{\pm uv}\,,$$
$$g^{uv}\,k_{ux}^{\pm}\,k_{vy}^{\mp} = \frac{1}{2}\left(\nabla_x\xi_z\nabla_y\xi^z - \frac{1}{4}\,g_{xy}\nabla_u\xi_v\nabla^u\xi^v\right)\,, \tag{2.5}$$
$$g^{uv}\,g^{xy}\,k_{ux}^{\pm}\,k_{vy}^{\mp} = 0\,,$$

valid irrespective of the nature of the space.

The self-duality condition (2.1) can be recast using (2.3) as

$$\nabla_x k_{uv}^- = 0\,, \tag{2.6}$$

leading to

$$\partial_x\,k_-^2 = 0 \implies k_-^2 = c\,, \tag{2.7}$$

where c is a non-negative constant. Consequently, in hyper-Kähler spaces, a Killing vector is translational if $k_{uv}^- = 0$, and rotational otherwise [17–20].

In order to clarify the meaning of translational versus rotational isometry, we evaluate the Lie derivative on the complex structures with respect to the Killing vector ξ:

$$(\mathcal{L}_\xi J_K)_{uv} = \begin{cases} \nabla_v\left((J_K)_u{}^w\,\xi_w\right) - \nabla_u\left((J_K)_v{}^w\,\xi_w\right) = \partial_v\left((J_K)_u{}^w\,\xi_w\right) - \partial_u\left((J_K)_v{}^w\,\xi_w\right)\,, \\ \nabla_u\xi_w\,(J_K)^w{}_v - (J_K)_u{}^w\,\nabla_w\xi_v = [\nabla\xi, J_K]_{uv}\,, \end{cases} \tag{2.8}$$

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[1]The (anti-)self-dual components of a 2-form A_{uv} are defined as $A_{uv}^{\pm} = \frac{1}{2}\left(A_{uv} \pm \frac{1}{2}\varepsilon_{uv}{}^{wz}A_{wz}\right)$.

where the bracket stands for the ordinary commutator of matrices (not to be confused with the Lie bracket). The latter expression trivializes for a translational isometry [20]

$$\mathcal{L}_\xi J_K = [k^+, J_K^-] = 0 \,, \tag{2.9}$$

because $[A^-, B^+] = 0$ for any pair of 2-forms A and B. Therefore a translational Killing vector ξ is *triholomorphic*, leaving the three complex structures invariant. In addition, eqs. (2.8) and (2.9) ensure the existence of a triplet of Killing potentials (moment maps) $\mathcal{K}_I$ defined as

$$(J_I)_u{}^w \xi_w = -\frac{1}{2}\partial_u \mathcal{K}_I \,. \tag{2.10}$$

For a rotational isometry, we can always find a basis of complex structures such that

$$\mathcal{L}_\xi J_1 = J_2 \,, \quad \mathcal{L}_\xi J_2 = -J_1 \,, \quad \mathcal{L}_\xi J_3 = 0 \,. \tag{2.11}$$

Consequently a rotational Killing vector ξ is *simply holomorphic* since only one complex structure remains invariant. In addition, (2.8) and (2.11) ensure the existence of a Killing potential $\mathcal{K}$ defined as

$$(J_3)_u{}^w \xi_w = -\frac{1}{2}\partial_u \mathcal{K} \implies \xi_u = \frac{1}{2}(J_3)_u{}^v \partial_v \mathcal{K} \,, \tag{2.12}$$

and the hyper-Kähler metric g_{uv}^{HK} satisfies the relation

$$g_{uv}^{\text{HK}} = \frac{1}{2}\left(\delta_u^w \delta_v^z + (J_\perp)_u{}^w (J_\perp)_v{}^z\right)\nabla_w \nabla_z \mathcal{K} \,, \tag{2.13}$$

where $J_\perp$ is any complex structure orthogonal to J_3. Diagonalizing the latter selects Kähler coordinates with

$$g_{ab}^{\text{HK}} = 0 \,, \quad g_{a\bar{b}}^{\text{HK}} = \nabla_a \nabla_{\bar{b}} \mathcal{K} = \partial_a \partial_{\bar{b}} \mathcal{K} = \mathcal{K}_{a\bar{b}} \,. \tag{2.14}$$

Thus the Killing potential of J_3 is the Kähler potential for $J_\perp$ [28].

Adapting a coordinate τ along the orbits of the Killing field as $\xi = \partial_\tau$, the hyper-Kähler metric reads:

$$\begin{aligned} \mathrm{d}s_{\text{HK}}^2 &= \frac{1}{V}(\mathrm{d}\tau + \omega)^2 + V\,\mathrm{d}\ell^2 \,, \\ \mathrm{d}\ell^2 &= \gamma_{ij}\,\mathrm{d}X^i \mathrm{d}X^j \,, \quad X^i = (X, Y, Z) \,. \end{aligned} \tag{2.15}$$

When ξ is a translational Killing vector, we can use the Gibbons-Hawking frame [17]

$$\gamma_{ij} = \delta_{ij} \,, \quad \nabla V = \nabla \times \omega \,. \tag{2.16}$$

The complex structures read in this case:

$$J_1 = -i\left(a \wedge b - \bar{a} \wedge \bar{b}\right), \quad J_2 = a \wedge b + \bar{a} \wedge \bar{b}, \quad J_3 = -i\left(a \wedge \bar{a} + b \wedge \bar{b}\right), \tag{2.17}$$

with

$$a = \frac{1}{\sqrt{2}}\left(\sqrt{V}\mathrm{d}z + \frac{i}{\sqrt{V}}(\mathrm{d}\tau + \omega)\right), \quad b = \frac{1}{\sqrt{2}}\sqrt{V}\,(\mathrm{d}x + i\,\mathrm{d}y)\,. \tag{2.18}$$

Alternatively, for a rotational Killing vector, the Gibbons-Hawking frame (2.16) is traded for the Boyer-Finley one [18–20]:[2]

$$\mathrm{d}\ell^2 = \mathrm{d}Z^2 + \mathrm{e}^\Psi \left(\mathrm{d}X^2 + \mathrm{d}Y^2\right),$$
$$V = \frac{1}{2}\,\Psi_Z\,, \qquad \omega = \frac{1}{2}\,\Psi_Y\,\mathrm{d}X - \frac{1}{2}\,\Psi_X\,\mathrm{d}Y\,, \qquad (2.19)$$
$$\Psi_{XX} + \Psi_{YY} + \left(\mathrm{e}^\Psi\right)_{ZZ} = 0\,.$$

The complex structures are given by (2.17) with

$$a = \frac{1}{\sqrt{2}}\left(\sqrt{V}\mathrm{d}z + \frac{i}{\sqrt{V}}\left(\mathrm{d}\tau + \omega\right)\right), \quad b = \frac{\mathrm{e}^{i\,\tau}}{\sqrt{2}}\,\sqrt{V}\,\mathrm{e}^{\frac{\Psi}{2}}\left(\mathrm{d}x + i\,\mathrm{d}y\right). \qquad (2.20)$$

Notice the explicit τ-dependence, necessary for (2.11) to hold. Finally, the Killing potential $\mathcal{K}$ for $\xi = \partial_\tau$ is obtained using (2.12), (2.17) and (2.20):

$$\mathcal{K} = 2Z\,. \qquad (2.21)$$

We now turn to quaternion-Kähler spaces. These are Einstein and conformally self-dual:[3]

$$W_{uvxy} - \frac{1}{2}\,\varepsilon_{uv}{}^{wz}\,W_{wzxy} = 0\,. \qquad (2.22)$$

We will here normalize the scalar curvature to $R = -12$. Assuming again the existence of an isometry and using eq. (2.3), the Einstein-space condition $R_{uv} = -3g_{uv}$ and (2.22), we find:

$$\nabla_x k_{vu}^- = -\frac{1}{2}\left(g_{ux}g_{vy} - g_{uy}g_{vx} - \varepsilon_{uvxy}\right)\xi^y\,, \qquad (2.23)$$

instead of (2.6). Contrary to what happens for hyper-Kähler spaces, the distinction between translational and rotational Killings is no longer relevant here, and the Gibbons-Hawking or the Boyer-Finley forms (2.15), (2.16) and (2.19) are replaced by the Przanowski-Tod frame [29–32], where[4]

$$\mathrm{d}s_{\mathrm{QK}}^2 = \frac{1}{Z^2}\left(\frac{1}{V}(\mathrm{d}\tau + \omega)^2 + V\left(\mathrm{d}Z^2 + \mathrm{e}^\Psi\left(\mathrm{d}X^2 + \mathrm{d}Y^2\right)\right)\right),$$
$$\mathrm{d}\omega = V_X\,\mathrm{d}Y \wedge \mathrm{d}Z + V_Y\,\mathrm{d}Z \wedge \mathrm{d}X + \left(V\,\mathrm{e}^\Psi\right)_Z\,\mathrm{d}X \wedge \mathrm{d}Y\,, \qquad (2.24)$$
$$\Psi_{XX} + \Psi_{YY} + \left(\mathrm{e}^\Psi\right)_{ZZ} = 0\,, \qquad 2V = 2 - Z\,\Psi_Z\,.$$

A straightforward computation shows that the coordinate Z is related to the anti-self-dual covariant derivative of the Killing field $\xi = \partial_\tau$:

$$\frac{1}{Z^2} = k_-^2 = k_{uv}^- k^{-uv}\,. \qquad (2.25)$$

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[2]Indices on scalar functions denote ordinary partial derivatives.

[3]Four-dimensional quaternion-Kähler spaces are in general not Kähler, but there are exceptions. These include $SU(3)/SU(2) \times U(1)$ and $SU(1,2)/SU(2) \times U(1)$ (see appendix A). The Kähler structure introduces a canonical orientation and self-duality is not equivalent with anti-self-duality. In particular, a four-dimensional Kähler metric which is Weyl *anti*-self-dual has vanishing scalar curvature [23].

[4]The integrability condition for ω, given by the linearized Toda equation $\left(\partial_X^2 + \partial_Y^2\right)V + \partial_Z^2\left(V\mathrm{e}^\Psi\right) = 0$, is actually a consequence of the last equations in (2.24).

3   The kinetic term and the rigid limits

As discussed in ref. [16], using any solution of the Toda equation, one can build both a quaternion-Kähler space with symmetry, expressed *à la* Przanowski-Tod, and a hyper-Kähler space in the Boyer-Finley frame. This sets a simple one-to-one relationship among these spaces.

Although this relationship sounds formal, it supports a deeper geometric interpretation: the hyper-Kähler member of the pair appears actually as a zooming-in of the quaternionic member around the fixed point of the isometry that supports the fiber in the Przanowski-Tod frame. This isometry survives in the hyper-Kähler space as a simply holomorphic symmetry. From a physical viewpoint, as we will see soon, the limiting procedure at hand goes along with the gravity decoupling limit.

In order to elaborate on the geometric picture of the above correspondence, we recall that the kinetic term reads:[5]

$$\mathcal{K} = \frac{1}{2k^2}\, G^{\mu\nu} g^{\mathrm{QK}}_{uw}\, \partial_\mu q^u\, \partial_\nu q^w \,. \tag{3.1}$$

Here, $G_{\mu\nu}$ is the spacetime metric and x^μ the associated coordinates, whereas g^{QK}_{uv} is the quaternionic target-space, coordinated with q^u. Notice that $q^u, g^{\mathrm{QK}}_{uv}$ and $G_{\mu\nu}$ are dimensionless, whereas ∂_μ have dimension one and $\mathcal{K}$ dimension four.

The gravity decoupling limit of the quaternion-Kähler space, reached at $k \to 0$, should be taken in a zoom-in manner in order to avoid the trivialization of the geometry into flat space. In that aim we introduce the following redefinitions:

$$
\begin{aligned}
Z &= \alpha\widehat{Z} - \delta\,, & V &= \delta\widehat{V}\,, & \tau &= \alpha\,\delta\,\widehat{\tau}\,, & \omega &= \alpha\,\delta\,\widehat{\omega}\,, \\
X &= \alpha\widehat{X}\,, & Y &= \alpha\widehat{Y}\,, & \Psi &= \alpha\widehat{\Psi}\,,
\end{aligned}
\tag{3.2}
$$

in the Przanowski-Tod metric (2.24), leading to

$$
\begin{aligned}
\mathrm{d}s^2_{\mathrm{QK}} &= \frac{\alpha^2\,\delta}{(\alpha\,\widehat{Z} - \delta)^2}\left(\frac{1}{\widehat{V}}\,(\mathrm{d}\widehat{\tau} + \omega)^2 + \widehat{V}\left(\mathrm{d}\widehat{Z}^2 + \mathrm{e}^{\alpha\widehat{\Psi}}\left(\mathrm{d}\widehat{X}^2 + \mathrm{d}\widehat{Y}^2\right)\right)\right), \\
\mathrm{d}\omega &= V_{\widehat{X}}\,\mathrm{d}\widehat{Y}\wedge\mathrm{d}\widehat{Z} + V_{\widehat{Y}}\,\mathrm{d}\widehat{Z}\wedge\mathrm{d}\widehat{X} + \left(V\,\mathrm{e}^{\alpha\widehat{\Psi}}\right)_{\widehat{Z}}\,\mathrm{d}\widehat{X}\wedge\mathrm{d}\widehat{Y}\,, \\
\widehat{V} &= \frac{1}{2}\,\Psi_{\widehat{Z}} + \frac{1}{2\delta}\left(2 - \alpha\widehat{Z}\,\Psi_{\widehat{Z}}\right),
\end{aligned}
\tag{3.3}
$$

and

$$k^2_- = k^-_{uv}k^{-uv} = \frac{\alpha^2\delta^2}{(\alpha\widehat{Z} - \delta)^2}\,. \tag{3.4}$$

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[5]Out of the full supergravity action, we consider the following part

$$S = \int \sqrt{-g}\,\mathrm{d}^4x\,(\mathcal{K} - \mathcal{V} + \mathcal{L}_{\mathrm{EH}})\,, \quad \mathcal{L}_{\mathrm{EH}} = \frac{1}{k^2}\left(\frac{\mathcal{R}}{2} - \Lambda\right),$$

where $\mathcal{K}$, $\mathcal{V}$ and $\mathcal{L}_{\mathrm{EH}}$ correspond to the kinetic, potential and Einstein-Hilbert terms respectively.

From Przanowski-Tod to Boyer-Finley. The kinetic term (3.1) with the insertion of (3.3), remains finite in the double-scaling limit

$$\alpha = 1, \quad k \to 0, \quad \delta \to \infty, \quad k^2 \delta = \frac{1}{\tilde{\mu}^2}, \tag{3.5}$$

where $\tilde{\mu}$ is an arbitrary finite mass scale. In this limit, (3.1) reads:

$$\mathcal{K} = \frac{\tilde{\mu}^2}{2} G^{\mu\nu} g_{uw}^{\text{HK}} \partial_\mu q^u \partial_\nu q^w, \tag{3.6}$$

where g_{uv}^{HK} is the hyper-Kähler space in Boyer-Finley frame (2.15) and (2.19), corresponding to the solution of the Toda equation used in the quaternionic metric g_{uw}^{QK} of (3.1). Furthermore, using (3.4) we find that k_-^2 remains non-vanishing in the double-scaling limit (3.5). Thus, the original quaternionic isometry is mapped onto a simply holomorphic Killing vector ∂_τ. Other isometries of the quaternionic metric also survive in the hyper-Kähler limit, if they commute with ∂_τ. Additional isometries may also exist.

Several remarks are in order here regarding the implementation of the double-scaling limit (3.5). This limit consists of zooming-in around $Z \to -\infty$ in the quaternion-Kähler space. The latter is described, in the Przanowski-Tod representation, by a solution $\Psi(X, Y, Z)$ of the Toda equation, and $Z \to -\infty$ is the fixed locus of the isometry generated by ∂_τ, as the norm of this Killing vanishes in that limit. The double-scaling limit (3.5) is consistent provided $\widehat{\Psi}(\widehat{X}, \widehat{Y}, \widehat{Z}) = \frac{1}{\alpha}\Psi(\alpha\widehat{X}, \alpha\widehat{Y}, \alpha\widehat{Z} - \delta)$ introduced in (3.2) makes sense when $\alpha \to 1$ and $\delta \to \infty$. Being a solution of a partial differential equation, $\Psi(X, Y, Z)$ is actually a function of $X + X_0, Y + Y_0, Z + Z_0$, where X_0, Y_0, Z_0 are arbitrary constants. This freedom makes it possible to tune Z_0 so as to absorb δ before the limit is taken. In this way, the limit does not affect $\Psi(X, Y, Z)$ and the very same function can thus be used on the two sides of the limit. This is why the double-scaling limit under consideration is equivalent to the one-to-one correspondence among quaternionic and hyper-Kähler spaces with symmetry set in the beginning of the present section, and based on the use of a given solution of the Toda equation. From this perspective, the relationship at hand can either be interpreted as a decoupling of gravity when starting from a quaternion-Kähler space, or as a coupling to gravity, when starting from a hyper-Kähler σ-model endowed with a simply holomorphic Killing vector sustaining a rigid $\mathcal{N} = 2$ model.

It should finally be stressed that the double-scaling limit under investigation can be taken for any isometry of the quaternion-Kähler space. This provides as many decoupling limits of gravity as symmetries in the σ-model, not all inequivalent though. We will come back to that in the examples of section 5 and appendix A, when discussing in particular the fate of the symmetries along the decoupling.

From Przanowski-Tod to Gibbons-Hawking. Starting with a quaternion-Kähler space with symmetry in the Przanowski-Tod representation, we can reach in some cases another hyper-Kähler space. This zoom-in limit is not necessarily taken in the same area of the manifold as the previous limit.[6] Instead of the double-scaling limit (3.5), we perform

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[6]See the discussion at the end of section 5.1.

the following triple-scaling limit on the kinetic term (3.1) with redefinitions (3.2):

$$k \to 0, \quad \alpha \to 0, \quad \delta \to \infty, \quad \frac{k^2 \delta}{\alpha^2} = \frac{1}{\tilde{\mu}^2}. \qquad (3.7)$$

The kinetic term is still given by (3.6), where now g_{uv}^{HK} is a hyper-Kähler space in the Gibbons-Hawking frame (2.15), (2.16). Hence, the original quaternionic isometry generated by ∂_τ becomes triholomorphic in the hyper-Kähler limit, where indeed k_-^2 and k_{uv}^- vanish (see (3.4)). Again, additional isometries may exist, as in the previous double-scaling limit.

The rigid limit under consideration is again a zooming-in around $Z \to -\infty$, where the norm of ∂_τ vanishes, further restricted to $X = Y = 0$. The limiting hyper-Kähler space exists as long as $\widehat{V} = \frac{1}{2\alpha} \partial_{\widehat{Z}} \Psi(\alpha \widehat{X}, \alpha \widehat{Y}, \alpha \widehat{Z})$ makes sense when $\alpha \to 0$. This requirement sets restrictions to the Przanowski-Tod geometries that allow for the triple-scaling limit, contrary to the previous double-scaling limit, which always exists. Despite that, non-trivial examples can be successfully worked out. In conclusion, although the original Przanowski-Tod isometry is generically simply holomorphic in the limiting hyper-Kähler space, the option for a triholomorphic limit exists in certain instances.

Alternative limit to Gibbons-Hawking. Before closing this section, we would like to mention an alternative possibility for reaching a hyper-Kähler Gibbons-Hawking space from a Przanowski-Tod quaternionic geometry. This limit is peculiar, because when it exists, it always leads to the same space, namely the unique hyper-Kähler invariant under Heisenberg $\ltimes$ U(1) symmetry [16, 34].

Assume that a line (X_0, Y_0, Z_0) exists in a quaternion-Kähler space of the Przanowski-Tod type (2.24), such that

$$\begin{cases} V(X_0, Y_0, Z_0) = 0 \Leftrightarrow Z\partial_Z \Psi|_0 = 2, \\ \omega_X|_0 = \omega_Y|_0 = \omega_Z|_0 = 0, \\ \Psi(X_0, Y_0, Z_0) = \psi_0, & \partial_Z (Z\partial_Z \Psi)|_0 = -2\psi_Z, \\ \partial_X (Z\partial_Z \Psi)|_0 = -2\psi_X \, \mathrm{e}^{\frac{\psi_0}{2}}, & \partial_Y (Z\partial_Z \Psi)|_0 = -2\psi_Y \, \mathrm{e}^{\frac{\psi_0}{2}}, \end{cases} \qquad (3.8)$$

with ψ_0, ψ_X, ψ_Y and ψ_Z finite, potentially vanishing constants. In the neighborhood of this line, we define the coordinates $\widehat{X}, \widehat{Y}, \widehat{Z}$:

$$X = X_0 + (k\tilde{\mu})^{\frac{2}{3}} \, \mathrm{e}^{-\frac{\psi_0}{2}} \, \widehat{X}, \quad Y = Y_0 + (k\tilde{\mu})^{\frac{2}{3}} \, \mathrm{e}^{-\frac{\psi_0}{2}} \, \widehat{Y}, \quad Z = Z_0 + (k\tilde{\mu})^{\frac{2}{3}} \, \widehat{Z}, \qquad (3.9)$$

and we expand V and ω at linear order, since we are ultimately interested in the scaling limit $k \to 0$. We find that

$$\begin{aligned} V &\approx (k\tilde{\mu})^{\frac{2}{3}} \left(\psi_X \widehat{X} + \psi_Y \widehat{Y} + \psi_Z \widehat{Z} \right) = (k\tilde{\mu})^{\frac{2}{3}} \widehat{V}, \\ \omega &\approx (k\tilde{\mu})^{\frac{4}{3}} \left(\psi_X \widehat{Y} \, \mathrm{d}\widehat{Z} + \psi_Y \widehat{Z} \, \mathrm{d}\widehat{X} + \psi_Z \widehat{X} \, \mathrm{d}\widehat{Y} \right) = (k\tilde{\mu})^{\frac{4}{3}} \widehat{\omega}. \end{aligned} \qquad (3.10)$$

Introducing finally

$$\tau = (k\tilde{\mu})^{\frac{4}{3}} \widehat{\tau}, \qquad (3.11)$$

we can proceed with the decoupling limit $k \to 0$ in the kinetic term (3.1) and we find the rigid limit (3.6) with the hyper-Kähler metric

$$\mathrm{d}s_{\mathrm{HK}}^2 = \frac{1}{Z_0^2} \left(\frac{1}{\widehat{V}} (\mathrm{d}\widehat{\tau} + \widehat{\omega})^2 + \widehat{V} \left(\mathrm{d}\widehat{X}^2 + \mathrm{d}\widehat{Y}^2 + \mathrm{d}\widehat{Z}^2 \right) \right), \tag{3.12}$$

where both $\widehat{V}$ and $\widehat{\omega}$ are linear in the hated coordinates with

$$\widehat{\nabla} \, \widehat{V} = \widehat{\nabla} \times \widehat{\omega} = \{\psi_X, \psi_Y, \psi_Z\} \,. \tag{3.13}$$

Thanks to the relation (3.13), it is always possible to trade the coordinates $\{\widehat{\tau}, \widehat{X}, \widehat{Y}, \widehat{Z}\}$ for $\{t, x, y, z\}$:

$$
\begin{aligned}
t &= Z_0^{\frac{-2}{3}} \, r_0^{\frac{1}{3}} \, \left(\cos \vartheta_0 \, \widehat{Z} + \sin \vartheta_0 \left(\cos \varphi_0 \, \widehat{X} + \sin \varphi_0 \, \widehat{Y} \right) \right), \\
x &= Z_0^{\frac{-2}{3}} \, r_0^{\frac{1}{3}} \, \left(-\sin \vartheta_0 \, \widehat{Z} + \cos \vartheta_0 \left(\cos \varphi_0 \, \widehat{X} + \sin \varphi_0 \, \widehat{Y} \right) \right), \\
y &= Z_0^{\frac{-2}{3}} \, r_0^{\frac{1}{3}} \, \left(-\sin \varphi_0 \, \widehat{X} + \cos \varphi_0 \, \widehat{Y} \right), \\
z &= Z_0^{\frac{-4}{3}} \, r_0^{\frac{-1}{3}} \, \left(\widehat{\tau} + f \right),
\end{aligned}
$$

where

$$f = \frac{1}{4} \, r_0 \, \cos \vartheta_0 \, \sin 2\varphi_0 \, \left(\widehat{X}^2 - \widehat{Y}^2 \right) + r_0 \, \widehat{Y} \left(\sin^2 \varphi_0 \cos \vartheta_0 \widehat{X} + \cos \varphi_0 \sin \vartheta_0 \widehat{Z} \right)$$

and $(r_0, \vartheta_0, \varphi_0)$ are constants defined as

$$\psi_X = r_0 \sin \vartheta_0 \cos \varphi_0 \,, \quad \psi_Y = r_0 \sin \vartheta_0 \sin \varphi_0 \,, \quad \psi_Z = r_0 \cos \vartheta_0 \,.$$

The metric thus reads:

$$\mathrm{d}s_{\mathrm{HK}}^2 = \frac{1}{t}(\mathrm{d}z + x \, \mathrm{d}y)^2 + t \left(\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 \right), \tag{3.14}$$

which is the unique hyper-Kähler space invariant under Heisenberg $\ltimes$ U(1) symmetry [16, 34], generated by $(\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}, \mathfrak{M})$ obeying

$$[\mathfrak{X}, \mathfrak{Y}] = \mathfrak{Z}, \quad [\mathfrak{M}, \mathfrak{X}] = \mathfrak{Y}, \quad [\mathfrak{M}, \mathfrak{Y}] = -\mathfrak{X} \,, \tag{3.15}$$

and realized as

$$\mathfrak{X} = \partial_x - y\partial_z \,, \quad \mathfrak{Y} = \partial_y \,, \quad \mathfrak{Z} = \partial_z \,, \quad \mathfrak{M} = y \, \partial_x - x \, \partial_y + \frac{1}{2} \left(x^2 - y^2 \right) \partial_z \,. \tag{3.16}$$

The Killing fields $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ are triholomorphic (translational) whereas $\mathfrak{M}$ is simply holomorphic (rotational).

If conditions (3.8) are met, the gravity decoupling limit under consideration provides the specific hyper-Kähler space (3.14). This occurs for example in the two-parameter family of U(1) × U(1)-symmetric quaternion-Kähler spaces obtained by quaternionic quotient based upon gauging $\mathfrak{Y}$ and $\mathfrak{Z}$ inside the Sp(2, 4) of the $\mathcal{N} = 2$ hypermultiplet manifold [16].

This family of quaternion-Kähler spaces contains the sub-family of the Heisenberg $\ltimes \mathrm{U}(1)$ spaces resulting from a $\mathcal{Z}$ gauging [3].

Finally, it is useful to exhibit Kähler coordinates for the hyper-Kähler space (3.14). There are at least two inequivalent Kähler coordinate systems adapted to the isometry at hand. In the first one, the action of $\mathcal{M}$ is not holomorphic:

$$
\begin{aligned}
&\Phi = t + i\,y\,, && T = -t\,x + i\,z\,, && K = \frac{\left(T + \overline{T}\right)^2}{\Phi + \overline{\Phi}} + \frac{1}{12}\left(\Phi + \overline{\Phi}\right)^3\,, \\
&\mathcal{X} = -\Phi\,\partial_T - \overline{\Phi}\,\partial_{\overline{T}}\,, && \mathcal{Y} = i\left(\partial_\Phi - \partial_{\overline{\Phi}}\right), && \mathcal{Z} = i\left(\partial_T - \partial_{\overline{T}}\right), \\
&\mathcal{M} = \frac{1}{2i}\left(\Phi - \overline{\Phi}\right)\mathcal{X} + \frac{T + \overline{T}}{\Phi + \overline{\Phi}}\,\mathcal{Y} + \frac{1}{2}\left(\left(\frac{T + \overline{T}}{\Phi + \overline{\Phi}}\right)^2 - \frac{1}{4}\left(\Phi - \overline{\Phi}\right)^2\right)\mathcal{Z}\,,
\end{aligned}
\tag{3.17}
$$

while it is holomorphic in the second:

$$
\begin{aligned}
&\Psi = x + iy\,, && U = \frac{1}{4}\left(2t^2 - x^2 - y^2\right) + i\left(\frac{1}{2}x\,y + z\right), \\
&Q = U + \overline{U} + \frac{1}{2}\Psi\overline{\Psi} = t^2\,, && K = \frac{4}{3}\,Q^{\frac{3}{2}}\,, \\
&\mathcal{X} = -\frac{1}{2}\Psi\partial_U - \frac{1}{2}\overline{\Psi}\partial_{\overline{U}} + \partial_\Psi + \partial_{\overline{\Psi}}\,, && \mathcal{Y} = \frac{i}{2}\left(\Psi\partial_U - \overline{\Psi}\partial_{\overline{U}}\right) + i\left(\partial_\Psi - \partial_{\overline{\Psi}}\right), \\
&\mathcal{Z} = i\left(\partial_U - \partial_{\overline{U}}\right), && \mathcal{M} = -i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right).
\end{aligned}
\tag{3.18}
$$

4 The scalar potential

4.1 Potential and spectrum

The scalar potential of $\mathcal{N} = 2$ supergravity theories is obtained by gauging one or several symmetries realized as isometries on the quaternion-Kähler geometry. These isometries act on the components of hyper- and vector multiplets. The gauging procedure involves in general the graviphoton and possibly other gauge fields in vector multiplets. Here, we will gauge isometries of the hypermultiplet σ-model using only the graviphoton as gauge field. Despite the obvious limitations of such a choice (e.g. partial breaking into $\mathcal{N} = 1$ is impossible without vector multiplets: the massive $\mathcal{N} = 1$ gravitino multiplet includes two massive spin-one fields), its analysis is rich and instructive, as we will see in the following. When considering extra vector multiplets, the output of the gauging depends on whether the isometry acts or not on the vectors at hand: when it does not, one commonly obtains a run-away behaviour, alternatively the scalar potential is more intricate and no generic conclusion can be drawn *a priori*. We leave this investigation for the future.

In $\mathcal{N} = 2$ supergravity, the choice of the symmetry to be gauged is free. In particular, the concept of translational versus rotational isometries is not pertinent in the quaternion-Kähler target space. This is in contrast with the global-supersymmetry case, and one of our purposes is to analyze how this distinction emerges in the gravity-decoupling limit. As we will see, it is intimately linked to the background spacetime geometry (Minkowski or AdS) dictated by the scalar potential.

The gauging procedure works as follows. The hypermultiplet metric is quaternion-Kähler and admits three complex structures satisfying the quaternionic algebra (2.2). For each isometry generated by a Killing field ξ, one defines the corresponding Killing prepotentials [21]:

$$P_I = -\frac{1}{4k^2}\left(J_I\right)^u{}_v \nabla_u \xi^v. \tag{4.1}$$

Once an isometry is selected, its gauging with the graviphoton produces a superpotential W expressed in terms of the Killing prepotentials as:

$$W^2 = \sum_{I=1}^{3} P_I P_I. \tag{4.2}$$

The corresponding scalar potential $\mathcal{V}_\xi$ takes the form [22]

$$\mathcal{V}_\xi = k^2 |X^0|^2 \left(-6W^2 + 4g^{uv}\partial_u W \partial_v W\right). \tag{4.3}$$

In the latter expression, X^0 is the compensator for dilation symmetry, gauge-fixed to $\frac{1}{k}$.

Using (4.1), (4.2), the quaternion algebra (2.2) and some of the identities (2.5), we find (the gravitational coupling k^2 should not be confused with the square of the anti-self-dual covariant derivative of the Killing k_-^2)

$$W^2 = \frac{1}{4k^4}\, k_-^2\,, \tag{4.4}$$

and its derivative[7]

$$\partial_x W^2 = \frac{1}{k^4} k_{xy}^- \xi^y\,. \tag{4.5}$$

Hence, we retrieve

$$g^{uv}\partial_u W \partial_v W = \frac{1}{4W^2}\, g^{uv}\partial_u W^2 \partial_v W^2 = \frac{1}{k^4 k_-^2}\, g^{uv}\, k_{ux}^-\, k_{vy}^- \xi^x \xi^y = \frac{1}{4k^4}\, g_{uv}\, \xi^u \xi^v\,. \tag{4.6}$$

Expressions (4.4) and (4.6) enable us to produce the following equivalent expressions for the scalar potential (4.3):

$$\mathcal{V}_\xi = \frac{1}{k^4}\begin{cases} k^4\left(-6W^2 + 4\, g^{uv}\, \partial_u W \partial_v W\right), \\ -6\, k^4 \sum\limits_{I=1}^{3} P_I P_I + g_{uv}\, \xi^u \xi^v\,, \\ -\frac{3}{2}\, k_-^2 + g_{uv}\, \xi^u \xi^v\,. \end{cases} \tag{4.7}$$

Given the scalar potential (4.7), it is straightforward to investigate its supersymmetric vacuums. Those obey $\langle \partial_u W \rangle = 0 \Leftrightarrow \langle \xi^u \rangle = 0$ (or equivalently $\langle g_{uv}\xi^u\xi^v \rangle = 0$ — the brackets mean as usual that the quantity under consideration is evaluated at the vacuum). The cosmological constant is related to the — generically non-vanishing — value of the potential at the extremum,

$$\frac{1}{k^2}\, \Lambda = \langle \mathcal{V}_\xi \rangle = -\frac{3}{2k^4}\, \langle k_-^2 \rangle. \tag{4.8}$$

---

[7]We have used (2.23): $\partial_x k_-^2 = \nabla_x\left(k_{uv}^- k^{-uv}\right) = 4k_{xy}^- \xi^y$.

Expanding around the vacuum allows to determine the mass matrix:

$$M^u{}_v = \frac{k^2}{2} \langle g^{ux} \partial_x \partial_v \, \mathcal{V}_\xi \rangle = \frac{1}{k^2} \langle k^{+uw} k^+_{vw} - 2 k^{-uw} k^-_{vw} - k^{+uw} k^-_{vw} \rangle \,. \tag{4.9}$$

Equation (4.8) shows that the spacetime has negative curvature $\mathcal{R} = 4\Lambda$, whereas eq. (4.9) describes the spectrum of an $\mathcal{N} = 2$ chiral multiplet in AdS$_4$ [24, 25]. Indeed, using the identities (2.5) the mass matrix can be diagonalized with eigenvalues

$$
\begin{aligned}
M_A^2 &= m^2 - 2\mu^2 - m\mu \,, & M_B^2 &= m^2 - 2\mu^2 + m\mu \,, \\
m^2 &= \frac{1}{2k^2} \langle k_+^2 \rangle \,, & \mu^2 &= \frac{1}{2k^2} \langle k_-^2 \rangle = -\frac{\Lambda}{3} \,.
\end{aligned}
\tag{4.10}
$$

Recapitulating, $\langle k_+^2 \rangle$ and $\langle k_-^2 \rangle$ control respectively the physical mass m of the chiral multiplet and the cosmological constant Λ of the anti-de Sitter spacetime. The described vacuum is stable as it is supersymmetric. Furthermore it satisfies the Breitenlohner-Freedman stability bound [24, 25]

$$M_{A,B}^2 \geqslant \frac{3}{4} \Lambda \quad \Longrightarrow \quad \left(m \pm \frac{\mu}{2} \right)^2 \geqslant 0 \,. \tag{4.11}$$

In (4.10), $M_{A,B}^2$ is the coefficient of the Lagrangian mass term. A field with a shift symmetry or a flat direction of the potential correspond to $M_A^2 = 0$ or $M_B^2 = 0$. This does not mean that the field is massless. In AdS$_4$ space, a field propagating on the lightcone has a Lagrangian mass term with coefficient $\frac{2\Lambda}{3} = -2\mu^2$. We may then have a hypermultiplet with two flat directions and two massless scalars, if $m = \pm\mu$.

4.2 The decoupling limits

So far our analysis has been confined in the supergravity framework, i.e. with coupling to gravity. We would like now to investigate the decoupling limit, and in particular the behaviour of the above scalar potential (4.7) in the rigid limits presented in section 3. In these decoupling limits as they emerge in the analysis of the kinetic term, the Killing field supporting the fiber of the quaternion-Kähler space in its Przanowski-Tod representation (2.24), ∂_τ, plays a preferred role. In particular, the zooming-in is triggered around distinguished points, where the norm of this vector is singular. However, in general, this specific Killing field needs not be the one that enters the gauging procedure. Consequently, the behaviour of the potential in the decoupling limits associated with the kinetic term is not unique and also depends on the isometry chosen for gauging.

In the present general analysis, we do not make any assumption regarding extra isometries in the quaternion-Kähler space. Hence, we limit our discussion to the gauging of the graviphoton along the shift isometry $\xi = g \, \partial_\tau$ fibering the Przanowski-Tod metric (2.24), with g a dimensionless gauge coupling. In section 5, when dealing with a specific quaternionic space, we will use the option of gauging isometries other than the one carrying the fiber and being responsible for the decoupling in the kinetic term.

Using (2.25) with $\xi = g \, \partial_\tau$, we find for the scalar potential (4.7):

$$\mathcal{V}_\xi = \frac{g^2}{k^4} \left(-\frac{3}{2Z^2} + \frac{1}{Z^2 V} \right) \,. \tag{4.12}$$

Our goal is to investigate the behaviour of the latter expression when $k \to 0$. As already discussed extensively in section 3 for the kinetic term, this rigid limit must be taken in a zoom-in manner. Performing the redefinitions as stated in (3.2), we obtain:

$$\mathcal{V}_\xi = \frac{\widehat{g}^2 \alpha^2 \, \delta^2}{k^4} \left(-\frac{3}{2(\alpha \, \widehat{Z} - \delta)^2} + \frac{1}{(\alpha \, \widehat{Z} - \delta)^2 \delta \widehat{V}} \right), \quad \widehat{g} = \frac{g}{\alpha \, \delta}. \tag{4.13}$$

From Przanowski-Tod to Boyer-Finley. Performing the double-scaling limit (3.5) on (4.13), assuming that $g \to \infty$ while keeping the coupling constant $\widehat{g}$ finite, yields the potential[8]

$$\mathcal{V}_{\text{flat}} = \frac{\widehat{g}^2 \tilde{\mu}^2}{k^2} \left(\frac{1}{\widehat{V}} - 3\widehat{Z} \right), \tag{4.14}$$

in the presence of a non-vanishing cosmological constant

$$\Lambda = -\frac{3\widehat{g}^2}{2k^2}. \tag{4.15}$$

Before proceeding with the alternative rigid limit, we would like to pause and make contact with the general results of Butter and Kuzenko on $\mathcal{N} = 2$ supersymmetric σ-models in AdS$_4$ [26, 27]. According to these authors, rigid $\mathcal{N} = 2$ supersymmetry in AdS$_4$ spacetime requires the target space of the σ-model be a *non-compact* hyper-Kähler manifold endowed with a simply holomorphic isometry. Let ξ be the simply holomorphic Killing field, and assume a basis for the complex structures obeying (2.11), so that the preserved one is J_3. The latter provides a *globally defined* Killing potential $\mathcal{K}$ as in (2.12). Following Butter and Kuzenko, the scalar potential reads:

$$\mathcal{V}_{\text{BK}} = \mu^2 \tilde{\mu}^2 \left(\frac{1}{2} g_{\text{HK}}^{uv} \, \partial_u \mathcal{K} \, \partial_v \mathcal{K} - 3\mathcal{K} \right), \quad \Lambda = -3\mu^2. \tag{4.16}$$

Consider a generic hyper-Kähler metric as in eqs. (2.15) and (2.19), with a rotational isometry $\xi = \partial_\tau$ and Killing potential (2.21). Inserting the latter into (4.16), we recover (4.14) with $\mu = \frac{\widehat{g}}{\sqrt{2}\,k}$.

From Przanowski-Tod to Gibbons-Hawking. We now turn to the triple-scaling limit (3.7), for which the Killing field becomes translational in the hyper-Kähler space. Performed on the scalar potential (4.13), while keeping $\widehat{g}$ finite, this limit yields

$$\mathcal{V}_{\text{flat}} = \frac{\widehat{g}^2 \tilde{\mu}^2}{k^2 \, \widehat{V}}. \tag{4.17}$$

The Killing ∂_τ is translational in the limit at hand, k_-^2 vanishes and so does Λ. The result (4.17) is in agreement with the potential of a hypermultiplet of global $\mathcal{N} = 2$ supersymmetry in Minkowski spacetime [33]. Along the same lines of thought, the alternative decoupling limit (3.9) on (4.12) yields

$$\mathcal{V}_{\text{flat}} = \frac{\widehat{g}^2 \tilde{\mu}^2}{k^2 \, Z_0^2 \, \widehat{V}}, \tag{4.18}$$

where $\widehat{g} = \frac{g}{(k\tilde{\mu})^{4/3}}$ is a finite coupling constant when $k \to 0$.

---

[8]The subscript "flat" refers to the Ricci-flat nature of the hypermultiplet target space — not to the spacetime.

5 Spaces with Heisenberg isometry

5.1 The kinetic term and its three distinct rigid limits

As already mentioned in the introduction, Heisenberg symmetry has a distinguished role. Firstly, only two hyper-Kähler spaces exist with this isometry group [34], and one quaternion-Kähler [16]. Secondly, the latter captures the type-II string one-loop perturbative corrections to the hypermultiplet scalar manifold [4], and can be derived through the quaternionic quotient method by gauging the $\mathcal{Z}$ isometry inside the $\mathrm{Sp}(2,4)$ of the $\mathcal{N} = 2$ hypermultiplet manifold [3]. The quaternion-Kähler space under consideration has actually extended Heisenberg $\ltimes \mathrm{U}(1)$ isometry and its metric reads:

$$\mathrm{d}s_{\mathrm{QK}}^2 = \frac{8\rho^2}{V_1 V_2^2} \left(\mathrm{d}\tau + \eta \mathrm{d}\varphi\right)^2 + \frac{2V_1}{V_2^2} \left(\mathrm{d}\rho^2 + \mathrm{d}\eta^2 + \mathrm{d}\varphi^2\right),$$
$$V_1 = \rho^2 + 2\sigma \geqslant 0, \quad V_2 = \rho^2 - 2\sigma, \quad \sigma = \text{constant}. \tag{5.1}$$

The Heisenberg $\ltimes \mathrm{U}(1)$ algebra is realized as

$$[\mathcal{X}, \mathcal{Y}] = \mathcal{Z}, \qquad [\mathcal{M}, \mathcal{X}] = \mathcal{Y}, \quad [\mathcal{M}, \mathcal{Y}] = -\mathcal{X},$$
$$\mathcal{X} = \partial_\eta - \varphi \, \partial_\tau, \qquad \mathcal{Y} = \partial_\varphi, \qquad \mathcal{Z} = \partial_\tau, \quad \mathcal{M} = \varphi \, \partial_\eta - \eta \, \partial_\varphi + \frac{1}{2} \left(\eta^2 - \varphi^2\right) \partial_\tau. \tag{5.2}$$

For vanishing σ, this isometry algebra is actually extended to $\mathrm{U}(1,2)$, and the quaternionic space becomes the non-compact $\frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2) \times \mathrm{U}(1)}$ (see appendix A). In the frame at hand, the space (5.1) is fibered over $\mathcal{Z}$.

It was noticed in [3] that the rigid limit of the kinetic term (3.1) with (5.1) is σ-dependent. For $\sigma \geqslant 0$ two trivial, flat-space rigid limits occur around $\rho^2 \sim 0$ and $\rho^2 - 2\sigma \sim 0$. For $\sigma < 0$ a non-trivial limit appears at $\rho^2 + 2\sigma \sim 0$. The latter case corresponds precisely to the hyper-Kähler metric (3.14), invariant under the Heisenberg $\ltimes \mathrm{U}(1)$ symmetry (3.15). These results can be summarized as follows:

- PARAMETRIC REGION $\sigma \geqslant 0$

  1. Around $\rho^2 \sim 0$ we zoom as:

  $$\rho = \sqrt{\sigma} \, k\tilde{\mu} \, t, \quad \eta = \sqrt{\sigma} \, k\tilde{\mu} \, x, \quad \varphi = \sqrt{\sigma} \, k\tilde{\mu} \, y, \quad \tau = \sigma z, \quad k \to 0, \tag{5.3}$$

  and this leads to flat space

  $$\mathrm{d}s_{HK}^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}t^2 + t^2 \mathrm{d}z^2, \tag{5.4}$$

  where the Killing supporting the fiber in the original quaternionic space, $\mathcal{Z} \propto \partial_z$ is here a simply holomorphic (rotational) Killing vector.

  2. Around $\rho^2 - 2\sigma \sim 0$ we zoom as:

  $$\rho^2 = 2\sigma + k\tilde{\mu} \left(1 + k\tilde{\mu} \, t\right), \qquad \eta = \frac{1}{\sqrt{8\sigma}} \left(k\tilde{\mu}\right)^2 x,$$
  $$\varphi = \frac{1}{\sqrt{8\sigma}} \left(k\tilde{\mu}\right)^2 y, \qquad \tau = \frac{1}{2} \left(k\tilde{\mu}\right)^2 z, \quad k \to 0, \tag{5.5}$$

and obtain again flat space

$$\mathrm{d}s^2_{\mathrm{HK}} = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}t^2 + \mathrm{d}z^2\,, \tag{5.6}$$

where $\mathcal{Z} \propto \partial_z$ is now a triholomorphic (translational) Killing vector.

- PARAMETRIC REGION $\sigma < 0$

Around $\rho^2 + 2\sigma \sim 0$ we zoom as:

$$\rho^2 = -2\sigma \left(1 + 2\,t(k\tilde{\mu})^{\frac{2}{3}}\right)\,, \qquad \eta = \sqrt{-2\sigma}x(k\tilde{\mu})^{\frac{2}{3}}\,,$$
$$\varphi = \sqrt{-2\sigma}y(k\tilde{\mu})^{\frac{2}{3}}\,, \qquad \tau = -2\sigma\, z(k\tilde{\mu})^{\frac{4}{3}}\,, \quad k \to 0\,, \tag{5.7}$$

such that the Kretschmann scalar of (5.1),

$$R_{xuvw}R^{xuvw} = 48\frac{(\rho^4 + 4\sigma^2)(\rho^8 + 56\rho^4\sigma^2 + 16\sigma^4)}{(\rho^2 + 2\sigma)^6}\,, \tag{5.8}$$

remains finite when $k \to 0$. This limiting procedure results to the hyper-Kähler space (3.14),

$$\mathrm{d}s^2_{\mathrm{HK}} = \frac{1}{t}(\mathrm{d}z + x\,\mathrm{d}y)^2 + t\left(\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2\right)\,, \tag{5.9}$$

which is invariant under the Heisenberg $\ltimes$ U(1) symmetry (3.15), (3.16). Therefore the decoupling limit (5.7) preserves all the isometries of the quaternionic ancestor metric (5.1). Along the process, the Killing supporting the fiber, $\mathcal{Z} \propto \partial_z$ becomes a triholomorphic (translational) Killing vector.

In order to make contact with the general developments presented in section 3, we should recast the quaternionic Heisenberg $\ltimes$ U(1)-invariant metric (5.1) in the Przanowski-Tod form (2.24). This is achieved by keeping τ unaltered, while trading ρ, η, φ for X, Y, Z as follows:

$$X = \eta\,, \quad Y = \varphi\,, \quad Z = \frac{V_2(\rho)}{2}\,, \tag{5.10}$$

and setting

$$\omega = \eta\,\mathrm{d}\varphi\,, \quad V = \frac{V_1(\rho)}{2\rho^2}\,, \quad \mathrm{e}^{\Psi} = \rho^2\,. \tag{5.11}$$

The fiber of the Przanowski-Tod is supported by the Killing field $\mathcal{Z} = \partial_\tau$, which turns rotational or translational in the hyper-Kähler, depending on the decoupling limit.

With these conventions, on the one hand, the rigid limits for $\sigma \geqslant 0$, $\rho^2 \sim 0$ and $\rho^2 - 2\sigma \sim 0$, correspond to (3.5) (Przanowski-Tod to Boyer-Finley i.e. hyper-Kähler limit with rotational Killing $\mathcal{Z}$) and (3.7) (Przanowski-Tod to Gibbons-Hawking i.e. hyper-Kähler limit with translational Killing $\mathcal{Z}$). On the other hand, the rigid limit $\rho^2 + 2\sigma \sim 0$ for $\sigma < 0$ is the alternative Przanowski-Tod to Gibbons-Hawking limit, (3.9) and (3.11), leading to the unique hyper-Kähler space with Heisenberg $\ltimes$ U(1) symmetry, with a translational Killing vector $\mathcal{Z}$ supporting the fiber.

Notice finally that in all cases, extra Killing fields survive the decoupling limit, either simply holomorphic or triholomorphic, which may be chosen to further recast the hyper-Kähler metric in another Boyer-Finley or Gibbons-Hawking frame. Such options can be exploited depending on the form of the original scalar potential, and its structure in the decoupling limit.

5.2 The scalar potential

The form of the scalar potential depends on which symmetry is gauged, i.e. which element is chosen inside the Heisenberg $\ltimes$ U(1) isometry group of the quaternionic metric (5.1). We will always use the graviphoton for the gauging, as already advertised, and the general formalism of section 4.1.

In our analysis, we will systematically investigate the rigid limits. There are always three distinct cases corresponding to the limits (5.3), (5.5) and (5.7), exhibited for the kinetic term (3.1) on the Heisenberg $\ltimes$ U(1)-invariant quaternionic space (5.1), and leading to the hyper-Kähler spaces (5.4), (5.6) and (5.9). They are associated with the Przanowski-Tod to Boyer-Finley limit (hyper-Kähler limit with rotational Killing), the Przanowski-Tod to Gibbons-Hawking limit (hyper-Kähler limit with translational Killing), and the Przanowski-Tod to Gibbons-Hawking limit with full Heisenberg $\ltimes$ U(1) symmetry.

Coming back to the potential term, two distinct situations arise depending on which isometry is gauged: (i) either the field carrying the Przanowski-Tod fiber $\mathcal{Z}$ — the potential is Heisenberg $\ltimes$ U(1) invariant, (ii) or any other Killing — the potential is not Heisenberg $\ltimes$ U(1) invariant. In the first instance, the properties of the potential and its associated spectrum at the supergravity level and in the decoupling limits will follow the general classification presented in section 4. For these limits, in particular, we will find (section 4.2) rigid $\mathcal{N} = 2$ supersymmetry with potential (4.14) and anti-de Sitter vacuum, or with potentials (4.17), (4.18) and Minkowski vacuum. The case (ii) is expected to be slightly different, and is worth presenting case by case: $\mathcal{M}$, $\mathcal{Z}$ plus $\mathcal{M}$, $\mathcal{Y}$. In the three rigid-supersymmetry limits, we find only anti-de Sitter vacuums for the first, and anti-de Sitter and Minkowski for the second and the third. The $\mathcal{Y}$ gauging exhibits a further peculiarity already at the supergravity level, namely a de Sitter $\mathcal{N} = 0$ vacuum, which stands outside of the general analysis of section 4.1, where supersymmetry was unbroken.

Gauging the $\mathcal{Z}$ isometry. Here, the Killing vector supporting the gauging is $\xi = g\mathcal{Z}$ (see (5.2)). The scalar potential is obtained thanks to the general formula (4.7):

$$\mathcal{V}_\xi = \frac{2g^2}{k^4} \frac{\rho^2 - 6\sigma}{V_1 V_2^2}, \tag{5.12}$$

which is invariant under the action of Heisenberg $\ltimes$ U(1). The potential at hand has an extremum at the origin $\rho = 0$, the fixed point of ξ, and two flat directions η, φ. In order to analyze this extremum, we move from polar to Cartesian coordinates

$$(\rho, \tau) \mapsto (q_1, q_2): \quad \rho = \sqrt{\sigma\left(q_1^2 + q_2^2\right)}, \quad \tau = \sigma \arctan\frac{q_2}{q_1}, \tag{5.13}$$

as $\rho = 0$ is a coordinate singularity of (5.1).

Expanding (5.12) at second order around the extremum:

$$\mathcal{V}_\xi \approx \frac{\widehat{g}^2}{k^4}\left(-\frac{3}{2} - \frac{q_1^2 + q_2^2}{2}\right), \tag{5.14}$$

where $\widehat{g} = g/\sigma$. From the latter, and normalizing with respect to the kinetic term, eqs. (3.1) and (5.1), we read off the mass terms and the cosmological constant:

$$M^2_{q_1, q_2} = -\frac{\widehat{g}^2}{k^2} = \frac{2}{3}\Lambda\,, \quad M^2_{\eta, \varphi} = 0\,, \quad \Lambda = -\frac{3\widehat{g}^2}{2k^2}\,. \tag{5.15}$$

These satisfy the Breitenlohner-Freedman stability bound, and fit the general form (4.10) of the mass spectrum of an $\mathcal{N} = 2$ chiral multiplet in AdS$_4$ spacetime, with $A = \{q_1, q_2\}$, $B = \{\eta, \varphi\}$ and $\mu = m = \frac{\widehat{g}}{\sqrt{2}\,k}$. Besides the flat directions η, φ, the fields $q_{1,2}$ are massless.

Next task in our agenda is to analyze the behaviour of the potential (5.12) in the three distinguished rigid limits (5.3), (5.5) and (5.7).

- PARAMETRIC REGION $\sigma \geqslant 0$

   1. Applying the rigid limit (5.3) to (5.12), we find the potential

      $$\mathcal{V}_{\text{flat}} = -\frac{\widehat{g}^2 \tilde{\mu}^2}{2k^2}\,t^2\,. \tag{5.16}$$

      To this end, we use the generic expression for the potential in the Przanowski-Tod to Boyer-Finley rigid limit, eq. (4.14), after rewriting the flat space (5.4) in the Boyer-Finley frame along the rotational isometry $\mathcal{Z} = \partial_\tau = \frac{1}{\sigma}\partial_z$, for which

      $$\frac{1}{\widehat{V}} = 2\widehat{Z} = t^2\,. \tag{5.17}$$

      In the present rigid limit, the vacuum is thus an AdS$_4$ spacetime, and the hyper-Kähler space describes a global $\mathcal{N} = 2$ hypermultiplet.

   2. The rigid limit under consideration is now (5.5). Applied to (5.12), it leads to a global $\mathcal{N} = 2$ hypermultiplet in Minkowski spacetime with an irrelevant constant potential

      $$\mathcal{V}_{\text{flat}} = -\frac{\tilde{g}^2 \tilde{\mu}^2}{2k^2}\,, \tag{5.18}$$

      where $\tilde{g} = \frac{2\sigma \widehat{g}}{(k\tilde{\mu})^2}$ is a finite coupling constant when $k \to 0$.

- PARAMETRIC REGION $\sigma < 0$

   The scalar potential (5.12) becomes

   $$\mathcal{V}_{\text{flat}} = \frac{\tilde{g}^2 \tilde{\mu}^2}{k^2\,t}\,, \tag{5.19}$$

   in the rigid limit (5.7), with $\tilde{g} = -\frac{\widehat{g}}{2(k\tilde{\mu})^{4/3}}$ a finite coupling constant at $k \to 0$. Again, the cosmological constant vanishes and we describe a global $\mathcal{N} = 2$ hypermultiplet in Minkowski spacetime.

Gauging the $\mathcal{M}$ isometry. Consider now the isometry of (5.1) generated by $\xi = g\,\mathcal{M}$. Using (4.7) we find the following scalar potential:

$$\mathcal{V}_\xi = \frac{g^2}{k^4} \left(-\frac{3}{2} - \frac{V_1}{V_2^2} \left(\eta^2 + \varphi^2 \right) + \frac{V_2 - 4\sigma}{2V_1 V_2^2} \left(\eta^2 + \varphi^2 \right)^2 \right), \tag{5.20}$$

invariant under the action of $\mathcal{M}$ and $\mathcal{Z}$. This potential has an extremum at the fixed point of ξ, $\eta = \varphi = 0$, and two flat directions ρ, τ. We can directly read off the mass terms and the cosmological constant (again normalization with respect to the kinetic term is required):

$$M_{\rho,\tau}^2 = 0\,, \quad M_{\eta,\varphi}^2 = -\frac{g^2}{k^2} = \frac{2}{3}\Lambda\,, \quad \Lambda = -\frac{3g^2}{2k^2}\,. \tag{5.21}$$

Comparing with the general expression (4.10), we identify the fields $A = \{\eta, \varphi\}$ and $B = \{\rho, \tau\}$, whereas $\mu = m = \frac{g}{\sqrt{2}\,k}$. We find again the spectrum of an $\mathcal{N} = 2$ chiral multiplet in AdS$_4$, with two massless fields ρ, τ and two massive fields η, φ.

Let us now consider the usual rigid limits when the dynamics is captured by the potential (5.20).

- PARAMETRIC REGION $\sigma \geqslant 0$

  The pattern goes as in the previous gauging. We first consider the rigid limits (5.3) or (5.5). We use again eq. (4.14) after rewriting the flat space (5.4) or (5.6) in the Boyer-Finley frame along the rotational isometry $\mathcal{M}$, with

  $$\frac{1}{\widehat{V}} = x^2 + y^2\,, \quad \widehat{Z} = \frac{1}{2}\left(x^2 + y^2 \right)\,. \tag{5.22}$$

  We thus find for both limits AdS$_4$ spacetime with $\Lambda = -\frac{3g^2}{2k^2}$. The hyper-Kähler space describes a a global $\mathcal{N} = 2$ hypermultiplet with potential

  $$\mathcal{V}_{\text{flat}} = -\frac{g^2 \tilde{\mu}^2}{2k^2} \left(x^2 + y^2 \right)\,. \tag{5.23}$$

- PARAMETRIC REGION $\sigma < 0$

  Applying the rigid limit (5.7) to (5.20), we utilize eq. (4.14) after rewriting (3.14) in the Boyer-Finley frame along the rotational isometry $\mathcal{M}$ [16]:

  $$\frac{1}{\widehat{V}} = \frac{1}{4t}(x^2 + y^2)^2 + t(x^2 + y^2)\,, \quad \widehat{Z} = \frac{1}{2}t(x^2 + y^2)\,. \tag{5.24}$$

  We now find a global $\mathcal{N} = 2$ hypermultiplet in AdS$_4$ spacetime with with potential

  $$\mathcal{V}_{\text{flat}} = -\frac{g^2 \tilde{\mu}^2}{2k^2} \left(t(x^2 + y^2) - \frac{1}{2t} \left(x^2 + y^2 \right)^2 \right)\,, \tag{5.25}$$

  and cosmological constant $\Lambda = -\frac{3g^2}{2k^2}$.

  A comment is worth here on the Killing potential $\mathcal{K}$. This is determined using (2.21) and (5.24):

  $$\mathcal{K} = 2\widehat{Z} = t(x^2 + y^2)\,. \tag{5.26}$$

In the Kähler coordinates (3.17), the Killing potential $\mathcal{K}$ reads:

$$\mathcal{K} = \frac{1}{2} K + W + \overline{W}, \quad W = -\frac{1}{6} \Phi^3. \tag{5.27}$$

Hence $\mathcal{K}$ appears as a Kähler potential and W is a Kähler transformation. Notice also that in the set of Kähler coordinates (3.17), the complex structure J_1 is diagonalized while $\mathcal{M}$ is not holomorphic.

Gauging $\mathcal{Z}$ and $\mathcal{M}$ isometries. This gauging is performed along $\xi = g_1 \mathcal{Z} + g_2 \mathcal{M}$. The scalar potential is determined with (4.7) and (5.1), and is invariant under the action of $\mathcal{M}$ and $\mathcal{Z}$:

$$\mathcal{V}_\xi = \frac{2g_1^2}{k^4} \frac{\rho^2 - 6\sigma}{V_1 V_2^2} + \frac{g_2^2}{k^4} \left(-\frac{3}{2} - \frac{V_1}{V_2^2} \left(\eta^2 + \varphi^2 \right) + \frac{V_2 - 4\sigma}{2V_1 V_2^2} \left(\eta^2 + \varphi^2 \right)^2 \right)$$
$$- 2g_1 g_2 \frac{\left(\eta^2 + \varphi^2 \right) \left(V_2 - 4\sigma \right) + 3V_1 V_2}{k^4 V_1 V_2^2}. \tag{5.28}$$

This potential has an extremum at $\rho = \eta = \varphi = 0$, the fixed point of ξ. To analyze this extremum, we change coordinates from polar to Cartesian as in (5.13), and we expand (5.28) at second-order around the extremum

$$\mathcal{V}_\xi \approx \frac{1}{k^4} \left(-\frac{3(\widehat{g}_1 - g_2)^2}{2} + \frac{1}{2} \widehat{g}_1 (3g_2 - \widehat{g}_1)(q_1^2 + q_2^2) + \frac{g_2}{2\sigma} (3\widehat{g}_1 - g_2)(\eta^2 + \varphi^2) \right), \tag{5.29}$$

where $\widehat{g}_1 = \frac{g_1}{\sigma}$. The mass terms and the cosmological constant are obtained as usual:

$$M_{q_1,q_2}^2 = \frac{g_1(3g_2 - \widehat{g}_1)}{k^2}, \quad M_{\eta,\varphi}^2 = \frac{g_2(3\widehat{g}_1 - g_2)}{k^2}, \quad \Lambda = -\frac{3(\widehat{g}_1 - g_2)^2}{2k^2}, \tag{5.30}$$

and satisfy the Breitenlohner-Freedman stability bound. For $\widehat{g}_1 \neq g_2$, this vacuum generically describes the spectrum of a hypermultiplet in AdS$_4$ and fits the general form (4.10) of the mass spectrum with $A = \{\eta, \varphi\}$, $B = \{q_1, q_2\}$ and

$$\mu = \frac{g_2 - \widehat{g}_1}{\sqrt{2}\, k}, \quad m = \frac{\widehat{g}_1 + g_2}{\sqrt{2}\, k}. \tag{5.31}$$

We now come to consider the rigid limits of the kinetic term (3.1), (5.1) on the potential (5.28).

- PARAMETRIC REGION $\sigma \geqslant 0$

1. Applying the rigid limit (5.3) to (5.28), we obtain:

$$\mathcal{V}_\xi = -\frac{3(\widehat{g}_1 - g_2)^2}{2k^4} - \frac{\tilde{\mu}^2}{2k^2} \left(\widehat{g}_1^2 t^2 + g_2^2(x^2 + y^2) - 3\widehat{g}_1 g_2(t^2 + x^2 + y^2) \right). \tag{5.32}$$

Two distinct cases should be examined, when $\widehat{g}_1 \neq g_2$ or $\widehat{g}_1 = g_2$:

(a) For $\widehat{g}_1 \neq g_2$, we find a global $\mathcal{N} = 2$ hypermultiplet in AdS$_4$ spacetime with $\Lambda = -\frac{3(\widehat{g}_1 - g_2)^2}{2k^2}$ and potential

$$\mathcal{V}_{\text{flat}} = -\frac{\tilde{\mu}^2}{2k^2} \left(\widehat{g}_1^2 t^2 + g_2^2 (x^2 + y^2) - 3\widehat{g}_1 g_2 (t^2 + x^2 + y^2) \right). \tag{5.33}$$

To prove this, we utilize (4.14) after rewriting the hyper-Kähler metric (5.4) in the Boyer-Finley frame along the rotational isometry $\xi = \widehat{g}_1 \mathcal{Z} + g_2 \mathcal{M}$. We find

$$\frac{1}{\widehat{V}} = \widehat{g}_1^2 t^2 + g_2^2 \left(x^2 + y^2 \right), \quad \widehat{Z} = \frac{1}{2} \left(\widehat{g}_1 - g_2 \right) \left(\widehat{g}_1 t^2 - g_2 \left(x^2 + y^2 \right) \right).$$

(b) For $\widehat{g}_1 = g_2$, we obtain the potential:[9]

$$\mathcal{V}_{\text{flat}} = \frac{\tilde{\mu}^2 \widehat{g}_1^2}{k^2} \left(t^2 + x^2 + y^2 \right), \tag{5.34}$$

which corresponds to a massive $\mathcal{N} = 2$ hypermultiplet in Minkowski spacetime. Indeed this can be easily seen by changing coordinates in (5.4) from polar (t, z) to Cartesian ones (q_1, q_2), as $t = 0$ is a coordinate singularity of the hyper-Kähler metric (5.4).

2. We now consider the rigid limit (5.5) to (5.28). We rewrite the hyper-Kähler metric (5.6) in the Boyer-Finley frame along the rotational isometry $\xi = \widetilde{g}_1 \mathcal{Z} + g_2 \mathcal{M}$ and use (4.14):

$$\frac{1}{\widehat{V}} = \widetilde{g}_1^2 + g_2^2 \left(x^2 + y^2 \right), \quad \widehat{Z} = \frac{1}{2} \left(\widetilde{g}_1^2 + g_2^2 (x^2 + y^2) - 2\widetilde{g}_1 g_2 t \right). \tag{5.35}$$

We obtain a global $\mathcal{N} = 2$ hypermultiplet in AdS$_4$ spacetime with potential

$$\mathcal{V}_{\text{flat}} = -\frac{\tilde{\mu}^2}{2k^2} \left(\widetilde{g}_1^2 - 6\widetilde{g}_1 g_2 t + g_2^2 (x^2 + y^2) \right), \tag{5.36}$$

and $\Lambda = -\frac{3g_2^2}{2k^2}$, where $\widetilde{g}_1 = \frac{2\sigma \widehat{g}_1}{(k\tilde{\mu})^2}$ is a finite coupling constant when $k \to 0$.

- PARAMETRIC REGION $\sigma < 0$

The last rigid limit is (5.7) applied to (5.28). Now, with (4.14), we express the hyper-Kähler metric (5.9) in the Boyer-Finley frame along the rotational isometry $\xi = \widetilde{g}_1 \mathcal{Z} + g_2 \mathcal{M}$:

$$\frac{1}{\widehat{V}} = \frac{4\widetilde{g}_1^2 - 4\widetilde{g}_1 g_2 (x^2 + y^2) + g_2^2 (x^2 + y^2)(4t^2 + x^2 + y^2)}{4t},$$
$$\widehat{Z} = \frac{1}{2} g_2 t \left(-2\widetilde{g}_1 + g_2 (x^2 + y^2) \right). \tag{5.37}$$

Again, we find a global $\mathcal{N} = 2$ hypermultiplet in AdS$_4$ spacetime with $\Lambda = -\frac{3g_2^2}{2k^2}$ and potential

$$\mathcal{V}_{\text{flat}} = \frac{\tilde{\mu}^2}{4tk^2} \left(4\widetilde{g}_1^2 + 4\widetilde{g}_1 g_2 (3t^2 - x^2 - y^2) - g_2^2 (2t^2 - x^2 - y^2)(x^2 + y^2) \right), \tag{5.38}$$

where $\widetilde{g}_1 = -\frac{\widehat{g}_1}{2(k\tilde{\mu})^{4/3}}$ is a finite coupling constant when $k \to 0$.

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[9]For $\widehat{g}_1 = g_2$, the isometry is translational, in agreement with a theorem in [20] on commuting rotational isometries, and the rigid limit yields a global $\mathcal{N} = 2$ hypermultiplet in Minkowski spacetime.

Gauging the $\mathcal{Y}$ isometry. This last gauging is based on the isometry $\xi = g\,\mathcal{Y}$ and leads to the scalar potential

$$\mathcal{V}_\xi = \frac{2g^2}{k^4}\,\frac{\eta^2(V_2 - 4\sigma) - 2V_1(V_2 + \sigma)}{V_1 V_2^2}\,, \tag{5.39}$$

invariant under the action of $\mathcal{Y}$ and $\mathcal{Z}$.

The potential (5.39) has an extremum at $\rho = \eta = 0$, and a flat direction φ. As usual, we change coordinates from polar to Cartesian, see eqs. (5.13), and we expand (5.39) at second order

$$\mathcal{V}_\xi \approx \frac{g^2}{\sigma k^4} - \frac{3g^2\eta^2}{2\sigma^2 k^4}\,, \tag{5.40}$$

where $\sigma > 0$, required by positivity of the metric (5.1) around the extremum at hand. Using the latter and normalizing with (5.1), we read off the mass terms and the cosmological constant:

$$M_\eta^2 = -\frac{3g^2}{\sigma k^2}\,, \quad M_\varphi^2 = M_{q_1}^2 = M_{q_2}^2 = 0\,, \quad \Lambda = \frac{g^2}{\sigma k^2}\,. \tag{5.41}$$

As advertised earlier on, this corresponds to an $\mathcal{N} = 0$ hypermultiplet in dS$_4$ spacetime with $\mathcal{R} = 4\Lambda$.

Our next task is the analysis of the usual rigid limits on the potential (5.39).

- **PARAMETRIC REGION $\sigma \geqslant 0$**

  1. The rigid limit (5.3) applied to (5.39) leads to a global $\mathcal{N} = 2$ hypermultiplet in Minkowski spacetime with constant potential $\mathcal{V}_{\text{flat}} = \frac{\widehat{g}^2 \tilde{\mu}^2}{k^2}$, where $\widehat{g} = \frac{g}{\sqrt{\sigma}k\tilde{\mu}}$ is a finite coupling constant when $k \to 0$.

  2. In the alternative rigid limit (5.5), we find again a global $\mathcal{N} = 2$ hypermultiplet in Minkowski spacetime (with another constant potential $\mathcal{V}_{\text{flat}} = -\frac{\widehat{g}^2 \tilde{\mu}^2}{2k^2}$, and $\widehat{g} = \frac{\sqrt{8\sigma}\,g}{(k\tilde{\mu})^2}$ a coupling constant, finite in the decoupling).

- **PARAMETRIC REGION $\sigma < 0$**

  Similarly the rigid limit (5.7) of the potential (5.39) gives a global $\mathcal{N} = 2$ hypermultiplet in Minkowski spacetime with potential

  $$\mathcal{V}_{\text{flat}} = \frac{\widehat{g}^2 \tilde{\mu}^2 (t^2 + x^2)}{k^2 t}\,, \tag{5.42}$$

  where $\widehat{g} = \frac{g}{\sqrt{-2\sigma}(k\tilde{\mu})^{4/3}}$ remains finite when $k \to 0$.

  It is worth mentioning, that this potential could be obtained from the $\mathcal{N} = 1$ expression

  $$\mathcal{V} = \frac{\tilde{\mu}^2}{k^2}\,K^{a\bar{b}} W_a \overline{W}_{\bar{b}}\,, \tag{5.43}$$

  expressed in terms of a linear holomorphic superpotential: $W = \widehat{g}\,T$, in the Kähler basis (3.17). This linear superpotential breaks supersymmetry as the kinetic term is non-canonical.

6 Conclusion

We can now highlight and summarize our results. The core of the present work is the investigation of various off-shell gravity decoupling limits of the $\mathcal{N} = 2$ scalar hypermultiplet. This concerns at the first place the kinetic term, based on a σ-model which has a quaternion-Kähler target space. Analyzing the decoupling limit, establishes various relationships between quaternionic and hyper-Kähler spaces with symmetry.

Equally important is the behaviour of the scalar potential, produced by gauging symmetries with specific vectors. For the general analysis, we have chosen the graviphoton, along a generic isometry of a quaternion-Kähler space of the Przanowski-Tod type. The rigid limits reveal two separate cases: a rigid $\mathcal{N} = 2$ theory on Minkowski or on AdS$_4$ spacetime, depending on whether the isometry is translational or rotational in the hyper-Kähler limit. These results are in agreement with previous results in the literature for global $\mathcal{N} = 2$ in Minkowski and AdS$_4$ spaces [2] and [26, 27].

In order to illustrate our general results, we analyzed extensively the quaternionic metric with Heisenberg $\ltimes$ U(1) isometry, eq. (5.1). The global $\mathcal{N} = 2$ limits of this space are found to be trivial (flat space) or the hyper-Kähler space (5.9), which is invariant under the Heisenberg $\ltimes$ U(1) symmetry (3.15), (3.16). We further derived the scalar potential by gauging the graviphoton along all possible isometries of the quaternion-Kähler space, $(\mathcal{Y}, \mathcal{Z}, \mathcal{M})$ and studied the vacuum structure of the scalar potential together with its on/off-shell rigid limits.

Interesting open questions remain at this stage, which have not been addressed in our work. An important one is to gauge isometries of the hypermultiplet σ-model using the graviphoton *and* a vector multiplet. The latter combination can give access to vacuum solutions which describe the spectrum of $\mathcal{N} = 1$ hypermultiplets in Minkowski or AdS$_4$ spacetimes.

A Pseudo-Fubini-Study metric

We shall utilize the results of sections 3 and 4 for the pseudo-Fubini-Study metric on $\widetilde{CP}_2 = \frac{\text{SU}(1,2)}{\text{SU}(2) \times \text{U}(1)}$

$$\mathrm{d}s^2_{\text{QK}} = 2g_{a\bar{b}}\,\mathrm{d}z^a \mathrm{d}\overline{z}^{\bar{b}}, \quad g_{a\bar{b}} = K_{a\bar{b}}, \quad K = -\ln\left(1 - |z|^2 - |w|^2\right), \quad z^a = (z, w), \quad \text{(A.1)}$$

This is a Kähler-Einstein-WSD space, with $R = -12$, which describes the universal hypermultiplet at string tree-level.

The Kähler potential is invariant under the action of

$$\xi = \alpha\,i\,(z\,\partial_z - \overline{z}\,\partial_{\overline{z}}) + \beta\,i\,(w\,\partial_w - \overline{w}\,\partial_{\overline{w}})\,, \quad \alpha, \beta \in \mathbb{R}\,, \tag{A.2}$$

and through eq. (4.7) we find the scalar potential

$$\mathcal{V}_\xi = \frac{(1 - r_2^2)(4r_1^2 + 3r_2^2 - 3)\alpha^2 + (1 - r_1^2)(4r_2^2 + 3r_1^2 - 3)\beta^2 + 2\alpha\beta(3 - 3r_1^2 - 3r_2^2 + r_1^2 r_2^2)}{2k^4(1 - r_1^2 - r_2^2)^2}\,, \tag{A.3}$$

with $r_1^2 = |z|^2$ and $r_2^2 = |w|^2$.

This potential has an extremum at $(z, w) = (0, 0)$, fixed point of ξ. Expanding it around this point we find at second order

$$\mathcal{V}_\xi \approx \frac{1}{k^4}\left(-\frac{3}{2}(\alpha - \beta)^2 + \alpha\,(3\beta - \alpha)\,r_1^2 + \beta\,(3\alpha - \beta)\,r_2^2\right)\,, \tag{A.4}$$

and we read off the masses and the cosmological constant which satisfy the Breitenlohner-Freedman stability bound

$$M_{r_1}^2 = \frac{\alpha\,(3\beta - \alpha)}{k^2}\,, \quad M_{r_2}^2 = \frac{\beta\,(3\alpha - \beta)}{k^2}\,, \quad \Lambda = -\frac{3(\alpha - \beta)^2}{2k^2}\,. \tag{A.5}$$

This vacuum generically describes the spectrum of a chiral multiplet in an AdS$_4$ spacetime, similarly to (4.10)

$$M_{r_1}^2 = m^2 - 2\mu^2 - m\mu\,, \quad M_{r_2}^2 = m^2 - 2\mu^2 + m\mu\,, \tag{A.6}$$

with $\Lambda = -3\mu^2$, where:

$$\mu = \frac{\alpha - \beta}{\sqrt{2}\,k}\,, \quad m = \frac{\alpha + \beta}{\sqrt{2}\,k}\,. \tag{A.7}$$

We can now analyze the rigid limit around $z, w \sim 0$. The kinetic term (3.1) corresponding to (A.1), has a unique trivial gravity decoupling limit around $z, w \sim 0$, where we zoom as:

$$z = k\tilde{\mu}\,\widehat{z}\,, \quad w = k\tilde{\mu}\,\widehat{w}\,, \quad k \to 0\,, \tag{A.8}$$

leading to flat space

$$ds_{\mathrm{HK}}^2 = 2\left(\mathrm{d}\widehat{z}\,\mathrm{d}\overline{\widehat{z}} + \mathrm{d}\widehat{w}\,\mathrm{d}\overline{\widehat{w}}\right)\,, \tag{A.9}$$

while the potential (A.3) truncates to:

$$\mathcal{V}_\xi = -\frac{3(\alpha - \beta)^2}{2k^4} + \frac{\tilde{\mu}^2}{k^2}\left(\alpha\,(3\beta - \alpha)\,|\widehat{z}|^2 + \beta\,(3\alpha - \beta)\,|\widehat{w}|^2\right)\,. \tag{A.10}$$

Two cases need to be considered, for $\alpha \neq \beta$ or $\alpha = \beta$:

1. For $\alpha \neq \beta$, we find a global $\mathcal{N} = 2$ hypermultiplet in AdS$_4$ spacetime with potential

$$\mathcal{V}_{\text{flat}} = \frac{\tilde{\mu}^2}{k^2} \left(\alpha \left(3\beta - \alpha \right) |\widehat{z}|^2 + \beta \left(3\alpha - \beta \right) |\widehat{w}|^2 \right). \tag{A.11}$$

and a non-vanishing cosmological constant $\Lambda = -\frac{3(\alpha - \beta)^2}{2k^2}$. To prove this, we utilize (4.14) after rewriting the hyper-Kähler metric (A.9) in the Boyer-Finley frame along the rotational isometry (A.2)

$$\frac{1}{\widehat{V}} = 2 \left(\alpha^2 |\widehat{z}|^2 + \beta^2 |\widehat{w}|^2 \right), \quad \widehat{Z} = (\alpha - \beta) \left(\alpha |\widehat{z}|^2 - \beta |\widehat{w}|^2 \right). \tag{A.12}$$

2. For $\alpha = \beta$, we have a global $\mathcal{N} = 2$ hypermultiplet in Minkowski spacetime with potential:

$$\mathcal{V}_{\text{flat}} = \frac{2\tilde{\mu}^2}{k^2} \alpha^2 \left(|\widehat{z}|^2 + |\widehat{w}|^2 \right), \tag{A.13}$$

which corresponds to a massive $\mathcal{N} = 2$ hypermultiplet in Minkowski spacetime.

B   Four-dimensional Kähler spaces with an isometry

We are interested in providing an alternative exhibition of generic, Ricci-flat, scalar-flat or Einstein four-dimensional Kähler spaces with a holomorphic isometry. Hyper-Kähler (Ricci-flat), scalar-flat or Einstein solutions appear as Gibbons-Hawking like metrics. For generic and Ricci-flat four-dimensional Kähler space with a holomorphic isometry, see also ref. [35].

B.1   Four-dimensional Kähler spaces

We begin with Kähler complex coordinates T, Φ, with an isometry acting as a shift of ImT and so the Kähler potential takes the form $K = K(T + \overline{T}, \Phi, \overline{\Phi})$. A simple rearrangement yields

$$\mathrm{d}s^2_{\text{Kähler}} = K_{T\overline{T}} \, \mathrm{d}T \mathrm{d}\overline{T} + K_{T\overline{\Phi}} \, \mathrm{d}T \mathrm{d}\overline{\Phi} + K_{\Phi\overline{T}} \, \mathrm{d}\Phi \mathrm{d}\overline{T} + K_{\Phi\overline{\Phi}} \, \mathrm{d}\Phi \mathrm{d}\overline{\Phi} \tag{B.1}$$

$$= K_{T\overline{T}} \left(\mathrm{dIm}\, T + \frac{i}{2K_{T\overline{T}}} \left(K_{T\overline{\Phi}} \mathrm{d}\overline{\Phi} - K_{\overline{T}\Phi} \mathrm{d}\Phi \right) \right)^2 + \frac{1}{K_{T\overline{T}}} \left(\frac{1}{4} \left(\mathrm{d}K_T \right)^2 + \mathcal{D} \, \mathrm{d}\Phi \mathrm{d}\overline{\Phi} \right),$$

where $\mathcal{D}$ is the determinant of the Kähler metric

$$\mathcal{D} := \det K_{a\overline{b}} = K_{T\overline{T}} K_{\Phi\overline{\Phi}} - K_{T\overline{\Phi}} K_{\overline{T}\Phi}. \tag{B.2}$$

Next we define a new coordinate Z as

$$K_T := \frac{1}{2} \left(Z + c \right), \tag{B.3}$$

where c is an integration constant that can be absorbed by a Kähler transformation

$$K \mapsto K + \frac{c}{2} \left(T + \overline{T} \right).$$

We may think of (B.3) defining a Legendre transformation as

$$H(Z + c, \Phi, \overline{\Phi}) = \frac{1}{2}\,(Z + c)\,(T + \overline{T}) - K(T + \overline{T}, \Phi, \overline{\Phi})\,, \tag{B.4}$$

with transformation relations

$$H_Z = \frac{1}{2}\,(T + \overline{T})\,, \quad H_\Phi = -K_\Phi\,, \quad H_{\overline{\Phi}} = -K_{\overline{\Phi}}\,, \tag{B.5}$$

which lead to[10]

$$K_{T\overline{T}} = \frac{1}{4H_{ZZ}}\,, \quad K_{T\overline{\Phi}} = -\frac{H_{Z\overline{\Phi}}}{2H_{ZZ}}\,, \quad K_{\overline{T}\Phi} = -\frac{H_{Z\Phi}}{2H_{ZZ}}\,, \quad K_{\Phi\overline{\Phi}} = \frac{H_{Z\Phi}H_{Z\overline{\Phi}}}{H_{ZZ}} - H_{\Phi\overline{\Phi}}\,, \tag{B.6}$$

and the determinant (B.2) equals

$$\mathcal{D} = -\frac{H_{\Phi\overline{\Phi}}}{4H_{ZZ}}\,. \tag{B.7}$$

Putting altogether we find that the line element (B.1) rewrites as

$$\begin{aligned}
\mathrm{d}s_{\text{Kähler}}^2 &= \frac{1}{4}\left(\frac{1}{V}\,(\mathrm{d}\tau + \omega)^2 + V\left(\mathrm{d}Z^2 + \mathrm{e}^\Psi\left(\mathrm{d}X^2 + \mathrm{d}Y^2\right)\right)\right), \\
V &= H_{ZZ}\,, \quad \omega = H_{ZY}\mathrm{d}X - H_{ZX}\mathrm{d}Y\,, \quad \mathrm{e}^\Psi = -\frac{H_{XX} + H_{YY}}{H_{ZZ}}\,, \\
\tau &= \mathrm{Im}T\,, \quad \Phi = X + i\,Y\,,
\end{aligned} \tag{B.8}$$

with[11]

$$\mathrm{d}\omega = V_X\,\mathrm{d}Y \wedge \mathrm{d}Z + V_Y\,\mathrm{d}Z \wedge \mathrm{d}X + \left(V\,\mathrm{e}^\Psi\right)_Z\,\mathrm{d}X \wedge \mathrm{d}Y\,. \tag{B.9}$$

These results apply to an arbitrary four-dimensional Kähler metric with an isometry.

B.2 Hyper-Kähler spaces

To describe a hyper-Kähler metric we impose Ricci-flatness on (B.8)

$$R_{a\bar{b}} = -\,(\ln\mathcal{D})_{a\bar{b}} = 0\,, \tag{B.10}$$

with general solution

$$\mathcal{D} = \mathrm{e}^{F(T,\Phi) + \overline{F}(\overline{T},\overline{\Phi})}\,. \tag{B.11}$$

Demanding invariance under the shift isometry of $\mathrm{Im}T$, we find:

$$\mathcal{D} = \mathrm{e}^{-\alpha\left(T + \overline{T}\right)}|f(\Phi)|^2\,, \quad \alpha = \text{constant}\,, \tag{B.12}$$

where $f(\Phi)$ can be eliminated by a holomorphic redefinition of Φ, here we use $f(\Phi) = \frac{1}{4}$.

Employing (B.5), (B.7) and (B.12), we find:

$$H_{XX} + H_{YY} + \mathrm{e}^{-2\alpha\,H_Z}\,H_{ZZ} = 0\,. \tag{B.13}$$

---

[10]Using the Jacobian matrix for the Legendre transformation $\left(T + \overline{T}, \Phi, \overline{\Phi}\right) \mapsto \left(Z, \Phi, \overline{\Phi}\right)$, implied by (B.5).

[11]Whose compatibility relation reads: $V_{XX} + V_{YY} + \left(V\mathrm{e}^\Psi\right)_{ZZ} = 0$.

Translational isometry. For $\alpha = 0$, eq. (B.13) simplifies to the Laplace equation in Cartesian coordinates (X, Y, Z)

$$H_{XX} + H_{YY} + H_{ZZ} = 0, \tag{B.14}$$

and the line element is given by

$$ds^2_{\text{HK}} = \frac{1}{4} \left(\frac{1}{V} (d\tau + \omega)^2 + V (dX^2 + dY^2 + dZ^2) \right),$$
$$\nabla V = \nabla \times \omega \implies \nabla^2 V = 0. \tag{B.15}$$

parameterizing a hyper-Kähler space with a translational isometry, the Gibbons-Hawking metric [17].

Rotational isometry. For $\alpha \neq 0$, we evaluate Ψ through (B.8) and (B.13)

$$\Psi = -2\alpha \, H_Z. \tag{B.16}$$

Next we differentiate (B.13) with respect to Z and we get the Toda equation

$$\Psi_{XX} + \Psi_{YY} + \left(e^\Psi \right)_{ZZ} = 0, \tag{B.17}$$

and the line element is given by

$$ds^2_{\text{HK}} = \frac{1}{4} \left(\frac{1}{V} (d\tau + \omega)^2 + V \left(dZ^2 + e^\Psi \left(dX^2 + dY^2 \right) \right) \right),$$
$$V = -\frac{1}{2\alpha} \Psi_Z, \quad \omega = -\frac{1}{2\alpha} \Psi_Y dX + \frac{1}{2\alpha} \Psi_X dY,$$
$$\Psi_{XX} + \Psi_{YY} + \left(e^\Psi \right)_{ZZ} = 0. \tag{B.18}$$

parameterizing a hyper-Kähler space with a rotational isometry, the Boyer-Finley metric [18, 19].

B.3 Scalar-flat spaces

To describe a scalar-flat space we impose vanishing scalar curvature on (B.8)[12]

$$R = 2g^{a\bar{b}} R_{a\bar{b}} = -2g^{a\bar{b}} (\ln \mathcal{D})_{a\bar{b}} = 0, \tag{B.19}$$

whose general solution reads

$$\mathcal{D} = e^{-\alpha(T + \overline{T}) + \Sigma_Z} |f(\Phi)|^2, \quad \alpha = \text{constant}, \tag{B.20}$$

where Σ_Z is a solution of $\nabla^2_{\text{4d}} \Sigma_Z = 0$ and we use $f(\Phi) = \frac{1}{4}$.

Utilizing (B.5), (B.7) and (B.20), we find:

$$H_{XX} + H_{YY} + e^{-2\alpha \, H_Z + \Sigma_Z} H_{ZZ} = 0. \tag{B.21}$$

---

[12]So it is Weyl anti-self-dual, according to footnote 3.

Using the above we obtain Ψ through (B.8)

$$\Psi = -2\alpha\, H_Z + \Sigma_Z \,. \tag{B.22}$$

Next we differentiate (B.21) with respect to Z and we get the Toda equation

$$\Psi_{XX} + \Psi_{YY} + \left(\mathrm{e}^\Psi\right)_{ZZ} = \frac{1}{4}\, V\, \mathrm{e}^\Psi\, \nabla^2_{\mathrm{4d}} \Sigma_Z = 0\,, \tag{B.23}$$

and the line element is given by

$$\begin{aligned}
\mathrm{d}s^2_{\mathrm{LeBrun}} &= \frac{1}{4}\left(\frac{1}{V}\left(\mathrm{d}\tau + \omega\right)^2 + V\left(\mathrm{d}Z^2 + \mathrm{e}^\Psi\left(\mathrm{d}X^2 + \mathrm{d}Y^2\right)\right)\right), \\
\mathrm{d}\omega &= V_X\,\mathrm{d}Y \wedge \mathrm{d}Z + V_Y\,\mathrm{d}Z \wedge \mathrm{d}X + \left(V\,\mathrm{e}^\Psi\right)_Z \mathrm{d}X \wedge \mathrm{d}Y\,, \\
\Psi_{XX} &+ \Psi_{YY} + \left(\mathrm{e}^\Psi\right)_{ZZ} = 0\,.
\end{aligned} \tag{B.24}$$

parameterizing a scalar-flat Kähler space with an isometry, the LeBrun metric [36].

B.4 Einstein spaces

To describe an Einstein metric we impose on (B.8)

$$R_{a\bar{b}} = -\left(\ln \mathcal{D}\right)_{a\bar{b}} = \Lambda\, K_{a\bar{b}}\,, \tag{B.25}$$

whose general solution reads

$$\mathcal{D} = \mathrm{e}^{-\alpha\left(T+\overline{T}\right) - \Lambda\, K}|f(\Phi)|^2\,, \quad \alpha = \text{constant}\,, \tag{B.26}$$

where we use $f(\Phi) = \frac{1}{4}$.

Employing (B.5), (B.7) and (B.26), we find:

$$H_{XX} + H_{YY} + \mathrm{e}^{-2\alpha\, H_Z - \Lambda\, K}\, H_{ZZ} = 0\,, \tag{B.27}$$

Using the above we derive Ψ through (B.8)

$$\Psi = -2\alpha\, H_Z - \Lambda\, K\,. \tag{B.28}$$

Using the latter and eqs. (B.3), (B.5), (B.8), we find

$$V = H_{ZZ} = -\frac{\Psi_Z}{\Lambda\left(Z+c\right) + 2\alpha}\,. \tag{B.29}$$

Then we differentiate (B.27) with respect to Z

$$\Psi_{XX} + \Psi_{YY} + \left(\mathrm{e}^\Psi\right)_{ZZ} = -\frac{1}{4}\,\Lambda\, V\, \mathrm{e}^\Psi\, \nabla^2_{\mathrm{4d}} K = -2\Lambda\, V\, \mathrm{e}^\Psi\,, \tag{B.30}$$

where we have used:

$$\nabla^2_{\mathrm{4d}} K = 2g^{a\bar{b}}\nabla_a\nabla_{\bar{b}} K = 2g^{a\bar{b}} K_{a\bar{b}} = 2\, g^{a\bar{b}} g_{a\bar{b}} = 8\,, \quad \Gamma_{a\bar{b}}{}^c = \Gamma_{a\bar{b}}{}^{\bar{c}} = 0\,. \tag{B.31}$$

Plugging (B.29) into (B.30) we find the Pedersen-Poon equation

$$\Psi_{XX} + \Psi_{YY} + \left(\mathrm{e}^{\Psi}\right)_{ZZ} = \frac{2\Lambda\,\Psi_Z\,\mathrm{e}^{\Psi}}{\Lambda\,(Z+c) + 2\alpha}\,, \tag{B.32}$$

and the line element is given by

$$\begin{aligned}
\mathrm{d}s^2_{\mathrm{PP}} &= \frac{1}{4}\left(\frac{1}{V}\,(\mathrm{d}\tau + \omega)^2 + V\left(\mathrm{d}Z^2 + \mathrm{e}^{\Psi}\left(\mathrm{d}X^2 + \mathrm{d}Y^2\right)\right)\right), \\
\mathrm{d}\omega &= V_X\,\mathrm{d}Y \wedge \mathrm{d}Z + V_Y\,\mathrm{d}Z \wedge \mathrm{d}X + \left(V\,\mathrm{e}^{\Psi}\right)_Z\,\mathrm{d}X \wedge \mathrm{d}Y\,, \\
V &= -\frac{\Psi_Z}{\Lambda\,(Z+c) + 2\alpha}\,, \quad \Psi_{XX} + \Psi_{YY} + \left(\mathrm{e}^{\Psi}\right)_{ZZ} = \frac{2\Lambda\,\Psi_Z\,\mathrm{e}^{\Psi}}{\Lambda\,(Z+c) + 2\alpha}\,.
\end{aligned} \tag{B.33}$$

parameterizing a Kähler-Einstein space with an isometry, the Pedersen-Poon metric [37].

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