

Yangian symmetry of smooth Wilson loops in $\mathcal{N} = 4$ super Yang-Mills theory

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ABSTRACT: We show that appropriately supersymmetrized smooth Maldacena-Wilson loop operators in $\mathcal{N} = 4$ super Yang-Mills theory are invariant under a Yangian symmetry $Y[\mathfrak{psu}(2, 2|4)]$ built upon the manifest superconformal symmetry algebra of the theory. The existence of this hidden symmetry is demonstrated at the one-loop order in the weak coupling limit as well as at leading order in the strong coupling limit employing the classical integrability of the dual $\text{AdS}_5 \times S^5$ string description. The hidden symmetry generators consist of a canonical non-local second order variational derivative piece acting on the superpath, along with a novel local path dependent contribution. We match the functional form of these Yangian symmetry generators at weak and strong coupling and find evidence for an interpolating function. Our findings represent the smooth counterpart to the Yangian invariance of scattering superamplitudes dual to light-like polygonal super Wilson loops in the $\mathcal{N} = 4$ super Yang-Mills theory.

KEYWORDS: Wilson, 't Hooft and Polyakov loops, AdS-CFT Correspondence, Conformal and W Symmetry

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1 Introduction and summary

The $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory is a distinguished four dimensional gauge theory that has been intensively studied in recent years. It is a finite, maximally supersymmetric and quantum superconformal four dimensional gauge theory which may be understood as an idealized version of QCD. In the planar limit the model with $SU(N)$ gauge group exhibits integrability, which manifests itself through an infinite dimensional extension of the superconformal symmetry algebra of Yangian type. Not being a symmetry

of the action integrability appears in gauge invariant observables of the theory with a non-trivial dependence on the 't Hooft coupling λ . Of course $\mathcal{N} = 4$ SYM theory is also the gauge theory with the best established string theory dual description in terms of the $\text{AdS}_5 \times S^5$ superstring. The string dynamics, described by a two-dimensional quantum field theory on the worldsheet, also appears to be integrable.

So far integrable structures have been detected in $\mathcal{N} = 4$ SYM for two- and three-point functions of local gauge invariant operators. Our understanding for the case of two-point functions which yield the local operator's scaling dimensions is the most mature. Exact results are obtained through a reformulation of the problem in terms of a dynamic super-spin chain, associated Bethe ansätze and their generalizations, see [1] for a comprehensive overview. This language was also successfully used recently for the study of three-point functions [2–7]. A further sector is that of scattering amplitudes in the gauge theory which are invariant under superconformal transformations, see [8] for an introduction. Here the discovery of a hidden dual superconformal symmetry [9], which combines with the conventional superconformal symmetry into a Yangian symmetry algebra [10] again points to an underlying integrability. The Yangian invariance of tree-level super-amplitudes in the theory was argued to extend to the loop-level integrands in [11]. The one-loop amplitudes enjoy a deformed symmetry [12, 13]. In $\mathcal{N} = 4$ SYM theory scattering amplitudes are dual to supersymmetrically extended Wilson loops with light-like polygonal boundaries [14–21]. A recent constructive application of integrability to the space-time S-matrix uses a decomposition of the dual Wilson loop into pentagon blocks and yields non-perturbative results in the form of an OPE-like expansion [22, 23]. All these developments point towards rich integrable structure behind amplitudes/null-polygon Wilson loops, which is not completely uncovered yet, partly due to breakdown or deformation of the superconformal and Yangian symmetries by the infrared (amplitudes) or ultraviolet (light-like Wilson loops) divergencies.

In this paper we turn to a prominent further class of observables in $\mathcal{N} = 4$ SYM and its string-dual being almost as old as the AdS/CFT correspondence proposal [24] itself: the Maldacena-Wilson loop operators [25, 26]. Here the loop-path variable couples next to the gauge field also to the adjoint scalars in the form [25]

$$W(C) = \frac{1}{N} \text{Tr} \mathcal{P} \exp \left(i \oint_C d\tau (A_\mu(x) \dot{x}^\mu + \Phi_i(x) |\dot{x}| n^i) \right) \quad \text{with } (n^i)^2 = 1. \quad (1.1)$$

The Maldacena-Wilson loop operators are locally 1/2 BPS symmetric, their expectation values are finite for smooth loops and are invariant under conformal transformations. The expectation value at strong coupling follows from the regularized minimal surface of an open string in anti-de-Sitter space ending on the four dimensional boundary on the curve C of the Wilson loop. As a direct consequence of integrability of the differential equations that determine the shape of the minimal surface, the minimal area satisfies a number of Ward identities of Yangian type which can be derived from the Hamilton-Jacobi formalism [27, 28] following an unpublished idea of Polyakov [29]. For the expectation value of the circular Maldacena-Wilson loop exact results to all orders in λ and $1/N$ are available [30, 31]. Given these properties it is natural to ask whether integrability in the sense of a hidden

Yangian symmetry exists for smooth Maldacena-Wilson loops both at weak and at strong coupling.¹ Our goal is to investigate possible Yangian symmetries of Wilson loops in detail.

As we will discuss the conformal symmetry for the Maldacena-Wilson loop is represented through functional derivative operators acting on the space of paths $x^\mu(\tau)$ e.g. for the special conformal transformations

$$\int ds k^\mu(s) \langle W(C) \rangle = 0, \quad \text{with } k^\mu(s) = x^2(s) \frac{\delta}{\delta x^\mu(s)} - 2x_\mu(s)x^\nu(s) \frac{\delta}{\delta x^\nu(s)},$$

and similarly for the dilatations $d(s)$ as well as Poincaré transformations. In order to establish the Yangian symmetry it turns out to be necessary to consider the supersymmetrization of the Maldacena-Wilson loop operator (1.1) describing a path in an *non*-chiral superspace $\{x^\mu(\tau), \theta_\alpha^A(\tau), \bar{\theta}_{A\dot{\alpha}}(\tau)\}$ with $\alpha, \dot{\alpha} = 1, 2$ and $A = 1, 2, 3, 4$. We establish this object up to second-order in anti-commuting path variables and show its superconformal invariance at leading order in perturbation theory. One can think of it as a smooth version of the light-like polygonal non-chiral super-Wilson loops of [20, 21] although we have not yet detailed the precise relation. At weak and strong coupling we show that a natural definition of the level-one generators of the Yangian algebra of $\mathfrak{psu}(2, 2|4)$ indeed annihilate the constructed super Maldacena-Wilson loops $\langle \mathcal{W}(C) \rangle$. Concretely for the level-one momentum generator $P_\mu^{(1)}$ we show

$$\left\{ \int_{s_1 < s_2} ds_1 ds_2 (d(s_1) \eta^{\mu\nu} - m^{\mu\nu}(s_1)) p_\nu(s_2) - \frac{i}{4} \bar{q}^{A\dot{\alpha}}(s_1) \bar{\sigma}_{\alpha\dot{\alpha}}^\mu q_A^\alpha(s_2) - (s_1 \leftrightarrow s_2) + f(\lambda) \int ds \left(\frac{\ddot{x}^2}{\dot{x}^4} - \frac{(\dot{x} \cdot \ddot{x})^2}{\dot{x}^6} \right) \dot{x}^\mu \right\} \langle \mathcal{W}(C) \rangle = 0 \quad (1.2)$$

at leading order in the weak or strong coupling expansion. Assuming that there are no fermionic corrections at the leading order of strong coupling expansion our results indicate that the function $f(\lambda)$ has limiting behavior

$$f(\lambda \ll 1) = \frac{7\lambda}{96\pi^2}, \quad \text{and} \quad f(\lambda \gg 1) = \frac{\lambda}{4\pi^2},$$

at weak and strong coupling. It would be interesting to understand the form of $f(\lambda)$ beyond these leading orders. Hence in this paper we provide good evidence for the existence of an infinite dimensional hidden symmetry of the super Maldacena-Wilson loops

$$J_a^{(n)} \langle \mathcal{W}(C) \rangle = 0, \quad J_a^{(n)} \in Y[\mathfrak{psu}(2, 2|4)], \quad n \in \mathbb{N}. \quad (1.3)$$

We note that this uncovered hidden symmetry (1.3) has strong similarities to a more than 30 year old speculation of Polyakov [32, 33] on the existence of a hidden symmetry for Wilson loops in pure Yang-Mills theory related to the integrability of the non-linear sigma model. In a sense (1.3) is a realization of this for $\mathcal{N} = 4$ SYM. Parallel to these works [32, 33] loop equations were proposed by Makeenko and Migdal [34, 35] in a related attempt to reformulate QCD as the dynamics of Wilson loops. Indeed generalizations of

¹This idea was jointly developed with N. Drukker, as well as independently with A. Sever and P. Vieira.

the loop equations to super Maldacena-Wilson loops in the AdS/CFT context were studied before in [27, 36, 37]. The Yangian symmetry generators (1.3) are somewhat different from the loop Laplacian appearing in the loop equations. First of all, the Yangian generators are non-local, unlike the Laplacian. They are also honest second-order variational operators and, in contradistinction to the loop Laplacian, do not satisfy the Leibniz rule. In the Makeenko-Migdal equations self-intersecting loops are very important. On the other hand Yangian and conformal symmetries in general get broken by cusps or self-intersections, and in this paper we only consider smooth, non-intersecting loops.

2 Maldacena-Wilson loop

The Maldacena-Wilson Loop operator in $\mathcal{N} = 4$ Super-Yang Mills Theory in Lorentzian signature $\eta^{\mu\nu} = \text{diag}(+, -, -, -)$ is given by (1.1), where $x^\mu(s) : [a, b] \rightarrow \mathbb{R}^{1,3}$ parametrizes the integration contour C , \mathcal{P} denotes path-ordering and n^i is a constant unit 6-vector which specifies a point on S^5 . Note also that we define the modulus as $|\dot{x}| := \sqrt{\dot{x}^2}$, hence for space-like velocity $|\dot{x}|$ is imaginary, i.e. $|\dot{x}| = i \|\dot{x}\|$. At leading order in perturbation theory one easily derives the correlation function

$$\langle W(C) \rangle = 1 - \frac{\lambda}{16\pi^2} \int d\tau_1 d\tau_2 \frac{\dot{x}_1 \cdot \dot{x}_2 - |\dot{x}_1| |\dot{x}_2|}{(x_1 - x_2)^2} + \dots, \tag{2.1}$$

where we have abbreviated $x_i := x(\tau_i)$, see appendix A and B for our conventions and propagators. In our discussion of the symmetries of the Maldacena-Wilson loop we will initially focus on the one-loop contribution for which we introduce the notation

$$\langle W(C) \rangle_{(1)} = -\frac{\lambda}{16\pi^2} \int d\tau_1 d\tau_2 I_{12} \quad \text{where} \quad I_{12} := \frac{\dot{x}_1 \dot{x}_2 - |\dot{x}_1| |\dot{x}_2|}{(x_1 - x_2)^2}. \tag{2.2}$$

2.1 Conformal invariance at one-loop

Before our discussion of possible hidden symmetries of the Maldacena-Wilson Loop we review its invariance under conformal transformations. We introduce the following natural functional derivative representation of the conformal algebra acting on the path $x^\mu(s)$ of the loop contour C

$$\begin{aligned} P_\mu &= \int ds p_\mu(s) = \int ds \frac{\delta}{\delta x^\mu(s)} \\ M_{\mu\nu} &= \int ds m_{\mu\nu}(s) = \int ds \left(x_\mu(s) \frac{\delta}{\delta x^\nu(s)} - x_\nu(s) \frac{\delta}{\delta x^\mu(s)} \right) \\ D &= \int ds d(s) = \int ds x^\mu(s) \frac{\delta}{\delta x^\mu(s)} \\ K_\mu &= \int ds k_\mu(s) = \int ds \left(x^2(s) \frac{\delta}{\delta x^\mu(s)} - 2x_\mu(s) x^\nu(s) \frac{\delta}{\delta x^\nu(s)} \right), \end{aligned} \tag{2.3}$$

introducing the momentum and angular momentum densities $p_\mu(s)$ and $m_{\mu\nu}(s)$ as well as the dilatation $d(s)$ and special conformal density $k_\mu(s)$. These generators satisfy the commutation relations (C.1).

Let us now prove the conformal invariance of the one-loop expectation value $\langle W(C) \rangle_{(1)}$. For this we note the functional derivatives

$$\begin{aligned} \frac{\delta x_\nu(\tau)}{\delta x^\mu(s)} &= \eta_{\mu\nu} \delta(\tau - s), \\ \frac{\delta \dot{x}_\mu(\tau)}{\delta x^\nu(s)} &= \eta_{\mu\nu} \partial_\tau \delta(\tau - s), \\ \frac{\delta |\dot{x}(\tau)|}{\delta x^\nu(s)} &= \frac{\dot{x}_\nu(\tau)}{|\dot{x}(\tau)|} \partial_\tau \delta(\tau - s). \end{aligned} \quad (2.4)$$

It comes as no surprise that already the integrand I_{12} of $\langle W(C) \rangle_{(1)}$ is translation invariant by acting on it with P_μ

$$\begin{aligned} P_\mu I_{12} &= \int ds \frac{\delta}{\delta x^\mu(s)} \frac{\dot{x}_1 \dot{x}_2 - |\dot{x}_1| |\dot{x}_2|}{(x_1 - x_2)^2} \\ &= \frac{1}{(x_1 - x_2)^2} \left[\left(\dot{x}_\mu(\tau_1) - \frac{|\dot{x}(\tau_1)|}{|\dot{x}(\tau_2)|} \dot{x}_\mu(\tau_2) \right) \int ds \partial_{\tau_2} \delta(\tau_2 - s) + (1 \leftrightarrow 2) \right] \\ &\quad - 2 \frac{\dot{x}_1 \dot{x}_2 - |\dot{x}_1| |\dot{x}_2|}{(x_1 - x_2)^4} (x_{1\mu} - x_{2\mu}) \int ds (\delta(\tau_1 - s) - \delta(\tau_2 - s)) \end{aligned}$$

By virtue of

$$\partial_{\tau_i} \int ds \delta(\tau_i - s) = 0 \quad \text{and} \quad \int ds (\delta(\tau_1 - s) - \delta(\tau_2 - s)) = 0 \quad (2.5)$$

we indeed find invariance at the integrand level

$$P_\mu I_{12} = 0.$$

A similar computation reveals that $M_{\mu\nu} I_{12} = 0$ and the scale invariance of I_{12} is manifest. For the generators K_μ of special conformal transformations the computation is a bit more involved. We first calculate the action of K_μ on the integrand I_{12} writing $x_{12} = x_1 - x_2$

$$\begin{aligned} K_\mu I_{12} &= \int ds (x^2(s) \delta_\mu^\nu - 2x_\mu(s) x^\nu(s)) \frac{\delta}{\delta x^\nu(s)} \frac{\dot{x}_1 \dot{x}_2 - |\dot{x}_1| |\dot{x}_2|}{(x_1 - x_2)^2} \\ &= -2 \left(\dot{x}_{1,\mu} \frac{\dot{x}_2 x_{12}}{x_{12}^2} - \dot{x}_{2,\mu} \frac{\dot{x}_1 x_{12}}{x_{12}^2} \right) = \dot{x}_{1,\mu} \partial_2 \ln(-x_{12}^2) + \dot{x}_{2,\mu} \partial_1 \ln(-x_{12}^2), \end{aligned}$$

which is a total derivative in each term. Therefore the integrated expression for a closed loop is invariant and we have

$$K_\mu \langle W(C) \rangle_{(1)} = 0,$$

as claimed. In fact the conformal invariance of $\langle W(C) \rangle$ may be shown via Ward identities beyond perturbation theory.

2.2 Towards a hidden Yangian symmetry

Inspired by the Yangian symmetry discovered for scattering amplitudes in $\mathcal{N} = 4$ SYM [10] which are dual to light-like supersymmetric Wilson loops [14–19, 21] it is natural to search for a parallel structure for the Maldacena-Wilson loop.

Yangian symmetries traditionally appear in 2d integrable field theories (see [38] for a review) and in fact our construction at strong coupling to be discussed in section 4 follows this. The Yangian algebra $Y(\mathfrak{g})$ of a simple Lie algebra \mathfrak{g} was introduced by Drinfeld [39, 40]. It is a deformation of the loop-algebra spanned by the generators $J_a^{(n)}$ with grading $n \in \mathbb{N}$. One demands the level-zero and level-one commutation relations

$$[J_a^{(0)}, J_b^{(0)}] = f_{ab}^c J_c^{(0)}, \quad [J_a^{(0)}, J_b^{(1)}] = f_{ab}^c J_c^{(1)}, \quad (2.6)$$

where we use mixed brackets $[\cdot, \cdot]$ to denote the graded commutator. In fact the first two level generators $J_a^{(0)}$ and $J_a^{(1)}$ span all of $Y(\mathfrak{g})$. In addition there is a set of representation dependent Serre relations, a generalized Jacobi-like identity. The higher level generators follow from commutators of the level-one generators. The distinguishing feature of the Yangian is a non-trivial co-product for the level-one generators

$$\Delta(J_a^{(0)}) = J_a^{(0)} \otimes \mathbf{1} + \mathbf{1} \otimes J_a^{(0)}, \quad \Delta(J_a^{(1)}) = J_a^{(1)} \otimes \mathbf{1} + \mathbf{1} \otimes J_a^{(1)} + f_a^{bc} J_b^{(0)} \otimes J_c^{(0)}. \quad (2.7)$$

Note that in the last term quadratic in $J_a^{(0)}$ the structure constant with ‘inverted’ indices appears. Indices are raised and lowered with the group metric $\frac{1}{2} \text{Tr}(J_{Ra}^{(0)} J_{Rb}^{(0)})$ with $J_{R,a}^{(0)}$ in the defining representation of \mathfrak{g} .

This is known as Drinfeld’s first realization of the Yangian. Importantly however, the Yangian is closely related to the Yang-Baxter equation which is central to the quantum inverse scattering method of integrability. $Y(\mathfrak{g})$ may be also given a Hopf algebra structure by introducing a co-unit and an antipode, but that will be of no relevance here.

An integrable 2d field theory realizes these structures in physics. The above co-products translate

$$J_a^{(0)} = \int ds j_a^{(0)}(s), \quad J_a^{(1)} = \int ds j_a^{(1)}(s) + f_a^{bc} \int_{s_1 < s_2} ds_1 ds_2 j_b^{(0)}(s_1) j_c^{(0)}(s_2). \quad (2.8)$$

where the level-one generators contain a non-local piece related to the non-trivial co-product of (2.7). The local contributions derive from a conserved current $j_{m,a}(s, \tau)$ with $\partial^m j_{m,a} = 0$ ($m = 0, 1$) in the sense of $j_a^{(0)}(s) = j_{0,a}(s, 0)$ and $j_a^{(1)}(s) = j_{1,a}(s, 0)$. Eqs. (2.6) are then obeyed at the classical level via Poisson-bracket relations.

For the case of our interest $Y(\mathfrak{psu}(2, 2|4))$ the level-zero generators of the superconformal group together with one level-one generator, e.g. the momentum $P_\mu^{(1)}$, span the entire Yangian

$$Y(\mathfrak{psu}(2, 2|4)) = \text{span} (P_\mu^{(0)}, M_{\mu\nu}^{(0)}, K_\mu^{(0)}, D^{(0)}, Q_A^\alpha{}^{(0)}, \bar{Q}^{A\dot{\alpha} (0)}; P_\mu^{(1)}). \quad (2.9)$$

The non-local terms in $J_a^{(1)}$ for $Y(\mathfrak{psu}(2, 2|4))$ have been constructed in a discrete representation in the context of super-amplitudes in [10]. For this representation the validity of the super-Serre relations was established [41].

It is then straightforward to translate the result of [10] to the continuous case. To begin with let us focus on the non-local contribution to the level-one momentum generator

which reads

$$P_{\text{nl}}^{(1)\mu} = \int ds_1 ds_2 \left\{ (d(s_1) \eta^{\mu\nu} - m^{\mu\nu}(s_1)) p_\nu(s_2) - \frac{i}{4} \bar{q}^{A\dot{\alpha}}(s_1) \bar{\sigma}_{\alpha\dot{\alpha}}^\mu q_A^\alpha(s_2) - (s_1 \leftrightarrow s_2) \right\} \theta(s_2 - s_1), \quad (2.10)$$

where $m^{\mu\nu}(s)$, $p^\mu(s)$ and $d(s)$ denote the densities of (2.3). Let us postpone the form of the super-charge densities $q_A^\alpha(s)$ and $\bar{q}^{A\dot{\alpha}}(s)$ for the moment and focus on the bosonic part of the level-one momentum generator denoted by $P_{\text{bos}}^{(1)\mu}$.

Before we embark on the explicit evaluation of $P_{\text{bos}}^{(1)\mu}$ acting on $\langle W(C) \rangle_{(1)}$ we need to discuss the regularization of (2.10). This is necessary as $P_{\text{bos}}^{(1)\mu}$ contains two functional derivatives which may act on the same point along the Maldacena-Wilson loop giving rise to ill defined terms such as $\delta(0)$. It is natural to introduce a point-splitting regulator ϵ by demanding that $s_2 > s_1 + \epsilon$ holds. However, this condition is not reparametrization invariant. For this one rather performs a point-splitting defined via a cut-off parametrized by ϵ of the arc-length via

$$s_1 < s_2 - d(s_2, \epsilon) \quad \text{with} \quad \int_{s_2-d(s_2,\epsilon)}^{s_2} ds \|\dot{x}(s)\| = \epsilon.$$

If one parametrizes the curve by arc-length, i.e. $\dot{x}^2 = -1$, as we always do in concrete calculations, this subtlety disappears and one simply has $d(s_2, \epsilon) = \epsilon$. The regularized level-one momentum generator follows from (2.10) by replacing

$$\theta(s_2 - s_1) \rightarrow \theta(s_2 - s_1 - d(s_2, \epsilon)).$$

Of course one may not confine oneself to arc length parametrization $\dot{x}^2 = -1$ before one has acted with the variational derivatives. After taking the derivatives one may then set $d(s_2, \epsilon) = \epsilon$. This being understood the bosonic part of the level-one momentum generator may be written as

$$\begin{aligned} P_{\text{bos}, \epsilon}^{(1)\mu} &= \int ds_1 ds_2 \left\{ (d(s_1) \eta^{\mu\nu} - m^{\mu\nu}(s_1)) p_\nu(s_2) - (s_1 \leftrightarrow s_2) \right\} \theta(s_2 - s_1 - \epsilon) \\ &= \int ds_1 ds_2 (d(s_1) \eta^{\mu\nu} - m^{\mu\nu}(s_1)) p_\nu(s_2) (\theta(s_2 - s_1 - \epsilon) + \theta(s_2 - s_1 + \epsilon)) \\ &\quad - \int ds_1 (d(s_1) \eta^{\mu\nu} - m^{\mu\nu}(s_1)) \int ds_2 p_\nu(s_2), \end{aligned}$$

where we have used $\theta(x) = 1 - \theta(-x)$ in the last step. The last term in the above factorizes into $(M^{\mu\nu} - D \eta^{\mu\nu}) P_\nu$ and we know already that it annihilates I_{12} defined in (2.2). Hence we only need to study the action of the generator

$$\tilde{P}_{\text{bos}, \epsilon}^{(1)\mu} = \int ds_1 ds_2 (d(s_1) \eta^{\mu\nu} - m^{\mu\nu}(s_1)) p_\nu(s_2) \theta(s_2 - s_1 - \epsilon) + (\epsilon \rightarrow -\epsilon) \quad (2.11)$$

on the vacuum expectation value of the Maldacena-Wilson loop. Details of this rather tedious calculation may be found in the appendix E. The final result we obtained reads

$$P_{\text{bos},\varepsilon}^{(1)\mu} \langle W(C) \rangle_{(1)} = \frac{\lambda}{16\pi^2} \left\{ \frac{1}{6} \int d\tau \dot{x}^\mu(\tau) \left(\frac{\dot{x}^2}{\dot{x}^4} - \frac{(\dot{x} \cdot \ddot{x})^2}{\dot{x}^6} \right) + 16 \int d\tau_1 d\tau_2 \frac{\dot{x}_1 \dot{x}_2 - |\dot{x}_1| |\dot{x}_2|}{(x_1 - x_2)^4} (x_1^\mu - x_2^\mu) \theta(\tau_2 - \tau_1 - d(\tau_2, \varepsilon)) \right\} \quad (2.12)$$

suppressing contributions proportional to $\delta(\varepsilon)$ as well as $\mathcal{O}(\varepsilon)$ terms, see (E.22) in the appendix for the full result.

We hence see that a bi-local expression remains under the action of $P_{\text{bos},\varepsilon}^{(1)\mu}$ which is to be expected as we have not taken into account the fermionic piece of the level-one momentum generator $P_{\text{ferm},\varepsilon}^{(1)\mu}$ in (2.10). The natural guess then is that the supersymmetric completion of the bosonic Maldacena-Wilson loop will be invariant under the full Yangian symmetry, as the functional form of the bi-local term above is that of a fermion-propagator in configuration space.

Hence the fermionic completion of the level-one momentum generator

$$P_{\text{ferm},\varepsilon}^{(1)\mu} = -\frac{i}{4} \int ds_1 ds_2 \bar{q}^{A\dot{\alpha}}(s_1) \bar{\sigma}_{\alpha\dot{\alpha}}^\mu q_A^\alpha(s_2) (\theta(s_2 - s_1 - \varepsilon) - \theta(s_1 - s_2 - \varepsilon)), \quad (2.13)$$

acting on the additional fermionic terms in the Maldacena-Wilson loop correlator to be established should cancel the unwanted bi-local term in (2.12).

3 Supersymmetric completion of the Maldacena-Wilson loop

The possibility of a supersymmetric completion of the Maldacena-Wilson loop (1.1) was already discussed in the early work on the subject [36]. Implicitly it was even constructed before that in [42] which established the super-connection for $\mathcal{N} = 1$ super Yang-Mills in 10d. The Maldacena-Wilson loop is then a specific light-like path in this higher dimensional superspace with 10d bosonic base. We will need the explicit form of the operator to higher orders in anticommuting coordinates. Also we work in a 4d formulation from the outset.

3.1 Construction of the super Maldacena-Wilson loop

The construction principle is clear: as we saw the bosonic Maldacena-Wilson loop is invariant under conformal transformations generated by operators acting as first-order functional derivatives in the space of bosonic loops $x^\mu(s)$. In order to supersymmetrize this we need to define paths in superspace. We choose a full non-chiral superspace parametrized by

$$x_{\dot{\alpha}\alpha}(s) = \sigma_{\dot{\alpha}\alpha}^\mu x_\mu(s), \quad \theta_\alpha^A(s), \quad \bar{\theta}_{A\dot{\alpha}}(s). \quad (3.1)$$

The need for a non-chiral superspace is easy to see: in order to cancel the bi-local contribution in (2.12) by acting with $P_{\text{ferm},\varepsilon}^{(1)\mu}$ on a fermionic correction to $\langle W(C) \rangle$ it is clear that this correction has to be of order $\theta\bar{\theta}$ as the supercharge densities \bar{q} and q start out as variational derivatives in $\bar{\theta}$ and θ respectively. Would we consider only a chiral superspace $\{x^\mu(s), \theta_\alpha^A(s)\}$ then the bi-local operator $P_{\text{ferm},\varepsilon}^{(1)\mu}$ would only give rise to terms of order θ^2 and the result of the bosonic action in (2.12) would not receive any purely bosonic corrections.

We then make the ansatz for the super Maldacena-Wilson loop

$$\mathcal{W}(C) = \frac{1}{N} \text{Tr} \mathcal{P} \exp (i I [A, \psi, \bar{\psi}, \phi; x, \theta, \bar{\theta}]) \quad (3.2)$$

with the exponent I possessing an expansion in Graßmann-odd variables

$$I [A, \psi, \bar{\psi}, \phi; x, \theta, \bar{\theta}] = \oint_C d\tau (\mathcal{I}_0 + \mathcal{I}_1 + \bar{\mathcal{I}}_1 + \mathcal{I}_{2m} + \mathcal{I}_2 + \bar{\mathcal{I}}_2 + \mathcal{O}(\{\bar{\theta}^i \theta^{3-i}\})). \quad (3.3)$$

Here \mathcal{I}_0 is the exponent of the usual Maldacena-Wilson loop (1.1). Consequently \mathcal{I}_n ($\bar{\mathcal{I}}_n$) are extensions of order n in θ ($\bar{\theta}$) and the mixed term \mathcal{I}_{2m} is of order $\theta\bar{\theta}$. Terms which contain higher orders of Graßmann odd coordinates can be neglected in this context, since corrections to the bosonic result (2.12) due to the fermionic part of the level-one momentum generator can only arise from terms of $\langle \mathcal{W}(C) \rangle_{(1)}$ which are of order $\theta\bar{\theta}$ as argued above. To be complete at order two in Graßmann odd coordinates we nevertheless will also derive the \mathcal{I}_2 and $\bar{\mathcal{I}}_2$ terms.

To proceed with the construction we note the relevant supersymmetry transformations of the fields generated by \mathfrak{Q}_A^α and $\bar{\mathfrak{Q}}^{A\dot{\alpha}}$

$$\mathfrak{Q}_A^\alpha (A^{\beta\dot{\beta}}) = 2i \epsilon^{\alpha\beta} \bar{\psi}_A^{\dot{\beta}} \quad \bar{\mathfrak{Q}}^{A\dot{\alpha}} (A^{\beta\dot{\beta}}) = -2i \epsilon^{\dot{\alpha}\dot{\beta}} \psi^{A\beta} \quad (3.4)$$

$$\mathfrak{Q}_A^\alpha (\bar{\phi}_{BC}) = \sqrt{2} i \epsilon_{ABCD} \psi^{D\alpha} \quad \bar{\mathfrak{Q}}^{A\dot{\alpha}} (\bar{\phi}_{BC}) = -\sqrt{2} i (\bar{\psi}_B^{\dot{\alpha}} \delta_C^A - \bar{\psi}_C^{\dot{\alpha}} \delta_B^A) \quad (3.5)$$

$$\mathfrak{Q}_A^\alpha (\psi^{B\beta}) = \frac{i}{2} F^{\alpha\beta} \delta_A^B + i \epsilon^{\beta\alpha} [\bar{\phi}_{AC}, \phi^{BC}] \quad \bar{\mathfrak{Q}}^{A\dot{\alpha}} (\psi^{B\beta}) = -\sqrt{2} D^{\beta\dot{\alpha}} \phi^{AB} \quad (3.6)$$

$$\mathfrak{Q}_A^\alpha (\bar{\psi}_B^{\dot{\beta}}) = -\sqrt{2} D^{\dot{\beta}\alpha} \bar{\phi}_{AB}$$

$$\bar{\mathfrak{Q}}^{A\dot{\alpha}} (\bar{\psi}_B^{\dot{\beta}}) = -\frac{i}{2} F^{\dot{\alpha}\dot{\beta}} \delta_B^A + i \epsilon^{\dot{\alpha}\dot{\beta}} [\phi^{AC}, \bar{\phi}_{BC}] \quad (3.7)$$

The construction principle for the supersymmetric Maldacena-Wilson loop is to require that the linearized supersymmetry field-transformations of the exponent term I may equally well be written as a supersymmetric transformation of the superpath $\{x_{\dot{\alpha}\alpha}(s), \theta_\alpha^A(s), \bar{\theta}_{A\dot{\alpha}}(s)\}$. For this we note the representation of the supersymmetry transformations Q_A^α and $\bar{Q}^{A\dot{\alpha}}$ acting in the path superspace

$$Q_A^\alpha = \int ds q_A^\alpha(s) = \int ds \left(-\frac{\delta}{\delta \theta_\alpha^A(s)} + i \bar{\theta}_{A\dot{\alpha}}(s) \frac{\delta}{\delta x_{\alpha\dot{\alpha}}(s)} \right) \quad (3.8)$$

$$\bar{Q}^{A\dot{\alpha}} = \int ds \bar{q}^{A\dot{\alpha}}(s) = \int ds \left(\frac{\delta}{\delta \bar{\theta}_{A\dot{\alpha}}(s)} - i \theta_\alpha^A(s) \frac{\delta}{\delta x_{\alpha\dot{\alpha}}(s)} \right). \quad (3.9)$$

The exponent I in (3.3) is now constructed in such a fashion to obey the key relations

$$\mathfrak{Q}_A^\alpha (I) = Q_A^\alpha (I) \quad \bar{\mathfrak{Q}}^{A\dot{\alpha}} (I) = \bar{Q}^{A\dot{\alpha}} (I), \quad (3.10)$$

i.e. the supersymmetry variations of the path equal the supersymmetry variations of the fields. This guarantees the invariance of the vacuum expectation value of the Maldacena-Wilson superloop seen by the following simple argument

$$0 = \langle \mathfrak{Q}_A^\alpha \mathcal{W}(C) \rangle = \frac{i}{N} \langle \text{Tr} \mathcal{P} \{ e^{iI} \mathfrak{Q}_A^\alpha I \} \rangle = \frac{i}{N} \langle \text{Tr} \mathcal{P} \{ e^{iI} Q_A^\alpha I \} \rangle = Q_A^\alpha \langle \mathcal{W}(C) \rangle, \quad (3.11)$$

where the zero on the left hand side follows from the invariance of the vacuum state.

Up to quadratic order in Grassmann variables we find the explicit expressions for the first few components of the Maldacena-Wilson loop exponent \mathcal{I} in (3.3)

$$\mathcal{I}_0 = \frac{1}{2} A^{\beta\dot{\beta}} \dot{x}_{\beta\dot{\beta}} - \frac{1}{2} \phi^{CD} \bar{\eta}_{CD} |\dot{x}| \quad (3.12)$$

$$\mathcal{I}_1 = i \theta^{B\dot{B}} \bar{\psi}_B^{\dot{\beta}} \dot{x}_{\beta\dot{\beta}} + \sqrt{2} i \theta_\beta^C \psi^{D\dot{\beta}} \bar{\eta}_{CD} |\dot{x}| \quad (3.13)$$

$$\bar{\mathcal{I}}_1 = -i \bar{\theta}_B^{\dot{\beta}} \psi^{B\dot{\beta}} \dot{x}_{\beta\dot{\beta}} - \sqrt{2} i \bar{\theta}_{C\dot{\beta}} \bar{\psi}_D^{\dot{\beta}} \eta^{CD} |\dot{x}| \quad (3.14)$$

$$\mathcal{I}_2 = -\frac{i}{\sqrt{2}} \theta_\gamma^C \theta^{B\dot{\beta}} (\partial^{\dot{\beta}\gamma} \bar{\phi}_{CB}) \dot{x}_{\beta\dot{\beta}} + \frac{1}{2\sqrt{2}} \theta_\beta^C \theta_\gamma^D F_{\text{lin}}^{\gamma\dot{\beta}} \bar{\eta}_{CD} |\dot{x}| + \sqrt{2} i \theta_\gamma^C \dot{\theta}^{B\dot{\beta}} \bar{\phi}_{CB} \quad (3.15)$$

$$\bar{\mathcal{I}}_2 = -\frac{i}{\sqrt{2}} \bar{\theta}_{C\dot{\gamma}} \bar{\theta}_B^{\dot{\beta}} (\partial^{\dot{\beta}\gamma} \phi^{CB}) \dot{x}_{\beta\dot{\beta}} - \frac{1}{2\sqrt{2}} \bar{\theta}_{C\dot{\beta}} \bar{\theta}_{D\dot{\gamma}} F_{\text{lin}}^{\dot{\beta}\gamma} \eta^{CD} |\dot{x}| + \sqrt{2} i \bar{\theta}_{C\dot{\gamma}} \dot{\theta}_B^{\dot{\beta}} \phi^{CB} \quad (3.16)$$

$$\begin{aligned} \mathcal{I}_{2m} = & \frac{1}{4} \theta_\gamma^B \bar{\theta}_B^{\dot{\beta}} F_{\text{lin}}^{\gamma\dot{\beta}} \dot{x}_{\beta\dot{\beta}} + \frac{1}{4} \theta^{B\dot{B}} \bar{\theta}_{B\dot{\gamma}} F_{\text{lin}}^{\dot{\beta}\gamma} \dot{x}_{\beta\dot{\beta}} + 2i \theta_\gamma^B \bar{\theta}_{C\dot{\beta}} (\partial^{\dot{\beta}\gamma} \bar{\phi}_{BE}) \eta^{CE} |\dot{x}| \quad (3.17) \\ & - \frac{i}{2} \theta_\gamma^B \bar{\theta}_{B\dot{\gamma}} (\partial^{\dot{\beta}\gamma} \phi^{CD}) \bar{\eta}_{CD} |\dot{x}| + \frac{i}{2} \dot{\theta}_\beta^B \bar{\theta}_{B\dot{\beta}} \phi^{CD} \bar{\eta}_{CD} \frac{\dot{x}^{\beta\dot{\beta}}}{|\dot{x}|} - \frac{i}{2} \theta_\beta^B \dot{\theta}_{B\dot{\beta}} \phi^{CD} \bar{\eta}_{CD} \frac{\dot{x}^{\beta\dot{\beta}}}{|\dot{x}|}. \end{aligned}$$

The details on the computation can be found in appendix E.2. At leading order in the θ expansion these terms coincide with the results spelled out in appendix C of [36] using a ten dimensional and euclidean notation.

It is now straightforward to compute the one-loop contribution to the vacuum expectation value of $\mathcal{W}(C)$. Using the conventions and propagators in appendix B we find

$$\begin{aligned} \langle \mathcal{W}(C) \rangle_{(1)} = & -\frac{\lambda}{4\pi^2} \int d\tau_1 d\tau_2 \left\{ \left(\frac{1}{4} - i(\bar{\theta}_2 \sigma_\mu \theta_1) \frac{x_{12}^\mu}{x_{12}^2} \right) \left(\frac{\dot{x}_1 \cdot \dot{x}_2 - |\dot{x}_1| |\dot{x}_2|}{(x_1 - x_2)^2} \right) \right. \\ & + (\bar{\theta}_2 \sigma_\mu \theta_1 - \bar{\theta}_2 \sigma_\mu \theta_2) \frac{\epsilon^{\mu\nu\rho\kappa} \dot{x}_{1\nu} \dot{x}_{2\rho} x_{12\kappa}}{(x_1 - x_2)^4} \\ & + \frac{i}{2} (\bar{\theta}_2 \sigma_\mu \dot{\theta}_1) \frac{\dot{x}_2^\mu}{x_{12}^2} - \frac{i}{2} (\dot{\theta}_2 \sigma_\mu \theta_1) \frac{\dot{x}_1^\mu}{x_{12}^2} \\ & \left. - \frac{i}{2} (\bar{\theta}_2 \sigma_\mu \dot{\theta}_2 - \dot{\theta}_2 \sigma_\mu \theta_2) \frac{1}{x_{12}^2} \frac{|\dot{x}_1|}{|\dot{x}_2|} \dot{x}_2^\mu \right\}. \quad (3.18) \end{aligned}$$

Taking this result we may indeed check the Maldacena-Wilson loop to be supersymmetric at one loop

$$Q_A^\alpha \langle \mathcal{W}(C) \rangle_{(1)} = 0 \quad \bar{Q}^{A\dot{\alpha}} \langle \mathcal{W}(C) \rangle_{(1)} = 0. \quad (3.19)$$

as is shown in appendix E.3.

Full superconformal invariance, especially invariance under $K_{\alpha\dot{\alpha}}$ at level-zero is, however, not yet expected. The reason for this is that our present choice of super-space $\{x_{\alpha\dot{\alpha}}, \theta_\alpha^A, \bar{\theta}_{A\dot{\alpha}}\}$ lacks the inclusion of bosonic coordinates y_A^B for the R -symmetry degrees of freedom. As is discussed in appendix D the closure of the $\mathfrak{su}(2, 2|4)$ algebra requires this inclusion in the form of derivatives in y_A^B for the representation of generators $S_A^\alpha, \bar{S}^{A\dot{\alpha}}$ and $K^{\alpha\dot{\alpha}}$. In this respect we have presently constructed the exponent $I(x, \theta, \bar{\theta}, y)$ of the super Maldacena-Wilson loop only for $y_A^B = 0$.

3.2 Yangian invariance of the super Maldacena-Wilson loop at weak coupling

Let us now turn to the key question of the potential level-one Yangian invariance of our result for $\langle \mathcal{W}(C) \rangle_{(1)}$ in (3.18). We recall the result of the action of the level-one momentum operator in the purely bosonic case in (2.12). The detailed evaluation of the corrections due to the presence of fermionic terms may be found in appendix E.4. The final result we obtained reads

$$P_{\text{ferm}, \varepsilon}^{(1)\mu} \langle \mathcal{W}(C) \rangle_{(1)} \Big|_{\substack{\theta=0 \\ \varepsilon=0}} = -\frac{\lambda}{16\pi^2} \left\{ 16 \int d\tau_1 d\tau_2 \frac{\dot{x}_1 \cdot \dot{x}_2 - |\dot{x}_1| |\dot{x}_2|}{(x_1 - x_2)^4} (x_1^\mu - x_2^\mu) \theta(\tau_2 - \tau_1 - d(\tau_2, \varepsilon)) \right. \\ \left. + \frac{4}{3} \int d\tau \dot{x}^\mu(\tau) \left(\frac{\ddot{x}^2}{\dot{x}^4} - \frac{(\dot{x} \cdot \ddot{x})^2}{\dot{x}^6} \right) \right\}. \tag{3.20}$$

Remarkably, the terms are of the same type as those we encountered in the bosonic result (2.12). In the above we have again dropped terms of order $\delta(\varepsilon)$ and $\mathcal{O}(\varepsilon)$ for the full result see (E.61).² Adding up the two contributions (2.12) and (3.20) yields the complete result

$$\lim_{\varepsilon \rightarrow 0} P_{\text{nl}, \varepsilon}^{(1)\mu} \langle \mathcal{W}(C) \rangle_{(1)} \Big|_{\theta=0} = -\frac{7\lambda}{96\pi^2} \int d\tau \left(\frac{\ddot{x}^2}{\dot{x}^4} - \frac{(\dot{x} \cdot \ddot{x})^2}{\dot{x}^6} \right) \dot{x}^\mu, \tag{3.21}$$

where all bi-local terms have canceled out! What remains after taking the limit $\varepsilon \rightarrow 0$ is a simple reparametrization invariant curve integral. This in fact defines the *local* contribution to the level-one momentum generator

$$P^{(1)\mu} := P_{\text{nl}}^{(1)\mu} + \frac{7\lambda}{96\pi^2} \int d\tau \left(\frac{\ddot{x}^2}{\dot{x}^4} - \frac{(\dot{x} \cdot \ddot{x})^2}{\dot{x}^6} \right) \dot{x}^\mu. \tag{3.22}$$

We have thus detected a local contribution to the level-one Yangian generators at the one-loop order, cf. equations (2.7) and (2.8). Then indeed up to this order in perturbation theory and at leading order in the θ -expansion we have uncovered a hidden symmetry of the appropriately supersymmetrized Maldacena-Wilson loop

$$P^{(1)\mu} \langle \mathcal{W}(C) \rangle = 0. \tag{3.23}$$

We shall now see that this symmetry may be also found at strong coupling.

4 Strong coupling analysis

At strong coupling the expectation value of the Wilson loop is determined by the regularized minimal area in AdS₅ [25, 26]:

$$\mathcal{W}(C) = e^{-\frac{\sqrt{\lambda} A(C)}{2\pi}}. \tag{4.1}$$

The area is computed by minimizing the string action:³

$$A(C) = \frac{1}{2} \int_{\text{reg}} d\tau ds \sqrt{h} h^{ab} \frac{1}{Z^2} (\partial_a X^\mu \partial_b X_\mu + \partial_a Z \partial_b Z) - \frac{L(C)}{\epsilon}. \tag{4.2}$$

²However, it is worth mentioning that the structure of the neglected $\delta(\varepsilon)$ -terms in both parts (i.e. the “bosonic part” and the “fermionic part”) of the calculation is the same, but their coefficients do not cancel out.

³We use the standard Poincaré-patch metric of AdS₅ and switch to the Euclidean signature both in target space and on the worldsheet.

The minimal surface is subject to the boundary conditions $Z(0, s) = 0$, $X^\mu(0, s) = x^\mu(s)$, where $x^\mu(s)$ parametrizes the contour C on the boundary. In this section we commit ourselves to the $\dot{x}^2 = 1$ gauge, and also fix the conformal gauge for the worldsheet metric: $h_{ab} = \delta_{ab}$.

As the area diverges at small Z , it has to be regulated by subtracting a boundary counter-term. Regularization consists in discarding the slice of the minimal surface with $Z < \epsilon$ for some small ϵ , subtracting a divergent counter-term proportional to the perimeter $L(C)$ of the Wilson loop, and then sending ϵ to zero.

Integrability of the string sigma-model in AdS_5 guarantees that Wilson loops satisfy Yangian Ward identities, just because the equations of motion for the minimal surface admit additional conservation laws. Following the idea of [29], we derive the Yangian identities for the minimal area from a higher analogue of the Hamilton-Jacobi equation.⁴

4.1 Integrability

The equations of motion of the sigma-model on AdS_5 are equivalent to the conservation law of the isometry current

$$\partial_i J^i = 0. \tag{4.3}$$

The current takes values in the $\mathfrak{so}(5, 1)$ isometry algebra

$$J_i = \partial_i X^m \hat{\Xi}_m, \quad \hat{\Xi}_m = \Xi_{ma} \hat{T}^a. \tag{4.4}$$

Here $i = \tau, s$ are the worldsheet Lorentz indices, $X^m = (X^\mu, Z)$ are the embedding coordinates of the string, Ξ_a^m , $a = 1 \dots 15$, are the AdS_5 Killing vectors, and \hat{T}^a are the generators of $\mathfrak{so}(5, 1)$ defined in appendix C. Because any cycle on the worldsheet is contractible, the charge associated with the isometry current equals to zero

$$Q \equiv \int_{\tau=\text{const}} ds J_\tau = 0, \tag{4.5}$$

see figure 1 for a pictorial argument. This equation is not an identity and is only valid on-shell, when the embedding coordinates satisfy the equations of motion. We will later derive conformal Ward identities for the minimal area from this equation.

The isometries of AdS_5 can be uplifted from the conformal transformations on the boundary. Indeed, if ξ_a^μ are the conformal Killing vectors on \mathbb{R}^4 satisfying

$$\partial_{(\mu} \xi_{\nu)a} = \frac{1}{2} \eta_{\mu\nu} \partial_\lambda \xi_a^\lambda, \tag{4.6}$$

then

$$\Xi_a^\mu = \xi_a^\mu - \frac{z^2}{8} \partial_\mu \partial_\nu \xi_a^\nu, \quad \Xi_a^z = \frac{z}{4} \partial_\mu \xi_a^\mu. \tag{4.7}$$

satisfy the Killing equation in the AdS metric

$$\nabla_{(m} \Xi_{n)a}^i = 0. \tag{4.8}$$

⁴The results in this section were obtained in collaboration with A. Sever and P. Vieira.

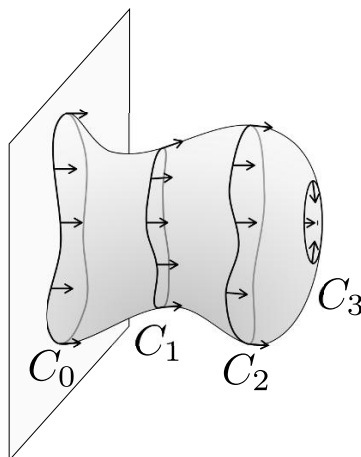


Figure 1. As any cycle C on the minimal surface worldsheet is contractible and the charges Q and $Q^{(1)}$ do not change under cycle deformations the initial cycle C_0 at the boundary may be shrunk to zero at the tip of the surface. This forces the charges Q and $Q^{(1)}$ to vanish.

Together with the equations of motion for the embedding coordinates, the Killing equation guarantees that the current (4.4) is conserved.

The explicit form of the conformal Killing vectors can be read off from

$$\hat{\xi}^\mu \equiv \xi_a^\mu \hat{T}^a = \hat{P}^\mu - \hat{M}^{\mu\nu} x_\nu + \hat{D}x^\mu + \hat{K}^\mu x^2 - 2\hat{K}^\nu x_\nu x^\mu. \quad (4.9)$$

The commutation relations (C.11)–(C.14) then imply⁵

$$[\hat{\Xi}_\mu, \hat{\Xi}_\nu] = -\frac{1}{2z^2} \partial_{[\mu} \hat{\xi}_{\nu]}, \quad [\hat{\Xi}_\mu, \hat{\Xi}_z] = -\frac{1}{z^3} \hat{\xi}_\mu - \frac{1}{8z} \partial_\mu \partial_\nu \hat{\xi}^\nu. \quad (4.10)$$

As a consequence of these equations together with (4.7), the current (4.4) is not only conserved but is also flat

$$\partial_i J_j - \partial_j J_i + 2[J_i, J_j] = 0. \quad (4.11)$$

The flatness condition is actually an identity, independent of whether the embedding coordinates of the string satisfy the equations of motion or not.

The existence of a flat conserved current is a hallmark of integrability. Such a current implies the existence of an infinite number of conserved charges, local or non-local depending on which basis one chooses. The Yangian symmetry is associated with the non-local charges. The first Yangian charge has the following form

$$Q^{(1)} = \frac{1}{2} \int ds_1 ds_2 \epsilon(s_1 - s_2) [J_\tau(s_1) J_\tau(s_2)] - \int ds J_s(s), \quad (4.12)$$

where $\epsilon(s) = \theta(s) - \theta(-s)$ is the anti-symmetric step function, cf. our discussion in section 2.2.

⁵The indices of $\hat{\xi}^\mu$ are raised and lowered with the flat Euclidean metric, while the indices of $\hat{\Xi}^m$ are transformed with the AdS₅ metric.

Usually, the Yangian charge is conserved only on an infinite line, while on a periodic interval the conservation condition acquires a boundary term. In our case the spacial coordinate on the worldsheet is periodic, but it turns out that the boundary term vanishes and the Yangian charge is exactly conserved. This can be understood from the following heuristic argument. A closed Wilson loop can be mapped to an open Wilson line passing through infinity by a global conformal transformation. The spacial coordinate will then have an infinite range and the Yangian charge will be automatically conserved. The conformal transformation that maps a finite loop to an infinite line is actually anomalous [31] (see also [43] for the string derivation), but we need not rely on this indirect argument, as the direct computation simply shows that the Yangian charge is conserved. Using the current conservation (4.3), we get

$$\partial_\tau Q^{(1)} = - \int ds (\partial_\tau J_s - \partial_s J_\tau + [J_\tau, J_s]) - [J_s(L) - J_s(0), Q] = 0. \quad (4.13)$$

The bulk term cancels due to the flatness condition (4.11), while the boundary contribution vanishes because the isometry charge Q is equal to zero. Therefore the Yangian charge is conserved, and in fact equals to zero

$$Q^{(1)} = 0, \quad (4.14)$$

by the same argument as in eq. (4.5) see again figure 1.

4.2 Conformal Ward identities

As a warm-up exercise we first derive conformal Ward identities for the Wilson loop at strong coupling from conservation of the isometry charge. The derivation relies on the Taylor expansion of the minimal surface near the boundary. Because the AdS metric is singular the first terms in this expansion are completely fixed by the equations of motion, the boundary conditions and the Virasoro constraints [27, 28]

$$\begin{aligned} X^\mu(s, \tau) &= x^\mu(s) + 0 \cdot \tau + \frac{1}{2} \ddot{x}^\mu(s) \tau^2 - \frac{1}{3} p^\mu(s) \tau^3 \\ &\quad - \left(\frac{1}{8} \ddot{\ddot{x}}^\mu + \frac{1}{3} \dot{x}^2 \ddot{x}^\mu + \frac{1}{3} \ddot{x}^\nu \ddot{x}_\nu \dot{x}^\mu \right) \tau^4 + \dots \end{aligned} \quad (4.15)$$

$$Z(s, \tau) = \tau + 0 \cdot \tau^2 - \frac{1}{3} \dot{x}^2(s) \tau^3 + \dots \quad (4.16)$$

The first coefficient that is not fixed by the boundary conditions at $\tau = 0$ is $p^\mu(s)$, but it can be related to the variational derivative of the minimal area [27, 28]

$$p^\mu(s) = \frac{\delta A}{\delta x_\mu(s)}. \quad (4.17)$$

Plugging the near-boundary expansion of the embedding coordinates into the isometry current (4.4) we get

$$J_\tau = (\dot{x}^\mu \hat{\xi}_\mu) \frac{1}{\tau} - \hat{\xi}^\mu \frac{\delta A}{\delta x^\mu} - \left[\frac{1}{24} \dot{x}^2 \partial_\mu \hat{\xi}^\mu + \left(\frac{1}{2} \ddot{x}^\mu + \frac{2}{3} \dot{x}^2 \dot{x}^\mu \right) \dot{\hat{\xi}}_\mu \right] \tau + O(\tau^2) \quad (4.18)$$

$$J_s = \dot{x}^\mu \hat{\xi}_\mu \frac{1}{\tau^2} + \left(\ddot{x}^\mu + \frac{2}{3} \dot{x}^2 \dot{x}^\mu \right) \hat{\xi}_\mu - \left(\frac{1}{2} \ddot{x}^\mu \hat{\xi}_\mu + \frac{1}{8} \partial_\mu \hat{\xi}^\mu \right) \dot{\hat{\xi}}^\mu + O(\tau). \quad (4.19)$$

All time-depend terms in J_τ^i are total derivatives and integrate to zero, as they should. This is just a consequence of charge conservation. As the total charge vanishes the zeroth-order term should also integrate to zero. This gives the constraint

$$\oint ds \xi_a^\mu \frac{\delta A}{\delta x^\mu} = 0, \tag{4.20}$$

being nothing but the conformal Ward identity for the regularized minimal area. We thus formally proved that the minimal area, and with it the Wilson loop at strong coupling are invariant under infinitesimal conformal transformations. It is the strong coupling counterpart of our discussion in section 2.1.

4.3 Yangian Ward identities

The Yangian Ward identity is derived in the same way, by expanding the condition $Q^{(1)} = 0$ at small τ . Using (4.18), (4.19) we find at order $O(1/\tau^2)$ and $O(1/\tau)$

$$\oint dx^\mu \left\{ \hat{\xi}_\mu + [\hat{\xi}_\mu, (\dot{x}^\nu \hat{\xi}_\nu)] \right\} = 0 \tag{4.21}$$

$$\oint dx^\mu [\hat{\xi}_\mu, \hat{\xi}^\nu] \frac{\delta A}{\delta x^\nu} = 0. \tag{4.22}$$

As expected these equations are identically satisfied by virtue of eqs. (C.11) and (C.12).

A non-trivial equation is obtained at the next order $O(\tau^0)$ term in $Q^{(1)}$

$$\frac{1}{2} \int ds_1 ds_2 \epsilon(s_1 - s_2) [\hat{\xi}_1^\mu, \hat{\xi}_2^\nu] \frac{\delta A}{\delta x_1^\mu} \frac{\delta A}{\delta x_2^\nu} - \int ds \hat{\xi}^\mu (\ddot{x}^2 \dot{x}_\mu + \ddot{x}_\mu) = 0. \tag{4.23}$$

In the course of the derivation we used the Killing vector identities from the appendix C, which greatly simplify the local term.

Finally, given the minimal area law (4.1), we find that the Wilson loop vacuum expectation value at strong coupling satisfies a second-order variational equation

$$Q_a^{(1)} \mathcal{W}(C) = 0 \tag{4.24}$$

with⁶

$$Q_a^{(1)} = f_a^{bc} \int ds_1 ds_2 \epsilon(s_1 - s_2) \xi_{1b}^\mu \xi_{2c}^\nu \frac{\delta^2}{\delta x_1^\mu \delta x_2^\nu} - \frac{\lambda}{2\pi^2} \int ds \xi_a^\mu (\ddot{x}^2 \dot{x}_\mu + \ddot{x}_\mu) \tag{4.25}$$

Projected onto the momentum generator, $P^{(1)\mu}$, this expression has exactly the same structure as the bosonic part of the Yangian generator at weak coupling (3.22), which was written in an ungauged fashion. Taking $\dot{x}^2 = 1$ in (3.22) one recovers the above local term except for the value of the coefficient of the local term that differs by a factor of 7/24.

⁶An additional second-derivative term that arises upon application of this operator to (4.1) has relative order $O(1/\sqrt{\lambda})$ and can be neglected in the $\lambda \rightarrow \infty$ limit.

5 Conclusions and outlook

In this work we have presented substantial evidence for the existence of a hidden Yangian symmetry for smooth supersymmetric Maldacena-Wilson loops in $\mathcal{N} = 4$ SYM theory. For this the level-one generators of the Yangian algebra were shown to annihilate the expectation value of the Wilson loop operator at leading order perturbation theory as well as at leading order in the strong coupling limit upon employing the classical AdS string description. While the classical AdS string analysis remained purely bosonic, on the weak coupling gauge theory side it was necessary to consider the supersymmetric completion of the original Maldacena-Wilson loop operator of [25]. This completion requires the definition of a loop operator coupling to all the fields of the $\mathcal{N} = 4$ multiplet to a path in an off-shell superspace coordinatized by $\{x_{\alpha\dot{\alpha}}, \theta_{\alpha}^A, \bar{\theta}_{A\dot{\alpha}}, y_A^B\}$. We have explicitly constructed this Wilson loop operator to quadratic order in anti-commuting coordinates and for $y_A^B = 0$. After computing the one-loop vacuum expectation value of this operator the invariance under the action of the Yangian level-one momentum generator was established. Here next to the canonical non-local piece a local contribution to the Yangian generator appeared. Compared to the Yangian generators annihilating super-amplitudes (or light-like polygonal super Wilson loops) the emergence of such a local term is novel, although it does appear in the spectral parameter deformed amplitudes of [44, 45]. Consistently the same variational symmetry generators were shown to also annihilate the minimal surface at strong coupling. The only difference here is a differing numerical coefficient in front of the local-piece of the level-one generator. It would be interesting to investigate the light-like limit of our construction and find its relation to the light-like polygonal super Wilson loops [17–19, 21]. We note that naively our local term is singular in the light-like limit.

A further issue is the above-mentioned coefficient in front of the local term. In general it should be a function of the coupling constant λ . Interestingly we find that in both limits — at leading order in weak and strong coupling — this function is of order λ . The two coefficients, however, do not agree. We can offer two possible explanations. For one it is natural to expect the existence of an interpolating function in front of the local term receiving corrections to the limits considered. That function happens to limit to a linear behavior in the weak and strong coupling limits. On the other hand it is intriguing that opposed to the weak coupling analysis it was not necessary at the strong coupling to include the fermionic degrees of freedom. From the perspective of the IIB superstring in $\text{AdS}_5 \times S^5$ a superpath is actually natural as the superstring ends on the trajectory of a superparticle on the boundary. Whether the inclusion of fermionic degrees of freedom on the string side will affect the purely bosonic local term and its coefficient is unclear to us at this point. We cannot exclude the possibility that switching on the y_A^B couplings on the weak-coupling side can also modify the result. In this context it would be also interesting to explore the consequences of κ -symmetry on the boundary.

One may wonder how to interpret a Wilson loop in superspace physically. A good way to think about it is that the $\theta_{\alpha}^A(s)$ and $\bar{\theta}_{A\dot{\alpha}}(s)$ parameters capture the polarizations of a (super)-particle carried along the loop in $\mathbb{R}^{1,3}$. If one is interested only in the standard Wilson loop one simply projects to the $\theta_{\alpha}^A(s) = \bar{\theta}_{A\dot{\alpha}}(s) = 0$ part. Nevertheless the consid-

ered extension may be useful in establishing new (exact) results on a general $\langle \mathcal{W}(C) \rangle$. One could draw a similarity to going super for the on-shell BCFW recursions for amplitudes in the theory [46–48], which led to a complete analytic solution [49] at tree-level.

In any case the hidden Yangian symmetry of Wilson loops should constrain the functional form of the vacuum expectation value. It will be important to understand the structure of invariants and the consequences for possible exact results. Finally, the question of how to include a spectral parameter into our considerations is an obvious one. On the string side equivalence classes of solutions with identical regularized areas ending on smooth contours have been constructed in [50] parametrized by a spectral parameter. For the polygonal light-like situation similar structures were identified in [51]. It is tempting to speculate on a relation to our findings.

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A Conventions

Minkowski space. We follow the conventions of [52]. Our signature is $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$. Spinor indices are raised and lowered according to

$$\begin{aligned} \lambda^\alpha &= \epsilon^{\alpha\beta} \lambda_\beta, & \lambda_\alpha &= \lambda^\beta \epsilon_{\beta\alpha}, & \lambda_{\dot{\alpha}} &= \epsilon_{\dot{\alpha}\dot{\beta}} \lambda^{\dot{\beta}}, & \lambda^{\dot{\alpha}} &= \lambda_{\dot{\beta}} \epsilon^{\dot{\beta}\dot{\alpha}} \\ \epsilon^{12} &= \epsilon_{12} = 1, & \epsilon^{i\dot{2}} &= \epsilon_{i\dot{2}} = -1 & \Rightarrow & \epsilon^{\alpha\beta} \epsilon_{\gamma\beta} &= \delta_\gamma^\alpha, & \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon_{\dot{\gamma}\dot{\beta}} &= \delta_{\dot{\gamma}}^{\dot{\alpha}} \end{aligned} \quad (\text{A.1})$$

We note

$$\sigma^{\mu\dot{\alpha}\beta} = (\mathbf{1}, \sigma), \quad \bar{\sigma}^\mu_{\alpha\dot{\beta}} = (\mathbf{1}, -\sigma) \quad (\text{A.2})$$

with the vector σ of Pauli matrices

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If not stated otherwise the index position of the matrices σ and $\bar{\sigma}$ is given by (A.2). They can be identified as follows:

$$\sigma^{\mu\dot{\alpha}\beta} = \epsilon^{\beta\gamma} \bar{\sigma}^\mu_{\dot{\gamma}\dot{\delta}} \epsilon^{\dot{\delta}\dot{\alpha}} = \bar{\sigma}^{\mu\beta\dot{\alpha}} \quad \bar{\sigma}^\mu_{\alpha\dot{\beta}} = \epsilon_{\dot{\beta}\dot{\gamma}} \sigma^{\mu\dot{\gamma}\delta} \epsilon_{\delta\alpha} = \sigma^\mu_{\dot{\beta}\alpha}.$$

Contractions of space-time or spinor indices give the following results:

$$\bar{\sigma}_{\alpha\dot{\beta}}^{\mu}\bar{\sigma}_{\mu\gamma\dot{\delta}} = -2\epsilon_{\alpha\gamma}\epsilon_{\dot{\beta}\dot{\delta}}, \quad \sigma_{\dot{\alpha}\beta}^{\mu}\sigma_{\mu\dot{\gamma}\delta} = -2\epsilon_{\beta\delta}\epsilon_{\dot{\alpha}\dot{\gamma}}, \quad \bar{\sigma}_{\alpha\dot{\alpha}}^{\mu}\bar{\sigma}^{\nu\alpha\dot{\alpha}} = 2\eta^{\mu\nu} = \sigma_{\dot{\alpha}\alpha}^{\mu}\sigma^{\nu\dot{\alpha}\alpha} \quad (\text{A.3})$$

To a space-time vector p^{μ} we assign a bi-spinor as follows:

$$p^{\alpha\dot{\alpha}} := \bar{\sigma}^{\mu\alpha\dot{\alpha}}p_{\mu} = \sigma^{\mu\dot{\alpha}\alpha}p_{\mu} =: p^{\dot{\alpha}\alpha}$$

The identities (A.3) imply that

$$p^{\mu} = \frac{1}{2}\bar{\sigma}^{\mu\alpha\dot{\alpha}}p_{\alpha\dot{\alpha}} = \frac{1}{2}\sigma^{\mu\dot{\alpha}\alpha}p_{\dot{\alpha}\alpha} \quad \text{and} \quad p_{\alpha\dot{\alpha}}k^{\alpha\dot{\alpha}} = 2p_{\mu}k^{\mu}.$$

Defining

$$\sigma^{\mu\nu\alpha\beta} := \frac{i}{2}(\bar{\sigma}^{\mu\alpha\dot{\delta}}\sigma^{\nu\dot{\gamma}\beta} - \bar{\sigma}^{\nu\alpha\dot{\delta}}\sigma^{\mu\dot{\gamma}\beta})\epsilon_{\dot{\gamma}\dot{\delta}}, \quad \bar{\sigma}^{\mu\nu\dot{\alpha}\dot{\beta}} := \frac{i}{2}(\sigma^{\mu\dot{\alpha}\gamma}\bar{\sigma}^{\nu\delta\dot{\beta}} - \sigma^{\nu\dot{\alpha}\gamma}\bar{\sigma}^{\mu\delta\dot{\beta}})\epsilon_{\delta\gamma},$$

we also assign bi-spinors to an antisymmetric 2-tensor $F_{\mu\nu}$:

$$F^{\alpha\beta} := F_{\mu\nu}\sigma^{\mu\nu\alpha\beta}, \quad F^{\dot{\alpha}\dot{\beta}} := F_{\mu\nu}\bar{\sigma}^{\mu\nu\dot{\alpha}\dot{\beta}}. \quad (\text{A.4})$$

These two bi-spinors associated to $F_{\mu\nu}$ can be related to $F^{\alpha\dot{\alpha}\beta\dot{\beta}} := F_{\mu\nu}\bar{\sigma}^{\mu\alpha\dot{\alpha}}\bar{\sigma}^{\nu\beta\dot{\beta}}$ by the following identity:

$$F^{\alpha\dot{\alpha}\beta\dot{\beta}} = \frac{i}{2}\epsilon^{\dot{\alpha}\dot{\beta}}F^{\alpha\beta} + \frac{i}{2}\epsilon^{\alpha\beta}F^{\dot{\alpha}\dot{\beta}} \quad (\text{A.5})$$

The bi-spinors associated to $F_{\mu\nu}$ are symmetric, $F^{\alpha\beta} = F^{\beta\alpha}$, $F^{\dot{\alpha}\dot{\beta}} = F^{\dot{\beta}\dot{\alpha}}$. For other bi-spinors we have the general symmetry property:

$$\Lambda_{(\alpha\beta)} = \Lambda_{\alpha\beta} + \frac{1}{2}\epsilon_{\alpha\beta}\Lambda^{\gamma}_{\gamma} \quad (\text{A.6})$$

$$\Lambda_{(\dot{\alpha}\dot{\beta})} = \Lambda_{\dot{\alpha}\dot{\beta}} + \frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\Lambda^{\dot{\gamma}}_{\dot{\gamma}} \quad (\text{A.7})$$

We note the Fierz identity

$$\bar{\xi}^{\dot{\alpha}}\xi^{\beta} = \frac{1}{2}\sigma^{\mu\dot{\alpha}\beta}(\bar{\xi}^{\dot{\gamma}}\sigma_{\mu\dot{\gamma}\delta}\xi^{\delta}), \quad (\text{A.8})$$

and some trace identities for the sigma matrices:

$$\frac{1}{2}\text{Tr}(\bar{\sigma}^{\mu}\sigma^{\nu}) = \eta^{\mu\nu}, \quad (\text{A.9})$$

$$\frac{1}{2}\text{Tr}(\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\sigma}^{\rho}\sigma^{\kappa}) = \eta^{\mu\nu}\eta^{\rho\kappa} + \eta^{\nu\rho}\eta^{\mu\kappa} - \eta^{\mu\rho}\eta^{\nu\kappa} - i\epsilon^{\mu\nu\rho\kappa}, \quad (\text{A.10})$$

$$\frac{1}{2}\text{Tr}(\sigma^{\mu}\bar{\sigma}^{\nu}\sigma^{\rho}\bar{\sigma}^{\kappa}) = \eta^{\mu\nu}\eta^{\rho\kappa} + \eta^{\nu\rho}\eta^{\mu\kappa} - \eta^{\mu\rho}\eta^{\nu\kappa} + i\epsilon^{\mu\nu\rho\kappa}, \quad (\text{A.11})$$

Graßmann functional derivatives are defined by:

$$\frac{\delta}{\delta\theta_{\alpha}^A(s)}\theta_{\beta}^B(\tau) = \delta_{\alpha}^B\delta_{\beta}^{\alpha}\delta(\tau - s), \quad \frac{\delta}{\delta\bar{\theta}_{A\dot{\alpha}}(s)}\bar{\theta}_{B\dot{\beta}}(\tau) = \delta_{B\dot{\beta}}^A\delta_{\dot{\beta}}^{\dot{\alpha}}\delta(\tau - s). \quad (\text{A.12})$$

Note also that our conventions imply that

$$\frac{\partial}{\partial x_{\alpha\dot{\alpha}}}x_{\beta\dot{\beta}} = 2\delta_{\beta}^{\alpha}\delta_{\dot{\beta}}^{\dot{\alpha}}. \quad (\text{A.13})$$

Six-dimensional space. Consider the vector space \mathbb{R}^6 with the metric $\eta_{IJ} = \text{diag}(-1, \dots, -1)$. To a vector $\phi_I \in \mathbb{R}^6$ we assign (4×4) -matrices by the prescription

$$\phi^{AB} := \frac{1}{\sqrt{2}} \Sigma^{IAB} \phi_I, \quad \bar{\phi}_{AB} := \frac{1}{\sqrt{2}} \bar{\Sigma}_{AB}^I \phi_I. \quad (\text{A.14})$$

The sigma matrices are given by

$$(\Sigma^{1AB}, \dots, \Sigma^{6AB}) = (\eta_{1AB}, \eta_{2AB}, \eta_{3AB}, i\bar{\eta}_{1AB}, i\bar{\eta}_{2AB}, i\bar{\eta}_{3AB}), \quad (\text{A.15})$$

$$(\bar{\Sigma}_{AB}^1, \dots, \bar{\Sigma}_{AB}^6) = (\eta_{1AB}, \eta_{2AB}, \eta_{3AB}, -i\bar{\eta}_{1AB}, -i\bar{\eta}_{2AB}, -i\bar{\eta}_{3AB}), \quad (\text{A.16})$$

$$\begin{aligned} \eta_{iAB} &:= \epsilon_{iAB4} + \delta_{iA}\delta_{4B} - \delta_{iB}\delta_{4A} \\ \bar{\eta}_{iAB} &:= \epsilon_{iAB4} - \delta_{iA}\delta_{4B} + \delta_{iB}\delta_{4A}. \end{aligned} \quad (\text{A.17})$$

The epsilon-tensor has the following contraction:

$$\epsilon_{DABC} \epsilon^{DKLM} = \delta_{ABC}^{KLM} + \delta_{ABC}^{MKL} + \delta_{ABC}^{LMK} - \delta_{ABC}^{LKM} - \delta_{ABC}^{MLK} - \delta_{ABC}^{KML}. \quad (\text{A.18})$$

We note the following identities:

$$\bar{\phi}_{AB} = \frac{1}{2} \epsilon_{ABCD} \phi^{CD} \quad \phi^{AB} = \frac{1}{2} \epsilon^{ABCD} \bar{\phi}_{CD} \quad X^{AB} \bar{Y}_{AB} = -2X^I Y_I. \quad (\text{A.19})$$

For a unit vector $n \in \mathbb{R}^6$, $n^I n_I = -1$, we have:

$$\bar{n}_{AB} n^{CB} = \frac{1}{2} \delta_A^C, \quad \bar{n}_{AB} n^{AB} = 2. \quad (\text{A.20})$$

B Propagators

We have with

$$A_\mu(x) = A_\mu^a(x) T^a \quad \phi^{aAB}(x) = \phi^{aAB}(x) T^a \quad \psi_\alpha^{aA}(x) = \psi_\alpha^A(x) T^a$$

and $\text{Tr}(T^a T^a) = \frac{1}{2} \delta^{ab}$ the propagators

$$\langle \phi^{aAB}(x_1) \phi^{bCD}(x_2) \rangle = -\frac{g^2}{4\pi^2} \frac{\epsilon^{ABCD} \delta^{ab}}{(x_1 - x_2)^2} \quad (\text{B.1})$$

$$\langle \psi_\alpha^{aA}(x_1) \bar{\psi}_{\dot{\alpha}B}^b(x_2) \rangle = \frac{ig^2}{2\pi^2} \delta^{ab} \delta_B^A \frac{(x_1 - x_2)_{\alpha\dot{\alpha}}}{(x_1 - x_2)^4} \quad (\text{B.2})$$

$$\langle A_\mu^a(x_1) A_\nu^b(x_2) \rangle = \frac{g^2}{4\pi^2} \frac{\eta_{\mu\nu} \delta^{ab}}{(x_1 - x_2)^2} \quad (\text{B.3})$$

C Conformal algebra and Killing vectors

We use the conventions of [10] for the conformal algebra. The generators $\{P_\mu, M_{\mu\nu}, D, K_\mu\}$ are collectively denoted by T_a , and satisfy the commutation relations:

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} \\ [P_\mu, P_\nu] &= 0 & [M_{\mu\nu}, P_\lambda] &= \eta_{\nu\lambda} P_\mu - \eta_{\mu\lambda} P_\nu \\ [D, P_\mu] &= -P_\mu & [D, K_\mu] &= K_\mu \\ [D, M_{\mu\nu}] &= 0 & [M_{\mu\nu}, K_\rho] &= \eta_{\nu\rho} K_\mu - \eta_{\mu\rho} K_\nu \\ [P_\mu, K_\nu] &= 2M_{\mu\nu} - 2\eta_{\mu\nu} D & [K_\mu, K_\nu] &= 0. \end{aligned} \quad (\text{C.1})$$

In the standard M_{MN} basis of $\mathfrak{so}(4, 2)$, where $M, N = \mu, 4, 4'$ with $\eta_{4'4'} = 1 = -\eta_{44}$,

$$D = M_{44'} \quad P_\mu = M_{\mu 4'} + M_{\mu 4} \quad K_\mu = M_{\mu 4'} - M_{\mu 4}. \quad (\text{C.2})$$

The Killing metric on the $\mathfrak{so}(4, 2)$ algebra is

$$\langle M_{MN}, M_{KL} \rangle = \eta_{LM}\eta_{NK} - \eta_{LN}\eta_{MK}. \quad (\text{C.3})$$

The dual basis of generators is defined by raising the indices with the inverse of the Killing metric

$$\hat{T}^a = G^{ab}T_a, \quad (\text{C.4})$$

such that

$$\langle \hat{T}^a, T_b \rangle = \delta_b^a. \quad (\text{C.5})$$

The structure constants that appear in the Yangian generators are those that arise in the commutation relations in the dual basis:

$$[\hat{T}^a, \hat{T}^b] = f_c^{ab}\hat{T}^c. \quad (\text{C.6})$$

Explicitly,

$$\hat{M}^{\mu\nu} = -\eta^{\mu\lambda}\eta^{\nu\rho}M_{\lambda\rho} \quad \hat{P}^\mu = -\frac{1}{2}\eta^{\mu\nu}K_\nu \quad \hat{K}^\mu = -\frac{1}{2}\eta^{\mu\nu}P_\nu \quad \hat{D} = D, \quad (\text{C.7})$$

and

$$\begin{aligned} [\hat{M}^{\mu\nu}, \hat{M}^{\rho\sigma}] &= \eta^{\mu\rho}\hat{M}^{\nu\sigma} + \eta^{\nu\sigma}\hat{M}^{\mu\rho} - \eta^{\mu\sigma}\hat{M}^{\nu\rho} - \eta^{\nu\rho}\hat{M}^{\mu\sigma} \\ [\hat{P}^\mu, \hat{P}^\nu] &= 0 & [\hat{M}^{\mu\nu}, \hat{P}^\lambda] &= \eta^{\mu\lambda}\hat{P}^\nu - \eta^{\nu\lambda}\hat{P}^\mu \\ [\hat{D}, \hat{P}^\mu] &= \hat{P}^\mu & [\hat{D}, \hat{K}^\mu] &= -\hat{K}^\mu \\ [\hat{D}, \hat{M}^{\mu\nu}] &= 0 & [\hat{M}^{\mu\nu}, \hat{K}^\rho] &= \eta^{\mu\rho}\hat{K}^\nu - \eta^{\nu\rho}\hat{K}^\mu \\ [\hat{P}^\mu, \hat{K}^\nu] &= \frac{1}{2}\eta^{\mu\nu}\hat{D} - \frac{1}{2}\hat{M}^{\mu\nu} & [\hat{K}^\mu, \hat{K}^\nu] &= 0. \end{aligned} \quad (\text{C.8})$$

The conformal Killing vectors $\xi_a^\lambda = \{\delta_\mu^\lambda, x_\mu\delta_\nu^\lambda - x_\nu\delta_\mu^\lambda, x^\lambda, x^2\delta_\mu^\lambda - 2x^\lambda x_\mu\}$ satisfy the commutation relations of the conformal algebra,

$$\{\xi_a, \xi_b\} = f_{ab}^c \xi_c, \quad (\text{C.9})$$

with respect to the Lie bracket

$$\{u, v\}^\mu = u^\nu\partial_\nu v^\mu - v^\nu\partial_\nu u^\mu. \quad (\text{C.10})$$

The following identities for the Lie-algebra-valued Killing vector (4.9):

$$[\hat{\xi}^\mu, \hat{\xi}^\nu] = 0 \quad (\text{C.11})$$

$$[\hat{\xi}^\mu, \partial_\nu \hat{\xi}^\lambda] = \eta^{\mu\lambda}\hat{\xi}_\nu - \delta_\nu^\mu \hat{\xi}^\lambda - \delta_\nu^\lambda \hat{\xi}^\mu \quad (\text{C.12})$$

$$[\hat{\xi}^\mu, \partial_\nu \partial_\lambda \hat{\xi}^\rho] = \eta_{\nu\lambda}\partial^\rho \hat{\xi}^\mu - \delta_\nu^\rho \partial_\lambda \hat{\xi}^\mu - \delta_\lambda^\rho \partial_\nu \hat{\xi}^\mu \quad (\text{C.13})$$

$$[\partial_\nu \hat{\xi}^\nu, \partial_\mu \partial_\lambda \hat{\xi}^\lambda] = -4\partial_\mu \partial_\nu \hat{\xi}^\nu. \quad (\text{C.14})$$

are used in checking the flatness condition of the worldsheet isometry current.

D Superconformal algebra in non-chiral superspace

In the study of superamplitudes in $\mathcal{N} = 4$ SYM one encounters a representation of the $\mathfrak{su}(2, 2|4)$ superalgebra in the form of differential operators on a chiral-superspace $\{x_{\alpha\dot{\alpha}}, \theta_{\alpha}^A\}$ [9]. Extending this algebra naively into a non-chiral superspace $\{x_{\alpha\dot{\alpha}}, \theta_{\alpha}^A, \bar{\theta}_{A\dot{\alpha}}\}$, where one simply conjugates the θ expressions to $\bar{\theta}$'s does not work since the resulting algebra will not close. From a coset construction [53] of the algebra it is clear that we need to introduce coordinates carrying R-symmetry indices y_A^B . Defining a set of coordinates and their variation under the algebra

$$X = \{x_{\alpha\dot{\alpha}}, \theta_{\alpha}^A, \bar{\theta}_{A\dot{\alpha}}, y_A^B\}, \quad \delta X = \{\delta x_{\alpha\dot{\alpha}}, \delta\theta_{\alpha}^A, \delta\bar{\theta}_{A\dot{\alpha}}, \delta y_A^B\}, \quad (\text{D.1})$$

we can understand the superspace in which our Wilson loop is constructed as a subsurface given by putting $y_A^B = 0$. But starting at a specific point on the constraint surface

$$X_0 = \{(x_0)_{\alpha\dot{\alpha}}, (\theta_0)_{\alpha}^A, (\bar{\theta}_0)_{A\dot{\alpha}}, 0\} \quad (\text{D.2})$$

general superconformal variations δX will yield $y_A^B \neq 0$. Therefore not all transformations are expected to be symmetries of our super Wilson loop.

The superconformal transformations (D.1) are generated by the following operators:

$$M_{\alpha\beta} = 2i x_{\dot{\gamma}(\alpha} \partial_{\beta)}^{\dot{\gamma}} + 4i \theta_{(\alpha}^A \partial_{\beta)A} \quad \bar{M}_{\dot{\alpha}\dot{\beta}} = 2i x^{\dot{\gamma}}_{(\dot{\alpha}} \partial_{\dot{\beta})\dot{\gamma}} - 4i \bar{\theta}_{A(\dot{\alpha}} \partial_{\dot{\beta})}^A \quad (\text{D.3})$$

$$D = \frac{1}{2} x_{\alpha\dot{\alpha}} \partial^{\alpha\dot{\alpha}} + \frac{1}{2} \theta_{\beta}^B \partial_B^{\beta} + \frac{1}{2} \bar{\theta}_{B\dot{\beta}} \partial^{B\dot{\beta}} \quad P^{\alpha\dot{\alpha}} = \partial^{\alpha\dot{\alpha}} \quad (\text{D.4})$$

$$K_{\alpha\dot{\alpha}} = -x_{\alpha\dot{\gamma}} x_{\dot{\alpha}\gamma} \partial^{\gamma\dot{\gamma}} - 2x_{\dot{\alpha}\gamma} \theta_{\alpha}^C \partial_C^{\gamma} - 2x_{\alpha\dot{\gamma}} \bar{\theta}_{C\dot{\alpha}} \partial^{C\dot{\gamma}} + 4i \theta_{\alpha}^A \bar{\theta}_{B\dot{\alpha}} \partial_A^B \quad (\text{D.5})$$

$$Q_A^{\alpha} = -\partial_A^{\alpha} + y_A^B \partial_B^{\alpha} + i \bar{\theta}_{A\dot{\alpha}} \partial^{\alpha\dot{\alpha}} \quad \bar{Q}^{A\dot{\alpha}} = \partial^{A\dot{\alpha}} + y_B^A \partial^{B\dot{\alpha}} - i \theta_{\alpha}^A \partial^{\alpha\dot{\alpha}} \quad (\text{D.6})$$

$$S_A^{\alpha} = (\delta_B^A + y_B^A)(x_{\dot{\alpha}\gamma} \partial^{B\dot{\gamma}} + 2i \theta_{\alpha}^C \partial_C^B) - i x_{\alpha\dot{\gamma}} \theta_{\beta}^A \partial^{\beta\dot{\gamma}} - 2i \theta_{\beta}^A \theta_{\alpha}^C \partial_C^{\beta} \quad (\text{D.7})$$

$$\bar{S}_{A\dot{\alpha}} = (-\delta_A^B + y_A^B)(x_{\dot{\alpha}\gamma} \partial_B^{\gamma} - 2i \bar{\theta}_{D\dot{\alpha}} \partial_B^D) + i x_{\dot{\alpha}\gamma} \bar{\theta}_{A\dot{\beta}} \partial^{\gamma\dot{\beta}} + 2i \bar{\theta}_{A\dot{\beta}} \bar{\theta}_{C\dot{\alpha}} \partial^{C\dot{\beta}} \quad (\text{D.8})$$

$$R'^A_B = 2i(-\delta_B^D + y_B^D)(\delta_C^A + y_C^A) \partial_D^C + 2i(-\delta_B^C + y_B^C) \theta_{\gamma}^A \delta_C^{\gamma} \\ + 2i(\delta_C^A + y_C^A) \bar{\theta}_{B\dot{\alpha}} \partial^{C\dot{\alpha}} + 2\bar{\theta}_{B\dot{\alpha}} \theta_{\alpha}^A \partial^{\alpha\dot{\alpha}} \quad (\text{D.9})$$

$$C = \frac{1}{4}(\theta_{\alpha}^D \partial_D^{\alpha} - \bar{\theta}_{C\dot{\alpha}} \partial^{C\dot{\alpha}} + i \theta_{\alpha}^A \bar{\theta}_{A\dot{\alpha}} \partial^{\alpha\dot{\alpha}} - \partial_A^A \\ + y_A^B \theta_{\alpha}^A \partial_B^{\alpha} + y_A^B \bar{\theta}_{B\dot{\alpha}} \partial^{A\dot{\alpha}} + y_A^C y_C^B \partial_B^A) \quad (\text{D.10})$$

where we used the shorthand notation

$$\partial^{\dot{\alpha}\alpha} = \frac{\partial}{\partial x_{\alpha\dot{\alpha}}}, \quad \partial_A^{\alpha} = \frac{\partial}{\partial \theta_{\alpha}^A}, \quad \partial^{\dot{\alpha}A} = \frac{\partial}{\partial \bar{\theta}_{A\dot{\alpha}}}, \quad \partial_A^B = \frac{\partial}{\partial y_B^A}. \quad (\text{D.11})$$

Note also that our conventions imply (A.13). The commutation relations of the above generators agree with (C.1) if the generators of the conformal algebra are translated as defined in appendix A.

The nontrivial part of the algebra, which is realized by these generators is given by

$$\begin{aligned}
\{Q_A^\alpha, \bar{Q}^{B\dot{\alpha}}\} &= 2i\delta_A^B P^{\dot{\alpha}\alpha} & \{S_\alpha^A, \bar{S}_{B\dot{\alpha}}\} &= -2i\delta_B^A K_{\alpha\dot{\alpha}} \\
[K_{\gamma\dot{\gamma}}, Q_A^\alpha] &= 2\delta_\gamma^\alpha \bar{S}_{A\dot{\gamma}} & [K_{\gamma\dot{\gamma}}, \bar{Q}^{B\dot{\alpha}}] &= 2\delta_\gamma^{\dot{\alpha}} S_\gamma^B \\
[P_{\alpha\dot{\alpha}}, K_{\beta\dot{\beta}}] &= i\epsilon_{\alpha\beta} \bar{M}_{\dot{\alpha}\dot{\beta}} + i\epsilon_{\dot{\alpha}\dot{\beta}} M_{\alpha\beta} + 4\epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} D & & (D.12) \\
\{Q_A^\alpha, S_\beta^B\} &= \delta_A^B M_\beta^\alpha + \delta_\beta^B R^B_A + 2i\delta_A^B \delta_\beta^\alpha (D+C) & [P^{\dot{\alpha}\alpha}, S_\beta^A] &= 2\delta_\beta^{\dot{\alpha}} \bar{Q}^{A\dot{\alpha}} \\
\{\bar{Q}^{A\dot{\alpha}}, \bar{S}_{B\dot{\beta}}\} &= -\delta_B^A \bar{M}_{\dot{\alpha}\dot{\beta}} - \delta_{\dot{\beta}}^{\dot{\alpha}} R^A_B + 2i\delta_B^A \delta_{\dot{\beta}}^{\dot{\alpha}} (D-C) & [P^{\dot{\alpha}\alpha}, \bar{S}_{\dot{\beta}}] &= 2\delta_{\dot{\beta}}^{\dot{\alpha}} Q_A^\alpha.
\end{aligned}$$

Making contact with our smaller superspace where $y_A^B = 0$, we see that P, M, \bar{M}, Q and \bar{Q} are well defined generators in the sense that they preserve the constraint surface $y_A^B = 0$. The remaining generators K, S, \bar{S}, R contain derivatives w.r.t. y_A^B and are therefore not expected to be symmetries of a constrained $y_A^B = 0$ Wilson loop. Note that the level-one momentum generator (2.10) is also well defined since it only depends on combinations of P, M, \bar{M}, Q and \bar{Q} .

E Details of the weak coupling computation

E.1 Non-local variation of $\langle W(C) \rangle$

In this appendix we calculate

$$P_{\text{bos}, \epsilon}^{(1)\mu} \langle W(C) \rangle_{(1)} = -\frac{\lambda}{16\pi^2} P_{\text{bos}, \epsilon}^{(1)\mu} \int d\tau_1 d\tau_2 I_{12}.$$

For computational purposes it is helpful to rewrite the generator in the following form:

$$\begin{aligned}
P_{\text{bos}, \epsilon}^{(1)\mu} &= \int ds_1 ds_2 (d(s_1) \eta^{\mu\nu} - m^{\mu\nu}(s_1)) p_\nu(s_2) (\theta(s_2 - s_1 - d(s_2, \epsilon)) + \theta(s_2 - s_1 + d(s_1, \epsilon))) \\
&= \int ds_1 ds_2 N^{\mu\nu\rho\sigma} x_\nu(s_1) \frac{\delta}{\delta x^\rho(s_1)} \frac{\delta}{\delta x^\sigma(s_2)} (\theta(s_2 - s_1 - d(s_2, \epsilon)) + \theta(s_2 - s_1 + d(s_1, \epsilon))), \\
N^{\mu\nu\rho\sigma} &:= \eta^{\mu\sigma} \eta^{\nu\rho} + \eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\mu\nu} \eta^{\rho\sigma}
\end{aligned}$$

Using this form, we start by calculating the double functional derivative of I_{12} multiplied with $x_\nu(s_1)$. Using (2.4) we find:

$$\begin{aligned}
 x_\nu(s_1) \frac{\delta}{\delta x^\rho(s_1)} \frac{\delta}{\delta x^\sigma(s_2)} \frac{\dot{x}(\tau_1)\dot{x}(\tau_2) - |\dot{x}(\tau_1)||\dot{x}(\tau_2)|}{(x(\tau_1) - x(\tau_2))^2} = \\
 \frac{x_\nu(s_1)}{(x_1 - x_2)^2} \left[\left(\eta_{\rho\sigma} - \frac{\dot{x}_\sigma(\tau_2)\dot{x}_\rho(\tau_1)}{|\dot{x}(\tau_2)||\dot{x}(\tau_1)|} \right) \partial_{\tau_1} \delta(\tau_1 - s_1) \partial_{\tau_2} \delta(\tau_2 - s_2) \right. \\
 + \left(\eta_{\rho\sigma} - \frac{\dot{x}_\sigma(\tau_1)\dot{x}_\rho(\tau_2)}{|\dot{x}(\tau_1)||\dot{x}(\tau_2)|} \right) \partial_{\tau_2} \delta(\tau_2 - s_1) \partial_{\tau_1} \delta(\tau_1 - s_2) \\
 + \left(\frac{\dot{x}_\sigma(\tau_1)\dot{x}_\rho(\tau_1)|\dot{x}(\tau_2)|}{|\dot{x}(\tau_1)|^3} - \eta_{\rho\sigma} \frac{|\dot{x}(\tau_2)|}{|\dot{x}(\tau_1)|} \right) \partial_{\tau_1} \delta(\tau_1 - s_1) \partial_{\tau_1} \delta(\tau_1 - s_2) \\
 \left. + \left(\frac{\dot{x}_\sigma(\tau_2)\dot{x}_\rho(\tau_2)|\dot{x}(\tau_1)|}{|\dot{x}(\tau_2)|^3} - \eta_{\rho\sigma} \frac{|\dot{x}(\tau_1)|}{|\dot{x}(\tau_2)|} \right) \partial_{\tau_2} \delta(\tau_2 - s_1) \partial_{\tau_2} \delta(\tau_2 - s_2) \right] \\
 - \frac{2x_\nu(s_1)}{(x_1 - x_2)^4} \left\{ (x_\rho(\tau_1) - x_\rho(\tau_2)) (\delta(\tau_1 - s_1) - \delta(\tau_2 - s_1)) \left[\dot{x}_\sigma(\tau_1) \partial_{\tau_2} \delta(\tau_2 - s_2) \right. \right. \\
 + \dot{x}_\sigma(\tau_2) \partial_{\tau_1} \delta(\tau_1 - s_2) - \frac{|\dot{x}(\tau_1)|}{|\dot{x}(\tau_2)|} \dot{x}_\sigma(\tau_2) \partial_{\tau_2} \delta(\tau_2 - s_2) - \frac{|\dot{x}(\tau_2)|}{|\dot{x}(\tau_1)|} \dot{x}_\sigma(\tau_1) \partial_{\tau_1} \delta(\tau_1 - s_2) \left. \right] \\
 + (x_\sigma(\tau_1) - x_\sigma(\tau_2)) (\delta(\tau_1 - s_2) - \delta(\tau_2 - s_2)) \left[\dot{x}_\rho(\tau_1) \partial_{\tau_2} \delta(\tau_2 - s_1) \right. \\
 + \dot{x}_\rho(\tau_2) \partial_{\tau_1} \delta(\tau_1 - s_1) - \frac{|\dot{x}(\tau_1)|}{|\dot{x}(\tau_2)|} \dot{x}_\rho(\tau_2) \partial_{\tau_2} \delta(\tau_2 - s_1) - \frac{|\dot{x}(\tau_2)|}{|\dot{x}(\tau_1)|} \dot{x}_\rho(\tau_1) \partial_{\tau_1} \delta(\tau_1 - s_1) \left. \right] \\
 \left. + (x_1 \dot{x}_2 - |\dot{x}_1||\dot{x}_2|) \eta_{\rho\sigma} (\delta(\tau_1 - s_2) - \delta(\tau_2 - s_2)) (\delta(\tau_1 - s_1) - \delta(\tau_2 - s_1)) \right\} \\
 + \frac{8x_\nu(s_1)}{(x_1 - x_2)^6} (x_1 \dot{x}_2 - |\dot{x}_1||\dot{x}_2|) (x_\sigma(\tau_1) - x_\sigma(\tau_2)) (x_\rho(\tau_1) - x_\rho(\tau_2)) \cdot \\
 \cdot (\delta(\tau_1 - s_2) - \delta(\tau_2 - s_2)) (\delta(\tau_1 - s_1) - \delta(\tau_2 - s_1)).
 \end{aligned}$$

We order the above result by the structure of the delta functions and derivatives that appear in it. To abbreviate, we use that the Wilson loop integral is symmetric under $(\tau_1 \leftrightarrow \tau_2)$. Also, we fix the parametrization to be of unit-speed, demanding that $|\dot{x}| = 1$. Then we get the following expression (where by $\hat{=}$ we mean that the expression on the right-hand side gives the same result when integrated over τ_1 and τ_2 in parametrization by arc-length):

$$\begin{aligned}
 x_\nu(s_1) \frac{\delta}{\delta x^\rho(s_1)} \frac{\delta}{\delta x^\sigma(s_2)} \frac{\dot{x}(\tau_1)\dot{x}(\tau_2) - |\dot{x}(\tau_1)||\dot{x}(\tau_2)|}{(x(\tau_1) - x(\tau_2))^2} \hat{=} \\
 x_\nu(s_1) \left\{ F_{\rho\sigma}^{(1)}(\tau_1, \tau_2) \partial_{\tau_1} \delta(\tau_1 - s_1) \partial_{\tau_1} \delta(\tau_1 - s_2) + F_{\rho\sigma}^{(2)}(\tau_1, \tau_2) \partial_{\tau_1} \delta(\tau_1 - s_1) \partial_{\tau_2} \delta(\tau_2 - s_2) \right. \\
 + F_{\rho\sigma}^{(3)}(\tau_1, \tau_2) (\delta(\tau_1 - s_1) - \delta(\tau_2 - s_1)) \partial_{\tau_1} \delta(\tau_1 - s_2) \\
 + F_{\sigma\rho}^{(3)}(\tau_1, \tau_2) (\delta(\tau_1 - s_2) - \delta(\tau_2 - s_2)) \partial_{\tau_1} \delta(\tau_1 - s_1) \\
 \left. + F_{\rho\sigma}^{(4)}(\tau_1, \tau_2) (\delta(\tau_1 - s_2) - \delta(\tau_2 - s_2)) (\delta(\tau_1 - s_1) - \delta(\tau_2 - s_1)) \right\} \quad (\text{E.1})
 \end{aligned}$$

Here, we defined:

$$F_{\rho\sigma}^{(1)}(\tau_1, \tau_2) := -\frac{2}{x_{12}^2}(\dot{x}_{1\rho}\dot{x}_{1\sigma} + \eta_{\rho\sigma}), \quad F_{\rho\sigma}^{(2)}(\tau_1, \tau_2) := \frac{2}{x_{12}^2}(\dot{x}_{1\rho}\dot{x}_{2\sigma} + \eta_{\rho\sigma}), \quad (\text{E.2})$$

$$F_{\rho\sigma}^{(3)}(\tau_1, \tau_2) := \frac{4}{x_{12}^4}x_{12\rho}\dot{x}_{12\sigma}, \quad F_{\rho\sigma}^{(4)}(\tau_1, \tau_2) := 2\frac{\dot{x}_1\dot{x}_2+1}{x_{12}^4}\left(4\frac{x_{12\rho}x_{12\sigma}}{x_{12}^2} - \eta_{\rho\sigma}\right). \quad (\text{E.3})$$

We will denote the contribution to $P_{\text{bos}, \varepsilon}^{(1)\mu}\langle W(C)\rangle_{(1)}$ of the above terms by C_i^μ , i.e.:

$$C_i^\mu := \int_0^L d\tau_1 d\tau_2 ds_1 ds_2 \{(\theta(s_2 - s_1 - \varepsilon)c_i^\mu(\tau_1, \tau_2, s_1, s_2)) + (\varepsilon \rightarrow -\varepsilon)\} \quad (\text{E.4})$$

and we define:

$$c_1^\mu(\tau_1, \tau_2, s_1, s_2) := N^{\mu\nu\rho\sigma}x_\nu(s_1)F_{\rho\sigma}^{(1)}(\tau_1, \tau_2)\partial_{\tau_1}\delta(\tau_1 - s_1)\partial_{\tau_1}\delta(\tau_1 - s_2) \quad (\text{E.5})$$

$$c_2^\mu(\tau_1, \tau_2, s_1, s_2) := N^{\mu\nu\rho\sigma}x_\nu(s_1)F_{\rho\sigma}^{(2)}(\tau_1, \tau_2)\partial_{\tau_1}\delta(\tau_1 - s_1)\partial_{\tau_2}\delta(\tau_2 - s_2) \quad (\text{E.6})$$

$$c_3^\mu(\tau_1, \tau_2, s_1, s_2) := N^{\mu\nu\rho\sigma}x_\nu(s_1)F_{\rho\sigma}^{(3)}(\tau_1, \tau_2)(\delta(\tau_1 - s_1) - \delta(\tau_2 - s_1))\partial_{\tau_1}\delta(\tau_1 - s_2) \quad (\text{E.7})$$

$$c_4^\mu(\tau_1, \tau_2, s_1, s_2) := N^{\mu\nu\rho\sigma}x_\nu(s_1)F_{\rho\sigma}^{(3)}(\tau_1, \tau_2)(\delta(\tau_1 - s_2) - \delta(\tau_2 - s_2))\partial_{\tau_1}\delta(\tau_1 - s_1) \quad (\text{E.8})$$

$$c_5^\mu(\tau_1, \tau_2, s_1, s_2) := N^{\mu\nu\rho\sigma}x_\nu(s_1)F_{\rho\sigma}^{(5)}(\tau_1, \tau_2)(\delta(\tau_1 - s_2) - \delta(\tau_2 - s_2)) \cdot (\delta(\tau_1 - s_1) - \delta(\tau_2 - s_1)) \quad (\text{E.9})$$

With these definitions, we have:

$$P_{\text{bos}, \varepsilon}^{(1)\mu}\langle W(C)\rangle_{(1)} = -\frac{\lambda}{16\pi^2}\sum_{i=1}^5 C_i^\mu,$$

We first discuss these terms separately, integrating out the delta-functions. For explicitness, we spell out the calculation for C_1^μ . Consider the integral

$$\begin{aligned} & \int_0^L d\tau_1 d\tau_2 F_{\rho\sigma}^{(1)}(\tau_1, \tau_2) \int_0^L ds_1 ds_2 \theta(s_2 - s_1 - \varepsilon) x_\nu(s_1) \partial_{\tau_1} \delta(\tau_1 - s_1) \partial_{\tau_1} \delta(\tau_1 - s_2) \\ &= -\int_0^L d\tau_1 d\tau_2 F_{\rho\sigma}^{(1)}(\tau_1, \tau_2) \int_0^L ds_1 x_\nu(s_1) \partial_{s_1} \delta(\tau_1 - s_1) \delta(\tau_1 - s_1 - \varepsilon) \\ &= \int_0^L d\tau_1 d\tau_2 F_{\rho\sigma}^{(1)}(\tau_1, \tau_2) \int_0^L ds_1 \delta(\tau_1 - s_1) \partial_{s_1} (x_\nu(s_1) \delta(\tau_1 - s_1 - \varepsilon)) \\ &= \int_0^L d\tau_1 d\tau_2 F_{\rho\sigma}^{(1)}(\tau_1, \tau_2) \dot{x}_\nu(\tau_1) \delta(-\varepsilon) \\ &\quad + \int_0^L d\tau_1 d\tau_2 F_{\rho\sigma}^{(1)}(\tau_1, \tau_2) \int_0^L ds_1 x_\nu(s_1) \delta(\tau_1 - s_1) \partial_\varepsilon \delta(\tau_1 - s_1 - \varepsilon) \\ &= \int_0^L d\tau_1 d\tau_2 F_{\rho\sigma}^{(1)}(\tau_1, \tau_2) \dot{x}_\nu(\tau_1) \delta(\varepsilon) + \partial_\varepsilon \int_0^L d\tau_1 d\tau_2 F_{\rho\sigma}^{(1)}(\tau_1, \tau_2) x_\nu(\tau_1) \delta(\varepsilon) \end{aligned}$$

Using that $\delta(\varepsilon) = \delta(-\varepsilon)$ and $\partial_\varepsilon \delta(\varepsilon) = -\partial_{-\varepsilon} \delta(-\varepsilon)$ we get:

$$C_1^\mu = 2 \int_0^L d\tau_1 d\tau_2 N^{\mu\nu\rho\sigma} F_{\rho\sigma}^{(1)}(\tau_1, \tau_2) \dot{x}_\nu(\tau_1) \delta(\varepsilon) \quad (\text{E.10})$$

Using the periodicity of the curves we consider, one may convince oneself that all boundary terms in the above calculation vanish. In a similar fashion we get the following results:

$$C_2^\mu = \int_0^L d\tau_1 d\tau_2 N^{\mu\nu\rho\sigma} F_{\rho\sigma}^{(2)}(\tau_1, \tau_2) \dot{x}_\nu(\tau_1) (\delta(\tau_2 - \tau_1 - \varepsilon) + \delta(\tau_2 - \tau_1 + \varepsilon)) \\ + \partial_\varepsilon \int_0^L d\tau_1 d\tau_2 N^{\mu\nu\rho\sigma} F_{\rho\sigma}^{(2)}(\tau_1, \tau_2) x_\nu(\tau_1) (\delta(\tau_2 - \tau_1 - \varepsilon) - \delta(\tau_2 - \tau_1 + \varepsilon)) \quad (\text{E.11})$$

$$C_3^\mu = 2 \int_0^L d\tau_1 d\tau_2 N^{\mu\nu\rho\sigma} F_{\rho\sigma}^{(3)}(\tau_1, \tau_2) x_\nu(\tau_1) \delta(\varepsilon) \\ - \int_0^L d\tau_1 d\tau_2 N^{\mu\nu\rho\sigma} F_{\rho\sigma}^{(3)}(\tau_1, \tau_2) x_\nu(\tau_1) (\delta(\tau_1 - \tau_2 - \varepsilon) + \delta(\tau_1 - \tau_2 + \varepsilon)) \quad (\text{E.12})$$

$$C_4^\mu = \int_0^L d\tau_1 d\tau_2 N^{\mu\nu\rho\sigma} F_{\sigma\rho}^{(3)}(\tau_1, \tau_2) \dot{x}_\nu(\tau_1) - 2 \int_0^L d\tau_1 d\tau_2 N^{\mu\nu\rho\sigma} F_{\sigma\rho}^{(3)}(\tau_1, \tau_2) x_\nu(\tau_1) \delta(\varepsilon) \\ + \int_0^L d\tau_1 d\tau_2 N^{\mu\nu\rho\sigma} F_{\sigma\rho}^{(3)}(\tau_1, \tau_2) x_\nu(\tau_1) (\delta(\tau_2 - \tau_1 - \varepsilon) + \delta(\tau_2 - \tau_1 + \varepsilon)) \\ - \int_0^L d\tau_1 d\tau_2 N^{\mu\nu\rho\sigma} F_{\sigma\rho}^{(3)}(\tau_1, \tau_2) \dot{x}_\nu(\tau_1) (\theta(\tau_2 - \tau_1 - \varepsilon) + \theta(\tau_2 - \tau_1 + \varepsilon)) \quad (\text{E.13})$$

$$C_5^\mu = -2 \int_0^L d\tau_1 d\tau_2 N^{\mu\nu\rho\sigma} F_{\rho\sigma}^{(4)}(\tau_1, \tau_2) (x_\nu(\tau_1) - x_\nu(\tau_2)) \theta(\tau_2 - \tau_1 - \varepsilon) \quad (\text{E.14})$$

Taking into account that $N^{\mu\nu\rho\sigma} A_{[\rho\sigma]\nu} = 0$, we can simplify $C_3^\mu + C_4^\mu$ to get:

$$C_3^\mu + C_4^\mu = \int_0^L d\tau_1 d\tau_2 N^{\mu\nu\rho\sigma} F_{\sigma\rho}^{(3)}(\tau_1, \tau_2) \dot{x}_\nu(\tau_1) \\ - \int_0^L d\tau_1 d\tau_2 N^{\mu\nu\rho\sigma} F_{\sigma\rho}^{(3)}(\tau_1, \tau_2) \dot{x}_\nu(\tau_1) (\theta(\tau_2 - \tau_1 - \varepsilon) + \theta(\tau_2 - \tau_1 + \varepsilon)) \\ = - \int_0^L d\tau_1 d\tau_2 N^{\mu\nu\rho\sigma} F_{\sigma\rho}^{(3)}(\tau_1, \tau_2) (\dot{x}_\nu(\tau_1) - \dot{x}_\nu(\tau_2)) (\theta(\tau_2 - \tau_1 - \varepsilon)) \quad (\text{E.15})$$

Inserting the contractions

$$N^{\mu\nu\rho\sigma} F_{\rho\sigma}^{(1)}(\tau_1, \tau_2) \dot{x}_{1\nu} = 6 \frac{\dot{x}_1^\mu}{x_{12}^2}, \quad N^{\mu\nu\rho\sigma} F_{\rho\sigma}^{(2)}(\tau_1, \tau_2) \dot{x}_{1\nu} = -2 \frac{2\dot{x}_1^\mu + \dot{x}_2^\mu}{x_{12}^2}, \quad (\text{E.16})$$

$$N^{\mu\nu\rho\sigma} F_{\rho\sigma}^{(2)}(\tau_1, \tau_2) x_{1\nu} = -\frac{2}{x_{12}^2} (2x_1^\mu + x_1^\mu (\dot{x}_1 \dot{x}_2) - \dot{x}_1^\mu (x_1 \dot{x}_2) - \dot{x}_2^\mu (x_1 \dot{x}_1)) \quad (\text{E.17})$$

$$N^{\mu\nu\rho\sigma} F_{\sigma\rho}^{(3)}(\tau_1, \tau_2) \dot{x}_{12\nu} = -8 \frac{\dot{x}_1 \dot{x}_2 + 1}{x_{12}^4} x_{12}^\mu,$$

$$N^{\mu\nu\rho\sigma} F_{\rho\sigma}^{(4)}(\tau_1, \tau_2) x_{12\nu} = 12 \frac{\dot{x}_1 \dot{x}_2 + 1}{x_{12}^4} x_{12}^\mu \quad (\text{E.18})$$

into (E.10)–(E.15) and using the $(\tau_1 \leftrightarrow \tau_2)$ -symmetry of the integral we arrive at:

$$\begin{aligned}
P_{\text{bos},\varepsilon}^{(1)\mu} \langle W(C) \rangle_{(1)} &= -\frac{\lambda}{16\pi^2} \sum_{i=1}^5 C_i^\mu = \frac{\lambda}{16\pi^2} \left\{ -6 \int_0^L d\tau_1 d\tau_2 \frac{\dot{x}_1^\mu + \dot{x}_2^\mu}{(x_1 - x_2)^2} \delta(\varepsilon) \right. \\
&+ 6 \int_0^L d\tau_1 d\tau_2 \frac{\dot{x}_1^\mu}{(x_1 - x_2)^2} (\delta(\tau_2 - \tau_1 - \varepsilon) + \delta(\tau_2 - \tau_1 + \varepsilon)) \\
&+ 2 \partial_\varepsilon \int_0^L d\tau_1 d\tau_2 \frac{2x_1^\mu + x_1^\mu (\dot{x}_1 \dot{x}_2) - \dot{x}_1^\mu (x_1 \dot{x}_2) - \dot{x}_2^\mu (x_1 \dot{x}_1)}{(x_1 - x_2)^2} (\delta(\tau_2 - \tau_1 - \varepsilon) - \delta(\tau_2 - \tau_1 + \varepsilon)) \\
&\left. + 16 \int_0^L d\tau_1 d\tau_2 \frac{\dot{x}_1 \dot{x}_2 + 1}{(x_1 - x_2)^4} (x_1 - x_2)^\mu \theta(\tau_2 - \tau_1 - \varepsilon) \right\} \quad (\text{E.19})
\end{aligned}$$

We now expand the two middle terms. Making use of

$$\dot{x}^2 \equiv -1 \Rightarrow \dot{x}\ddot{x} \equiv 0 \Rightarrow \ddot{x}^2 + \dot{x}\ddot{x} \equiv 0$$

we find:

$$\begin{aligned}
6 \int_0^L d\tau_1 d\tau_2 \frac{\dot{x}_1^\mu}{(x_1 - x_2)^2} (\delta(\tau_2 - \tau_1 - \varepsilon) + \delta(\tau_2 - \tau_1 + \varepsilon)) &= \\
- \frac{12}{\varepsilon^2} \int d\tau \dot{x}^\mu(\tau) + \int_0^L d\tau \dot{x}^\mu(\tau) \ddot{x}(\tau)^2 + \mathcal{O}(\varepsilon) &\quad (\text{E.20})
\end{aligned}$$

and

$$\begin{aligned}
2\partial_\varepsilon \int_0^L d\tau_1 d\tau_2 \frac{2x_1^\mu + x_1^\mu (\dot{x}_1 \dot{x}_2) - \dot{x}_1^\mu (x_1 \dot{x}_2) - \dot{x}_2^\mu (x_1 \dot{x}_1)}{(x_1 - x_2)^2} (\delta(\tau_2 - \tau_1 - \varepsilon) - \delta(\tau_2 - \tau_1 + \varepsilon)) &= \\
- \frac{2}{\varepsilon^2} \int d\tau \dot{x}^\mu(\tau) + \frac{4}{3} \int_0^L d\tau \ddot{x}^\mu(\tau) - \frac{5}{6} \int_0^L d\tau \dot{x}^\mu(\tau) \ddot{x}^2(\tau) + \mathcal{O}(\varepsilon) &\quad (\text{E.21})
\end{aligned}$$

Inserting (E.20) and (E.21) into (E.19) and using that

$$\int d\tau \dot{x}^\mu(\tau) = 0 = \int_0^L d\tau \ddot{x}^\mu(\tau),$$

which certainly holds for smooth periodic curves, we get:

$$\begin{aligned}
P_{\text{bos},\varepsilon}^{(1)\mu} \langle W(C) \rangle_{(1)} &= \frac{\lambda}{16\pi^2} \left\{ \frac{1}{6} \int_0^L d\tau \dot{x}^\mu(\tau) \ddot{x}^2(\tau) - 6 \int_0^L d\tau_1 d\tau_2 \frac{\dot{x}_1^\mu + \dot{x}_2^\mu}{(x_1 - x_2)^2} \delta(\varepsilon) \right. \\
&\left. + 16 \int_0^L d\tau_1 d\tau_2 \frac{\dot{x}_1 \dot{x}_2 + 1}{(x_1 - x_2)^4} (x_1 - x_2)^\mu \theta(\tau_2 - \tau_1 - \varepsilon) + \mathcal{O}(\varepsilon) \right\} \quad (\text{E.22})
\end{aligned}$$

For any finite value of ε we have $\delta(\varepsilon) = 0$ and we therefore drop the $\delta(\varepsilon)$ term. Moreover in the above result, the parametrization of the curve is still fixed to arc-length. Lifting this constraint we have the reparametrization invariant result

$$\begin{aligned}
P_{\text{bos},\varepsilon}^{(1)\mu} \langle W(C) \rangle_{(1)} &= \frac{\lambda}{16\pi^2} \left\{ \frac{1}{6} \int d\tau \dot{x}^\mu(\tau) \left(\frac{\ddot{x}^2}{\dot{x}^4} - \frac{(\dot{x} \cdot \ddot{x})^2}{\dot{x}^6} \right) + \mathcal{O}(\varepsilon) \right. \\
&\left. + 16 \int d\tau_1 d\tau_2 \frac{\dot{x}_1 \dot{x}_2 - |\dot{x}_1| |\dot{x}_2|}{(x_1 - x_2)^4} (x_1 - x_2)^\mu \theta(\tau_2 - \tau_1 - d(\tau_2, \varepsilon)) \right\} \quad (\text{E.23})
\end{aligned}$$

E.2 Supersymmetric completion of the Maldacena-Wilson loop

As we argued in section 3 it will be enough for our purpose to require that the equations

$$\mathfrak{Q}_A^\alpha(I) = Q_A^\alpha(I) \quad \bar{\mathfrak{Q}}^{A\dot{\alpha}}(I) = \bar{Q}^{A\dot{\alpha}}(I) \quad (\text{E.24})$$

with the supersymmetrically completed exponent

$$I[A, \psi, \bar{\psi}, \phi; x, \theta, \bar{\theta}] = \oint_C d\tau (\mathcal{I}_0 + \mathcal{I}_1 + \bar{\mathcal{I}}_1 + \mathcal{I}_{2m} + \mathcal{I}_2 + \bar{\mathcal{I}}_2 + \mathcal{O}(\bar{\theta}^i \theta^{3-i})) \quad (\text{E.25})$$

hold true up to order θ and $\bar{\theta}$, respectively. Furthermore, we will only focus on the terms which are linear in the fields since only those will contribute to the 1-loop expectation value $\langle \mathcal{W}(C) \rangle_{(1)}$. Therefore, the equations (E.24) can be split into the following set of equations:

$$\mathfrak{Q}_A^\alpha(I_0) = Q_A^\alpha(I_1)|_{\bar{\theta}=0} \quad (\text{E.26})$$

$$\mathfrak{Q}_A^\alpha(I_1) = Q_A^\alpha(I_2)|_{\bar{\theta}=0} \quad (\text{E.27})$$

$$\mathfrak{Q}_A^\alpha(\bar{\mathcal{I}}_1) = Q_A^\alpha(I_0 + I_{2m})|_{\theta=\dot{\theta}=0} \quad (\text{E.28})$$

$$\bar{\mathfrak{Q}}^{A\dot{\alpha}}(I_0) = \bar{Q}^{A\dot{\alpha}}(\bar{\mathcal{I}}_1)|_{\theta=0} \quad (\text{E.29})$$

$$\bar{\mathfrak{Q}}^{A\dot{\alpha}}(\bar{\mathcal{I}}_1) = \bar{Q}^{A\dot{\alpha}}(\bar{\mathcal{I}}_2)|_{\theta=0} \quad (\text{E.30})$$

$$\bar{\mathfrak{Q}}^{A\dot{\alpha}}(I_1) = \bar{Q}^{A\dot{\alpha}}(I_0 + I_{2m})|_{\bar{\theta}=\dot{\bar{\theta}}=0} \quad (\text{E.31})$$

where we introduced the notation

$$I_x = \oint_C d\tau \mathcal{I}_x. \quad (\text{E.32})$$

Let us start by calculating how \mathcal{I}_0 transforms under supersymmetry transformations generated by \mathfrak{Q}_A^α and $\bar{\mathfrak{Q}}^{A\dot{\alpha}}$, respectively. In order to have a more compact notation we will mostly consider the equations (E.26)–(E.31) on the level of the integrand and only write the integral if we use integration by parts. With the basic field transformations given by (3.4)–(3.7) one finds

$$\mathfrak{Q}_A^\alpha(\mathcal{I}_0) = i\epsilon^{\alpha\beta} \bar{\psi}_A^{\dot{\beta}} \dot{x}_{\beta\dot{\beta}} - \sqrt{2}i \psi^{D\alpha} \bar{\eta}_{AD} |\dot{x}| \quad (\text{E.33})$$

$$\bar{\mathfrak{Q}}^{A\dot{\alpha}}(\mathcal{I}_0) = -i\epsilon^{\dot{\alpha}\dot{\beta}} \psi^{A\dot{\beta}} \dot{x}_{\beta\dot{\beta}} - \sqrt{2}i \bar{\psi}_D^{\dot{\alpha}} \eta^{AD} |\dot{x}|. \quad (\text{E.34})$$

It can easily be seen that the equations (E.26) and (E.29) are satisfied if we choose \mathcal{I}_1 and $\bar{\mathcal{I}}_1$ as follows:

$$\mathcal{I}_1 = i\theta^{B\beta} \bar{\psi}_B^{\dot{\beta}} \dot{x}_{\beta\dot{\beta}} + \sqrt{2}i\theta_\beta^C \psi^{D\beta} \bar{\eta}_{CD} |\dot{x}| \quad (\text{E.35})$$

$$\bar{\mathcal{I}}_1 = -i\bar{\theta}_B^{\dot{\beta}} \psi^{B\beta} \dot{x}_{\beta\dot{\beta}} - \sqrt{2}i\bar{\theta}_{C\dot{\beta}} \bar{\psi}_D^{\dot{\beta}} \eta^{CD} |\dot{x}| \quad (\text{E.36})$$

Since we know how \mathcal{Q}_A^α and $\bar{\mathcal{Q}}^{A\dot{\alpha}}$ act on fields we can directly write down how \mathcal{I}_1 and $\bar{\mathcal{I}}_1$ transform under supersymmetry transformations.

$$\mathcal{Q}_A^\alpha(\mathcal{I}_1) = \sqrt{2}i\theta^{B\beta}(\partial^{\dot{\beta}\alpha}\bar{\phi}_{AB})\dot{x}_{\beta\dot{\beta}} + \frac{1}{\sqrt{2}}\theta_\beta^C F_{\text{lin}}^{\alpha\beta}\bar{\eta}_{CA}|\dot{x}| \quad (\text{E.37})$$

$$\mathcal{Q}_A^\alpha(\bar{\mathcal{I}}_1) = -\frac{1}{2}\bar{\theta}_A^{\dot{\beta}}F_{\text{lin}}^{\alpha\beta}\dot{x}_{\beta\dot{\beta}} - 2i\bar{\theta}_{C\dot{\beta}}(\partial^{\dot{\beta}\alpha}\bar{\phi}_{AB})\eta^{CB}|\dot{x}| \quad (\text{E.38})$$

$$\bar{\mathcal{Q}}^{A\dot{\alpha}}(\mathcal{I}_1) = -\frac{1}{2}\theta^{A\beta}F_{\text{lin}}^{\dot{\alpha}\beta}\dot{x}_{\beta\dot{\beta}} + 2i\theta_\beta^C(\partial^{\dot{\beta}\alpha}\phi^{AB})\bar{\eta}_{CB}|\dot{x}| \quad (\text{E.39})$$

$$\bar{\mathcal{Q}}^{A\dot{\alpha}}(\bar{\mathcal{I}}_1) = -\sqrt{2}i\bar{\theta}_B^{\dot{\beta}}(\partial^{\dot{\beta}\alpha}\phi^{AB})\dot{x}_{\beta\dot{\beta}} + \frac{1}{\sqrt{2}}\bar{\theta}_{C\dot{\beta}}F_{\text{lin}}^{\dot{\alpha}\beta}\eta^{CA}|\dot{x}| \quad (\text{E.40})$$

In these equations $F_{\text{lin}}^{\alpha\beta}$ and $F_{\text{lin}}^{\dot{\alpha}\beta}$ denote the parts of (A.4) which are linear in the gauge fields. The parts \mathcal{I}_2 and $\bar{\mathcal{I}}_2$ can be constructed by imposing that the equations (E.27) and (E.30) hold true. The result reads:

$$\mathcal{I}_2 = -\frac{i}{\sqrt{2}}\theta_\gamma^C\theta^{B\beta}(\partial^{\dot{\beta}\gamma}\bar{\phi}_{CB})\dot{x}_{\beta\dot{\beta}} + \frac{1}{2\sqrt{2}}\theta_\beta^C\theta_\gamma^D F_{\text{lin}}^{\gamma\beta}\bar{\eta}_{CD}|\dot{x}| + \sqrt{2}i\theta_\gamma^C\dot{\theta}^{B\gamma}\bar{\phi}_{CB} \quad (\text{E.41})$$

$$\bar{\mathcal{I}}_2 = -\frac{i}{\sqrt{2}}\bar{\theta}_{C\dot{\gamma}}\bar{\theta}_B^{\dot{\beta}}(\partial^{\dot{\beta}\dot{\gamma}}\phi^{CB})\dot{x}_{\beta\dot{\beta}} - \frac{1}{2\sqrt{2}}\bar{\theta}_{C\dot{\beta}}\bar{\theta}_{D\dot{\gamma}}F_{\text{lin}}^{\dot{\gamma}\dot{\beta}}\eta^{CD}|\dot{x}| + \sqrt{2}i\bar{\theta}_{C\dot{\gamma}}\dot{\bar{\theta}}_B^{\dot{\gamma}}\phi^{CB} \quad (\text{E.42})$$

Since the calculations which show that the equations (E.27) and (E.30) are indeed satisfied are a little bit more involved we will give some details on at least one of them. Applying the θ -derivative of Q_A^α to \mathcal{I}_2 yields:

$$\begin{aligned} Q_A^\alpha(\mathcal{I}_2)|_{\dot{\theta}=0} &= \int d\tau \left(-\sqrt{2}i\theta_\beta^B(\partial^{\dot{\beta}(\alpha}\bar{\phi}_{AB})\dot{x}^{\beta)})_{\dot{\beta}} + \frac{1}{\sqrt{2}}\theta_\beta^C F_{\text{lin}}^{\alpha\beta}\bar{\eta}_{CA}|\dot{x}| - \sqrt{2}i\dot{\theta}^{B\alpha}\bar{\phi}_{AB} \right) \\ &= \int d\tau \left(\sqrt{2}i\theta^{B\beta}(\partial^{\dot{\beta}\alpha}\bar{\phi}_{AB})\dot{x}_{\beta\dot{\beta}} - \frac{i}{\sqrt{2}}\theta^{B\alpha}(\partial^{\dot{\beta}\beta}\bar{\phi}_{AB})\dot{x}_{\beta\dot{\beta}} \right. \\ &\quad \left. + \frac{1}{\sqrt{2}}\theta_\beta^C F_{\text{lin}}^{\alpha\beta}\bar{\eta}_{CA}|\dot{x}| - \sqrt{2}i\dot{\theta}^{B\alpha}\bar{\phi}_{AB} \right) \\ &= \int d\tau \left(\sqrt{2}i\theta^{B\beta}(\partial^{\dot{\beta}\alpha}\bar{\phi}_{AB})\dot{x}_{\beta\dot{\beta}} + \frac{1}{\sqrt{2}}\theta_\beta^C F_{\text{lin}}^{\alpha\beta}\bar{\eta}_{CA}|\dot{x}| \right) \end{aligned} \quad (\text{E.43})$$

In order to get the second line we used the identity (A.6). We note that the last term in the second line can be rewritten as a derivative with respect to the curve parameter τ acting on $\bar{\phi}_{AB}$. Using integration by parts we see that the rewritten term cancels the $\dot{\theta}$ -term. In a similar manner it can be shown that $\bar{\mathcal{I}}_2$ satisfies equation (E.30). Let us now turn to the construction of \mathcal{I}_{2m} . While \mathcal{I}_2 and $\bar{\mathcal{I}}_2$ are not necessarily needed for our purpose since their contractions do not contribute to the bosonic order after having applied the level-one momentum generator, this does not apply to \mathcal{I}_{2m} . In contrast to the construction of \mathcal{I}_1 , $\bar{\mathcal{I}}_1$, \mathcal{I}_2 and $\bar{\mathcal{I}}_2$ we do now have two equations for one expression and it is not clear that they are compatible with each other.

We will start by calculating how Q_A^α acts on I_0 .

$$\begin{aligned}
 Q_A^\alpha(I_0) &= \int ds d\tau i \bar{\theta}_{A\dot{\alpha}}(s) \frac{\delta}{\delta x_{\alpha\dot{\alpha}}(s)} \left(\frac{1}{2} A^{\beta\dot{\beta}} \dot{x}_{\beta\dot{\beta}} - \frac{1}{2} \phi^{CD} \bar{\eta}_{CD} |\dot{x}| \right) \\
 &= \int d\tau \left(\frac{i}{2} \bar{\theta}_{A\dot{\alpha}} (\partial^{\alpha\dot{\alpha}} A^{\beta\dot{\beta}}) \dot{x}_{\beta\dot{\beta}} + i \dot{\bar{\theta}}_{A\dot{\alpha}} A^{\alpha\dot{\alpha}} - \frac{i}{2} \bar{\theta}_{A\dot{\alpha}} (\partial^{\alpha\dot{\alpha}} \phi^{CD}) \bar{\eta}_{CD} |\dot{x}| - \frac{i}{2} \dot{\bar{\theta}}_{A\dot{\alpha}} \phi^{CD} \bar{\eta}_{CD} \frac{\dot{x}^{\alpha\dot{\alpha}}}{|\dot{x}|} \right) \\
 &= \int d\tau \left(\frac{i}{2} \bar{\theta}_{A\dot{\alpha}} (\partial^{\alpha\dot{\alpha}} A^{\beta\dot{\beta}} - \partial^{\beta\dot{\beta}} A^{\alpha\dot{\alpha}}) \dot{x}_{\beta\dot{\beta}} - \frac{i}{2} \bar{\theta}_{A\dot{\alpha}} (\partial^{\alpha\dot{\alpha}} \phi^{CD}) \bar{\eta}_{CD} |\dot{x}| - \frac{i}{2} \dot{\bar{\theta}}_{A\dot{\alpha}} \phi^{CD} \bar{\eta}_{CD} \frac{\dot{x}^{\alpha\dot{\alpha}}}{|\dot{x}|} \right) \\
 &= \int d\tau \left(-\frac{1}{4} \bar{\theta}_A^{\dot{\beta}} F_{\text{lin}}^{\alpha\beta} \dot{x}_{\beta\dot{\beta}} - \frac{1}{4} \bar{\theta}_{A\dot{\alpha}} F_{\text{lin}}^{\dot{\alpha}\dot{\beta}} \dot{x}_{\dot{\beta}}^\alpha - \frac{i}{2} \bar{\theta}_{A\dot{\alpha}} (\partial^{\alpha\dot{\alpha}} \phi^{CD}) \bar{\eta}_{CD} |\dot{x}| - \frac{i}{2} \dot{\bar{\theta}}_{A\dot{\alpha}} \phi^{CD} \bar{\eta}_{CD} \frac{\dot{x}^{\alpha\dot{\alpha}}}{|\dot{x}|} \right)
 \end{aligned}$$

First we applied the functional derivative to I_0 and integrated out the δ -functions by evaluating the generator integral. In order to get to the third line we used integration by parts in the second term. The last line follows by using the identity (A.5). The calculation including $\bar{Q}^{A\dot{\alpha}}(I_0)$ works completely analogously.

$$\bar{Q}^{A\dot{\alpha}}(I_0) = \int d\tau \left(-\frac{1}{4} \theta_\alpha^A F_{\text{lin}}^{\alpha\beta} \dot{x}_\beta^{\dot{\alpha}} - \frac{1}{4} \theta^{AB} F_{\text{lin}}^{\dot{\alpha}\dot{\beta}} \dot{x}_{\beta\dot{\beta}} + \frac{i}{2} \theta_\alpha^A (\partial^{\alpha\dot{\alpha}} \phi^{CD}) \bar{\eta}_{CD} |\dot{x}| + \frac{i}{2} \dot{\theta}_\alpha^A \phi^{CD} \bar{\eta}_{CD} \frac{\dot{x}^{\alpha\dot{\alpha}}}{|\dot{x}|} \right)$$

By requiring that equation (E.28) holds true, \mathcal{I}_{2m} can be determined (up to the term including $\dot{\theta}$) to be

$$\begin{aligned}
 \mathcal{I}_{2m} &= \frac{1}{4} \theta_\gamma^B \bar{\theta}_B^{\dot{\beta}} F_{\text{lin}}^{\gamma\beta} \dot{x}_{\beta\dot{\beta}} + \frac{1}{4} \theta^{B\beta} \bar{\theta}_{B\dot{\gamma}} F_{\text{lin}}^{\dot{\gamma}\dot{\beta}} \dot{x}_{\beta\dot{\beta}} + 2i \theta_\gamma^B \bar{\theta}_{C\dot{\beta}} (\partial^{\beta\dot{\gamma}} \bar{\phi}_{BE}) \eta^{CE} |\dot{x}| \\
 &\quad - \frac{i}{2} \theta_\gamma^B \bar{\theta}_{B\dot{\gamma}} (\partial^{\gamma\dot{\gamma}} \phi^{CD}) \bar{\eta}_{CD} |\dot{x}| + \frac{i}{2} \dot{\theta}_\beta^B \bar{\theta}_{B\dot{\beta}} \phi^{CD} \bar{\eta}_{CD} \frac{\dot{x}^{\beta\dot{\beta}}}{|\dot{x}|} - \frac{i}{2} \theta_\beta^B \dot{\bar{\theta}}_{B\dot{\beta}} \phi^{CD} \bar{\eta}_{CD} \frac{\dot{x}^{\beta\dot{\beta}}}{|\dot{x}|}.
 \end{aligned} \tag{E.44}$$

The application of Q_A^α to \mathcal{I}_{2m} yields

$$\begin{aligned}
 Q_A^\alpha(\mathcal{I}_{2m})|_{\dot{\theta}=\dot{\theta}=0} &= -\frac{1}{4} \bar{\theta}_A^{\dot{\beta}} F_{\text{lin}}^{\alpha\beta} \dot{x}_{\beta\dot{\beta}} + \frac{1}{4} \bar{\theta}_{A\dot{\alpha}} F_{\text{lin}}^{\dot{\alpha}\dot{\beta}} \dot{x}_{\dot{\beta}}^\alpha - 2i \bar{\theta}_{C\dot{\alpha}} (\partial^{\alpha\dot{\alpha}} \bar{\phi}_{AB}) \eta^{CB} |\dot{x}| \\
 &\quad + \frac{i}{2} \bar{\theta}_{A\dot{\alpha}} (\partial^{\alpha\dot{\alpha}} \phi^{CD}) \bar{\eta}_{CD} |\dot{x}| + \frac{i}{2} \dot{\bar{\theta}}_{A\dot{\alpha}} \phi^{CD} \bar{\eta}_{CD} \frac{\dot{x}^{\alpha\dot{\alpha}}}{|\dot{x}|},
 \end{aligned} \tag{E.45}$$

from which we immediately see that equation (E.28) is indeed satisfied. We will now show that (E.44) also solves equation (E.31). Therefore we calculate:

$$\begin{aligned}
 \bar{Q}^{A\dot{\alpha}}(\mathcal{I}_{2m})|_{\bar{\theta}=\dot{\bar{\theta}}=0} &= +\frac{1}{4} \theta_\alpha^A F_{\text{lin}}^{\alpha\beta} \dot{x}_\beta^{\dot{\alpha}} - \frac{1}{4} \theta^{A\alpha} F_{\text{lin}}^{\dot{\alpha}\dot{\beta}} \dot{x}_{\dot{\beta}}^\alpha - 2i \theta_\alpha^B (\partial^{\alpha\dot{\alpha}} \bar{\phi}_{BC}) \eta^{AC} |\dot{x}| \\
 &\quad + \frac{i}{2} \theta_\alpha^A (\partial^{\alpha\dot{\alpha}} \phi^{CD}) \bar{\eta}_{CD} |\dot{x}| - \frac{i}{2} \dot{\theta}_\alpha^A \phi^{CD} \bar{\eta}_{CD} \frac{\dot{x}^{\alpha\dot{\alpha}}}{|\dot{x}|}
 \end{aligned} \tag{E.46}$$

The third term can be rewritten as follows

$$\begin{aligned}
 2i \theta_\alpha^B (\partial^{\alpha\dot{\alpha}} \bar{\phi}_{BC}) \eta^{AC} |\dot{x}| &= \frac{i}{2} \theta_\alpha^B (\partial^{\alpha\dot{\alpha}} \phi^{KL}) \bar{\eta}_{MN} |\dot{x}| \epsilon_{BCKL} \epsilon^{ACMN} \\
 &= i \theta_\alpha^A (\partial^{\alpha\dot{\alpha}} \phi^{CD}) \bar{\eta}_{CD} |\dot{x}| - 2i \theta_\alpha^C (\partial^{\alpha\dot{\alpha}} \phi^{AB}) \bar{\eta}_{CB} |\dot{x}|,
 \end{aligned} \tag{E.47}$$

where we employed the identity (A.18). Inserting (E.47) in (E.46) yields:

$$\begin{aligned}
 \bar{Q}^{A\dot{\alpha}}(\mathcal{I}_{2m})|_{\bar{\theta}=\dot{\bar{\theta}}=0} &= +\frac{1}{4} \theta_\alpha^A F_{\text{lin}}^{\alpha\beta} \dot{x}_\beta^{\dot{\alpha}} - \frac{1}{4} \theta^{AB} F_{\text{lin}}^{\dot{\alpha}\dot{\beta}} \dot{x}_{\beta\dot{\beta}} - \frac{i}{2} \theta_\alpha^A (\partial^{\alpha\dot{\alpha}} \phi^{CD}) \bar{\eta}_{CD} |\dot{x}| \\
 &\quad + 2i \theta_\alpha^C (\partial^{\alpha\dot{\alpha}} \phi^{AB}) \bar{\eta}_{CB} |\dot{x}| - \frac{i}{2} \dot{\theta}_\alpha^A \phi^{CD} \bar{\eta}_{CD} \frac{\dot{x}^{\alpha\dot{\alpha}}}{|\dot{x}|}
 \end{aligned} \tag{E.48}$$

If we combine this equation with the result for $\bar{Q}^{A\dot{\alpha}}(I_0)$ we note that (E.31) also holds true.

E.3 Check of supersymmetry at one-loop order

The result (3.18) should be supersymmetric by construction. Nevertheless it is a straightforward check to see if Q_A^α ($\bar{Q}^{A\dot{\alpha}}$) annihilate $\langle \mathcal{W}(C) \rangle_{(1)}$. We note that having computed the vacuum expectation value to order $\theta\bar{\theta}$ only allows us to check the invariance of $\langle \mathcal{W}(C) \rangle_{(1)}$ at order $\bar{\theta}$ (θ) for Q (\bar{Q}). Focusing on Q for the moment, we verify that

$$-\int ds \frac{\delta}{\delta\theta_A^\alpha(s)} \langle \mathcal{W}(C) \rangle_{(1)} + i \int ds \bar{\theta}_{A\dot{\alpha}} \frac{\delta}{\delta x_{\alpha\dot{\alpha}}(s)} (\langle \mathcal{W}(C) \rangle_{(1)})_{\theta=0} = 0. \quad (\text{E.49})$$

Combining the results of the individual terms

$$\begin{aligned} -\int ds \frac{\delta}{\delta\theta_A^\alpha(s)} \langle \mathcal{W}(C) \rangle_{(1)} &= \frac{i\lambda}{4\pi^2} \int d\tau_1 d\tau_2 \left\{ \bar{\theta}_{A\dot{\alpha}}(\tau_2) \frac{\dot{x}_1 \dot{x}_2 - |\dot{x}_1| |\dot{x}_2|}{(x_1 - x_2)^4} x_{12}^{\alpha\dot{\alpha}} \right. \\ &\quad \left. + \frac{\dot{\bar{\theta}}_{A\dot{\alpha}}(\tau_2) \dot{x}_1^{\alpha\dot{\alpha}}}{2 x_{12}^2} - \frac{\dot{\bar{\theta}}_{A\dot{\alpha}}(\tau_2) |\dot{x}_1|}{2x_{12}^2 |\dot{x}_2|} \dot{x}_2^{\alpha\dot{\alpha}} \right\}, \\ i \int ds \bar{\theta}_{A\dot{\alpha}} \frac{\delta}{\delta x_{\alpha\dot{\alpha}}(s)} (\langle \mathcal{W}(C) \rangle_{(1)})_{\theta=0} &= -\frac{i\lambda}{4\pi^2} \int d\tau_1 d\tau_2 \left\{ \bar{\theta}_{A\dot{\alpha}}(\tau_2) \frac{\dot{x}_1 \dot{x}_2 - |\dot{x}_1| |\dot{x}_2|}{(x_1 - x_2)^4} x_{12}^{\alpha\dot{\alpha}} \right. \\ &\quad \left. + \frac{\dot{\bar{\theta}}_{A\dot{\alpha}}(\tau_2) \dot{x}_1^{\alpha\dot{\alpha}}}{2 x_{12}^2} - \frac{\dot{\bar{\theta}}_{A\dot{\alpha}}(\tau_2) |\dot{x}_1|}{2x_{12}^2 |\dot{x}_2|} \dot{x}_2^{\alpha\dot{\alpha}} \right\} \end{aligned} \quad (\text{E.50})$$

we find the expected result

$$Q_A^\alpha \langle \mathcal{W}(C) \rangle_{(1)} \Big|_{\bar{\theta}} = 0. \quad (\text{E.51})$$

The calculation for $\bar{Q}^{A\dot{\alpha}} \langle \mathcal{W}(C) \rangle_{(1)} \Big|_{\theta} = 0$ can be repeated equally.

E.4 Non-local variation of the fermionic contributions to $\langle \mathcal{W}(C) \rangle$

According to (2.13), the fermionic part of the generator $P_\varepsilon^{(1)\mu}$ is given by

$$P_{\text{ferm}, \varepsilon}^{(1)\mu} = -\frac{i}{4} \int_0^L ds_1 ds_2 \bar{q}^{A\dot{\alpha}}(s_2) \bar{\sigma}_{\alpha\dot{\alpha}}^\mu q_A^\alpha(s_1) (\theta(s_1 - s_2 - \epsilon) - \theta(s_2 - s_1 - \epsilon)), \quad (\text{E.52})$$

with

$$q_A^\alpha(s) = -\frac{\delta}{\delta\theta_A^\alpha(s)} + i \bar{\theta}_{A\dot{\alpha}}(s) \frac{\delta}{\delta x_{\alpha\dot{\alpha}}(s)} \quad (\text{E.53})$$

$$\bar{q}^{A\dot{\alpha}}(s) = \frac{\delta}{\delta\bar{\theta}_{A\dot{\alpha}}(s)} - i \theta_\alpha^A(s) \frac{\delta}{\delta x_{\alpha\dot{\alpha}}(s)}. \quad (\text{E.54})$$

We will only be interested in the fermionic correction to the bosonic result (E.23). Therefore we are looking for contributions where the final result does not depend on the Grassmann variables θ and $\bar{\theta}$. This means that the only correction can come from the action of

$$\hat{P}_{\text{ferm}, \varepsilon}^{(1)\mu} = \frac{i}{4} \int_0^L ds_1 ds_2 \frac{\delta}{\delta\bar{\theta}_{A\dot{\alpha}}(s_2)} \bar{\sigma}_{\alpha\dot{\alpha}}^\mu \frac{\delta}{\delta\theta_\alpha^A(s_1)} (\theta(s_1 - s_2 - \epsilon) - \theta(s_2 - s_1 - \epsilon)) \quad (\text{E.55})$$

on objects that have a $\theta\bar{\theta}$ component.

To simplify the calculation it is useful to take a look at the $\theta\bar{\theta}$ structure of the one loop result (3.18) and write down the action of the interesting part of the fermionic generator

$$\hat{P}_{\text{ferm},\varepsilon}^{(1)\mu} \bar{\theta}_B(\tau_2) \bar{\sigma}_\nu \theta^B(\tau_1) = -i 2\delta_\nu^\mu [\theta(\tau_1 - \tau_2 - \varepsilon) - \theta(\tau_2 - \tau_1 - \varepsilon)] \quad (\text{E.56})$$

$$\hat{P}_{\text{ferm},\varepsilon}^{(1)\mu} \bar{\theta}_B(\tau_2) \bar{\sigma}_\nu \dot{\theta}^B(\tau_1) = -i 2\delta_\nu^\mu [\delta(\tau_1 - \tau_2 - \varepsilon) + \delta(\tau_2 - \tau_1 - \varepsilon)] \quad (\text{E.57})$$

and the corollaries

$$\hat{P}_{\text{ferm},\varepsilon}^{(1)\mu} \bar{\theta}_B(\tau_1) \bar{\sigma}_\nu \theta^B(\tau_1) = 0 \quad (\text{E.58})$$

$$\hat{P}_{\text{ferm},\varepsilon}^{(1)\mu} \bar{\theta}_B(\tau_2) \bar{\sigma}_\nu \dot{\theta}^B(\tau_1) = -\hat{P}_{\text{ferm},\varepsilon}^{(1)\mu} \dot{\bar{\theta}}_B(\tau_2) \bar{\sigma}_\nu \theta^B(\tau_1) \quad (\text{E.59})$$

$$\hat{P}_{\text{ferm},\varepsilon}^{(1)\mu} (\dot{\bar{\theta}}_B(\tau_1) \bar{\sigma}_\nu \theta^B(\tau_1) - \bar{\theta}_B(\tau_1) \bar{\sigma}_\nu \dot{\theta}^B(\tau_1)) = i 8 \delta(-\varepsilon) \delta_\nu^\mu. \quad (\text{E.60})$$

Using these relations we get

$$\begin{aligned} & \hat{P}_{\text{ferm},\varepsilon}^{(1)\mu} \langle \mathcal{W}(C) \rangle_{(1)} \\ &= -\frac{\lambda}{4\pi^2} \int_0^L d\tau_1 d\tau_2 \left\{ \left(2 \frac{\dot{x}_1 \dot{x}_2 + 1}{x_{12}^4} x_{12}^\mu + 2i \frac{\epsilon^{\mu\nu\rho\kappa} \dot{x}_{1\nu} \dot{x}_{2\rho} x_{12\kappa}}{x_{12}^4} \right) [\theta(\tau_1 - \tau_2 - \varepsilon) - \theta(\tau_2 - \tau_1 - \varepsilon)] \right. \\ & \quad \left. - \frac{\dot{x}_1^\mu + \dot{x}_2^\mu}{x_{12}^2} [\delta(\tau_1 - \tau_2 - \varepsilon) + \delta(\tau_2 - \tau_1 - \varepsilon)] - 4 \frac{\dot{x}_2^\mu}{x_{12}^2} \delta(\varepsilon) \right\} \\ &= -\frac{\lambda}{4\pi^2} \left\{ \int_0^L d\tau_1 d\tau_2 \left(4 \frac{\dot{x}_1 \dot{x}_2 + 1}{(x_1 - x_2)^4} x_{12}^\mu \theta(\tau_2 - \tau_1 - \varepsilon) - 2 \frac{\dot{x}_1^\mu + \dot{x}_2^\mu}{x_{12}^2} \delta(\varepsilon) \right) \right. \\ & \quad \left. + \frac{1}{3} \int_0^L d\tau \dot{x}^\mu(\tau) \ddot{x}^2(\tau) + \mathcal{O}(\varepsilon) \right\} \quad (\text{E.61}) \end{aligned}$$

where in the last step we replaced $(\tau_1 \leftrightarrow \tau_2)$ where appropriate and took the results of (E.20) into account. Writing this again in a reparametrization invariant form and noting that $\delta(\varepsilon) = 0$, we arrive at:

$$\begin{aligned} P_{\text{ferm},\varepsilon}^{(1)\mu} \langle \mathcal{W}(C) \rangle_{(1)} \Big|_{\substack{\theta=0 \\ \bar{\theta}=0}} &= -\frac{\lambda}{16\pi^2} \left\{ 16 \int d\tau_1 d\tau_2 \frac{\dot{x}_1 \dot{x}_2 - |\dot{x}_1| |\dot{x}_2|}{(x_1 - x_2)^4} (x_1 - x_2)^\mu \theta(\tau_2 - \tau_1 - d(\tau_2, \varepsilon)) \right. \\ & \quad \left. + \frac{4}{3} \int d\tau \dot{x}^\mu(\tau) \left(\frac{\ddot{x}^2}{\dot{x}^4} - \frac{(\dot{x} \cdot \ddot{x})^2}{\dot{x}^6} \right) + \mathcal{O}(\varepsilon) \right\}. \quad (\text{E.62}) \end{aligned}$$

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