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Self-completeness and the generalized uncertainty principle

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Abstract: The generalized uncertainty principle discloses a self-complete characteristic of gravity, namely the possibility of masking any curvature singularity behind an event horizon as a result of matter compression at the Planck scale. In this paper we extend the above reasoning in order to overcome some current limitations to the framework, including the absence of a consistent metric describing such Planck-scale black holes. We implement a minimum-size black hole in terms of the extremal configuration of a neutral non-rotating metric, which we derived by mimicking the effects of the generalized uncertainty principle via a short scale modified version of Einstein gravity. In such a way, we find a self-consistent scenario that reconciles the self-complete character of gravity and the generalized uncertainty principle.

Keywords: Models of Quantum Gravity, Black Holes

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1 Introduction

It is a foregone conclusion that our classical understanding of gravitation is not applicable in the quantum regime. A number of resolutions to this inadequacy involving modifications to spacetime structure have been proposed, including string inspired models and spin-loop networks. A noted feature that has gained much traction over the last decade is the necessity of a minimal length scale that sets the quantum gravity threshold. This provides a natural platform for self-regularization of quantum field theories [1], and furthermore allows for quantum gravity to be realizable in $(3+1)$-dimensions.

Along these lines, it has been shown [2–5] that gravity may be considered *self-complete*, in the sense that there exists a minimum horizon scale hiding curvature singularities. Specifically, this distance is defined by the confluence of the classical Schwarzschild radius r_{H} and the Compton wavelength λ_{C},

$$r_{\mathrm{H}} = \lambda_{\mathrm{C}} \quad \Longrightarrow \quad \frac{2GM_{\mathrm{BH}}}{c^2} = \frac{h}{cM_{\mathrm{BH}}} \ . \tag{1.1}$$

This gives the mass of the lightest black hole

$$M_{\mathrm{BH}} \geq \sqrt{\frac{hc}{2G}} = \sqrt{\pi} M_{\mathrm{P}} \tag{1.2}$$

and, at the same time, the mass of heaviest quantum mechanical particle. As a result the Planck scale $M_{\mathrm{P}} = \sqrt{\hbar c/G}$ corresponds to the energy at which matter undergoes a transition from a particle phase to a black hole one. By looking at the corresponding length scale, one learns that the Planck length $\ell_{\mathrm{P}} \equiv M_{\mathrm{P}}^{-1}$ is the minimal size for both particles and black holes, which makes ℓ_{P} the smallest resolvable scale. From this perspective, the sub-Planckian world is dominated by light objects described by quantum mechanics, while the trans-Planckian world is dominated by classical objects described by GR.

The essence of self-completeness is also encoded in the generalized uncertainty principle (GUP). A simple way to understand the GUP is by considering a light pulse traveling some

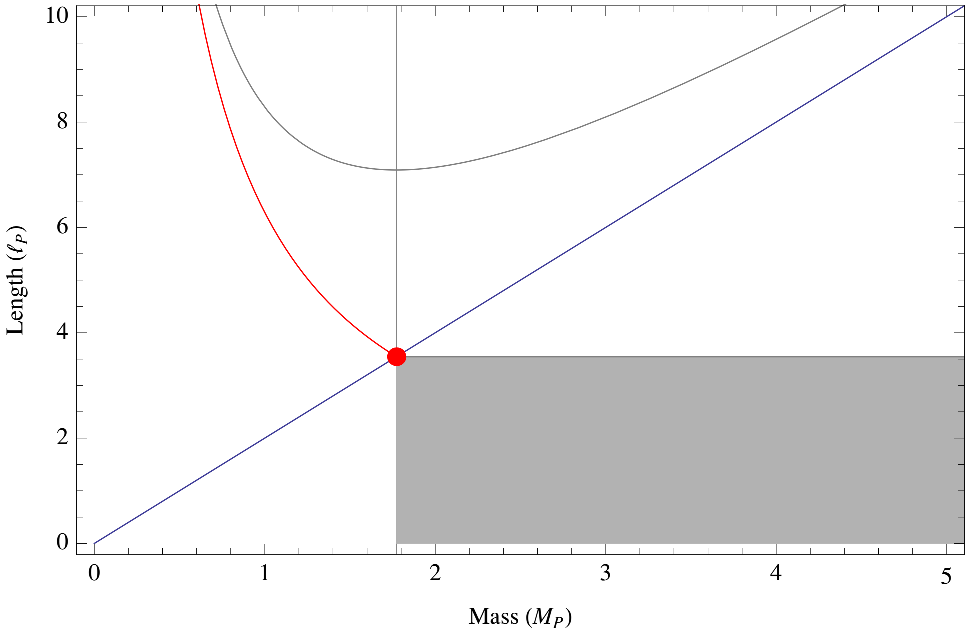

Figure 1. Length vs. mass for standard Schwarzschild solution. The Compton wavelength (red) and horizon radius (blue) curves intersect at $M = \sqrt{\pi}M_{\mathrm{P}}$, $l = 2\sqrt{\pi}\ell_{\mathrm{P}}$ (dot). Equation (1.3) approximates the behavior of both these curves (gray). The shaded area is excluded from experiment, while sub-planckian black holes are allowed.

distance l. The physical measurement of l is affected by an uncertainty $\Delta l_{\mathrm{w}} \sim \lambda$, where λ is the wavelength of the photon.

The energy associated with the light pulse can, however, distort the background spacetime. The measure of l will correspondingly change by an amount $\Delta l_{\mathrm{g}} \sim l(|\phi|/c^2)$, where ϕ is the Newton potential due to a photon of energy $\sim \hbar\nu$, and c is the speed of light. As a result of the above additional uncertainty, one can conclude that the total uncertainty of l is given by $\Delta l \sim \Delta l_{\mathrm{w}} + \Delta l_{\mathrm{g}} \sim \lambda + \ell_{\mathrm{P}}^2/\lambda$. Such a relation can be derived in several additional *Gedankenexperimente* [6–12] and is corroborated by string theory [13–15]. One can additionally extend this line of reasoning to generic particles of mass M to get

$$\Delta x \sim \frac{\hbar}{Mc} + \frac{GM}{c^2} \tag{1.3}$$

where Δx is the position uncertainty (see figure 1). By minimizing the above expression with respect to the mass, one discovers that the Planck length is again the minimal achievable length scale and that it clearly separates particles (whose size is governed by the Compton wavelength $\sim \hbar/Mc$) from black holes (whose size is governed by the Schwarzschild radius $\sim GM/c^2$).

The fact that black holes cannot be smaller than the Planck length and accordingly cannot be lighter than the Planck mass has repercussions on their emission spectra. The Hawking temperature can be obtained in terms of the energy of the emitted particles as $T \sim Mc^2$. By assuming in the vicinity of the black hole the uncertainty relation $M \sim \hbar/c\Delta x$ with $\Delta x \sim GM/c^2$, one can readily reproduce the Hawking result. Taking

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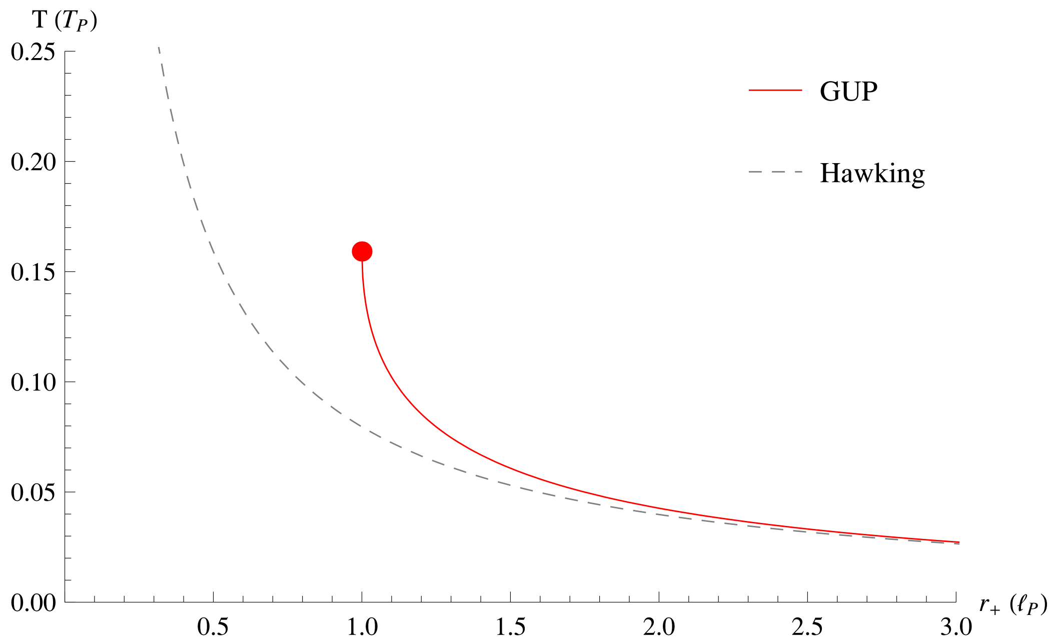

Figure 2. Black hole temperature vs. radius in a GUP framework (solid red) eq. (1.4) and Hawking temperature for a regular Schwarzschild black hole (dashed gray). The presence of a hot remnant is indicated by a red dot.

into consideration the relation (1.3), however, the Hawking temperature turns out to be

$$T \sim \frac{\hbar c}{2\pi}\left(\frac{\Delta x}{\ell_{\mathrm{P}}^2}\right)\left(1 \pm \sqrt{1-\frac{\ell_{\mathrm{P}}^2}{\Delta x^2}}\right). \tag{1.4}$$

The above equation reproduces the Hawking result in the limit $\Delta x \gg \ell_{\mathrm{P}}$ if the negative sign is chosen. Equation (1.4) shows relevant modifications when approaching scales $\sim \ell_{\mathrm{P}}$ and implies the existence of hot Planck scale black hole remnants, as shown in figure 2 [16].

Despite its virtues, the above analysis is handicapped by several weak points. For instance, we implicitly assume that quantum gravity effects can be treated semi-classically at scales on the order of the Planck length. On the contrary, one expects that deviations from the classical Schwarzschild radius should occur before the Planck scale, i.e. when one reaches energies $\lesssim M_{\mathrm{P}}$. This possibility is supported by the inadequacy of the Schwarzschild metric as an accurate description of the sub-Planckian spacetime.

In the particle phase, i.e. at energies $< M_{\mathrm{P}}$, matter is not sufficiently compressed to collapse into a black hole. The Schwarzschild metric, however, allows for black holes of any mass and size — even for $M_{\mathrm{BH}} < M_{\mathrm{P}}$ and $r_{\mathrm{H}} < \ell_{\mathrm{P}}$ — in sharp contrast to the aforementioned self-complete character of gravity. Such limitations of the Schwarzschild metric become more severe by noting that the temperature (1.4) cannot be derived by its surface gravity or from that of any known black hole solution of GR.

The GUP additionally introduces an ambiguity of the sign in eq. (1.4), whose positive sign choice has no physical meaning. Lastly, the resulting black hole remnants have been conjectured as a natural cold dark matter component. As mentioned above, these

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"remnants" do not have a vanishing temperature, as one would expect, but a Planckian temperature. These issues consequently cast doubts about the stability of such black hole remnants. By inspecting the heat capacity associated with eq. (1.4), i.e., $C = dM/dT$, one finds that it is negative and asymptotically vanishes for $r_{\rm H} \to \ell_{\rm P}$. This means that the system is suffering from the equivalent instabilities of conventional black hole evaporation. The emission persists as a runaway divergent process up to the Planckian regime. When $M_{\rm BH} \sim T \sim M_{\rm P}$, however, the Schwarzschild metric cannot longer describe the system "black hole + radiation" due to relevant quantum back reaction on the metric itself.

A viable solution to the above problems is offered by those families of quantum gravity improved black hole metrics that admit an extremal configuration even in the neutral, non-rotating case. Such metrics are inspired by a variety of formulations, including non-commutative geometry (NCG) [17–19], non-local gravity [20, 21], asymptotically-safe gravity [22], loop quantum gravity [23–26], vector ungravity [27] and Bardeen-like, short scale, quantum gravity effects [28, 29]. The degeneracy of the horizon allows for a minimum-size extremal black hole and lets one circumvent the above inconsistencies of the Schwarzschild metric.

As a by-product, the self-complete character of gravity is preserved in the case of black hole decay through Hawking emission. Contrary to the Schwarzschild metric, in which the curvature singularity can be exposed in the final stage of the evaporation, extremal configurations are zero temperature black holes also stable evaporation remnants. In this spirit, NCG-inspired black holes have been exploited to improve the self-completeness paradigm [30]. More recently, a Schwarzschild-like self-complete metric admitting horizon extremisation has been derived solely in the realm of GR without invoking additional principles like NCG, GUP, etc. [31]. In addition, such a new metric can pave the way to a solution of the recently-uncovered incompatibility between self-completeness and another widely expected character of quantum gravity, namely the spontaneous dimensional reduction of spacetime at the Planck scale [32].

In this paper, we further the above line of research and reconcile the ideas of GUP with the self-complete character of gravity in a consistent way. Rather than considering wavelength corrections as in (1.3), we follow the route of implementing a minimal resolution length $\sqrt{\beta}$ at the level of canonical commutators. Taking advantage of the resulting modifications of integration measures in momentum space, we derive a non-local version of the Schwarzschild geometry. We then exploit the properties of this new metric to draw further conclusions about self-completeness and GUP with special attention to resulting corrections at the Planck scale.

2 Generalized uncertainty principle

In regular quantum mechanics, the cannonical commutator,

$$[\mathbf{x}, \mathbf{p}] = i\hbar\,, \tag{2.1}$$

results in Heisenberg's well-known uncertainty relation between position and momentum

$$\Delta x \Delta p \geq \frac{\hbar}{2}\,. \tag{2.2}$$

However, if additional momentum dependent terms are added to eq. (2.1),

$$[\mathbf{x}^i, \mathbf{p}_j] = i\delta^i_j \hbar (1 + \beta \mathbf{p}^2)\,, \tag{2.3}$$

($\beta > 0$) this will result in a modified uncertainty relation of the form

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left(1 + \beta (\Delta p)^2\right)\ . \tag{2.4}$$

Such modification is known in the literature as the GUP. In turn, eq. (2.4) introduces a non-zero commutation between the coordinate operators

$$[\mathbf{x}_i, \mathbf{x}_j] = 2i\hbar\beta\left(\mathbf{p}_i \mathbf{x}_j - \mathbf{p}_j \mathbf{x}_i\right)\ . \tag{2.5}$$

Because this commutator is non-vanishing unless $\beta = 0$, the GUP introduces a non-zero minimal uncertainty in position, which translates into the existence of a minimal length $\sqrt{\beta}$ (for recent reviews on the huge literature in this field see [33, 34]). This implies that position eigenstates cannot exist and it is necessary to work with momentum eigenstates or limit ourselves to minimal-uncertainty position states [1]. Furthermore, this results in a momentum integration measure

$$\int \frac{d^n p}{1 + \beta \vec{p}^2} |p\rangle \langle p| = 1\,, \tag{2.6}$$

which presents a UV cutoff of $\sqrt{\beta}$, where n is the Euclidean space dimension [1].

GUP approaches have found a myriad of applications in high energy physics and quantum systems, including quantum field theory [35, 36], gauge theories [37], cosmology [38, 39] and particle physics [40]. Applications to black hole thermodynamics are of particular interest in the present context, and the interested reader is referred to [8, 10, 41–48] and references therein.

For our purposes, the implementation of GUP effects in the gravitational field requires certain discussion [21]. The suppression of the UV sector corresponds to a non-local deformation of the integration measure due to the action of a infinite number of derivative terms. As a result, GUP deformations can be encoded in non-local gravity actions. Such actions have been proposed with the goal of formulating a perturbative, super-renormalizable, UV finite approach to quantum gravity. In [49, 50], the following non-local Lagrangian has been proposed:

$$\begin{aligned}\mathcal{L}_{\mathcal{G}} = \sqrt{-g}\,\Bigg\{ &\frac{\beta}{\kappa^2} R - \beta_2 \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) + \beta_0 R^2 + \tilde{\lambda} \\ &+ \left(R_{\mu\nu}\, h_2\left(-\frac{\tilde{\Box}}{\Lambda^2}\right) R^{\mu\nu} - \frac{1}{3} R\, h_2\left(-\frac{\tilde{\Box}}{\Lambda^2}\right) R \right) - R\, h_0(-\frac{\tilde{\Box}}{\Lambda^2})\, R \Bigg\} \\ &- \frac{1}{2\xi} f^\mu[g] w\left(-\frac{\nabla^2}{\Lambda^2}\right) f_\mu[g] + \bar{c}^\mu M_{\mu\nu} c^\nu,\end{aligned} \tag{2.7}$$

where $\tilde{\Box} = \nabla^\mu \nabla_\mu$ and ∇^2 respectively denote the covariant and ordinary D'Alembertian, $f_\mu[g]$ is the gauge-fixing function with gauge-term weight w, $\bar{c}^\mu M_{\mu\nu} c^\nu$ is the Faddeev-Popov term, $\kappa^2 = 16\pi G$, Λ is some energy scale, $\tilde{\lambda}$ is the cosmological constant and h_0,

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h_2 are non-polynomial entire functions. The theory has been recently re-proposed in [51] and applied to massive gravity [52], the Starobinksi model [53] and to resolve the initial cosmological singularity [54]. A complementary formulation leading to the most general covariant, ghost-free gravitational action has been presented in [55].

Gravitation is widely expected to be asymptotically-safe [60]. This implies that, at the fixed point of the theory, interaction terms turn out to be negligible. One can, as a result, employ truncated versions of the Lagrangian (2.7) and derive the corresponding field equations by considering just the effects of the modified propagator [56–59]. From functional variation of the total action

$$S = S_{\mathrm{G}} + S_{\mathrm{M}}\,, \tag{2.8}$$

one finds

$$\mathcal{A}^2(\Box)\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 8\pi G T_{\mu\nu}\,, \tag{2.9}$$

where $\mathcal{A}(\Box)$ is a non-polynomial entire function (deriving from h_0 and h_2) of the dimensionless generally covariant D'Alambertian operator, $\Box = \ell^2 g_{\mu\nu}\nabla^\mu\nabla^\nu$, with $\ell \equiv 1/\Lambda$.

Following [21], the above equations can be cast in a more familiar form as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G \mathcal{T}_{\mu\nu}\,, \tag{2.10}$$

with $\mathcal{T}_{\mu\nu} \equiv \mathcal{A}^{-2}(\Box)T_{\mu\nu}$. In such a form, non-local effects are encoded into a non-standard source term couple to ordinary Einstein gravity. In the case of a static, spherically symmetric source, the conventional energy-momentum tensor displays an energy density peaked at the origin [27, 61], i.e.,

$$T_0^0 = -\frac{M}{4\pi r^2}\delta(r)\,, \tag{2.11}$$

where $\delta(r)$ is the Dirac delta function. The line element solving (2.10) will be static and spherically symmetric as usual:

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2, \tag{2.12}$$

$$f(r) = 1 - \frac{2G\mathcal{M}(r)}{r}\,, \tag{2.13}$$

with the unknown function $\mathcal{M}(r)$,

$$\mathcal{M}(r) = -4\pi\int_0^r dr' r'^2\, \mathcal{T}^0{}_0\,, \tag{2.14}$$

accounting for all non-local effects and necessarily satisfying $\mathcal{M}(r) \to M$ for $r \gg \ell$, where M is total mass-energy of the system.

In order to find $\mathcal{M}(r)$, it is necessary to choose a particular $\mathcal{A}(\Box)$. Unfortunately, there is to date no experimental information about quantum gravity and we possess no experimental restrictions on $\mathcal{A}$. We can nevertheless postulate the profile of the cuf-off function by invoking some reasonable physical principle. Along this line of reasoning, one can model the effect of the GUP by requiring the action of $\mathcal{A}^{-2}$ on T_0^0 to be given by

$$\mathcal{A}^{-2}(\Box)\delta(\vec{x}) = (2\pi)^{-3}\int \frac{d^3p}{1+\beta\vec{p}^2}e^{i\vec{x}\cdot\vec{p}}\,, \tag{2.15}$$

where $\vec{x}$ are free-falling, Cartesian-like coordinates, provided that $\beta = \ell^2$. From (2.15) it follows that the profile of $\mathcal{A}$ must be

$$\mathcal{A}(\Box) = (1 - \Box)^{1/2} \,. \tag{2.16}$$

By means of the Schwinger representation, the exponentiation of a generic differential operator $\hat{\Delta}$ can be written as

$$\hat{\Delta}^{\alpha} = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} ds\ s^{-\alpha-1}\ e^{-s\hat{\Delta}} \,. \tag{2.17}$$

As a consequence, by setting $\hat{\Delta} = 1 - \Box$ and $\alpha = 1/2$, one can represent $\mathcal{A}$ as

$$(1 - \Box)^{1/2} = -\frac{1}{2\sqrt{\pi}} \int_0^{\infty} ds\ s^{-3/2}\ e^{-s} e^{s\Box}. \tag{2.18}$$

The above expression reconciles the GUP and non-local gravity: it is evident that $\mathcal{A}$ acts as a non-polynomial entire function. Accordingly, $\mathcal{A}^{-2}$ can be obtained from the case $\alpha = -1$.

It is now straightforward to compute the energy density by applying the operator on the standard stress-energy tensor:

$$\mathcal{T}_0^0 = -M\mathcal{A}^{-2}(\Box)\delta(\vec{x}) = -\frac{M}{\beta}\frac{e^{-|\vec{x}|/\sqrt{\beta}}}{4\pi|\vec{x}|} \,. \tag{2.19}$$

Finally, integrating (2.19) we find

$$\mathcal{M}(r)/M = 1 - e^{-r/\sqrt{\beta}} - (r/\sqrt{\beta})e^{-r/\sqrt{\beta}} \,, \tag{2.20}$$

which means, by substitution in (2.12), that the GUP inspired metric is given by

$$ds^2 = -\left(1 - 2\frac{GM}{c^2 r}\gamma\left(2; r/\sqrt{\beta}\right)\right)dt^2 - \left(1 - 2\frac{GM}{c^2 r}\gamma\left(2; r/\sqrt{\beta}\right)\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{2.21}$$

where $\gamma(s;x) = \int_0^x t^{s-1}e^{-t}dt$ is the lower incomplete gamma function. The spacetime (2.21) matches the Schwarzschild metric at large distances, $(r \gg \sqrt{\beta})$. However the horizon structure is different. The corresponding metric coefficient is shown in figure 3.

By studying the horizon equation $g_{rr}^{-1} = 0$ we can distinguish three cases depending on the value of the total mass M with respect to a mass scale M_0:

i) for $M_{\rm BH} = M > M_0$ we have two horizons $r_{\pm}$. In the limit when $M \gg M_0$, the outer radius coincides with the standard value $(r_+ \to 2GM/c^2)$, while the inner one vanishes $(r_- \to 0)$;

ii) for $M_{\rm BH} = M = M_0$ the two horizons coalesce into a single degenerate horizon $r_+ = r_- = r_0$, corresponding to an extremal black hole solution;

iii) for $M < M_0$ the horizon equation cannot be solved and one has a horizon-less geometry.

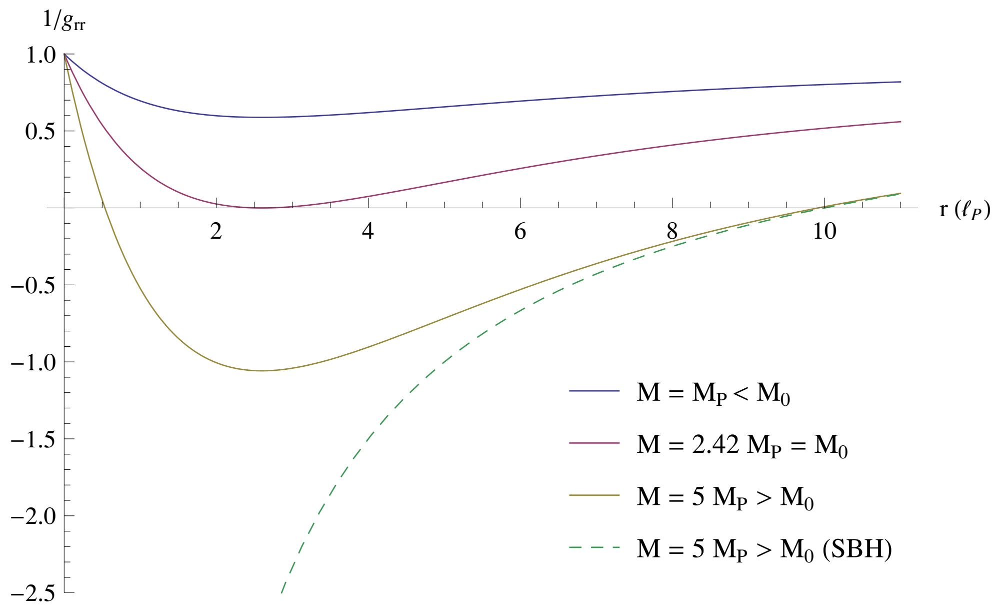

Figure 3. Metric coefficient for GUP metric (2.21) with $\sqrt{\beta} = 1.45\ell_{\mathrm{P}}$. Notice naked singularity, extremal and regular black hole cases. The Schwarzschild (SBH) case for $M = 5M_{\mathrm{P}}$ is showed for comparison. The minimum of the extremal case takes place at $M_0 \approx 1.67\sqrt{\beta}c^2/G$ and a $r_0 \approx 1.79\sqrt{\beta}$ for all values of β.

Note that the extremal configuration has a mass $M_0 \approx 1.66\sqrt{\beta}c^2/G$ and a radius $r_0 \approx 1.73\sqrt{\beta}$ (figure 3). Finally, at short scale, $r \approx 0$, the curvature singularity is softened but persists. This means that the vacuum energy associated to the virtual graviton exchange is divergent or, in other words, that the graviton propagator is not UV finite. One can verify this by looking at the short scale behavior of the energy density in (2.19): GUP effects can spread the Dirac into a distribution that is less pathological but still divergent as r^{-1}. We note that an unpleasant drawback of this is the exposure of the (naked) singularity in the horizon-less geometry case ($M < M_0$).

The above results do not come as a surprise. The UV finiteness of any non-local theory like that in (2.7) is guaranteed at any order only for a certain degree of convergence of the entire function $\mathcal{A}$. According to the definition given in [62, 63], such a global convergence occurs for entire functions of order higher than 1/2. As an example, NCG inspired black holes [17–19, 64–69] and the associated quantum field theory [70–72] are non-local formulations employing such a kind of entire function [20]. At the level of free fields the convergence is achieved also in the case of order 1/2. However, one can show that the GUP is represented by an entire function (2.15) of order lower than 1/2, *de facto* failing to improve the classical spacetime geometry [21]. For a full analysis of the geometry and the thermodynamics of the solution (2.21) see [21].

Despite the fact that the GUP inspired gravity fails to be UV finite, we wonder whether it may be at least self-complete, i.e. whether it is “always” able to mask this bad short-distance behaviour behind an event horizon. If this were the case also the previously raised issue of the naked singularity would turn to be circumvented.

3 Self-completeness

The metric (2.21) is an important step forward *en route* to a reconciliation between GUP and self-completeness. The presence of an extremal configuration naturally prevents the existence of black holes smaller than r_0. Furthermore, in the case of Hawking emission the usual black hole temperature definition $T = \kappa/2\pi$, where κ is the surface gravity of the metric (2.21), gives

$$T = \frac{\hbar c}{4\pi r_+}\left(1 - \frac{r_+^2}{\beta}\frac{e^{-r_+/\sqrt{\beta}}}{\gamma(2; r/\sqrt{\beta})}\right). \tag{3.1}$$

This temperature improves the result in (1.4), which cannot be associated to any surface gravity. (3.1) possesses a zero for a finite, positive value of r_+. Such a zero implies the existence of an evaporation remnant and has to coincide with the radius r_0 of the extremal configuration according to a general property of the horizon extremisation. This is a first step in the direction of self-completeness: one cannot probe the curvature singularity during the process of black hole decay.

We notice that such an evaporation end-point exhibits intriguing new properties. At $r_+ = r_{\max} \approx 4.20\sqrt{\beta}$ the temperature admits a maximum $T_{\max} \equiv T(r_{\max}) \approx 1.35 \times 10^{-2}\hbar c/\sqrt{\beta}$. This fact has important repercussions for the stability of the evaporation remnant. By examining the form of the heat capacity

$$C = \frac{\partial M}{\partial r_+}\left(\frac{\partial T}{\partial r_+}\right)^{-1}$$

one can distinguish three regimes: $C < 0$ for $r_+ > r_{\max}$, $C \to \pm\infty$ for $r_+ \to (r_{\max})_\mp$ and $C > 0$ for $r_0 < r_+ < r_{\max}$. The profile of C is controlled by the derivative of the temperature (sign and extremal points), being $\partial M/\partial r_+ 0$ positive and finite for $r_+ > r_0$ (see figure 4). From the above analysis one can conclude that, at the maximum temperature $T_{\max}$, the system undergoes a transition from an unstable negative heat capacity phase to a stable positive heat capacity cooling down towards a cold extremal configuration. The latter is characterized by both vanishing temperature and vanishing heat capacity ($\partial M/\partial r_+ = 0$ for $r_+ = r_0$) becoming a reliable candidate for cold dark matter component. We stress that during the process no relevant quantum back reaction occurs and no further short scale corrections have to be taken into account for the metric (2.21). This can be seen by noting that $T \ll M_{\rm BH}$ during all the evaporation, being $T/M_{\rm BH} < T_{\max}/M_0 \approx 8.06 \times 10^{-3} G\hbar/(\beta c)$.

To prove that the above scenario correctly describes the self-complete character of gravity, however, we need to show how the transition "particle $\leftrightarrow$ black hole" takes place.

Following the prescription outlined in [30], we start by deriving the radius of the extremal configuration. From the horizon condition $1/g_{rr} = 0$ one can define the mass parameter M as a function of the radius r_+,

$$M \equiv M_{\rm BH}(r_+) = \frac{c^2}{2G}\frac{r_+}{\gamma(2; r_+/\sqrt{\beta})}. \tag{3.2}$$

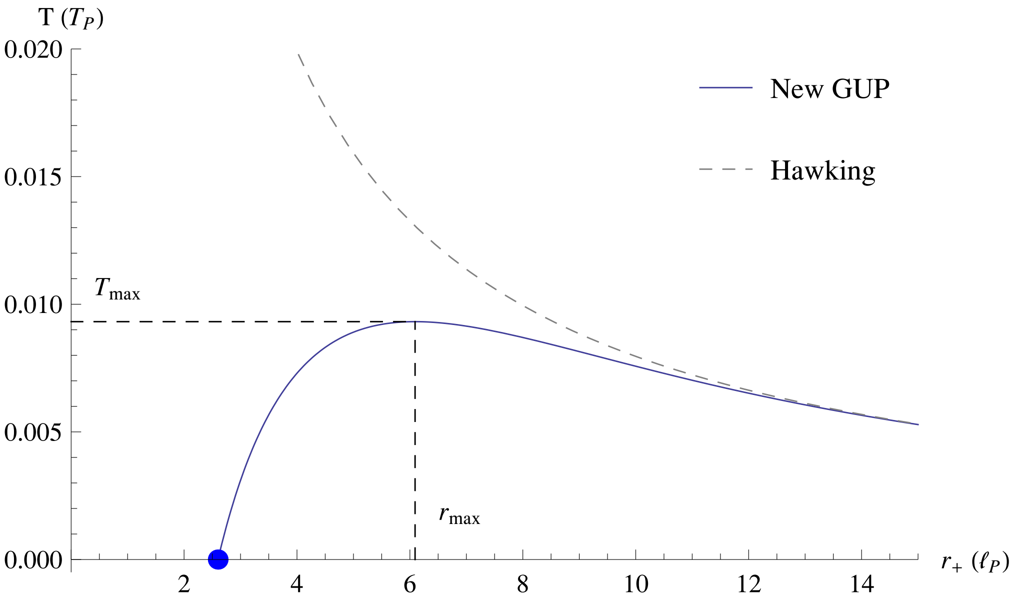

Figure 4. New GUP black hole temperature eq. (3.1) for $\sqrt{\beta} = 1.45\ell_\mathrm{P}$ (solid blue) and the regular Hawking temperature (dashed gray). The black hole achieves a maximum temparture $T_\mathrm{max} \approx 9.34 \times 10^{-3} T_\mathrm{P}$ at $r_\mathrm{max} \approx 4.20\sqrt{\beta}$. Unlike the old GUP temperature (cf. figure 2), our new solution yields a cold remnant (blue dot).

The minimum of this function can be calculated by considering $dM(r_+)/dr_+ = 0$, whose solution r_0, given by

$$\gamma(2; r_0/\sqrt{\beta}) - \left(\frac{r_0}{\sqrt{\beta}}\right)^2 e^{-r_0/\sqrt{\beta}} = 0, \tag{3.3}$$

identifies the radius of the extremal configuration for which the temperature (3.1) vanishes as expected.

Black holes can have radii $r_+ \geq r_0$, while at shorter scales the horizon equation has no solutions. That is: for $r_+ \leq r_0$ only quantum mechanical particles can exist. As a result, we assume that r_0 is the transition point between the two aforementioned phases. This fact is summarized in a the condition

$$\frac{h}{cM_0} = r_0. \tag{3.4}$$

where $M_0 \equiv M_\mathrm{BH}(r_0)$. We note that eq. (3.3) is independent of the parameter β and can be solved in terms of the dimensionless quantity $x_0 \equiv r_0/\sqrt{\beta}$. This allows us to fix the value of the parameter β in order to fulfil eq. (3.4) as

$$\beta = 4\pi \, \frac{\gamma(2; x_0)}{x_0^2} \, \ell_\mathrm{P}^2. \tag{3.5}$$

By introducing the dimensionless quantity $m_0 \equiv M_0 G/\sqrt{\beta} c^2$, one can write the above relation as $\beta = (2\pi/x_0 m_0)\ell_\mathrm{P}^2$. Accordingly we obtain

$$r_0 = \sqrt{\frac{2\pi x_0}{m_0}} \, \ell_\mathrm{P} \quad M_0 = \sqrt{\frac{2\pi m_0}{x_0}} \, M_\mathrm{P}. \tag{3.6}$$

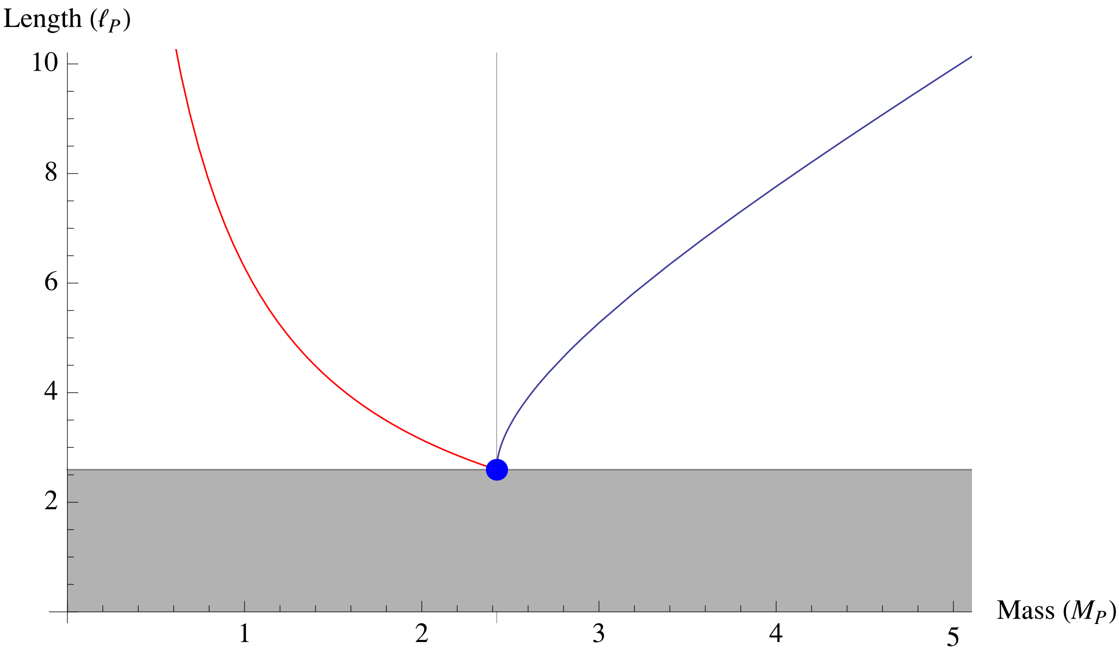

Figure 5. Plot of length vs. mass including new GUP corrections for $\sqrt{\beta} \approx 1.45\ell_{\rm P}$. The Compton wavelength (red) and horizon radius (blue) curves intersect at (M_0, r_0), marked by a dot. The shaded area is excluded from experiment, meaning there can never be an exposed singularity.

Recalling that numerical estimates give $x_0 \approx 1.79$ and $m_0 \approx 1.68$, we obtain $\sqrt{\beta} \approx 1.45\ell_{\rm P}$, $r_0 \approx 2.59\ell_{\rm P}$ and $M_0 \approx 2.42M_{\rm P}$ (see figure 5).

From here on, we can promote r_0 and M_0 as the new "fundamental scales". Indeed, these parameters identify a consistent transition between the two phases in both directions, i.e. during the compression ("particle $\rightarrow$ black hole") and during the decay ("particle $\leftarrow$ black hole"). We stress that the decay is correctly described in terms of thermal emission at the temperature associated with the surface gravity of the metric (2.21) without any ambiguity. In addition, the singularity can never be exposed during any of the two aforementioned processes, a fact that virtually eliminates the threat of a of naked singularity for $M < M_0$.

3.1 Wavelength correction

In light of the above results, we are now ready to re-formulate the *Gedankenexperiment* described in the introductory section. By writing eq. (2.21) as

$$ds^2 = -(1 + 2\phi_{\rm GUP})dt^2 + (1 + 2\phi_{\rm GUP})^{-1}dr^2 + r^2 d\Omega^2 \tag{3.7}$$

one obtains an improved Newtonian potential $\phi_{\rm GUP}$ that linearly vanishes at the origin, $\phi_{\rm GUP} \sim -GMr/\beta$, and matches the standard Newtonian potential, $\phi_{\rm GUP} \approx -GM/r$ at large distances. Such a quantity allows us to estimate the local spacetime distortion in terms of the GUP inspired non-local gravity, rather than in terms of standard Einstein gravity. As a result, one obtains a gravitational uncertainty $\Delta\lambda_{\rm g} = 2\pi\ell_{\rm P}^2\gamma(2, 2\pi\hbar G/c\lambda\sqrt{\beta})/\lambda$. By

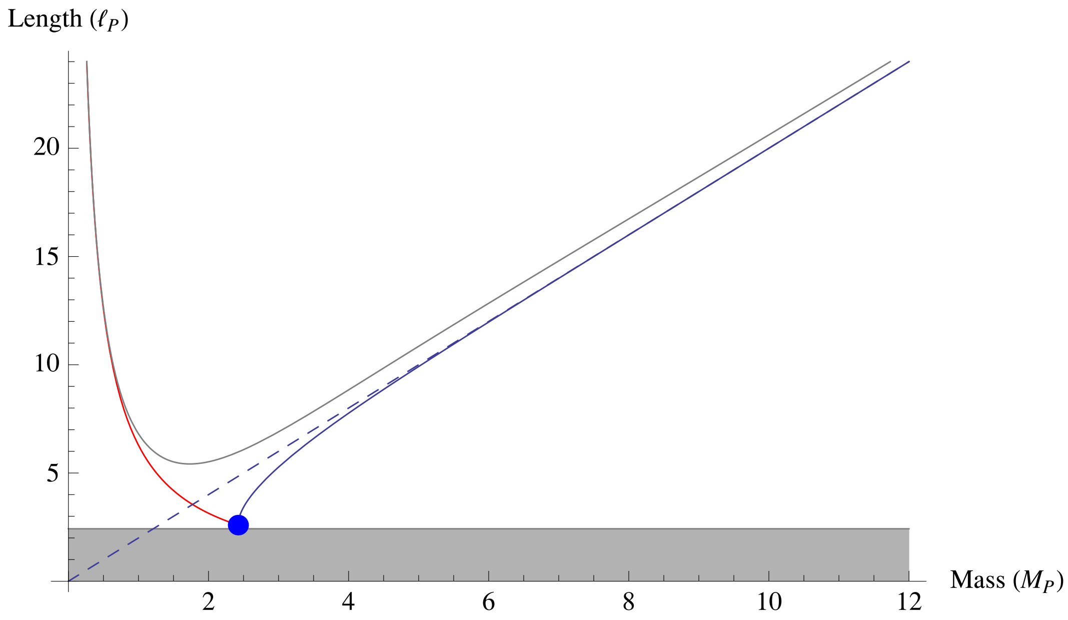

Figure 6. Length vs. mass plot including GUP corrections for $\sqrt{\beta} \approx 1.45\ell_{\rm P}$. Relation (3.8), shown in gray, approximates the behavior of both the Compton wavelength (red) and the GUP horizon radius (blue). The shaded area is excluded from experiment, meaning there can never be an exposed singularity.

considering the full uncertainty for an arbitrary massive particle, one can write

$$\Delta x \sim 2\pi \frac{\hbar}{Mc} + 2\frac{GM}{c^2}\gamma\left(2; \Delta x/\sqrt{\beta}\right) \tag{3.8}$$

in place of (1.3). As shown in figure 6, away from the Planck scale, the above relation works as (1.3), namely $\Delta x \approx 2\pi\frac{\hbar}{Mc}$ for quantum particles ($M \ll M_{\rm P}$), and $\Delta x \approx \frac{GM}{c^2}$ for classical black holes ($M \gg M_{\rm P}$). At the Planck scale ($M \sim M_{\rm P}$), however, the gamma function in (3.8) departs from unity, $0 < \gamma\left(2; \Delta x(M_{\rm P})/\sqrt{\beta}\right) < 1$. This corresponds to accounting for a crucial non-local gravity effect, namely the minimal black hole mass M_0. One then finds that for $M \sim M_0$

$$\Delta x \approx 2\pi \frac{\hbar}{M_0 c} + 2\frac{GM_0}{c^2}\gamma\left(2; r_0/\sqrt{\beta}\right) = 2r_0 \tag{3.9}$$

We stress that, contrary to the case in (1.3), the scale M_0 is corroborated by the corresponding metric. In this sense (3.8) provides a Planck scale completion of (1.3).

As a related comment we note that (3.8) is not in conflict with the uncertainty relations in (2.4). Rather, it is the "translation" of the deformed integration measure in (2.15) from locally flat coordinates to curvilinear ones. In such a transformation, the GUP inspired non-local gravity works in a more complicated way than Einstein gravity, by introducing nontrivial terms like the incomplete gamma function.

4 Conclusions

In this paper we showed how to reconcile the self-complete character of gravity with the GUP. We started by stressing that the conventional ideas at the heart of the GUP fail to be accurate at the Planck scale. Among these various limitations, the GUP implies the existence of black hole remnants that are not compatible with a neutral, classical metric like the Schwarzschild geometry. As a result, one ends up with an ambiguity between particles and black holes in the sub-Planckian regime.

Against this background, we exploited the idea of GUP at the level of integration measure in momentum space in order to construct a non-local version of Einstein's equations. By deriving the corresponding static, neutral black hole solution, we showed that Planck scale black hole remnants naturally emerge from the metric coefficients as extremal zero temperature configurations. This fact paves the way to a consistent scenario for the self-completeness that overcomes the standard case limitations. Black holes form as a result of matter compression to sizes of the order of the radius of the extremal configuration ($\sim \ell_{\rm P}$). A further increase of energy leads to bigger black holes that approach classical solutions of GR.

The reverse process is also free from pathologies. A black hole cannot endlessly decay. The evaporation end-point is represented again in terms of the aforementioned extremal configuration, which fulfils the special and unique feature of being at the same time the heaviest quantum particle and the lightest black hole. In addition, they enjoy the property of having both zero temperature and zero heat capacity, thus becoming a reliable candidate for dark matter component.

In principle GUP deformations of the integration measure in momentum space could be exploited to account for further corrections to the spectra of particles emitted by the black hole. Preliminary studies in this direction concerning the case of NCG-inspired black holes, however, show that these kind of corrections lead only to sub-leading effects [73]. Such a result is consistent with the general property of metrics admitting a maximum black hole temperature: the nature of the radiation is of secondary concern being the quantum backreaction negligible during the complete evaporation process.

Lastly, we considered a *Gedenkenexperiment* that summarizes the above results and improves the conventional reasoning. We introduced a new GUP that improves the conventional relations presented in [6–11] with non-local gravity corrections at the Planck scale. This satisfies all limiting cases for the expected black hole behavior by replacing standard Einstein gravity with the GUP inspired version.

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