Partition functions of superconformal Chern-Simons theories from Fermi gas approach

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ABSTRACT: We study the partition function of three-dimensional $\mathcal{N} = 4$ superconformal Chern-Simons theories of the circular quiver type, which are natural generalizations of the ABJM theory, the worldvolume theory of M2-branes. In the ABJM case, it was known that the perturbative part of the partition function sums up to the Airy function as $Z(N) = e^A C^{-1/3} \text{Ai}[C^{-1/3}(N - B)]$ with coefficients C, B and A and that for the non-perturbative part the divergences coming from the coefficients of worldsheet instantons and membrane instantons cancel among themselves. We find that many of the interesting properties in the ABJM theory are extended to the general superconformal Chern-Simons theories. Especially, we find an explicit expression of B for general $\mathcal{N} = 4$ theories, a conjectural form of A for a special class of theories, and cancellation in the non-perturbative coefficients for the simplest theory next to the ABJM theory.

Contents

1 Introduction and summary

There is no doubt that M-theory is one of the most important achievements in theoretical physics, though, at the same time, it is one of the most mysterious theories. It is a famous result from the AdS/CFT correspondence [1] that the number of degrees of freedom of the stack of N M2-branes is $N^{3/2}$ and that of the stack of M5-branes is N^3. With these novel large N behaviors which are in contrast to the intuitive behavior N^2 of D-branes, it is obvious that these M-theoretical branes deserve intensive studies.

The M2-brane worldvolume theory on the flat spacetime was explored by supersymmetrizing the topological Chern-Simons theory [2] and finally it was proposed [3] that the worldvolume theory of N M2-branes on the geometry $\mathbb{C}^4/\mathbb{Z}_k$ is described by $\mathcal{N} = 6$ supersymmetric Chern-Simons theory with gauge group $U(N)_k \times U(N)_{-k}$ and bifundamental matters between them, which is dubbed ABJM theory. Here the subscript k and $-k$ are the Chern-Simons levels associated to each $U(N)$ factor.

Following recent progress of localization techniques [4–7], it was found that for the partition function and vacuum expectation values of supersymmetric quantities in the ABJM

theory, the infinite-dimensional path integral in defining these quantities is reduced to a finite-dimensional matrix integration. Furthermore, due to the large supersymmetries, many interesting properties of this ABJM matrix model are discovered [8–26]: the perturbative part of the partition function sums up to the Airy function [13]; the divergences in the coefficients of membrane instantons and those of worldsheet instantons cancel among themselves [20]; the non-perturbative part of the partition function is expressed in terms of the refined topological string [24].

Here we briefly review these results on the partition function of the ABJM theory. First let us consider the perturbative part.[1] It was predicted from the gravity dual that the 't Hooft coupling $\lambda = N/k$ should be shifted as $\lambda_{\text{eff}} = \lambda - 1/24 + 1/(24k^2)$ [27] and (except an inconsistency in the coefficient of the k^{-2} correction) this shift was captured from the study of the matrix model [10, 12]. With the shift of the 't Hooft coupling in mind, the all genus perturbative corrections of the partition function sum up to the Airy function [13]

$$Z_{\text{pert}}(N) = e^A C^{-1/3} \operatorname{Ai}\left[C^{-1/3}(N - B)\right], \tag{1.1}$$

using the relation with the topological string theory on local $\mathbb{P}^1 \times \mathbb{P}^1$ [9, 10]. This result was later beautifully rederived [15] by rewriting the ABJM partition function into the partition function of a Fermi gas system, without mentioning the relation with the topological string. Here the N-independent constants C and B are given by simple functions of k

$$C_{\text{ABJM}}(k) = \frac{2}{\pi^2 k}, \quad B_{\text{ABJM}}(k) = \frac{1}{3k} + \frac{k}{24}, \tag{1.2}$$

while A is a very complicated function

$$A_{\text{ABJM}}(k) = -\frac{1}{6} \log \frac{k}{4\pi} + 2\zeta'(-1) - \frac{\zeta(3)}{8\pi^2} k^2 + \frac{1}{3} \int \frac{dx}{e^{kx} - 1} \left(\frac{3}{x \sinh^2 x} - \frac{3}{x^3} + \frac{1}{x} \right), \tag{1.3}$$

which was obtained by taking the Borel sum of the constant map contribution [16].

The non-perturbative effects have a more drastic structure. It turns out that there are two types of non-perturbative effects. One is called worldsheet instanton [10, 28] which corresponds to the string worldsheet wrapping the holomorphic cycle $\mathbb{CP}^1$ in $\mathbb{CP}^3$, while the other is called membrane instanton [12] which corresponds to the D2-brane wrapping the Lagrangian submanifold $\mathbb{RP}^3$ of $\mathbb{CP}^3$, where $\mathbb{CP}^3$ comes from the string theory limit $k \to \infty$ of $\mathbb{C}^4/\mathbb{Z}_k$. It was found [20] that the coefficients of both instanton effects are actually divergent at certain levels k, though the divergences are cancelled perfectly, if we include all of the non-perturbative effects including worldsheet instantons, membrane instantons and their bound states. This cancellation mechanism helps us to determine the whole non-perturbative effects [24].

It is interesting to ask whether the beautiful structures in the ABJM theory persist in other theories. Before arriving at the ABJM theory, a large class of supersymmetric Chern-Simons theories were found. For $\mathcal{N} = 3$, the supersymmetric Chern-Simons theories are

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[1]The partition function can be studied in the perturbation of $1/N$ with the 't Hooft coupling $\lambda = N/k$ fixed from the stringy regime or with the M-theory background k fixed from the M-theory regime. The perturbation can be understood in either sense.
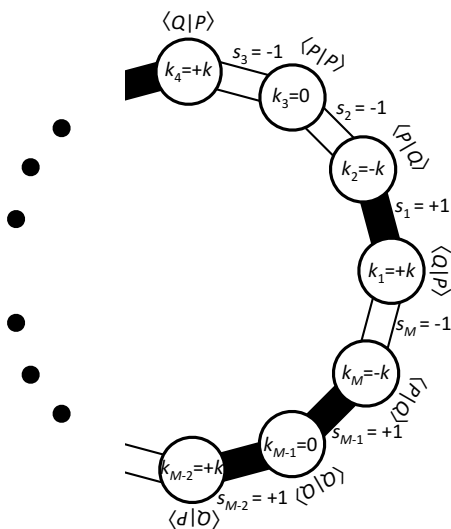
Figure 1. Circular quiver with $\{s_a\}_{a=1}^{M} = \{(+1), (-1)^2, \cdots, (+1)^2, (-1)\}$, which is associated to the $\mathcal{N} = 4$ superconformal Chern-Simons theories. Here we paint the edges assigned with $s_a = +1$ black, and those assigned with $s_a = -1$ white.

constructed [29–31] for general gauge groups and general matter contents. Especially, the theory has the conformal symmetry when the gauge group is $\prod_{a=1}^{M} \mathrm{U}(N)_{k_a}$ with $\sum_{a=1}^{M} k_a = 0$ and the matter fields are in the bifundamental representation of $\mathrm{U}(N)_{k_a}$ and $\mathrm{U}(N)_{k_{a+1}}$ [32, 33]. These theories can be expressed by a circular quiver taking the same form as the Dynkin diagram of $\widehat{A}_{M-1}$, where each vertex denotes the $\mathrm{U}(N)$ factor of the gauge group and each edge denotes the bifundamental matter. The Chern-Simons theory with less supersymmetries is believed to describe the worldvolume theory of multiple M2-branes on a geometry with less supersymmetries.

It was found that among others when the number of the $\mathrm{U}(N)$ factors is even and the levels are k and $-k$ appearing alternatively, the number of the supersymmetries is enhanced to $\mathcal{N} = 4$ [34, 35] and the background geometry is interpreted to be an orbifold [36–38]. As pointed out in [39], the $\mathcal{N} = 4$ enhancement is not restricted to the case of alternating levels. In fact, as long as the levels are expressed as

$$k_a = \frac{k}{2}(s_a - s_{a-1}), \quad s_a = \pm 1, \tag{1.4}$$

the supersymmetry of all these theories extends to $\mathcal{N} = 4$. Since these theories are characterized by s_a which are associated to the edges of the quivers and take only two signs, it is more suitable to assign two colors to the edges, rather than to paint the vertices. See figure 1.

With its simplicity in derivation, the authors of [15] were able to further argue that, for the large class of general $\mathcal{N} = 3$ superconformal circular quiver Chern-Simons theories (associated with a hermitian Hamiltonian, as explained later), the perturbative partition function is always given by the same form (1.1) with some coefficients C, B and A. Also, the study of the large N behavior (the coefficient C) in many theories can be found in

earlier works [40–54]. Especially it is worthwhile to mention that, according to [15] the expression of the coefficient C was given a geometrical interpretation as the classical volume inside the Fermi surface. Moreover, recently in [55] the special $\mathcal{N} = 4$ case with the gauge group $[\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}]^r$, whose quiver is the r-ple repetition of that of the ABJM theory, was studied carefully including the instanton effect using the relation to the original ABJM theory. Alongside, the authors found that if the circular quiver is the r-ple repetition of a "fundamental" circular quiver, the grand potential of the repetitive theory is given explicitly by that of the "fundamental" theory. Especially, it was found that the perturbative coefficients of the r-ple repetitive theory $[C]_r$, $[B]_r$, $[A]_r$ are related to $[C]_1$, $[B]_1$, $[A]_1$ by

$$[C]_r = \frac{1}{r^2}[C]_1, \quad [B]_r = [B]_1 - \frac{\pi^2}{3}\left(1 - \frac{1}{r^2}\right)[C]_1, \quad [A]_r = r[A]_1. \tag{1.5}$$

However, the coefficients B, A and the non-perturbative corrections for general $\mathcal{N} = 3$ theories have not been known so far.

In this paper, we extend the previous studies on the ABJM theory to the more general $\mathcal{N} = 4$ cases with the levels (1.4). Especially we hope that after figuring out the cancellation mechanism among all of the instanton effects, the instanton moduli space of the membrane theories will become clearer. We first concentrate on the perturbative part. Using the Fermi gas formalism, we give an explicit formula for B, which is deeply related to the redefinition of the 't Hooft coupling. We have found that, when the edges are assigned with

$$\{s_a\}_{a=1}^M = \{(+1)^{q_1}, (-1)^{p_1}, (+1)^{q_2}, (-1)^{p_2}, \cdots, (+1)^{q_m}, (-1)^{p_m}\}, \tag{1.6}$$

where the expression denotes a sequence consisting of q_1 elements of $+1$, p_1 elements of -1 and so on in this ordering, the coefficient B is given by

$$B = \frac{B^{(0)}}{k} + kB^{(2)}, \tag{1.7}$$

with

$$\begin{aligned}
B^{(0)} &= -\frac{1}{6}\left[\frac{\Sigma(p)}{\Sigma(q)} + \frac{\Sigma(q)}{\Sigma(p)} - \frac{4}{\Sigma(q)\Sigma(p)}\right], \\
B^{(2)} &= \frac{1}{24}\left[\Sigma(q)\Sigma(p) - 12\left(\frac{\Sigma(q,p,q)}{\Sigma(q)} + \frac{\Sigma(p,q,p)}{\Sigma(p)} - \frac{\Sigma(q,p)\Sigma(p,q)}{\Sigma(q)\Sigma(p)}\right)\right].
\end{aligned} \tag{1.8}$$

Here we adopt the notation of $\Sigma(L)$, with L denoting an alternating sequence of q and p, whose definition is given by

$$\begin{aligned}
\Sigma(q) &= \sum_{a=1}^m q_a, & \Sigma(p) &= \sum_{a=1}^m p_a, \\
\Sigma(q,p) &= \sum_{1 \le a \le b \le m} q_a p_b, & \Sigma(p,q) &= \sum_{1 \le a < b \le m} p_a q_b, \\
\Sigma(q,p,q) &= \sum_{1 \le a \le b < c \le m} q_a p_b q_c, & \Sigma(p,q,p) &= \sum_{1 \le a < b \le c \le m} p_a q_b p_c.
\end{aligned} \tag{1.9}$$

Note that the condition in each summation can be restated as the requirement that we choose q_a's and p_a's out of $q_1, p_1, q_2, p_2, \cdots, q_m, p_m$ by respecting its ordering. We stress that the result (1.7) with (1.8) is encoded suitably in this notation. After we introduce this notation, the proof of the result is quite straightforward.

It is still difficult to obtain the general expression of the coefficient A with the current technology. For the special case when the edges of $s_a = +1$ and those of $s_a = -1$ are clearly separated

$$\{s_a\}_{a=1}^M = \{(+1)^q, (-1)^p\}, \tag{1.10}$$

we conjecture that the coefficient A is given in terms of the coefficient of the ABJM case $A_{\text{ABJM}}(k)$ (1.3) by

$$A(k) = \frac{1}{2}\big(p^2 A_{\text{ABJM}}(qk) + q^2 A_{\text{ABJM}}(pk)\big). \tag{1.11}$$

Later we shall provide evidences for this conjecture using the WKB expansion (6.33) and numerical data (table 2).

After determining the perturbative part, we continue to the non-perturbative part.[2] To fully understand the non-perturbative instanton effects, we still need lots of future studies. We shall concentrate on the separative case (1.10) with $q = 2$, $p = 1$, that is, $\{s_a\}_{a=1}^3 = \{(+1)^2, (-1)\}$, which is the simplest case other than the ABJM theory. Using the WKB expansion of the Fermi gas formalism, we can study the membrane instanton order by order in $\hbar = 2\pi k$. We have found that the first membrane instanton is consistent with

$$J_{\text{np}}^{\text{MB}}(\mu) = -\frac{2}{\tan\frac{\pi k}{2}} e^{-\mu} + \mathcal{O}(e^{-2\mu}), \tag{1.12}$$

up to the $\mathcal{O}(k^5)$ term in the $\hbar = 2\pi k$ expansion. On the other hand, using the numerical coefficients of the grand potential for $k = 3, 4, 5, 6$, we conjecture that the first worldsheet instanton is given by

$$J_{\text{np}}^{\text{WS}}(\mu) = \frac{4\cos\frac{\pi}{k}}{\sin^2\frac{2\pi}{k}} e^{-\frac{2}{k}\mu} + \mathcal{O}(e^{-\frac{4}{k}\mu}). \tag{1.13}$$

We can see that the coefficients of both the first membrane instanton (1.13) and the first worldsheet instanton (1.12) are divergent at $k = 2$ and the remaining finite part after cancelling the divergences matches perfectly with the numerical coefficients at $k = 2$.

The remaining part of this paper is organized as follows. In section 2, we shall demonstrate the Fermi gas formalism for general $\mathcal{N} = 4$ superconformal Chern-Simons theories. Then in section 3 we shall proceed to derive the expression of B for general $\mathcal{N} = 4$ circular quivers. We shall shortly see the consistency with the transformation under the repetition in section 4 and see the possible generalization to the $\mathcal{N} = 3$ cases in section 5. After that,

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[2]The interpretation of these non-perturbative instanton effects in the gravity dual still awaits to be studied carefully. In this paper we call these non-perturbative instanton effects membrane instanton when the exponent is proportional to μ while we call them worldsheet instanton when the exponent is proportional to μ/k.

we shall turn to the WKB expansion of the grand potential in section 6, where not only the consistency with the expression of B but also further information on the coefficient A and the instantons are found. In section 7 we shall study the non-perturbative instanton effects for the special case of $\{s_a\}_{a=1}^3 = \{(+1)^2, (-1)\}$. Finally in section 8 we conclude with some future directions.

Note added. As our work had been completed and we were in the final stage of checking the draft, Hatsuda and Okuyama submitted their work [56], where they also used the Fermi gas formalism to study the N_f matrix model [53]. Although the original theories are different, in terms of the Fermi gas formalism, the density matrix (2.4) in [56] is reproduced if we restrict our setup to the separative case $\{s_a\}_{a=1}^M = \{(+1)^{N_f}, (-1)\}$ and put $k = 1$. Their results also have some overlaps with ours. For example, our conjectural form of the coefficient A (1.11) reduces to their conjecture (3.12) in [56] under this restriction.

2 $\mathcal{N} = 4$ Chern-Simons matrix model as a Fermi gas

In this section we shall show that the partition functions of $\mathcal{N} = 4$ superconformal circular quiver Chern-Simons theories, with gauge group $\prod_{a=1}^M \mathrm{U}(N)_{k_a}$ and Chern-Simons levels chosen to be (1.4), can be regarded as the partition functions of N-particle ideal Fermi gas systems governed by non-trivial Hamiltonians. Although this structure was already proved in [15] for more general $\mathcal{N} = 3$ superconformal circular quiver Chern-Simons theories without the restriction of levels (1.4), we shall repeat the derivation since the special simplification occurs for $\mathcal{N} = 4$ theories with the levels (1.4). In particular we find that, corresponding to the colors of edges $\{s_a\}_{a=1}^M$ (1.6), the Hamiltonian of the associated Fermi gas system is given by

$$
e^{-\widehat{H}} = \left[2\cosh\frac{\widehat{Q}}{2} \right]^{-q_1} \left[2\cosh\frac{\widehat{P}}{2} \right]^{-p_1} \cdots \left[2\cosh\frac{\widehat{Q}}{2} \right]^{-q_m} \left[2\cosh\frac{\widehat{P}}{2} \right]^{-p_m}. \tag{2.1}
$$

Let us begin with the partition function of an $\mathcal{N} = 4$ circular quiver Chern-Simons theory with gauge group $[\mathrm{U}(N)]^M$ and levels (1.4),

$$
Z(N) = \frac{1}{(N!)^M} \int \left(\prod_{a=1}^M \prod_{i=1}^N D\lambda_{a,(i)} \right) \left(\prod_{a=1}^M \frac{\prod_{i<j}^N \left(2\sinh\frac{\lambda_{a,(i)} - \lambda_{a,(j)}}{2} \right)^2}{\prod_{i,j}^N 2\cosh\frac{\lambda_{a+1,(i)} - \lambda_{a,(j)}}{2}} \right), \tag{2.2}
$$

obtained by localization technique [5]. Here M is the number of vertices and the integration measure is given by

$$
D\lambda_{a,(i)} = \frac{d\lambda_{a,(i)}}{2\pi} \exp\left(\frac{ik_a}{4\pi} \lambda_{a,(i)}^2 \right), \tag{2.3}
$$

with k_a being the Chern-Simons level for the a-th $\mathrm{U}(N)$ factor of the gauge group $[\mathrm{U}(N)]^M$.

Using the Cauchy identity

$$
\frac{\prod_{i<j}^N (x_i - x_j) \prod_{i<j}^N (y_i - y_j)}{\prod_{i,j}(x_i + y_j)} = \det_{i,j} \frac{1}{x_i + y_j}, \tag{2.4}
$$

and the integration formula [57]

$$\frac{1}{N!}\int\left(\prod Dx_k\right)\det_{i,k}(\phi_i(x_k))\det_{j,k}(\psi_j(x_k)) = \det_{i,j}\left(\int Dy\phi_i(y)\psi_j(y)\right), \qquad (2.5)$$

we find that the partition function is

$$Z(N) = \frac{1}{N!}\int\left(\prod_{i=1}^{N}D\lambda_{1,(i)}\right)\sum_{\sigma\in S_N}(-1)^\sigma\rho(\lambda_{1,(\sigma(i))},\lambda_{1,(i)}), \qquad (2.6)$$

where the density matrix $\rho(x,y)$ is given by

$$\rho(x,y) = \int\left(\prod_{a=2}^{M}D\lambda_a\right)\frac{1}{2\cosh\frac{x-\lambda_M}{2}}\left(\prod_{a=2}^{M-1}\frac{1}{2\cosh\frac{\lambda_{a+1}-\lambda_a}{2}}\right)\frac{1}{2\cosh\frac{\lambda_2-y}{2}}. \qquad (2.7)$$

If we introduce the grand potential $J(\mu)$ as

$$e^{J(\mu)} = 1 + \sum_{N=1}^{\infty}e^{\mu N}Z(N), \qquad (2.8)$$

the sum over the permutation in (2.6) simplifies into

$$J(\mu) = \operatorname{tr}\log(1 + e^\mu\rho). \qquad (2.9)$$

Here both the multiplication among ρ and the trace are performed with $D\lambda_1$, just as the multiplication within ρ (2.7) which is performed with $D\lambda_a$ ($a = 2,\cdots,M$). Introducing the Fourier transformation ($\lambda_{M+1} = \lambda_1$)

$$\frac{1}{2\cosh\frac{\lambda_{a+1}-\lambda_a}{2}} = \int\frac{d\Lambda_a}{2\pi}\frac{e^{\frac{i}{2\pi}(\lambda_{a+1}-\lambda_a)\Lambda_a}}{2\cosh\frac{\Lambda_a}{2}}, \qquad (2.10)$$

for all a, we find that the integration associated with λ_a in $\operatorname{tr}\rho^n$ is given by

$$\int\cdots\frac{d\Lambda_a}{2\pi}\frac{1}{2\cosh\frac{\Lambda_a}{2}}\frac{d\Lambda_{a-1}}{2\pi}\frac{1}{2\cosh\frac{\Lambda_{a-1}}{2}}\cdots\int\frac{d\lambda_a}{2\pi}\exp\left[\frac{ik_a\lambda_a^2}{4\pi} - \frac{i(\Lambda_a - \Lambda_{a-1})\lambda_a}{2\pi}\right]. \qquad (2.11)$$

If we introduce the coordinate variables $\Lambda_a = Q_a$ for $s_a = +1$ and the momentum variables $\Lambda_a = P_a$ for $s_a = -1$, we find that, up to an irrelevant numerical factor which will be cancelled out finally, this integration essentially gives that

$$\int\frac{d\lambda_a}{2\pi}\exp\left[\frac{ik_a\lambda_a^2}{4\pi} - \frac{i(\Lambda_a - \Lambda_{a-1})\lambda_a}{2\pi}\right] \simeq \langle\Lambda_a|\Lambda_{a-1}\rangle, \qquad (2.12)$$

because the inner products of the coordinate and momentum eigenstates are given by

$$\langle Q_a|Q_{a-1}\rangle = 2\pi\delta(Q_a - Q_{a-1}), \qquad \langle P_a|P_{a-1}\rangle = 2\pi\delta(P_a - P_{a-1}),$$
$$\langle Q_a|P_{a-1}\rangle = \frac{1}{\sqrt{k}}e^{iQ_aP_{a-1}/(2\pi k)}, \qquad \langle P_a|Q_{a-1}\rangle = \frac{1}{\sqrt{k}}e^{-iP_aQ_{a-1}/(2\pi k)}. \qquad (2.13)$$

Finally the integration in $\operatorname{tr}\rho^n$ is given by

$$\int \cdots \frac{d\Lambda_a}{2\pi}\frac{d\Lambda_{a-1}}{2\pi} \cdots \frac{1}{2\cosh\frac{\Lambda_a}{2}}\langle\Lambda_a|\Lambda_{a-1}\rangle\frac{1}{2\cosh\frac{\Lambda_{a-1}}{2}} \cdots . \tag{2.14}$$

This means that, if we define the position and momentum operator $\widehat{Q}, \widehat{P}$ obeying the canonical commutation relation

$$[\widehat{Q}, \widehat{P}] = i\hbar, \tag{2.15}$$

with $\hbar = 2\pi k$, the Hamiltonian $\widehat{H}(\widehat{Q}, \widehat{P})$ is given as (2.1) for the ordering (1.6) (see figure 1). Therefore, the grand potential $J(\mu)$ can be interpreted as the grand potential of the ideal Fermi gas system whose one-particle Hamiltonian $\widehat{H}$ is given by (2.1), where μ is the chemical potential dual to the number of particles.

3 Fermi surface analysis

In the previous section we have constructed the Fermi gas formalism for $\mathcal{N} = 4$ superconformal Chern-Simons theories by rewriting the partition function into that of non-interacting N-particle Fermi gas systems with non-trivial Hamiltonians (2.1).

Note that the Hamiltonian (2.1) is non-hermitian. In some particular cases, including the ABJM theory, however, we can choose it to be hermitian by redefining the Hamiltonian by

$$e^{-\widehat{H}} \to \left(2\cosh\frac{\widehat{Q}}{2}\right)^x e^{-\widehat{H}}\left(2\cosh\frac{\widehat{Q}}{2}\right)^{-x}, \tag{3.1}$$

with a real number x, which does not affect the trace. Below, we shall restrict ourselves to these cases.

It was argued in [15] that, for a large class of general $\mathcal{N} = 3$ superconformal circular quiver Chern-Simons theories associated to a hermitian Hamiltonian $\widehat{H}$ in the above sense, the number $n(E)$ of states whose eigenvalue of $\widehat{H}$ is smaller than E is universally given as

$$n(E) = CE^2 + n(0) + \text{non-pert}, \tag{3.2}$$

with C and $n(0)$ being constants depending on k and "non-pert" standing for non-perturbative corrections. From this form the authors showed that the perturbative part of the grand potential is given by a cubic polynomial

$$J_{\text{pert}}(\mu) = \frac{C}{3}\mu^3 + B\mu + A, \tag{3.3}$$

where the coefficient B is given by

$$B = n(0) + \frac{\pi^2 C}{3}. \tag{3.4}$$

However, the explicit forms of $n(0)$ and A for the general circular quivers were not known.

In this section we shall calculate $n(0)$ and C explicitly for the class of $\mathcal{N} = 4$ super-conformal circular quiver Chern-Simons theories, from the study of the Fermi surface as in [15]. The results are, for the quiver (1.6),

$$C = \frac{2}{\pi^2 k \Sigma(q) \Sigma(p)}, \tag{3.5}$$

$$n(0) = -\frac{1}{6k}\left(\frac{\Sigma(p)}{\Sigma(q)} + \frac{\Sigma(q)}{\Sigma(p)}\right) + kB^{(2)}, \tag{3.6}$$

where $B^{(2)}$ is defined in (1.8). Using (3.4) we can read off the expression of B (1.7) directly from this result.

3.1 The strategy

We follow the strategy of [15] in calculation. The concrete definition of the number of states $n(E)$ is

$$n(E) = \operatorname{tr} \theta(E - \widehat{H}). \tag{3.7}$$

If we introduce the Wigner transformation $\widehat{A} \to (\widehat{A})_{\mathrm{W}}$ with

$$(\widehat{A})_{\mathrm{W}} = \int \frac{dQ'}{2\pi} \left\langle Q - \frac{Q'}{2} \left| \widehat{A} \right| Q + \frac{Q'}{2} \right\rangle e^{\frac{iQ'P}{\hbar}}, \tag{3.8}$$

similarly to the case of ABJM theory [15], $n(E)$ is approximated by

$$n(E) \simeq \int \frac{dQ\,dP}{2\pi\hbar} \theta(E - H_{\mathrm{W}}), \tag{3.9}$$

up to non-perturbative corrections in E for large E. Here we have introduced the abbreviation $H_{\mathrm{W}} = (\widehat{H})_{\mathrm{W}}$. This means that, up to the non-perturbative corrections, $n(E)$ is given by the volume inside the Fermi surface of the semiclassical Wigner Hamiltonian,

$$n(E) \simeq \frac{1}{2\pi\hbar} \operatorname{vol}\{(Q,P) \in \mathbb{R}^2 | H_{\mathrm{W}}(Q,P) \leq E\}. \tag{3.10}$$

Here H_{W} is calculated from (2.1) by using the following property of the Wigner transformation

$$(\widehat{A}\widehat{B})_{\mathrm{W}} = (\widehat{A})_{\mathrm{W}} \star (\widehat{B})_{\mathrm{W}}, \tag{3.11}$$

with the star product given by

$$\star = \exp\left[\frac{i\hbar}{2}\left(\overleftarrow{\partial}_Q \overrightarrow{\partial}_P - \overleftarrow{\partial}_P \overrightarrow{\partial}_Q\right)\right], \tag{3.12}$$

which follows from the definition of the Wigner transformation (3.8).

Before going on, we shall argue some general properties of the Fermi surface (3.10). The Wigner Hamiltonian H_{W} obtained from the quantum Hamiltonian (2.1) is a sum of the classical part

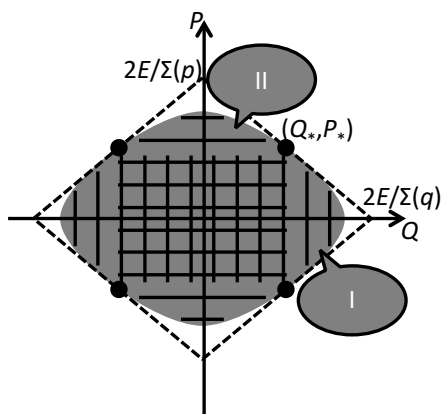$$H_{\mathrm{W}}^{(0)} = \Sigma(q)U + \Sigma(p)T, \tag{3.13}$$

Figure 2. The Fermi surface of the $\mathcal{N} = 4$ superconformal circular quiver Chern-Simons theory. We depict region I ($|P| \leq P_*$) by the region shaded by the vertical lines, while region II ($|Q| \leq Q_*$) denotes that shaded by the horizontal lines.

with

$$U(Q) = \log 2 \cosh \frac{Q}{2}, \quad T(P) = \log 2 \cosh \frac{P}{2}, \tag{3.14}$$

and $\hbar$ corrections which consist of derivatives of U and T. Also, from the behavior of $U(Q)$ and $T(P)$ in the limit of $|Q| \to \infty$ and $|P| \to \infty$,

$$U = \frac{|Q|}{2} + \mathcal{O}(e^{-|Q|}), \quad U' = \frac{\text{sgn}(Q)}{2} + \mathcal{O}(e^{-|Q|}), \quad U'' = \mathcal{O}(e^{-|Q|}),$$

$$T = \frac{|P|}{2} + \mathcal{O}(e^{-|P|}), \quad T' = \frac{\text{sgn}(P)}{2} + \mathcal{O}(e^{-|P|}), \quad T'' = \mathcal{O}(e^{-|P|}), \tag{3.15}$$

it follows that the Fermi surface is approaching to

$$\Sigma(q)|Q| + \Sigma(p)|P| = 2E, \tag{3.16}$$

as $E \to \infty$.

From this property, if we choose a point (Q_*, P_*) on the Fermi surface which is distant only by $\mathcal{O}(e^{-E})$ from the midpoint $(E/\Sigma(q), E/\Sigma(p))$ of the edge of (3.16), the total volume inside the Fermi surface is decomposed as

$$\text{vol} = \text{vol(I)} + \text{vol(II)} - 2Q_* \cdot 2P_*, \tag{3.17}$$

where region I denotes the $|P| \leq P_*$ part inside the Fermi surface while region II denotes the $|Q| \leq Q_*$ part. See figure 2.

3.2 Semiclassical Wigner Hamiltonian

Now let us start concrete calculations. The quantum Hamiltonian (2.1) is

$$e^{-\widehat{H}} = e^{-(q_1-x)\widehat{U}} e^{-p_1\widehat{T}} e^{-q_2\widehat{U}} e^{-p_2\widehat{T}} \cdots e^{-x\widehat{U}}, \tag{3.18}$$

where $\widehat{U} = U(\widehat{Q})$ and $\widehat{T} = T(\widehat{P})$. Here we have introduced a constant x deliberately, which does not change the trace of operators, to make $\widehat{H}$ hermitian. Let us compute this Hamiltonian using the Baker-Campbell-Hausdorff formula,

$$e^X e^Y = \exp\left[X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]] + \cdots \right]. \tag{3.19}$$

For the computation, we prepare the following formula which holds up to higher brackets[3]

$$e^{-q_1 \widehat{U}} e^{-p_1 \widehat{T}} e^{-q_2 \widehat{U}} e^{-p_2 \widehat{T}} \cdots e^{-q_{m+1} \widehat{U}}$$
$$= \exp\left[-\Sigma_{m+1}(q)\widehat{U} - \Sigma_m(p)\widehat{T} + \left(\Sigma_m(q,p) - \frac{1}{2}\Sigma_{m+1}(q)\Sigma_m(p) \right)[\widehat{U},\widehat{T}] \right.$$
$$- \frac{1}{2}\left(\Sigma_{m+1}(q,p,q) - \frac{1}{6}\Sigma_{m+1}(q)^2\Sigma_m(p) \right)[\widehat{U},[\widehat{T},\widehat{U}]]$$
$$\left. - \frac{1}{2}\left(\Sigma_m(p,q,p) - \frac{1}{6}\Sigma_{m+1}(q)\Sigma_m(p)^2 \right)[\widehat{T},[\widehat{U},\widehat{T}]] \right], \tag{3.22}$$

and substitute $q_1 - x$ into q_1 and x into q_{m+1}. Here we write explicitly the index m in the definition of $\Sigma(L)$ in (1.9) to avoid confusion. As we shall see below, higher brackets are irrelevant to the perturbative coefficients C and B.

We shall choose x to be

$$x = \frac{\Sigma_m(q,p)}{\Sigma_m(p)} - \frac{\Sigma_m(q)}{2}, \tag{3.23}$$

so that the coefficient of a non-hermitian operator $[\widehat{U},\widehat{T}]$ vanishes

$$\left. \Sigma_m(q,p) - \frac{1}{2}\Sigma_{m+1}(q)\Sigma_m(p) \right|_{\substack{q_1 \to q_1 - x \\ q_{m+1} \to x}} = \Sigma_m(q,p) - x\Sigma_m(p) - \frac{1}{2}\Sigma_m(q)\Sigma_m(p) = 0. \tag{3.24}$$

---

[3]One can prove the formula (3.22) by induction along with its "dual" formula

$$e^{-q_1 \widehat{U}} e^{-p_1 \widehat{T}} e^{-q_2 \widehat{U}} e^{-p_2 \widehat{T}} \cdots e^{-p_m \widehat{T}}$$
$$= \exp\left[-\Sigma_m(q)\widehat{U} - \Sigma_m(p)\widehat{T} + \left(\Sigma_m(q,p) - \frac{1}{2}\Sigma_m(q)\Sigma_m(p) \right)[\widehat{U},\widehat{T}] \right.$$
$$- \frac{1}{2}\left(\Sigma_m(q,p,q) - \frac{1}{6}\Sigma_m(q)^2\Sigma_m(p) \right)[\widehat{U},[\widehat{T},\widehat{U}]]$$
$$\left. - \frac{1}{2}\left(\Sigma_m(p,q,p) - \frac{1}{6}\Sigma_m(q)\Sigma_m(p)^2 \right)[\widehat{T},[\widehat{U},\widehat{T}]] \right], \tag{3.20}$$

up to higher brackets. Multiplying $e^{-q_{m+1}\widehat{U}}$ to this from the right and applying the Baker-Campbell-Hausdorff formula (3.19), one obtains the relation (3.22). Also, multiplying $e^{-p_{m+1}\widehat{T}}$ from the right further, and using an identity

$$\Sigma_m(q,p) + \Sigma_{m(+1)}(p,q) = \Sigma_{m(+1)}(q)\Sigma_m(p), \tag{3.21}$$

one obtains the relation (3.20) with m replaced with $m+1$. Combining with the fact that both of the formulae hold for $m = 1$, we complete the proof of both formulae by induction.

Then, the coefficients of $[\widehat{U}, [\widehat{T}, \widehat{U}]]$ and $[\widehat{T}, [\widehat{U}, \widehat{T}]]$ become

$$
\begin{aligned}
c^T &= -\frac{1}{2}\left(\Sigma_{m+1}(q,p,q) - \frac{1}{6}\Sigma_{m+1}(q)^2\Sigma_m(p)\right)\Bigg|_{\substack{q_1 \to q_1 - x \\ q_{m+1} \to x}} \\
&= -\frac{1}{2}\left(\Sigma_m(q,p,q) - x\Sigma_m(p,q) + x\Sigma_m(q,p) - x^2\Sigma_m(p) - \frac{1}{6}\Sigma_m(q)^2\Sigma_m(p)\right) \\
&= -\frac{1}{2}\left(\Sigma_m(q,p,q) + \frac{1}{12}\Sigma_m(q)^2\Sigma_m(p) - \frac{\Sigma_m(q,p)\Sigma_m(p,q)}{\Sigma_m(p)}\right),
\end{aligned}
\tag{3.25}
$$

and

$$
\begin{aligned}
c^U &= -\frac{1}{2}\left(\Sigma_m(p,q,p) - \frac{1}{6}\Sigma_{m+1}(q)\Sigma_m(p)^2\right)\Bigg|_{\substack{q_1 \to q_1 - x \\ q_{m+1} \to x}} \\
&= -\frac{1}{2}\left(\Sigma_m(p,q,p) - \frac{1}{6}\Sigma_m(q)\Sigma_m(p)^2\right),
\end{aligned}
\tag{3.26}
$$

where we have used (3.21) in the computation.

The Wigner Hamiltonian H_{W} is obtained by replacing the operators $\widehat{U}$, $\widehat{T}$ with the functions U, T and the operator product with the $\star$-product. Then, we find that the $\hbar$-expansion of the Wigner Hamiltonian

$$
H_{\mathrm{W}} = \sum_{s=0}^{\infty} \hbar^s H_{\mathrm{W}}^{(s)},
\tag{3.27}
$$

is given by $H_{\mathrm{W}}^{(0)}$ in (3.13) and

$$
H_{\mathrm{W}}^{(2)} = -c^T(U')^2 T'' - c^U(T')^2 U'',
\tag{3.28}
$$

up to higher order terms. The higher order terms in $\hbar$ in (3.27) comes from both higher brackets and higher derivatives from the $\star$-products. General form of such terms is

$$
\sum_{n \geq 3}\left[c_n^T (U')^n T^{(n)} + c_n^U (T')^n U^{(n)}\right] + \sum_{m,n \geq 2} U^{(m)} T^{(n)}(\cdots),
\tag{3.29}
$$

with c_n^T and c_n^U being some constants. Since (Q,P) on the Fermi surface always satisfies either $|Q| \geq Q_*$ or $|P| \geq P_*$, the third terms are always non-perturbative according to the asymptotic behavior of U and T in (3.15). As we see below, the first two terms do not affect the volume (3.17) up to non-perturbative corrections either.

3.3 Volume inside the Fermi surface

Now that the Wigner Hamiltonian with quantum corrections is obtained to the required order, let us calculate the volume inside the Fermi surface (3.10), following the decomposition (3.17). First we consider the region I. Since $|Q| \geq Q_* \sim E$ holds for the parts of the Fermi surface surrounding this region, we can use the approximation (3.15) for U. Then the points on the Fermi surface $H_{\mathrm{W}} = E$ are parametrized as $(Q_\pm(P), P)$ with

$$
Q_\pm(P) = \pm\frac{2}{\Sigma_m(q)}\left[E - \Sigma_m(p)T + \frac{\hbar^2}{4}c^T T'' - \sum_{n \geq 3}\left(\pm\frac{1}{2}\right)^n c_n^T T^{(n)}\right] + \text{non-pert},
\tag{3.30}
$$

with which the volume of region I is

$$\text{vol(I)} = \int_{-P_*}^{P_*} dP \int_{Q_-(P)}^{Q_+(P)} dQ$$
$$= \frac{4}{\Sigma_m(q)} \left[2EP_* - \Sigma_m(p) \left(\frac{P_*^2}{2} + \frac{\pi^2}{6} \right) + \frac{\hbar^2}{4} c^T \right] + \text{non-pert}. \qquad (3.31)$$

The contribution from $T^{(n)}$ with $n \geq 3$ is the surface term $T^{(n-1)}$, which just gives non-perturbative effects due to (3.15) when evaluated at $P = \pm P_*$. Similarly, the volume of region II is evaluated, using the approximation (3.15) for $T(P)$, as

$$\text{vol(II)} = \frac{4}{\Sigma_m(p)} \left[2EQ_* - \Sigma_m(q) \left(\frac{Q_*^2}{2} + \frac{\pi^2}{6} \right) + \frac{\hbar^2}{4} c^U \right] + \text{non-pert}. \qquad (3.32)$$

Summing up all the contributions to (3.17), one obtains the total volume. After substituting the volume into (3.10), the number of states $n(E)$ is written as (3.2), with C and $n(0)$ given by (3.5) and (3.6).

4 Repetition invariance

As explained in (1.5), it was found in [55] that, if the circular quiver is the r-ple repetition of another fundamental circular quiver, the coefficients C, B and A of the repetitive theory are related to those of the fundamental theory. This implies that the quantity $n(0)$ (3.4) is invariant under repetition,

$$[n(0)]_r = [n(0)]_1. \qquad (4.1)$$

In this section we show this property explicitly for the result (3.6) we have obtained in the previous section for general $\mathcal{N} = 4$ circular quivers.

Suppose that the circular quiver (1.6) is the r-ple repetition of a fundamental circular quiver ($M = r\tilde{M}$, $m = r\tilde{m}$)

$$\{s_a\}_{a=1}^{\tilde{M}} = \{(+1)^{\tilde{q}_1}, (-1)^{\tilde{p}_1}, \cdots, (+1)^{\tilde{q}_{\tilde{m}}}, (-1)^{\tilde{p}_{\tilde{m}}}\}. \qquad (4.2)$$

To study how $n(0)$ changes under the repetition, let us first consider its building block $\Sigma_m(L)$ defined in (1.9). For this purpose, we shall decompose the label a of q_a and p_a into two integers $(\alpha, \tilde{a})$ by

$$a = (\alpha - 1)\tilde{m} + \tilde{a}, \qquad (4.3)$$

with $1 \leq \alpha \leq r$ and $1 \leq \tilde{a} \leq \tilde{m}$, which implies

$$q_a = \tilde{q}_{\tilde{a}}, \quad p_a = \tilde{p}_{\tilde{a}}, \qquad (4.4)$$

Then we find that the relation $a < b$ (or $a \leq b$) appearing in the summation in (1.9) is represented as

$$\text{``}\alpha < \beta\text{''}, \qquad \text{or} \qquad \text{``}\alpha = \beta \quad \text{and} \quad \tilde{a} < \tilde{b} \quad (\text{or} \quad \tilde{a} \leq \tilde{b})\text{''}, \qquad (4.5)$$

if we decompose a and b into $(\alpha, \tilde{a})$ and $(\beta, \tilde{b})$ respectively. This means that we can decompose $\Sigma_m(L)$ for the repetitive quiver into the products of $\Sigma_{\tilde{m}}(L_i)$ for the fundamental ones with different α,

$$\Sigma_m(L) = \sum_{s=1}^{r} \sum_{L_1, L_2, \cdots, L_s} F_s(r) \prod_{i=1}^{s} \Sigma_{\tilde{m}}(L_i), \tag{4.6}$$

with a combinatorial factor $F_s(r)$. Here the sum is taken over all possible partitions of L, $L = L_1 L_2 \cdots L_s$. The combinatorial factor $F_s(r)$ is given by counting possible combinations of $\{\alpha_i\}_{i=1}^{s}$ satisfying the inequality $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_s \leq r$,

$$F_s(r) = \#\{(\alpha_1, \cdots, \alpha_t) | 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_s \leq r\} = \binom{r}{s}. \tag{4.7}$$

For example, the condition $1 \leq a \leq b < c \leq m$ in defining $\Sigma_m(q, p, q)$ (1.9) is decomposed as

$$(1, 1) \leq (\alpha, \tilde{a}) \leq (\beta, \tilde{b}) < (\gamma, \tilde{c}) \leq (r, \tilde{m}), \tag{4.8}$$

where the inequalities are understood in the sense of (4.5). This implies that $\Sigma_m(q, p, q)$ can be decomposed into $\Sigma_{\tilde{m}}(q, p, q)$, $\Sigma_{\tilde{m}}(q, p)\Sigma_{\tilde{m}}(q)$, $\Sigma_{\tilde{m}}(q)\Sigma_{\tilde{m}}(p, q)$ or $\Sigma_{\tilde{m}}(q)^2\Sigma_{\tilde{m}}(p)$ respectively when $\alpha = \beta = \gamma$, $\alpha = \beta < \gamma$, $\alpha < \beta = \gamma$ or $\alpha < \beta < \gamma$. The combinatorial factor of decomposing $\Sigma_m(q, p, q)$ into $\Sigma_{\tilde{m}}(q, p)\Sigma_{\tilde{m}}(q)$ is computed by choosing two different elements $\alpha = \beta$ and γ out of $\{1, 2, \cdots, r\}$. In this way, we find several formulae

$$\Sigma_m(q) = \binom{r}{1} \Sigma_{\tilde{m}}(q),$$

$$\Sigma_m(q, p) = \binom{r}{1} \Sigma_{\tilde{m}}(q, p) + \binom{r}{2} \Sigma_{\tilde{m}}(q) \Sigma_{\tilde{m}}(p), \tag{4.9}$$

$$\Sigma_m(q, p, q) = \binom{r}{1} \Sigma_{\tilde{m}}(q, p, q) + \binom{r}{2} (\Sigma_{\tilde{m}}(q) \Sigma_{\tilde{m}}(p, q) + \Sigma_{\tilde{m}}(q, p) \Sigma_{\tilde{m}}(q)) + \binom{r}{3} \Sigma_{\tilde{m}}(q)^2 \Sigma_{\tilde{m}}(p),$$

as well as those with the role of q and p switched. With these relations and (3.21), one can prove that $n(0)$ in (3.6) satisfies

$$[n(0)]_r = [n(0)]_1. \tag{4.10}$$

5 A preliminary study on $\mathcal{N} = 3$ quivers

Having obtained the expression of the coefficient B for the $\mathcal{N} = 4$ superconformal circular quiver Chern-Simons theories in section 3 and checked the repetition invariance in section 4, in this section we shall make a digression to comment on possible generalization of the analysis to the $\mathcal{N} = 3$ cases. It was already shown in [15] that the partition function of $\mathcal{N} = 3$ Chern-Simons matrix models can also be rewritten into that of a Fermi gas system and the sum of the perturbative terms is given by the Airy function (1.1). Here the one-particle Hamiltonian of the Fermi gas system is given as

$$e^{-\widehat{H}} = e^{-\widehat{U}_1} e^{-\widehat{U}_2} \cdots e^{-\widehat{U}_M}, \tag{5.1}$$

with $\widehat{U}_a$ defined by

$$\widehat{U}_a = \log 2 \cosh \frac{\widehat{P} - \nu_a \widehat{Q}}{2}, \tag{5.2}$$

and ν_a given by the Chern-Simons levels $k_a = k n_a$ as

$$\nu_a = \sum_{b=1}^{a} n_b, \tag{5.3}$$

which implies $\nu_M = 0$. In this section, we shall apply the analysis in section 3 to this theory and calculate $n(E)$ at the most leading part in $\hbar$ expansion. Since the Hamiltonian (5.1) is symmetric under exchange among ν_a's at the most leading order, for the later convenience, let us replace ν_a with $\nu_{\sigma(a)}$ so that the new ν_a satisfies

$$\nu_a \le \nu_{a+1}, \tag{5.4}$$

for all a. The conclusion is that the coefficients C and B in the Airy function (1.1) are given by[4]

$$C = \frac{2}{\pi \hbar} \sum_{a=1}^{M} \frac{|\nu_{a+1} - \nu_a|}{\sum_{b=1}^{M} |\nu_{a+1} - \nu_b| \sum_{c=1}^{M} |\nu_a - \nu_c|} + \mathcal{O}(\hbar), \tag{5.5}$$

$$B = \frac{2\pi}{3\hbar} \sum_{a=1}^{M} \frac{|\nu_{a+1} - \nu_a|}{\sum_{b=1}^{M} |\nu_{a+1} - \nu_b| \sum_{c=1}^{M} |\nu_a - \nu_c|} - \frac{\pi}{3\hbar} \sum_{a=1}^{M} \frac{1}{\sum_{b=1}^{M} |\nu_b - \nu_a|} + \mathcal{O}(\hbar), \tag{5.6}$$

with $\nu_{M+1} = \nu_1$

The idea of calculation is similar to the one used in section 3 and [15]. At this order, the Wigner Hamiltonian is given as the classical one

$$H_{\mathrm{W}}^{(0)} = \sum_{a=1}^{M} U_a, \tag{5.7}$$

with

$$U_a = \log 2 \cosh \frac{P - \nu_a Q}{2}. \tag{5.8}$$

To obtain the total volume inside the Fermi surface

$$n(E) = \frac{1}{2\pi \hbar} \mathrm{vol}\left\{ (Q, P) \Big| \sum_{i=a}^{M} \log 2 \cosh \frac{P - \nu_a Q}{2} \le E \right\} + \mathcal{O}(\hbar), \tag{5.9}$$

below we consider its deviation from the volume inside the convex $2M$-gon[5]

$$\frac{1}{2\pi \hbar} \mathrm{vol}\left\{ (Q, P) \Big| \sum_{a=1}^{M} |P - \nu_a Q| \le 2E \right\}, \tag{5.10}$$
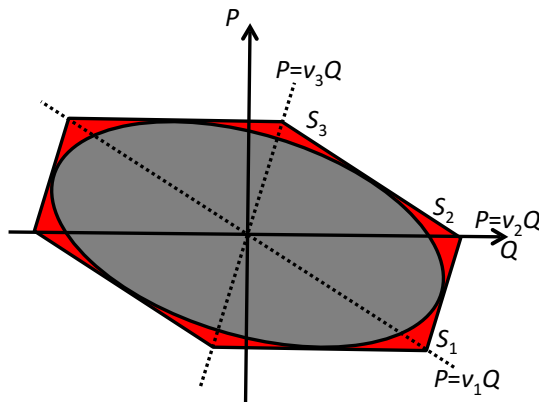
– 15 –

Figure 3. The Fermi surface of the $\mathcal{N} = 3$ superconformal circular quiver Chern-Simons theory. The outer polygon is the limiting convex $2M$-gon (5.10) and the inner closed curve is the Fermi surface.

where the Fermi surface (5.9) is approaching in the limit $E \to \infty$ as in section 3. (See figure 3.)

Now let us calculate the volume of the deviation, the red region in figure 3. Since both $H_{\mathrm{W}}(Q, P)$ and the polygon are invariant under $(Q, P) \to (-Q, -P)$, we can restrict ourselves to $Q > 0$. Hereafter, we shall denote as S_a the region around the vertex with $P - \nu_a Q = 0$ and $Q > 0$, surrounded by the curve $H_{\mathrm{W}} = E$ and the two edges of the polygon ending on this vertex. Since S_a is distant at order E from the lines $P - \nu_b Q = 0$ with $b \neq a$, on S_a the Hamiltonian can be approximated up to non-perturbative corrections in E as

$$H_{\mathrm{W}} \simeq H_{W,a} = \sum_{b(\neq a)}^{M} \frac{|P - \nu_b Q|}{2} + \log 2 \cosh \frac{P - \nu_a Q}{2}. \tag{5.11}$$

There are further simplification of calculation due to the invariance of the volume under an affine transformation $(Q, P) \to (Q, P - \nu_a Q)$ on each S_a. See figure 4. After this affine transformation, if we denote the points on the edge of the polygon as $(Q(P), P)$ and those on the Fermi surface $H_{W,a} = E$ as $(Q'(P), P)$, we find that

$$Q(P) - Q'(P) = \frac{1}{\sum_{b=1}^{M} |\nu_b - \nu_a|} \left(2 \log 2 \cosh \frac{P}{2} - |P| \right) + \text{non-pert}. \tag{5.12}$$

Therefore the volume of the region S_a is

$$\mathrm{vol}(S_a) = \int_{P_{a-}}^{P_{a+}} dP \frac{1}{\sum_{b=1}^{M} |\nu_b - \nu_a|} \left(2 \log 2 \cosh \frac{P}{2} - |P| \right) + \text{non-pert}. \tag{5.13}$$

---

[4]Although we neglect the $\mathcal{O}(\hbar)$ corrections to B and C in our analysis in this section, the formula for C (5.5) is actually correct without any $\hbar$ corrections, as already known in [15] (see also [44]).

[5]For simplicity, we assume the generic case $\nu_a \neq \nu_b$ ($a \neq b$) in the following argument, though we can justify the final results (5.5) and (5.6).
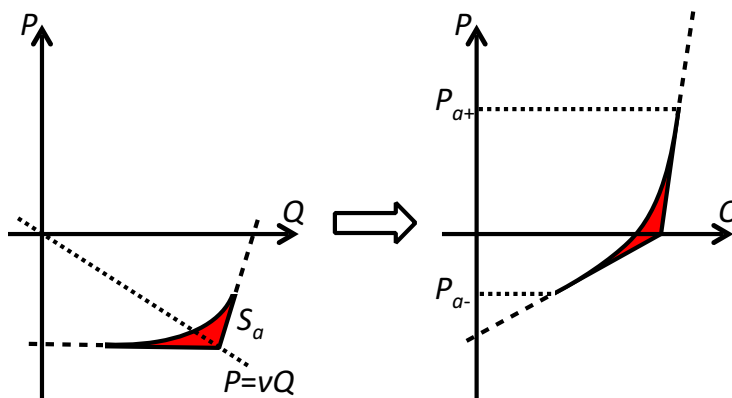
Figure 4. The left figure shows the region S_a, the part of the colored region, and the right figure shows its affine transformation.

Here we have denoted by $P_\pm$ the value of the P-coordinate at the midpoints of the currently considered edges of the polygon, where the Fermi surface and the edge of polygon coalesce up to $\mathcal{O}(e^{-E})$. Since the integrand is $\mathcal{O}(e^{-E})$ at $|P| \sim E$, one can extend the domain of integration to $(-\infty, \infty)$ and obtains

$$\mathrm{vol}(S_a) = \frac{\pi^2}{3 \sum_{b=1}^{M} |\nu_b - \nu_a|} + \text{non-pert}. \tag{5.14}$$

Subtracting them from the volume inside the polygon, one finally obtains

$$n(E) = \frac{1}{2\pi\hbar} \left(4E^2 \sum_{a=1}^{M} \frac{|\nu_{a+1} - \nu_a|}{\sum_{b=1}^{M} |\nu_{a+1} - \nu_b| \sum_{c=1}^{M} |\nu_a - \nu_c|} - \sum_{a=1}^{M} \frac{2\pi^2}{3 \sum_{b=1}^{M} |\nu_b - \nu_a|} \right)$$
$$+ \mathcal{O}(\hbar) + \text{non-pert}. \tag{5.15}$$

If one choose the Chern-Simons levels as (1.4) so that the supersymmetry enhances to $\mathcal{N} = 4$, the values of ν_a's (before rearranged as (5.4)) are

$$\{\nu_a\}_{a=1}^{M} = \{(+1)^{q_1}, (0)^{p_1}, (+1)^{q_2}, (0)^{p_2}, \cdots, (+1)^{q_r}, (0)^{p_r}\}, \tag{5.16}$$

and the classical limit of the results for $\mathcal{N} = 4$ theories (3.5) and (3.6) are recovered.

Note that, although the hermiticity of the Hamiltonian is crucial in discussing the physical Fermi surface in section 3, in the $\mathcal{N} = 3$ cases the trick making Hamiltonian hermitian by unitary transformation works only for very restricted cases. We hope, however, to extend our results on $\mathcal{N} = 3$ to higher corrections in $\hbar$ by, for example, the method in section 6 in future works.

6 WKB expansion of grand potential

In this section, we shall calculate the grand potential $J(\mu)$ at the first few leading orders in $\hbar$ including the non-perturbative term in μ. We find that all these computations are consistent with our perturbative result of the coefficient B obtained in section 3. Besides,

we have obtained several new insights on the coefficient A and the non-perturbative terms, which enable us to conjecture the expression of the coefficient A (1.11) for the case when the edges of $s_a = +1$ and those of $s_a = -1$ are separated, and the expression of the first membrane instanton (1.12) for the case of $\{s_a\}_{a=1}^3 = \{(+1)^2, (-1)\}$.

Again, the computation is parallel to [15]. We write $J(\mu)$ as

$$J(\mu) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{\mu n} \int \frac{dQ dP}{2\pi \hbar} (e^{-n\widehat{H}})_{\mathrm{W}}, \tag{6.1}$$

and evaluate the integral for each n by expanding $(e^{-n\widehat{H}})_{\mathrm{W}}$ order by order in $\hbar$. Then, we can substitute the results back to $J(\mu)$ and resum the series to obtain the $\hbar$-expansion,

$$J(\mu) = \sum_{s=0}^{\infty} \hbar^{s-1} J^{(s)}(\mu). \tag{6.2}$$

As stressed in [15], there are two kinds of $\hbar$ corrections to $(e^{-n\widehat{H}})_{\mathrm{W}}$. One is from the correction to H_{W} itself from $H_{\mathrm{W}}^{(0)}$, which is partly discussed in section 3. The other comes from the fact that $H_{\mathrm{W}} \star H_{\mathrm{W}} \neq H_{\mathrm{W}}^2$. Keeping in mind the decomposition $e^{-n\widehat{H}} = e^{-nH_{\mathrm{W}}} e^{-n(\widehat{H} - H_{\mathrm{W}})}$, the latter contributions can be systematically treated by introducing

$$\mathcal{G}_t = ((\widehat{H} - H_{\mathrm{W}})^t)_{\mathrm{W}}. \tag{6.3}$$

The first few non-trivial examples of $\mathcal{G}_t$ are given by

$$\mathcal{G}_2 = H_{\mathrm{W}} \star H_{\mathrm{W}} - H_{\mathrm{W}}^2, \quad \mathcal{G}_3 = H_{\mathrm{W}} \star H_{\mathrm{W}} \star H_{\mathrm{W}} - 3H_{\mathrm{W}}(H_{\mathrm{W}} \star H_{\mathrm{W}}) + 2H_{\mathrm{W}}^3, \quad \cdots. \tag{6.4}$$

It was shown in [15] that, apart from $\hbar$ corrections to H_{W} itself, the $\hbar$ expansion of $\mathcal{G}_t$ is

$$\mathcal{G}_t = \sum_{s=2\left[\frac{t+2}{3}\right]}^{\infty} \hbar^s \mathcal{G}_t^{(s)}, \tag{6.5}$$

with $\mathcal{G}_t^{(s)} = 0$ for any odd s. With these contributions, $(e^{-n\widehat{H}})_{\mathrm{W}}$ is written as

$$\left(e^{-n\widehat{H}}\right)_{\mathrm{W}} = e^{-nH_{\mathrm{W}}^{(0)}} \exp\left[-n \sum_{s=2}^{\infty} \hbar^s H_{\mathrm{W}}^{(s)}\right] \times \left(1 + \sum_{t=2}^{\infty} \frac{(-n)^t}{t!} \mathcal{G}_t\right), \tag{6.6}$$

expanding the second and third factor, one obtains the parts which contribute to each $J^{(s)}(\mu)$.

Below we perform these studies for $J^{(0)}(\mu)$ and $J^{(2)}(\mu)$. Then restricting to the class of separative quivers, that is, $\{s_a\}_{a=1}^M = \{(+1)^q, (-1)^p\}$, we calculate $J^{(4)}(\mu)$. Note that $J^{(s)}(\mu)$ vanishes for any odd s since the integrand is always an odd function with respect to Q or P at this order. In our computation the following quantity appears frequently,

$$\mathcal{F}(a, \alpha, b, \beta, \mu) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{\mu n} \int \frac{dQ dP}{2\pi} \frac{1}{\left(2 \cosh \frac{Q}{2}\right)^{an+\alpha}} \frac{1}{\left(2 \cosh \frac{P}{2}\right)^{bn+\beta}}. \tag{6.7}$$

This quantity can be computed by integrating each term with the formula

$$\int_{-\infty}^{\infty} dx \frac{1}{\left(2\cosh\frac{x}{2}\right)^n} = \frac{\sqrt{4\pi}}{2^n} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}, \tag{6.8}$$

and using the multiplication theorem of the gamma function

$$\Gamma(mx) = \frac{m^{mx}}{\sqrt{(2\pi)^{m-1}m}} \prod_{i=0}^{m-1} \Gamma\left(x + \frac{i}{m}\right), \tag{6.9}$$

for $m \in \mathbb{N}$ and $x \in \mathbb{R}$, so that we can use the Pochhammer's generalized hypergeometric function

$$_pF_q(a_1, \cdots a_p; b_1, \cdots b_q; z) = \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{i=1}^p \Gamma(a_i)} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p \Gamma(a_i+n)}{\prod_{j=1}^q \Gamma(b_j+n)} \frac{z^n}{n!}. \tag{6.10}$$

Then we find that this function can be expressed as

$$\mathcal{F}(a,\alpha,b,\beta,\mu) = -\frac{1}{2^{\alpha+\beta}\sqrt{ab}} \left[\left(\frac{-e^\mu}{2^{a+b}}\right)^2 \frac{\Gamma(1)}{\Gamma(2)} \prod_{i=0}^{a-1} \frac{\Gamma(1+\frac{\alpha}{2a}+\frac{i}{a})}{\Gamma(1+\frac{\alpha+1}{2a}+\frac{i}{a})} \prod_{j=0}^{b-1} \frac{\Gamma(1+\frac{\beta}{2b}+\frac{j}{b})}{\Gamma(1+\frac{\beta+1}{2b}+\frac{j}{b})} \right.$$

$$\times {}_{a+b+2}F_{a+b+1}\left(\left\{1+\frac{\alpha}{2a}+\frac{i}{a}\right\}_{i=0}^{a-1}, \left\{1+\frac{\beta}{2b}+\frac{j}{b}\right\}_{j=0}^{b-1}, 1, 1; \right.$$

$$\left. \left\{1+\frac{\alpha+1}{2a}+\frac{i}{a}\right\}_{i=0}^{a-1}, \left\{1+\frac{\beta+1}{2b}+\frac{j}{b}\right\}_{j=0}^{b-1}, 2; \left(\frac{-e^\mu}{2^{a+b}}\right)^2 \right)$$

$$-\frac{e^\mu}{2^{a+b}} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})} \prod_{i=0}^{a-1} \frac{\Gamma(\frac{1}{2}+\frac{\alpha}{2a}+\frac{i}{a})}{\Gamma(\frac{1}{2}+\frac{\alpha+1}{2a}+\frac{i}{a})} \prod_{j=0}^{b-1} \frac{\Gamma(\frac{1}{2}+\frac{\beta}{2b}+\frac{j}{b})}{\Gamma(\frac{1}{2}+\frac{\beta+1}{2b}+\frac{j}{b})}$$

$$\times {}_{a+b+2}F_{a+b+1}\left(\left\{\frac{1}{2}+\frac{\alpha}{2a}+\frac{i}{a}\right\}_{i=0}^{a-1}, \left\{\frac{1}{2}+\frac{\beta}{2b}+\frac{j}{b}\right\}_{j=0}^{b-1}, \frac{1}{2}, 1; \right.$$

$$\left. \left. \left\{\frac{1}{2}+\frac{\alpha+1}{2a}+\frac{i}{a}\right\}_{i=0}^{a-1}, \left\{\frac{1}{2}+\frac{\beta+1}{2b}+\frac{j}{b}\right\}_{j=0}^{b-1}, \frac{3}{2}; \left(\frac{-e^\mu}{2^{a+b}}\right)^2 \right) \right]. \tag{6.11}$$

In the following three subsections, we shall first compute the grand potential order by order in $\hbar$ and express the final result using the function $\mathcal{F}(a,\alpha,b,\beta,\mu)$. Then, we choose several specific types of quivers $\{s_a\}_{a=1}^M$ to study the grand potentials in the large μ expansion and guess the general behavior of the perturbative and non-perturbative parts, $J^{(s)}(\mu) = J^{(s)}_{\text{pert}}(\mu) + J^{(s)}_{\text{np}}(\mu)$.

6.1 $J^{(0)}(\mu)$

First we consider the most leading part, $J^{(0)}(\mu)$. At this order, $(e^{-n\widehat{H}})_{\text{W}}$ is simply $e^{-nH_{\text{W}}^{(0)}}$. Since $H_{\text{W}}^{(0)}$ is given as (3.13), the quantity is nothing but the one computed previously[6]

$$J^{(0)}(\mu) = \mathcal{F}(\Sigma(q), 0, \Sigma(p), 0, \mu). \tag{6.12}$$

---

[6]For the r-ple repetition of the ABJM quiver, this result was also obtained by Masazumi Honda by similar techniques (private note).

Studying the asymptotic behavior of $J^{(0)}(\mu)$ at $\mu \to \infty$ for $1 \leq \Sigma(q) \leq 4$ and $1 \leq \Sigma(p) \leq 4$, we have found that the perturbative part coincides with the following expression

$$J_{\mathrm{pert}}^{(0)}(\mu) = \frac{4\mu^3}{3\pi\Sigma(q)\Sigma(p)} + \left[\frac{4\pi}{3\Sigma(q)\Sigma(p)} - \frac{\pi}{3}\left(\frac{\Sigma(p)}{\Sigma(q)} + \frac{\Sigma(q)}{\Sigma(p)}\right)\right]\mu$$
$$+ \frac{2\zeta(3)}{\pi}\left(\frac{\Sigma(p)^2}{\Sigma(q)} + \frac{\Sigma(q)^2}{\Sigma(p)}\right). \tag{6.13}$$

The μ dependent part is consistent with the results obtained in section 3. We have also found that, the non-perturbative corrections consist of terms proportional to

$$\exp\left[-\frac{2n\mu}{\Sigma(q)}\right], \quad \text{or} \quad \exp\left[-\frac{2n\mu}{\Sigma(p)}\right], \tag{6.14}$$

with $n \geq 1$ but not their bound states. For example, for $\Sigma(q) = 1, \Sigma(p) = 2$ we obtain

$$J_{\mathrm{np}}^{(0)}(\mu) = -8e^{-\mu} + \left[-\frac{12\mu^2 - 28\mu - 28}{\pi} + \pi\right]e^{-2\mu} + \mathcal{O}(e^{-3\mu}), \tag{6.15}$$

while for $\Sigma(q) = 2, \Sigma(p) = 3$ we find

$$J_{\mathrm{np}}^{(0)}(\mu) = -\frac{160\pi^2}{9\sqrt{3}\,\Gamma\left(-\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{8}{3}\right)}e^{-\frac{2\mu}{3}} - 64e^{-\mu} + \frac{9\cdot 2^{\frac{2}{3}}\pi^{\frac{3}{2}}}{\Gamma\left(-\frac{5}{3}\right)\Gamma\left(\frac{7}{6}\right)}e^{-\frac{4\mu}{3}} + \mathcal{O}(e^{-2\mu}), \tag{6.16}$$

without e.g. the bound state $e^{-\frac{5}{3}\mu}$ of $e^{-\frac{2}{3}\mu}$ and $e^{-\mu}$.

6.2 $J^{(2)}(\mu)$

Collecting the relevant terms in the expansion of $(e^{-n\widehat{H}})_{\mathrm{W}}$ (6.6), $J^{(2)}(\mu)$ is given as

$$J^{(2)}(\mu) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}e^{\mu n} \int \frac{dQ dP}{2\pi} e^{-nH_{\mathrm{W}}^{(0)}}\left[-nH_{\mathrm{W}}^{(2)} + \frac{n^2}{2}\mathcal{G}_2^{(2)}(H_{\mathrm{W}}^{(0)}) - \frac{n^3}{6}\mathcal{G}_3^{(2)}(H_{\mathrm{W}}^{(0)})\right], \tag{6.17}$$

where $\mathcal{G}_t^{(s)}$ is defined by (6.4) and (6.5), whose several relevant terms are given explicitly by

$$\mathcal{G}_2^{(2)}(H_{\mathrm{W}}^{(0)}) = -\frac{1}{4}\Sigma(q)\Sigma(p)U''T'',$$
$$\mathcal{G}_3^{(2)}(H_{\mathrm{W}}^{(0)}) = -\frac{\Sigma(q)^2\Sigma(p)}{4}(U')^2T'' - \frac{\Sigma(q)\Sigma(p)^2}{4}(T')^2U''. \tag{6.18}$$

Using the integration by parts

$$\int dQ\, e^{-nH_{\mathrm{W}}^{(0)}}U'g(Q,P) = \int dQ\, e^{-nH_{\mathrm{W}}^{(0)}}\frac{1}{n\Sigma(q)}\frac{\partial g}{\partial Q},$$
$$\int dP\, e^{-nH_{\mathrm{W}}^{(0)}}T'g(Q,P) = \int dP\, e^{-nH_{\mathrm{W}}^{(0)}}\frac{1}{n\Sigma(p)}\frac{\partial g}{\partial P}, \tag{6.19}$$

for an arbitrary function $g(Q,P)$, one can replace

$$(U')^2 \to \frac{1}{n\Sigma(q)}U'', \quad (T')^2 \to \frac{1}{n\Sigma(p)}T'', \tag{6.20}$$

in the integrand in (6.17). After these replacements, we can use our formula (6.11) directly to obtain

$$J^{(2)}(\mu) = \left[B^{(2)} - \frac{1}{24} \Sigma(q) \Sigma(p) \partial_\mu^2 \right] \mathcal{F}(\Sigma(q), 2, \Sigma(p), 2, \mu). \tag{6.21}$$

Again we calculate the asymptotic behavior of $J^{(2)}(\mu)$ for $1 \le \Sigma(q) \le 4$, $1 \le \Sigma(p) \le 4$ and obtain the perturbative parts expressed as

$$J^{(2)}_{\text{pert}}(\mu) = \frac{B^{(2)}\mu}{2\pi} - \frac{B^{(2)}(\Sigma(q) + \Sigma(p))}{2\pi}, \tag{6.22}$$

where the term proportional to μ is consistent with the result obtained in section 3. We have also found that each term in the non-perturbative part exhibits the same behavior (6.14) as those in $J^{(0)}(\mu)$. For $\{s_a\}_{a=1}^3 = \{(+1)^2, (-1)\}$, for example, we find that

$$J^{(2)}_{\text{np}}(\mu) = \frac{1}{6} e^{-\mu} + \left[\frac{\mu^2 - 11\mu/3 - 1/2}{2\pi} - \frac{\pi}{24} \right] e^{-2\mu} + \mathcal{O}(e^{-3\mu}). \tag{6.23}$$

Remarkably, the exponents appearing in this expression depends only on $(\Sigma(q), \Sigma(p))$, not on the ordering of $\{s_a\}_{a=1}^M$. For example, for $\{s_a\}_{a=1}^4 = \{(+1)^2, (-1)^2\}$, we find that

$$J^{(2)}_{\text{np}}(\mu) = \left[\frac{2\mu + 1}{3\pi} \right] e^{-\mu} + \left[\frac{\mu^2 - 17\mu/3 + 7/6}{\pi} - \frac{\pi}{3} \right] e^{-2\mu} + \mathcal{O}(e^{-3\mu}), \tag{6.24}$$

while, for $\{s_a\}_{a=1}^4 = \{(+1), (-1), (+1), (-1)\}$, we find

$$J^{(2)}_{\text{np}}(\mu) = \left[-\frac{\mu^2 - 10\mu/3 - 2/3}{8\pi} + \frac{\pi}{24} \right] e^{-\mu} + \left[\frac{5\mu^2 - 77\mu/3 + 7/6}{4\pi} - \frac{5\pi}{12} \right] e^{-2\mu} + \mathcal{O}(e^{-3\mu}). \tag{6.25}$$

Both of these last two examples share the same instanton exponents with different polynomial coefficients.

6.3 $J^{(4)}(\mu)$ for separative models

The terms in (6.6) which are relevant to $J^{(4)}(\mu)$ are $H_{\text{W}}^{(2)}$, $H_{\text{W}}^{(4)}$ and $\mathcal{G}_t$ with $2 \le t \le 6$. Here we shall restrict ourselves to the case $m = 1$, that is, $\{s_a\}_{a=1}^M = \{(+1)^q, (-1)^p\}$, since $H_{\text{W}}^{(4)}$ for general circular quivers is still obscure. In this case $H_{\text{W}}^{(4)}$ is given as

$$H_{\text{W}}^{(4)} = \frac{qp^2}{144} T' T^{(3)} U^{(4)} - \frac{q^2 p}{288} U' U^{(3)} T^{(4)} - \frac{q^3 p^2}{240} (U')^2 U'' (T'')^2 + \frac{q^2 p^3}{60} (T')^2 T'' (U'')^2$$
$$- \frac{q^3 p^2}{80} (U')^2 U'' T' T^{(3)} + \frac{q^2 p^3}{120} (T')^2 T'' U' U^{(3)} + \frac{7q^4 p}{5760} (U')^4 T^{(4)} - \frac{qp^4}{720} (T')^4 U^{(4)}. \tag{6.26}$$

Though the result contains a lot of terms, it is again simplified by using the following replacements

$$(U')^4 \rightarrow \frac{1}{(nq)^2} \left(9(U'')^2 - \frac{3}{2} U'' \right), \qquad (T')^4 \rightarrow \frac{1}{(np)^2} \left(9(T'')^2 - \frac{3}{2} T'' \right),$$
$$(U')^2 U'' \rightarrow \frac{1}{nq} \left(3(U'')^2 - \frac{1}{2} U'' \right), \qquad (T')^2 T'' \rightarrow \frac{1}{np} \left(3(T'')^2 - \frac{1}{2} T'' \right), \tag{6.27}$$

which are allowed by the integrating by parts (6.19) and the definition of U and T (3.14). One finally obtains

$$J^{(4)}(\mu) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} e^{\mu n} \int \frac{dQ dP}{2\pi} e^{-nH_W^{(0)}}$$
$$\times \frac{(qp)^2 (1-n^2)}{5760} \left[-(9-n^2) \left((U'')^2 + \frac{1}{2} U'' \right) \left((T'')^2 + \frac{1}{2} T'' \right) + (4-n^2) U'' T'' \right]. \quad (6.28)$$

After processing the integral and the sum over n in the same way as in $J^{(0)}(\mu)$ and $J^{(2)}(\mu)$, one can write $J^{(4)}(\mu)$ as

$$J^{(4)}(\mu) = \frac{(qp)^2}{5760} \left[-(1-\partial_\mu^2)(9-\partial_\mu^2) f_{41} + (1-\partial_\mu^2)(4-\partial_\mu^2) f_{42} \right], \quad (6.29)$$

with

$$f_{41} = \mathcal{F}(q,4,p,4,\mu) + \frac{1}{2}\mathcal{F}(q,2,p,4,\mu) + \frac{1}{2}\mathcal{F}(q,4,p,2,\mu) + \frac{1}{4}\mathcal{F}(q,2,p,2,\mu),$$
$$f_{42} = \mathcal{F}(q,2,p,2,\mu). \quad (6.30)$$

We calculate its asymptotic behavior at $\mu \to \infty$ for small q, p and find that the results are consistent with the following expression

$$J_{\text{pert}}^{(4)}(\mu) = -\frac{(q+p)(qp)^2}{69120\pi}. \quad (6.31)$$

Also, we calculate the non-perturbative effect and find

$$J_{\text{np}}^{(4)}(\mu) = \frac{1}{1440} e^{-\mu} + \left[-\frac{\mu^2 - 49\mu/15 + 34/15}{96\pi} + \frac{\pi}{1152} \right] e^{-2\mu} + \mathcal{O}(e^{-3\mu}), \quad (6.32)$$

for $\{s_a\}_{a=1}^3 = \{(+1)^2, (-1)\}$.

6.4 Implication of WKB analysis

In the above subsections we have studied the WKB expansion order by order and guess the general form of the perturbative part of $J^{(0)}(\mu)$, $J^{(2)}(\mu)$ and $J^{(4)}(\mu)$ for general $\mathcal{N} = 4$ circular quivers. Collecting the cubic and linear terms in $J^{(0)}(\mu)$ and $J^{(2)}(\mu)$, it is straightforward to see that the results match respectively with C and B in our Fermi surface studies in section 3. If we collect the constant terms for the separated model from (6.13), (6.22) and (6.31), we find

$$A = \frac{4\zeta(3)}{\pi} \frac{1}{2} \left(\frac{p^2}{q\hbar} + \frac{q^2}{p\hbar} \right) - \frac{1}{24\pi} \frac{p^2 q\hbar + q^2 p\hbar}{2} - \frac{1}{34560\pi} \frac{p^2 (q\hbar)^3 + q^2 (p\hbar)^3}{2} + \mathcal{O}(\hbar^5). \quad (6.33)$$

This result leads us to conjecture that the coefficient A is given in terms of that of the ABJM theory by (1.11). Also, if we collect the first instanton term for the case of $\{s_a\}_{a=1}^3 =$

$\{(+1)^2, (-1)\}$, we find

$$
\begin{aligned}
J_{\mathrm{np}}^{\mathrm{MB}}(\mu) = {} & \left[-\frac{4}{\pi k} + \frac{\pi k}{3} + \frac{(\pi k)^3}{180} + \mathcal{O}(k^5) \right] e^{-\mu} + \frac{1}{\pi} \left[\left(-\frac{6}{\pi k} + \pi k - \frac{(\pi k)^3}{12} + \mathcal{O}(k^5) \right) \mu^2 \right. \\
& + \left(\frac{14}{\pi k} - \frac{11 \pi k}{3} + \frac{49 (\pi k)^3}{180} + \mathcal{O}(k^5) \right) \mu \\
& + \left. \left(\frac{14}{\pi k} + \frac{\pi}{2k} - \frac{\pi k}{2} - \frac{\pi^3 k}{12} - \frac{17 (\pi k)^3}{90} + \frac{\pi^5 k^3}{144} + \mathcal{O}(k^5) \right) \right] e^{-2\mu} + \mathcal{O}(e^{-3\mu}).
\end{aligned}
\tag{6.34}
$$

This is consistent with the series expansion of (1.12). In the next section, we shall see a strong numerical evidence for these conjectures (1.11) and (1.12) for the $\{s_a\}_{a=1}^3 = \{(+1)^2, (-1)\}$ case.

If we restrict ourselves to the separative case $\{s_a\}_{a=1}^3 = \{(+1)^2, (-1)\}$, we can proceed further with the instanton expansion. We find that the instanton takes the form

$$
J_{\mathrm{np}}^{\mathrm{MB}}(\mu) = \sum_{\ell=1}^{\infty} \left[c_{2\ell-1} e^{-(2\ell-1)\mu} + (a_{2\ell} \mu^2 + b_{2\ell} \mu + c_{2\ell}) e^{-2\ell\mu} \right].
\tag{6.35}
$$

As in [22], we can define the functions

$$
J_a(\mu) = \sum_{\ell=1}^{\infty} a_{2\ell} e^{-2\ell\mu}, \quad J_b(\mu) = \sum_{\ell=1}^{\infty} b_{2\ell} e^{-2\ell\mu}, \quad J_c(\mu) = \sum_{\ell=1}^{\infty} c_\ell e^{-\ell\mu},
\tag{6.36}
$$

and rewrite the sum of the perturbative part and the membrane instanton part

$$
J_{\mathrm{pert}}(\mu) + \mu^2 J_a(\mu) + \mu J_b(\mu) + J_c(\mu) = J_{\mathrm{pert}}(\mu_{\mathrm{eff}}) + \mu_{\mathrm{eff}} \widetilde{J}_b(\mu_{\mathrm{eff}}) + \widetilde{J}_c(\mu_{\mathrm{eff}}),
\tag{6.37}
$$

in terms of the effective chemical potential

$$
\mu_{\mathrm{eff}} = \mu + \frac{J_a(\mu)}{C}.
\tag{6.38}
$$

Then we find that the two coefficients $\widetilde{b}_{2\ell}$ and $\widetilde{c}_{2\ell}$ defined by

$$
\widetilde{J}_b(\mu_{\mathrm{eff}}) = \sum_{\ell=1}^{\infty} \widetilde{b}_{2\ell} e^{-2\ell\mu_{\mathrm{eff}}}, \quad \widetilde{J}_c(\mu_{\mathrm{eff}}) = \sum_{\ell=1}^{\infty} \widetilde{c}_\ell e^{-\ell\mu_{\mathrm{eff}}},
\tag{6.39}
$$

satisfy the derivative relation

$$
\widetilde{c}_{2\ell} = -k^2 \frac{\partial}{\partial k} \frac{\widetilde{b}_{2\ell}}{2\ell k}.
\tag{6.40}
$$

We have checked it for $1 \le \ell \le 4$. This structure [22] was important in the ABJM case for the result to be expressed in terms of the refined topological string [24]. This makes us to expect the theory to be solved as in the ABJM case.

7 Cancellation mechanism beyond ABJM

In the previous sections, we have studied mainly the perturbative part of the general $\mathcal{N} = 4$ superconformal circular quiver Chern-Simons theories. Here we shall look more carefully into the non-perturbative effects by restricting ourselves to a certain model. Aside from the ABJM matrix model, which has a dual description of the topological string theory on local $\mathbb{P}^1 \times \mathbb{P}^1$, the next-to-simplest case would probably be the separated one with $\{s_a\}_{a=1}^3 = \{(+1)^2, (-1)\}$. We shall see explicitly the first sign that this theory has a similar interesting structure in the instanton expansion. Namely, both the coefficients of the worldsheet instanton and the membrane instanton contain poles at certain coupling constants, though the poles are cancelled in the sum. First, let us note that the membrane instanton effect of this model has been fixed to be (1.12) in (6.34) and is divergent when k is an even number $k = k_{\text{even}}$,

$$J_{\text{np}}^{\text{MB}}(\mu) \sim -\frac{4}{\pi(k - k_{\text{even}})} e^{-\mu}. \tag{7.1}$$

Hereafter, we shall see that the divergence at $k = 2$ is cancelled by the first worldsheet instanton.

We also determine the total non-perturbative effects by following the strategy of [20]. We first compute the exact values of the partition function $Z(N)$ up to a certain number $N_{\max}$ [18–20]. We have computed them for $(k, N_{\max}) = (1, 20), (2, 13), (3, 7), (4, 9), (5, 3), (6, 7)$. Several examples are listed in table 1.

Then, we assume the polynomial expression for the instanton coefficient in the grand potential to be the same form as that in the ABJM case and fit the data of the exact values in table 1 with the corresponding expression of the partition function to find out the unknown coefficients. We can then determine the coefficients from those with larger contribution in $\mu \to \infty$ one by one. For example, if the grand potential is given by

$$J^{k=4}(\mu) = \frac{C}{3}\mu^3 + B\mu + A + \gamma_1 e^{-\frac{1}{2}\mu} + (\alpha_2\mu^2 + \beta_2\mu + \gamma_2)e^{-\mu} + \gamma_3 e^{-\frac{3}{2}\mu} + \mathcal{O}(e^{-2\mu}), \tag{7.2}$$

We fit the exact values of $Z(N)$ against the function

$$\begin{aligned} Z(N) = e^A C^{-1/3} \bigg(&\text{Ai}\Big[C^{-1/3}(N - B)\Big] + \gamma_1 \,\text{Ai}\Big[C^{-1/3}\Big(N + \frac{1}{2} - B\Big)\Big] \\ &+ \Big(\alpha_2\partial_N^2 - \beta_2\partial_N + \gamma_2 + \frac{1}{2}\gamma_1^2\Big)\text{Ai}\Big[C^{-1/3}\Big(N + 1 - B\Big)\Big] \\ &+ \Big(\gamma_3 + \gamma_1\big(\alpha_2\partial_N^2 - \beta_2\partial_N + \gamma_2\big) + \frac{1}{6}\gamma_1^3\Big)\text{Ai}\Big[C^{-1/3}\Big(N + \frac{3}{2} - B\Big)\Big]\bigg), \end{aligned} \tag{7.3}$$

with the six unknown coefficients $A, \gamma_1, \alpha_2, \beta_2, \gamma_2, \gamma_3$. We can first confirm the coincidence between the numerical value of A and our expected value of A (1.11). After that, we plug in the expected exact value (1.11) and repeat the same fitting to determine γ_1. Note that, unlike the ABJM matrix model, since the exponential decay is rather slow, we find a better accuracy if we include coefficients of the higher instanton effects into fitting.

$$Z_2(1) = \frac{1}{8\pi}, \quad Z_2(2) = \frac{-8+\pi^2}{1024\pi^2}, \quad Z_2(3) = \frac{-600+61\pi^2}{368640\pi^3}, \quad Z_2(4) = \frac{960-9424\pi^2+945\pi^4}{94371840\pi^4},$$

$$Z_2(5) = \frac{2479680 - 1928080\pi^2 + 169899\pi^4}{237817036800\pi^5},$$

$$Z_2(6) = \frac{14999040 + 110004160\pi^2 - 118324488\pi^4 + 10843875\pi^6}{91321742131200\pi^6},$$

$$Z_3(1) = \frac{1}{12\pi}, \quad Z_3(2) = \frac{-864+89\pi^2}{31104\pi^2}, \quad Z_3(3) = \frac{-21384 + 13311\pi^2 - 2048\sqrt{3}\pi^3}{10077696\pi^3},$$

$$Z_3(4) = \frac{614304 - 1821312\pi^2 - 32768\sqrt{3}\pi^3 + 196297\pi^4}{1934917632\pi^4},$$

$$Z_3(5) = \frac{339072480 - 997174800\pi^2 + 44236800\sqrt{3}\pi^3 + 936266499\pi^4 - 158617600\sqrt{3}\pi^5}{15672832819200\pi^5},$$

$$Z_3(6) = (-5845063680 + 55396185120\pi^2 + 530841600\sqrt{3}\pi^3 - 110714929056\pi^4$$
$$- 2124840960\sqrt{3}\pi^5 + 11796983935\pi^6)/(2708265511157760\pi^6),$$

$$Z_4(1) = \frac{1}{16\pi}, \quad Z_4(2) = \frac{-48+5\pi^2}{8192\pi^2}, \quad Z_4(3) = \frac{-2640 + 833\pi^2 - 180\pi^3}{5898240\pi^3},$$

$$Z_4(4) = \frac{6400 - 15776\pi^2 - 4864\pi^3 + 3081\pi^4}{402653184\pi^4},$$

$$Z_4(5) = \frac{48625920 - 83759200\pi^2 + 11894400\pi^3 + 38045661\pi^4 - 10773000\pi^5}{30440580710400\pi^5},$$

$$Z_4(6) = (-1157345280 + 10549584640\pi^2 + 5902848000\pi^3 - 17773668432\pi^4$$
$$- 9397728000\pi^5 + 4494764925\pi^6)/(46756731971174400\pi^6),$$

$$Z_5(1) = \frac{1}{20\pi}, \quad Z_5(2) = \frac{-7000 + (3145 - 1088\sqrt{5})\pi^2}{400000\pi^2},$$

$$Z_5(3) = \frac{-300000 + (367025 - 14400\sqrt{5})\pi^2 - 18432\sqrt{50 - 10\sqrt{5}}\pi^3}{360000000\pi^3},$$

$$Z_6(1) = \frac{1}{24\pi}, \quad Z_6(2) = \frac{-3240 + 331\pi^2}{746496\pi^2}, \quad Z_6(3) = \frac{-495720 + 287037\pi^2 - 43520\sqrt{3}\pi^3}{2418647040\pi^3},$$

$$Z_6(4) = \frac{459794880 - 1161396144\pi^2 - 320716800\sqrt{3}\pi^3 + 289774225\pi^4}{50153065021440\pi^4},$$

$$Z_6(5) = (572595791040 - 1548287349840\pi^2 + 122276044800\sqrt{3}\pi^3 + 1331543069217\pi^4$$
$$- 229345715200\sqrt{3}\pi^5)/(1137471514686259200\pi^5),$$

$$Z_6(6) = (-9765317657088 + 73750628879424\pi^2 + 30831120875520\sqrt{3}\pi^3$$
$$- 143992509769800\pi^4 - 81529317310464\sqrt{3}\pi^5 + 57069728465365\pi^6)$$
$$/(786220310951142359040\pi^6).$$

Table 1. Exact values of the partition function $Z_k(N)$ of the model $\{s_a\}_{a=1}^3 = \{(+1)^2, (-1)\}$.

		numerical values	expected exact values
$k = 2$	A	-0.3103048519	-0.3103048520
	α_1	0.0000001660	0
	β_1	0.6366192705	$2/\pi \simeq 0.6366197724$
	γ_1	0.6366205469	$2/\pi \simeq 0.6366197724$
$k = 3$	A	-0.8115986816	-0.8115986811
	γ_1	2.666666666	$8/3 \simeq 2.666666667$
$k = 4$	A	-1.368159992	-1.368159992
	γ_1	2.828426464	$2\sqrt{2} \simeq 2.828427125$
$k = 5$	A	-2.014160398	-2.014179117
	γ_1	3.577692778	$8/\sqrt{5} \simeq 3.577708764$
$k = 6$	A	-2.762757648	-2.762757648
	γ_1	4.618802104	$8/\sqrt{3} \simeq 4.618802154$

Table 2. Comparison of numerical values obtained from fitting and expected exact values for the perturbative coefficient A and the non-perturbative ones. The expected exact values for A is given in (1.11) written in terms of the ABJM value (1.3) while the expect values for the first instanton effects are given in (7.4).

Finally we find that, from the numerical studies of the partition function of the separative model with $\{s_a\}_{a=1}^3 = \{(+1)^2, (-1)\}$, the grand potential is given by

$$J_{\mathrm{np}}^{k=2}(\mu) = \left[\frac{2\mu+2}{\pi}\right]e^{-\mu} + \mathcal{O}(e^{-2\mu}), \qquad J_{\mathrm{np}}^{k=3}(\mu) = \frac{8}{3}e^{-\frac{2}{3}\mu} + \mathcal{O}(e^{-\frac{4}{3}\mu}),$$

$$J_{\mathrm{np}}^{k=4}(\mu) = 2\sqrt{2}e^{-\frac{1}{2}\mu} + \mathcal{O}(e^{-\mu}), \qquad J_{\mathrm{np}}^{k=5}(\mu) = \frac{8}{\sqrt{5}}e^{-\frac{2}{5}\mu} + \mathcal{O}(e^{-\frac{4}{5}\mu}),$$

$$J_{\mathrm{np}}^{k=6}(\mu) = \frac{8}{\sqrt{3}}e^{-\frac{1}{3}\mu} + \mathcal{O}(e^{-\frac{2}{3}\mu}). \tag{7.4}$$

The comparison of these exact values with the numerical values obtained from fitting can be found in table 2. Note that, although we only display the first several exact values of the partition function in table 1, we have used our full set of exact values in obtaining the numerical values. Aside from the case of $k = 5$ where we have only a few data, as a whole we find a very good match.

Since there are no other contributions than the worldsheet instanton in the first instanton effect in $J^{k=3,4,5,6}(\mu)$, we expect that these coefficients should be explained by the first worldsheet instanton. We find a good interpolating function for it as in (1.13). Note that this factor is divergent at integers $k = 1, 2$. At $k = 2$, we find that

$$J_{\mathrm{np}}^{\mathrm{WS}}(\mu) \sim \left[\frac{4}{\pi(k-2)} + \frac{2(\mu+1)}{\pi}\right]e^{-\mu}, \tag{7.5}$$

where the divergence cancels completely with (7.1) which is coming from the membrane instanton (1.12) and the finite part reproduces the numerical study (7.4). This is a non-trivial consistency check of our conjecture of (1.12) and (1.13).

8 Discussion

In this paper we have studied the partition functions of superconformal Chern-Simons theories of the circular quiver type using the Fermi gas formalism. Aside from the preliminary study in section 5, our main target is the cases where the supersymmetry is enhanced to $\mathcal{N} = 4$. Following the argument that the perturbative part should sums up to the Airy function (1.1) in this case as well, we have explicitly determined the perturbative coefficient B (1.7) for the general $\mathcal{N} = 4$ cases. We also find a conjectural form (1.11) of the coefficient A for the special case where the two colors of edges in the circular quiver diagram are separated, i.e. $\{s_a\}_{a=1}^{M} = \{(+1)^q, (-1)^p\}$. We further restrict ourselves to the case of $\{s_a\}_{a=1}^{3} = \{(+1)^2, (-1)\}$, which is the simplest case next to the ABJM case, and study the non-perturbative effects. We find that the non-perturbative effects enjoy the similar cancellation mechanism as in the ABJM case. Both the coefficients of the worldsheet instanton and those of the membrane instanton are divergent at certain levels, though the divergences are cancelled completely.

We would like to stress that our study is one of the first signals that it is possible to generalize the success in the ABJM theory to more general theories whose relation with the topological string theory is not so clear. Namely, after finding out that for the ABJM theory the cancellation of divergences in coefficients [20] helps to determine the grand potential in terms of the refined topological string theory on local $\mathbb{P}^1 \times \mathbb{P}^1$ [24], the ABJ theory [58, 59] was studied carefully in [57, 60–64] using its relation to the topological string theory [9, 10]. Here for the general $\mathcal{N} = 4$ superconformal theories of the circular quiver type, though the direct relation to the topological strings is still unclear, our study suggests that most of the methods used in the ABJ(M) theory are also applicable. The final result may correspond to some deformations of the topological string theories and [65] may be helpful along this line.

We hope to extend the results on the ABJM theories to the class of models with $\{s_a\}_{a=1}^{M} = \{(+1)^q, (-1)^p\}$, which we believe that it is appropriate to call the "(q,p)-minimal model" in $\mathcal{N} = 3$ quiver Chern-Simons theories. Even more, maybe we can finally solve all of the $\mathcal{N} = 4$ or $\mathcal{N} = 3$ Chern-Simons theories and understand the whole moduli space by studying the cancellation mechanism among various instanton effects.

Before it, there are many basic points to be fixed firstly. For example, in this paper we have restricted ourselves to the theories with hermitian Hamiltonians in the Fermi gas formalism. We believe, however, that our result (1.7) works for the non-hermitian cases to some extent by the following two observations. First, the result (3.6) from the Fermi surface analysis is consistent with that from the WKB analysis (6.22) where we do not refer to the hermiticity. Second, the formal expression associated to the non-hermitian higher commutators reduces finally to vanishing non-perturbative terms (3.29). It is desirable to give a more concrete argument for the non-hermitian cases. Also, though we have given a few non-trivial evidences for our conjecture of the coefficient A for the separative models, it is desirable to prove it rigorously and write down a formula for the general case.

In this paper we have displayed the coefficients of the membrane instanton (1.12) and the worldsheet instanton (1.13) for the next-to-simplest (2,1) separative model, $\{s_a\}_{a=1}^{3} =$

$\{(+1)^2, (-1)\}$. Actually we can continue to the coefficients of higher instantons. We can find an exact function expression for the second membrane instanton coefficient which is consistent with the WKB expansion in section 6. Also, we can repeat the numerical fitting in section 7 to higher instantons as in the ABJM case [20, 22] to find an exact function expression for the second and third worldsheet instanton coefficients. It seems that the cancellation mechanism works as well. However, we decide not to display them because the evidences are not enough yet.

It is also interesting to observe that the $k = 1$ and $k = 2$ grand potentials in the (2,1) model resemble respectively to the $k = 2$ grand potential in the ABJM theory and that in [56] with $N_f = 4$. This implies that in general the $k = 1$ grand potential in the $(2q, 1)$ model is related to the $k = 2$ grand potential in the $(q, 1)$ model with the signs of the odd instanton terms reversed. Using the results in [56], we have checked this relation also for $q = 3, 4, 6$.

Obviously it is interesting to reproduce many of our prediction from the gravity side. Let us list several discussions. First we have seen the shift of the coefficient B (1.7), which implies the shift of the 't Hooft coupling constant

$$\lambda_{\text{eff}} = \lambda - B^{(2)} - \frac{B^{(0)}}{k^2}. \tag{8.1}$$

We would like to see its origin in the gravity dual along the line of [27]. Next the result of the WKB expansion (6.14) implies that the membrane instanton can wrap on the Lagrangian submanifolds which have the volume divided by the factors of $\Sigma(q)$ and $\Sigma(p)$. It would be interesting to reproduce these effects from the gravity dual. It was known [66] that the ordering of (1.6) corresponds to the extra discrete torsion in the orbifold background. In this sense, we find it natural that this effect appears only in the shift of the 't Hooft coupling and in the coefficient polynomials as in (6.24) and (6.25). We would like to understand this effect better. Along the line of the interpretation in the gravity dual, it is very interesting to note that the coefficient of the one-loop log term was studied from the gravity side [67] and the match with the expansion of the Airy function (1.1) was found. Also, very recently, the Airy function was reproduced from the localization computation in the gauged supergravity [68].

Finally, though we have used the matrix model (2.2) obtained after localization for the partition functions of superconformal Chern-Simons theories, it would be interesting to study the non-perturbative instanton effects directly from the field-theoretical viewpoints.

Acknowledgments

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