Real space renormalization group for twisted lattice $\mathcal{N} = 4$ super Yang-Mills

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ABSTRACT: A necessary ingredient for our previous results on the form of the long distance effective action of the twisted lattice $\mathcal{N} = 4$ super Yang-Mills theory is the existence of a real space renormalization group which preserves the lattice structure, both the symmetries and the geometric interpretation of the fields. In this brief article we provide an explicit example of such a blocking scheme and illustrate its practicality in the context of a small scale Monte Carlo renormalization group calculation. We also discuss the implications of this result, and the possible ways in which to use it in order to obtain further information about the long distance theory.

KEYWORDS: Extended Supersymmetry, Lattice Gauge Field Theories, Renormalization Group, Lattice Quantum Field Theory

ARXIV EPRINT: 1408.7067

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1 Introduction

There has been significant recent progress on the lattice discretization of $\mathcal{N} = 4$ super Yang-Mills (SYM) [1–5]. (For alternative approaches see [6–11].) One motivation for such efforts is that it is highly desirable to test the AdS/CFT correspondence at a finite number of colors N, and for moderate values of the 't Hooft coupling $\lambda = g^2 N \sim 1$. Indeed, results in this regime would, in theory, open the way to nonperturbative results for quantum gravity. Another reason to study $\mathcal{N} = 4$ SYM on the lattice is that the continuum theory is an interacting conformal field theory at all scales, unlike the situation with theories inside the conformal window, which only approach a conformal fixed point in the infrared (IR).

Originally, the lattice construction was discovered by orbifolding a matrix model in such a way as to preserve a nilpotent supercharge [1, 12]. Subsequently, the same construction was arrived at by discretizing not the usual theory but a *topologically twisted* cousin. In flat space this corresponds merely to an exotic change of variables — one more suited to discretization. In the case of $\mathcal{N} = 4$ SYM there are three independent topological twists of the theory and the one that is employed in the lattice work is the Marcus or Geometric-Langlands twist [13, 14]. The resulting lattice action takes the form

$$S = \frac{1}{2g^2}(\mathcal{Q}\lambda + S_{\text{closed}})$$

$$\lambda = \sum_x a^4 \operatorname{Tr}\left(\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a - \frac{1}{2}\eta d\right)$$

$$S_{\text{closed}} = -\frac{1}{4}\sum_x a^4 \epsilon_{abcde}\chi_{de}\overline{\mathcal{D}}_c^{(-)}\chi_{ab}(x) \tag{1.1}$$

where we include the appropriate factors of the lattice spacing a and the explicit expressions for the terms involving covariant derivatives are given by

$$\mathcal{F}_{ab}(x) = \mathcal{D}_a^{(+)}\mathcal{U}_b(x) = \mathcal{U}_a(x)\mathcal{U}_b(x+e_a) - \mathcal{U}_b(x)\mathcal{U}_a(x+e_b)$$

$$\overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(x) = \mathcal{U}_a(x)\overline{\mathcal{U}}_a(x) - \overline{\mathcal{U}}_a(x-e_a)\mathcal{U}_a(x-e_a)$$

$$\epsilon_{abcde}\chi_{de}\overline{\mathcal{D}}_c^{(-)}\chi_{ab}(x) = \epsilon_{abcde}\chi_{de}(x+e_a+e_b)\big[\chi_{ab}(x)\overline{\mathcal{U}}_c(x-e_c)$$
$$- \overline{\mathcal{U}}_c(x-e_c+e_a+e_b)\chi_{ab}(x-e_c)\big]. \tag{1.2}$$

Notice that these expressions involve fields which are associated to the links of an A_4^* lattice which possesses five (linearly dependent) basis vectors and an associated S^5 point group symmetry. To complete the specification of the action we also need the action of $\mathcal{Q}$ on the lattice fields, which is given by

$$\mathcal{Q}\mathcal{U}_a = \psi_a, \qquad \mathcal{Q}\psi_a = 0, \qquad \mathcal{Q}\overline{\mathcal{U}}_a = 0$$
$$\mathcal{Q}\chi_{ab}(x) = -\overline{\mathcal{F}}_{ab}(x) \equiv \overline{\mathcal{U}}_b(x+e_a)\overline{\mathcal{U}}_a(x) - \overline{\mathcal{U}}_a(x+e_b)\overline{\mathcal{U}}_b(x)$$
$$\mathcal{Q}\eta = d, \qquad \mathcal{Q}d = 0. \tag{1.3}$$

It can be checked that the classical continuum limit of this lattice action yields the usual Marcus twist of $\mathcal{N} = 4$ SYM if the lattice fields are decomposed into their irreducible components under the S^5 symmetry (see [12]) and the link fields expanded according to[1]

$$\mathcal{U}_a(x) = \frac{1}{a} + \mathcal{A}_a(x), \quad \overline{\mathcal{U}}_a(x) = \frac{1}{a} - \overline{\mathcal{A}}_a(x). \tag{1.4}$$

As an example of this argument consider the A_4^* term $\sum \chi_{ab}\mathcal{D}_{[a}\psi_{b]}$ which emerges after carrying out the $\mathcal{Q}$-variation of the $\chi_{ab}\mathcal{F}_{ab}$ term above. Decompose the lattice fields (in a fixed gauge) into their S^5 irreducible components via the relations

$$\chi_{ab} = P_{a\mu}P_{b\nu}\chi_{\mu\nu} + P_{a5}P_{b\nu}\bar{\psi}_\nu \tag{1.5}$$

$$\psi_b = P_{b\lambda}\psi_\lambda + P_{b5}\overline{\eta} \tag{1.6}$$

$$\mathcal{D}_a = P_{a\rho}\mathcal{D}_\rho + P_{a5}\phi. \tag{1.7}$$

The 5×5 orthogonal matrix P that appears in these expressions is introduced in [12] and serves as the bridge between the fields occurring on the A_4^* lattice and their continuum cousins.[2] In these expressions Greek indices run from one to four while Latin cover the range from one through five. If we substitute these decompositions into this A_4^* fermion term we find

$$\chi_{ab}\mathcal{D}_a\psi_b = P_{a\mu}P_{b\nu}P_{a\rho}P_{b\lambda}\chi_{\mu\nu}\mathcal{D}_\rho\psi_\lambda + P_{a5}P_{b\nu}P_{b\lambda}P_{a5}\bar{\psi}_\nu\phi\psi_\lambda + \dots \tag{1.8}$$

$$\chi_{ab}\mathcal{D}_b\psi_a = P_{a\mu}P_{b\nu}P_{b\rho}P_{a\lambda}\chi_{\mu\nu}\mathcal{D}_\rho\psi_\lambda + P_{a5}P_{b\nu}P_{a5}P_{b\rho}\bar{\psi}_\nu\mathcal{D}_\rho\overline{\eta} + \dots. \tag{1.9}$$

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[1] We work with antihermitian generators of the SU(N) gauge group.

[2] It is crucial for these arguments that in fact the lowest lying irreducible representations of the S^5 (strictly its A^5 subgroup) match those of the continuum twisted SO(4) group.

Using the orthogonal properties of the matrix P all other terms vanish since they involve contractions of the type $P_{a\mu}P_{a5} = 0$ and the expression simplifies to

$$\sum \chi_{ab}\mathcal{D}_{[a}\psi_{b]} \rightarrow \chi_{\mu\nu}\mathcal{D}_{[\mu}\psi_{\nu]} + \bar{\psi}_\mu \mathcal{D}_\mu \bar{\eta} + \bar{\psi}_\nu[\phi, \psi_\nu]. \tag{1.10}$$

These terms match precisely some of those appearing in the continuum Marcus twist of $\mathcal{N} = 4$ SYM. Similar reductions occur for all terms in the A_4^* action and confirm that the lattice theory does indeed target $\mathcal{N} = 4$ SYM in the naive continuum limit. Notice that the action has an additional U(1) ghost number[3] symmetry under which the fields $\left(\eta, \psi, \chi, \bar{\psi}, \bar{\eta}, \phi, \bar{\phi}, \mathcal{A}\right)$ carry charges $(1, -1, 1, -1, 1, 2, -2, 0)$. This symmetry is hidden in the original A_4^* lattice formulation and is visible only when the A_4^* fields are decomposed into their irreducible representations under the S^5 lattice symmetry. It will be important in our later analysis.

2 Blocking transformation

The original lattice Λ may be described by $\Lambda = \left\{a\sum_{\mu=1}^4 n_\mu e_\mu | n \in \mathbf{Z}^4\right\}$, where the e_μ are the first four of the five (degenerate) basis vectors of the A_4^* lattice. The blocked lattice will merely be doubled in every direction: $\Lambda' = \left\{2a\sum_{\mu=1}^4 n_\mu e_\mu | n \in \mathbf{Z}^4\right\}$. From this point forward we will work in lattice units, setting $a = 1$. The blocked fields will be denoted by primes and must begin and end on sites of the blocked lattice Λ'. The trick is to come up with a blocking transformation such that the $\mathcal{Q}$ algebra is preserved (maintenance of S_5 symmetry will be straightforward), with the geometric intepretation also surviving. For example, $\chi'_{ab}(x)$ must begin on site $x + 2e_a + 2e_b$ and end on site x since the original field $\chi_{ab}(x)$ begins on $x + e_a + e_b$ and ends on x. One choice that achieves this is the following:

$$\mathcal{U}'_a(x) = \xi\mathcal{U}_a(x)\mathcal{U}_a(x + e_a), \qquad \overline{\mathcal{U}}'_a(x) = \xi\overline{\mathcal{U}}_a(x + e_a)\overline{\mathcal{U}}_a(x)$$

$$d'(x) = d(x), \qquad\qquad \eta'(x) = \eta(x)$$

$$\psi'_a(x) = \xi\left[\psi_a(x)\mathcal{U}_a(x + e_a) + \mathcal{U}_a(x)\psi_a(x + e_a)\right]$$

$$\chi'_{ab}(x) = \frac{\xi^2}{2}\left[\overline{\mathcal{U}}_a(x + e_a + 2e_b)\overline{\mathcal{U}}_b(x + e_a + e_b)\chi_{ab}(x) + \overline{\mathcal{U}}_b(x + 2e_a + e_b)\overline{\mathcal{U}}_a(x + e_a + e_b)\chi_{ab}(x)\right]$$

$$+ \xi^2\left[\overline{\mathcal{U}}_a(x + e_a + 2e_b)\chi_{ab}(x + e_b)\overline{\mathcal{U}}_b(x) + \overline{\mathcal{U}}_b(x + 2e_a + e_b)\chi_{ab}(x + e_a)\overline{\mathcal{U}}_a(x)\right]$$

$$+ \frac{\xi^2}{2}\left[\chi_{ab}(x + e_a + e_b)\overline{\mathcal{U}}_a(x + e_b)\overline{\mathcal{U}}_b(x) + \chi_{ab}(x + e_a + e_b)\overline{\mathcal{U}}_b(x + e_a)\overline{\mathcal{U}}_a(x)\right]. \tag{2.1}$$

Because the link variables, being elements of $GL(N, \mathbf{C})$, are non-compact we have allowed for the possibility that they are rescaled by a factor ξ under the transformation. (This will become important when we perform the two-lattice matching in our Monte Carlo renormalization group (MCRG) analysis of section 4, and in this context ξ becomes a blocking parameter similar to those in other schemes. Indeed, it is typical in MCRG to tune a blocking parameter in order to achieve matching.) The following parts of the algebra are obvious upon inspection: $\mathcal{Q}\overline{\mathcal{U}}' = 0$, $\mathcal{Q}\eta' = d'$, $\mathcal{Q}d' = 0$. In particular note that for the

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[3]This is referred to as a ghost number because $\mathcal{Q}$ is used as a BRST symmetry in the construction of a topological field theory from the Marcus twist.

η, d system we have simply utilized decimation. It is not difficult to also see that $\mathcal{Q}\mathcal{U}'_a = \psi'_a$ by making use of the orginal algebra $\mathcal{Q}\mathcal{U}_a = \psi_a$. The fact that $\mathcal{Q}\psi'_a = 0$ then follows from the minus sign that comes in when $\mathcal{Q}$ is pushed past $\psi_a(x)$:

$$\begin{aligned} \mathcal{Q}\psi'_a(x) &= \xi \mathcal{Q} \left[\psi_a(x)\mathcal{U}_a(x + e_a) + \mathcal{U}_a(x)\psi_a(x + e_a) \right] \\ &= -\xi\psi_a(x)\psi_a(x + e_a) + \xi\psi_a(x)\psi_a(x + e_a) = 0 \,. \end{aligned} \tag{2.2}$$

To demonstrate that $\mathcal{Q}\chi'_{ab} = -\overline{\mathcal{F}}'_{ab}$ we first note that the logical definition of the field strength in terms of the blocked fields is a straightforward transcription of the original expression:

$$\overline{\mathcal{F}}'_{ab}(x) = - \left[\overline{\mathcal{U}}'_b(x + 2e_a)\overline{\mathcal{U}}'_a(x) - \overline{\mathcal{U}}'_a(x + 2e_b)\overline{\mathcal{U}}'_b(x) \right]. \tag{2.3}$$

Then applying $\mathcal{Q}$ to the expression for χ'_{ab} in terms of the original fields, one indeed obtains the desired expression after a few steps of algebra. At this point one immediately recognizes that the nilpotency $\mathcal{Q}^2 = 0$ has also been maintained. It is also easy to see that the properties under the symmetric group S^5 have been preserved: any invariant of the original fields is also S^5 invariant when expressed in terms of the blocked fields. For instance, $\sum_a \mathcal{U}'_a\overline{\mathcal{U}}'_a$ is obviously invariant under permutations of the indices.

3 Renormalization

What we are interested in is the number of counterterms that must be fine-tuned in order to obtain the desired long distance effective theory — i.e., one whose classical continuum limit is nothing but $\mathcal{N} = 4$ SYM. The strategy is to enumerate the lattice operators that could possibly be generated under renormalization group flow with the blocking scheme given above. Lattice operators that give relevant or marginal operators in the continuum limit are the ones that would correspond to counterterms which must be fine-tuned. Of course some operators can be given their canonical coefficients simply by a rescaling of the fields; this is something that we will also describe below. The remaining coefficients, which are determined by the flow from the ultraviolet theory (UV), would have to be fine-tuned by adjusting corresponding coefficients in that UV theory. If we can write down two lattice operators that both give the same relevant/marginal operator and only differ by irrelevant operators in the continuum limit, then we can count them as a single counterterm for the purpose of fine-tuning, and we only need write one of them for our description of the "most general long distance effective action." This is because this long distance action is defined up to irrelevant operators, which do not affect the counting of counterterms that must be fine-tuned.

In the continuum theory, the $\mathcal{Q}$ closed term that appears in the action is the unique renormalizable operator with this property. Hence we know that on the lattice the $\mathcal{Q}$ closed term is also unique. Thus what remains is to enumerate the $\mathcal{Q}$ exact operators that are renormalizable. These must all take the form $\mathcal{Q} \operatorname{Tr} \left[\Psi f \left(\mathcal{U}, \overline{\mathcal{U}}, d \right) \right]$ or $\mathcal{Q} \left\{ \operatorname{Tr} \eta \operatorname{Tr} f \left(\mathcal{U}, \overline{\mathcal{U}}, d \right) \right\}$, where Ψ is one of the fermion fields. Cubic or higher powers of fermions would be nonrenormalizable, and the quantity that $\mathcal{Q}$ acts on must be fermionic so that the action is bosonic. Only η can be used in a double trace operator, because a

field must be a site field in order for its trace to be gauge invariant. Thus, beginning with $\Psi = \eta$, we have the following possible terms:

$$
\begin{aligned}
&\mathcal{Q}\,\text{Tr}\left[\eta(x)\overline{\mathcal{U}}_a(x-e_a)\mathcal{U}_a(x-e_a)\right], \qquad \mathcal{Q}\,\text{Tr}\,(\eta d) \\
&\mathcal{Q}\,\text{Tr}\left[\eta(x)\mathcal{U}_a(x)\overline{\mathcal{U}}_a(x)\right], \qquad\qquad Q\text{Tr}\eta, \\
&\mathcal{Q}\left[\text{Tr}\,\eta\,\text{Tr}\left(\mathcal{U}_a\overline{\mathcal{U}}_a\right)\right].
\end{aligned}
\tag{3.1}
$$

However, the original action is invariant under the shift symmetry

$$
\eta(x) \to \eta(x) + c\mathbf{1}_N
\tag{3.2}
$$

where c is an arbitrary constant Grassmann parameter. This symmetry restricts the above terms to the following combinations:

$$
\begin{aligned}
&\mathcal{Q}\,\text{Tr}\left[\eta\overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a\right], \qquad \mathcal{Q}\,\text{Tr}\,(\eta d) \\
&\mathcal{Q}\,\text{Tr}\left(\eta\mathcal{U}_a\overline{\mathcal{U}}_a\right) - \frac{1}{N}\mathcal{Q}\left\{\text{Tr}\,\eta\,\text{Tr}\left(\mathcal{U}_a\overline{\mathcal{U}}_a\right)\right\}.
\end{aligned}
\tag{3.3}
$$

Thus we only find one term that is not already present in the original action; this will be the so-called "β term" below.

As far as $\Psi = \psi_a$ is concerned, one gauge invariant combination that we can write down is $\text{Tr}\,\psi_a\overline{\mathcal{U}}_a$. However, $\mathcal{Q}$ acting on this vanishes identically. Another operator that is allowed by the symmetries is

$$
\Delta S = \beta_2 \sum_x a^4\left(a\mathcal{Q}\,\text{Tr}\,\psi_a\mathcal{U}_a\overline{\mathcal{U}}_a\overline{\mathcal{U}}_a\right) = \beta_2 \sum_x a^4\left[a\,\text{Tr}\,\psi_a(x)\psi_a(x+e_a)\overline{\mathcal{U}}_a(x+e_a)\overline{\mathcal{U}}_a(x)\right]
\tag{3.4}
$$

where β_2 is a dimensionless constant generated under the renormalization group (RG) flow, and the power of a in front of the operator is dictated by the mass dimensions of the fields, according to the way in which we normalized the links in (1.4): $[\psi_a] = 3/2$, $[\mathcal{U}_a] = 1$. Of course the factor of a^4 simply represents the measure d^4x in the continuum limit, as in the original action above. The explicit power of a in front of the operator makes it appear as if this is an irrelevant operator in the continuum limit; however, this is not the case because of the factors of $1/a$ that arise from (1.4). Explicitly:

$$
\begin{aligned}
&a\,\text{Tr}\,\psi_a(x)\psi_a(x+e_a)\overline{\mathcal{U}}_a(x+e_a)\overline{\mathcal{U}}_a(x) \\
&= a\,\text{Tr}\left\{\psi_a(x)\left[\psi_a(x) + a\partial_a\psi_a(x) + \mathcal{O}(a^2)\right]\left[\frac{1}{a} - \overline{\mathcal{A}}_a(x) + \mathcal{O}(a)\right]\left[\frac{1}{a} - \overline{\mathcal{A}}_a(x)\right]\right\} \\
&= \text{Tr}\,\psi_a\overline{\mathcal{D}}_a\psi_a + \mathcal{O}(a).
\end{aligned}
\tag{3.5}
$$

Thus at leading order there is a marginal operator coming from this term. It violates the Euclidean SO(4) Lorentz symmetry, but is consistent with the S^5 point group symmetry of the lattice. However, in fact this operator is prohibited by the U(1) symmetry described earlier and hence $\beta_2 = 0$ in the renormalized theory.

For the fermion choice of $\Psi = \chi_{ab}$, we can form the operators

$$
\mathcal{Q}\,\text{Tr}\,(\chi_{ab}\mathcal{U}_a\mathcal{U}_b), \qquad \mathcal{Q}\,\text{Tr}\,(\chi_{ab}\mathcal{U}_b\mathcal{U}_a).
\tag{3.6}
$$

However, the antisymmetry $\chi_{ab} = -\chi_{ba}$ requires that these be combined with a minus sign, leading to the operator

$$\mathcal{Q} \,\mathrm{Tr}\left(\chi_{ab}\mathcal{D}_a^{(+)}\mathcal{U}_b\right) \tag{3.7}$$

which is already present in the action. As before, adding additional powers of $\mathcal{U}_a\overline{\mathcal{U}}_a$ merely leads to the same marginal operator in the continuum limit; leaving them out only changes irrelevant operators — something that we are not interested in as far as counting counterterms is concerned.

It is clear that the blocked fields must have the same geometric interpretation on the lattice Λ' in order for these arguments to hold. This dictates the structure of the site arguments of the fields, for instance appearing in (3.7), such that the same term as in the original action appears in the long distance effective theory. It is also important that the blocking preserves the S^5 symmetry, so that this restriction on operators will be present. Without it, we would have generated many other possibilities in the above analysis.

Thus the most general long distance effective action is[4]

$$\mathcal{Q}\,\mathrm{Tr}\left\{\alpha_1\chi_{ab}\mathcal{F}_{ab} + \alpha_2\eta\left[\overline{\mathcal{D}}_a, \mathcal{D}_a\right] - \frac{\alpha_3}{2}\eta d\right\} - \frac{\alpha_4}{4}\epsilon_{abcde}\,\mathrm{Tr}\,\chi_{de}\overline{\mathcal{D}}_c\chi_{ab}$$
$$+ \beta Q\left\{\mathrm{Tr}\,\eta\mathcal{U}_a\overline{\mathcal{U}}_a - \frac{1}{N}\,\mathrm{Tr}\,\eta\,\mathrm{Tr}\,\mathcal{U}_a\overline{\mathcal{U}}_a\right\} \tag{3.8}$$

where we have suppressed an overall $\sum_x a^4$ factor. Acting with $\mathcal{Q}$, followed by a rescaling of fields

$$\eta \to \lambda_\eta\eta, \quad \chi_{ab} \to \lambda_\chi\chi_{ab}, \quad \psi_a \to \lambda_\psi\psi_a, \quad d \to \lambda_d d \tag{3.9}$$

we obtain

$$\mathrm{Tr}\left\{-\alpha_1\overline{\mathcal{F}}_{ab}\mathcal{F}_{ab} - \alpha_1\lambda_\chi\lambda_\psi\chi_{ab}\mathcal{D}_{[a}\psi_{b]} + \alpha_2\lambda_d d\left[\overline{\mathcal{D}}_a, \mathcal{D}_a\right] - \alpha_2\lambda_\eta\lambda_\psi\eta\overline{\mathcal{D}}_a\psi_a\right.$$
$$\left. - \frac{\alpha_3}{2}\lambda_d^2 d^2 - \frac{\alpha_4}{4}\lambda_\chi^2\epsilon_{abcde}\chi_{de}\overline{\mathcal{D}}_c\chi_{ab}\right\} + \beta\left\{\lambda_d\,\mathrm{Tr}\left(d\mathcal{U}_a\overline{\mathcal{U}}_a\right) - \lambda_\eta\lambda_\psi\,\mathrm{Tr}\left(\eta\psi_a\overline{\mathcal{U}}_a\right)\right.$$
$$\left. - \frac{1}{N}\lambda_d\,\mathrm{Tr}\,d\,\mathrm{Tr}\left(\mathcal{U}_a\overline{\mathcal{U}}_a\right) + \frac{1}{N}\lambda_\eta\lambda_\psi\,\mathrm{Tr}\,\eta\,\mathrm{Tr}\left(\psi_a\overline{\mathcal{U}}_a\right)\right\}. \tag{3.10}$$

Using the freedom in the four rescaling factors, we can simultaneously impose four constraints,

$$\alpha_1\lambda_\chi\lambda_\psi = \alpha_1, \quad \alpha_2\lambda_d = \alpha_1, \quad \alpha_2\lambda_\eta\lambda_\psi = \alpha_1, \quad \alpha_4\lambda_\chi^2 = \alpha_1 \tag{3.11}$$

which sets many of the coefficients above to α_1. Solving this system one obtains

$$\lambda_\eta = \sqrt{\frac{\alpha_1^3}{\alpha_4\alpha_2^2}}, \quad \lambda_\chi = \frac{1}{\lambda_\psi} = \sqrt{\frac{\alpha_1}{\alpha_4}}, \quad \lambda_d = \frac{\alpha_1}{\alpha_2}. \tag{3.12}$$

It is also convenient to define

$$\alpha_3' = \alpha_3\left(\frac{\alpha_1}{\alpha_2}\right)^2, \quad \beta' = \beta\frac{\alpha_1}{\alpha_2}. \tag{3.13}$$

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[4]Actually there is one further operator that can be added to the A_4^* action $\mathcal{O} = \sum_x \epsilon_{abcde}$ $\mathrm{Tr}\left(\overline{\mathcal{U}}_a(x)\overline{\mathcal{U}}_b(x+a)\overline{\mathcal{U}}_c(x+a+b)\overline{\mathcal{U}}_d(x+a+b+c)\overline{\mathcal{U}}_e(x+a+b+c+d)\right)$. However this operator (which is $\mathcal{Q}$ exact) yields only the usual topological term $\int \epsilon_{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}$ in the continuum limit.

Then the action takes the form

$$
\begin{aligned}
\mathrm{Tr} \Big\{ &- \alpha_1 \overline{\mathcal{F}}_{ab} \mathcal{F}_{ab} - \alpha_1 \chi_{ab} \mathcal{D}_{[a} \psi_{b]} + \alpha_1 d \left[\overline{\mathcal{D}}_a, \mathcal{D}_a \right] - \alpha_1 \eta \overline{\mathcal{D}}_a \psi_a \\
&- \frac{\alpha_3'}{2} d^2 - \frac{\alpha_1}{4} \epsilon_{abcde} \chi_{de} \overline{\mathcal{D}}_c \chi_{ab} \Big\} + \beta' \Big\{ \mathrm{Tr} \left(d \mathcal{U}_a \overline{\mathcal{U}}_a \right) - \mathrm{Tr} \left(\eta \psi_a \overline{\mathcal{U}}_a \right) \\
&- \frac{1}{N} \mathrm{Tr} \, d \, \mathrm{Tr} \left(\mathcal{U}_a \overline{\mathcal{U}}_a \right) + \frac{1}{N} \mathrm{Tr} \, \eta \, \mathrm{Tr} \left(\psi_a \overline{\mathcal{U}}_a \right) \Big\}.
\end{aligned}
\tag{3.14}
$$

In fact it is remarkable that the β term does not bifurcate into multiple coefficients; this is a consequence of $\lambda_d = \lambda_\eta \lambda_\psi$.

At this point, after rescaling of the fields, we find that a total of at most two fine-tunings will be required: $\alpha_3' \to \alpha_1$ and $\beta' \to 0$. (The overall factor of α_1 just corresponds to the renormalized gauge coupling, which does not need to be fine-tuned since the continuum theory is conformal.) This is drastically superior to the case of a naive implementation such as Wilson fermions with SO(4) symmetry (the SO(6) symmetry cannot be preserved because it is chiral), where there are eight fine-tunings (see appendix A).

However, there is another tool at our disposal. In [3] it was shown that no effective potential was generated for the bosonic fields at any order in lattice perturbation theory. Thus the moduli space is not lifted by perturbative radiative corrections. If this is also true of nonperturbative effects, then the β term is forbidden, since it includes trilinear coupling of the scalars, which would lift the moduli space (see appendix B). This would mean that under the RG flow, $\beta \equiv 0$ is maintained. Any deviation from this would have to arise from nonperturbative phenomena. It would be interesting to study the effects of instantons in the lattice theory in order to see whether or not they generate an effective potential.

These arguments reveal that a *single fine tuning of a marginal operator* $c2 = \alpha_3'/\alpha_1$ is all that should be required to target $\mathcal{N} = 4$ SYM in the continuum limit defined by $L \to \infty$ with g^2 held fixed. The situation is similar to the case of lattice QCD with Wilson fermions where the bare mass must be fine-tuned to achieve the chiral limit. In actuality our situation is somewhat better because we do not need to tune the bare coupling in order to achieve the desired lattice spacing. This is a consequence of the fact that the continuum theory is conformal at all scales.

We should also comment on our recent work [4] involving the restoration of R symmetries, which in the continuum is a global SU(4) symmetry that does not commute with supersymmetry. It was found that restoration of even a discrete version of the R symmetry, denoted by R_a and R_{ab}, is sufficient to guarantee the correct continuum limit. It has the effect of setting $\beta \equiv 0$ and all of the α_i coefficients equal to each other. Thus in a Monte Carlo renormalization group analysis (see next section) using the above blocking scheme, it should be seen that blocked observables are related to each other by R_a symmetry after a sufficient number of steps. The R_a symmetry was tested for 1×1 Wilson loops in [5] and it was found to be violated by $\mathcal{O}(10)\%$. It would be of interest to repeat this measurement after a few blocking steps and check whether or not the violation is reduced.

4 Monte Carlo renormalization group

The strategy here is in principle relatively simple, though in practice rather challenging. One simulates the theory on a fine lattice of size L^4, obtaining configurations of the fine lattice fields. This ensemble of fine lattice fields is then blocked according to the procedure outlined in section 2 to produce an ensemble of blocked lattices of size $(L/2)^4$. Observables are then computed from these blocked fields. These could include $m \times n$ Wilson loops, or mesonic correlation functions using blocked fermions in the interpolating operators. The so-called "gluino-glue" state would also be of interest. One then simulates a coarse lattice of size $(L/2)^4$, but with the more general action given in the preceding section with a single additional coupling $c2$ (we set $\beta = \beta_2 = 0$ following the arguments in section 3). The same class of observables are now computed directly on the coarser lattice. For instance, if an $m \times n$ Wilson loop was computed on the blocked lattice, then a $m \times n$ Wilson loop is computed on the coarse lattice. The coupling $c2$ of the coarse lattice action is then tuned until there is a match between the observables. Notice that this matching is done at the same lattice volume and hence the leading finite size effects are removed. This gives one MCRG step. Similar to more conventional MCRG blocking schemes used for lattice QCD we have an adjustable blocking parameter, the scaling factor ξ, that can be tuned to optimize the matching between different observables.

As a preliminary step in this direction, we have performed a blocking step $8^4 \to 4^4$. The scaling parameter is determined from holding the 1×1 Wilson loop constant. I.e., if $W(1,1)_{\text{b.f.}} = \xi^4 W(2,2)_{\text{fine}}$ and $W(1,1)_{\text{coarse}}$ are set equal, then ξ is determined. Here "b.f." indicates the fine lattice blocked using the RG blocking transformations above. We show the determination of this parameter in figure 1. These simulations utilize auxiliary parameters μ and κ to regulate the flat directions and suppress the U(1) sector [the gauge group is U(N) not SU(N)] — we refer the reader to [5] for details. The current simulations employ $\mu = 1.0$ and $\kappa = 0.5$. In addition, on both fine and coarse lattices the coupling $c2$ is set to its classical value $c2 = 1.0$ and the gauge coupling on both coarse and fine lattices are set equal. According to the above discussion,

$$\xi^4 = \frac{W(1,1)_{\text{coarse}}}{W(2,2)_{\text{fine}}}. \tag{4.1}$$

Taking this rescaling into account for other Wilson loops, we show the matching of $W(2,1)_{\text{b.f.}}$ and $W(2,2)_{\text{b.f.}}$ to $W(2,1)_{\text{coarse}}$ and $W(2,2)_{\text{coarse}}$ in figures 2 and 3 respectively. It can be seen that this simple rescaling factor is quite sufficient to give a matching of Wilson loop observables. There is no need to tune the coupling $c2$ on the coarse lattices to achieve a good matching which implies that the system already lies close to a fixed point of the RG transformation. Of course in the continuum this is to be expected, since the beta function vanishes for all gauge couplings, but it is quite a startling result for the lattice theory we are studying. One important point to make about this result is that it suggests that the β term is not generated nonperturbatively, since we did not need to add it to the coarse lattice theory in order to obtain matching. Of course the current lattices are small, our statistics are limited and the number and type of operators used in the analysis is very
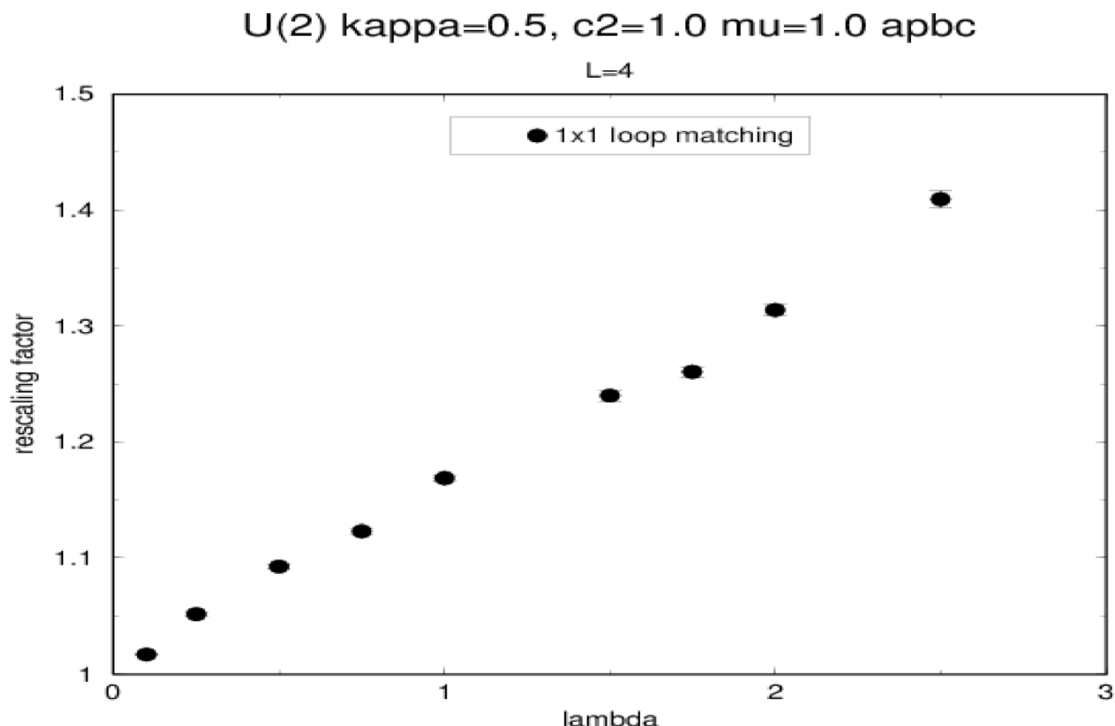
Figure 1. Determination of the scaling parameter ξ. Plotted on the vertical axis is ξ^4, the rescaling factor needed to match the 1×1 Wilson loop measured on the blocked lattices to its value on the coarse lattice.

small. We postpone a more detailed analysis to a followup paper and regard the results presented here as merely a proof of principle for this new blocking scheme.

5 Conclusions

In this article we have exhibited a RG blocking scheme for $\mathcal{N} = 4$ lattice SYM that preserves the symmetries and structure of the original lattice formulation: $\mathcal{Q}$ supersymmetry, S^5 point group symmetry, η shift symmetry, $U(N)$ gauge symmetry, the hidden $U(1)$ ghost number, and the spacetime realization of the fields in terms of 0-forms, 1-forms and 2-forms with corresponding site, link, and diagonal gauge transformation properties. The existence of such a real space RG transformation is necessary to our arguments in [3] about the form of the long distance effective action, and the number of fine-tunings that are required in order to recover the full symmetry group of the target continuum theory.

We have also shown that rescalings of the lattice fields reduces the number of counterterms that must be adjusted in this procedure. In addition, we have argued that the so-called β term lifts the moduli space, whereas the results of [3] prove that the moduli space is not lifted to all orders in perturbation theory. We therefore conclude that $\beta \equiv 0$, so that there is one less fine-tuning. Thus we finally arrive at a rather encouraging result: only a single parameter must be manipulated in order to obtain the desired continuum limit. This is comparable to the tuning required in Wilson quark simulations of lattice QCD.
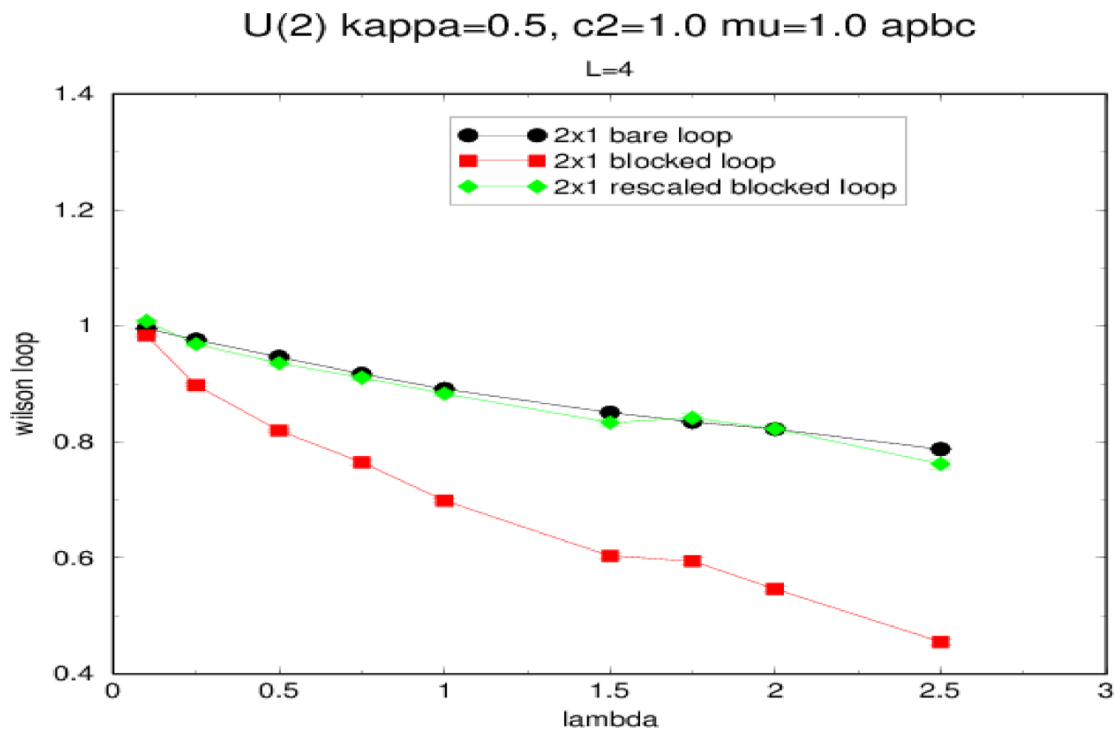
Figure 2. A comparison of $W(2,1)_{\text{b.f.}}$ and $W(2,1)_{\text{coarse}}$ with the rescaling factor taken into account.

These results have led us to a preliminary implementation of MCRG. We find that using the rescaling freedom in the blocked link fields we are able to obtain a matching of Wilson loops without any fine-tuning or flow of couplings at all. This is consistent with an approximately conformal theory.

Follow-up work will include MCRG on larger lattices, and the inclusion of matching observables that involve fermions. This is important because symmetry restoration must be checked in all sectors, not just the bosonic. As mentioned above, our tests of R_a symmetry in [5] have been limited to Wilson loops, and this is not a sufficient test to establish the full restoration of R_a symmetry, since fermionic observables should also be symmetric if the lattice action is properly tuned. It is somewhat surprising that [5] found an $\mathcal{O}(10)\%$ violation of the R_a symmetry but that in the present study we see no evidence for flow of couplings. One possibility is that the violation of R_a symmetry is not having a significant effect on conformality. Another possibility is that R_a symmetry tests are more sensitive to deviations from the desired $\mathcal{N}=4$ behavior.
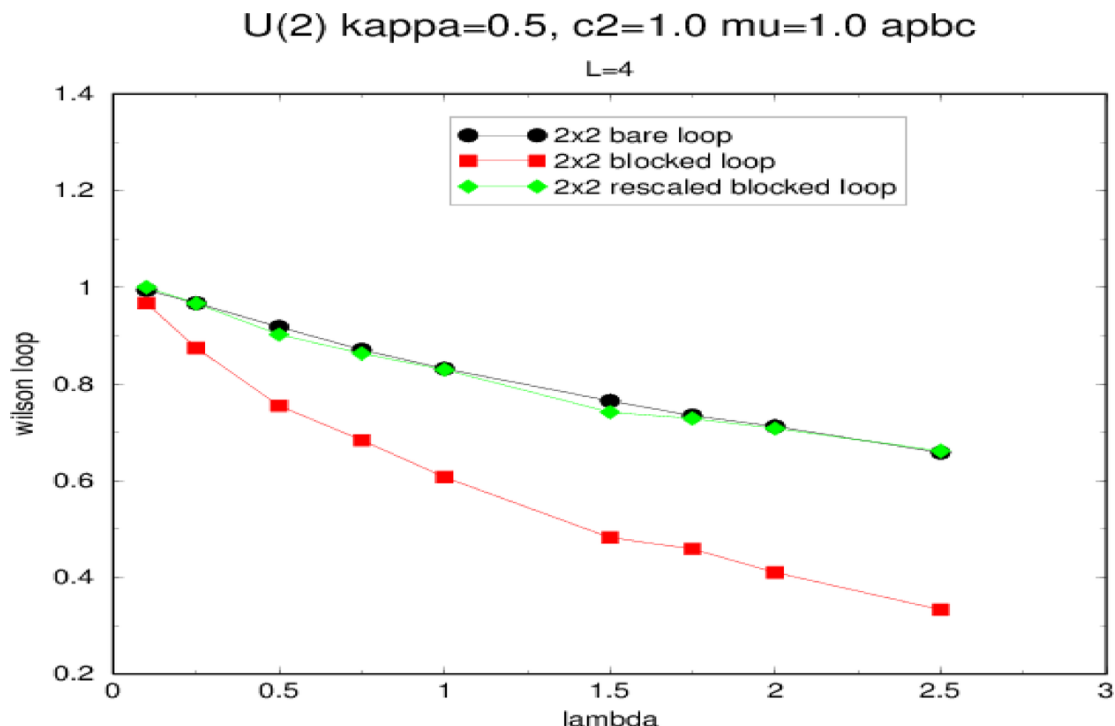
Acknowledgments

Figure 3. A comparison of $W(2,2)_{\text{b.f.}}$ and $W(2,2)_{\text{coarse}}$ with the rescaling factor taken into account.

A Wilson fermion action

Here we enumerate the fine-tunings that would have to be performed if Wilson fermions were used for the fermion discretization of lattice $\mathcal{N} = 4$ SYM. In the case of Wilson fermions, chiral symmetry is explicitly broken by the regulator. Thus one cannot preserve the $SU(4)_R$ of the continuum theory. However, the $SO(4)$ subgroup can be preserved. Under this subgroup, the fermions λ_i, $i = 1, \ldots, 4$ (we use a two-component notation in terms of Weyl fermions) transform as a **4** and the scalars ϕ_m, $m = 1, \ldots, 6$ transform as a **6**, or antisymmetric representation, which we can make explicit by mapping to $\phi_{ij} = -\phi_{ji}$, $i, j = 1, \ldots, 4$. Then the most general long distance effective action consistent with the symmetries of the lattice theory is

$$
\begin{aligned}
S = \int d^4x \ \text{Tr} \ \Bigg\{ &\frac{1}{2g_r^2} F_{\mu\nu} F_{\mu\nu} + \frac{i}{g_r^2} \overline{\lambda}_i \overline{\sigma}^\mu D_\mu \lambda_i + \frac{1}{g_r^2} D_\mu \phi_m D_\mu \phi_m + m_\phi^2 \phi_m \phi_m \\
&+ m_\lambda \left(\lambda_i \lambda_i + \overline{\lambda}_i \overline{\lambda}_i \right) + \kappa_1 \phi_m \phi_m \phi_n \phi_n + \kappa_2 \phi_m \phi_n \phi_m \phi_n + y_1 \left(\lambda_i \left[\phi_{ij}, \lambda_j \right] + \overline{\lambda}_i \left[\phi_{ij}, \overline{\lambda}_j \right] \right) \\
&+ y_2 \epsilon_{ijkl} \left(\lambda_i \left[\phi_{jk}, \lambda_l \right] + \overline{\lambda}_i \left[\phi_{jk}, \overline{\lambda}_l \right] \right) \Bigg\} \\
&+ \int d^4x \ \Bigg\{ \kappa_3 (\text{Tr} \ \phi_m \phi_m)^2 + \kappa_4 \ \text{Tr} \ \phi_m \phi_n \ \text{Tr} \ \phi_m \phi_n \Bigg\}.
\end{aligned}
\tag{A.1}
$$

The coefficients of the first three terms were achieved by rescaling the fields. The other eight coefficients will be determined by the renormalization group flow, and must be fine-tuned by adjusting corresponding UV coefficients in the lattice theory.

B Continuum limit of the β term

To arrive at the continuum limit of the β term discussed in the main text, we apply the link expansion (1.4) and keep the terms that are not $\mathcal{O}(a)$ suppressed. This leads to

$$\int d^4x\ \beta\frac{1}{a}\bigg\{\sum_a \mathrm{Tr}\left(d\left(\mathcal{A}_a-\overline{\mathcal{A}}_a\right)\right)-\frac{1}{N}\mathrm{Tr}\ d\sum_a\mathrm{Tr}\left(\mathcal{A}_a-\overline{\mathcal{A}}_a\right)$$
$$-\sum_a\mathrm{Tr}\ \eta\psi_a+\frac{1}{N}\mathrm{Tr}\ \eta\sum_a\mathrm{Tr}\ \psi_a+\mathcal{O}(a)\bigg\}. \tag{B.1}$$

Now we recall that

$$\mathcal{A}_a=A_a+iB_a,\quad \overline{\mathcal{A}}_a=A_a-iB_a \tag{B.2}$$

where A_a gives rise to the ordinary gauge fields and one scalar, and B_a lead to the other five scalars of $\mathcal{N}=4$ SYM. Thus $\mathcal{A}_a-\overline{\mathcal{A}}_a=2iB_a$. Also we decompose the U(N) generators into $T^0=(i/\sqrt{2N})\mathbf{1}_N$ and $T^A\in su(N)$ with $A=1,\ldots,N^2-1$. We will normalize the SU(N) generators to $\mathrm{Tr}\ T^AT^B=(1/2)\delta^{AB}$. What we find is that all of the U(1) fields disappear from the above expression and we are left with:

$$\int d^4x\ \frac{\beta}{2}\frac{1}{a}\bigg\{2id^A\sum_a B_a^A-\eta^A\sum_a\psi_a^A+\mathcal{O}(a)\bigg\}. \tag{B.3}$$

Since the mass dimension of the auxiliary field is $[d]=2$, what we see is that we have two dimension three operators, with a coefficient with mass dimension one. Absent fine-tuning (or our moduli space argument), the size of this coefficient is $\mathcal{O}(1/a)$.

It is now desirable to eliminate the auxiliary field. For this we need all of the terms in the action that involve d. We evaluate

$$\left[\overline{\mathcal{D}}_a,\mathcal{D}_a\right]=2iD_aB_a=2i(\partial_aB_a+[A_a,B_a]). \tag{B.4}$$

Then the terms in the action with the auxiliary field are

$$\mathrm{Tr}\left(2i\alpha_2dD_aB_a-\frac{\alpha_3}{2}d^2\right)+\frac{\beta}{2a}2id^A\sum_a B_a^A. \tag{B.5}$$

Solving the auxiliary equations of motion yields

$$d^0=2i\frac{\alpha_2}{\alpha_3}\partial_aB_a^0$$
$$d^A=2i\frac{\alpha_2}{\alpha_3}(D_aB_a)^A+2i\frac{\beta}{\alpha_3}\frac{1}{a}\sum_a B_a^A. \tag{B.6}$$

Substituting these back into (B.5) yields

$$-\frac{\alpha_2^2}{\alpha_3}\left(\partial_aB_a^0\right)^2-\frac{\alpha_2^2}{\alpha_3}(D_aB_a)^A(D_bB_b)^A-\frac{2\alpha_2\beta}{\alpha_3}\frac{1}{a}(D_aB_a)^A\sum_b B_b^A-\frac{\beta^2}{\alpha_3}\frac{1}{a^2}\sum_a B_a^A\sum_b B_b^A. \tag{B.7}$$

Thus we see that the SU(N) scalar mode $\sum_a B_a^A$ gets an $\mathcal{O}\left(1/a^2\right)$ mass term. In addition, we have a cubic interaction $[\partial_a+A_a,B_a]\sum_b B_b$ in the SU(N) sector. Both of these would lift the moduli space, which we have shown previously in [3] does not occur to any order in perturbation theory. Thus unless nonperturbative effects lift the moduli space, $\beta\equiv 0$. (It can also be seen from (B.3) that for $\beta\neq 0$ the SU(N) fermions would get a mass term $\eta\sum_a\psi_a$.)

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