

# Analytic results for two-loop master integrals for Bhabha scattering I

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**ABSTRACT:** We evaluate analytically the master integrals for one of two types of planar families contributing to massive two-loop Bhabha scattering in QED. As in our previous paper, we apply a recently suggested new strategy to solve differential equations for master integrals for families of Feynman integrals. The crucial point of this strategy is to use a new basis of the master integrals where all master integrals are pure functions of uniform weight. This allows to cast the differential equations into a simple canonical form, which can straightforwardly be integrated order by order in  $\epsilon$ . The boundary conditions are also particularly transparent in this setup. We identify the class of functions relevant to this problem to all orders in  $\epsilon$ . We present the results up to weight four for all except one integrals in terms of a subset of Goncharov polylogarithms, which one may call two-dimensional harmonic polylogarithms. For one integral, and more generally at higher weight, the solution is written in terms of Chen iterated integrals.

**KEYWORDS:** NLO Computations

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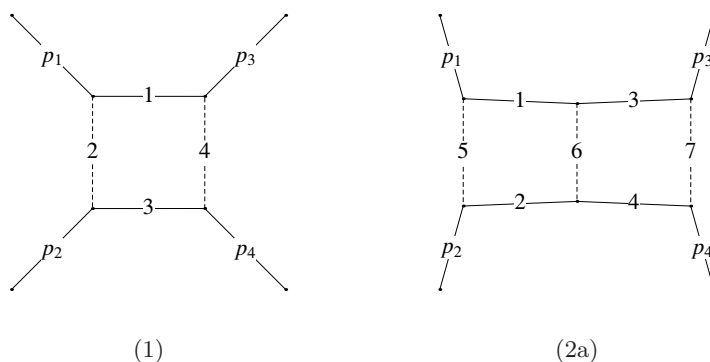
## 1 Introduction

Two-loop Feynman integrals contributing to Bhabha scattering were extensively studied because of the corresponding high experimental precision achieved at  $e^+e^-$  colliders. Planar two-loop box integrals with one-loop insertion were calculated in [1] by the method of differential equations (DE) [2–4], see [5, 6] for reviews. Then all the two-loop integrals involving a closed fermion loop were evaluated in [7]. The first result for a massive two-loop double box was obtained in [8] by the method of Mellin-Barnes (MB) representation, see chapter 5 of [6] for a review. Then a similar result for a master integral with a numerator was obtained [9] and pole parts in the dimensional regularization parameter  $\epsilon = (4 - D)/2$  for some other master integrals were evaluated.

An attempt to evaluate all the master integrals for Bhabha scattering, i.e. integrals belonging to two families of planar diagrams and one family of non-planar diagrams was undertaken in [10–14], with the help of DE.<sup>1</sup> However, starting from master integrals with

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<sup>1</sup>Analytic results for other mass configurations were also obtained using this method, see e.g. [15–17] and references therein.



**Figure 1.** One-loop and two-loop family of integrals (2a) for Bhabha scattering. Solid lines indicate massive propagators, and dashed lines massless ones.

five propagators, the resulting DE involved highly non-trivial coupled sets of equations, which complicated the situation considerably. In those cases it was also found that the determination of the boundary constants was a complicated task. Finally, it remained unclear what precise class of functions would be sufficient to express all master integrals.

One should mention that for phenomenological applications to Bhabha scattering sometimes the small mass limit is sufficient. This was investigated in [18] based on an IR regularization scheme conversion from the dimensionally regularized massless two-loop Bhabha amplitudes of [19], and in refs. [20–27]. (The magnitude of the mass corrections was well numerically studied in [28].) However, the problem of analytical evaluation of all the master integrals for Bhabha scattering remained open up to now. Moreover, its analytic expressions are obviously desirable in order to have control over errors in approximations, especially at intermediate energies where the latter may not always be justified. Furthermore, understanding the structure of loop integrals more generally is an interesting and important challenge. For example, until now it was not known which class of special functions one ultimately needs in order to describe the  $\epsilon$  expansion of the integrals. This is an important question, and, as we will see, understanding the answer to it will turn out to be almost tantamount to obtaining the full analytic solution.

In this paper we present a solution of this problem for the family of planar diagrams of the first type contributing to massive two-loop Bhabha scattering — see figure 1(2a). For the other planar and non-planar family, see e.g. figure 1 of [9]. Explicitly, we are dealing with dimensionally regularized integrals

$$\begin{aligned}
 G_{a_1, a_2, \dots, a_9}(s, t, m^2; D) &= \int \int \frac{d^D k_1 d^D k_2}{(-k_1^2 + m^2)^{a_1} [-(k_1 + p_1 + p_2)^2 + m^2]^{a_2} [-k_2^2 + m^2]^{a_3}} \\
 &\times \frac{[-(k_2 + p_1)^2]^{-a_8} [-(k_1 - p_3)^2]^{-a_9}}{[-(k_2 + p_1 + p_2)^2 + m^2]^{a_4} [-(k_1 + p_1)^2]^{a_5} [-(k_1 - k_2)^2]^{a_6} [-(k_2 - p_3)^2]^{a_7}} \quad (1.1)
 \end{aligned}$$

labelled by seven indices for propagators and two indices for irreducible numerators. In other words, the  $a_i$  can take any integer values, with  $a_8$  and  $a_9$  being zero or negative. The external legs are on-shell,  $p_i^2 = m^2$ , for  $i = 1, 2, 3, 4$  and the Mandelstam variables are

$s = (p_1 + p_2)^2$  and  $t = (p_1 + p_3)^2$ , and  $u = (p_2 + p_3)^2 = -s - t + 4m^2$ . In the following, we will often suppress the explicit dependence of the integrals on the kinematical variables.

We will show that one can resolve all issues mentioned above within the DE approach. The crucial point is to apply a new strategy of solving DE for master integrals which was suggested [29] by one of the authors of the present paper and already applied to the evaluation of all the planar three-loop four-point massless on-shell master integrals [30]. The key ingredient of this strategy is to turn, after revealing master integrals by integration-by-parts (IBP) relations [31], to a much more convenient basis of master integrals consisting of *pure* functions of uniform *weight*.<sup>2</sup> Let us recall that for iterated integrals over logarithmic differential forms, i.e. Chen iterated integrals [32] (see also the recent lecture notes [33, 34] and ref. [35]), the weight is defined as the number of integrations. For rational arguments of the logarithms, these functions are the commonly used Goncharov polylogarithms [36], or closely related hyperlogarithms. We will say that a linear combination of such functions has uniform (i.e. homogeneous) weight if all its summands have the same weight. Finally, a function is called pure if only  $\mathbb{Q}$ -linear combinations are allowed. As a consequence given such a pure function, the weight of its differential is lowered by one unit. This property is essential for such functions to be able to satisfy simple differential equations.

The first step of the method of DE is to take derivatives of a given master integral with respect to kinematical invariants and masses. The result of this differentiation is then expressed in terms of Feynman integrals of the given family and, according to the known IBP reduction,<sup>3</sup> in terms of the master integrals. In this way, one obtains a system of first-order differential equations for the master integrals, and can then try to solve this system with appropriate boundary conditions.

We find a basis choice that puts the differential equations into the canonical form proposed in ref. [29],

$$df = \epsilon d\tilde{A} f. \quad (1.2)$$

Here  $f(s/m^2, t/m^2; \epsilon)$  is a list of master integrals for the family defined in eq. (1.1),  $\tilde{A}$  is a matrix and  $d\tilde{A}$  which is a logarithmic one-form, independent of  $\epsilon$ .

This implies that the general solution, to all orders in  $\epsilon$  can immediately be written down in terms of Chen iterated integrals,<sup>4</sup> where the integration kernels follow from the one-form  $d\tilde{A}$ . We will discuss these functions and their relation to functions more commonly used in the physics literature in more detail in the main text. Moreover, the arguments of the logarithms in  $\tilde{A}$  completely specify the class of functions needed to describe the Feynman integrals to all orders in the  $\epsilon$  expansion. At low orders in this expansion, up to weight four which is relevant to NNLO computations, only a subset of these functions is needed, and we will describe this in details. Another feature of eq. (1.2) is that it makes the singularity structure of the integrals completely transparent. Among other things, this

<sup>2</sup>In the physics literature, this is often referred to as uniform degree of “transcendentality”. We prefer the more concise notion of weight for iterated integrals.

<sup>3</sup>In our calculations, we use FIRE [37, 38] to perform an IBP reduction.

<sup>4</sup>It is well known that Feynman integrals can also contain elliptic functions, a particular example being the two-loop massive sunrise integrals. It is an interesting question how to extend the present approach to these cases.

is extremely helpful when determining the boundary constants of integration. The latter follow from simple physical considerations, such as the absence of certain branch cuts in unphysical channels, and are straightforward to implement in this approach, especially when thinking about the solution in the general picture of Chen iterated integrals.<sup>5</sup>

Our results show that, contrary to common belief, two-loop integrals involving masses need not be complicated, if thought about in the right way. We present the results up to degree four for all except one integrals in terms of a subset of Goncharov polylogarithms, which one may call two-dimensional harmonic polylogarithms. For one integral, and more generally for higher degree, the solution is written in terms of Chen iterated integrals. We discuss one degree four example in detail and explain how to evaluate this integral numerically. For the Goncharov polylogarithms, standard tools for numerical evaluation are available [39, 40].

This paper is organized as follows. In section 2 we discuss the one-loop case in detail within our method. The reason is that with small changes the two-loop case can be treated in the same way. Also, this allows to expand results for one-loop integrals to any order in  $\epsilon$ , if they are needed for future computations. We discuss in detail several different analytic representations of the answer, using Chen iterated integrals, Goncharov polylogarithms, and multiple polylogarithms. In section 3, we apply the same technique to two loops. We discuss the class of functions needed and give an explicit analytic solution for all integrals up to transcendental weight four. For reasons of brevity, we only show explicitly the result for two of the most complicated integrals, and provide the formulas for the remaining integrals in electronic form in an ancillary file. In section 4, we conclude and discuss future directions. Technical details are delegated to the appendix.

## 2 Master integrals and differential equations at one loop

Here we discuss the one-loop case as a useful pedagogical example and as a preparation for the two-loop analysis. As we will see, most of the analysis can be directly carried over to the two-loop case. We start by defining the family of one-loop integrals, as well as convenient variables. We then discuss the choice of integral basis, the expected analytic structure of the integrals, and the differential equations they satisfy. We explain how to solve the latter to all orders in the  $\epsilon$  expansion and discuss various possibilities for representing the answer in analytic form.

### 2.1 Definitions

We will discuss the family of one-loop integrals displayed in figure 1(1). In other words, we have

$$G_{a_1, \dots, a_4} = \int \frac{d^D k}{\left[-k^2 + m^2\right]^{a_1} \left[-(k+p_1)^2\right]^{a_2} \left[-(k+p_1+p_2)^2 + m^2\right]^{a_3} \left[-(k-p_3)^2\right]^{a_4}}, \quad (2.1)$$

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<sup>5</sup>After having chosen a particular contour in the Chen iterated integrals, e.g. in order to obtain a particular form of the answer in terms of Goncharov polylogarithms, it can happen that individual terms diverge in the limit relevant for the boundary conditions, and this has to be treated with care.

for any integer values of the  $a_i$ . We can always make integrals dimensionless by multiplying them by appropriate factors. Therefore we can parametrize them by following dimensionless variables, which we choose as

$$\frac{-s}{m^2} = \frac{(1-x)^2}{x}, \quad \frac{-t}{m^2} = \frac{(1-y)^2}{y}. \quad (2.2)$$

The new variables  $x$  and  $y$  have the advantage of resolving some square roots typical for massive scattering processes. It is useful to define all functions in the Euclidean region  $s < 0, t < 0$ , where the results for planar integrals are manifestly real, and analytically continue those formulas to the physical region.

We may note that the definition (2.2) is invariant under inversions of  $x$  and  $y$ . Depending on the  $x$ -dependent normalization factors, all functions will have definite antisymmetry or symmetry properties under these transformations. This symmetry also implies that we can take  $|x| < 1, |y| < 1$  without loss of generality.

There are several special values for  $x$  and  $y$  corresponding to important physical limits of the integrals. Focusing on the variable  $s$  (or, equivalently,  $x$ ), we have

$$x = 0 \leftrightarrow s = \infty, \quad x = 1 \leftrightarrow s = 0 \quad x = -1 \leftrightarrow s = 4m^2, \quad (2.3)$$

and similarly for  $s \leftrightarrow t$ . Physically, the limiting cases of eq. (2.3) correspond to the Regge limit, the soft limit, and the threshold of creating a pair of massive particles, respectively. Moreover,  $u = 0$  corresponds to  $x = -y$ . Other special configurations are related to the above by the  $x \rightarrow 1/x$  and  $y \rightarrow 1/y$  symmetry.

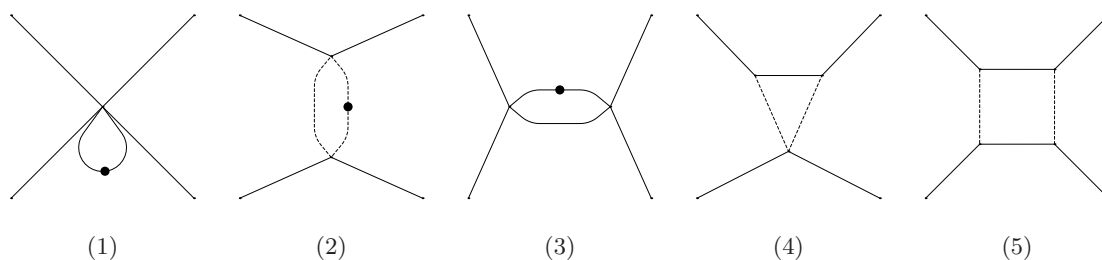
## 2.2 Analytic structure of the integrals

As we will see, the integrals of the family (2.1) will give rise to two-variable generalizations of multi-valued functions similar to polylogarithms. When writing down the answer for these integrals, it is important to understand their branch cut structure. This knowledge can also be helpful for determining integration constants. For instance, the absence of singularities at certain kinematical points can lead to analytic constraints determining boundary constants. This will be particularly transparent in the canonical form (1.2) of the differential equations.

When thinking about the analytic structure of the answer it is useful to have the example of the one-loop box function in mind, see figure 1(1), which has the Feynman parametrization

$$G_{1,1,1,1} = \Gamma(1 + \epsilon) \int_0^\infty \frac{\left(\prod_{i=1}^4 d\alpha_i\right) \delta(1 - \sum_{i=1}^4 \alpha_i)}{[\alpha_1\alpha_3(-s) + \alpha_2\alpha_4(-t) + m^2(\alpha_1 + \alpha_3)^2 - i0]^{2+\epsilon}}. \quad (2.4)$$

The  $-i0$  prescription can be absorbed into  $m^2$ , or by addition  $+i0$  to  $s$  and  $t$ . From the physical point of view it is clear that  $G_{1,1,1,1}$  has branch cuts in the  $t$ -channel starting at  $t = 0$  and extending along the positive  $t$  axis, corresponding to two massless particles in the  $t$ -channel unitarity cut. Likewise, one expects a branch cut starting at  $s = 4m^2$  (and extending along the positive  $s$  axis) from the  $s$ -channel cut through two massive lines. One can arrive at the same conclusions by analyzing possible singularities of eq. (2.4), using Landau equations [41]. Moreover, for planar graphs, we expect there to be no branch cuts



**Figure 2.** Master integrals for the integral family of one-loop Bhabha integrals.

at  $u = 0$ , where  $s + t + u = 4m^2$ , and hence  $x = -y$ . Another interesting limit is  $s \rightarrow 0$ , or equivalently  $x \rightarrow 1$ . Analyzing the above Feynman parametrization (2.4) we expect that this limit does not introduce new divergences (unlike e.g. the  $t \rightarrow 0$  limit), and therefore should commute with the  $\epsilon \rightarrow 0$  expansion, and give a finite answer.

Finally, it is clear that the answer must be real-valued in the Euclidean region  $s < 0, t < 0$ . As already mentioned, the above information about the analytic structure of the integrals will be extremely useful in determining the boundary constants.

The explicit results we present will be written in a form where each term is real-valued in the Euclidean region  $0 < x < 1, 0 < y < 1$ . One can extend them to the physical region by analytic continuation, see e.g. [1, 42].

### 2.3 Choice of integral basis

We choose candidate integrals that are expected to be pure functions of uniform weight. The general ideas how to find such a basis were outlined in ref. [29], and a number of pedagogical examples were given in ref. [30]. The main idea is to consider (generalized) unitarity cuts of integrals. Moreover, in the case of simple subintegrals, one can also analyze explicit Feynman parametrizations, and in this way deduce that a given integral (or linear combination of integrals) will have uniform weight, without having to fully compute the integrals. In the present case, the masses slightly change the technical implementation of these ideas, but the general method remains unchanged.

This analysis leads us to the following choice, depicted in figure 2,

$$f_i = (m^2)^\epsilon e^{2\epsilon\gamma_E} g_i \tag{2.5}$$

with

$$g_1 = \epsilon G_{2,0,0,0}, \tag{2.6}$$

$$g_2 = \epsilon t G_{0,2,0,1}, \tag{2.7}$$

$$g_3 = \epsilon \sqrt{(-s)(4m^2 - s)} G_{2,0,1,0}, \tag{2.8}$$

$$g_4 = -2\epsilon^2 (4m^2 - t)(-t) G_{1,1,0,1}, \tag{2.9}$$

$$g_5 = -2\epsilon^2 \sqrt{(-s)(4m^2 - s)} t G_{1,1,1,1}. \tag{2.10}$$

We normalized all integrals such that they have a Taylor expansion around  $\epsilon = 0$ , i.e.

$$f_i = \sum_{k \geq 0} \epsilon^k f_i^{(k)}. \tag{2.11}$$

Of course, the tadpole and bubble integrals are elementary (see e.g. [6]),

$$f_1 = \epsilon \Gamma(\epsilon) e^{\epsilon \gamma_E}, \tag{2.12}$$

$$f_2 = -\epsilon \Gamma(1 - \epsilon) \Gamma(-\epsilon) \Gamma(1 + \epsilon) / \Gamma(1 - 2\epsilon) \left( \frac{y}{(1 - y)^2} \right)^\epsilon e^{\epsilon \gamma_E}. \tag{2.13}$$

They will provide simple boundary conditions to all orders in  $\epsilon$ . One may observe that they have uniform weight in their expansion in  $\epsilon$ .

Let us now analyze the system of differential equations that these integrals satisfy.

### 2.4 One loop differential equations and integration

With the choice of integral basis given in eq. (2.6), we find the following system of differential equations. We have eq. (1.2), with the matrix  $\tilde{A}$  given by

$$\begin{aligned} \tilde{A} = & \left[ \begin{aligned} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \end{pmatrix} \log x + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & 0 & -2 \end{pmatrix} \log(1 + x) + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \end{pmatrix} \log y + \\ & + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \log(1 + y) + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix} \log(1 - y) + \\ & + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 1 \end{pmatrix} \log(x + y) + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -2 & 1 \end{pmatrix} \log(1 + xy) \end{aligned} \right]. \tag{2.14} \end{aligned}$$

Let us analyze the logarithmic differential forms  $d\tilde{A}$  appearing in the differential equation (1.2). One may note that there is no  $d \log(1 - x)$  term, which is related to the boundary condition at  $x = 1$  (cf. the discussion in section 2.2.) Note however, that the finiteness of the functions  $f_i$  at  $x = 1$  does not imply the absence of such a term in the differential equations, as we will see in the two-loop analysis. Therefore, for more generality, let us allow for such a term in our discussion.

As was pointed out in ref. [29], a differential equation of the type (1.2) makes it manifest that the solution is simple. The latter can always be written down in terms of Chen iterated integrals that have manifest transcendental weight. Moreover, experience shows that the boundary conditions for planar integrals usually can be obtained from simple physical conditions. Indeed, we can demand that the solution have the properties discussed in section 2.2. It can be seen that these conditions, together with the elementary bubble and tadpole integrals of eqs. (2.12) and (2.13), determine all boundary constants. Therefore one can claim that the problem is essentially solved when a form (1.2) of the differential equations is found.



To some extent the choice of how to present the explicit solution, or what is meant by explicit, is a matter of taste and depends also on the applications one has in mind. Let us elaborate on the various possibilities in the following subsections.

## 2.5 Symbol of the solution

Before writing down explicit formulas for the solution, let us mention that as a corollary of the differential equations (1.2), the symbol [43, 44] of the answer is immediately determined. The reader unfamiliar with symbols may think of the symbol as encoding the differential form  $d\tilde{A}$  needed to define the functions through iterated integrals, as discussed in the next section. In the present case, the symbol alphabet is seen to be

$$\{x, 1 \pm x, y, 1 \pm y, x + y, 1 + xy\} \quad (2.15)$$

where we again included  $1 - x$  in view of the two-loop case, although it was not needed at one loop.

The boundary conditions discussed in section 2.2 and coming from the elementary integrals of eqs. (2.12) and (2.13) give us

$$f^{(0)} = \{1, 1, 0, 0, 0\}. \quad (2.16)$$

This completely specifies the symbol. It is simply obtained by iterations of the matrix  $\tilde{A}$ . Let us take the box integral  $f_5$  of eq. (2.5) as a specific example. Using the notation [...] for a symbol, e.g.  $\mathcal{S}(\text{Li}_2(x)) = -[1 - x, x]$ , one obtains, in the first few orders of the  $\epsilon$  expansion,

$$\mathcal{S}(f_5^{(0)}) = 0, \quad (2.17)$$

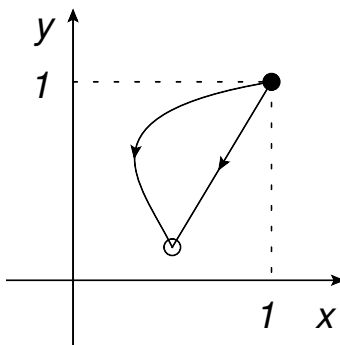
$$\mathcal{S}(f_5^{(1)}) = 4[x], \quad (2.18)$$

$$\mathcal{S}(f_5^{(2)}) = -8[x, 1 - y] + 4[x, y] - 8[1 - y, x] + 4[y, x], \quad (2.19)$$

$$\begin{aligned} \mathcal{S}(f_5^{(3)}) = & 8[x, x, 1 + x] + 4[x, x, y] - 4[x, x, x + y] - 4[x, x, 1 + xy] \\ & - 16[x, 1 + x, 1 + x] - 8[x, 1 + x, y] + 8[x, 1 + x, x + y] \\ & + 8[x, 1 + x, 1 + xy] + 16[x, 1 - y, 1 + x] + 16[x, 1 - y, 1 - y] \\ & - 8[x, 1 - y, x + y] - 8[x, 1 - y, 1 + xy] - 8[x, y, 1 + x] \\ & - 8[x, y, 1 - y] + 4[x, y, x + y] + 4[x, y, 1 + xy] + 16[1 - y, x, 1 + x] \\ & + 16[1 - y, x, 1 - y] - 8[1 - y, x, x + y] - 8[1 - y, x, 1 + xy] \\ & + 16[1 - y, 1 - y, x] - 8[1 - y, y, x] + 8[1 - y, y, x + y] \\ & - 8[1 - y, y, 1 + xy] - 8[y, x, 1 + x] - 8[y, x, 1 - y] \\ & + 4[y, x, x + y] + 4[y, x, 1 + xy] - 8[y, 1 - y, x]x + 4[y, y, x] \\ & - 4[y, y, x + y] + 4[y, y, 1 + xy]. \end{aligned} \quad (2.20)$$

This can obviously be continued to any desired order. This is very useful because it specifies exactly which classes of integral functions are needed to a given order in  $\epsilon$ .

The attentive reader might be surprised about the previous sentence, since equation (1.2), together with the explicit logarithmic form of  $\tilde{A}$  in (2.14), already specify the function class required to all orders in  $\epsilon$ . However, it frequently happens that the first few



**Figure 3.** Two integration paths are shown connecting the base point (black dot) and the argument of the function (circle). The Euclidean region for real  $x, y$  with  $0 < x < 1, 0 < y < 1$  is delimited by dashed lines. Note that this picture is a  $\mathbb{R}^2$  slice of  $\mathbb{C}^2$ . Chen iterated integrals are homotopy invariant (on the space where the singular points are removed).

orders of the  $\epsilon$  expansion of the master integrals involve only a subset of such functions, as a consequence of the specific start of the symbol iteration, see eq. (2.16). (Recall that the latter follows from the boundary conditions.) We will see an explicit example of this at two loops, where it turns out that all except one functions at weight four require a smaller function class compared to the all-order solution in  $\epsilon$ .

### 2.6 General solution in terms of Chen iterated integrals

In general, an iterative solution in  $\epsilon$  of an equation of the type of eq. (1.2) can be written in terms of Chen iterated integrals. We have

$$f(x, y, \epsilon) = \mathbb{P} e^{\epsilon \int_{\mathcal{C}} d\tilde{A}} g(\epsilon), \tag{2.21}$$

where  $g(\epsilon)$  is constant w.r.t.  $x$  and  $y$ , and represents the boundary conditions.  $\mathcal{C}$  is a contour in the (in general complex) space of kinematical variables connecting a chosen base point  $(x_0, y_0)$  to  $(x, y)$ .  $\mathbb{P}$  stands for path ordering of the matrix exponential. We see that this defines  $f$  at each order  $\epsilon^n$  as an  $n$ -fold integral of an  $n$ -form. The latter is determined from the matrix  $\tilde{A}$ . Note that the integrals are homotopy invariant, as long as  $\mathcal{C}$  stays away from the singularities of  $d\tilde{A}$ . Moreover, the monodromy properties of the answer are completely transparent, as they are encoded into  $\tilde{A}$ . (See [33, 34] for more details.) As we already mentioned, the boundary conditions can be obtained from physical considerations.

Some choices of contours  $\mathcal{C}$  are illustrated in figure 3 for real  $x, y$  in the Euclidean region  $1 > x > 0, 1 > y > 0$ . However, one should keep in mind that the full picture should be in  $\mathbb{C}^2$ . A concrete application of this will be discussed in section 3.5.

From a mathematical point of view, eq. (2.21) completely solves the problem of computing the master integrals in analytic form. This way of writing the solution is also very natural for analytic continuation. It is analogous to the definition of the logarithm for complex argument. Choosing specific integration contours  $\mathcal{C}$  one often recovers function classes more frequently used in the physics literature, as we discuss presently. We should caution the reader that these other representations, while useful in various regards, often

obscure properties that are manifest in the form (2.21), and suffer from the fact that they can be represented in many different but equivalent ways, essentially as a consequence of the homotopy invariance of eq. (2.21).

### 2.7 Solution in terms of Goncharov polylogarithms

Let us discuss relations of the representation of the general solution of the previous subsection to functions more frequently used in physics. We may notice that the arguments of logarithms in eq. (2.14) are linear in the variables, by virtue of the parametrization (2.2). (The same is not the case when using  $-s/m^2$  and  $-t/m^2$  as independent variables.) This fact allows us to immediately rewrite the general solution discussed above in terms of Goncharov polylogarithms. The latter are defined as follows,

$$G(a_1, \dots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \dots, a_n; t), \tag{2.22}$$

with

$$G(a_1; z) = \int_0^z \frac{dt}{t - a_1}, \quad a_1 \neq 0. \tag{2.23}$$

For  $a_1 = 0$ , we have  $G(\vec{0}_n; x) = 1/n! \log^n(x)$ . The total differential of a general Goncharov polylogarithm is

$$\begin{aligned} dG(a_1, \dots, a_n; z) &= G(\hat{a}_1, a_2, \dots, a_n; z) d \log \frac{z - a_1}{a_1 - a_2} \\ &+ G(a_1, \hat{a}_2, a_3, \dots, a_n; z) d \log \frac{a_1 - a_2}{a_2 - a_3} + \dots + \\ &+ G(a_1, \dots, a_{n-1}, \hat{a}_n; z) d \log \frac{a_{n-1} - a_n}{a_n}, \end{aligned} \tag{2.24}$$

where  $\hat{a}$  means that this element is omitted. As discussed in subsection 2.6, one can choose various integration contours  $\mathcal{C}$  in the solution (2.21). It is clear that upon choosing some linear parametrization one can represent all terms in eq. (2.21) as Goncharov polylogarithms with certain parameters  $a_i$ . Note that different choices of contour can give representations of the same iterated integral in terms of different sets of Goncharov polylogarithms. In this sense, the formulation in terms of Chen iterated integrals is more universal.

In the specific case at hand, we found it useful to follow an even more pedestrian approach, which we found to give a convenient form of the answer, as we discuss presently. At degree one, we obviously have the set of functions

$$g^{(1)} = \{\log x, \log(1 \pm x), \log y, \log(1 \pm y), \log(1 + xy), \log(x + y)\}. \tag{2.25}$$

Now, consider a degree two function  $f(x, y)$ . We can consider the  $\partial_y f(x, y)$  derivative. Its general form is dictated by the symbol alphabet. Integrating back, we find

$$f(x, y) = \int_0^y c_{i,j} \sum_{i,j} \left( \partial_y g_i^{(1)} \right) g_j^{(1)} + h(x). \tag{2.26}$$

For  $j = 1, 2, 3$  we have a factorized integral, so we exclude these cases, without loss of generality. In all other cases, we immediately see that we have the definition of a Goncharov

polylogarithm, with  $z = y$ , and indices  $a_j$  drawn from the set  $\{0, \pm 1, -x, -1/x\}$ . Moreover,  $h(x)$  should be given by harmonic polylogarithms (HPL) [45] (cf. appendix A.) It is easy to see that at higher orders, we will at most have  $\mathbb{Q}$ -linear combinations of products of the above types of functions. (Of course, one can derive an equivalent representation by focusing on the  $\partial_x$  derivatives.)

Note that this yields a representation that is manifestly real-valued for  $0 < x < 1, 0 < y < 1$ , i.e. in the Euclidean region. This is easy to see: thanks to  $0 < y < 1$ , the integration variables  $t$  satisfy  $0 < t < 1$ , and, given the set of allowed  $a_i$  relevant here, one always had  $t - a_i > 0$ .

The set of functions is slightly more general than the 2dHPLs introduced in [46, 47]. The reason is that in these references, the index  $-1$  was not required (also, w.r.t. their notation we have  $x \rightarrow -x, y \rightarrow -y$ ). However, this is a very natural generalization, just like in the case of HPLs, and it seems natural to continue using the name 2dHPLs for this slightly larger class of functions. The fact that these special functions appear in different problems makes them even more useful.

For example, here is our result for  $f_5$  up to degree three for illustration.

$$\begin{aligned}
 f_5 = & \epsilon \left[ 4H_0(x) \right] + \epsilon^2 \left[ 4G_0(y)H_0(x) - 8G_1(y)H_0(x) \right] \\
 & + \epsilon^3 \left[ -8G_0(y)H_{-1,0}(x) + 4G_0(y)H_{0,0}(x) - 8H_0(x)G_{1,0}(y) + 16H_0(x)G_{1,1}(y) \right. \\
 & + 4H_0(x)G_{-\frac{1}{x},0}(y) - 8H_0(x)G_{-\frac{1}{x},1}(y) + 4H_0(x)G_{-x,0}(y) - 8H_0(x)G_{-x,1}(y) \\
 & + 8H_{-1,0}(x)G_{-\frac{1}{x}}(y) + 8H_{-1,0}(x)G_{-x}(y) - 4H_{0,0}(x)G_{-\frac{1}{x}}(y) - 4H_{0,0}(x)G_{-x}(y) \\
 & + 4G_{-\frac{1}{x},0,0}(y) - 8G_{-\frac{1}{x},0,1}(y) - 4G_{-x,0,0}(y) + 8G_{-x,0,1}(y) + 8H_{-2,0}(x) \\
 & - 16H_{-1,-1,0}(x) + 8H_{-1,0,0}(x) - 4H_{0,0,0}(x) + \frac{10}{3}\pi^2 G_{-\frac{1}{x}}(y) - 2\pi^2 G_{-x}(y) \\
 & \left. - \frac{2}{3}\pi^2 G_0(y) - \frac{4}{3}\pi^2 H_{-1}(x) - \frac{7}{3}\pi^2 H_0(x) + 8\zeta_3 \right] + \mathcal{O}(\epsilon^4). \tag{2.27}
 \end{aligned}$$

We note that such formulas lend themselves well to numerical evaluation [39]. We therefore expect that this form of the answer should be sufficient for physical applications. Nevertheless, before closing this discussion, we wish to comment on another possibility of presenting the answer in yet another form.

### 2.8 Solution to a given weight in terms of multiple polylogarithms

The attentive reader will have noticed that some of the Goncharov polylogarithms in eq. (2.27) are just harmonic polylogarithms, up to signs. More generally, when writing out the solution to a given order in  $\epsilon$ , one may try to rewrite the answer in terms of a minimal number of functions. For example, to degree four, one can write the answers in terms of classical polylogarithms and one additional function  $\text{Li}_{2,2}$ , of certain arguments. Such a rewriting is not difficult, and examples can be found in many places in the literature, see e.g. [1]. A simple example at weight two is

$$G(-1/x, 0; y) = \log y \log(1 + xy) + \text{Li}_2(-xy). \tag{2.28}$$

In more complicated cases, the Hopf algebra structure of the integrals is helpful for finding such rewritings, see e.g. [48, 49]. We will not perform such a rewriting at fixed order in  $\epsilon$  here since it would distract from the general pattern of the solution. We do remark however that when physical quantities are computed, additional simplifications may occur, e.g. due to symmetries or other properties, and then such an analysis may be more worthwhile.

In the present case, such a rewriting is relatively straightforward for all except one functions at two loops up to weight four. We believe however that the formulation in terms of Goncharov polylogarithms, or more precisely the subset of 2dHPLs, of section 2.7 is more useful in practice. See also the related discussion in section 4.

### 3 Master integrals and differential equations at two loops

Having discussed the one-loop case in great detail in the previous section, we will find the two-loop analysis relatively straightforward.

#### 3.1 Choice of master integrals

Let us start by presenting our basis choice for the family of integrals corresponding to figure 1(2a). Explicitly, we have  $f_i = (m^2)^{2\epsilon} e^{\epsilon\gamma_E} g_i$ , where

$$g_1 = \epsilon^2 G_{2,0,2,0,0,0,0,0,0}, \tag{3.1}$$

$$g_2 = \epsilon^2 t G_{0,0,0,0,2,2,1,0,0}, \tag{3.2}$$

$$g_3 = \frac{\epsilon^2(4\epsilon + 1)}{\epsilon + 1} m^2 G_{0,0,1,0,2,2,0,0,0}, \tag{3.3}$$

$$g_4 = \epsilon^2 \sqrt{-s} \sqrt{4m^2 - s} G_{1,2,2,0,0,0,0,0,0}, \tag{3.4}$$

$$g_5 = \frac{1}{2} \epsilon^2 \sqrt{-s} \sqrt{4m^2 - s} (2G_{0,1,2,0,0,2,0,0,0} + G_{0,2,2,0,0,1,0,0,0}), \tag{3.5}$$

$$g_6 = \epsilon^2 s G_{0,2,2,0,0,1,0,0,0}, \tag{3.6}$$

$$g_7 = -2\epsilon^3 \sqrt{-s} \sqrt{4m^2 - s} G_{0,0,1,1,1,2,0,0,0}, \tag{3.7}$$

$$g_8 = -2\epsilon^3 \sqrt{-s} \sqrt{4m^2 - s} G_{0,1,2,0,1,1,0,0,0}, \tag{3.8}$$

$$g_9 = -2\epsilon^3 \sqrt{-t} \sqrt{4m^2 - t} G_{1,0,0,0,1,1,2,0,0}, \tag{3.9}$$

$$g_{10} = \epsilon^2 s (4m^2 - s) G_{1,2,1,2,0,0,0,0,0}, \tag{3.10}$$

$$g_{11} = 4\epsilon^4 \sqrt{-s-t} \sqrt{4m^2 - s-t} G_{0,1,1,0,1,1,1,0,0}, \tag{3.11}$$

$$g_{12} = -2\epsilon^3 \sqrt{-s} \sqrt{4m^2 - s} t G_{0,1,1,0,1,2,1,0,0} \tag{3.12}$$

$$g_{13} = -2\epsilon^3 m^2 \sqrt{-t} \sqrt{4m^2 - t} G_{0,2,1,0,1,1,1,0,0}, \tag{3.13}$$

$$g_{14} = -\frac{1}{2} \epsilon^2 t (\epsilon G_{0,1,1,0,1,2,1,0,0} (2m^2 - s) - 2m^2 (m^2 G_{0,2,2,0,1,1,1,0,0} - 2\epsilon G_{0,2,1,0,1,1,1,0,0})), \tag{3.14}$$

$$g_{15} = 4\epsilon^4 \sqrt{-s} \sqrt{4m^2 - s} G_{1,0,1,1,1,1,0,0,0}, \tag{3.15}$$

$$g_{16} = -\frac{1}{4} \epsilon^2 (4m^2 - s) \left( -8\epsilon^2 G_{1,0,1,1,1,1,0,0,0} + 8\epsilon m^2 G_{1,0,1,2,1,1,0,0,0} + 8\epsilon m^2 G_{1,0,2,1,1,1,0,0,0} + 4\epsilon G_{0,0,1,1,1,2,0,0,0} - 2\epsilon G_{0,1,2,0,1,1,0,0,0} + 2G_{0,1,2,0,0,2,0,0,0} + G_{0,2,2,0,0,1,0,0,0} \right), \tag{3.16}$$

$$g_{17} = 4\epsilon^4 \sqrt{-t} \sqrt{4m^2 - t} G_{1,0,1,0,1,1,1,0,0}, \tag{3.17}$$

$$g_{18} = \epsilon \left[ t (2\epsilon^3 G_{1,0,1,0,1,1,1,0,0} + (2\epsilon + 1) m^4 G_{1,0,2,0,2,1,1,0,0}) + 2\epsilon^2 G_{1,0,0,0,1,1,2,0,0} (3m^2 - t) + \epsilon m^2 G_{0,0,1,0,2,2,0,0,0} \right], \quad (3.18)$$

$$g_{19} = -2\epsilon^3 \sqrt{-s} \sqrt{4m^2 - st} G_{1,1,0,0,1,1,2,0,0}, \quad (3.19)$$

$$g_{20} = 4\epsilon^4 \sqrt{-s} \sqrt{-t} \sqrt{4m^2 - s} \sqrt{4m^2 - t} G_{1,1,1,0,1,1,1,0,0}, \quad (3.20)$$

$$g_{21} = \epsilon^2 \sqrt{-s} \sqrt{4m^2 - s} \left( -8\epsilon^2 t G_{1,1,1,0,1,1,1,0,0} + 8\epsilon^2 G_{0,1,1,0,1,1,1,0,0} - 4\epsilon(2\epsilon - 1) G_{1,1,1,0,1,1,1,-1,0} \right), \quad (3.21)$$

$$g_{22} = 4\epsilon^4 st (4m^2 - s) G_{1,1,1,1,1,1,1,0,0}, \quad (3.22)$$

$$g_{23} = 4\epsilon^4 s (4m^2 - s) G_{1,1,1,1,1,1,1,-1,0}. \quad (3.23)$$

The corresponding Feynman diagrams can be found in figure 4.

### 3.2 System of differential equations

We find that the basis choice of eqs. (3.1)–(3.23) puts the differential equations into canonical form, i.e.

$$df = \epsilon d\tilde{A} f. \quad (3.24)$$

Here  $f$  contains 23 integrals, and  $\tilde{A}$  is a  $23 \times 23$  matrix. The matrix  $\tilde{A}$  can be put into the following form,

$$\begin{aligned} \tilde{A} = & B_1 \log(x) + B_2 \log(1+x) + B_3 \log(1-x) + B_4 \log(y) + B_5 \log(1+y) \\ & + B_6 \log(1-y) + B_7 \log(x+y) + B_8 \log(1+xy) \\ & + B_9 \log(x+y-4xy+x^2y+xy^2) + B_{10} \log\left(\frac{1+Q}{1-Q}\right) \\ & + B_{11} \log\left(\frac{(1+x)+(1-x)Q}{(1+x)-(1-x)Q}\right) + B_{12} \log\left(\frac{(1+y)+(1-y)Q}{(1+y)-(1-y)Q}\right). \end{aligned} \quad (3.25)$$

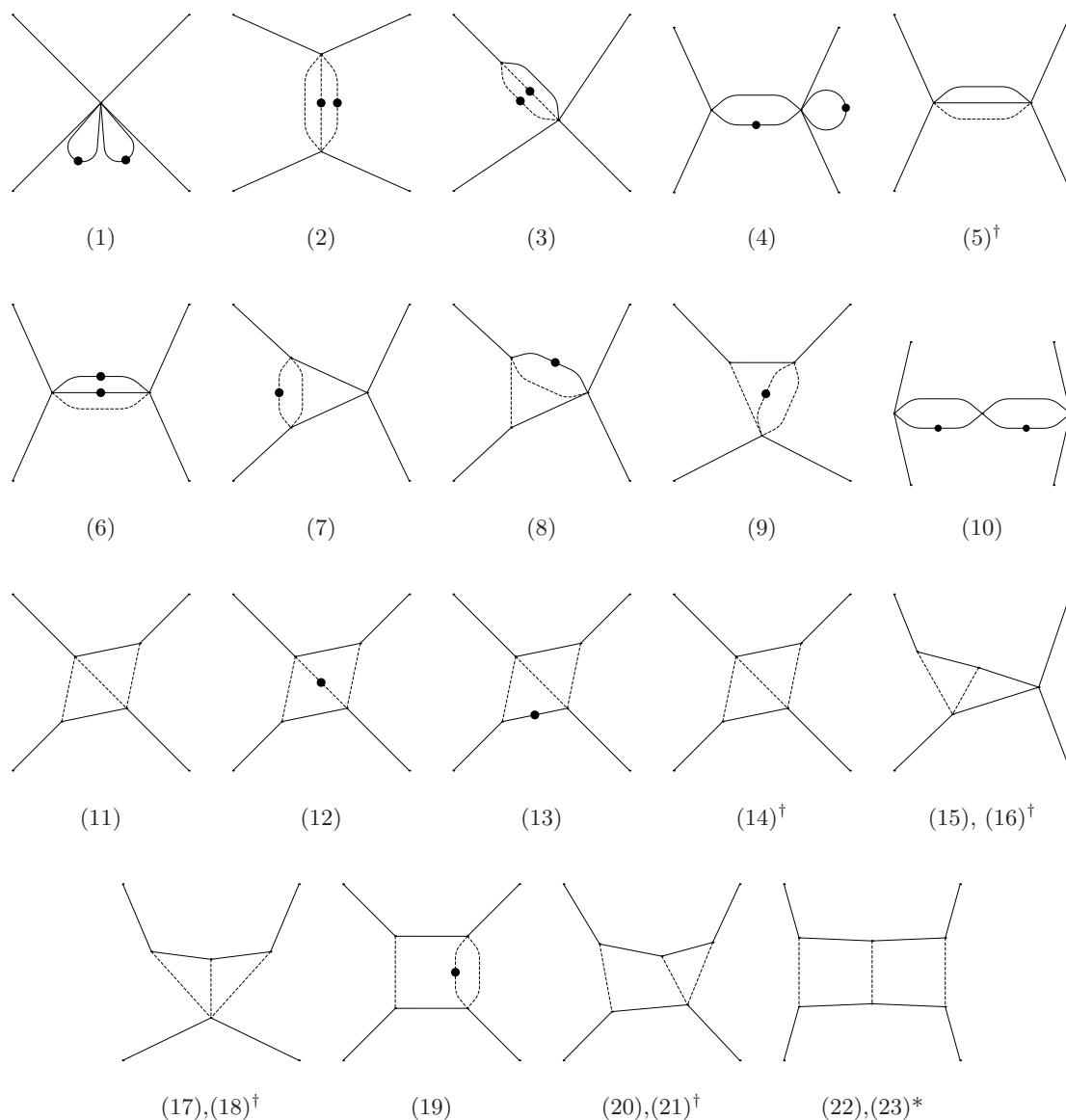
Here

$$Q = \sqrt{\frac{(x+y)(1+xy)}{x+y-4xy+x^2y+xy^2}}, \quad (3.26)$$

and the  $B_i$  are constant matrices, see appendix B.

It is important to note that the matrix  $\tilde{A}$  only contains logarithmic singularities, so that we can expect the answer to be expressed in terms of iterated integrals.

We see that w.r.t. the one-loop case, the four terms in the last two lines of eq. (3.25) are new. Analyzing the arguments of the logarithms we find that most of them only have singularities at  $x = 0$ ,  $y = 0$ ,  $x = 1$ ,  $y = 1$ . However, the coefficient of  $B_9$  has additional singularities that take a more complicated form in terms of  $x$  and  $y$ . In their original variables, they correspond to  $s = -t$ . It is worth pointing out that the matrix  $B_9$  only has one non-zero entry, see eq. (B.9), and this symbol entry will appear first at degree five for integral 11, and at higher degrees for other integrals.



**Figure 4.** Master integrals for integral family (2a) of Bhabha integrals. Dots denote doubled propagators. An asterisk indicates numerators not shown in the picture, and a dagger indicates that the master integral is a linear combinations of integrals (possibly with dots), that are not shown.

### 3.3 Solution at symbol level

We find that we can determine all integration constants from physical conditions, up to a few trivial propagator-type integrals. In particular, we find for the first order in the  $\epsilon$  expansion

$$f^{(0)} = \{1, -1, -1/2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1/2, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0\}. \quad (3.27)$$

This, together with the differential equation (3.24) and (3.25), completely specifies the symbol.

Before writing down the explicit solution for the basis integrals, we find it useful to analyze the symbols of the functions. The reason is that although one would expect that all terms appearing in eq. (3.25) eventually enter the r.h.s. of the differential equations for the integrals with many propagators, the matrices  $B_i$  are sparse matrices with many zero entries. It is therefore possible that, at least to some low order in the  $\epsilon$  expansion, not all of the structures present in general will be needed.

Having physical applications at NNLO in mind, we explicitly analyze this question up to degree four. We can make the following structural observations:

- At order  $\epsilon, \epsilon^2$  and  $\epsilon^3$ , the symbol alphabet remains the same as at one loop.
- At order  $\epsilon^4$  all functions except  $f_{11}$  have the same symbol alphabet as at one loop.

In other words, most of the new symbol entries, corresponding to matrices  $B_9, B_{10}, B_{11}, B_{12}$  given in eqs. (B.9)–(B.12), are not needed for physical two-loop results! (It would be interesting to know whether in our basis, the weight four part of integral 11 is needed in physical calculations.)

### 3.4 Explicit results in terms of Goncharov polylogarithms

Having observed that the symbol alphabet for all functions except for  $f_{11}$  is the same as at one loop, all the observations made there apply immediately to this case. In particular, as explained in section 2.7, we can write the analytic answer in terms of certain Goncharov polylogarithms. One may call these functions two-dimensional harmonic polylogarithms.

All analytic results are collected in the ancillary file `Results_2a.m` which is supplied with the paper. Here, for illustration, we present the result for integral  $f_{23}$ , up to weight four. It reads

$$\begin{aligned}
 f_{23} = & \epsilon^2 \left[ -12H_{0,0}(x) \right] + \epsilon^3 \left[ -16G_0(y)H_{0,0}(x) + 32G_1(y)H_{0,0}(x) + 8H_{2,0}(x) \right. \\
 & \left. + 16H_{-1,0,0}(x) - 4H_{0,0,0}(x) + \frac{4}{3}\pi^2 H_0(x) + 4\zeta_3 \right] + \epsilon^4 \left[ 32G_0(y)H_{-2,0}(x) \right. \\
 & - 32H_{-2,0}(x)G_{-\frac{1}{x}}(y) - 32H_{-2,0}(x)G_{-x}(y) + 64G_{1,0}(y)H_{0,0}(x) - 128G_{1,1}(y)H_{0,0}(x) \\
 & - 32H_{0,0}(x)G_{-\frac{1}{x},0}(y) + 64H_{0,0}(x)G_{-\frac{1}{x},1}(y) - 32H_{0,0}(x)G_{-x,0}(y) \\
 & + 64H_{0,0}(x)G_{-x,1}(y) - 16H_0(x)G_{-\frac{1}{x},0,0}(y) + 32H_0(x)G_{-\frac{1}{x},0,1}(y) \\
 & + 16H_0(x)G_{-x,0,0}(y) - 32H_0(x)G_{-x,0,1}(y) + 64G_0(y)H_{-1,0,0}(x) \\
 & - 64H_{-1,0,0}(x)G_{-\frac{1}{x}}(y) - 64H_{-1,0,0}(x)G_{-x}(y) \\
 & - 48G_0(y)H_{0,0,0}(x) + 48H_{0,0,0}(x)G_{-\frac{1}{x}}(y) + 48H_{0,0,0}(x)G_{-x}(y) - 120H_{-3,0}(x) \\
 & + \frac{52}{3}\pi^2 H_{0,0}(x) + 48H_{3,0}(x) + 128H_{-2,-1,0}(x) - 120H_{-2,0,0}(x) - 48H_{-2,1,0}(x) \\
 & + 64H_{-1,-2,0}(x) - 32H_{-1,2,0}(x) - 48H_{2,-1,0}(x) + 32H_{2,0,0}(x) + 16H_{2,1,0}(x) \\
 & + 64H_{-1,-1,0,0}(x) - 80H_{-1,0,0,0}(x) + 76H_{0,0,0,0}(x) + \frac{8}{3}\pi^2 G_0(y)H_0(x) \\
 & - \frac{40}{3}\pi^2 H_0(x)G_{-\frac{1}{x}}(y) + 8\pi^2 H_0(x)G_{-x}(y) - 16\zeta_3 H_{-1}(x) - 28\zeta_3 H_0(x) \\
 & \left. + \frac{8}{3}\pi^2 H_{-2}(x) - \frac{4}{3}\pi^2 H_2(x) - \frac{4\pi^4}{15} \right] + \mathcal{O}(\epsilon^5) . \tag{3.28}
 \end{aligned}$$



This formula may be evaluated numerically using standard tools available in the literature, see e.g. [39].

We checked our results against various formulas available in the literature, e.g. [10]. A particularly non-trivial test the agreement with the result of [8] for integral  $f_{22}$ , since the differential equations for this integral involve most other integrals. We also performed numerical checks in the Euclidean region by the implementation of sector decomposition [50, 51] in FIESTA [52, 53], and analytic checks in asymptotic limits that were obtained from Mellin-Barnes representations.

### 3.5 Explicit parametrization for Chen iterated integral

In the previous paragraph, we have written down an explicit solution for all integrals except for  $f_{11}$  up to degree four. We also explained, in section 2.6, how to write the general solution to the two-loop problem in terms of Chen iterated integrals.

Let us exemplify this using  $f_{11}$ . The basis integral is finite as  $\epsilon \rightarrow 0$ , which explains why it first appears at order  $\epsilon^4$  in our normalization.

Using the explicit results for all other integrals, and expanding eq. (3.24) to fourth order in  $\epsilon$ , we find that  $f_{11}^{(4)}$  satisfies the following differential equation,

$$d f_{11}^{(4)} = g_1 d \log \left( \frac{1-Q}{1+Q} \right) + g_2 d \log \left( \frac{(1+x)+(1-x)Q}{(1+x)-(1-x)Q} \right) + g_3 d \log \left( \frac{(1+y)+(1-y)Q}{(1+y)-(1-y)Q} \right), \quad (3.29)$$

where the three functions  $g_1, g_2, g_3$  are given explicitly by

$$g_1 = -16G_0(y)H_{0,0}(x) + 32G_1(y)H_{0,0}(x) + 8G_{0,0,0}(y) - 16G_{0,0,1}(y) - 32H_{-2,0}(x) + 8H_{0,0,0}(x) - 16H_{1,0,0}(x) + \frac{8}{3}\pi^2 G_0(y) + \frac{16}{3}\pi^2 G_1(y) - \frac{8}{3}\pi^2 H_0(x) - 32\zeta_3, \quad (3.30)$$

$$g_2 = -16H_{0,0,0}(x) - \frac{8}{3}\pi^2 H_0(x), \quad (3.31)$$

$$g_3 = 8H_0(x)G_{-\frac{1}{x},0}(y) - 16H_0(x)G_{-\frac{1}{x},1}(y) - 8H_0(x)G_{-x,0}(y) + 16H_0(x)G_{-x,1}(y) + 16H_{-1,0}(x)G_{-\frac{1}{x}}(y) - 16H_{-1,0}(x)G_{-x}(y) - 8G_0(y)H_{0,0}(x) - 8H_{0,0}(x)G_{-\frac{1}{x}}(y) + 8H_{0,0}(x)G_{-x}(y) + 8G_{-\frac{1}{x},0,0}(y) - 16G_{-\frac{1}{x},0,1}(y) + 8G_{-x,0,0}(y) - 16G_{-x,0,1}(y) - 16G_{-1,0,0}(y) + 32G_{-1,0,1}(y) - 16H_{-2,0}(x) + 8H_{0,0,0}(x) + \frac{20}{3}\pi^2 G_{-\frac{1}{x}}(y) + 4\pi^2 G_{-x}(y) - \frac{32}{3}\pi^2 G_{-1}(y) - \frac{4}{3}\pi^2 H_0(x) - 24\zeta_3. \quad (3.32)$$

They can also be written in terms of classical polylogarithms, in a form that is manifestly real-valued in the region  $0 < x < 1, 0 < y < 1$ .

At  $x = 1$ , equation (3.29) simplifies, and integrating back we find

$$f_{11}^{(4)}(x = 1, y) = \frac{8}{3}\pi^2 H_{0,0}(y) + 8H_{0,0,0,0}(y) - \frac{16}{3}\pi^2 H_2(y) + 16H_4(y) + \frac{32\pi^4}{45}. \quad (3.33)$$

Note that in particular, at  $y = 1$ , we have  $f_{11}^{(4)}(x = 1, y = 1) = 0$ . We may also note that  $f^{(4)}$  vanishes for  $x = -y$ .

As we reviewed in section 2.6, we can always write the solution to an equation like (3.29) in terms of Chen iterated integrals [32].<sup>6</sup> In order to do this, we choose a base point (corresponding to a boundary condition) in the kinematical space.

We may choose different base points depending on the application we have in mind. For example, for numerical evaluation in the Euclidean region  $0 < x < 1, 0 < y < 1$ , choosing  $(x, y) = (1, 1)$  as a base point is natural, and we have  $f_{11}^{(4)}(x = 1, y = 1) = 0$ .

To evaluate  $f_{11}^{(4)}(x_f, y_f)$  at some point  $(x_f, y_f)$ , we integrate the differential form of eq. (3.29) along a contour. The contour is chosen such that it avoids singularities and branch cuts. For values  $0 < x_f < 1, 0 < y_f < 1$ , a possible choice for the contour  $\mathcal{C}$  is, cf. figure 3,

$$x(\tau) = (1 - \tau) + \tau x_f, \quad y(\tau) = (1 - \tau) + \tau y_f. \quad (3.34)$$

Then, we have

$$f_{11}^{(4)}(x_f, y_f) = \int_{\mathcal{C}} d f_{11}^{(4)}. \quad (3.35)$$

Of course, one can also choose more complicated contours.

In this way, we can easily obtain numerical values in the region  $0 < x < 1, 0 < y < 1$ . (For other region, one has to define a contour that avoids the branch cuts.) For example, we find

$$f_{11}^{(4)}(1/3, 2/5) = 108.10505928259012, \quad (3.36)$$

$$f_{11}^{(4)}(6/7, 3/5) = 46.46787346208666, \quad (3.37)$$

$$f_{11}^{(4)}(3/8, 7/9) = 130.0624797134173. \quad (3.38)$$

For numerical evaluation in the physical region  $x > 0, y < 0$  (or  $x > 0, y < 0$ ), it is convenient to choose a different base point.  $x = -y$ , e.g.  $x = 1, y = -1$  is a good choice. In this way, we straightforwardly obtain numerical values in that region.

A similar type of integral representation was used in ref. [54] for the evaluation of two-loop six-point amplitudes in  $\mathcal{N} = 4$  SYM.

In this context, it may be interesting to note the approach of ref. [55], where differential equations for loop integrals are integrated numerically. The canonical form of the differential equations (1.2) that we derived can also be beneficial for increased speed and stability in such a purely numerical approach. We stress again that eq. (1.2) makes the singularity structure of the integrals completely transparent, and this can be useful in order to avoid spurious singularities.

It would be interesting to know whether the weight four function  $f_{11}^{(4)}$  can be rewritten in terms of classical polylogarithms and  $\text{Li}_{2,2}(x, y)$ , see the discussion of section 2.8. Unfortunately, even if such a representation does exist, it is by no means guaranteed that the individual terms in such an expression have a symbol alphabet given by eq. (3.25). Instead, it could be that ‘spurious’ symbol entries are needed for such a representation, with the latter canceling in the complete answer. In that case, the main difficulty lies in finding these spurious symbol entries in a systematic way.

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<sup>6</sup>Strictly speaking, the answer below is given in a hybrid formulation, since we are using the 2dHPL form of the functions  $g_1, g_2, g_3$ . We could of course represent the latter also as integrals over logarithmic differential forms.

## 4 Discussion and outlook

In this paper, we analytically computed the master integrals for all Feynman integrals of the family of eq. (1.1), corresponding to figure 1(2a). This completes the previous partial results for some of the integrals available in the literature. We wrote our results explicitly up to weight four, which is sufficient for NNLO applications. We also identified the class of functions needed to any order in  $\epsilon$ , and higher order results can be obtained by expanding the general solution of eq. (2.21), taking into account the boundary conditions discussed in section 2.2. Moreover, as a corollary, the symbol at any order in  $\epsilon$  can be trivially obtained, as explained in section 3.3.

We would like to discuss further the nature of our results for the master integrals. Let us observe that, usually, a result for some function, e.g. a Feynman integral, is characterized as analytical if it is explicitly expressed in terms of known functions. By known functions we usually mean elementary functions, and sets of special functions whose properties are well studied, and that arise in many different problems. For a discussion of this topic, see [56]. The latter criteria are obviously important, because otherwise one could introduce a new special function for each problem, and declare the problem solved. Clearly, this would not be satisfactory. What can we say about our results for the master integrals from this point of view?

We would like to argue that Chen iterated integrals can be considered as a class of special functions that appear in many physical problems, particularly in Feynman integrals. They are well studied in the mathematical literature, and important properties, such as their singularity structure, monodromies, differential equations, functional relations, can be studied in a simple way. They also appear in other areas, such as the theory of knot invariants [57]. In the case where the logarithmic integration forms have rational arguments, one obtains functions that arise frequently in quantum field theory computations, such as, in decreasing order of complexity, Goncharov polylogarithms, 2dHPLs, HPLs, polylogarithms, and logarithms.

Just like many other special functions, these functions appear as the solution of a class of differential equations. This is also the context in which they appear here. Let us emphasize that these iterated integrals form exactly the class of functions needed to describe the given family of Feynman integrals, to all orders in  $\epsilon$ . Therefore they are the natural language to discuss them.

We should mention, as discussed in section 2.8, that if one is interested in writing down the solution to a given order in  $\epsilon$ , one can try to write the solution in terms of a minimal number of functions, such as classical polylogarithms for degree two, at the cost of having more complicated arguments of those functions. While it is certainly desirable to write an answer in terms of the smallest possible number of building blocks, we wish to point out that such a rewriting may make less obvious some of the properties that are transparent in the more general language of iterated integrals. In particular, there is no canonical way of presenting the answer due to numerous identities between polylogarithms of different arguments, see e.g. [58]. This is probably the reason why in the theoretical physics literature on Feynman integrals, such a rewriting is often not performed, even in the cases where it is rather straightforward.

Given the physical relevance of the class of functions defined from the symbol alphabet (2.15) in this and other problems, it would be interesting to construct a set of single-valued functions of this type. This is a topic that is very well-studied in the mathematical literature, see [33, 34] and references therein. Likewise, one could study the generalization to (3.25) from the same point of view. As we emphasized, this case is the appropriate class of functions for the solution to all orders in  $\epsilon$ .

In this context we wish to mention that there is a choice of variables that suggests that this class of functions is natural. As in [7] we can introduce a third variable  $z$  in analogy with eq. (2.2), namely  $-u/m^2 = (1-z)^2/z$ . The variables  $x, y, z$  are of course constrained due to  $u = -s - t + 4m^2$ , which becomes

$$\frac{(1-x)^2}{x} + \frac{(1-y)^2}{y} + \frac{(1-z)^2}{z} + 4 = 0. \tag{4.1}$$

Using these constrained variables, the symbol alphabet takes the following simple form,

$$\{x, 1 \pm x, y, 1 \pm y, z, 1 \pm z, x + y, 1 + xy, x + z, 1 + xz, y + z, 1 + yz\}. \tag{4.2}$$

Finally, we wish to mention that yet another equivalent representation is obtained by letting  $x = (1-z_1)/(1+z_1), y = (1-z_2)/(1+z_2), z = (1-z_3)/(1+z_3)$ . In this case, the constraint (4.1) takes the following form,

$$1 + \sum_{i=1}^3 \frac{z_i^2}{(1-z_i^2)} = 0 \tag{4.3}$$

and the symbol alphabet becomes

$$\{z_i, 1 \pm z_i, 1 \pm z_i z_j\}, \tag{4.4}$$

for  $i, j \in \{1, 2, 3\}$ . It will be interesting to see whether this more general case is also relevant in other situations.

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## A Definition of harmonic polylogarithms

The harmonic polylogarithms [45]  $H_{a_1, a_2, \dots, a_n}(x)$  (HPL), with  $a_i \in \{1, 0, -1\}$ , are defined recursively by

$$H_{a_1, a_2, \dots, a_n}(x) = \int_0^x f_{a_1}(t) H_{a_2, \dots, a_n}(t) dt, \tag{A.1}$$

where

$$f_{\pm 1}(x) = \frac{1}{1 \mp x}, \quad f_0(x) = \frac{1}{x}, \tag{A.2}$$

$$H_{\pm 1}(x) = \mp \log(1 \mp x), \quad H_0(x) = \log x, \tag{A.3}$$

and at least one of the indices  $a_i$  is non-zero. For all  $a_i = 0$ , one has

$$H_{\underbrace{0,0,\dots,0}_n}(x) = \frac{1}{n!} \log^n x. \quad (\text{A.4})$$

## B Definition of the matrices $B_i$

Here we present the non-zero elements of the matrices  $B_i$  of eq. (3.25). It is worth noting that all matrices are sparse, i.e. most of their entries are zero. For brevity of notation, we will denote their entries for each case by  $a_{ij}$ , hoping that this will not lead to confusion. For  $B_1$ , we have:

$$\begin{aligned} a_{4,1} &= -1, a_{4,4} = 1, a_{5,1} = 1, a_{5,5} = 4, a_{5,6} = 3, a_{6,5} = -2, a_{6,6} = -1, a_{7,3} = -2, \\ a_{7,7} &= 2, a_{8,1} = -2, a_{8,3} = -4, a_{8,6} = -4, a_{8,8} = -2, a_{10,4} = 2, a_{10,10} = 2, a_{11,11} = -6, \\ a_{12,2} &= 6, a_{12,12} = 2, a_{12,14} = -8, a_{13,13} = -2, a_{14,8} = 3/2, a_{14,12} = 1/2, a_{15,6} = -2, \\ a_{15,8} &= 1, a_{15,15} = -1, a_{15,16} = 4, a_{16,4} = 1, a_{16,5} = -1, a_{16,7} = 1, a_{16,8} = 1, \\ a_{16,15} &= -1/2, a_{16,16} = 2, a_{19,2} = 3, a_{20,13} = 4, a_{21,1} = -4, a_{21,2} = -8, a_{21,3} = -16, \\ a_{21,4} &= -4, a_{21,6} = -16, a_{21,8} = 4, a_{21,14} = 16, a_{22,12} = 4, a_{22,19} = -8, a_{23,2} = 6, \\ a_{23,3} &= 4, a_{23,4} = -4, a_{23,5} = 4, a_{23,7} = -4, a_{23,8} = 4, a_{23,10} = -4, a_{23,14} = -8, \\ a_{23,15} &= 2, a_{23,16} = 4, a_{23,18} = -4, a_{23,19} = 8/3, a_{23,21} = -2, a_{23,22} = 1, a_{23,23} = 3. \end{aligned} \quad (\text{B.1})$$

For  $B_2$ :

$$\begin{aligned} a_{4,4} &= -2, a_{5,5} = -6, a_{7,7} = -4, a_{8,8} = 2, a_{10,10} = -4, a_{12,5} = -8, a_{12,12} = -6, \\ a_{15,15} &= 2, a_{16,16} = -4, a_{19,7} = -6, a_{19,19} = -2, a_{20,20} = -2, a_{21,15} = 8, a_{21,21} = -2, \\ a_{22,10} &= -16, a_{22,16} = 16, a_{22,22} = -4, a_{22,23} = -4, a_{23,10} = 16, a_{23,23} = -4. \end{aligned} \quad (\text{B.2})$$

For  $B_3$ :

$$\begin{aligned} a_{5,5} &= -2, a_{6,6} = 2, a_{8,8} = 2, a_{11,11} = -4, a_{12,12} = -2, a_{13,9} = -3, \\ a_{13,13} &= 6, a_{15,8} = -2, a_{21,8} = -4. \end{aligned} \quad (\text{B.3})$$

For  $B_4$ :

$$\begin{aligned} a_{2,2} &= 2, a_{9,2} = -1, a_{9,3} = 2, a_{9,9} = 1, a_{11,11} = -6, a_{12,5} = -4, a_{13,1} = 1, \\ a_{13,2} &= 3/2, a_{13,3} = 3, a_{13,6} = 3, a_{13,9} = -3/2, a_{13,13} = 4, a_{13,14} = -4, a_{14,9} = 3/2, \\ a_{14,13} &= -2, a_{14,14} = 2, a_{17,1} = -2, a_{17,2} = 2, a_{17,17} = -2, a_{17,18} = -4, a_{18,3} = -2, \\ a_{18,9} &= -1, a_{18,17} = 1, a_{18,18} = 2, a_{19,7} = -3, a_{19,19} = 1, a_{20,7} = 2, a_{20,15} = 2, \\ a_{20,19} &= -2/3, a_{20,20} = 2, a_{20,21} = -1, a_{21,4} = -8, a_{21,15} = 4, a_{21,20} = 4, a_{21,21} = -2, \\ a_{22,10} &= -8, a_{22,16} = 8, a_{22,23} = -2, a_{23,10} = 8. \end{aligned} \quad (\text{B.4})$$

For  $B_5$ :

$$a_{9,9} = 2, a_{13,13} = -2, a_{17,17} = 2, a_{20,20} = -2. \quad (\text{B.5})$$

For  $B_6$ :

$$\begin{aligned}
 a_{2,2} &= -4, a_{9,9} = -4, a_{11,11} = -4, a_{12,12} = -4, a_{13,13} = -4, a_{14,14} = -4, \\
 a_{17,17} &= 2, a_{18,3} = 4, a_{18,18} = -4, a_{19,19} = -4, a_{20,20} = -4, a_{21,4} = 8, \\
 a_{21,8} &= 4, a_{21,21} = 2, a_{22,22} = -4, a_{23,2} = 12, a_{23,3} = 8, a_{23,10} = -8, \\
 a_{23,14} &= -16, a_{23,16} = 8, a_{23,18} = -8, a_{23,22} = 2, a_{23,23} = 2.
 \end{aligned} \tag{B.6}$$

For  $B_7$ :

$$\begin{aligned}
 a_{11,11} &= 3, a_{12,5} = 4, a_{12,9} = 6, a_{12,12} = 2, a_{12,13} = -4, a_{13,5} = 1, \\
 a_{13,9} &= 3/2, a_{13,12} = 1/2, a_{13,13} = -1, a_{19,7} = 3, a_{19,9} = 3, a_{19,19} = 1, \\
 a_{20,2} &= -3, a_{20,3} = -2, a_{20,14} = 4, a_{20,16} = -4, a_{20,18} = 2, a_{20,20} = 1, \\
 a_{21,4} &= 4, a_{21,8} = -2, a_{21,15} = -4, a_{21,17} = -2, a_{21,21} = 1, a_{22,10} = 8, \\
 a_{22,16} &= -8, a_{22,20} = -4, a_{22,22} = 2, a_{22,23} = 2, a_{23,2} = -6, a_{23,3} = -4, \\
 a_{23,10} &= -4, a_{23,14} = 8, a_{23,16} = -4, a_{23,18} = 4, a_{23,20} = 4, \\
 a_{23,22} &= -1, a_{23,23} = -1.
 \end{aligned} \tag{B.7}$$

For  $B_8$ :

$$\begin{aligned}
 a_{11,11} &= 3, a_{12,5} = 4, a_{12,9} = -6, a_{12,12} = 2, a_{12,13} = 4, a_{13,5} = -1, \\
 a_{13,9} &= 3/2, a_{13,12} = -1/2, a_{13,13} = -1, a_{19,7} = 3, a_{19,9} = -3, a_{19,19} = 1, \\
 a_{20,2} &= 3, a_{20,3} = 2, a_{20,14} = -4, a_{20,16} = 4, a_{20,18} = -2, a_{20,20} = 1, \\
 a_{21,4} &= 4, a_{21,8} = -2, a_{21,15} = -4, a_{21,17} = 2, a_{21,21} = 1, a_{22,10} = 8, \\
 a_{22,16} &= -8, a_{22,20} = 4, a_{22,22} = 2, a_{22,23} = 2, a_{23,2} = -6, a_{23,3} = -4, \\
 a_{23,10} &= -4, a_{23,14} = 8, a_{23,16} = -4, a_{23,18} = 4, a_{23,20} = -4, \\
 a_{23,22} &= -1, a_{23,23} = -1.
 \end{aligned} \tag{B.8}$$

For  $B_9$ :

$$a_{11,11} = 5. \tag{B.9}$$

For  $B_{10}$ :

$$a_{11,1} = 2, a_{11,2} = 4, a_{11,3} = 4, a_{11,6} = 4, a_{11,14} = -8, a_{14,11} = 3/2. \tag{B.10}$$

For  $B_{11}$ :

$$a_{11,8} = -2, a_{21,11} = 6. \tag{B.11}$$

Finally, for  $B_{12}$ , we have:

$$a_{11,9} = -2, a_{11,13} = 4, a_{13,11} = -3/2. \tag{B.12}$$

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