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Critical points and number of master integrals

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Abstract: We consider the question about the number of master integrals for a multiloop Feynman diagram. We show that, for a given set of denominators, this number is totally determined by the critical points of the polynomials entering either of the two representations: the parametric representation and the Baikov representation. In particular, for the parametric representation the corresponding polynomial is just the sum of Symanzik polynomials. The relevant topological invariant is the sum of the Milnor numbers of the proper critical points. We present a *Mathematica* package `Mint` to automatize the counting of the master integrals for the typical case, when all critical points are isolated.

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Contents

1 Introduction

Accurate theoretical predictions for the scattering amplitudes in Standard Model and beyond require perturbative calculations at high order. The feasibility of these calculations crucially depends on our ability to evaluate the multiloop integrals. Remarkably, multiloop integrals provide a fruitful ground for the application and development of methods coming from various fields of mathematics, such as complex analysis, differential and difference equations theory, algebraic geometry etc. In a few last decades, an enormous progress in the calculation of the multiloop integrals has been made. Deep insights into the analytic and geometric nature of the multiloop integrals have been gained. Many methods of the calculation have been invented. However, each step up in the loop order is connected with a jump in the computational complexity, so there is always a demand in new, yet more powerful tools for the calculation of multiloop integrals.

One of the important tools that is relevant nowadays is the integration-by-part (IBP) identities introduced in refs. [1, 2]. Using these identities, it is possible to reduce the calculation of any multiloop integral with a given set of denominators to the calculation of a finite number of the master integrals. An important feature of this reduction is the possibility to construct differential and difference equations for the master integrals. From the computational viewpoint, the IBP reduction is known to be quite complicated problem. The reason is the absence of the general effective algorithm for this reduction. Almost all publicly available programs, like `FIRE` [3] and `Reduze` [4, 5], heavily rely on the Laporta algorithm, which includes a brute-force search of the reduction rules. Recently one of the authors has presented `LiteRed` package [6] which performs a heuristic search of the reduction rules. Nevertheless, up to now the problem of the IBP reduction has not been solved.

This paper can be considered as a little step towards the construction of the effective reduction algorithm. We present a simple recipe to determine the number of the master integrals in the given sector.

In the next section we show that it is possible to rewrite the parametric representation in the form where the Symanzik polynomials F and U enter only in the combination $F+U$. The integrals of the similar form have been considered in ref. [7]. In that paper the rank of the corresponding cohomology group has been expressed via the volume of Newton polytope under some non-degeneracy assumptions using the results of ref. [8]. In ref. [8] the volume of Newton polytope has been related to a certain topological invariant of a critical point, called Milnor number (see the definition below). In refs. [9–11] the homology group, connected with the Laplace integral, has been considered. In these papers the independent cycles were related to the steepest descent contours of the critical points of the Laplace integral exponent. We combine the ideas of these papers to devise a simple algorithm for counting the master integrals. Loosely speaking, it turns out that this number is equal to the number of critical points of the sum of Symanzik polynomials $F+U$. We demonstrate the efficiency of this recipe on the example of a family of 4-loop $g-2$ integrals.

2 Parametric and Baikov representation

Suppose that we are interested in the calculation of the L-loop integral with M denominators in d dimensions depending on E external momenta

$$J(\mathbf{n}) = J(n_1, n_2, \ldots, n_M) = \int \prod_{i=1}^{L} \frac{d^d l_i}{\pi^{d/2}} \prod_{\alpha=1}^{M} D_\alpha^{-n_\alpha}\,,$$
$$D_\alpha = A_\alpha^{ij} l_i \cdot l_j + 2B_\alpha^{ik} l_i \cdot p_k + C_\alpha\,. \tag{2.1}$$

Here $p_1, \ldots, p_E$ are linearly independent external momenta, A_α are $L\times L$ matrices, B_α are $L\times E$ matrices, and C_α are some constants.

Parametric representation. The parametric representation of $J(\mathbf{n})$ has the form

$$J(\mathbf{n}) = \frac{\Gamma(|\mathbf{n}| - Ld/2)}{\prod_\alpha \Gamma(n_\alpha)} \int \prod_\alpha dz_\alpha z_\alpha^{n_\alpha-1} \delta\left(1-\sum z\right) \frac{F^{Ld/2-|\mathbf{n}|}}{U^{(L+1)d/2-|\mathbf{n}|}}\,, \tag{2.2}$$

where $|\mathbf{n}| = \sum_{\alpha=0}^{M} n_\alpha$, U and F are the homogeneous polynomials of degrees L and $L+1$, respectively. These polynomials can be expressed in terms of quantities

$$A^{ij} = \sum_\alpha z_\alpha A_\alpha^{ij}, \quad B^i = \sum_\alpha z_\alpha B_\alpha^{ij} p_j, \quad C = \sum_\alpha z_\alpha C_\alpha \tag{2.3}$$

as follows

$$U = \det(A)\,, \quad F = \det(A)\, C - \left(A^{\mathrm{Adj}}\right)^{ij} B^i \cdot B^j, \tag{2.4}$$

where $A^{\mathrm{Adj}} = \det(A)\, A^{-1}$ is the adjoint matrix.

Remarkably, it is possible to rewrite (2.2) in the form, which contains U and F only in the combination $F+U$. Indeed, it is easy to show that the following representation holds:

$$J(\mathbf{n}) = \frac{\Gamma(d/2)}{\Gamma((L+1)d/2 - |\mathbf{n}|)\prod_\alpha \Gamma(n_\alpha)} \int_0^\infty \ldots \int_0^\infty \prod_\alpha dz_\alpha z_\alpha^{n_\alpha-1} G^{-d/2}\,, \tag{2.5}$$
$$G = F + U\,. \tag{2.6}$$

In order to pass from (2.5) to (2.2) it is sufficient to insert $1 = \int ds\delta(s - \sum z)$, scale $z \to sz$ and integrate over s. Usually, in addition to the denominators $D_1, \ldots, D_M$, one considers also the irreducible numerators $D_{M+1}, \ldots, D_N$, see the next subsection. However, it can be shown that the master integrals can be chosen to have no numerators. Therefore, the representation (2.5), which describes the integrals without numerators, can be used for the determination of the master integrals.

Let us consider the identities which appear when the monomial $\prod_\alpha z^{\alpha-1}$ in the representation (2.5) is replaced by some suitable polynomial in such a way that the whole integrand is a total derivative. The explicit form of these relations is not important for the present consideration. These identities are the counterparts of the IBP identities in the momentum representation, and it is natural to expect that they lead to the same number of master integrals. We only note that, in contrast to the momentum representation, the integration domain in eq. (2.5) has a boundary (where some variables are equal to zero). Thus the integration of a total derivative gives, in general, some surface terms. These terms are expressed via the integrals in simpler sectors (the integrals with some denominators missing). Therefore, if we want to determine the master integrals in the given sector, we can safely neglect these surface terms.

Baikov representation. So far we considered parametric representation (2.5) of the multiloop integrals. Another representation for the multiloop integrals has been introduced in ref. [18]. Let us fix the notation

$$\begin{aligned} s_{ij} &= q_i \cdot q_j\,, \\ q_i &= \begin{cases} l_i, & i \leqslant L \\ p_{i-L} & i > L \end{cases} \end{aligned}$$

Then, the simplest way to derive the Baikov representation is to pass from the integration over the loop momenta to the integration over

$$s_{ij}, \quad 1 \leqslant i \leqslant L, \quad i \leqslant j \leqslant L + E\,, \tag{2.7}$$

as described in ref. [19]. The total number of new variables is $N = L(L+1)/2 + LE$. Assuming that the denominators $D_1, \ldots, D_M$ in eq. (2.5) are linearly independent, we can choose $N - M$ irreducible numerators $D_{M+1}, \ldots D_N$. The resulting formula reads

$$\begin{aligned} J(\mathbf{n}) = & \frac{\pi^{(L-N)/2} S_E^{(E+1-d)/2}}{\Gamma\left[(d-E-L+1)/2, \ldots, (d-E)/2\right]} \\ & \times \int \left(\prod_{i=1}^{L} \prod_{j=i}^{L+E} ds_{ij} \right) S^{(d-E-L-1)/2} \prod_{\alpha=1}^{N} D_\alpha^{-n_\alpha}, \end{aligned}$$

where now $\mathbf{n} = (n_1, \ldots n_N)$ and $n_{k>M} < 0$. The quantities S and S_E have the form

$$S = \det\left\{ s_{ij}|_{i,j=1\ldots L+E} \right\}, \quad S_E = \det\left\{ s_{ij}|_{i,j=L+1\ldots L+E} \right\}.$$

The functions D_α are linear functions of the variables (2.7), so that $\prod_{i=1}^{L}\prod_{j=i}^{L+E} ds_{ij} \propto dD_1 \dots dD_N$. Thus, we have

$$J(\mathbf{n}) \propto \int \left(\prod_{\alpha=1}^{N} D_\alpha^{-n_\alpha} dD_\alpha\right) P^{(d-E-L-1)/2},$$

where $P(D_1, \dots D_N)$ is obtained from S by expressing s_{ij} via $D_1, \dots D_N$. This representation is very similar to (2.5), except that now the variables $D_1, \dots, D_M$ are raised to the negative powers. Following the Baikov's original idea, we choose the contours of integration over these variables as sufficiently small circles around the origin of the complex plane. In fact, this choice of the contours corresponds to the maximal unitary cut of the integral. After this prescription, the integrals in the subsectors are all vanishing. Taking the integrals over $D_1, \dots, D_M$ by residues, we are left with the integrals of the form

$$\int \left(\prod_{\alpha=M+1}^{N} D_\alpha^{-n_\alpha} dD_\alpha\right) P_0^{(d-I)/2}, \tag{2.8}$$

where $P_0(D_{M+1}, \dots D_N) = P(0, \dots, 0, D_{M+1}, \dots, D_N)$, and I is some integer number.

So, one can see that both the parametric and Baikov representations can be written in the form depending on a single polynomial, G and P_0, respectively.

3 Number of master integrals, basis of M-cycles and critical points

For definiteness, let us consider here the parametric representation. The integration-by-part identities determine equivalence in the space of $J(\mathbf{n})$, and master integrals represent the basis in the quotient space, which is known to be finite dimensional [12]. Naturally, the question about the number of master integrals arises. Due to the well-known duality between the homology and cohomology groups, the dimensionality of this quotient space, i.e., the number of master integrals, is equal to the number of independent "contours" of integration, generating no surface terms (and providing the convergence of the integral). The homology group of these cycles has been considered by Pham in refs. [10, 11] and is equivalent to the relative homology $H_M\left(\mathbb{C}^M \backslash \mathcal{Z}, \mathcal{B}\right)$. Here $\mathbb{C}^M \backslash \mathcal{Z}$ is a $2M$-dimensional real variety obtained from $\mathbb{C}^M$ by removing algebraic variety $\mathcal{Z} = \left\{z \in \mathbb{C}^M, G(z) = 0\right\}$ and $\mathcal{B} = \left\{z \in \mathbb{C}^M, |G(z)| \geqslant B\right\}$ ($B > 0$ is large enough) is a set of points in $\mathbb{C}^M$ where $|G(z)|$ is large enough. The number of master integrals is the rank of $H_M\left(\mathbb{C}^M \backslash \mathcal{Z}, \mathcal{B}\right)$. In this section we specify the correspondence between the basis cycles of $H_M\left(\mathbb{C}^M \backslash \mathcal{Z}, \mathcal{B}\right)$ and the critical points of the polynomial G. One of the consequences of this correspondence is the equality of the number of master integrals and the sum of Milnor numbers of the proper critical points (see definition below).

The above consideration is also valid for the Baikov representation with the replacement $G \to P_0$. In ref. [20] a criterion of the existence of master integral in a given sector has been formulated. In our notations, this criterion states, that if the polynomial P_0 has no proper critical points, there is no master integrals in the sector. It also states that

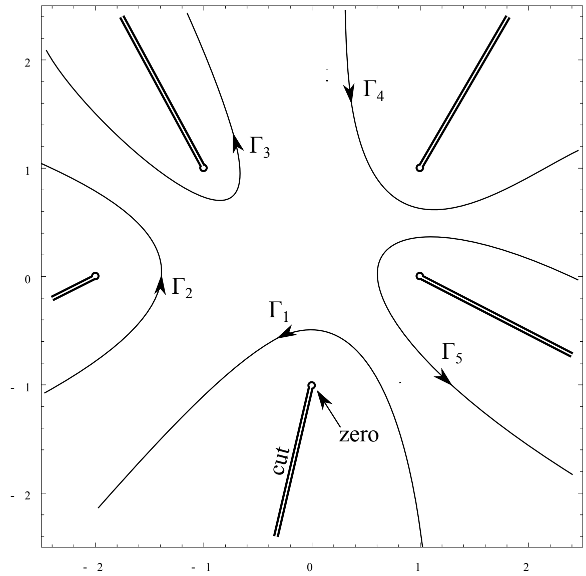

Figure 1. Contour basis in the cut plane. Out of 5 contours $\Gamma_1, \ldots \Gamma_5$ only 4 are independent, e.g. $\Gamma_5 = -\Gamma_1 - \Gamma_2 - \Gamma_3 - \Gamma_4$

the number of master integrals is bounded from below by the number of the nondegenerate isolated proper critical points of the polynomial P_0. As far as it concerns the Baikov representation, our counting recipe can be considered as a developement of ref. [20].

One-dimensional case. We consider the integral

$$\int \frac{dz\ z^{n-1}}{G(z)^{\nu}}, \tag{3.1}$$

where $G(z)$ is some polynomial of p-th degree of a single variable z. The integrand is defined in the cut plane with cuts starting at zeros of $G(z)$ and going to infinity. We want to determine the number of independent contours of integration which do not give rise to the surface terms. Obviously, for large enough positive ν those contours should start and end at infinity, embracing one or more cuts. Of course, the result is known in advance: this number is one less than the number of distinct zeros of the polynomial $G(z)$. This statement is demonstrated in figure 1. However, we would like to describe an approach which can be generalized to the case of many variables.

Let $z_0^{(1)}, \ldots, z_0^{(k)}$ are distinct zeros of G with degeneracies $p^{(1)}, \ldots, p^{(k)}$, so that $\sum_i p^{(i)} = p$. Then, obviously, $z_0^{(i)}$ is also zero of $\partial_z G$ with degeneracy $p^{(i)} - 1$ if $p^{(i)} > 1$. If $z_0^{(i)}$ is not degenerate ($p^{(i)} = 1$), then necessarily $\partial_z G\left(z_0^{(i)}\right) \neq 0$. Then, out of $p-1$ zeros of $\partial_z G$ (called *critical points* of G in what follows) there are exactly $p-1-\sum_i \left(p^{(i)} - 1\right) = k-1$ critical points which are not zeros of G. In what follows we will call them *proper critical points*. So, the number of independent contours is equal to the number of proper critical points of G (including degeneracy). This simple observation hints to a deep connection between independent contours and proper critical points.

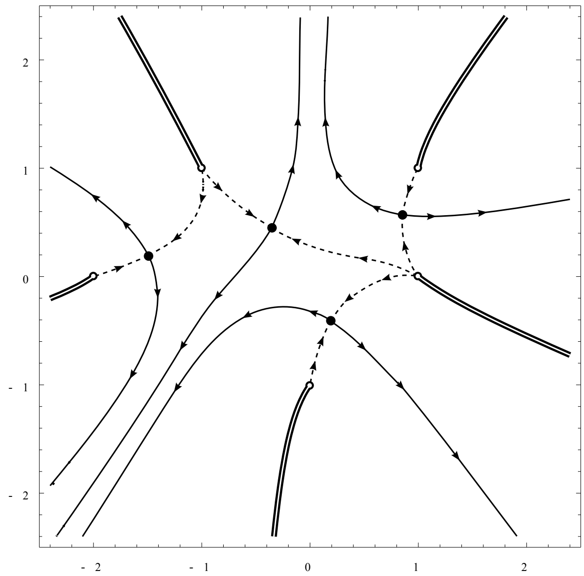

Figure 2. Saddle-point contours $\Gamma_+\left(z^{(i)}\right)$ and $\Gamma_-\left(z^{(i)}\right)$ in the cut plane (respectively, solid and dashed curves with arrows).

To reveal this connection, let us assume that all proper critical points $z^{(1)},\ldots z^{(k-1)}$ are non-degenerate and all *critical phases* $\phi^{(i)}=\arg G\left(z^{(i)}\right)$ are distinct. Let us first consider the curves in the complex plane of z defined by the condition $\arg G\left(z\right)=\phi$, where ϕ is some noncritical phase. From the Cauchy-Riemann condition, these curves are gradient flow curves of $h\left(z\right)=\ln\left|G\left(z\right)\right|$. For given ϕ one can draw such a curve starting from each zero and going to infinity. These curves provide a natural choice for the cuts. Let us now consider the curves defined by the condition $\arg G\left(z\right)=\phi^{(i)}$, where $\phi^{(i)}$ is the critical phase. The corresponding critical point z^i is a saddle point of $h\left(z\right)$, i.e. it is an intersection of the curves of the steepest descent $\Gamma_-\left(z^{(i)}\right)$ and of the steepest ascent $\Gamma_+\left(z^{(i)}\right)$. The curve $\Gamma_-\left(z^{(i)}\right)$ obviously ends at zeros, while $\Gamma_+\left(z^{(i)}\right)$ is going to infinity never passing through zero. Let us consider a superposition $\Gamma=\sum c_i\Gamma_+\left(z^{(i)}\right)$. The integer coefficient c_i is equal to the intersection index of Γ with the contour $\Gamma_-\left(z^{(i)}\right)$ for a suitable choice of the orientation of $\Gamma_\pm\left(z^{(i)}\right)$. The intersection index is topological invariant, i.e., it can not be changed by continuous deformations (in the cut plane). Therefore, $\Gamma\sim 0$ (is contractible) only when all c_i are zero, which means that the contours

$$\Gamma_1=\Gamma_+\left(z^{(1)}\right),\ldots,\Gamma_{k-1}=\Gamma_+\left(z^{(k-1)}\right)$$

are independent. In one-dimensional case the completeness of this set is obvious and therefore this system forms a basis. Note that for negative ν the basis of contours can be chosen as the set of $\Gamma_-\left(z^{(i)}\right)$.

Multidimensional case. Let us now briefly consider the multidimensional case. We have the integral (3.1), where now $z=(z_1,\ldots,z_M)$ and $dzz^{n-1}=dz_1z_1^{n_1-1}\ldots dz_Mz_M^{n_1-1}$. We want to determine the number of independent multidimensional "contours" of integration which are M-cycles in $\mathbb{C}^M$ space.

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Zeros of $G(z)$ are no more isolated points, but hypersurfaces of $M-1$ complex dimensions. Remarkably, the solution of M complex equations for gradient $\partial G/\partial z_\alpha = 0$ consists, in non-degenerate case, of some isolated points, which we again call critical points of $G(z)$. We consider non-degenerate case in the following sense:

1. There is a finite number of proper critical points $z^{(1)}, \ldots, z^{(k-1)}$ defined as the solutions of

$$\begin{aligned} \partial G/\partial z_\alpha = 0, &\quad \alpha = 1, \ldots, N \\ G(z) \neq 0 & \end{aligned} \tag{3.2}$$

2. The Hessian matrix $\frac{\partial^2 G}{\partial z_\alpha \partial z_\beta}\left(z^{(i)}\right)$ at each critical point is invertible.

3. The critical phases $\phi^{(i)} = \arg G\left(z^{(i)}\right)$ are all distinct.

Now the consideration of the previous subsection can be easily generalized. We can follow a usual construction of Morse theory using $h(z) = \ln |G(z)|$ as a Morse function and $x_\alpha = \Re z_\alpha$, $y_\alpha = \Im z_\alpha$ as coordinates. The Morse theory is formulated for a Riemannian manifold, which in our case has a flat metrics. Then the gradient flow equations have the form

$$\begin{aligned} \frac{dx_\alpha}{dt} &= \frac{\partial h}{\partial x_\alpha}, \\ \frac{dy_\alpha}{dt} &= \frac{\partial h}{\partial y_\alpha}. \end{aligned} \tag{3.3}$$

This system can be written in the form

$$\frac{dz_\alpha}{dt} = \frac{\partial h}{\partial \bar{z}_\alpha}. \tag{3.4}$$

We determine $\Gamma_\pm\left(z^{(i)}\right)$ as the union of the trajectories of eqs. (3.4) subject to the condition $z(t) \overset{t\to\mp\infty}{\longrightarrow} z^{(i)}$. The varieties $\Gamma_\pm\left(z^{(i)}\right)$ are nothing but the Lefschetz thimbles [11, 13], see also ref. [14]. Due to Cauchy-Riemann conditions, the phase of $G(z)$ on $\Gamma_\pm\left(z^{(i)}\right)$ remains constant and is equal to ϕ_i, so that the contours $\Gamma_+\left(z^{(i)}\right)$ and $\Gamma_-\left(z^{(j)}\right)$ intersect only for $i = j$. Then the independence of $\Gamma_+\left(z^{(i)}\right)$ can be proved in the same way as in 1d case. In fact, it is known, that Lefschetz thimbles constitute the basis of the relative homology $H_M\left(\mathbb{C}^M \backslash \mathcal{Z}, \mathcal{B}\right)$, see ref. [11]. In particular, it means that the rank of the relative homology group is equal to the number of critical points if conditions 1-3 are satisfied.

If the conditions 2,3 are not fulfilled for G, we can perform a small perturbations $G(z) \to G_\epsilon(z) = G(z) + \epsilon g(z)$, where $g(z)$ is some suitable polynomial and consider only those critical points of $G_\epsilon(z)$, which are close to $z^{(i)}$. The number of the critical points of $G_\epsilon(z)$ in the vicinity of $z^{(i)}$ is the "multiplicity" of $z^{(i)}$, an invariant called Milnor number of $G(z)$ at $z = z^{(i)}$, see, e.g. ref. [15]. So, we come to the following conclusion: *If $G(z)$ has only isolated proper critical points, the number of independent contours of integration (M-cycles) is equal to the sum of Milnor numbers of the proper critical points of $G(z)$.*

Zeros of the polynomial G are the branching points of the integrand in eq. (2.5). The cuts can be chosen as $(2M-1)$ real dimensional variety C_ϕ determined by the condition

$\arg G(z) = \phi$, where ϕ is a fixed noncritical value of the phase. Note that the contours $\Gamma_+\left(z^{(i)}\right)$ do not intersect the cuts.

Let us consider now the case when $G(z)$ has non-isolated proper critical points. It means that the set of points, where eq. (3.2) is satisfied, forms a critical variety of dimension ≥ 1. In practical applications, as illustrated in section 5, non-isolated proper critical points are rather rare. In this case one can still construct the basis of M-cycles, but it requires somewhat more work. The solution of eq. (3.2) is a union of several algebraic varieties — the irreducible parts. Some of these irreducible parts may be isolated points, and they should be treated as explained above.

Let $\mathcal{V}$ be an irreducible component of dimension $s > 0$. In order to construct the M-cycles passing through $\mathcal{V}$ one has to consider the compact s-cycles of $\mathcal{V}$ (the elements of the middle homology group). For each s-cycle from the basis of this homology group one has to consider the union of upward gradient flow lines Γ_+ starting on the points of that cycle. For each point of the cycle, these lines form a variety of dimension $M - s$. Altogether, these lines form the M-cycle, a member of the basis we are looking for. This consideration assumes non-degeneracy of $\mathcal{V}$. However, it can happen, that the critical variety $\mathcal{V}$ is degenerate, that is the Hessian matrix has zero modes which are not tangent to $\mathcal{V}$. In this case there are several M-cycles per each s-cycle. To sum up, the number of the independent M-cycles is equal to the sum of Milnor numbers of isolated proper critical points plus the number of independent s-cycles, on the s-dimensional components ($s > 0$) of the critical set (counted with multiplicity, if the component is degenerate). Let us iterate, that in the applications to multiloop integrals the non-isolated critical points appear very rarely, and the more so do the degenerate non-isolated critical points.

In conclusion of this section we note that similar ideas appeared earlier in refs. [16, 17]. The difference with our approach is that in refs. [16, 17] the critical points of Symanzik polynomials U and F were studied separately. These points tend to be non-isolated, which makes their treatment more difficult. In order to apply this approach to counting the master integrals, it is necessary to consider the critical points of the map (U, F): $\mathbb{C}^M \to \mathbb{C}^2$, with the additional condition $U = 1$. It can be shown that this approach, up to some details, is equivalent to one presented above.

Pedagogical example. From the above consideration it follows that we can count the number of master integrals in a given sector by counting the number of proper critical points (accounting for their possible multiplicity) of the polynomial $F+U$. As an example, let us consider the following family of sunrise integrals

$$J(n_1, n_2, n_3) = \int \frac{\mathrm{d}^d l_1 \mathrm{d}^d l_2}{\left(i\pi^{d/2}\right)^2} \left[l_1^2+1\right]^{-n_1} \left[l_2^2+1\right]^{-n_2} \left[(l_1+l_2-p)^2+1\right]^{-n_3}$$

```
In[1]:= << LiteRed`

In[2]:= Declare[{l1, l2, p}, Vector];
        SetDim[d];

In[4]:= NewBasis[s2,
          {1 + sp[l1], 1 + sp[l2], 1 + sp[l1 + l2 - p], sp[l1, p], sp[l2, p]}, {l1, l2}];
        GenerateIBP[
         s2]

In[6]:= SolvejSector[js[s2, 1, 1, 1, 0, 0], SR → False]

        Sector js[s2, 1, 1, 1, 0, 0]

           Master integrals found: j[s2, 1, 1, 1, 0, 0],
         j[s2, 1, 1, 2, 0, 0], j[s2, 1, 2, 1, 0, 0], j[s2, 2, 1, 1, 0, 0].
           jRules[s2, 1, 1, 1, 0, 0]   reduction rules for the sector.
           MIs[s2]   updated list of the masters.

Out[6]= 4
```

Figure 3. Finding reduction rules for $J\,(n_1, n_2, n_3)$ with `LiteRed`.

where $n_i \in \mathbb{N}$. Note that we need not introduce irreducible numerators. We have

$$J(n_1,n_2,n_3) = \frac{\Gamma\,(d/2) \iiint\limits_0^\infty dz_1 z_1^{n_1-1} dz_2 z_2^{n_2-1} dz_3 z_3^{n_3-1} G^{-d/2}}{\Gamma\,(3d/2 - n_1 - n_2 - n_3, n_1, n_2, n_3)}\,,$$
$$G = F + U = z_1 z_2 + z_1 z_3 + z_2 z_3 + z_1^2 z_3 + z_2^2 z_3 + z_1 z_2^2 + z_1 z_3^2 + z_2 z_3^2 + z_1^2 z_2 + \left(p^2+3\right) z_1 z_2 z_3\,.$$

The polynomial G has eight critical points (the solutions of $\nabla G = 0$)

$$z^{(1)} = -\frac{\left(p^2-1,1,1\right)}{3\,(p^2+1)},\quad z^{(2)} = -\frac{\left(1,p^2-1,1\right)}{3\,(p^2+1)},\quad z^{(3)} = -\frac{\left(1,1,p^2-1\right)}{3\,(p^2+1)},\quad z^{(4)} = -\frac{2\,(1,1,1)}{p^2+9},$$
$$z^{(5)} = (0,0,-1),\qquad z^{(6)} = (0,-1,0),\qquad z^{(7)} = (-1,0,0),\qquad z^{(8)} = (0,0,0)$$

of which the first four are proper. All four proper critical points $z^{(1)},\ldots,z^{(4)}$ are non-degenerate, therefore there are four master integrals in this sector. Indeed, running the simple *Mathematica* program using `LiteRed` package, see figure 3 reveals four master integrals.

Note the option **SR** → **False**, which forbids `LiteRed` to use symmetries of the integral with respect to permutations of indices. If we used the default setting **SR** → **True** instead, `LiteRed` would clearly find only two independent integrals $J\,(1,1,1)$ and $J\,(2,1,1)$. It is easy to account for the permutation symmetries also for critical points $z^{(1)},\ldots,z^{(4)}$. Namely, there are two orbits of the permutation group acting on the critical points: $\{z^{(4)}\}$ and $\{z^{(1)}, z^{(2)}, z^{(3)}\}$. We see that the number of orbits gives the number of master integrals with the account of symmetries.

In order to use the Baikov representation, we need to introduce two irreducible numerators D_4, D_5. This can be done quite arbitrarily, provided that, together with

$D_1 = l_1^2 + 1$, $D_2 = l_2^2 + 1$, $D_3 = (l_1 + l_2 - p)^2 + 1$, they form a complete basis. We choose

$$D_4 = (l_1 - p)^2 , \quad D_5 = (l_2 - p)^2$$

We have

$$P_0 = \frac{D_4 D_5}{4} \left(p^2 - 3 - D_4 - D_5\right) + \frac{\left(p^2+1\right)^2}{4}$$

There are four critical points $z^{(i)} = \left(D_4^{(i)}, D_5^{(i)}\right)$:

$$z^{(1)} = (0,0) \, , \; z^{(2)} = \left(0, p^2 - 3\right) , \; z^{(3)} = \left(p^2 - 3, 0\right) , z^{(4)} = \frac{1}{3}\left(p^2 - 3, p^2 - 3\right) .$$

Each point is non-degenerate and proper, so we again conclude that there are 4 master integrals if we neglect the symmetry relations.

The account of the symmetry relations is somewhat less obvious in this representation than in the parametric representation. The reason is that the numerators are transformed one-to-many upon the symmetry, in contrast to the one-to-one transformation of the denominators. Nevertheless, we can find the action of the symmetries on D_4 and D_5 from the corresponding mapping of loop momenta. In the linear subspace determined by $D_1 = D_2 = D_3 = 0$ they read

$$\begin{aligned}
&1. D_4 \to D_4,\; D_5 \to D_5\\
&2. D_4 \to D_5,\; D_5 \to D_4\\
&3. D_4 \to D_4,\; D_5 \to p^2 - 3 - D_4 - D_5\\
&4. D_4 \to D_5,\; D_5 \to p^2 - 3 - D_4 - D_5\\
&5. D_4 \to p^2 - 3 - D_4 - D_5,\; D_5 \to D_5\\
&6. D_4 \to p^2 - 3 - D_4 - D_5,\; D_5 \to D_4
\end{aligned}$$

Now it is trivial to find the orbits of this symmetry group: they are $\{z^{(4)}\}$ and $\{z^{(1)}, z^{(2)}, z^{(3)}\}$. Therefore, we again conclude, that after the account of the symmetries, there are two master integrals.

4 Algebraic treatment

There is a well-known correspondence between algebraic varieties and ideals in the polynomial rings. Due to this correspondence, in order to find the sum of Milnor numbers of proper critical points, one need not explicitly solve the polynomial system of equations (3.2). Instead, one may calculate the dimensionality of the quotient ring of the polynomial ideal we will describe in a moment. If we were interested in the sum of Milnor numbers of all critical points (including non-proper ones), we would choose Jacobian ideal, which is generated by $\partial G/\partial z_1, \ldots, \partial G/\partial z_M$. The condition $G \neq 0$ can be taken into account by introducing an extra variable z_0 and considering the ideal

$$\mathcal{I} = \langle \partial G/\partial z_1, \ldots, \partial G/\partial z_M, z_0 G - 1 \rangle . \tag{4.1}$$

Choosing some monomial ordering and constructing the Groebner basis, we can determine the set of irreducible monomials. Then the dimensionality of the quotient space is just the number of those monomials.

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```
In[1]:= << Mint`

In[2]:= Declare[{k1, k2, k3, k4, p}, Vector];
        sp[p, p] = 1;

In[4]:= Timing[CountMIs[{-sp[k1], -sp[k2], -sp[k3],
           -sp[k4], 1 - sp[k1 + p], 1 - sp[k1 + k2 + p], 1 - sp[k1 + k2 + k3 + p],
           1 - sp[k2 + k3 + k4 + p], 1 - sp[k3 + k4 + p], 1 - sp[k4 + p]}, {k1, k2, k3, k4}]]

Out[4]= {3.468750, 2}
```

Figure 4. Example of using `Mint`.

Symmetries. The above method gives us the number of master integrals without the account of possible symmetry relations between them. In terms of the polynomial G, those symmetries are such permutations of the variables $z_1, \ldots z_M$, which leave G intact. They form a permutation group P.

The most straightforward way to take those symmetries into account is the following. For each irreducible monomial $m = z_1^{n_1} z_2^{n_2} \ldots z_M^{n_M}$ and for each permutation $p = (p_1, p_2, \ldots, p_M) \in P$ we construct and reduce with respect to $\mathcal{I}$ the polynomial $m - pm = z_1^{n_1} z_2^{n_2} \ldots z_M^{n_M} - z_{p_1}^{n_1} z_{p_2}^{n_2} \ldots z_{p_M}^{n_M}$. Let us denote as r the number of linearly independent remainders. Then, the number of the master integrals surviving the symmetry relations is just the number of irreducible monomials minus r.

Non-isolated critical points. The existence of non-isolated proper critical points can be easily seen as infinite dimensionality of the quotient ring of $\mathcal{I}$. Then one needs to determine the irreducible components of the critical set, together with their possible degeneracy. This problem can be naturally solved with algebraic approach by performing the primary decomposition of $\mathcal{I}$, i.e., finding the decomposition

$$\mathcal{I} = \bigcap \mathcal{I}_i \,,$$

where each $\mathcal{I}_i$ is a primary ideal corresponding to some irreducible component. If $\mathcal{I}_i$ is also prime, the corresponding critical variety is non-degenerate. The number of middle-dimensional cycles in this critical variety can be also determined by the algebraic method (see next section), but the discussion of the most general case is beyond the scope of this paper.

***Mathematica* package `Mint`.** From the above consideration it follows, that, apart from the case of non-isolated critical points, the problem of determination of the number of independent M-cycles (equal to the number of master integrals) can be easily solved on any modern computer algebra system. We have developed a simple *Mathematica* package `Mint`, [21], which finds the number of the master integrals with a given set of denominators if the proper critical points of the corresponding polynomial $F + U$ are isolated. If it is not the case, the program returns *Indeterminate*. The example of its usage is shown in figure 4 Note that, knowing the number M of master integrals (with account of symmetry) in a given sector, one can select as masters almost any M integrals, provided that no two of

```
In[5]:= Timing[FindMIs[{-sp[k1], -sp[k2], -sp[k3],
        -sp[k4], 1 - sp[k1 + p], 1 - sp[k1 + k2 + p], 1 - sp[k1 + k2 + k3 + p],
        1 - sp[k2 + k3 + k4 + p], 1 - sp[k3 + k4 + p], 1 - sp[k4 + p]}, {k1, k2, k3, k4}]]
Out[5]= {4.343750, {{1, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Figure 5. Finding the master integrals with **FindMIs**.

```
In[1]:= << LiteRed`
        << Mint`
In[3]:= Declare[{k1, k2, k3, k4, p}, Vector];
        sp[p, p] = 1;
In[5]:= NewBasis[gnp,
          {-2 sp[k1, k2], -2 sp[k3, k4], -2 sp[k2, k3], -sp[k1], -sp[k2], -sp[k3], -sp[k4],
           1 - sp[k1 + p], 1 - sp[k1 + k2 + p], 1 - sp[k1 + k2 + k3 + p], 1 - sp[k1 + k2 + k3 + k4 + p],
           1 - sp[k2 + k3 + k4 + p], 1 - sp[k3 + k4 + p], 1 - sp[k4 + p]}, {k1, k2, k3, k4}];
In[6]:= CountMIs[js[gnp, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1]]
Out[6]= 2
In[7]:= FindMIs[js[gnp, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1]]
Out[7]= {j[gnp, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1],
          j[gnp, 0, 0, 0, 2, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1
```

Figure 6. Example of using `Mint` together with `LiteRed`.

them are equal due to symmetry relation. The accidental linear dependency between them, though may happen in principle, is very unlikely. The `Mint` package contains a procedure **FindMIs** which suggests the simplest integrals which can be chosen as masters. Its output is a list of multi-indices, corresponding to a possible choice of the master integrals, see figure 5. If `Mint` is loaded after `LiteRed`, the procedures **CountMIs** and **FindMIs** can be called directly for the sectors, see the example in figure 6

By default, the `Mint` package uses parametric representation to count the master integrals. If used with `LiteRed`, it can also rely on the Baikov representation. Presumably, this approach should be useful for higher sectors, when the number of numerators is small. The corresponding call of the procedures **CountMIs** and **FindMIs** should include option **Method** $\to$ "**GramP**". Before this call, the `LiteRed`'s procedure **FindSymmetries** should be called in order to determine the symmetries of the numerators. We should notice that both methods, the one based on parametric representation and the one based on the Baikov representation worked equally effective for the complicated cases, such as the one described in the next section. Moreover, the non-isolated critical points seem to appear simultaneously in both approaches.

5 Example: 4-loop onshell $g-2$ integrals

As a nontrivial example of the application of the above method, let us consider the family of the integrals shown in figure 7.

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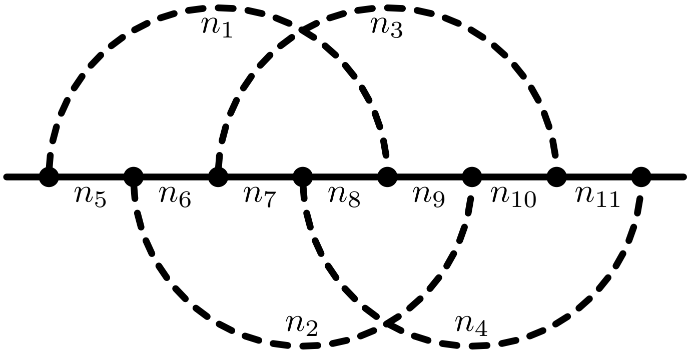

Figure 7. The family of integrals considered.

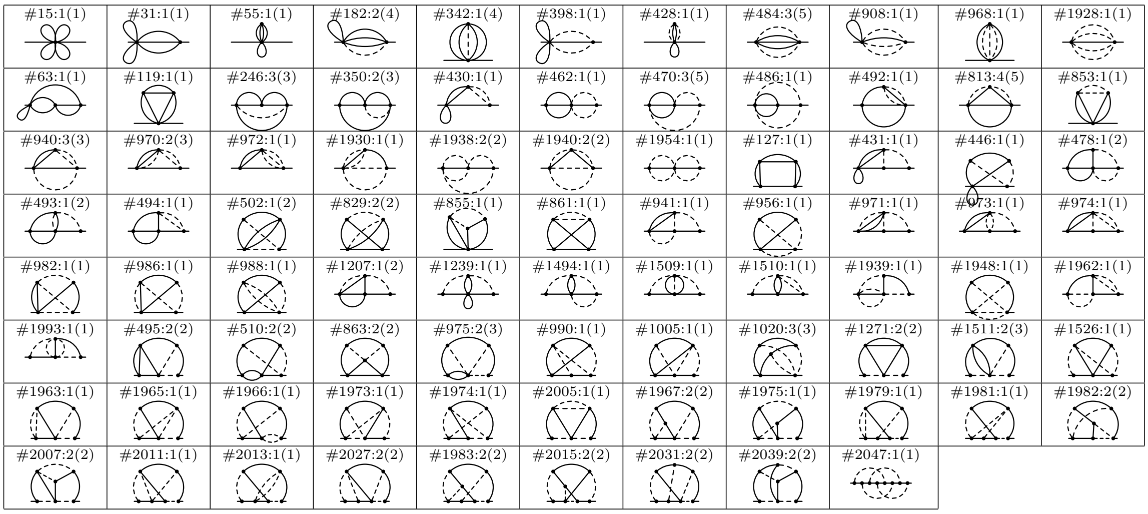

Figure 8. Master integrals.

There are 261 nonzero nonequivalent sectors in this family. Application of our counting method gives 84 sectors containing in total 119 master integrals. The graphs for each sector, together with the number of master integrals, are shown in figure 8. The sectors are numbered in the following way: for each sector its number is the string of indices of its simplest integral understood as binary number. E.g., $\#350 = 00101011110_2$ corresponds to the integrals with denominators $d_3, d_5, d_7, d_8, d_9, d_{10}$. The label above each diagram should be read as follows: $\#350 : 2(3)$ means that the sector #350 has 2 masters (3 masters) if the symmetries are used (not used).

Treating non-isolated critical points. Almost for all nonzero nonequivalent sectors the polynomials $G = F + U$ have isolated proper critical points. Out of 261 sectors there are 7 exceptions: sectors #246, #350, #414, #429, #821, #924, and #969. In each case there is a 1-dimensional critical variety. Let us explain how we determined the number of independent M-cycles for these cases on the example of sector #350. For this sector we

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have

$$
\begin{aligned}
G = F + U &= (z_{2356} + 1)\,(y_{1235} + y_{1236} + y_{1256} + y_{1356} + y_{2356}) \\
&+ z_4 z_{25}\,(z_{23456} + 1)\,(y_{13} + y_{16} + y_{36})\ ,
\end{aligned}
$$

where we have used the abbreviations $z_{ij\ldots k} = z_i + z_j + \ldots z_k$, $y_{ij\ldots k} = z_i z_j \ldots z_k$. The quotient algebra for the ideal $\mathcal{I}$, eq. (4.1), is infinite dimensional, which indicates non-isolated critical points. Then we calculate the primary decomposition of $\mathcal{I}$, e.g., by using `Sage` [22]. We have

$$
\mathcal{I} = \mathcal{I}_1 \cap \mathcal{I}_2,
$$

where

$$
\begin{aligned}
\mathcal{I}_1 &= \left\langle 5z_5 + 1,\ 5z_4 + 1,\ 5z_3 + 5z_6 + 1,\ 5z_2 + 1,\ 5z_1 - 1,\ 4z_0 + 3125,\ 25z_6^2 + 5z_6 - 1 \right\rangle\ , \\
\mathcal{I}_2 &= \langle 10z_6 + 3,\ 10z_3 + 3,\ 5z_2 + 5z_5 + 1,\ 20z_1 - 3, 27z_0 + 50000, \\
&\quad 100z_4^2 + 100z_5^2 + 20z_4 + 20z_5 - 3 \big\rangle\ .
\end{aligned}
$$

Both ideals are prime. The quotient space of the first ideal is 2-dimensional, in accordance with the fact, that the corresponding polynomial system has two solutions:

$$
\begin{aligned}
z^{(1)} &= \frac{1}{5}\,(1, -1, 1/\varphi, -1, -1, -\varphi)\ , \\
z^{(2)} &= \frac{1}{5}\,(1, -1, -\varphi, -1, -1, 1/\varphi)\ ,
\end{aligned}
$$

where $\varphi = \left(\sqrt{5} + 1\right)/2$. The quotient space of $\mathcal{I}_2$ is infinite dimensional. The middle homology of the algebraic variety determined by $\mathcal{I}_2$ obviously coincides with that of the variety in $\mathbb{C}^2$ determined by the equation $\tilde{G}\,(z_4, z_5) = 0$, where

$$
\tilde{G}\,(z_4, z_5) = 100z_4^2 + 100z_5^2 + 20z_4 + 20z_5 - 3\,.
$$

Remarkably, the basis of this homology can be found by exactly the same method that we used before, see, e.g. [15]. In fact, the homology basis is formed by the cycles (called the vanishing cycles) which are the intersection of the Lefschetz thimbles with the variety determined by $\tilde{G}\,(z_4, z_5) = 0$. We simply calculate the dimensionality of the quotient space of the ideal

$$
\tilde{\mathcal{I}} = \langle \partial\tilde{G}/\partial z_4, \partial\tilde{G}/\partial z_5, z_0\tilde{G} - 1\rangle\ .
$$

This dimensionality is equal to 1, which corresponds to one solution

$$
\tilde{z}^{(3)} = -\frac{1}{10}\,(1, 1)\ .
$$

In total we have two M-cycles passing through $z^{(1)}$ and $z^{(2)}$ and one M-cycle passing through the algebraic variety, corresponding to $\mathcal{I}_2$. Therefore, before taking the symmetry into account, there are 3 independent M-cycles, which corresponds to 3 master integrals. The symmetry of the integral $z_3 \leftrightarrow z_6$ results in $z^{(1)} \leftrightarrow z^{(2)}$, therefore, the two contours passing through $z^{(1)}$ and $z^{(2)}$ are symmetry equivalent. Thus, the account of symmetry relations leaves us with 2 master integrals.

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6 Conclusion

We have shown that the number of master integrals with a given set of denominators can be determined by examining the critical set of the polynomial $G = U + F$, where U and F are two Symanzik polynomials entering the parametric representation. Alternatively, one can consider critical set of the polynomial $P(D_1 = 0, \ldots D_M = 0, D_{M+1}, \ldots D_N)$ entering Baikov representation. In the case of isolated proper critical points, the number of master integrals is just the number of proper critical points counted with multiplicity. This equality follows from the construction of the independent integration contours as Lefschetz thimbles attached to the critical points. It seems that this geometrical construction should have some other applications beyond a simple counting of the master integrals. We have presented a simple *Mathematica* package `Mint` which automatically finds the number of master integrals with a given set of denominators.

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References

[1] K. Chetyrkin and F. Tkachov, *Integration by Parts: The Algorithm to Calculate β-functions in 4 Loops*, Nucl. Phys. **B 192** (1981) 159 [INSPIRE].

[2] F. Tkachov, *A Theorem on Analytical Calculability of Four Loop Renormalization Group Functions*, Phys. Lett. **B 100** (1981) 65 [INSPIRE].

[3] A. Smirnov, *Algorithm FIRE – Feynman Integral REduction*, JHEP **10** (2008) 107 [arXiv:0807.3243] [INSPIRE].

[4] A. von Manteuffel and C. Studerus, *Reduze 2 - Distributed Feynman Integral Reduction*, arXiv:1201.4330 [INSPIRE].

[5] C. Studerus, *Reduze-Feynman Integral Reduction in C++*, Comput. Phys. Commun. **181** (2010) 1293 [arXiv:0912.2546] [INSPIRE].

[6] R. Lee, *Presenting LiteRed: a tool for the Loop InTEgrals REDuction*, arXiv:1212.2685 [INSPIRE].

[7] A. Adolphson and S. Sperber, *On twisted de Rham cohomology*, *Nagoya Math. J.* **146** (1997) 55.

[8] A. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*, *Inv. Math.* **32** (1976) 1.

[9] M. Fedoryuk, *The Method of Steepest Descent* (in Russian), Nauka, Moscow (1977).

[10] F. Pham, *Vanishing homologies and the n variable saddlepoint method*, in *Singularities: Proceedings of the Summer Institute on Singularities, Humboldt, California, 1981* (1983).

[11] F. Pham, *La descente des cols par les onglets de Lefschetz, avec vues sur Gauss- Manin*, in *Systèmes différentiels et singularités, Colloq. Luminy/France 1983, Astérisque 130* (1985).

[12] A. Smirnov and A. Petukhov, *The Number of Master Integrals is Finite*, *Lett. Math. Phys.* **97** (2011) 37 [arXiv:1004.4199] [INSPIRE].

[13] S. Lefschetz, *L'analysis situs et la géométrie algébrique*, Gauthier-Villars, Paris France (1924).

[14] E. Witten, *Analytic Continuation Of Chern-Simons Theory*, arXiv:1001.2933 [INSPIRE].

[15] V. Arnold, S. Gusein-Zade and A. Varchenko, *Singularities of differentiable maps. Volume II: Monodromy and Asymptotics of Integrals*, Birkhäuser (1988).

[16] M. Marcolli, *Motivic renormalization and singularities*, in *Quanta of maths. Conference on non commutative geometry in honor of Alain Connes*, Paris, France, March 29–April 6, 2007, pp. 409–458.

[17] M. Marcolli, *Feynman Motives*, World Scientific, Singapore (2010).

[18] P. Baikov, *Explicit solutions of the multiloop integral recurrence relations and its application*, *Nucl. Instrum. Meth.* **A 389** (1997) 347 [hep-ph/9611449] [INSPIRE].

[19] R. Lee, *Calculating multiloop integrals using dimensional recurrence relation and D-analyticity*, *Nucl. Phys. Proc. Suppl.* **205-206** (2010) 135 [arXiv:1007.2256] [INSPIRE].

[20] P. Baikov, *A Practical criterion of irreducibility of multi-loop Feynman integrals*, *Phys. Lett.* **B 634** (2006) 325 [hep-ph/0507053] [INSPIRE].

[21] The `Mint` package can be downloaded from http://www.inp.nsk.su/~lee/programs/LiteRed/#utils.

[22] W. Stein et al., *Sage Mathematics Software (Version 5.11)*, The Sage Development Team, 2013, http://www.sagemath.org.