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## A note on oscillating strings in $\text{AdS}_3 \times S^3$ with mixed three-form fluxes

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**ABSTRACT:** We present a detailed study of the pulsating string solutions in  $\text{AdS}_3 \times S^3$  supported by both RR and NS-NS fluxes. This background has recently been proved to be integrable. We find the dispersion relation between the energy, oscillation number and other conserved charges when the NS-NS flux turned on is small. We further discuss the fate of the string solutions in pure RR and NS-NS cases.

**KEYWORDS:** AdS-CFT Correspondence, Bosonic Strings

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**1 Introduction**

The AdS/CFT duality between the  $\mathcal{N} = 4$  Supersymmetric Yang-Mills (SYM) theory in four dimensions and type IIB superstring in the compactified  $\text{AdS}_5$  space [1] has been the playground for research for more than fifteen years. Although finding exact matching between string states and dual operators from both sides is a highly non-trivial problem, studying classical string solutions in various  $\text{AdS} \times S$  backgrounds have played an important role along that direction. The anomalous dimensions of particular field theory operators having large charges can be obtained in the string side simply by looking at the dispersion relation among charges of classical strings. The fact that the dilatation operator of the  $\mathcal{N} = 4$  SYM theory in one loop can be written as the hamiltonian of the integrable Heisenberg  $XXX_{1/2}$  spin chain [2] has been proved to be a key concept in connecting integrability, spin chains and string theory in the context of classical solutions. This observation helped cast the strong coupling problem of the SYM theory into solving the algebraic bethe ansatz equations for the quantum spin chain [3]. Also, in the string theory side, the integrability of the  $\text{AdS}_5 \times S^5$  string sigma model based on the supercoset  $\frac{\text{PSU}(2,2|4)}{\text{SO}(4,1) \times \text{SO}(5)}$  has been established as the equations of motion for the superstring from this sigma model can be recast in the zero curvature form [4]. This ensures the existence of an infinite number of conserved quantities associated to string motion in this background. To speak precisely, the integrability associated with both sides of the duality has improved the understanding of the equivalence between the Bethe equation for the spin chain and the corresponding realization of worldsheet symmetries of the classical  $\text{AdS}_5 \times S^5$  string sigma model [5, 6]. The relevant Bethe equations are based on the knowledge of the S-matrix which focusses on the scattering of world-sheet excitations of the gauge-fixed string sigma model, or equivalently, the excitations of a certain spin chain in the dual gauge theory [7–11].

Over the years, a lot of studies have been done on the semiclassical rotating string solutions arising from the  $\text{AdS}_5 \times S^5$  string sigma model. This includes well known solutions like giant magnons [12], spiky strings [13] and most importantly folded spinning string

solutions [14]. However unlike their rotating counterparts, the pulsating string solutions [15] are less explored even though the pulsating-rotating solutions offer better stability than the pure rotating ones [16]. These solutions are time-dependent as opposed to the usual rigidly rotating string solutions. They are expected to be dual to highly excited states in terms of operators. For example the most general pulsating string in  $S^5$  charged under the isometry group  $SO(6)$  will have a dual operator of the form  $\text{Tr}(X^{J_1}Y^{J_2}Z^{J_3})$ . Here the  $X, Y$  and  $Z$  are the chiral scalars and  $J_i$ 's are the R-charges from the SYM theory. Pulsating string solutions have been thoroughly generalized in [17–22], and have been studied in a number of backgrounds having varying degrees of supersymmetry [23–26]. Simultaneously rotating and oscillating strings were presented in [27], and the generalisation of this was done with extra angular momenta in [28]. The one loop corrections to pulsating string spectrum in AdS have been discussed in [29]. Recently these type of strings have been used to probe integrable deformations of AdS string models like the Lunin-Maldacena background [30] and the one-parameter deformed AdS backgrounds [31, 32]. Also such pulsating solutions in non-local geometries have been looked at in [33].

The other famous example of a holographic dual pair is that of string propagation in  $\text{AdS}^3 \times S^3 \times T^4$  background and the  $\mathcal{N}=(4,4)$  superconformal field theory coming from the  $D1 - D5$  brane system. This duality has also been well explored from both sides and various semiclassical solutions in this background [34–43] has been studied in the context of integrability.<sup>1</sup> String theory in this background supported by NS-NS type flux can be described in terms of a  $SL(2, \mathbb{R})$  WZW model. It has been suggested recently that in  $\text{AdS}^3 \times S^3$  background supported by both NS-NS and RR type fluxes ( $H_3 = dB_2$  and  $F_3 = dC_2$ ), the string theory is integrable [45, 46]. The integrable structure of this theory, including the S-matrix, has been discussed at length in [47–53], and various semiclassical string solutions have been studied in [50, 54–58]. The NS-NS flux in this background depends on a parameter  $q$  with  $0 \leq q \leq 1$ , while the RR flux will be dependent on a parameter  $\hat{q} = \sqrt{1 - q^2}$ . This model then interpolates between a pure RR background, for which one can find the spectrum using the integrability based approaches, and the pure NS-NS case where the WZW approach suffices. But it has been clear that for a intermediate value of  $q$ , none of the approaches will be suitable enough. Classical string solutions in this mixed flux background might be helpful to bridge the two models together.

In the present note, we address the question of finding pulsating string solutions in this ‘mixed-flux’ background. We solve the F-string equations of motion with NS-NS flux and find the dispersion relations among various conserved quantities perturbatively upto the  $\mathcal{O}(q^2)$ , provided the flux turned on is small. To find the quantized spectrum of the string, we will use a Bohr-Sommerfeld quantization. Since the motion of the string is (quasi)periodic, we can use the oscillation number  $N = \oint p dq$  and the Noether charges to characterize its dynamics. The oscillation number is an adiabatic invariant quantity and we find that in terms of elliptic functions, which leads to the energy of the string. We will see that when the NS-NS flux is switched off, our results match exactly that of already known ones in the literature. We also discuss the dynamics of such strings in pure NS-NS background in both small and large energy regimes.

<sup>1</sup>For a review of integrability in  $\text{AdS}_3/\text{CFT}_2$  one can see [44] and references therein.

The rest of the note is organized as follows. In section 2, we will discuss about the strings pulsating near the centre of the sphere on the  $\mathbb{R} \times S^3$  with mixed flux. In section 3, we will talk about this class of strings in AdS<sub>3</sub> with a background mixed flux. We will also sketch a generalisation of the solution with an angular momentum in  $S^3$  and comment on the effect of its inclusion. In section 4 we conclude and present our outlook.

## 2 Pulsating string in $\mathbb{R} \times S^3$ with mixed flux

In this section we will study the semiclassical quantization of a closed string which undergoes pulsating motion in the  $\mathbb{R} \times S^3$  subsector of the total geometry. Let us start with the metric and the NS-NS flux<sup>2</sup> supporting the background,

$$\begin{aligned} ds^2 &= -dt^2 + d\theta^2 + \sin^2 \theta d\phi_1^2 + \cos^2 \theta d\phi_2^2 \\ b_{\phi_1 \phi_2} &= -q \cos^2 \theta \end{aligned} \tag{2.1}$$

Since we are interested in F-string motion, the RR flux will not be relevant for us. In what follows throughout the paper, we will take the value of  $q$  to be small and use it as a perturbation parameter as in [54]. Now to study string solutions in this background, we use the polyakov action

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau [\sqrt{-\gamma} \gamma^{\alpha\beta} g_{MN} \partial_\alpha X^M \partial_\beta X^N - \epsilon^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N b_{MN}], \tag{2.2}$$

where  $\sqrt{\lambda}$  is the 't Hooft coupling,  $\gamma^{\alpha\beta}$  is the worldsheet metric and  $\epsilon^{\alpha\beta}$  is the antisymmetric tensor defined as  $\epsilon^{\tau\sigma} = -\epsilon^{\sigma\tau} = 1$ . Variation of the action with respect to  $X^M$  gives us the following equation of motion

$$\begin{aligned} 2\partial_\alpha (\eta^{\alpha\beta} \partial_\beta X^N g_{KN}) - \eta^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N \partial_K g_{MN} - 2\partial_\alpha (\epsilon^{\alpha\beta} \partial_\beta X^N b_{KN}) \\ + \epsilon^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N \partial_K b_{MN} = 0, \end{aligned} \tag{2.3}$$

and variation with respect to the metric gives the two Virasoro constraints,

$$g_{MN} (\partial_\tau X^M \partial_\tau X^N + \partial_\sigma X^M \partial_\sigma X^N) = 0, \tag{2.4}$$

$$g_{MN} (\partial_\tau X^M \partial_\sigma X^N) = 0. \tag{2.5}$$

We use the conformal gauge (i.e.  $\sqrt{-\gamma} \gamma^{\alpha\beta} = \eta^{\alpha\beta}$ ) with  $\eta^{\tau\tau} = -1$ ,  $\eta^{\sigma\sigma} = 1$  and  $\eta^{\tau\sigma} = \eta^{\sigma\tau} = 0$ ) to solve the equations of motion. Let us propose an ansatz for studying the pulsating string in the form,

$$t = t(\tau), \quad \theta = \theta(\tau), \quad \phi_1 = m\sigma, \quad \phi_2 = \phi_2(\tau). \tag{2.6}$$

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<sup>2</sup>There is a gauge freedom in the choice of the two-form B-field for the Wess-Zumino term since the supergravity equations of motion depend on the three-form field strength instead. In the presence of a boundary one can incorporate a boundary term to parameterise this ambiguity. For details one can look at [50] where the 2-form field is written as  $-\frac{q}{2}(\cos 2\theta + c)$ , with  $c$  as the ambiguity term. This  $c$  can be fixed via natural physical requirements.

With this embedding, the equations of motion of  $\theta$  and  $\phi_2$  give

$$\begin{aligned}\ddot{\theta} &= -\cos\theta\sin\theta(m^2 + \dot{\phi}_2^2 + 2mq\dot{\phi}_2), \\ \ddot{\phi}_2 &= 2\tan\theta\dot{\theta}(mq + \dot{\phi}_2).\end{aligned}\tag{2.7}$$

While the  $t$  equation is satisfied simply by

$$t = \kappa\tau,\tag{2.8}$$

with  $\kappa$  being a constant. One can check that the virasoro constraint (2.4) is given by

$$\dot{\theta}^2 = \dot{t}^2 - m^2\sin^2\theta - \cos^2\theta\dot{\phi}_2^2,\tag{2.9}$$

while the (2.5) is trivially satisfied. To check the consistency of the equation of motion with the virasoro, we can integrate (2.7) to get

$$\begin{aligned}\dot{\theta}^2 &= -m^2(1 - q^2)\sin^2\theta - \frac{C_1^2}{\cos^2\theta} + C_2, \\ \dot{\phi}_2 &= \frac{C_1}{\cos^2\theta} - mq,\end{aligned}\tag{2.10}$$

where  $C_1$  and  $C_2$  are integration constants. It can be easily verified that the above equations are consistent with (2.9) with the choice

$$C_2 = \kappa^2 - m^2q^2 + 2mqC_1.\tag{2.11}$$

Now, looking at the isometries of the background (2.1) we find the conserved Noether charges are the energy and angular momentum of the string.

$$E = \frac{\sqrt{\lambda}}{2\pi} \int t d\sigma, \quad J = \frac{\sqrt{\lambda}}{2\pi} \int (\cos^2\theta\dot{\phi}_2 + q\cos^2\theta\phi_1') d\sigma,\tag{2.12}$$

The above can be rewritten as

$$\mathcal{E} = \dot{t}, \quad \mathcal{J} = \cos^2\theta\dot{\phi}_2 + mq\cos^2\theta,\tag{2.13}$$

since the integrands are not functions of  $\sigma$  we can perform the integration accordingly. Also, we have used  $\mathcal{E} = \frac{E}{\sqrt{\lambda}}$  and  $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$  as the ‘semiclassical’ values of the charges. It is worth noting that due to the gauge freedom in choosing the B-field one could add a constant term to the definition of the 2-form in (2.1). This would have made the charge  $\mathcal{J}$  ambiguous too with extra terms proportional to  $q$  appearing in the definition. We can avoid this by choosing the boundary term accordingly so that the field exactly is given by (2.1).

Using the above we can write the Virasoro constraint in the following suggestive form,

$$\dot{\theta}^2 = \mathcal{E}^2 - m^2\sin^2\theta - \frac{\mathcal{J}^2}{\cos^2\theta} - m^2q^2\cos^2\theta + 2m\mathcal{J}q.\tag{2.14}$$

It can be seen that as  $\theta$  varies between 0 to  $\frac{\pi}{2}$ , the  $\dot{\theta}^2$  varies between  $\mathcal{E}^2 - (\mathcal{J} - mq)^2$  to infinity. This looks like the equation of motion of a particle moving in an effective potential

$V(\theta)$ , where  $\theta$  rotates between a minimal and a maximal value. Note that the  $\theta$  equation can be written in the form,

$$\ddot{x} = m^2(1 - q^2) [-(R_+ + R_-)x + 2x^3], \quad (2.15)$$

where  $x = \sin \theta$ . The  $R_{\pm}$  are roots of the polynomial

$$f(z) = (\mathcal{E}^2 - m^2q^2 + 2mq\mathcal{J})(1 - z) - m^2z(1 - z)(1 - q^2) - \mathcal{J}^2, \quad (2.16)$$

with  $z = x^2$ . This can be written as

$$\dot{x}^2 = m^2(1 - q^2)(x^2 - R_-)(x^2 - R_+), \quad (2.17)$$

With arbitrary coefficients, the above equation of motion would represent a undamped Duffing oscillator. But with proper scaling of the variables, we can write the solution of (2.17) in terms of standard Jacobi elliptic function<sup>3</sup> provided the initial condition  $x(0) = 0$ :

$$\sin \theta(\tau) = \sqrt{R_-} \operatorname{sn} \left( m\sqrt{1 - q^2}\sqrt{R_+}\tau \middle| \frac{R_-}{R_+} \right). \quad (2.18)$$

Using the property of Jacobi functions that  $\operatorname{sn}(z|m) = \operatorname{sn}(z + 4\mathbb{K}(m)|m)$  and as usual taking only the real period, we can find that the condition for time-periodic solution for  $\theta$  is,

$$\frac{R_-}{R_+} < 1. \quad (2.19)$$

This translates to the following inequality

$$(\mathcal{E}^2 - m^2)^2 + 4\mathcal{J}^2m^2 + 4\mathcal{J}mq(\mathcal{E}^2 - m^2) > 0, \quad (2.20)$$

which gives a constraint on the conserved charges so that the string has a pulsating motion. Since the above quantity is the discriminant of  $R_{\pm}$ , this guarantees that the roots are real and not equal. It is also important to mention that for a periodic solution both  $R_{\pm}$  have to be positive, which leads to the following,

$$\mathcal{E}^2 - (\mathcal{J} - mq)^2 > 0. \quad (2.21)$$

This is in tune with our observation earlier about the limits of oscillation in the  $\dot{\theta}$  equation.

We can also find the dynamics of the string along the  $\phi_2$  direction by integrating  $\frac{d\phi_2}{d\theta}$  from (2.10) which has the form

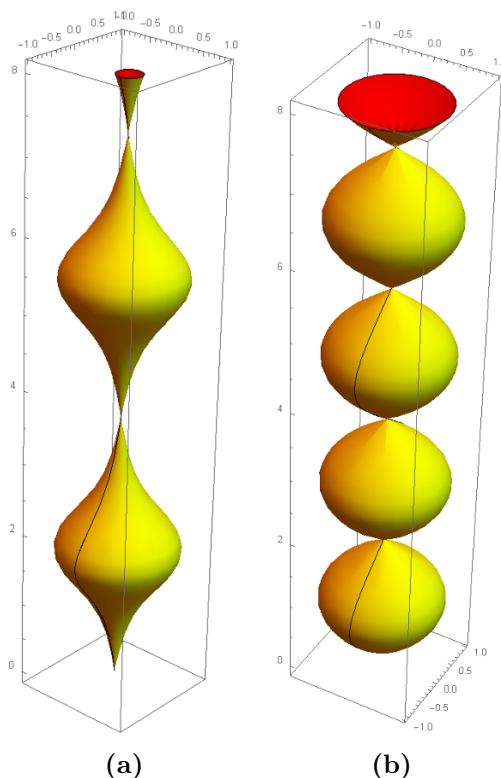
$$\frac{d\phi_2}{d\theta} = \frac{\mathcal{J} - mq \cos^2 \theta}{m\sqrt{1 - q^2} \cos \theta \sqrt{(\sin^2 \theta - R_-)(\sin^2 \theta - R_+)}}. \quad (2.22)$$

This can be integrated to find  $\phi_2$  in terms of standard elliptic integrals,

$$\begin{aligned} \phi_2(\tau) = & \frac{1}{m\sqrt{1 - q^2}} \left[ \frac{\mathcal{J}}{\sqrt{R_+}} \Pi \left( R_-, \sin^{-1} \left( \frac{1}{\sqrt{R_-}} \sin \theta(\tau) \right), \frac{R_-}{R_+} \right) \right. \\ & \left. - \frac{mq}{\sqrt{R_+}} \mathbb{F} \left( \sin^{-1} \left( \frac{1}{\sqrt{R_-}} \sin \theta(\tau) \right), \frac{R_-}{R_+} \right) \right] \end{aligned} \quad (2.23)$$

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<sup>3</sup>In the notation we follow  $\operatorname{sn}(z|m)$  is the solution of  $w'(z)^2 = (1 - w^2(z))(1 - mw^2(z))$ .



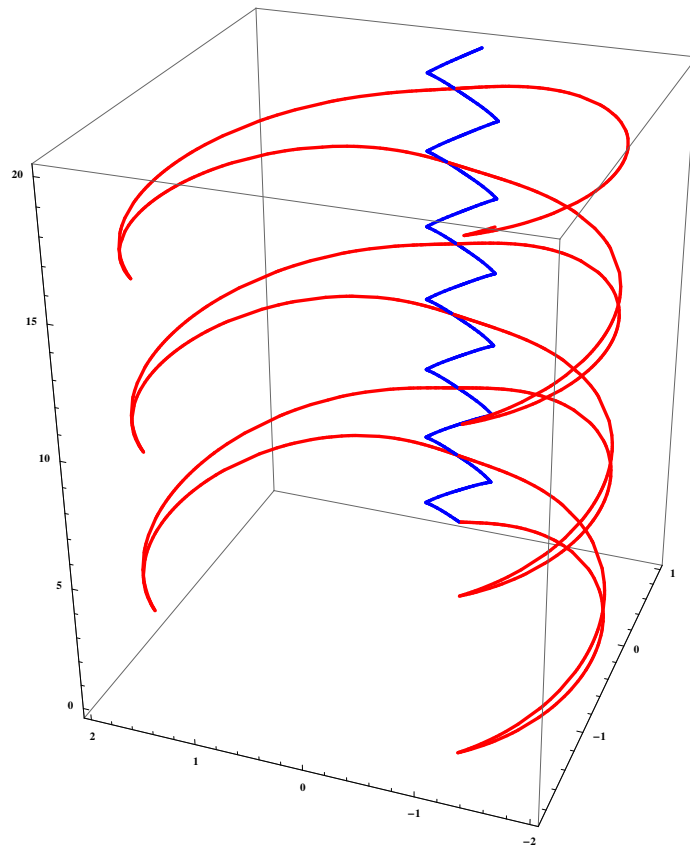
**Figure 1.** Pulsating ‘short’ circular strings in  $\mathbb{R} \times S^3$  as introduced in (2.24) evolving in the  $(X^0, X^1, X^2)$  plane. Both the strings have  $m = 2$ ,  $\mathcal{E} = \mathcal{J} = 0.2$  with a)  $q = 0.25$  and b)  $q = 0.99$ . The vertical direction coincides with the worldsheet time  $\tau$ .

Now since the solutions we are looking for is that of a circular string, we can parameterise the worldsheet in a simplified way using the variables

$$\begin{aligned}
 X^0 &= \tau, \\
 X^1 &= \sin \theta(\tau) \cos(m\sigma), \\
 X^2 &= \sin \theta(\tau) \sin(m\sigma), \\
 X^3 &= \cos \theta(\tau) \cos \phi_2(\tau), \\
 X^4 &= \cos \theta(\tau) \sin \phi_2(\tau), \quad \sigma \in [0, 2\pi].
 \end{aligned}
 \tag{2.24}$$

These are just some naive representations of the hypersurface traced out by the string as it moves forward in  $\tau$  (in analogy with the flat space pulsating solutions obtained in [15]), the actual worldsheet in a curved background can be very complex. The string solutions in  $(X^0, X^1, X^2)$  plane are plotted in for various values of the parameters in figure 1. The motion along  $(X^0, X^3, X^4)$  plane has been visualised in figure 2 to demonstrate how the  $\phi_2$  coordinate evolves with time.

Now we will use Bohr-Sommerfeld like quantization procedure for the pulsating strings in this background. The oscillation number (the adiabatic invariant associated to  $\theta$ ) can



**Figure 2.** Plot of temporal evolution of  $\phi_2(\tau)$  as described by eq. (2.23) in the  $(X^0, X^3, X^4)$  plane. Both the trajectories have  $m = 2$ ,  $\mathcal{E} = 0.2$ ,  $\mathcal{J} = 0.5$ . Blue one corresponds to  $q = 0.02$  and Red one corresponds to  $q = 0.95$ . The trajectories zig-zag along as the strings evolve. The vertical axis represents the worldsheet time  $\tau$ .

be written using the canonical momenta conjugate to  $\theta$  as follows,

$$\begin{aligned}
 N &= \sqrt{\lambda} \mathcal{N} = \frac{\sqrt{\lambda}}{2\pi} \oint d\theta \Pi_\theta \\
 &= \frac{\sqrt{\lambda}}{2\pi} \oint d\theta \sqrt{\mathcal{E}^2 - m^2 \sin^2 \theta - \frac{\mathcal{J}^2}{\cos^2 \theta} - m^2 q^2 \cos^2 \theta + 2m\mathcal{J}q}. \quad (2.25)
 \end{aligned}$$

Again, taking  $\sin \theta = x$  we can choose the proper limits and transform the above integral to

$$\mathcal{N} = \frac{2m\sqrt{1-q^2}}{\pi} \int_0^{\sqrt{R_-}} \frac{dx}{(1-x^2)} \sqrt{(x^2 - R_+)(x^2 - R_-)}. \quad (2.26)$$

We can directly compute the integral to find,

$$\begin{aligned}
 \mathcal{N} &= \frac{2m\sqrt{1-q^2}}{\pi\sqrt{R_+ - R_-}} \left[ (R_+ - R_-) \mathbb{E} \left( \frac{-R_-}{R_+ - R_-} \right) + (R_- - 1) \mathbb{K} \left( \frac{-R_-}{R_+ - R_-} \right) \right. \\
 &\quad \left. - (R_+ - 1) \mathbb{E} \left( \frac{R_-}{R_- - 1}, \frac{-R_-}{R_+ - R_-} \right) \right]. \quad (2.27)
 \end{aligned}$$



Instead of working with this, we can make the expressions a little simpler by taking the partial derivative of (2.26) with respect to  $m$  leading to

$$\frac{\partial \mathcal{N}}{\partial m} = I_1 - I_2. \quad (2.28)$$

The integrals  $I_1$  and  $I_2$  are given by

$$\begin{aligned} I_1 &= \frac{2q(\mathcal{J} - mq)}{\pi m \sqrt{1 - q^2}} \int_0^{\sqrt{R_-}} \frac{dx}{\sqrt{(x^2 - R_+)(x^2 - R_-)}}, \\ I_2 &= \frac{2\sqrt{1 - q^2}}{\pi} \int_0^{\sqrt{R_-}} \frac{x^2 dx}{\sqrt{(x^2 - R_+)(x^2 - R_-)}}. \end{aligned} \quad (2.29)$$

The integrals evaluate to complete elliptic functions of the first and second kind, and gives the expression

$$\frac{\partial \mathcal{N}}{\partial m} = \frac{2\sqrt{R_+}\sqrt{1 - q^2}}{\pi} \left[ \mathbb{E} \left( \frac{R_-}{R_+} \right) - \mathbb{K} \left( \frac{R_-}{R_+} \right) \right] + \frac{2q(\mathcal{J} - mq)}{\pi m \sqrt{1 - q^2} \sqrt{R_+}} \mathbb{K} \left( \frac{R_-}{R_+} \right). \quad (2.30)$$

In short string limit, i.e. when both energy and angular momentum of the string are small, we can expand the above expression keeping upto  $\mathcal{O}(q^2)$  correction terms. After expanding, we can integrate to get the series for  $\mathcal{N}$  as,

$$\begin{aligned} \mathcal{N} &= \left[ -\frac{mq^2}{2} + \left( -\frac{1}{2m} + \frac{3q^2}{8m} \right) \mathcal{J}^2 - \frac{3q\mathcal{J}^3}{4m^2} + \left( \frac{5}{16m^3} - \frac{225q^2}{128m^3} \right) \mathcal{J}^4 + \mathcal{O}(\mathcal{J}^5) \right] \\ &+ \left[ \left( \frac{1}{2m} + \frac{q^2}{8m} \right) + \frac{q\mathcal{J}}{4m^2} + \left( -\frac{3}{8m^3} + \frac{27q^2}{64m^3} \right) \mathcal{J}^2 - \frac{45q\mathcal{J}^3}{32m^4} \right. \\ &\left. + \left( \frac{105}{128m^5} - \frac{2625q^2}{512m^6} \right) \mathcal{J}^4 + \mathcal{O}(\mathcal{J}^5) \right] \mathcal{E}^2 + \mathcal{O}(\mathcal{E}^4). \end{aligned} \quad (2.31)$$

We can now invert the series to find the expression for the string energy in terms of the conserved quantities

$$\mathcal{E} = \sqrt{2m\mathcal{L}} \mathcal{A}_1(\mathcal{J}) \left[ 1 + \mathcal{B}_1(\mathcal{J})\mathcal{L} + \mathcal{O}(\mathcal{L}^2) \right], \quad (2.32)$$

where

$$\begin{aligned} \mathcal{L} &= \mathcal{N} + \frac{mq^2}{2} + \left( \frac{1}{2m} - \frac{3q^2}{8m} \right) \mathcal{J}^2 + \frac{3q\mathcal{J}^3}{4m^2} - \left( \frac{5}{16m^3} - \frac{225q^2}{128m^3} \right) \mathcal{J}^4 + \mathcal{O}(\mathcal{J}^5), \\ \mathcal{A}_1(\mathcal{J}) &= \left[ \left( 1 + \frac{q^2}{4} \right) + \frac{q\mathcal{J}}{2m} - \left( \frac{3}{4m^2} - \frac{27q^2}{32m^2} \right) \mathcal{J}^2 \right. \\ &\quad \left. - \frac{45q\mathcal{J}^3}{16m^3} + \left( \frac{105}{64m^4} - \frac{2625q^2}{256m^4} \right) \mathcal{J}^4 + \mathcal{O}(\mathcal{J}^5) \right]^{-1/2}, \\ \mathcal{B}_1(\mathcal{J}) &= \left( -\frac{1}{8m} + \frac{q^2}{64m} \right) - \frac{5q\mathcal{J}}{32m^2} + \left( \frac{33}{64m^3} - \frac{195q^2}{256m^3} \right) \mathcal{J}^2 \\ &+ \frac{327q\mathcal{J}^3}{128m^4} - \left( \frac{933}{512m^5} - \frac{102963q^2}{8192m^5} \right) \mathcal{J}^4 + \mathcal{O}(\mathcal{J}^5). \end{aligned}$$

As expected this expression is perfectly regular. One can explicitly check that for  $q = 0$ , i.e. without any flux the leading orders of string energy (2.32) reduces to that presented in [28]. Also with  $\mathcal{J} = 0$  the expression reduces to the result obtained in [29] for pulsating strings in  $\mathbb{R} \times S^2$ , which reads

$$\mathcal{E}_{\mathbb{R} \times S^2} = \sqrt{2m\mathcal{N}} \left[ 1 - \frac{\mathcal{N}}{8m} + \mathcal{O}(\mathcal{N}^2) \right]. \quad (2.33)$$

**The case of  $q = 1$ : pure NS-NS flux.** As is obvious, the description of pulsating strings we just presented will not be applicable to the case where  $q = 1$ , i.e. there is only NS-NS flux present. We have to start from the level of equations of motion and modify the dynamics accordingly.

A simple calculation gives the equation of motion for  $\theta$  for this case,

$$\dot{\theta}^2 = \mathcal{E}^2 - m^2 - \frac{\mathcal{J}^2}{\cos^2 \theta} + 2m\mathcal{J}. \quad (2.34)$$

With  $\sin \theta = x$ , the above equation can be transformed into

$$\dot{x}^2 = p^2(r^2 - x^2), \quad (2.35)$$

where  $p = \sqrt{\mathcal{E}^2 - m^2 + 2m\mathcal{J}}$  and  $r = \sqrt{\frac{\mathcal{E}^2 - (\mathcal{J} - m)^2}{p^2}}$ . The solution of the equation can be written as

$$x(\tau) = \pm r \sin(p\tau) \quad (2.36)$$

This is a simple pulsating string solution in the regime where  $p$  is real which cannot be true for a short string (small energy) with large  $m$ . So, a short string pulsating in  $\mathbb{R} \times S^3$  must have small ( $\mathcal{O}(1)$ ) number of windings to keep its motion intact. Such a solution has been visualised in figure 3. It is worth noting that the reality condition of  $p$  follows directly from (2.20) if we simply put  $q = 1$  and this in turn guarantees the reality of  $r$  since the numerator under root is nothing but (2.21) with  $q = 1$ . Now, the oscillation number for such a solution can be written as

$$\mathcal{N} = \frac{2}{\pi} \int_0^r \frac{dx}{1-x^2} \sqrt{p^2(1-x^2) - \mathcal{J}^2}. \quad (2.37)$$

With a transformation of the variable  $x \rightarrow rx$ , we can write this as

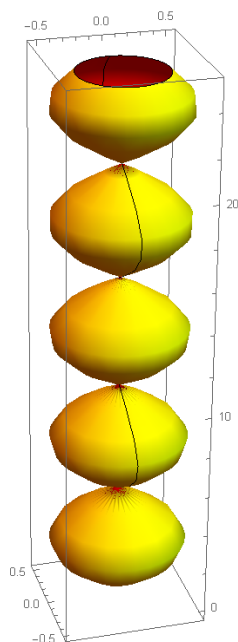
$$\mathcal{N} = \frac{2}{\pi} r^2 p \int_0^1 \frac{dx}{1-r^2 x^2} \sqrt{1-x^2} = (p - \mathcal{J}). \quad (2.38)$$

This leads to a simplified expression for string energy

$$\mathcal{E} = \sqrt{m^2 - 2m\mathcal{J} + (\mathcal{N} + \mathcal{J})^2}. \quad (2.39)$$

Expanding this in the short string limit, we get,

$$\mathcal{E} = (m - \mathcal{J}) + \left( \frac{\mathcal{J}}{m} + \frac{\mathcal{J}^2}{m^2} + \mathcal{O}(\mathcal{J}^3) \right) \mathcal{N} + \left( \frac{1}{2m} + \frac{\mathcal{J}}{2m^2} + \mathcal{O}(\mathcal{J}^3) \right) \mathcal{N}^2 + \mathcal{O}(\mathcal{N}^3). \quad (2.40)$$



**Figure 3.** String pulsating in  $\mathbb{R} \times S^3$  supported by pure NS-NS flux as described by eq. (2.36). Here  $\mathcal{E} = 0.6$ ,  $\mathcal{J} = 0.5$  and  $m = 1$ . The vertical axis represents  $\tau$ .

### 3 Pulsating string in $\text{AdS}_3$ with mixed flux

In the current section, we will concentrate on a string which pulsates in  $\text{AdS}_3$  with mixed flux. The string contracts to a point in  $\rho = 0$  and then expands up to reach a maximal size  $\rho = \rho_{\text{max}}$  to contract again and so on. Later we will also include an angular momentum from the sphere in the solution. We will study the semiclassical quantization of the classes of strings which rotates near the centre of  $\text{AdS}$  and also long objects that go up to the boundary. One can also do a Hamiltonian analysis of this system by reducing it to one-dimensional dynamical system. For a simple sketch of this point of view, refer to the appendix. Let us start with the metric and NS-NS flux of the relevant background.

$$\begin{aligned} ds^2 &= -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2, \\ b_{t\phi} &= q \sinh^2 \rho. \end{aligned} \tag{3.1}$$

We chose the pulsating string ansatz for this configuration as

$$t = t(\tau), \quad \rho = \rho(\tau), \quad \phi = m\sigma. \tag{3.2}$$

The polyakov action for the given background is given by

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left[ -\cosh^2 \rho \dot{t}^2 + \dot{\rho}^2 - m^2 \sinh^2 \rho + 2\dot{t}m q \sinh^2 \rho \right]. \tag{3.3}$$

Equation of motion for  $t$  and  $\rho$  are given by

$$\begin{aligned} 2 \cosh \rho \sinh \rho \dot{\rho} \dot{t} + \cosh^2 \rho \ddot{t} - 2mq \cosh \rho \sinh \rho \dot{\rho} &= 0, \\ \ddot{\rho} + \cosh \rho \sinh \rho (\dot{t}^2 + m^2 - 2\dot{t}mq) &= 0. \end{aligned} \tag{3.4}$$

Also from the Virasoro constraints we get

$$m^2 \sinh^2 \rho - \cosh^2 \rho \dot{t}^2 + \dot{\rho}^2 = 0. \quad (3.5)$$

Now, as in the previous section we can integrate the equations of motion to get

$$\begin{aligned} \dot{t} &= \frac{C_3}{\cosh^2 \rho} + mq, \\ \dot{\rho}^2 &= -m^2(1 - q^2) \sinh^2 \rho + \frac{C_3^2}{\cosh^2 \rho} + C_4, \end{aligned} \quad (3.6)$$

where  $C_3$  and  $C_4$  are integration constants. We can use the above expression and the (3.5) to show that the equations of motion and the Virasoro are completely self-consistent with the choice,

$$C_4 = 2C_3mq + m^2q^2. \quad (3.7)$$

The energy of the oscillating string is given by the conserved charge

$$\mathcal{E} = \cosh^2 \rho \dot{t} - mq \sinh^2 \rho. \quad (3.8)$$

And the canonical momentum conjugate to  $\rho$  is

$$\Pi_\rho = \dot{\rho}. \quad (3.9)$$

Using the equations (3.6) and (3.8), we can write the  $\rho$  equation in the form

$$\dot{\rho}^2 = \frac{(\mathcal{E} + mq \sinh^2 \rho)^2}{\cosh^2 \rho} - m^2 \sinh^2 \rho. \quad (3.10)$$

The above may be interpreted as an equation for a particle moving in an effective potential which grows to infinity at  $\rho \rightarrow \infty$ . The coordinate  $\rho(\tau)$  thus oscillates between 0 and a maximal  $\rho$  value ( $\rho_{\max}$ ). The equation of motion for the string can easily be recast using  $\sinh \rho = x$  as

$$\dot{x}^2 = m^2(1 - q^2)(x^2 - R_-)(R_+ - x^2), \quad (3.11)$$

where  $R_\pm$  are the roots of the polynomial

$$g(z) = (\mathcal{E} - mq)^2 - m^2(1 - q^2)z(1 + z) + mq(2\mathcal{E} - mq)(1 + z). \quad (3.12)$$

The solution can again be written in terms of the Jacobi elliptic function<sup>4</sup>

$$\sinh \rho(\tau) = \sqrt{\frac{-R_+ R_-}{R_+ - R_-}} \operatorname{sd} \left( m \sqrt{1 - q^2} \sqrt{R_+ - R_-} \tau \middle| \frac{R_+}{R_+ - R_-} \right). \quad (3.13)$$

As in the previous section, to have a time-periodic pulsating solution we must have

$$\frac{R_+}{R_+ - R_-} < 1. \quad (3.14)$$

---

<sup>4</sup>In the notation we follow,  $\operatorname{sd}(z|m)$  is the solution to  $w''(z) + w(z)(2m(1 - m)w^2(z) - 2m + 1) = 0$ .

The above condition translates to the following inequality,

$$-(m - 2q\mathcal{E}) - \sqrt{m^2 - 4mq\mathcal{E} + 4\mathcal{E}^2} < 0, \quad (3.15)$$

which gives the constraint on the solution to have a pulsating nature. Notice that this also takes into account the condition on the roots that  $R_- < 0$ . Again we can parameterise a circular string motion in terms of,

$$\begin{aligned} X^0 &= \tau, \\ X^1 &= \sinh \rho(\tau) \cos(m\sigma), \\ X^2 &= \sinh \rho(\tau) \sin(m\sigma), \quad \sigma \in [0, 2\pi]. \end{aligned} \quad (3.16)$$

We plot the solution for large value of  $\mathcal{E}$  and different values of  $q$  to explore the qualitative behaviour of the string dynamics in figure 4. It can be seen here that with large values of  $\mathcal{E}$  and  $q \rightarrow 1$ , the pulsating motion of the string is lost, as it progresses towards violating the inequality (3.15). To be clear, it can be shown that in the limit  $\mathcal{E} \rightarrow \infty$  and  $q \rightarrow 1$ , we can write

$$\frac{R_+}{R_+ - R_-} = 1 + \frac{q - 1}{2} + \mathcal{O}\left(\frac{1}{\mathcal{E}}\right), \quad (3.17)$$

which justifies the claim that in this limit the string solutions lose the pulsating motion. We will talk about it more in a later subsection where we discuss the case with pure NS-NS flux.

We can now define the oscillation number associated with string motion along  $\rho$  direction as,

$$\begin{aligned} N &= \sqrt{\lambda} \mathcal{N} = \frac{\sqrt{\lambda}}{2\pi} \oint d\rho \Pi_\rho \\ &= \frac{2\sqrt{\lambda}}{\pi} \int_0^{\rho_{\max}} d\rho \sqrt{\frac{(\mathcal{E} + mq \sinh^2 \rho)^2}{\cosh^2 \rho} - m^2 \sinh^2 \rho}. \end{aligned} \quad (3.18)$$

Changing the variable to  $x = \sinh \rho$  as before we integrate the above to get

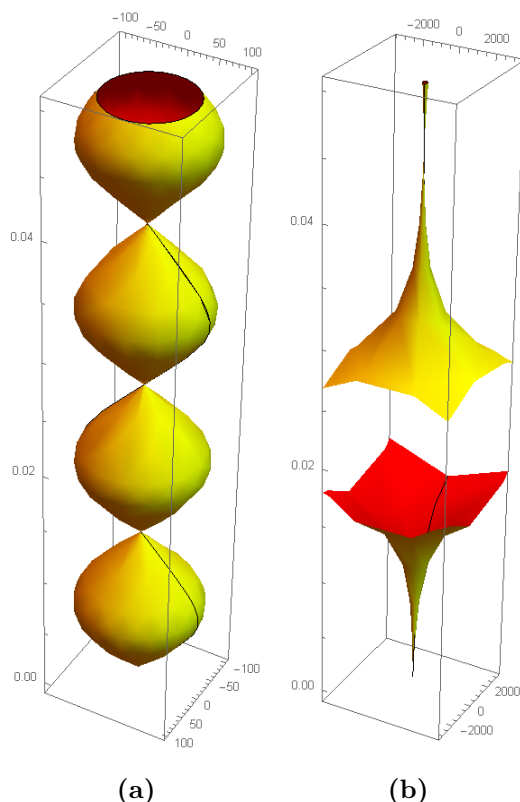
$$\begin{aligned} \mathcal{N} &= \frac{2m\sqrt{1-q^2}}{\pi} \int_0^{\sqrt{R_+}} \frac{dx}{1+x^2} \sqrt{(R_+ - x^2)(x^2 - R_-)} \\ &= \frac{2m\sqrt{1-q^2}}{\pi} \frac{1}{\sqrt{-R_-}} \left[ R_- \mathbb{E}\left(\frac{R_+}{R_-}\right) + (1+R_+) \left[ \mathbb{K}\left(\frac{R_+}{R_-}\right) - (1+R_-) \Pi\left(-R_+, \frac{R_+}{R_-}\right) \right] \right]. \end{aligned} \quad (3.19)$$

In the short string limit, the string moves near the centre of AdS, where  $\mathcal{E}$  and  $\mathcal{N}$  are small. Remember since the oscillation number is counterpart of the oscillator number ( $N_L + N_R$ ) in flat space, these short strings are not highly excited. In this case we get by expanding in  $\mathcal{E}$  and keeping upto  $\mathcal{O}(q^2)$  terms in the coefficients,

$$\mathcal{N} = \frac{\mathcal{E}^2}{2m} + \frac{q\mathcal{E}^3}{2m^2} + \frac{5(-1+3q^2)\mathcal{E}^4}{16m^3} - \frac{21q\mathcal{E}^5}{16m^4} + \frac{21(6-55q^2)\mathcal{E}^6}{256m^5} + \mathcal{O}(\mathcal{E}^7). \quad (3.20)$$

Inverting the series we get the expression for the string energy,

$$\mathcal{E} = \sqrt{2m\mathcal{N}} - q\mathcal{N} - \frac{5(-1+q^2)\mathcal{N}^{\frac{3}{2}}}{4\sqrt{2m}} + \frac{3q\mathcal{N}^2}{2m} + \frac{7(-11+24q^2)\mathcal{N}^{\frac{5}{2}}}{64\sqrt{2m}^{\frac{3}{2}}} + \mathcal{O}(\mathcal{N}^3). \quad (3.21)$$



**Figure 4.** Circular ‘Long’ strings in AdS<sub>3</sub> plotted according to eq. (3.13). Both of these have  $m = 2$  and  $\mathcal{E} = 2 \times 10^4$ . While a) corresponds to  $q = 0.1$  and shows perfect pulsating motion, b) corresponds to  $q = 0.9999$  and does not oscillate, giving unphysical structure as the (3.15) is almost saturated.

In the long string limit, the strings are highly excited and can reach the boundary of AdS. In this case  $\mathcal{E}$  will be large. For this regime, we can expand the oscillation number in  $\mathcal{E}$  and collect the coefficients order by order.

$$\mathcal{N} = \frac{\mathcal{E}}{2} + \mathcal{C}_1(q)\sqrt{m}\sqrt{\mathcal{E}} - \frac{mq}{2} + \mathcal{C}_2(q)\frac{m^{\frac{3}{2}}}{\sqrt{\mathcal{E}}} + \mathcal{O}(\mathcal{E}^{-\frac{3}{2}}) \quad (3.22)$$

Where  $\mathcal{C}_i(q)$ ’s have complicated expressions involving elliptic functions of  $q$ . Keeping in tune with rest of the paper we can write each of the coefficients as series in powers of  $q$ ,

$$\begin{aligned} \mathcal{C}_1(q) &= -0.38138 + 0.41731q + 0.14302q^2 + \dots \\ \mathcal{C}_2(q) &= 0.069552 + 0.143017q - 0.13041q^2 + \dots \end{aligned} \quad (3.23)$$

Inverting the series, the long string energy is obtained as,

$$\mathcal{E} = 2\mathcal{N} + \tilde{\mathcal{C}}_1(q)\sqrt{m\mathcal{N}} + \tilde{\mathcal{C}}_2(q)m + \tilde{\mathcal{C}}_3(q)\frac{m^{\frac{3}{2}}}{\sqrt{\mathcal{N}}} + \mathcal{O}(\mathcal{N}^{-\frac{3}{2}}). \quad (3.24)$$

Again here we can expand and write the coefficients as,

$$\begin{aligned} \tilde{\mathcal{C}}_1(q) &= 1.07871 - 1.18034q - 0.404514q^2 + \dots \\ \tilde{\mathcal{C}}_2(q) &= 0.29090 - 1.63662q + 0.13013q^2 + \dots \\ \tilde{\mathcal{C}}_3(q) &= -0.05914 - 0.600694q + 0.57628q^2 + \dots \end{aligned} \quad (3.25)$$

We now have eq. (3.24) as the semiclassical energy expressions for pulsating string configuration in AdS<sub>3</sub> supported by mixed flux. As usual, we have taken the amount of NS-NS flux turned on to be small and kept upto only  $\mathcal{O}(q^2)$  terms in the expressions we have considered. To avoid any confusion we must note that in the above results all the coefficients have been found exactly and then the values were numerically written down by expanding them in  $q$ . For example  $\tilde{\mathcal{C}}_1(0) = \frac{4\sqrt{2}}{\pi} [\mathbb{E}(-1) - \mathbb{K}(-1)] \approx 1.07871$ . It also stresses on the issue that after substituting  $q = 0$ , we can get back the exact results found in [27] and [29].

**Pulsating string in (AdS<sub>3</sub> × S<sup>1</sup>) with mixed flux.** Similarly as before we would start with the metric of AdS<sub>3</sub> with an added  $S^1 \subset S^3$  and the NS-NS flux

$$\begin{aligned} ds^2 &= -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2 + d\psi^2 \\ b_{t\phi} &= q \sinh^2 \rho. \end{aligned} \tag{3.26}$$

We chose the embedding ansatz for the pulsating string configuration as

$$t = t(\tau), \quad \rho = \rho(\tau), \quad \phi = m\sigma, \quad \psi = \psi(\tau). \tag{3.27}$$

The polyakov action for the given configuration is given by

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left[ -\cosh^2 \rho \dot{t}^2 + \dot{\rho}^2 - m^2 \sinh^2 \rho + \dot{\psi}^2 + 2\dot{t}mq \sinh^2 \rho \right]. \tag{3.28}$$

The energy and angular momentum of the oscillating string are given by

$$\mathcal{E} = \cosh^2 \rho \dot{t} - mq \sinh^2 \rho, \quad \mathcal{J} = \dot{\psi} \tag{3.29}$$

Using the equations of motion and the Virasoro constraints, we can get

$$\dot{\rho}^2 - \frac{(\mathcal{E} + mq \sinh^2 \rho)^2}{\cosh^2 \rho} + m^2 \sinh^2 \rho + \mathcal{J}^2 = 0, \tag{3.30}$$

which solves the same form of function as in (3.13). The qualitative difference between the solution in (3.13) and the one with inclusion of angular momentum can be discussed accordingly from the solution and the constraints on pulsating solution can be found. We note here that due to the inclusion of the extra angular momentum the condition (3.15) changes to.

$$-\mathcal{J}^2 - m^2 + 2mq\mathcal{E} - \sqrt{\mathcal{J}^4 + 2\mathcal{J}^2 m (-m + 2mq^2 - 2q\mathcal{E}) + m^2 (m^2 - 4mq\mathcal{E} + 4\mathcal{E}^2)} < 0. \tag{3.31}$$

For this case, oscillation number then becomes

$$\mathcal{N} = \frac{2m\sqrt{1-q^2}}{\pi} \frac{1}{\sqrt{-\tilde{R}_-}} \left[ \tilde{R}_- \mathbb{E} \left( \frac{\tilde{R}_+}{\tilde{R}_-} \right) + (1 + \tilde{R}_+) \left[ \mathbb{K} \left( \frac{\tilde{R}_+}{\tilde{R}_-} \right) - (1 + \tilde{R}_-) \Pi \left( -\tilde{R}_+, \frac{\tilde{R}_+}{\tilde{R}_-} \right) \right] \right].$$

Where the  $\tilde{R}_\pm$  are roots of the polynomial

$$h(z) = (\mathcal{E} - mq)^2 - m^2(1 - q^2)z(1 + z) + (2\mathcal{E}mq - m^2q^2 - \mathcal{J}^2)(1 + z). \tag{3.32}$$

It is obvious here that the condition (3.31) actually comes from  $\frac{\tilde{R}_+}{\tilde{R}_+ - \tilde{R}_-} < 1$ .

Similarly as before, we can expand the oscillation number in different limits. For the ‘long’ string limit, where  $\mathcal{E} \gg \mathcal{J}$ , we can write the oscillation number and invert it to get the string energy. It would include corrections to (3.24) in the powers of  $\mathcal{J}$ . We do not present the detailed expressions here for brevity. As an example let us write the pulsating string energy for ‘long’ strings in this background

$$\mathcal{E} = 2\mathcal{N} + \tilde{\mathcal{C}}_1(q)\sqrt{m\mathcal{N}} + \tilde{\mathcal{C}}_2(q)m + \left[ \tilde{\mathcal{C}}_3(q) + \tilde{\mathcal{C}}_4(q)\frac{\mathcal{J}^2}{m^2} \right] \frac{m^{\frac{3}{2}}}{\sqrt{\mathcal{N}}} + \mathcal{O}(\mathcal{N}^{-\frac{3}{2}}). \quad (3.33)$$

Where all other terms are same as in the previous subsection and the first correction of order  $\mathcal{J}^2$  is given by the power series in  $q$  as,

$$\tilde{\mathcal{C}}_4(q) = 0.29509 + 0.06742q + 0.03688q^2 + \dots \quad (3.34)$$

It is worth mentioning that at the  $\mathcal{O}(\frac{1}{\sqrt{\mathcal{N}}})$  there are no corrections in higher powers of  $\mathcal{J}$ . They start to appear in the following order and are negligible in the large  $\mathcal{N}$  regime.

**String solutions with  $q = 1$ .** Let us start with the pulsating string in  $\text{AdS}_3$  supported by pure NS-NS flux. The equation of motion for  $\rho$  takes the following form in this case

$$\dot{\rho}^2 - \frac{(\mathcal{E} - m)^2}{\cosh^2 \rho} + 2\mathcal{E}m - m^2 = 0. \quad (3.35)$$

This combined with  $x = \sinh \rho$  leads to the equation of motion

$$\dot{x}^2 = \mathcal{E}^2 + (2\mathcal{E}m - m^2)x^2. \quad (3.36)$$

The solution now depends on the value of  $d^2 = (2\mathcal{E}m - m^2)$ . For  $d^2 > 0$  (as for example with  $\mathcal{E} \gg m$  or, the long string limit) it can be written using the initial condition  $x(0) = 0$ ,

$$\sinh \rho(\tau) = \pm \left[ \frac{e^{d\tau}}{2d^2} - \frac{\mathcal{E}^2}{2} e^{-d\tau} \right] \quad (3.37)$$

which does not give rise to pulsation. On the other hand  $d^2 < 0$  clearly gives an oscillatory solution of the form

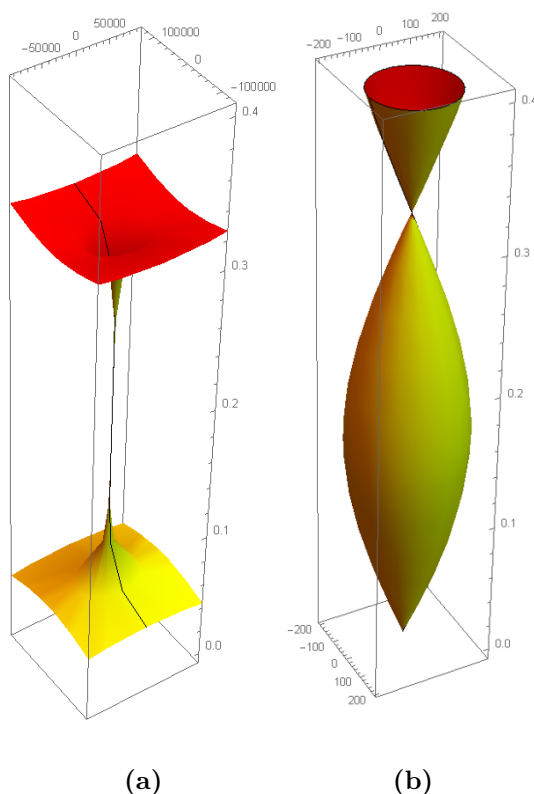
$$\sinh \rho(\tau) = \pm \frac{\mathcal{E}}{\tilde{d}} \sin \tilde{d}\tau, \quad \tilde{d} = id. \quad (3.38)$$

We can note here that if we put  $q = 1$  in (3.15), the condition reduces to  $(2\mathcal{E} - m) < 0$ , which is the same constraint on pulsating motion we have just found. Now, our problem is that a string with large  $\mathcal{E}$  (long) and small  $m$  will not show pulsating behaviour. This problem can be solved, for example, by making large number of windings of the string. So for pure NS-NS background, only the large winding strings with high energy can become a ‘long’ string. In a related case, the string solution in  $\text{AdS}_3 \times S^1$  with  $q = 1$  solves the equation

$$\dot{x}^2 = \mathcal{E}^2 + (2\mathcal{E}m - m^2 - \mathcal{J}^2)x^2. \quad (3.39)$$

For the long string, inclusion of a large angular momentum will again lead to an oscillatory pulsating solution. The trade-off however is, with larger angular momentum attached to





**Figure 5.** Circular string solutions in  $\text{AdS}_3$  with pure NS-NS flux as described by eq. (3.36) and (3.39). Both solutions have  $m = 1$  and  $\mathcal{E} = 2 \times 10^3$ . While a) has  $\mathcal{J} = 0$  and does not show desired behaviour, b) has  $\mathcal{J} = 64$  and re-gains pulsating motion due to the inclusion of angular momenta, however the pulsating strings are significantly smaller in size.

the string, it will try to collapse onto itself, thereby reducing the effective size of the string. However, the NS flux will continue trying to expand the string [34], still resulting in finitely smaller size. This idea is illustrated in figure 5, but it requires a deeper understanding to be commented on conclusively. The oscillation number for the small strings in  $\text{AdS}_3$  reads

$$\mathcal{N} = \frac{2}{\pi} \int_0^r \frac{dx}{1+x^2} \sqrt{\mathcal{E}^2 + (2\mathcal{E}m - m^2)x^2}. \quad (3.40)$$

Here  $r = \sqrt{\frac{\mathcal{E}^2}{m^2 - 2\mathcal{E}m}}$  is only real for small energies. With a change of variable  $x \rightarrow rx$ , we can write this as

$$\mathcal{N} = \frac{2}{\pi} r^2 d \int_0^1 \frac{dx}{1+r^2x^2} \sqrt{1-x^2} = -\mathcal{E} + m - \sqrt{m^2 - 2\mathcal{E}m}. \quad (3.41)$$

Which has a small  $\mathcal{E}$  expansion

$$\mathcal{N} = \frac{\mathcal{E}^2}{2m} + \frac{\mathcal{E}^3}{2m^2} + \frac{5\mathcal{E}^4}{8m^3} + \mathcal{O}(\mathcal{E}^5). \quad (3.42)$$

And gives the exact relation for short string energy

$$\mathcal{E} = \sqrt{2m\mathcal{N}} - \mathcal{N}. \quad (3.43)$$

As expected, the oscillation number for long strings becomes imaginary in this case.

## 4 Conclusion

In this note, we have discussed pulsating string solutions in subsectors of  $\text{AdS}_3 \times S^3$  with mixed flux and presented their dispersion relations. As expected, we could show the usual dispersion relation in terms of the oscillation number and other conserved quantities receives corrections due to the NS flux in the background. We have found out these corrections upto a certain order in the the parameter governing the strength of the NS-NS flux for both large and short string behaviour of the pulsating string. For a complete understanding into these string solutions, one would look for extract the dual gauge theory operators for such strings in mixed flux background. Unfortunately, the dual gauge theory of such a string background is not properly understood yet. The circular pulsating string solutions in  $\text{AdS}_5$  and the anomalous dimensions calculated from it played a significant part in understanding relevant sectors of the dual SYM theory. We hope our solutions can be useful in that regard also.

The straightforward extension of our work would be to find  $(p, q)$  string solutions in this mixed flux background which will be affected by both RR and NS-NS fluxes. This probe brane dynamics should be interesting. The other way would be to verify our solutions by properly reducing the oscillating string sigma model to that of a Neumann-Rosochatius integrable system [57, 58] using a pulsating type ansatz. Also, the  $q = 1$  pulsating solutions should be obtainable from a WZW model perspective. We hope to report on some of these issues in near future.

## Acknowledgments

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## A Hamiltonian dynamics and pulsating strings

We want to observe the phase space of the pulsating string moving on the  $\text{AdS}_3$  background supported by NS-NS flux. We will sketch an outline of this discussion here but will not talk about details. The lagrangian of such a pulsating string would be given by

$$\mathcal{L} = -\frac{1}{2} [-\cosh^2 \rho \dot{t}^2 + \dot{\rho}^2 + m^2 \sinh^2 \rho - 2imq \sinh^2 \rho], \quad (\text{A.1})$$

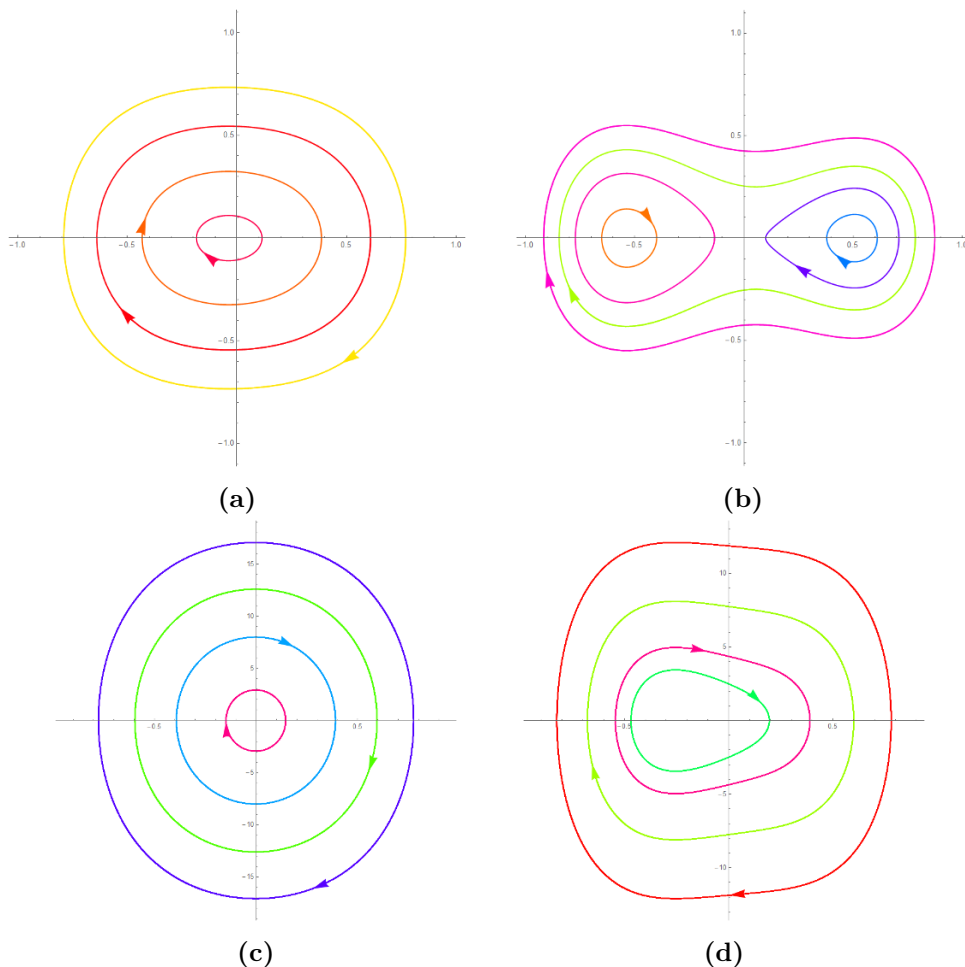
with proper choice of units. The corresponding hamiltonian can be written in terms of the conjugate momenta as following,

$$\mathcal{H} = \frac{1}{2} \left[ P_\rho^2 + m^2 \sinh^2 \rho + \frac{(\mathcal{E} + mq \sinh^2 \rho)}{\cosh^2 \rho} (3\mathcal{E} - mq \sinh^2 \rho) \right]. \quad (\text{A.2})$$

The hamiltonian equation of motion for  $\rho$  can then be written as,

$$\ddot{\rho} = \frac{mq\mathcal{E}}{\cosh \rho} - m^2 \cosh \rho \sinh \rho + \frac{1}{\cosh^3 \rho} [(m^2 q^2 + 3\mathcal{E}^2) \sinh \rho - 2mq\mathcal{E}]. \quad (\text{A.3})$$

The phase portraits corresponding to the above hamiltonian has been given in figure 6 and 7 for various value of the parameters. We also have dscussed their behaviour qualitatively. As a comparison, remember the equation of motion for this pulsating string could be cast

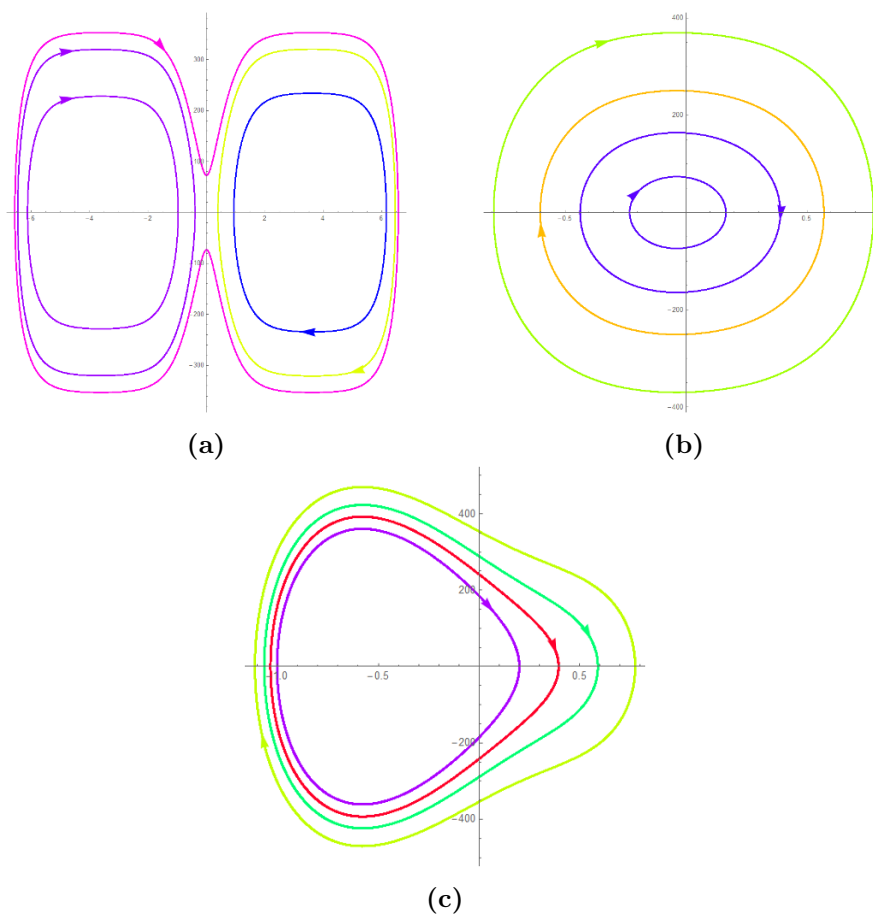


**Figure 6.** Phase portraits  $(P_\rho - \rho)$  for the short string from (A.2). a) is characterized by  $\mathcal{E} = 0.4$ ,  $m = 1$  and  $q = 0.05$  and shows nearly harmonic oscillator behaviour. As we increase the energy to 0.75 in b) the classical symmetry breaks and the additional equilibria occur. The original equilibrium is regained in c) as we take the winding number to  $m = 20$ . This equilibrium is maintained throughout even as we make the  $q = 0.999$  in d) as only the expansive nature of the NS-NS flux is not enough to disturb it.

into that of a non-linear oscillator as we have already seen in section 3. The system has a form of

$$\ddot{z} = \alpha z + \beta z^3. \tag{A.4}$$

In general this equation is that of undamped Duffing type oscillator (for  $\alpha < 0$ ) without any external forcing. With pulsating string ansatz, the coefficients  $\alpha$  and  $\beta$  turn out to be of useful form so that the solution can be written explicitly in terms of Jacobi elliptic functions as we got in our detailed discussion. Depending on the values of the parameters  $m$ ,  $q$  and the energy  $\mathcal{E}$ , the nature of the potential term changes, and so does the phase portraits when we talk in terms of point particles. For sufficiently small  $\beta$  the potential remains in the linear response regime and gives slightly deformed harmonic oscillator phase portraits.



**Figure 7.** Phase portraits  $(P_\rho - \rho)$  for the long string from (A.2). a) is characterized by  $\mathcal{E}=200$ ,  $m = 1$  and  $q = 0.05$  and shows highly deformed oscillator-like behaviour around the two minima. As we increase the winding number to  $m = 500$  in b) the original equilibrium is regained, however in a highly distorted form. This equilibrium is maintained throughout even as we make the  $q = 0.999$  in c).

Now looking at the potential, we can always say that  $z = 0$  is a equilibrium point, but when  $\alpha\beta < 0$ , the classical symmetry of the system is broken and two stable equilibria occur at  $z = \pm\sqrt{-\frac{\alpha}{\beta}}$ . The stability analysis of these equilibria can be done using the usual eigenvalue method. From a string point of view we can see that as the pulsating string attains more and more energy, the motion prefers the new minima of the potential. However if we increase the winding number suitably, the original equilibrium is regained as the state becomes ‘heavy’. If we increase the NS-NS flux, it tries to expand the string, but is prohibited by the large tension. A remarkable fact here is that there is no forcing term, which guarantees that the chaotic attractor type situation does not occur here for any value of the parameters.

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